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# Optimal Weighted Poincaré and Log-Sobolev Inequalities for Cauchy Measures <sup>\*</sup>

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## Abstract

In this paper, We establish the weighted Poincaré inequalities and Log-Sobolev inequalities for Cauchy distributions with optimal weight functions.

**Keywords:** Cauchy measure, weighted Poincaré inequality, weighted Log-Sobolev inequality.

**AMS Classification Subjects 2000:** 60E15 39B62 26Dxx

## 1 Introduction

A Borel probability  $\mu$  on  $\mathbb{R}^n$  is said to satisfy a weighted *Poincaré inequality* with weight function  $\omega^2$  (where  $\omega$  is a fixed non-negative Borel measurable function), if there exists a constant  $C > 0$  such that for every smooth function  $f : \mathbb{R}^n \rightarrow R$  with gradient  $\nabla f$ ,

$$\text{Var}_\mu(f) \leq C \int |\nabla f|^2 \omega^2 d\mu,$$

where the variance of  $f$  w.r.t.  $\mu$  is defined by

$$\text{Var}_\mu(f) = \int \left( f - \int f d\mu \right)^2 d\mu.$$

Similarly, we say that a Borel probability  $\mu$  on  $\mathbb{R}^n$  satisfies a weighted *logarithmic Sobolev inequality* with weight function  $\omega^2$ , if there exists a constant  $C > 0$  such that for every smooth function  $f : \mathbb{R}^n \rightarrow R$ ,

$$\text{Ent}_\mu(f^2) \leq C \int |\nabla f|^2 \omega^2 d\mu,$$

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where the entropy is defined by

$$\text{Ent}_\mu(f^2) = \int f^2 \log f^2 d\mu - \left( \int f^2 d\mu \right) \log \left( \int f^2 d\mu \right).$$

The weighted functional inequalities are relatively weak inequalities. Although they can't deduce the exponential concentration of a probability measure for sure, they can also show some degree of decay for this measure, as can be seen in [5].

Consider the Cauchy measures:

$$d\mu_\beta(x) = \frac{(1 + |x|^2)^{-\beta}}{c(n, \beta)} dx$$

where  $\beta > \frac{n}{2}$ , and  $c(n, \beta)$  is the normalizing constant.

In [5], Bobkov and Ledoux proved that the Cauchy measures  $\mu_\beta$  admit a weighted Poincaré inequality with the weight  $1 + |x|^2$  and the constant  $\frac{(\sqrt{1+\frac{2}{\beta-2}} + \sqrt{\frac{2}{\beta-2}})^2}{2(\beta-1)}$ , as well as a weighted log-Sobolev inequalities with the weight  $(1 + |x|^2)^2$  and the constant  $\frac{1}{\beta-1}$ .

Comparing with the results in [5], the better weight functions are found by Hebisch and Zegarliński (in [13]). However, the authors just prove that the constants exist and are independent of  $\beta$ .

The aim of this paper is to give the optimal weight functions and the corresponding constants of the inequalities for the one-dimensional Cauchy measures in a different way. But it's a pity that we can't reach the similar results for the multi-dimensional case in the same way.

## 2 Main result

**m1** **Theorem 2.1. (one-dimensional weighted Poincaré inequality)** For any  $\beta > 1/2$ , the probability measure  $\mu_\beta$  on  $\mathbb{R}$  satisfy the following weighted Poincaré inequality: for any smooth function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$\text{Var}_{\mu_\beta}(f) \leq C_\beta \int_{\mathbb{R}} |f'(x)|^2 (1 + x^2) d\mu_\beta(x),$$

where  $C_\beta$  has the same order with  $\frac{1}{\beta}$ . Moreover, the weight function is optimal in the sense of order.

**m2** **Theorem 2.2. (one-dimensional weighted log-Sobolev inequality)** For any  $\beta > 1$ , the probability measure  $\mu_\beta$  on  $\mathbb{R}$  satisfy the following weighted log-Sobolev inequality: for any smooth function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$\text{Ent}_{\mu_\beta}(f^2) \leq C_\beta \int_{\mathbb{R}} |f'(x)|^2 (1 + x^2) \log(e + x^2) d\mu_\beta(x),$$

where  $C_\beta$  has the same order with  $\frac{1}{\beta-1}$ . Moreover, the weight function is optimal in the sense of order, that is, for any other function  $\omega^2(x)$ , if  $\lim_{x \rightarrow +\infty} \frac{\omega^2(x)}{\log(e+x^2)} = 0$ , then the Cauchy measure  $\mu_\beta$  doesn't satisfy the weighted log-Sobolev inequality with the weight function  $(1 + x^2)\omega^2(x)$ .

**thm3** **Theorem 2.3.** (multi-dimensional weighted log-Sobolev inequality) For any  $n \geq 6$ ,  $\beta > n/2$ , the probability measure  $\mu_\beta$  on  $\mathbb{R}^n$  satisfy the following weighted log-Sobolev inequality: for any smooth function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$\text{Ent}_{\mu_\beta}(f^2) \leq C \int_{\mathbb{R}^n} |\nabla f(x)|^2 (1 + |x|^2) d\mu_\beta(x),$$

where  $C = \frac{2}{n - 4 - \frac{4}{\sqrt{n}} - \frac{1}{n}}$  is independent of  $\beta$ .

**Remark :** The weight function is of course optimal. Because it has already been optimal for Poincaré type inequality.

Now we state the previous relative works by Bobkov and Ledoux ([5]), Hebisch and Żegarliński ([13]) :

**thm2** **Theorem 2.4.** ([5]) For  $\beta \geq n$ , and any smooth bounded  $f$  on  $\mathbb{R}^n$ ,

$$\text{Var}_{\mu_\beta}(f) \leq \frac{\left(\sqrt{1 + \frac{2}{\beta-2}} + \sqrt{\frac{2}{\beta-2}}\right)^2}{2(\beta-1)} \int |\nabla f(x)|^2 (1 + |x|^2) \mu_\beta(x)$$

**thm3** **Theorem 2.5.** ([5]) If  $\beta \geq (n+1)/2$ ,  $\beta > 1$ , for any smooth bounded  $f$  on  $\mathbb{R}^n$ ,

$$\text{Ent}_{\mu_\beta}(f^2) \leq \frac{1}{\beta-1} \int |\nabla f(x)|^2 (1 + |x|^2)^2 \mu_\beta(x)$$

**thm4** **Theorem 2.6.** ([13]) Assume  $\mu_\beta$  is a probability measure on a  $n$ -dimensional manifold with metric  $d$ , and  $d\mu_\beta = e^{-\beta \log(1+d)dx/Z}$  with  $\beta \geq n$ ,  $\beta > 1$ . Suppose  $\text{Ric} \geq 0$ . Then for any  $q \geq 1$ , there are constants  $M_q, c_q \in (0, +\infty)$ , such that

$$M_q \mu_\beta(|f - \mu_\beta(f)|^q) \leq \mu_\beta((1+d)^q |\nabla f|^q)$$

and

$$\mu_\beta \left( |f|^q \log \frac{|f|^q}{\mu_\beta(|f|^q)} \right) \leq c_q \mu_\beta((1+d)^q \log(e+d) |\nabla f|^q)$$

**Theorem 2.1** does not give a better result than **Theorem 2.4** does. we just adapt another different way, and tell that the weight function is optimal.

From the comparison between **Theorem 2.2** with **Theorem 2.5**, it's clear that our result gives a better weight function. Moreover, the order of the constant isn't changed.

Contrast to **Theorem 2.6**, our results give the estimate of order for the constants.

In [13], their results still can derive a uniform weighted log-Sobolev inequality with weight  $1 + |x|^2$  and constant  $c_q$  independent of  $\beta$ . In fact, we can also get the same result by Bakry-Émery criterion, i.e. getting  $\Gamma_2 \geq \rho \Gamma$  for some  $\rho > 0$ .

### 3 Proofs of main results

#### 3.1 One-dimensional weighted Poincaré inequality

**thm01**

**Theorem 3.1.** ([16]) Let  $\mu, \nu$  be Borel measures on  $\mathbb{R}$  with  $\mu(\mathbb{R}) = 1$  and  $d\nu(x) = n(x)dx$ . Let  $m$  be a median of  $\mu$ . Let  $C_P$  be the optimal constant such that for every smooth function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , one has

$$\text{Var}_\mu(f) \leq C_P \int f'^2 d\nu.$$

Then  $\max(b, B) \leq C_P \leq 4 \max(b, B)$ , where

$$b = \sup_{x < m} \mu((-\infty, x]) \int_x^m \frac{1}{n}, \quad B = \sup_{x > m} \mu([x, \infty)) \int_m^x \frac{1}{n}.$$

**Proof of Theorem 3.1:** By the symmetry of the measure  $\mu_\beta$ , the median  $m$  of  $\mu_\beta$  is equal to 0. Define

$$b(\beta) := \sup_{\alpha \in (-\infty, 0)} \left( \int_{-\infty}^{\alpha} (1 + y^2)^{-\beta} dy \right) \left( \int_{\alpha}^0 (1 + y^2)^{\beta-1} dy \right)$$

$$B(\beta) := \sup_{\alpha \in (0, +\infty)} \left( \int_{\alpha}^{+\infty} (1 + y^2)^{-\beta} dy \right) \left( \int_0^{\alpha} (1 + y^2)^{\beta-1} dy \right).$$

Clearly, by **Theorem 3.2** and symmetry, we just need to give an upper bound on  $B(\beta)$ . Since the point 0 doesn't make trouble in our calculation, we can reduce the estimate on  $B(\beta)$  to that on  $\tilde{B}(\beta)$ ,

By the following simple estimate,

$$\int_x^{+\infty} (1 + y^2)^{-\beta} dy = 2 \int_{x^2}^{+\infty} (1 + t)^{-\beta} t^{-1/2} dt \leq \frac{2}{x} \frac{(1 + x^2)^{-\beta+1}}{\beta - 1},$$

and

$$\int_0^x (1 + y^2)^{\beta-1} dy = 2 \int_0^{x^2} (1 + t)^{\beta-1} t^{-1/2} dt \leq (1 + x^2)^{\beta-1} x,$$

w get  $B(\beta) \leq \frac{2}{\beta-1}$ .

However,

$$B(\beta) \geq \left( \int_{\frac{1}{\sqrt{\beta}}}^{\frac{2}{\sqrt{\beta}}} (1 + y^2)^{-\beta} dy \right) \left( \int_{\frac{1}{2\sqrt{\beta}}}^{\frac{1}{\sqrt{\beta}}} (1 + y^2)^{\beta-1} dy \right) \sim \frac{1}{\beta} \quad (\beta \rightarrow +\infty).$$

Thus we get the right order  $\frac{1}{\beta}$  of  $C_\beta$ .

For any other even function  $\omega^2(x)$  increasing in  $x > 0$  (or increasing in  $x \geq M$  for some

$M > 0$ ), and  $\lim_{x \rightarrow +\infty} \frac{\omega^2(x)}{1+x^2} = 0$ ,

$$\begin{aligned}
B_{\omega^2}(\beta) &:= \sup_{\alpha \in (0, +\infty)} \left( \int_x^{+\infty} (1+y^2)^{-\beta} dy \right) \left( \int_0^x \frac{(1+y^2)^\beta}{\omega^2(y)} dy \right) \\
&\geq \sup_{x \in (M, +\infty)} \left( \int_x^{+\infty} (1+y)^{-2\beta} dy \right) \left( \int_M^x \frac{y^{2\beta}}{\omega^2(y)} dy \right) \\
&\geq \sup_{x \in (M, +\infty)} \frac{(1+x)^{1-2\beta}}{2\beta-1} \frac{1}{\omega^2(x)} \frac{x^{2\beta+1} - M^{2\beta+1}}{2\beta+1} \\
&= +\infty.
\end{aligned}$$

Therefore, the weight function  $1+x^2$  is optimal in the sense of order.

### 3.2 One-dimensional weighted Log-Sobolev inequality

Here we'll make use of the refined characterization from Barthe and Roberto [4]. Of course, we may also use that one from Bobkov, and Götze [6], after all, we can't give a sharp estimate on the logarithmic Sobolev constants. The refined characterization is stated as follows.

**thm1**

**Theorem 3.2.** ([4]) *Let  $\mu, \nu$  be Borel measures on  $\mathbb{R}$  with  $\mu(\mathbb{R}) = 1$  and  $d\nu(x) = n(x)dx$ . Let  $m$  be a median of  $\mu$ . Let  $C$  be the optimal constant such that for every smooth function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , one has*

$$\text{Ent}_\mu(f^2) \leq C \int f'^2 d\nu.$$

Then  $\max(b_-, b_+) \leq C \leq 4 \max(B_-, B_+)$ , where

$$\begin{aligned}
b_+ &= \sup_{x > m} \mu([x, \infty)) \log \left( 1 + \frac{1}{2\mu([x, \infty))} \right) \int_m^x \frac{1}{n}, \\
B_+ &= \sup_{x > m} \mu([x, \infty)) \log \left( 1 + \frac{e^2}{\mu([x, \infty))} \right) \int_m^x \frac{1}{n}, \\
b_- &= \sup_{x < m} \mu((-\infty, x]) \log \left( 1 + \frac{1}{2\mu((-\infty, x])} \right) \int_x^m \frac{1}{n}, \\
B_- &= \sup_{x < m} \mu((-\infty, x]) \log \left( 1 + \frac{e^2}{\mu((-\infty, x])} \right) \int_m^x \frac{1}{n}.
\end{aligned}$$

**Proof of Theorem 2.1:** By the symmetry of the measure  $\mu_\beta$ , the median  $m$  of  $\mu_\beta$  is equal to 0. Moreover, in order to get our result, by **Theorem 3.2** we just need to give an upper bound on the following quantity,

$$\begin{aligned}
S(\beta) &:= \sup_{x \in (0, +\infty)} \left( \int_x^{+\infty} (1+y^2)^{-\beta} dy \right) \log \left( 1 + \frac{c(1, \beta)}{\int_x^{+\infty} (1+y^2)^{-\beta} dy} \right) \\
&\quad \cdot \left( \int_0^x \frac{(1+y^2)^\beta dy}{(1+y^2) \log(e+y^2)} \right).
\end{aligned}$$

Clearly, the superior can't be obtained on the point 0. If the superior is taken on  $x_\beta \in (0, +\infty)$ . Let

$$I_1 = \int_{x_\beta}^{+\infty} (1+y^2)^{-\beta} dy$$

$$I_2 = \int_0^{x_\beta} \frac{(1+y^2)^{\beta-1} dy}{\log(e+y^2)}$$

Then we'll discuss  $S(\beta)$  under the following three sorts of situations by  $\lim_{\beta \rightarrow +\infty} (1+(\frac{1}{\sqrt{\beta}})^2)^\beta = e$ :  
 Convention: in the following we'll use the signal " $\sim$ " to denote the same order as  $\beta \rightarrow +\infty$ .

**Case 1:** If  $x_\beta = o(\frac{1}{\sqrt{\beta}})$ , i.e.  $\lim_{\beta \rightarrow +\infty} \frac{x_\beta}{1/\sqrt{\beta}} = 0$ :

$$I_1 \sim \frac{1}{\sqrt{\beta}}$$

$$I_2 \sim \int_0^{x_\beta} (1+y^2)^{\beta-1} dy \sim x_\beta$$

By the monotonicity of  $x \log(1 + \frac{C}{x})$  in  $x > 0$ , we have immediately

$$S(\beta) \sim \frac{x_\beta}{\sqrt{\beta}} = o(\frac{1}{\beta}).$$

**Case 2:** If  $x_\beta = O(\frac{1}{\sqrt{\beta}})$ ,

$$I_1 \sim \frac{1}{\sqrt{\beta}}$$

$$I_2 \sim \int_0^{x_\beta} (1+y^2)^{\beta-1} dy \sim \frac{1}{\sqrt{\beta}}$$

$$S(\beta) \sim \frac{1}{\beta}$$

**Case 3:** If  $\frac{1}{\sqrt{\beta}} = o(x_\beta)$ ,

$$I_1 \sim \frac{1}{\beta-1} \cdot \frac{1}{x_\beta(1+x_\beta^2)^{\beta-1}}$$

$$I_2 \sim \begin{cases} \frac{(1+x_\beta^2)^\beta}{\beta x_\beta} & \text{if } \{x_\beta\} \text{ is bounded} \\ \frac{1}{\log(e+x_\beta^2)} \frac{(1+x_\beta^2)^\beta}{\beta x_\beta} & \text{if } \{x_\beta\} \text{ is unbounded} \end{cases} \quad (3.1)$$

$$S(\beta) \sim \frac{1}{\beta}$$

Now there is the last case left that the superior is got as  $x \rightarrow +\infty$ . For that we might as well do it by reducing the estimate on  $S(\beta)$  to that on  $\tilde{S}(\beta)$ , defined by

$$\begin{aligned}\tilde{S}(\beta) &:= \sup_{x \in (1, +\infty)} \left( \int_x^{+\infty} y^{-2\beta} dy \right) \log \left( 1 + \frac{1/(2\beta-1)}{\int_x^{+\infty} y^{-2\beta} dy} \right) \left( \int_1^x \frac{y^{2(\beta-1)} dy}{\log(e+y^2)} \right) \\ &= \sup_{x \in (1, +\infty)} \frac{x^{1-2\beta}}{2\beta-1} \log \left( 1 + x^{2\beta-1} \right) \int_1^x \frac{y^{2(\beta-1)} dy}{\log(e+y^2)}.\end{aligned}$$

By the basic formula of differential and integral, one can get readily

$$\begin{aligned}\frac{1}{2\beta-1} \frac{x^{2\beta-1}}{\log(e+x)} &\geq \frac{1}{2\beta-1} \left[ \frac{y^{2\beta-1}}{\log(e+y)} \right]_1^x \\ &= \int_1^x \frac{y^{2(\beta-1)}}{\log(e+y)} \left( 1 - \frac{y}{(2\beta-1)(e+y)} \frac{1}{\log(e+y)} \right) dy \\ &\geq \int_1^x \frac{y^{2(\beta-1)}}{\log(e+y)} \left( 1 - \frac{1}{2\beta-1} \frac{1}{\log(e+y)} \right) dy\end{aligned}$$

When  $2\beta-1 > 1$ , i.e.  $\beta > 1$ , we have

$$\begin{aligned}\int_1^x \frac{y^{2(\beta-1)} dy}{\log(e+y^2)} &\leq \int_1^x \frac{y^{2(\beta-1)} dy}{\log(e+y)} \\ &\leq \frac{1}{2\beta-2} \frac{x^{2\beta-1}}{\log(e+x)}\end{aligned}$$

Therefore,

$$\tilde{S}(\beta) \leq \sup_{x \in (1, +\infty)} \frac{1}{(2\beta-1)(2\beta-2)} \frac{\log(1+x^{2\beta-1})}{\log(e+x)} \leq \frac{1}{2\beta-2}$$

From the discussion above,  $\frac{1}{\beta}$  is the right order of log-Sobolev constants.

Moreover, for any other even function  $\omega^2(x)$  increasing in  $x > M > 0$ , and

$$\lim_{x \rightarrow +\infty} \frac{\omega^2(x)}{\log(e+x^2)} = 0,$$

we have

$$\begin{aligned}\tilde{S}_{\omega^2}(\beta) &:= \sup_{x \in (1, +\infty)} \left( \int_x^{+\infty} y^{-2\beta} dy \right) \log \left( 1 + \frac{1/(2\beta-1)}{\int_x^{+\infty} y^{-2\beta} dy} \right) \left( \int_1^x \frac{y^{2(\beta-1)} dy}{\omega^2(y)} \right) \\ &\geq \sup_{x \in (1, +\infty)} \frac{x^{1-2\beta}}{2\beta-1} \log \left( 1 + x^{2\beta-1} \right) \frac{1}{\omega^2(x)} \int_1^x y^{2(\beta-1)} dy \\ &= +\infty.\end{aligned}$$

As a result, the logarithmic-type weight function is optimal in the sense of order.

### 3.3 Multi-dimensional weighted log-Sobolev inequality

The Cauchy distribution can be represented in the following form:

$$d\mu_\beta(x) = \frac{e^{-\beta \log(1+|x|^2)} dx}{c(n, \beta)}, \quad x \in \mathbb{R}^n, \beta > n/2.$$

Let

$$V(x) := \beta \log(1 + |x|^2), \quad \omega^2(x) := 1 + |x|^2,$$

we have

$$\nabla V = \frac{2\beta x}{1 + |x|^2}, \quad \nabla^2 V = \frac{2\beta}{1 + |x|^2} - \frac{4\beta x \otimes x}{(1 + |x|^2)^2}$$

and

$$\nabla \omega^2 = 2x, \quad \nabla^2 \omega^2 = 2, \quad \Delta \omega = 2n.$$

The generator, associated with the measure  $\mu_\beta$  and the weight  $\omega^2$ , is

$$\mathcal{L}_\omega := \omega^2 \Delta + (\nabla \omega^2 - \omega^2 \nabla V) \nabla. \quad (3.2) \quad \boxed{\text{diffusion}}$$

Note that  $\mu_\beta$  is the invariant measure of the diffusion operator  $\mathcal{L}_\omega$ . We have the "Carré du champ" operator

$$\begin{aligned} \Gamma(f, g) &:= \frac{1}{2} (\mathcal{L}_\omega(fg) - f \mathcal{L}_\omega g - g \mathcal{L}_\omega f) \\ &= \omega^2 \nabla f \nabla g. \end{aligned}$$

Define the  $\Gamma_2$  curvature, see [2, 1, 14]

$$\Gamma_2(f, f) := \frac{1}{2} (\mathcal{L}_\omega \Gamma(f, f) - 2\Gamma(f, \mathcal{L}_\omega f)).$$

**Proposition 3.3.** *Assume that the dimension  $n \geq 6$ , we have the dimension curvature inequality holds, i.e. there exists positive constants  $\rho$ , such that*

$$\Gamma_2(f, f) \geq \rho \Gamma(f, f), \quad (3.3) \quad \boxed{\text{CD}}$$

for all  $f \in C_0^\infty(\mathbb{R}^n)$ .  $\rho$  can be chosen to be  $n - 4 - \frac{4}{\sqrt{n}} - \frac{1}{n}$  if necessary.

*Proof.* By the definition of  $\Gamma_2$  curvature,

$$\begin{aligned}
\Gamma_2(f, f) &= \omega^4 |\nabla \nabla f|^2 + \frac{\omega^2}{2} |\nabla f|^2 \Delta \omega^2 + 2\omega^2 \nabla f \nabla^2 f \nabla \omega^2 + \frac{1}{2} |\nabla f|^2 |\nabla \omega^2|^2 \\
&\quad - \frac{1}{2} \omega^2 |\nabla f|^2 \nabla V \nabla \omega^2 - \omega^2 \Delta f \nabla f \nabla \omega^2 - \omega^2 \nabla f \nabla^2 \omega^2 \nabla f \\
&\quad + \omega^2 (\nabla V \nabla f) (\nabla \omega^2 \nabla f) + \omega^4 \nabla f \nabla^2 V \nabla f \\
&= (1 + |x|^2)^2 |\nabla \nabla f|^2 + (2(\beta - 1) + n(1 + |x|^2)) |\nabla f|^2 \\
&\quad + 4(1 + |x|^2) \nabla f \cdot \nabla \nabla f \cdot x - 2(1 + |x|^2) \Delta f x \cdot \nabla f \\
&\stackrel{(i)}{\geq} (1 + |x|^2)^2 |\nabla \nabla f|^2 - 2 \left( 2 + \frac{1}{\sqrt{n}} \right) (1 + |x|^2) |\nabla \nabla f| (|x| |\nabla f|) \\
&\quad + (2(\beta - 1) + n(1 + |x|^2)) |\nabla f|^2 \\
&= \left( (1 + |x|^2) |\nabla \nabla f| - \left( 2 + \frac{1}{\sqrt{n}} \right) |x| |\nabla f| \right)^2 \\
&\quad + \left( \left( n - 4 - \frac{4}{\sqrt{n}} - \frac{1}{n} \right) |x|^2 + (2\beta - 2 + n) \right) |\nabla f|^2 \\
&\geq \min \left\{ n - 4 - \frac{4}{\sqrt{n}} - \frac{1}{n}, 2\beta - 2 + n \right\} (1 + |x|^2) |\nabla f|^2,
\end{aligned}$$

where  $|\nabla \nabla f|$  is the Hilbert-Schmidt norm of  $\nabla \nabla f$  and (i) follows from

$$(\Delta f)^2 \leq \frac{|\nabla \nabla f|^2}{n}.$$

If  $n \geq 6$ , there exists a positive constant  $\rho := \rho(n) = n - 4 - \frac{4}{\sqrt{n}} - \frac{1}{n} > 0$  (independent on  $\beta$ ), such that the dimension curvature inequality  $CD(\rho, \infty)$ , i.e. (3.3) holds.  $\square$

By the above proposition, Theorem 2.3 is wellknown, see [2], also [17, 1, 14].

## 4 Appendix

**a0** **Lemma 4.1.** *The normalized constant  $c(1, \beta)$  has the same order with  $\frac{1}{\sqrt{\beta}}$  as  $\beta$  goes to infinity.*

*Proof.*

$$\begin{aligned}
c(1, \beta) &= \int_{\mathbb{R}} (1 + x^2)^{-\beta} dx \\
&= 2 \int_0^{+\infty} (1 + x^2)^{-\beta} dx \\
&= \int_0^{+\infty} (1 + \tau)^{-\beta} \tau^{-1/2} d\tau \\
&= \int_0^{\frac{1}{\beta}} (1 + \tau)^{-\beta} \tau^{-1/2} d\tau + \int_{\frac{1}{\beta}}^{+\infty} (1 + \tau)^{-\beta} \tau^{-1/2} d\tau \sim \frac{1}{\sqrt{\beta}}.
\end{aligned}$$

This is because of the following facts:

$$\int_0^{\frac{1}{\beta}} (1 + \tau)^{-\beta} \tau^{-1/2} d\tau \sim \int_0^{\frac{1}{\beta}} \tau^{-1/2} d\tau = \frac{2}{\sqrt{\beta}}.$$

and

$$\int_{\frac{1}{\beta}}^{+\infty} (1 + \tau)^{-\beta} \tau^{-1/2} d\tau \leq \sqrt{\beta} \int_{\frac{1}{\beta}}^{+\infty} (1 + \tau)^{-\beta} d\tau \sim \frac{1}{\sqrt{\beta}}.$$

**Remark:** Let  $C(n, \beta) := \int_0^{+\infty} (1+r^2)^{-\beta} r^{n-1} dr$ , then  $c(n, \beta)$  has the same order with  $C(n, \beta)$  as  $\beta \rightarrow +\infty$ . Next we give an estimate for  $C(n, \beta)$ .

$$C(n, \beta) = \frac{1}{2} \int_0^{+\infty} (1 + \tau)^{-\beta} \tau^{\frac{n-2}{2}} d\tau$$

By the similar method, we can get

$$C(n, \beta) \sim \frac{1}{\beta^{\frac{n}{2}}}.$$

That is  $c(n, \beta)$  has the order of  $\frac{1}{\beta^{\frac{n}{2}}}$  as  $\beta$  large enough. □

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