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Memory Resilient Gain-scheduled State-Feedback Control of Uncertain LTI/LPV Systems with Time-Varying Delays

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Abstract

The stabilization of uncertain LTI/LPV time delay systems with time varying delays by state-feedback controllers is addressed. At the difference of other works in the literature, the proposed approach allows for the synthesis of resilient controllers with respect to uncertainties on the implemented delay. It is emphasized that such controllers unify memoryless and exact-memory controllers usually considered in the literature. The solutions to the stability and stabilization problems are expressed in terms of LMIs which allow to check the stability of the closed-loop system for a given bound on the knowledge error and even optimize the uncertainty radius under some performance constraints; in this paper, the $\mathcal{H}_\infty$ performance measure is considered. The interest of the approach is finally illustrated through several examples.

Keywords: Time delay systems; Controller Delay-resilience; Linear parameter varying systems; Robust LMIs; Relaxation

1. Introduction

Since several years, time-delay systems [24, 25, 38, 17, 18, 16, 14, 30, 21, 3] have attracted more and more interest since they arise in various problems [22, 25] such as chemical processes, biological systems, economic systems, etc. The presence of delays, in the equations describing the process, is often responsible of destabilizing effects and performance deterioration. Indeed, in fast systems, even a small time-delay may have a very harmful effect, and thus cannot be neglected. This has motivated the development of many types of stability tests and matched controller design techniques [40, 13, 36, 12, 8]. Nevertheless, while the theories for stability analysis and stabilization of LTI systems with constant delays are well established, the case of time-varying delays is still not well understood [23, 26].

On the other hand, over the past recent years, LPV systems [2, 1, 34, 27, 20] have been heavily studied since they offer a very general approach for the modeling and the control of complex systems such as nonlinear systems. This fresh upsurge of gain-scheduling based techniques is mainly due to the emergence of LMIs [6], which provide a powerful formalism for the expression of solutions of many problems arising in systems and control theory. It is important to note that many problems in LPV framework remain open and major improvements are still expected for stability analysis as well as for control law synthesis.

The stability analysis of LPV time-delay systems is still an open problem and is discussed for instance in [39, 37, 9, 7] while the control of LPV time-delay systems have also been studied in [35, 39, 37, 7, 11, 10] but still remains sporadic. These systems belong to the intersection of two families and hence inherit from the difficulties of each one. Additionally, new difficulties arise, for instance several robust control tools which are used to deal with LPV systems (such as the projection lemma [15] or the dualization lemma [27]) are difficult to apply to LPV time-delay

systems. Indeed, the stability analysis of such systems cannot be tackled using classical Lyapunov functions (as for finite dimensional LPV systems) but must be analyzed using adapted tools, namely Lyapunov-Razumikhin functions and Lyapunov-Krasovskii functionals, which increase the number of decision variables.

The main contribution of the paper concerns the synthesis of non-fragile controllers with respect to an uncertain knowledge of the implemented delay. Several papers mention non-fragility of observers/controllers but the robust stability analysis is done only after the synthesis [33, 28]. In such a case, the non-fragility radius (maximal tolerable delay uncertainty) is difficult to guarantee or optimize. In the proposed approach, the non-fragility radius can be fixed by the designer or even maximized. Moreover, since the approach is based on LMIs, time-varying and uncertain systems can be also easily handled at the difference of latter results which were based on frequency domain methods, thus restricted to LTI systems and difficult to generalize to the uncertain case. A close result but notably different is also provided in [19] where the control of a time-delay system with constant delay is performed using a controller with a different delay. However, no relationship between the delays is considered and thus the maximal admissible error cannot be analyzed from this result. Results on resilience of controllers with delay uncertainties are also provided in [10, 11] in the framework on *delay-scheduled controllers* where the delay acts as a gain-scheduling parameter on the controller expression. In the current paper, the initial time-delay system system structure for both the system and the controller is kept, in order to develop a stabilization result in this domain. The resulting problem can be equivalently represented as a stabilization problem of a system with two delays where the delays are coupled through an algebraic inequality. Thus the problem reduces in a correct and efficient accounting of this inequality in the LMI-based stabilization result.

It is worth mentioning that almost all the literature addresses the stabilization problem with memoryless (conservative) or exact-memory (non-implementable) controllers. The approach of the paper is more pragmatic and aims at designing controller with delays different from the system one. Delay estimation techniques, as in [4], could be used in order to determine the controller delay. Such a controller class find applications in the control of physical systems with delay on states such as the ones in [22].

The goal of this article is not to provide better results on the stability of time-delay systems by introducing new Lyapunov-Krasovskii functionals but is to show that it is possible to consider uncertainties of the delay knowledge and take it into account in an efficient way in the synthesis. The proposed approach is very general and can be extended to many types of (less conservative) different functionals.

For a real square matrix M we define $M^{\mathcal{S}} := M + M^T$ where M^T is its transpose. The space of signals with finite energy is denoted by $\mathcal{L}_2$ and the energy of $v \in \mathcal{L}_2$ is $||v||_{\mathcal{L}_2} := \left(\int_0^{+\infty} v^*(t)v(t)dt\right)^{1/2}$. $\mathbb{S}^k_{++}$ denotes the cone of real symmetric positive definite matrices of dimension k. $\mathbb{R}_+$ ($\mathbb{R}_{++}$) denotes the set of nonnegative (positive) real numbers. $\star$ denotes symmetric terms in symmetric matrices and in quadratic forms. $\otimes$ and $\times$ denote the Kronecker and the cartesian product respectively. $co\{S\}$ stands for the convex hull of the set S. $col_i(\lambda_i)$ is the column vector with components λ_i.

The paper is structured as follows. In Section 2, definitions and objectives of the paper are defined. In Section 3 we provide several delay-dependent stability results for uncertain LPV time-delay systems with time-varying delays. Section 4 is devoted to the development of constructive sufficient conditions to the existence of three types of parameter dependent state-feedback controllers. Finally, in Section 5, examples and discussions on the proposed approach are provided.

2. Definitions and Objectives

The following class of systems will be considered in the paper:

$$\begin{array}{rcl} \dot{x}(t) & = & A(\lambda,\rho)x(t) + A_h(\lambda,\rho)x_h(t) + B(\lambda,\rho)u(t) \\ & & +E(\lambda,\rho)w(t) \\ z(t) & = & C(\lambda,\rho)x(t) + C_h(\lambda,\rho)x_h(t) + D(\lambda,\rho)u(t) \\ & & +F(\lambda,\rho)w(t) \\ x(\eta) & = & \phi(\eta),\ \eta \in [-h_M, 0] \end{array} \tag{1}$$

where $x \in \mathbb{R}^n$, $x_h(t) = x(t-h(t)) \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^p$, $z \in \mathbb{R}^q$ and $\phi(\cdot)$ are respectively the system state, the delayed state, the control input, the exogenous input, the controlled output and the functional initial condition. The system matrices are defined by

$$\begin{bmatrix} A & A_h & B & E \\ C & C_h & D & F \end{bmatrix}(\lambda,\rho) = \sum_{i=1}^{N} \lambda_i \begin{bmatrix} A_i & A_{hi} & B_i & E_i \\ C_i & C_{hi} & D_i & F_i \end{bmatrix}(\rho) \tag{2}$$

where $\lambda = col_i(\lambda_i)$ is time-invariant and belongs to the unitary simplex Λ defined by

$$\Lambda := \left\{ \lambda_i \geq 0 : \ \sum_{i=1}^{N} \lambda_i = 1,\ i = 1, \ldots, N \right\} \tag{3}$$

The delay is assumed to belong to the set

$$\mathscr{H} := \{h : \mathbb{R}_+ \to [0, h_M], h \leq \mu < 1\} \tag{4}$$

with $h_M < +\infty$. The vector of parameters ρ belongs to

$$\mathscr{P} := \left\{\rho : \mathbb{R}_+ \to U_\rho \subset \mathbb{R}^{N_p}, \dot{\rho} \in co\{U_\nu\}\right\} \tag{5}$$

where $N_p > 0$ is the number of parameters, U_ρ is a connected compact subset of $\mathbb{R}^{N_p}$. U_ν is the set of vertices of the convex set in which the derivative of the parameters evolves and is defined by

$$U_\nu := \times_{i=1}^{N_p} \{\underline{\nu}_i, \bar{\nu}_i\} \tag{6}$$

where $\underline{\nu}_i$ and $\bar{\nu}_i$ denote respectively the upper and lower bound of $\dot{\rho}_i$; hence we have $\dot{\rho} \in co\{U_\nu\}$.

The aim of the current paper is to find a control law based on a parameter dependent state-feedback of the form

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - d(t)) \tag{7}$$

where the gains $K_0(\rho)$ and $K_h(\rho)$ are sought such that the controller stabilizes the uncertain LPV system (1). Note that the delay $d(t)$ involved in the control law is allowed to be different from the system delay $h(t)$. First, the usual case $d(t) = h(t)$ will be considered and then the more general case $d(t) = h(t) + \eta(t)$ with $|\eta(t)| \leq \delta$ will be solved in turn. To this aim, the following set is introduced

$$\mathscr{D}_\delta := \{d : \mathbb{R}_+ \to [0, h_M], |d(t) - h(t)| \leq \delta,\ h \in \mathscr{H}\} \tag{8}$$

and defines the set of controller delays.

Definition 2.1. *The following terminology is used for the controllers (7):*

- *If $K_h(\cdot) = 0$ the controller is referred to as a* ***memoryless controller****;*
- *If $d(t) = h(t)$ for all $t \geq 0$ (i.e. $\delta = 0$ in (8)) then the controller is referred to as an* ***exact memory controller****;*
- *If $|d(t) - h(t)| \leq \delta$ for some $\delta > 0$ then the controller is referred to as a δ-**memory resilient controller**.*

The set $\mathscr{D}_\delta$ is parameterized by the uncertainty radius $\delta \geq 0$. Note that when $\delta = 0$ the equality $d(t) = h(t)$ holds for all $t \geq 0$ and hence the δ-memory resilient controller coincides with exact memory controller. Note also that if $\delta = h_M$ then the implemented delay may take any value inside $[0, h_M]$ independently of the value of $h(t)$. In such a case, $h(t)$ can be considered as unknown, resulting then in the particular case where the memoryless and the h_M-memory resilient controllers are actually quite near, structurally speaking. It will be illustrated in the examples that δ-memory resilient controllers connect together the well-known memoryless and exact-memory controllers by providing a unique and generalized expression for all controllers.

With this in mind, it is possible to state the main problem of the paper:

Problem 2.1. *Find a parameter dependent δ-memory resilient state-feedback controller (7) which*

1. *Robustly asymptotically stabilizes system (1): $x(t) \to 0$ as $t \to +\infty$ with $w(t) = 0$, for all parameter trajectories $\rho \in \mathscr{P}$, for all delays $(h, d) \in \mathscr{H} \times \mathscr{D}_\delta$ and for all $\lambda \in \Lambda$.*
2. *Provides a guaranteed $\mathcal{L}_2$ performance attenuation gain from w to z satisfying $||z||_{\mathcal{L}_2} < \gamma ||w||_{\mathcal{L}_2}$ with $x(\eta) = 0$, $\eta \in [-h_M, 0]$, $w(t) \neq 0$ and for all parameter trajectories $\rho \in \mathscr{P}$, for all delays $(h, d) \in \mathscr{H} \times \mathscr{D}_\delta$ and for all $\lambda \in \Lambda$.*

3. A Control Oriented Delay-Dependent Stability Result

This section is devoted to the stability analysis of LPV time-delay systems of the form (1). The results are based on the extension of [18, 16] to uncertain LPV time-delay systems with time-varying delays. As we shall see later, the immediate LMI conditions derived from the Lyapunov-Krasovskii theorem are not well suited for stability and synthesis problems due to:

- multiple products between system matrices and decision variables; and
- quadratic terms on the polytope variable λ.

The second part of the proof is then devoted to the relaxation of such a result in order to both reduce the number of these products (to one) and make the dependence on the polytope variable λ affine. This makes the derivation of stabilization result easier and overall more efficient than approaches based on relaxations made after substitution of the closed-loop system matrices into the LMI.

Another approach based on the projection lemma [2] and on adjoint systems [5] was considered in [9]. This approach led to an equivalent problem independent of the controller matrices involving a nonlinear matrix inequality which was then solved using an iterative LMI algorithm. The current approach avoids such a computational complexity by tolerating an increase of conservatism. So, following this new idea, the following stability analysis result is stated:

Theorem 3.1. *System (1) with no control input (ie. $u(t) \equiv 0$) is robustly asymptotically stable for all $(h, \rho, \lambda) \in \mathscr{H} \times \mathscr{P} \times \Lambda$ if there exist continuously differentiable matrix functions $P_i : U_\rho \to \mathbb{S}^n_{++}$ for all $i \in \{1, \ldots, N\}$, a matrix function $X : U_\rho \to \mathbb{R}^{n \times n}$, N constant matrices $Q_i, R_i \in \mathbb{S}^n_{++}$ and a scalar $\gamma > 0$ such that the parameter dependent LMIs*

$$\Theta_i = \begin{bmatrix} -X(\rho)^{\mathcal{S}} & \Phi_{12i} & \Phi_{13i} & \Phi_{14i} & 0 & X(\rho)^T & h_M R_i \\ \star & \Phi_{22i} & R_i & 0 & C_i(\rho)^T & 0 & 0 \\ \star & \star & \Phi_{33i} & 0 & C_{hi}(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_p & F_i(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_q & 0 & 0 \\ \star & \star & \star & \star & \star & -P_i(\rho) & -h_M R_i \\ \star & \star & \star & \star & \star & \star & -R_i \end{bmatrix} \prec 0 \tag{9}$$

hold for all $(\rho, \nu, i) \in U_\rho \times co\{U_\nu\} \times \{1, \ldots, N\}$ with

$$\Phi_{22i} = \frac{\partial P_i(\rho)}{\partial \rho} \nu - P_i(\rho) + Q_i - R_i$$

$$\begin{bmatrix} [A(\lambda,\rho)^T P(\lambda,\rho)]^{\mathcal{S}} + Q(\lambda) - R(\lambda) & P(\lambda,\rho)A(\lambda,\rho) + R(\lambda) & P(\lambda,\rho)E(\lambda,\rho) & C(\lambda,\rho)^T & h_M A(\lambda,\rho)^T R(\lambda) \\ \star & -(1-\mu)Q(\lambda) - R(\lambda) & 0 & C_h(\lambda,\rho)^T & h_M A_h(\lambda,\rho)^T R(\lambda) \\ \star & \star & -\gamma I_p & F(\lambda,\rho)^T & h_M E(\lambda,\rho)^T R(\lambda) \\ \star & \star & \star & -\gamma I_q & 0 \\ \star & \star & \star & \star & -R(\lambda) \end{bmatrix} \prec 0 \tag{14}$$

$$\begin{array}{lcllcl} \Phi_{12i} & = & P_i(\rho) + X(\rho)^T A_i(\rho) & \Phi_{13i} & = & X(\rho)^T A_{hi}(\rho) \\ \Phi_{14i} & = & X(\rho)^T E_i(\rho) & \Phi_{33i} & = & -(1-\mu)Q_i - R_i \end{array}$$

In such a case, system (1) satisfies $||z||_{\mathcal{L}_2} < \gamma ||w||_{\mathcal{L}_2}$ *for all* $(\rho, h, \lambda) \in \mathscr{P} \times \mathscr{H} \times \Lambda$.

Proof : The choice of the functional V is inspired from [16, 18] and extended to the case of LPV time-delay systems with time-varying delays as in [9] and the supply rate $s(w,z)$ is considered:

$$\begin{array}{rcl} V & = & x^T(t)P(\lambda,\rho)x(t) + \displaystyle\int_{t-h(t)}^{t} x^T(\theta)Q(\lambda)x(\theta)d\theta \\ & & + h_M \displaystyle\int_{-h_M}^{0}\int_{t+\theta}^{t} x^T(\eta)R(\lambda)x(\eta)d\eta d\theta \\ s(w(t),z(t)) & = & \gamma w^T(t)w(t) - \gamma^{-1} z(t)^T z(t) \end{array} \tag{10}$$

with $P(\lambda,\rho) = \sum_{i=1}^{N} \lambda_i P_i(\rho)$, $Q(\lambda) = \sum_{i=1}^{N} \lambda_i Q_i$, $R(\lambda) = \sum_{i=1}^{N} \lambda_i R_i$. The supply-rate $s(w,z)$ characterizes the $\mathcal{L}_2$-gain from w to z. Define the function H to be

$$H = V - \int_0^t s(w(\theta), z(\theta))d\theta \tag{11}$$

The derivative of the function can be bounded from above by

$$\begin{array}{rcl} \dot{H} & \leq & x^T(t)\dfrac{\partial P(\lambda,\rho)}{\partial \rho}\dot{\rho}(t)x(t) + x(t)^T \left[A^T(\lambda,\rho)P(\lambda,\rho)\right]^{\mathcal{S}} x(t) \\ & & +2x_h^T(t)A_h(\lambda,\rho)^T P(\lambda,\rho)x(t) + 2w^T(t)E(\lambda,\rho)^T P(\lambda,\rho)x(t) \\ & & +x(t)^T Q(\lambda)x(t) - (1-\dot{h})x_h(t)^T Q(\lambda)x_h(t) \\ & & +h_M^2 \dot{x}(t)^T R(\lambda)\dot{x}(t) + \mathcal{I} - \gamma w(t)^T w(t) \\ & & +\gamma^{-1} z(t)^T z(t) \\ z(t) & = & C(\lambda,\rho)x(t) + C_h(\lambda,\rho)x(t-h(t)) + F(\lambda,\rho)w(t) \\ \mathcal{I} & = & -h_M \displaystyle\int_{t-h(t)}^{t} \dot{x}(\theta)^T R \dot{x}(\theta)d\theta \end{array} \tag{12}$$

Note that $-(1-\dot{h}) \leq -(1-\mu)$ and using Jensen's inequality [17] on $\mathcal{I}$ we obtain

$$\mathcal{I} \leq -\left(\int_{t-h(t)}^{t} \dot{x}(s)ds\right)^T R(\lambda)(\star)^T \tag{13}$$

Then expanding the expression of $s(w,z)$ we get LMI (14) after Schur complements. Due to products between decision matrices and system matrices, it is clear that the LMI is not linear in λ (e.g. $A(\lambda,\rho)P(\lambda,\rho)$). Moreover, the structure of (14) is not adapted to the controller design due to the presence of multiple products terms PA, PA_h, RA and RA_h preventing to find a linearizing change of variable even after congruence transformations. A relaxed version of (14) is then expected to remove the multiple products preventing the change of variables and limit the increase of conservatism. A similar approach has been used in [31, 32, 7].

In view of relaxing the latter result, define $\Theta = \sum_{i=1}^{N} \lambda_i \Theta_i$ where Θ_i is given in (9). Below, it is proved that $\Theta \prec 0$ implies the feasibility of (14). First note that Θ can be written as (where the dependency on λ, ρ and $\dot{\rho}$ are dropped for clarity):

$$\Theta|_{X=0} + Z_1^T X Z_2 + Z_2^T X^T Z_1 \prec 0 \tag{15}$$

where $Z_1 = \begin{bmatrix} -I & A & A_h & E & 0 & 0 \end{bmatrix}$ and $Z_2 = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Then invoking the projection lemma [15], the feasibility of $\Theta \prec 0$ implies the feasibility of the underlying LMI problem

$$\mathcal{N}_1^T \, \Theta|_{X=0} \, \mathcal{N}_1 \prec 0 \tag{16a}$$

$$\mathcal{N}_2^T \, \Theta|_{X=0} \, \mathcal{N}_2 \prec 0 \tag{16b}$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are basis of the null-space of Z_1 and Z_2 respectively. Note that since X only depends on ρ (and neither on $\dot{\rho}$ nor λ) then equivalence equivalence between (15) and (16) is lost and reduces then to an implication from (15) to (16) only.

After some tedious manipulations, it is possible to show that LMI (16a) is equivalent to (14) and thus that $\Theta \prec 0$ implies (14). Thus the conservatism of the approach is characterized by LMI (16b) and by the sole dependence of X on ρ. $\square$

LMIs (9) do not involve any multiple products and hence can be easily used for design purpose. The removal of multiple products has been allowed through the introduction of a 'slack' variable $X(\rho)$. The additional conservatism and the (slight) increase of the computational complexity are the price to pay to get easily tractable conditions for the stabilization problem.

4. Delay-Dependent Stabilization by State-Feedback

4.1. Robust Stabilization using Memoryless and Exact-Memory State-Feedback Controllers

In this part, stabilizing control laws of the form

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t)) \tag{17}$$

are sought. The closed loop system obtained from the interconnection of system (1) and controller (17) is given by

$$\begin{array}{rcl} \dot{x}(t) & = & A_{cl}(\lambda,\rho)x(t) + A_{hcl}(\lambda,\rho)x_h(t) + E(\lambda,\rho)w(t) \\ z(t) & = & C_{cl}(\lambda,\rho)x(t) + C_{hcl}(\lambda,\rho)x_h(t) + F(\lambda,\rho)w(t) \end{array} \tag{18}$$

with $A_{cl}(\lambda,\rho) = A(\lambda,\rho)+B(\lambda,\rho)K_0(\rho)$, $C_{cl}(\lambda,\rho) = C(\lambda,\rho)+D(\lambda,\rho)K_0(\rho)$, $A_{hcl}(\lambda,\rho) = A_h(\lambda,\rho)+B(\lambda,\rho)K_h(\rho)$ and $C_{hcl}(\lambda,\rho) = C_h(\lambda,\rho) + D(\lambda,\rho)K_h(\rho)$. The following theorem on robust stabilization is obtained:

Theorem 4.1. *There exists a parameter dependent state-feedback control of the form (17) which robustly asymptotically stabilizes system (1) for all $(h,\rho,\lambda) \in \mathscr{H}\times\mathscr{P}\times\Lambda$ if there exist continuously differentiable matrix functions $P_i : U_\rho \to \mathbb{S}^n_{++}$, constant matrices $Q_i, R_i \in \mathbb{S}^n_{++}$ for $i = 1,\ldots,N$, $X \in \mathbb{R}^{n\times n}$, matrix functions $Y_0, Y_h : U_\rho \to \mathbb{R}^{m\times n}$ and a scalar $\gamma > 0$ such that the parameter dependent LMIs*

$$\begin{bmatrix} \Xi_{11i} & \Xi_{12i} & \Xi_{13i} & E_i(\rho) & 0 & X & h_M R_i \\ \star & \Xi_{22i} & R & 0 & \Xi_{24i} & 0 & 0 \\ \star & \star & \Xi_{33i} & 0 & \Xi_{34i} & 0 & 0 \\ \star & \star & \star & -\gamma I_p & F_i(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_q & 0 & 0 \\ \star & \star & \star & \star & \star & -P_i(\rho) & -h_M R_i \\ \star & \star & \star & \star & \star & \star & -R_i \end{bmatrix} \prec 0 \tag{19}$$

hold for all $(\rho,\nu,i) \in U_\rho \times co\{U_\nu\} \times \{1,\ldots,N\}$ where

$$\begin{array}{rcl} \Xi_{11} & = & -X^{\mathcal{S}} \\ \Xi_{12i} & = & P_i(\rho) + A_i(\rho)X + B_i(\rho)Y_0(\rho) \\ \Xi_{23i} & = & A_{hi}(\rho)X + B_i(\rho)Y_h(\rho) \\ \Xi_{22i} & = & \dfrac{\partial P_i(\rho)}{\partial \rho}\nu - P_i(\rho) + Q_i - R_i \\ \Xi_{33i} & = & -(1-\mu)Q_i - R_i \\ \Xi_{24i} & = & [C_i(\rho)X + D_i(\rho)Y_0(\rho)]^T \\ \Xi_{34i} & = & [C_{hi}(\rho)X + D_i(\rho)Y_h(\rho)]^T \end{array}$$

In such a case, a suitable control law is given by (17) with gains $K_0(\rho) = Y_0(\rho)X^{-1}$ *and* $K_h(\rho) = Y_h(\rho)X^{-1}$*. Moreover, the closed-loop system satisfies* $||z||_{\mathcal{L}_2} < \gamma||w||_{\mathcal{L}_2}$ *for all* $(h, \rho, \lambda) \in \mathscr{H} \times \mathscr{P} \times \Lambda$*.*

Proof : Substitute the closed-loop system (18) into inequality (9) and set X to be a constant matrix. X is enforced to be constant in order to allow for the use of congruence transformations, otherwise, nonlinear terms would appear, making the solution to the problem difficult to solve (i.e. non LMI). Then performing a congruence transformation with respect to matrix

$$\mathrm{diag}(I_3 \otimes X^{-1}, I_{p+q}, I_2 \otimes X^{-1})$$

and applying the following linearizing change of variable

$$\begin{array}{lcllcl} X & \leftarrow & X^{-1} & P_i & \leftarrow & X^{-T}P_iX^{-1} \\ Q_i & \leftarrow & X^{-T}Q_iX^{-1} & R_i & \leftarrow & X^{-T}R_iX^{-1} \\ Y_0 & \leftarrow & K_0X^{-1} & Y_h & \leftarrow & K_hX^{-1} \end{array}$$

yields LMIs (19). □

Remark 4.1. *The latter result can be used to both synthesize exact-memory and memoryless control laws. Memoryless structures can be obtained setting* $Y_h(\cdot) = 0$*.*

4.2. Robust Stabilization using δ-Memory Resilient State-Feedback Controllers

This part establishes a new result on the stabilization of time-delay systems where the strong constraint on exact delay knowledge is relaxed. In the following, we will show that it is also possible to give stabilization conditions even in presence on time-varying uncertainties on the delay knowledge. We will see, as a direct consequence of the method, that Theorem 4.1 can be retrieved from Theorem 4.2, the main result of this section.

In the sequel, the following control law will be considered:

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - d(t)) \tag{20}$$

where the approximate value of the delay $d(t) \in \mathscr{D}_\delta$ is used. To the authors' knowledge, the only work on such control laws using LMI techniques is [19]. However, the provided approach only considers constant time-delays and does not consider any relationship between the delays. In the current approach, time-varying delays are allowed and the delay knowledge maximal error is explicitly taken into account in the stabilization conditions. Moreover, with the provided approach, it is easy to guarantee a given bound on the error or optimize it, at the difference of [33, 28].

The closed-loop system given by the interconnection of the control law (20) and system (1) is governed by the expressions:

$$\begin{array}{rcl} \dot{x}(t) & = & A_{cl}(\lambda, \rho)x(t) + A_h(\lambda, \rho)x_h(t) \\ & & +B(\lambda, \rho)K_h(\rho)x_d(t) + E(\lambda, \rho)w(t) \\ z(t) & = & C_{cl}(\lambda, \rho)x(t) + C_h(\lambda, \rho)x_h(t) \\ & & +D(\lambda, \rho)K_h(\rho)x_d(t) + F(\lambda, \rho)w(t) \end{array} \tag{21}$$

where $x_h(t) = x(t - h(t))$, $x_d(t) = x(t - d(t))$, $A_{cl}(\lambda, \rho) = A(\lambda, \rho) + B(\lambda, \rho)K_0(\rho)$ and $C_{cl}(\lambda, \rho) = C(\lambda, \rho) + D(\lambda, \rho)K_0(\rho)$. It is worth noting that this system is not a classical system with two delays. Indeed, the difficulty lies in the fact that both delays satisfy an algebraic inequality which constrains the trajectories of $d(t)$ to evolve within a ball, of radius δ, centered around the trajectory of $h(t)$. This additional information has to be taken into account for an efficient stability and performance analysis of the system (21). To this aim, the following preliminary result from [29] is used:

Proposition 4.1. *Let us define the operator* $\Delta(\cdot)$ *as*

$$\Delta(z_0(t)) \quad = \quad \frac{2}{\sqrt{7}\delta}\int_{t-d(t)}^{t-h(t)} z_0(\tau)d\tau \quad \text{with } (h,d) \in \mathscr{H} \times \mathscr{D}_\delta \tag{22}$$

For any input signal $\xi \in \mathcal{L}_2$*, the output* $\Delta(\xi)$ *is also in* $\mathcal{L}_2$ *and we have* $||\Delta(\xi)||_{\mathcal{L}_2} \leq ||\xi||_{\mathcal{L}_2}$.

Using this operator, we can turn system (21) into an uncertain single-delay system (i.e. with $d(t)$ only) depending explicitly on the delay error bound δ:

$$\begin{array}{rcl} \dot{x}(t) & = & A_{cl}(\lambda,\rho)x(t) + A_{hcl}(\lambda,\rho)x_d(t) \\ & & +E(\lambda,\rho)w(t) + \frac{\sqrt{7}}{2}\delta A_h(\lambda,\rho)w_0(t) \\ z(t) & = & C_{cl}(\lambda,\rho)x(t) + C_{hcl}(\lambda,\rho)x_d(t) \\ & & +F(\lambda,\rho)w(t) + \frac{\sqrt{7}}{2}\delta C_h(\lambda,\rho)x_d(t) \\ z_0(t) & = & \dot{x}(t) \\ w_0(t) & = & \Delta(z_0(t)) \end{array} \tag{23}$$

where $x_d(t) = x(t-d(t))$, $A_{cl}(\lambda,\rho) = A(\lambda,\rho)+B(\lambda,\rho)K_0(\rho)$, $A_{hcl}(\lambda,\rho) = A_h(\lambda,\rho)+B(\lambda,\rho)K_h(\rho)$, $C_{cl}(\lambda,\rho) = C(\lambda,\rho) + D(\lambda,\rho)K_0(\rho)$ and $C_{hcl}(\lambda,\rho) = C_h(\lambda,\rho) + D(\lambda,\rho)K_h(\rho)$.

Finally, according to the previous discussion, the main result of the paper is given below:

Theorem 4.2. *There exists a parameter dependent state-feedback control of the form (20) which robustly asymptotically stabilizes system (2) for all* $(h,d,\rho,\lambda) \in \mathscr{H} \times \mathscr{D}_\delta \times \mathscr{P} \times \Lambda$ *if there exist continuously differentiable matrix functions* $P_i : U_\rho \to \mathbb{S}^n_{++}$*, matrix functions* $S_i : U_\rho \to \mathbb{S}^n_{++}$*, constant matrices* $Q_i, R_i \in \mathbb{S}^n_{++}$*,* $X \in \mathbb{R}^{n\times n}$*,* $Y_0, Y_h : U_\rho \to \mathbb{R}^{m\times n}$ *and a scalar* $\gamma > 0$ *such that the parameter dependent LMIs*

$$\begin{bmatrix} \Omega_{1i} & \Omega_{2i} \\ \star & \Omega_{3i} \end{bmatrix} \prec 0 \tag{24}$$

hold for all $(\rho,\nu,i) \in U_\rho \times co\{U_\nu\} \times \{1,\dots,N\}$ *and where*

$$\begin{array}{rcl} \Omega_{1i} & = & \begin{bmatrix} \Omega_{11i} & \Omega_{12i} & \Omega_{13i} & \Omega_{14i} & E_i(\rho) & 0 \\ \star & \Omega_{22i} & R_i & \frac{\sqrt{7}}{2}\delta R_i & 0 & \Omega_{26i} \\ \star & \star & \Omega_{33i} & \frac{\sqrt{7}}{2}\delta\Omega_{33i} & 0 & \Omega_{36i} \\ \star & \star & \star & \Omega_{44i} & 0 & \frac{\sqrt{7}}{2}\delta C_{hi}(\rho)^T \\ \star & \star & \star & \star & -\gamma I_p & F_i(\rho)^T \\ \star & \star & \star & \star & \star & -\gamma I_q \end{bmatrix} \\ \Omega_{2i} & = & \begin{bmatrix} X & S_i(\rho) & h_M R_i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \Omega_{3i} & = & \begin{bmatrix} -P_i(\rho) & -S_i(\rho) & -h_M R_i \\ \star & -S_i(\rho) & 0 \\ \star & \star & -R_i \end{bmatrix} \\ \Omega_{11i} & = & -X^{\mathcal{S}} \\ \Omega_{12i} & = & P_i(\rho) + A_i(\rho)X + B_i(\rho)Y_0(\rho) \\ \Omega_{13i} & = & A_{hi}(\rho)X + B_i(\rho)Y_h(\rho) \\ \Omega_{14i} & = & \frac{\sqrt{7}}{2}\delta A_{hi}(\rho)X \\ \Omega_{22i} & = & \frac{\partial P_i}{\partial \rho}\nu - P_i(\rho) + Q_i - R_i \\ \Omega_{33i} & = & -(1-\mu)Q_i - R_i \\ \Omega_{26i} & = & (C_i(\rho)X + D_i(\rho)Y_0(\rho))^T \\ \Omega_{36i} & = & (C_{hi}(\rho)X + D_i(\rho)Y_h(\rho))^T \\ \Omega_{44i} & = & \frac{7}{4}\delta^2\Omega_{33i} - S_i(\rho) \end{array}$$

In such a case, the controller gains can be computed using $K_0(\rho) = L_0(\rho)X^{-1}$ *and* $K_h(\rho) = L_h(\rho)X^{-1}$*. Moreover the closed-loop system satisfies* $||z||_{\mathcal{L}_2} < \gamma||w||_{\mathcal{L}_2}$ *for all* $(\rho, h, d, \lambda) \in \mathscr{P} \times \mathscr{H} \times \mathscr{D}_\delta \times \Lambda$*.*

Proof : The proof is given in Appendix A. □

Remark 4.2. *Here, the derivative of the error is not restricted and is allowed to be arbitrarily large. If for some reason, the derivative is bounded from above by one, the latter results can be refined. Indeed, a sharper bound on* $||\Delta||_{\mathcal{L}_2-\mathcal{L}_2}$ *can be shown to be 1. Moreover, the derivative bound can also be taken into account through the introduction of another operator (the delay operator) similarly as in [17].*

It is important to note that when the delay is exactly known (i.e. $\delta = 0$), then Theorem 4.2 reduces to Theorem 4.1. This is stated in the following proposition:

Proposition 4.2. *When* $\delta = 0$*, then LMIs (24) is equivalent to LMIs (19) provided that the matrix* $S(\lambda, \rho) \succ 0$ *is chosen sufficiently small (e.g. according to the 2-norm).*

Proof : The proof is only sketched since it relies on simple arguments and easy calculations. First, set $\delta = 0$ in (24), this creates 0 entries on the 4^{th} row and column of (24) except for the diagonal value which is $-S_i(\rho)$. Since $S_i(\rho)$ is positive definite we can remove the 4^{th} row and column from (24). Now, we just need to analyze the impact of the remaining terms depending on $S_i(\rho)$ which are located on the 8^{th} row and column of (24). A Schur complement on the block $(8, 8)$ leads to a matrix sum of the form $\Lambda_i(\rho) + \Upsilon_i(\rho) \prec 0$ where $\Lambda_i(\rho)$ is exactly LMI (19) and

$$\Upsilon_i(\rho) = \begin{bmatrix} S_i(\rho) & 0 & \dots & 0 & -S_i(\rho) & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 \\ -S_i(\rho) & 0 & \dots & 0 & S_i(\rho) & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

It can be shown that this matrix has n positive eigenvalues (those of $S_i(\rho)$) and $\alpha - n$ zero eigenvalues where α is the dimension of $\Upsilon_i(\rho)$. Thus, choosing $S_i(\rho)$ as small as necessary, it is possible to approximate arbitrarily well LMI (19) by LMI (24) with $\delta = 0$. □

This shows that the main result embeds naturally (by construction) the case of controller with exact memory. Another important fact is when $\delta = h_M$, we get a result which is close to the memoryless case. This will be illustrated in the examples.

Remark 4.3. *The LMI conditions of Theorems 3.1, 4.1 and 4.2 must be satisfied for all* $(\rho, \nu) \in U_\rho \times co\{U_\nu\}$*. However, it is possible to reduce the computational complexity through the reduction of parameter set from* $U_\rho \times co\{U_\nu\}$ *to* $U_\rho \times U_\nu$*. This is done using the following particular structure for the matrix* $P(\lambda, \rho) = P_0(\rho) + \sum_{i=1}^{N} \lambda_i P_i$*. In this case, the resulting LMI conditions involve no product between* λ *and* $\dot{\rho}$ *and hence LMIs have to be checked only on the set* $U_\rho \times co\{U_\nu\}$*.*

5. Examples

This section is devoted to examples and discussions on the provided approach. It will be illustrated that the current approach improves result of the literature in the control of LPV time-delay systems. Moreover, the connection between memoryless and exact-memory controllers

μ	0	0.5	0.9	0.99
h_M [37]	1.2	1.2	1.2	~ 1
h_M [9]	–	79.1511	–	–
h_M Theorem 4.1	929.1372	371.0928	6.9218	2.9325

Table 1: Evolution of the delay margin h_M with respect to the bound on the delay derivative μ for a maximal allowable $\gamma < 10$

through δ-memory resilient controllers will be illustrated. Let us consider the following system borrowed from [35] and modified in [37]:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1+\phi\sin(t) \\ -2 & -3+\sigma\sin(t) \end{bmatrix} x(t) + \begin{bmatrix} \phi\sin(t) & 0.1 \\ -0.2+\sigma\sin(t) & -0.3 \end{bmatrix} x(t-h(t)) \\ &\quad + \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} w(t) + \begin{bmatrix} \phi\sin(t) \\ 0.1+\sigma\sin(t) \end{bmatrix} u(t) \\ z(t) &= \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t) \end{aligned} \tag{25}$$

where $\phi = 0.2$ and $\sigma = 0.1$. Define $\rho(t) := \sin(t)$, $h_M = 0.5$ and $\mu = 0.5$ as in [37].

5.1. Example 1: Memoryless State-Feedback (Small delay)

First, a memoryless control law is computed using Theorem 4.1. The parameter dependent decision matrices are chosen to be

$$\begin{aligned} P(\rho) &= P^0 + P^1\rho + P^2\rho^2 \\ Y_0(\rho) &= Y_0^0 + Y_0^1\rho + Y_0^2\rho^2 \end{aligned}$$

Verifying the LMI of Theorem 4.1 over a grid of $N_g = 41$ points yields a minimal value $\gamma^* = 1.9089$ which is better than all results obtained before. In [37], a minimal value of $\gamma = 3.09$ is found while in [9] an optimal value of $\gamma = 2.27$ is obtained (using a nonlinear approach). The controller computed using Theorem 4.1 is given by

$$K_0(\rho) = \begin{bmatrix} -1.0535 - 2.9459\rho + 1.9889\rho^2 \\ -1.1378 - 2.6403\rho + 1.9260\rho^2 \end{bmatrix}^T$$

It is worth noting that the results are even better while the controller has smaller coefficients than in the other approaches [37, 9]. It is hence expected to have a smaller control input which should remain within acceptable bounds, even in presence of disturbances.

The influence of μ on the delay-margin (with $\mathcal{L}_2$ performance constraint) is detailed in Table 1 where, as expected, the delay margin decreases as the value of μ increases. Moreover, the results of [37] are more conservative than those obtained using Theorem 4.1 since the delay-margin is always smaller than 1.4 (for $\gamma < 10$) for any value of μ.

As a final remark, the maximal value for h_M obtained for $\mu = 0$ is large and this suggests that the system might be delay-independent stabilizable in the constant delay case.

5.2. Example 2: State Feedback with Exact Memory (Large delay)

Still considering system (25) but with $h_M = 10$ and $\mu = 0.9$, an instantaneous state-feedback controller of the form $u(t) = K_0(\rho)x(t)$ is sought. Theorem 4.1 yields:

$$K_0(\rho) = \begin{bmatrix} 0.5724 - 6.3679\rho - 1.4898\rho^2 \\ -0.7141 - 4.1617\rho - 0.8425\rho^2 \end{bmatrix}^T \tag{26}$$

Exact memory	$\gamma = 4.1641$	Memoryless	$\gamma = 12.8799$
0-resilient	$\gamma = 4.1658$	10-resilient	$\gamma = 13.0604$

Table 2: Comparison of the results obtained using Theorem 4.2

which ensures a closed-loop $\mathcal{L}_2$ input/output performance level of 12.8799. Now an exact-memory state-feedback control law $u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t))$ is computed using Theorem 4.1 and the obtained controller gains are given by

$$\begin{aligned} K_0(\rho) &= \begin{bmatrix} 1.0524 - 2.8794\rho - 0.4854\rho^2 \\ -0.7731 - 1.8859\rho + 0.1181\rho^2 \end{bmatrix}^T \\ K_h(\rho) &= \begin{bmatrix} -0.6909 + 0.5811\rho + 0.1122\rho^2 \\ -0.0835 + 0.3153\rho + 0.0689\rho^2 \end{bmatrix}^T . \end{aligned}$$

This controller ensures a closed-loop $\mathcal{L}_2$ performance level of 4.1641. The gain of performance resulting from the use of a controller involving a delayed term is evident. However, this controller is non implementable due to the practical impossibility of knowing the exact delay value at any time. This motivates the synthesis of a memory-resilient controller.

5.3. Example 3: δ-Memory Resilient Controller Synthesis (Large delay)

Finally, a δ-memory resilient state-feedback controller stabilizing system (25) is sought. Using Theorem 4.2, the achieved minimal closed-loop $\mathcal{L}_2$ performance with respect to δ is plotted in Figure 1. As expected the minimal $\mathcal{L}_2$ performance level grows as the delay uncertainty radius increases. This illustrates that the achieved closed-loop performance is deteriorated when the delay is badly known.

Moreover, there are two remarkable values for the worst-case $\mathcal{L}_2$ gain, respectively obtained for $\delta = 0$ and $\delta = h_M$. For these particular values we have:

$$\begin{aligned} \gamma|_{\delta=0} &= 4.1658 \\ K_0(\rho)|_{\delta=0} &= \begin{bmatrix} 1.0542 - 2.8895\rho - 0.4827\rho^2 \\ -0.7714 - 1.8912\rho + 0.1216\rho^2 \end{bmatrix}^T \\ K_h(\rho)|_{\delta=0} &= \begin{bmatrix} -0.6885 + 0.5849\rho + 0.1116\rho^2 \\ -0.0817 + 0.3148\rho + 0.0667\rho^2 \end{bmatrix}^T \end{aligned} \tag{27}$$

$$\begin{aligned} \gamma|_{\delta=10} &= 13.0604 \\ K_0(\rho)|_{\delta=10} &= \begin{bmatrix} 0.4422 - 6.3469\rho - 1.3619\rho^2 \\ -0.9475 - 4.1219\rho - 0.6140\rho^2 \end{bmatrix}^T \\ K_h(\rho)|_{\delta=10} &= \begin{bmatrix} -0.0163 - 0.0005\rho + 0.0127\rho^2 \\ -0.0007 - 0.0006\rho + 0.0011\rho^2 \end{bmatrix}^T \end{aligned} \tag{28}$$

When the delay is exactly known (i.e. $\delta = 0$), the $\mathcal{L}_2$ performance index and the controller are very close (quite identical) to the ones obtained using Theorem 4.1 which considers exact-memory controllers. This illustrates well Remark 4.2.

On the other hand, when $\delta = h_M$, it could be considered that the delay is actually unknown since for any value for $h(t)$, the implemented value $d(t)$ may take any value into $[0, h_M]$. Hence, this means that the results with such a value for δ should be close to the results obtained using a memoryless control law. Comparing (28) and (26) it is possible to remark that the obtained closed-loop performance level is very near to the one obtained with a memoryless control law. Moreover, the matrix gain $K_h(\rho)$ in (28) has a small norm making it almost identical to a memoryless controller. The delayed action is highly penalized due to a too large uncertainty on the delay knowledge. The results are summarized in Table 2.

The above discussion illustrates that δ-memory resilient controllers define the intermediary behavior of the closed-loop system between the two extremal controllers: the memoryless and the exact-memory controllers. The emphasis of this continuity between memoryless and exact memory

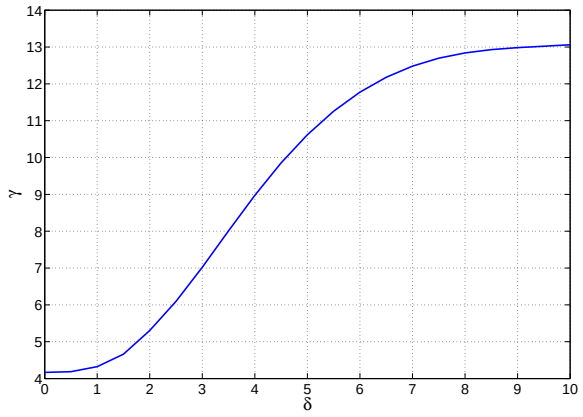


Figure 1: Evolution of the worst-case $\mathcal{L}_2$ gain of the closed-loop system with respect to maximal delay uncertainty δ

controller through δ-memory resilient controllers constitutes one of the main contribution of the paper. Moreover, such controllers are also more realistic, from a practical point of view, than exact-memory controllers.

6. Conclusion

The current paper introduces a new approach to the stabilization of LPV time-delay systems using parameter dependent state-feedback controllers. First, a delay-dependent stability test with $\mathcal{L}_2$ performance analysis for LPV time-delay systems with time-varying delays is provided in terms of parameter dependent LMIs. This result is obtained from the use of a parameter dependent Lyapunov-Krasovskii functional used along with the Jensen's inequality, an approach which has proven its efficiency. Since this result is not well suited for design purpose due to multiple products between decision matrices and system matrices, a relaxed version of the result is developed. This version involves an additional 'slack' (or 'lifting') variable and avoids any nonlinear terms (multiple products). This allows to find a linearizing change of variable for the stabilization problem.

A first stabilization result is then provided and characterizes both memoryless and exact-memory controllers. However, due to the difficulty of estimating delays, latter controllers are generally non-implementable in practice. This has motivated the development of another type of controllers, called 'δ-memory resilient controllers' where the delay implementation error is taken into account in the design. It turns out that this new class of controllers connect memoryless and exact-memory controllers together, through a unique formulation. Indeed, by acting the error bound δ between the delays, it is possible to recover exact-memory ($\delta = 0$), δ-memory resilient ($\delta \in (0, h_M)$) and memoryless ($\delta \sim h_M$) control laws successively. This part constitutes the main contribution of the paper and is illustrated in the examples.

Appendix A. Proof of theorem 4.2

Let us consider the closed-loop system (21) and the Lyapunov-Krasovskii functional V given in (10). Computing the derivative of (10) along the trajectories solutions of system (21) we get

$$\begin{array}{lcl} \dot{V} & \leq & X(t)^T \Psi(\lambda, \dot{\rho}, \rho) X(t) \\ \Psi(\lambda, \rho, \dot{\rho}) & = & \Pi(\lambda, \rho, \dot{\rho}) + h_M^2 \Gamma_1(\lambda, \rho)^T R(\lambda) \Gamma_1(\lambda, \rho) \end{array} \tag{A.3}$$

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} + \dot{P} + Q - R & P A_{hcl} + R & \frac{\sqrt{7}}{2}\delta P A_h + \delta R & PE & C_{cl}^T & h_M A_{cl}^T R & A_{cl}^T L \\ \star & -Q_\mu - R & -\sqrt{2}\delta(Q_\mu + R) & 0 & C_{hcl}^T & h_M A_{hcl}^T R & A_{hcl}^T L \\ \star & \star & -\frac{7}{4}\delta^2(Q_\mu + R) - L & 0 & \frac{\sqrt{7}}{2}\delta C_h^T & \frac{\sqrt{7}}{2} h_M \delta A_h^T R & \frac{\sqrt{7}}{2}\delta A_h^T L \\ \star & \star & \star & -\gamma I_p & F^T & h_M E^T R & E^T L \\ \star & \star & \star & \star & -\gamma I_q & 0 & 0 \\ \star & \star & \star & \star & \star & -R & 0 \\ \star & \star & \star & \star & \star & \star & -L \end{bmatrix} \prec 0 \tag{A.1}$$

$$\begin{bmatrix} -(X + X^T) & P + X^T A_{cl} & X^T A_{hcl} & \frac{\sqrt{7}}{2}\delta X^T A_h & X^T E & 0 & X^T & L & h_M R \\ \star & \dot{P} - P + Q - R & R & \frac{\sqrt{7}}{2}\delta R & 0 & C_{cl}^T & 0 & 0 & 0 \\ \star & \star & -Q_\mu - R & -\frac{\sqrt{7}}{2}\delta(Q_\mu + R) & 0 & C_{hcl}^T & 0 & 0 & 0 \\ \star & \star & \star & -\frac{7}{4}\delta^2(Q_\mu + R) - L & 0 & \frac{\sqrt{7}}{2}\delta C_h^T & 0 & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_p & F^T & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\gamma I_q & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -P & -L & -h_M R \\ \star & \star & \star & \star & \star & \star & \star & -L & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{A.2}$$

where $X(t) = col(x(t), x_h(t), x_d(t), w(t))$, Π is defined by

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & P(\lambda,\rho)E(\lambda,\rho) \\ \star & -(1-\mu)Q(\lambda) - R(\lambda) & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & 0 \end{bmatrix}$$

with

$$\begin{aligned} \Pi_{11} &= \left[A_{cl}(\lambda,\rho)^T P(\lambda,\rho)\right]^{\mathcal{S}} + \frac{\partial P(\lambda,\rho)}{\partial \rho}\dot{\rho} + Q(\lambda) - R(\lambda) \\ \Pi_{12} &= P(\lambda,\rho)A_h(\lambda,\rho) + R(\lambda) \\ \Pi_{13} &= P(\lambda,\rho)B(\lambda,\rho)K_h(\rho) \end{aligned}$$

and $\Gamma_1(\rho) = \begin{bmatrix} A_{cl}(\rho) & A_h(\rho) & B(\rho)K_h(\rho) & E(\rho) \end{bmatrix}$. Now according to the relation $w_0 = \Delta(\dot{x}) = 2(x_h(t) - x_d(t))/\sqrt{7}\delta$ we have

$$X(t) = \underbrace{\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & \frac{\sqrt{7}\delta}{2} I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} x(t) \\ x_d(t) \\ w_0(t) \\ w(t) \end{bmatrix}}_{Y(t)} \tag{A.4}$$

and thus

$$\dot{V} \leq Y(t)^T M^T \Psi(\lambda, \dot{\rho}, \rho) M Y(t) \tag{A.5}$$

In order to consider the uncertain norm-bounded operator Δ and input/output $\mathcal{L}_2$ performance, the following quadratic supply-rate $s(\zeta(t)) = \zeta(t)^T \Phi(\lambda,\rho)\zeta(t)$ defined by

$$\Phi(\lambda,\rho) = \text{diag}(-L(\lambda,\rho), L(\lambda,\rho), -\gamma I_p, \gamma I_q) \tag{A.6}$$

is added to (A.5) where $\zeta(t) = col(w_0(t), z_0(t), w(t), z(t))$ and $w_0(t) = \Delta[z_0(t)]$. Here, $L(\lambda,\rho) = L(\lambda,\rho)^T \succ 0$ is a parameter dependent D-scaling and $\gamma > 0$ characterizes the upper bound on the

$\mathcal{L}_2$ gain of the transfer $w \to z$. This leads to inequality (A.1) where the dependence on ρ and λ has been dropped for clarity. Then applying the same relaxation procedure as in the proof of Theorem 3.1 we get the new inequality (A.2) with $A_{hcl}(\cdot) = A_h(\cdot) + B(\cdot)K_h(\cdot)$. Finally performing a congruence transformation with respect to

$$diag(I_4 \otimes X^{-1}, I_{p+q}, I_3 \otimes X^{-1})$$

along with the change of variables

$$\begin{array}{lcllcl} X & \leftarrow & X^{-1} & P_i(\rho) & \leftarrow & X^{-T}P_i(\rho)X^{-1} \\ Q_i & \leftarrow & X^{-T}Q_iX^{-1} & R & \leftarrow & X^{-T}R_iX^{-1} \\ S_i(\rho) & \leftarrow & X^{-T}S_i(\rho)X^{-1} & Y_0(\rho) & \leftarrow & K_0(\rho)X^{-1} \\ Y_h(\rho) & \leftarrow & K_h(\rho)X^{-1} & & & \end{array} \tag{A.7}$$

yields LMI (24). This concludes the proof.

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