SOBOLEV AND HARDY-LITTLEWOOD-SOBOLEV INEQUALITIES: DUALITY AND FAST DIFFUSION

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Abstract. In the euclidean space, Sobolev and Hardy-Littlewood-Sobolev inequalities can be related by duality. In this paper, we investigate how to relate these inequalities using the flow of a fast diffusion equation in dimension $d \geq 3$. The main consequence is an improvement of Sobolev's inequality when $d \geq 5$, which involves the various terms of the dual Hardy-Littlewood-Sobolev inequality. In dimension $d = 2$, Onofri's inequality plays the role of Sobolev's inequality and can also be related to its dual inequality, the logarithmic Hardy-Littlewood-Sobolev inequality, by a super-fast diffusion equation.

1. Introduction

In dimension $d \geq 3$, it is well known since E. Lieb's paper [21] that Hardy-Littlewood-Sobolev inequalities are dual of Sobolev's inequalities. We will investigate this duality using the flow of a fast diffusion equation which has been considered in [14]. In dimension $d = 2$, Onofri's inequality plays the role of Sobolev's inequality and is the dual of the logarithmic Hardy-Littlewood-Sobolev inequality according to [7, 2, 6]. Based on [12, 28], we will investigate this duality using a variant of the logarithmic diffusion equation, also known as the super-fast diffusion equation.

In a recent paper, [8], E. Carlen, J.A. Carrillo and M. Loss noticed that Hardy-Littlewood-Sobolev inequalities in dimension $d \geq 3$ (or the logarithmic Hardy-Littlewood-Sobolev inequality if $d = 2$) and some special Gagliardo-Nirenberg inequalities can be related through another fast diffusion equation. Understanding the differences between the two approaches is one of the motivations of this paper.

Consider Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \leq \mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, u \in \mathcal{D}^{1,2}(\mathbb{R}^d) \,, \tag{1}$$

where S_d is the Aubin-Talenti constant (see [1, 26]) and $2^* = \frac{2\,d}{d-2}$. The space $\mathcal{D}^{1,2}(\mathbb{R}^d)$ is defined as the completion of smooth solutions with compact support w.r.t. the norm $w \mapsto \|w\| := (\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)})^{1/2}$. The Hardy-Littlewood-Sobolev inequality

$$\mathsf{S}_d \, \|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} \geq \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} v \, dx \quad \forall\, v \in \mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d) \tag{2}$$

involves the same optimal constant, S_d.

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As it has been noticed in [21], (1) and (2) are dual of each other, in the following sense. To a convex functional F, we may associate the functional F^* defined by Legendre's duality as

$$F^*[v] := \sup\left(\int_{\mathbb{R}^d} u\,v\,dx - F[u]\right)\,. \tag{3}$$

For instance, to $F_1[u] = \frac12\,\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)}$ defined on $\mathrm{L}^p(\mathbb{R}^d)$, we henceforth associate $F_1^*[v] = \frac12\,\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}$ on $\mathrm{L}^q(\mathbb{R}^d)$ where p and q are Hölder conjugate exponents: $1/p + 1/q = 1$. The supremum can be taken for instance on all functions in $\mathrm{L}^p(\mathbb{R}^d)$, or, by density, on the smaller space of the functions $u \in \mathrm{L}^p(\mathbb{R}^d)$ such that $\nabla u \in \mathrm{L}^2(\mathbb{R}^d)$. Similarly, to $F_2[u] = \frac12\,\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, we associate $F_2^*[v] = \frac12\,\mathsf{S}_d^{-1}\int_{\mathbb{R}^d} v\,(-\Delta)^{-1}v\,dx$ where $(-\Delta)^{-1}v = G_d * v$ with $G_d(x) = \frac{1}{d-2}\,|\mathbb{S}^{d-1}|^{-1}\,|x|^{2-d}$. As a straightforward consequence of Legendre's duality, if we have a functional inequality of the form $F_1[u] \le F_2[u]$, then we have the dual inequality $F_1^*[v] \ge F_2^*[v]$.

In this paper, we investigate the duality of (1) and (2) using a nonlinear diffusion equation. If v is a positive solution of the following *fast diffusion* equation:

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d\,, \tag{4}$$

and if we define $\mathsf{H}(t) := \mathsf{H}_d[v(t,\cdot)]$, with

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1}v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}\,,$$

then we observe that

$$\frac12\,\mathsf{H}' = -\int_{\mathbb{R}^d} v^{m+1}\,dx + \mathsf{S}_d\left(\int_{\mathbb{R}^d} v^{\frac{2\,d}{d+2}}\,dx\right)^{\frac2d}\int_{\mathbb{R}^d}\nabla v^m\cdot\nabla v^{\frac{d-2}{d+2}}\,dx$$

where $v = v(t,\cdot)$ is a solution of (4). With the choice $m = \frac{d-2}{d+2}$, we find that $m+1 = \frac{2\,d}{d+2}$, so that the above identity can be rewritten with $u = v^m$ as follows.

Proposition 1.1. *Assume that $d \ge 3$ and $m = \frac{d-2}{d+2}$. If v is a solution of (4) with nonnegative initial datum in $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, then*

$$\frac12\,\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1}v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}\right]$$
$$= \left(\int_{\mathbb{R}^d} v^{m+1}\,dx\right)^{\frac2d}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right] \ge 0\,.$$

As a consequence, one can prove that (2), which amounts to $\mathsf{H} \le 0$, is a consequence of (1), that is $\mathsf{H}' \ge 0$, by showing that $\limsup_{t>0}\mathsf{H}(t) = 0$. In this way, we also recover the property that $u = v^m$ is an optimal function for (1) if v is optimal for (2). By integrating along the flow defined by (4), we can actually obtain optimal integral remainder terms which improve on the usual Sobolev inequality (1), but only when $d \ge 5$ for integrability reasons: see Theorem 2.3. A slightly weaker result is the following improved Sobolev inequality, which relates (1) and (2).

Theorem 1.2. *Assume that $d \ge 5$ and let $q = \frac{d+2}{d-2}$. There exists a positive constant $\mathcal{C}$ such that, for any $w \in \mathcal{D}^{1,2}(\mathbb{R}^d)$, we have*

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \\ \le \mathcal{C} \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right] .$$

Moreover, we know that $\mathcal{C} \le \left(1 + \frac{2}{d}\right) \left(1 - e^{-d/2}\right) \mathsf{S}_d$.

Considerable efforts have been devoted to obtain improvements of Sobolev's inequality starting with [5, 4]. On the whole euclidean space, nice results based on rearrangements have been obtained in [10] and we refer to [9] for an interesting review of the various improvements that have been established over the years. It has to be noted that they are all of different nature than the inequality in Theorem 1.2, which is the main result of this paper.

Our approach is based on the evolution equation (4). Many papers have been devoted to the study of the asymptotic behaviour of the solutions, in bounded domains: [3, 16, 25], or in the whole space: [19, 24, 17]. In particular, the Cauchy-Schwarz inequality has been repeatedly used, for instance in [3, 25], and turns out to be a key tool in the proof of Theorem 1.2, as well as the solution *with separation of variables,* which is related to the Aubin-Talenti optimal function for (1). The novelty in our approach is to consider the problem from the point of view of the functional associated to (2) using(4) with $m = (d-2)/(d+2)$.

We now turn our attention to the case of the dimension $d = 2$. As we shall see in Section 3, Onofri's inequality [23]

$$\log \left(\int_{\mathbb{R}^2} e^{\,g} \, d\mu \right) - \int_{\mathbb{R}^2} g \, d\mu \le \frac{1}{16\,\pi} \int_{\mathbb{R}^2} |\nabla g|^2 \, dx \quad \forall \, g \in \mathcal{D}(\mathbb{R}^d) \tag{5}$$

plays the role of Sobolev's inequality in higher dimensions. Here the probability measure $d\mu$ is defined by

$$d\mu(x) := \mu(x)\, dx \quad \text{with} \quad \mu(x) := \frac{1}{\pi \, (1 + |x|^2)^2} \quad \forall \, x \in \mathbb{R}^2.$$

As for Sobolev's inequality, duality can also be used. The dual of Onofri's inequality is the logarithmic Hardy-Littlewood-Sobolev inequality: for any $f \in \mathrm{L}^1_+(\mathbb{R}^2)$ with $M = \int_{\mathbb{R}^2} f \, dx$, such that $f \, \log f$, $(1 + \log |x|^2) \, f \in \mathrm{L}^1(\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2} f \, \log \left(\frac{f}{M}\right) dx + \frac{2}{M} \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) \, f(y) \, \log |x - y| \, dx \, dy + M \, (1 + \log \pi) \ge 0 \, . \tag{6}$$

The duality has been established on the two-dimensional sphere in [2, 7] and directly on the euclidean space $\mathbb{R}^2$ in [6]. The euclidean case can also be recovered from the inequality on the sphere using the stereographic projection. For completeness, we shall give a proof of the inequality (with optimal constants) in the $\mathbb{R}^2$ case: see Proposition 3.2. The logarithmic Hardy-Littlewood-Sobolev inequality has recently attracted lots of attention in connection with the Keller-Segel model or in geometry: see for instance [6, 15, 22].

We may now proceed in the case $d = 2$ as we did for $d \ge 3$. Let

$$\mathsf{H}_2[v] := \int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\,dx - \frac{1}{4\,\pi}\int_{\mathbb{R}^2} v\,\log\left(\frac{v}{\mu}\right)\,dx\,.$$

Assume that v is a positive solution of

$$\frac{\partial v}{\partial t} = \Delta\log\left(\frac{v}{\mu}\right) \quad t>0\,,\quad x\in\mathbb{R}^2\,, \tag{7}$$

which replaces (4). Then we have the analog of Proposition 1.1.

Proposition 1.3. *Assume that $d = 2$. If v is a solution of (7) with nonnegative initial datum v_0 in $\mathrm{L}^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} v_0\,dx = 1$, $v_0\,\log v_0 \in \mathrm{L}^1(\mathbb{R}^2)$ and $v_0\,\log\mu \in \mathrm{L}^1(\mathbb{R}^2)$, then*

$$\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)] = \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx - \int_{\mathbb{R}^2}\left(e^{\frac{u}{2}}-1\right)u\,d\mu$$

with $\log(v/\mu) = u/2$.

The right hand side is nonnegative by Onofri's inequality:

$$\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)] \ge \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx + \int_{\mathbb{R}^2} u\,d\mu - \log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right) \ge 0\,.$$

See Lemma 3.3 and Corollary 3.4 for details, and Section 3.4 for further considerations on the two-dimensional case.

2. Improved Sobolev inequalities

This section is devoted to the proof of Theorem 1.2. We shall assume that

$$m = \frac{d-2}{d+2} \quad\text{and}\quad d \ge 3\,.$$

From the computations of Section 1, it is clear that the maximum of $\mathsf{H}_d[v]$ is achieved if and only if $u = v^m$ is an extremal for Sobolev's inequality. This is of course consistent with the fact that extremal points for Hardy-Littlewood-Sobolev inequalities and Sobolev inequalities are related through Legendre's duality precisely by the relation $u = v^m$.

It is also straightforward to check that (4) admits special solutions with *separation of variables* such that, for any $T > 0$,

$$\overline{v}_T(t,x) = c\,(T-t)^\alpha\,(F(x))^{\frac{d+2}{d-2}} \quad \forall\,(t,x)\in(0,T)\times\mathbb{R}^d\,,$$

where $\alpha = (d+2)/4$, $c^{1-m} = 4\,m\,d$, $m = \frac{d-2}{d+2}$, $p = d/(d-2)$ and F is the Aubin-Talenti solution of $-\Delta F = d\,(d-2)\,F^{(d+2)/(d-2)}$. Such a solution vanishes at $t = T$ and this behaviour is generic in a large class of solutions. Define

$$\|v\|_* := \sup_{x\in\mathbb{R}^d}(1+|x|^2)^{d+2}\,|v(x)|\,.$$

Lemma 2.1. [14, 27] *For any solution v of (4) with nonnegative, continuous, not identically zero initial datum $v_0 \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, there exists $T > 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$ such that $v(t,\cdot) \not\equiv 0$ for any $t \in (0,T)$ and*

$$\lim_{t\to T_-}(T-t)^{-\frac{1}{1-m}}\,\|v(t,\cdot)/\overline{v}(t,\cdot)-1\|_* = 0$$

with $\overline{v}(t,x) = \lambda^{(d+2)/2}\,\overline{v}_T(t,(x-x_0)/\lambda)$.

We shall refer to such a solution as a *solution vanishing at time* T. The above result has been established first in [14] when $\|v_0\|_*$ is finite and extended to solutions corresponding to any initial data $v_0 \in \mathrm{L}^{2d/(d+2)}_+(\mathbb{R}^d)$ in [27, Theorem 7.10]. In this framework, it is easy to establish further *a priori* estimates as follows.

Lemma 2.2. *Let $d \ge 3$ and $m = (d-2)/(d+2)$. If v is a solution of (4) vanishing at time $T > 0$ with initial datum $v_0 \in \mathrm{L}^{2d/(d+2)}_+(\mathbb{R}^d)$, then for any $t \in (0,T)$ we have the estimates*

$$\left(\tfrac{4\,(T-t)}{(d+2)\,\mathsf{S}_d}\right)^{\frac{d}{2}} \le \int_{\mathbb{R}^d} v^{m+1}(t,x)\,dx \le \int_{\mathbb{R}^d} v_0^{m+1}\,dx\,,$$
$$\|\nabla v^m(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \mathsf{S}_d^{-1}\left(\tfrac{4\,(T-t)}{d+2}\right)^{\frac{d}{2}-1}\,,$$

and the vanishing time T is bounded by

$$T \le \frac{1}{4}\,(d+2)\,\mathsf{S}_d\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{\frac{2}{d}}\,.$$

If additionnally $d \ge 5$, then $T \ge \frac{d+2}{2\,d}\,\int_{\mathbb{R}^d} v_0^{m+1}\,dx\,\|\nabla v_0^m\|^{-2}_{\mathrm{L}^2(\mathbb{R}^d)}$ and

$$\int_{\mathbb{R}^d} v^{m+1}(t,x)\,dx \ge \int_{\mathbb{R}^d} v_0^{m+1}\,dx - \tfrac{2\,d}{d+2}\,t\,\|\nabla v_0^m\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\,,$$
$$\|\nabla v^m(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le \|\nabla v_0^m\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\,.$$

Proof. By definition of the vanishing time, we find that $\mathsf{J}(t) := \int_{\mathbb{R}^d} v(t,x)^{m+1}\,dx$ satisfies $\mathsf{J}(t) > 0$ for any $t \in (0,T)$, $\mathsf{J}(T) = 0$ and, using the equation and integrating by parts,

$$\mathsf{J}' = -(m+1)\,\|\nabla v^m\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{1-\frac{2}{d}}\,.$$

If $d \ge 5$, then we also have

$$\mathsf{J}'' = 2\,m\,(m+1)\int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \ge 0\,.$$

It is easy to check that such an estimate makes sense if $v = \overline{v}_T$. For a general solution, this is also true as can be seen by rewriting the problem on $\mathbb{S}^d$ as in [14]. It is then clear that integrability conditions for v are exactly the same as for $\overline{v}_T$.

By integrating the first inequality from t to T, we find that

$$-\mathsf{J}(t)^{\frac{2}{d}} = \mathsf{J}(T)^{\frac{2}{d}} - \mathsf{J}(t)^{\frac{2}{d}} \le -\tfrac{4\,(T-t)}{(d+2)\,\mathsf{S}_d} \quad \forall\, t \in [0,T)\,,$$

which gives an upper bound for T by choosing $t = 0$. The second inequality shows the decay of $t \mapsto \|\nabla v^m(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ and gives the estimate $0 = \mathsf{J}(T) \ge \mathsf{J}(0) + T\,\mathsf{J}'(0)$, thus providing a lower bound for T. □

For later purpose, let us notice that

$$\frac{\mathsf{J}'}{\mathsf{J}} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{-\frac{2}{d}} \le -\kappa \quad \text{with} \quad \kappa := \frac{2\,d}{d+2}\,\frac{1}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-\frac{2}{d}} \le \frac{d}{2\,T}\,. \tag{8}$$

Next, we are going to compute the second derivative of $\mathsf{H}(t)=\mathsf{H}_d[v(t,\cdot)]$ w.r.t. t along the flow of (4). For this purpose, we assume that $d\ge5$ and notice that by the Cauchy-Schwarz inequality, we have

$$\|\nabla v^m\|^4_{\mathrm{L}^2(\mathbb{R}^d)}=\left(\int_{\mathbb{R}^d}v^{(m-1)/2}\,\Delta v^m\cdot v^{(m+1)/2}\,dx\right)^2\\ \le\int_{\mathbb{R}^d}v^{m-1}\,(\Delta v^m)^2\,dx\int_{\mathbb{R}^d}v^{m+1}\,dx\;.$$

As a consequence, we get that $\mathsf{Q}(t):=\|\nabla v^m(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d}v^{m+1}(t,x)\,dx\right)^{-(d-2)/d}$ is monotone decreasing. More precisely, with $m=(d-2)/(d+2)$,

$$\Lambda(t):=\frac{\int_{\mathbb{R}^d}|\nabla(v(t,x))^m|^2\,dx}{\int_{\mathbb{R}^d}(v(t,x))^{m+1}\,dx}\quad\text{and}\quad\mathsf{G}(t_1,t_2):=\exp\left[(m+1)\int_{t_1}^{t_2}\Lambda(s)\,ds\right]\;,\tag{9}$$

we get that $\mathsf{Q}'=-2\,m\,\mathsf{J}^{\frac{2}{d}-1}\,\mathsf{K}$ with $\mathsf{K}:=\int_{\mathbb{R}^d}v^{m-1}\,|\Delta v^m+\Lambda\,v|^2\,dx$ and, from

$$\mathsf{H}'=2\,\mathsf{J}\,(\mathsf{S}_d\,\mathsf{Q}-1)\;,$$

we deduce that

$$\mathsf{H}''=\frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}'+2\,\mathsf{J}\,\mathsf{S}_d\,\mathsf{Q}'=-(m+1)\,\Lambda\,\mathsf{H}'-4\,m\,\mathsf{S}_d\,\mathsf{J}^{\frac{2}{d}}\,\mathsf{K}\;.\tag{10}$$

As a consequence, the standard Sobolev inequality can be improved by *an integral remainder term.*

Theorem 2.3. *Assume that $d\ge5$ and let $q=\frac{d+2}{d-2}$. For any $w\in\mathcal{D}^{1,2}(\mathbb{R}^d)$ such that $\|w^q\|_*<\infty$, we have*

$$\mathsf{S}_d\,\|w^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}-\int_{\mathbb{R}^d}w^q\,(-\Delta)^{-1}w^q\,dx+\frac{4}{q}\,\mathsf{S}_d\int_0^T dt\int_0^t\mathsf{J}^{\frac{2}{d}}(s)\,\mathsf{K}(s)\,\mathsf{G}(t,s)\,ds\\ =2\,\|w\|^{\frac{4}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)}-\mathsf{S}_d\,\|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right]\int_0^T\mathsf{G}(t,0)\,dt\;.$$

Here v is the solution of (4) with $v_0=w^q$, $m=(d-2)/(d+2)=1/q$, T is the vanishing time, Λ and G are defined by (9), and we recall that $\mathsf{J}(t)=\int_{\mathbb{R}^d}v^{m+1}(t,x)\,dx$, $\mathsf{K}(t)=\int_{\mathbb{R}^d}v^{m-1}(t,x)\,|\Delta v^m(t,x)+\Lambda(t)\,v(t,x)|^2\,dx$.

Proof. The identity follows from (10) after an integration from 0 to t and another one from 0 to T. Details are left to the reader. □

Recall that the standard Hardy-Littlewood-Sobolev inequality (2) amounts to

$$\mathsf{S}_d\,\|w^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}-\int_{\mathbb{R}^d}w^q\,(-\Delta)^{-1}w^q\,dx=-\mathsf{H}_d[w^q]\ge0\;.$$

The main drawback of Theorem 2.3 is that several quantities can be computed only through the evolution equation and are therefore non explicit. With simple estimates, we can however get rid of such quantities. This is the purpose of Theorem 1.2.

Proof of Theorem 1.2. Notice that $\|w^q\|^2_{\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)}=\|w\|^{1+2/d}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$, so that the inequality holds in the space $\mathcal{D}^{1,2}(\mathbb{R}^d)$. Theorem 1.2 can be established first in the setting of smooth functions such that $\|w^q\|_*<\infty$ and then arguing by density.

From (8) and (10), we know that

$$\mathsf{H}'' \le -\kappa\,\mathsf{H}' \quad \text{with} \quad \kappa = \frac{2\,d}{d+2}\,\frac{1}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-2/d} .$$

By writing that $-\mathsf{H}(0) = \mathsf{H}(T) - \mathsf{H}(0) \le \mathsf{H}'(0)\,(1-e^{-\kappa\,T})/\kappa$ and using the estimate $\kappa\,T \le d/2$, we obtain the result. □

3. The two-dimensional case

3.1. From Gagliardo-Nirenberg inequalities to Onofri's inequality. Consider the following sub-family of Gagliardo-Nirenberg inequalities

$$\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \le \mathsf{C}_{p,d}\,\|\nabla f\|^{\theta}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|f\|^{1-\theta}_{\mathrm{L}^{p+1}(\mathbb{R}^d)} \tag{11}$$

with $\theta = \theta(p) := \frac{p-1}{p}\,\frac{d}{d+2-p\,(d-2)}$, $1 < p \le \frac{d}{d-2}$ if $d \ge 3$ and $1 < p < \infty$ if $d = 2$. Such an inequality holds for any smooth function f with sufficient decay at infinity and, by density, for any function $f \in \mathrm{L}^{p+1}(\mathbb{R}^d)$ such that ∇f is square integrable. We shall assume that $\mathsf{C}_{p,d}$ is the best possible constant. In [13], it has been established that equality holds in (11) if $f = F_p$ with

$$F_p(x) = (1+|x|^2)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d \tag{12}$$

and that all extremal functions are equal to F_p up to a multiplication by a constant, a translation and a scaling. If $d \ge 3$, the limit case $p = d/(d-2)$ corresponds to Sobolev's inequality and one recovers the results of T. Aubin and G. Talenti in [1, 26], with $\theta = 1$: the optimal functions for (1) are, up to scalings, translations and multiplications by a constant, all equal to $F_{d/(d-2)}(x) = (1+|x|^2)^{-(d-2)/2}$, and

$$\mathsf{S}_d = \left(\mathsf{C}_{d/(d-2),\,d}\right)^2 .$$

When $p \to 1$, the inequality becomes an equality, so that we may differentiate both sides with respect to p and recover the euclidean logarithmic Sobolev inequality in optimal scale invariant form (see [18, 29, 13] for details). For completeness, let us mention that the fast diffusion equation (4) has deep connection with (11) when $p = 1/(2\,m-1)$, but this is out of the scope of the present paper: see [13] for more details on this issue.

We now investigate the limit $p \to \infty$ in (11) when $d = 2$: Onofri's inequality (5) can be obtained in the limit. As an endpoint case of the family of the Gagliardo-Nirenberg inequalities (11), it plays the role of Sobolev's inequality in dimension $d \ge 3$.

Proposition 3.1. *Assume that $g \in \mathcal{D}(\mathbb{R}^d)$ is such that $\int_{\mathbb{R}^2} g\,d\mu = 0$ and let $f_p := F_p(1+\frac{g}{2p})$, where F_p is defined by (12). Then we have*

$$1 \le \lim_{p\to\infty} \mathsf{C}_{p,2}\,\frac{\|\nabla f\|^{\theta(p)}_{\mathrm{L}^2(\mathbb{R}^2)}\,\|f\|^{1-\theta(p)}_{\mathrm{L}^{p+1}(\mathbb{R}^2)}}{\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^2)}} = \frac{e^{\frac{1}{16\,\pi}\,\int_{\mathbb{R}^2}|\nabla g|^2\,dx}}{\int_{\mathbb{R}^2} e^{\,g}\,d\mu} .$$

We recall that $\mu(x) := \frac{1}{\pi}\,(1+|x|^2)^{-2}$, and $d\mu(x) := \mu(x)\,dx$.

Proof. We can rewrite (11) as

$$\frac{\int_{\mathbb{R}^2}|f|^{2p}\,dx}{\int_{\mathbb{R}^2}|F_p|^{2p}\,dx} \le \left(\frac{\int_{\mathbb{R}^2}|\nabla f|^2\,dx}{\int_{\mathbb{R}^2}|\nabla F_p|^2\,dx}\right)^{\frac{p-1}{2}}\,\frac{\int_{\mathbb{R}^2}|f|^{p+1}\,dx}{\int_{\mathbb{R}^2}|F_p|^{p+1}\,dx}$$

and observe that, with $f = f_p$, we have:
(i) $\lim_{p\to\infty} \int_{\mathbb{R}^2} |F_p|^{2p}\,dx = \int_{\mathbb{R}^2} \frac{1}{(1+|x|^2)^2}\,dx = \pi$ and

$$\lim_{p\to\infty} \int_{\mathbb{R}^2} |f_p|^{2p}\,dx = \int_{\mathbb{R}^2} F_p^{2p}\,(1+\tfrac{g}{2p})^{2p}\,dx = \int_{\mathbb{R}^2} \frac{e^g}{(1+|x|^2)^2}\,dx$$

so that the $\int_{\mathbb{R}^2} |f|^{2p}\,dx / \int_{\mathbb{R}^2} |F_p|^{2p}\,dx$ converges to $\int_{\mathbb{R}^2} e^{\,g}\,d\mu$ as $p\to\infty$,
(ii) $\int_{\mathbb{R}^2} |F_p|^{p+1}\,dx = (p-1)\,\pi/2$, $\lim_{p\to\infty} \int_{\mathbb{R}^2} |f_p|^{p+1}\,dx = \infty$, but

$$\lim_{p\to\infty} \frac{\int_{\mathbb{R}^2} |f_p|^{p+1}\,dx}{\int_{\mathbb{R}^2} |F_p|^{p+1}\,dx} = 1\,,$$

(iii) expanding the square and integrating by parts, we find that

$$\int_{\mathbb{R}^2} |\nabla f_p|^2\,dx = \frac{1}{4p^2}\int_{\mathbb{R}^2} F_p^2\,|\nabla g|^2\,dx - \int_{\mathbb{R}^2} (1+\tfrac{g}{2p})^2\,F_p\,\Delta F_p\,dx$$
$$= \frac{1}{4p^2}\int_{\mathbb{R}^2} |\nabla g|^2\,dx + \frac{2\pi}{p+1} + o(p^{-2})\,.$$

Here we have used $\int_{\mathbb{R}^2} |\nabla F_p|^2\,dx = \frac{2\pi}{p+1}$ and the condition $\int_{\mathbb{R}^2} g\,d\mu = 0$ in order to discard one additional term of the order of p^{-2}. On the other hand, we find that

$$\left(\frac{\int_{\mathbb{R}^2} |\nabla f|^2\,dx}{\int_{\mathbb{R}^2} |\nabla F_p|^2\,dx}\right)^{\frac{p-1}{2}} \sim \left(1 + \frac{p+1}{8\pi\,p^2}\int_{\mathbb{R}^2} |\nabla g|^2\,dx\right)^{\frac{p-1}{2}} \sim e^{\frac{1}{16\,\pi}\,\int_{\mathbb{R}^2} |\nabla g|^2\,dx}$$

as $p\to\infty$. Collecting these estimates concludes the proof. □

3.2. Legendre duality. Now we study the duality which relates Onofri's inequality (5) and the logarithmic Hardy-Littlewood-Sobolev inequality (6) in $\mathbb{R}^2$. With

$$F_1[u] := \log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right) \quad\text{and}\quad F_2[u] := \frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2\,dx + \int_{\mathbb{R}^2} u\,\mu\,dx\,,$$

Onofri's inequality amounts to $F_1[u] \le F_2[u]$. As for the case $d \ge 3$, we use Legendre's transformation (3).

Proposition 3.2. *For any $v \in \mathrm{L}^1_+(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} v\,dx = 1$, such that $v\log v$ and $(1+\log|x|^2)\,v \in \mathrm{L}^1(\mathbb{R}^2)$, we have*

$$F_1^*[v] - F_2^*[v] = \int_{\mathbb{R}^2} v\,\log\left(\frac{v}{\mu}\right)dx - 4\,\pi\int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\,dx \ge 0\,.$$

This result has been observed by several authors: see for instance [2, 7] for the duality argument on the two-dimensional sphere, which gives the above inequality on $\mathbb{R}^2$ by the stereographic projection. For completeness, let us give a proof. We essentially follow the computation of [6, Appendix] but explicitly compute the constants.

Proof. Taking the infimum in (3) on the Orlicz space

$$\left\{u \in \mathrm{L}^1_{\rm loc}(\mathbb{R}^2)\,:\,\nabla u \in \mathrm{L}^2(\mathbb{R}^2)\,,\,\int_{\mathbb{R}^2} e^u\,d\mu < \infty\right\}\,,$$

we find that

$$F_1^*[v] = \int_{\mathbb{R}^2} u\,v\,dx - \log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right) \quad \text{with} \quad \log v = \log\mu + u - \log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right),$$

$$F_2^*[v] = \int_{\mathbb{R}^2} u\,(v-\mu)\,dx - \frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2\,dx \quad \text{with} \quad -\Delta u = 8\,\pi\,(v-\mu)\,,$$

which proves the result. □

A useful observation is the fact that $-\Delta\log\mu = 8\,\pi\,\mu$ can be inverted as

$$(-\Delta)^{-1}\mu = \frac{1}{8\,\pi}\,\log\mu + C\,.$$

It is then easy to check that $u := (-\Delta)^{-1}\mu = G_2 * \mu$, with $G_2(x) = -\frac{1}{2\pi}\log|x|$, is such that $u(0) = 0$, which determines $C = \frac{1}{8\,\pi}\log\pi$. Hence with the notations of Proposition 3.2, we may observe that

$$F_1^*[v] = \int_{\mathbb{R}^2} v\,\log\left(\frac{v}{\mu}\right)dx\,,$$

$$F_2^*[v] = 4\,\pi\int_{\mathbb{R}^2} v\,(-\Delta)^{-1}v\,dx - \int_{\mathbb{R}^2} v\,\log\mu\,dx - 1 - \log\pi\,.$$

Collecting these observations, we find that

$$F_1^*[v] - F_2^*[v] = \int_{\mathbb{R}^2} v\,\log v\,dx - 4\,\pi\int_{\mathbb{R}^2} v\,(-\Delta)^{-1}v\,dx + 1 + \log\pi \ge 0\,.$$

For any $f \in \mathrm{L}^1_+(\mathbb{R}^2)$ with $M = \int_{\mathbb{R}^2} f\,dx$, such that $f\,\log f$, $(1+\log|x|^2)\,f \in \mathrm{L}^1(\mathbb{R}^2)$, this inequality, written for $v = f/M$, is nothing else than the logarithmic Hardy-Littlewood-Sobolev inequality (5).

3.3. Logarithmic diffusion equation: Onofri and logarithmic Hardy-Littlewood-Sobolev inequalities. The computation of $\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)]$ in Section 1 is formal but can easily be justified after noticing that the image w of v by the inverse stereographic projection on the sphere $\mathbb{S}^2$, up to a scaling, solves the equation

$$\frac{\partial w}{\partial t} = \Delta_{\mathbb{S}^2} w\,.$$

More precisely, if $x = (x_1, x_2) \in \mathbb{R}^2$, then u and w are related by

$$w(t,y) = \frac{u(t,x)}{4\,\pi\,\mu(x)}\,, \quad y = \left(\tfrac{2\,(x_1,x_2)}{1+|x|^2}, \tfrac{1-|x|^2}{1+|x|^2}\right) \in \mathbb{S}^2\,.$$

See [27, Section 8.2] for a review of some known results for the logarithmic diffusion equation, or super-fast diffusion equation, $\frac{\partial v}{\partial t} = \Delta\log v$, and [11, 28] for earlier results. Necessary adaptations to the case of (7) are left to the reader. Striking properties are the facts that the solution of (7) globally exists and its mass is preserved. For simplicity, we shall therefore assume that $1 = \int_{\mathbb{R}^2} v_0\,dx = \int_{\mathbb{R}^2} v(t,x)\,dx$ for any $t \ge 0$ and recall that $\int_{\mathbb{R}^2}\mu\,dx = 1$. It turns out that $\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)]$ has a sign because of Onofri's inequality.

Lemma 3.3. *For any $u \in \mathcal{D}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^2} e^{\frac{u}{2}}\,d\mu = 1$, we have*

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2\,dx \ge \int_{\mathbb{R}^2} \left(e^{\frac{u}{2}} - 1\right) u\,d\mu\,.$$

This inequality can of course be extended by density to the natural Orlicz space for which all integrals are well defined.

Proof. Inspired by [20], let us define

$$h(t) := \log\left(\int_{\mathbb{R}^2} e^{t\,u}\,d\mu\right)\,.$$

Claim 1: h is convex and h' is convex. Let us observe that

$$\left(\int_{\mathbb{R}^2} e^{t\,u}\,d\mu\right)^2 h''(t) = \int_{\mathbb{R}^2} e^{t\,u}\,d\mu \int_{\mathbb{R}^2} u^2\,e^{t\,u}\,d\mu - \left(\int_{\mathbb{R}^2} u\,e^{t\,u}\,d\mu\right)^2 \ge 0$$

by the Cauchy-Schwarz inequality. With one more derivation, we find that

$$\begin{aligned}\left(\int_{\mathbb{R}^2} e^{t\,u}\,d\mu\right)^3 & h'''(t)\\ &= \int_{\mathbb{R}^2} u^3\,e^{t\,u}\,d\mu\left(\int_{\mathbb{R}^2} e^{t\,u}\,d\mu\right)^2 - 3\int_{\mathbb{R}^2} u^2\,e^{t\,u}\,d\mu\int_{\mathbb{R}^2} u\,e^{t\,u}\,d\mu\int_{\mathbb{R}^2} e^{t\,u}\,d\mu\\ &\qquad + 2\left(\int_{\mathbb{R}^2} u\,e^{t\,u}\,d\mu\right)^3\,.\end{aligned}$$

Let us prove that $h'''(t) > 0$ for $t \in (0,1)$ a.e. If $\int_{\mathbb{R}^2} u\,e^{t\,u}\,d\mu > 0$, define

$$d\nu_t := \frac{e^{t\,u}}{\int_{\mathbb{R}^2} e^{t\,u}\,d\mu}\,d\mu \quad\text{and}\quad v := \frac{u}{\int_{\mathbb{R}^2} u\,d\nu_t}\,,$$

so that $\int_{\mathbb{R}^2} v\,d\nu_t = 1$. Then $h'''(t)$ has the sign of $\int_{\mathbb{R}^2}(v^3 - 3\,v^2 + 2)\,d\nu_t$. Taking the constraint into account, a direct optimization shows that an optimal function v takes two values, $1 \pm a$ for some $a > 0$ and moreover, $\nu_t(\{v = 1 + a\}) = \nu_t(\{v = 1 - a\})$ because of the condition $\int_{\mathbb{R}^2} v\,d\nu_t = 1$. In such a case, it is straightforward to check that $\int_{\mathbb{R}^2}(v^3 - 3\,v^2 + 2)\,d\nu_t = 0$, thus proving that $h'''(t) \ge 0$.

If $\int_{\mathbb{R}^2} u\,e^{t\,u}\,d\mu < 0$, a similar computation with $v := -u/\int_{\mathbb{R}^2} u\,d\nu_t$ shows that $-h'''(t)$ has the sign of $\int_{\mathbb{R}^2}(-v^3 - 3\,v^2 + 2)\,d\nu_t$ under the condition $\int_{\mathbb{R}^2} v\,d\nu_t = -1$, thus proving again that $h'''(t) \ge 0$.

Since $h'(t) = \int_{\mathbb{R}^2} u\,d\nu_t$ is monotone increasing, $\int_{\mathbb{R}^2} u\,e^{t\,u}\,d\mu = 0$ occurs for at most one $t \in [0,1]$ and we can conclude that h' is convex.

Claim 2: $h(1) \ge h'(1/2)$. We know that $h(0) = h(1/2) = 0$ and, by convexity of h', h'' is monotone nondecreasing. Let $\kappa := h''(1/2)$ and $\mathsf{p} := h'(1/2)$. On the one hand, we have $h''(t) \le \kappa$ for any $t \in [0,1/2]$, which means that, after integrating from $t < 1/2$ to $1/2$, we have

$$\mathsf{p} - h'(t) \le \kappa\,(\tfrac12 - t) \quad \forall\, t \in [0, \tfrac12]\,.$$

Hence, one more integration from 0 to $1/2$ gives

$$\frac{\mathsf{p}}{2} = \left[\mathsf{p}\,t - h(t)\right]_0^{1/2} \le \left[\kappa\,\tfrac{t}{2}\,(1-t)\right]_0^{1/2} = \frac{\kappa}{8}\,.$$

On the other hand, we have $h''(t) \ge \kappa$ for any $t \in [1/2,1]$, which means that, after integrating from $1/2$ to $t > 1/2$, we have

$$h'(t) - \mathsf{p} \ge \kappa\,(t - \tfrac12) \quad \forall\, t \in [\tfrac12, 1]\,.$$

One more integration from 1/2 to 1 gives

$$h(1)-\frac{\mathsf{p}}{2}\geq\left[\kappa\,\tfrac{t}{2}\,(t-1)\right]_0^{1/2}=\frac{\kappa}{8}\,.$$

Collecting the two estimates, we find that $h(1)\geq\mathsf{p}$, which proves the claim.

Using $h(0)=\log\left(\int_{\mathbb{R}^2}\,d\mu\right)=0$ and $h(1/2)=\log\left(\int_{\mathbb{R}^2}e^{u/2}\,d\mu\right)=0$, we have found that $\mathsf{p}=h'(1/2)=\int_{\mathbb{R}^2}u\,e^{u/2}\,d\mu\leq h(1)=\log\left(\int_{\mathbb{R}^2}e^u\,d\mu\right)$. This proves that

$$\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)]\geq\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx+\int_{\mathbb{R}^2}u\,d\mu-\log\left(\int_{\mathbb{R}^2}e^u\,d\mu\right)\geq 0\,,$$

where the last inequality is nothing else than Onofri's inequality. □

Corollary 3.4. *Consider a solution of* (7) *a nonnegative initial datum* v_0 *such that* $\int_{\mathbb{R}^2}v_0\,dx=1$. *Then* $\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)]\geq 0$.

3.4. Towards an improved Onofri inequality ? At least at a formal level, we can observe that computations of Section 2 for $d\geq 3$ do not carry to the case $d=2$. With $u=2\,\log(v/\mu)$, Equation (7) becomes

$$\mu\,\frac{\partial u}{\partial t}=e^{-\frac{u}{2}}\,\Delta u\,,$$

so that $\mathsf{J}:=2\int_{\mathbb{R}^2}v\,\left(\log\left(v/\mu\right)-1\right)\,dx$ takes the form

$$\mathsf{J}=\int_{\mathbb{R}^2}e^{\frac{u}{2}}\,(u-2)\,d\mu$$

and, when computing along the flow, we get

$$\mathsf{J}'=-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2\,dx\quad\text{and}\quad\mathsf{J}''=\int_{\mathbb{R}^2}(\Delta u)^2\,e^{-\frac{u}{2}}\,\frac{dx}{\mu}\,.$$

If we consider the Cauchy-Schwarz inequality as in Section 2, namely

$$\begin{aligned}4\,\mathsf{J}'^2&=\left(\int_{\mathbb{R}^2}|\nabla u|^2\,dx\right)^2=\left(\int_{\mathbb{R}^2}\Delta u\,u\,dx\right)^2\\&=\left(\int_{\mathbb{R}^2}\Delta u\,e^{-u/4}\,\frac{1}{\sqrt{\mu}}\cdot u\,e^{u/4}\,\sqrt{\mu}\,dx\right)^2\leq\mathsf{J}''\int_{\mathbb{R}^2}u^2\,e^{\frac{u}{2}}\,d\mu\,,\end{aligned}$$

it clearly turns out that the above r.h.s. cannot be controlled by $\mathsf{J}\,\mathsf{J}''$ using a simple Cauchy-Schwarz inequality. The scheme of the proof of Theorem 1.2 for improving Sobolev's inequality cannot be directly applied to Onofri's inequality.

4. Concluding remarks

In [8], E. Carlen, J.A. Carrillo and M. Loss noticed that Hardy-Littlewood-Sobolev inequalities and Gagliardo-Nirenberg inequalities can be related through the fast diffusion equation (4) with exponent $m=d/(d+2)$, when $d\geq 3$. The key computation goes as follows:

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt}\mathsf{H}_d[v(t,\cdot)]&=\frac{1}{2}\,\frac{d}{dt}\left[\int_{\mathbb{R}^d}v\,(-\Delta)^{-1}v\,dx-\mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}\right]\\&=\tfrac{d\,(d-2)}{(d-1)^2}\,\mathsf{S}_d\,\|u\|^{4/(d-1)}_{\mathrm{L}^{q+1}(\mathbb{R}^d)}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}-\|u\|^{2q}_{\mathrm{L}^{2q}(\mathbb{R}^d)}\end{aligned}$$

with $u = v^{(d-1)/(d+2)}$ and $q = (d+1)/(d-1)$. An explicit computation shows that

$$\tfrac{d\,(d-2)}{(d-1)^2}\,\mathsf{S}_d = (\mathsf{C}_{q,\,d})^{2q}\,,$$

thus proving that $\frac{d}{dt}\mathsf{H}_d[v(t,\cdot)]$ has a sign because of the Gagliardo-Nirenberg inequalities (11). Using the fact that the asymptotic behaviour of the solutions of the fast diffusion equation (4) with $m = d/(d+2)$ is governed by the Barenblatt self-similar solutions (see [13]), an integral remainder term (obtained by integrating along the flow of (4)) has been established, which improves on Hardy-Littlewood-Sobolev inequalities (2).

A similar result holds for the logarithmic Hardy-Littlewood-Sobolev inequality in dimension $d = 2$. Computing along the flow of (4) with exponent $m = 1/2$, it turns out that

$$\frac{\|v\|_{\mathrm{L}^1(\mathbb{R}^2)}}{8}\,\frac{d}{dt}\left[\frac{4\,\pi}{\|v\|_{\mathrm{L}^1(\mathbb{R}^2)}}\int_{\mathbb{R}^2} v\,(-\Delta)^{-1}v\;dx - \int_{\mathbb{R}^2} v\,\log v\;dx\right]$$
$$= \|u\|^4_{\mathrm{L}^4(\mathbb{R}^2)}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^2)} - \pi\,\|v\|^6_{\mathrm{L}^6(\mathbb{R}^2)}\,,$$

which is again one of the Gagliardo-Nirenberg inequalities (11), the one corresponding to $d = 2$ and $q = 3$, which is moreover such that $\pi\,(\mathsf{C}_{3,\,2})^6 = 1$.

The results of [8] relate Hardy-Littlewood-Sobolev inequalities (the logarithmic Hardy-Littlewood-Sobolev inequality if $d = 2$) with non-critical Gagliardo-Nirenberg inequalities through an evolution equation, although the Barenblatt self-similar solutions associated to the diffusion equation are not optimal for such Gagliardo-Nirenberg inequalities. In such a setting as well as in the setting considered in Sections 1–3, a nonlinear flow allows to improve on well known inequalities, with rather straightforward computations. These two examples suggest that much more can be done using flows of nonlinear diffusion equations.

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References

[1] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, J. Differential Geometry **11** (1976), no. 4, 573–598.

[2] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. of Math. (2) **138** (1993), no. 1, 213–242.

[3] J. G. Berryman and C. J. Holland, *Stability of the separable solution for fast diffusion*, Arch. Rational Mech. Anal. **74** (1980), no. 4, 379–388.

[4] G. Bianchi and H. Egnell, *A note on the Sobolev inequality*, J. Funct. Anal. **100** (1991), no. 1, 18–24.

[5] H. Brezis and E. H. Lieb, *Sobolev inequalities with remainder terms*, J. Funct. Anal. **62** (1985), no. 1, 73–86.

[6] V. Calvez and L. Corrias, *The parabolic-parabolic Keller-Segel model in* $\mathbb{R}^2$, Commun. Math. Sci. **6** (2008), no. 2, 417–447.

[7] E. Carlen and M. Loss, *Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on* S^n, Geom. Funct. Anal. **2** (1992), no. 1, 90–104.

[8] E. A. Carlen, J. A. Carrillo, and M. Loss, *Hardy-Littlewood-Sobolev inequalities via fast diffusion flows*, Proceedings of the National Academy of Sciences **107** (2010), no. 46, 19696–19701.
[9] A. Cianchi, *Quantitative Sobolev and Hardy inequalities, and related symmetrization principles*, in Sobolev spaces in mathematics. I, Vol. 8 of *Int. Math. Ser. (N. Y.)*, 87–116, Springer, New York (2009).
[10] A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli, *The sharp Sobolev inequality in quantitative form*, J. Eur. Math. Soc. (JEMS) **11** (2009), no. 5, 1105–1139.
[11] P. Daskalopoulos and M. A. del Pino, *On fast diffusion nonlinear heat equations and a related singular elliptic problem*, Indiana Univ. Math. J. **43** (1994), no. 2, 703–728.
[12] ———, *On a singular diffusion equation*, Comm. Anal. Geom. **3** (1995), no. 3-4, 523–542.
[13] M. del Pino and J. Dolbeault, *Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions*, J. Math. Pures Appl. (9) **81** (2002), no. 9, 847–875.
[14] M. del Pino and M. Sáez, *On the extinction profile for solutions of* $u_t = \Delta u^{(N-2)/(N+2)}$, Indiana Univ. Math. J. **50** (2001), no. 1, 611–628.
[15] J. Dolbeault and B. Perthame, *Optimal critical mass in the two-dimensional Keller-Segel model in* $\mathbb{R}^2$, C. R. Math. Acad. Sci. Paris **339** (2004), no. 9, 611–616.
[16] V. A. Galaktionov and J. R. King, *Fast diffusion equation with critical Sobolev exponent in a ball*, Nonlinearity **15** (2002), no. 1, 173–188.
[17] V. A. Galaktionov and L. A. Peletier, *Asymptotic behaviour near finite-time extinction for the fast diffusion equation*, Arch. Rational Mech. Anal. **139** (1997), no. 1, 83–98.
[18] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1975), no. 4, 1061–1083.
[19] J. King, *Self-similar behaviour for the equation of fast nonlinear diffusion*, Philosophical Transactions of the Royal Society of London. Series A: Physical and Engineering Sciences **343** (1993), no. 1668, 337.
[20] R. Latała and K. Oleszkiewicz, *Between Sobolev and Poincaré*, in Geometric aspects of functional analysis, Vol. 1745 of *Lecture Notes in Math.*, 147–168, Springer, Berlin (2000).
[21] E. H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math. (2) **118** (1983), no. 2, 349–374.
[22] K. Okikiolu, *Extremals for logarithmic Hardy-Littlewood-Sobolev inequalities on compact manifolds*, Geom. Funct. Anal. **17** (2008), no. 5, 1655–1684.
[23] E. Onofri, *On the positivity of the effective action in a theory of random surfaces*, Comm. Math. Phys. **86** (1982), no. 3, 321–326.
[24] M. A. Peletier and H. F. Zhang, *Self-similar solutions of a fast diffusion equation that do not conserve mass*, Differential Integral Equations **8** (1995), no. 8, 2045–2064.
[25] G. Savaré and V. Vespri, *The asymptotic profile of solutions of a class of doubly nonlinear equations*, Nonlinear Anal. **22** (1994), no. 12, 1553–1565.
[26] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) **110** (1976) 353–372.
[27] J. L. Vázquez, Smoothing and decay estimates for nonlinear diffusion equations, Vol. 33 of *Oxford Lecture Series in Mathematics and its Applications*, Oxford University Press, Oxford (2006).
[28] J. L. Vázquez, J. R. Esteban, and A. Rodríguez, *The fast diffusion equation with logarithmic nonlinearity and the evolution of conformal metrics in the plane*, Adv. Differential Equations **1** (1996), no. 1, 21–50.
[29] F. B. Weissler, *Logarithmic Sobolev inequalities for the heat-diffusion semigroup*, Trans. Amer. Math. Soc. **237** (1978) 255–269.

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