HJB equations for the optimal control of differential equations with delay in the control variable

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Abstract

We study a class of optimal control problems with state constraint, where the state equation is a differential equation with delays in the control variable. This class of problems arises in some economic applications, in particular in optimal advertising problems. The optimal control problem is embedded in a suitable Hilbert space and the associated Hamilton-Jacobi-Bellman (HJB) equation is considered in this space. It is proved that the value function is continuous with respect to a weak norm and that it solves in the viscosity sense the associated HJB equation. The main result is the proof of a directional C^1 regularity for the value function. This result represents the starting point to define a feedback map in classical sense going towards a verification theorem and the construction of optimal feedback controls for the problem.

Keywords: Hamilton-Jacobi-Bellman equation, optimal control, delay equations, viscosity solutions, regularity.

A.M.S. Subject Classification: 34K35, 49L25, 49K25.

1 Introduction

This paper is devoted to study a class of state constrained optimal control problems with distributed delay in the control variable and the associated Hamilton-Jacobi-Bellman (HJB) equation. The main result is the proof of a C^1 directional regularity for the value function associated to the control problem, which is the starting point to prove a smooth verification theorem.

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The study of control problems with delays is motivated by economic (see, e.g., [1, 2, 3, 12, 29, 23, 32, 33, 39, 42]) and engineering applications (see, e.g., [35]). Concerning the economic applications, which are the main motivation of our study, we observe that there is a wide variety of models with memory structures considered by the economic literature. We refer, for instance, to models where the memory structure arises in the state variable, as growth models with time-to-build in production (see [1, 2, 3]); to models where the memory structure arises in the control variable, as vintage capital models (see [12, 23]); advertising models (see [29, 32, 33, 39, 42]); growth models with time-to-build in investment (see [36, 43]).

From a mathematical point of view, our aim is to study the optimal control of the delay differential equation

$$\begin{cases} y'(t) = ay(t) + b_0 u(t) + \int_{-r}^{0} b_1(\xi)u(t+\xi)d\xi; \\ y(0) = y_0; \quad u(s) = \delta(s), \ s \in [-r, 0); \end{cases}$$

subject to the state constraint $y(\cdot) > 0$ and to the control constraint $u(\cdot) \in U \subset \mathbb{R}$. We remark that the case of state constraint $y(\cdot) \geq 0$ can be treated with similar arguments. The objective is to maximize a functional of the form

$$\int_0^{+\infty} e^{-\rho t}\Big(g_0(y(t)) - h_0(u(t))\Big)dt,$$

where $\rho > 0$ is a discount factor and $g_0 : \mathbb{R}^+ \to \mathbb{R}$, $h_0 : U \to \mathbb{R}$ are measurable functions[1]. Our ultimate goal is to get the synthesis of optimal controls for this problem, i.e. to produce optimal feedback controls by means of the dynamic programming approach.

The presence of the delay in the state equation (1) renders not possible to apply the dynamic programming techniques to the problem in its current form. A general way to tackle control problems with delay consists in representing the controlled system in a suitable infinite dimensional space (see [11, Part II, Ch. 1] for a general theory). In this way the delay is absorbed in the infinite dimensional state, but, on the other hand, the price to pay is that the resulting system is infinite dimensional and so is the value function V. Then the core of the problem becomes the study of the associated infinite dimensional HJB equation for V: the optimal feedback controls will be given in terms of the first space derivatives of V through the so-called verification theorem.

Sometimes explicit solutions to the (infinite dimensional) HJB equation are available (see [3, 23, 28] and Section 8). In this case the optimal feedback controls are explicitly given and the verification theorem is easy to prove. However in most cases the explicit solutions are not available and then one has to try to prove some regularity result for the solutions of the HJB equation in order to be able to define formally optimal feedback controls and check its optimality through the verification theorem. This is due to the fact that, to obtain an optimal control in feedback form, one needs the existence of an appropriately defined gradient of the solution. It is possible to prove verification theorems and representation of optimal feedbacks in the framework of viscosity solutions, even if the gradient is not defined in classical sense (see e.g. [9, 45] in finite dimension and [24, 37] in infinite dimension), but this is usually not satisfactory in applied problems since the closed loop equation becomes very hard to treat in such cases. The C^1 regularity of solutions to HJB equations is particularly important in infinite dimension, since in this case verification theorems in the framework of viscosity solutions contained in the above

[1] In economic applications typically they are respectively a utility and a cost function.

references are rather weak. For this reason, the main goal of the present paper is to prove a C^1 regularity result for the value function V of our problem.

To the best of our knowledge, C^1 regularity for first order HJB equation was proved first by Barbu and Da Prato [4] using methods of convex regularization and then developed by various authors (see e.g. [5, 6, 7, 19, 20, 25, 30, 31]) in the case without state constraints and without applications to problems with delay. In the papers [15, 16, 26] a class of optimal control problems with state constraints is treated using again methods of convex regularization, but the C^1 regularity is not proved. To our knowledge, the only paper proving a C^1 type regularity result for the solutions to HJB equations arising in optimal control problems with delay and state constraints is [27]. There a method introduced in finite dimension by Cannarsa and Soner [17] (see also [9]) and based on the concept of viscosity solution has been generalized in infinite dimension to get an *ad hoc* directional regularity result for viscosity solutions of the HJB equation.

In our paper we want to exploit further the method of [17] to get a C^1 type regularity result for our problem. The main difference of our paper with respect to [27] is that therein the delay is in the state variable, while here the delay is in the control variable. The case of problems with delay in the control variable is harder to treat. First of all, if we tried a standard infinite dimensional representation as in the case of state delay problems, we would get a *boundary control problem* in infinite dimension (see [34]). However, this first difficulty can be overcome when the original state equation is linear using a suitable transformation leading to the construction of the so called *structural state* (see [44]) and this is why, differently from [27], here we only treat the case of a linear state equation. But once we have done that, if we try to follow the approach of [27] to prove a C^1 regularity result for the value function, it turns out that much more care is needed in the choice of the space where to perform the infinite dimensional representation. While in [27] the product space $\mathbb{R} \times L^2$ is used to represent the delay system, we need here to use the more regular product space $\mathbb{R} \times W^{1,2}$ for reasons that are explained in the crucial Remark 5.5. We observe that the theory of the infinite dimensional representation of delay systems has been developed mainly in spaces of continuous function or in product spaces of type $\mathbb{R} \times L^2$ (see the above mentioned reference [11]). Therefore we restate the infinite dimensional representation in $\mathbb{R} \times W^{1,2}$ and carefully adapt the regularity method of [27] in such context. So we get the desired C^1 type regularity result (Theorem 6.11), which exactly as in [27] just allows to define an optimal feedback map in classical sense (see Corollary 6.13 and (60)). So, it is the starting point to construct optimal feedbacks for the problem as in [28].

The paper is structured as follows. In Section 2 we just give the definition of some spaces and state some notations. In Section 3 we give a precise formulation of the optimal control problem. In Section 4 we rephrase the problem in infinite dimension and state the equivalence with the original one (Theorem 4.5). In Section 5 we prove that the value function is continuous in the interior of its domain with respect to a weak norm (Proposition 5.9). In Section 6 we show that the value function solves in the viscosity sense the associated HJB equation (Theorem 6.9) and then we provide the main result, i.e. we prove that it has continuous classical derivative along a suitable direction in the space $\mathbb{R} \times W^{1,2}$ (Theorem 6.11). In Section 7 we show how our setting may cover advertising models[2]. In Section 8, we provide an example (not known in literature) with explicit solution to the HJB equation in the linear-quadratic case.

[2]It is worth to stress that, even if we focus to the application to optimal advertising problems, the same approach can be applied to other contexts such as vintage capital models (see [12, 23]) or growth models with time-to-build in investment (see [36, 43]).

2 Spaces and notations

Throughout paper we use the Lebesgue space

$$L_r^2 := L^2([-r,0];\mathbb{R}),$$

endowed with inner product

$$\langle f,g\rangle_{L_r^2} := \int_{-r}^{0} f(\xi)g(\xi)d\xi,$$

which renders it a Hilbert space. Also we use the Sobolev spaces

$$W_r^{k,2} := W^{k,2}([-r,0];\,\mathbb{R}), \quad k = 1,2,\dots$$

endowed with inner products

$$\langle f,g\rangle_{W_r^{k,2}} := \sum_{i=0}^{k}\int_{-r}^{0} f^{(i)}(\xi)g^{(i)}(\xi)d\xi, \quad k = 1,2,\,\dots,$$

which render them Hilbert spaces. The well-known Sobolev inclusions imply that

$$W_r^{k,2} \hookrightarrow C^{k-1}([-r,0];\mathbb{R}), \quad k = 1,2,\dots$$

with continuous embedding. Throughout the paper we will confuse the elements of $W_r^{k,2}$, which are classes of equivalence of functions, with their (unique) representatives in $C^{k-1}([-r,0];\mathbb{R})$, which are pointwise well defined functions. Given that, we define the spaces

$$W_{r,0}^{k,2} := \{f \in W_r^{k,2} \mid f^{(i)}(-r) = 0,\ \forall i = 0,1,...,k-1\} \subset W_r^{k,2}, \quad k = 1,2,\dots$$

We notice that in our definition of $W_{r,0}^{k,2}$ the boundary condition is only required at $-r$. The spaces $W_{r,0}^{k,2}$ are Hilbert spaces as closed subsets of the Hilbert spaces $W_r^{k,2}$. However, on these spaces we can also consider the inner products

$$\langle f,g\rangle_{W_{r,0}^{k,2}} := \int_{-r}^{0} f^{(k)}(\xi)g^{(k)}(\xi)d\xi, \quad k = 1,2,\,\dots$$

It is easy to see that, due to the boundary condition in the definition of the subspaces $W_{r,0}^{k,2}$, the inner products $\langle\cdot,\cdot\rangle_{W_{r,0}^{k,2}}$ are equivalent to the original inner products $\langle\cdot,\cdot\rangle_{W_r^{k,2}}$ on $W_{r,0}^{k,2}$, in the sense that they induce equivalent norms. For this reason, dealing with topological concepts, we will consider the simpler inner products (2) on the spaces $W_{r,0}^{k,2}$.

Also we consider the space

$$X := \mathbb{R}\times L_r^2.$$

This is a Hilbert space when endowed with the inner product

$$\langle \eta,\zeta\rangle := \eta_0\zeta_0 + \langle \eta_1,\zeta_1\rangle_{L_r^2},$$

where $\eta = (\eta_0,\eta_1(\cdot))$ is the generic element of X. The norm on this space is defined as

$$\|\eta\|_X^2 = |\eta_0|^2 + \|\eta_1\|_{L_r^2}^2.$$

Finally we consider the space $H \subset X$ defined as

$$H := \mathbb{R} \times W^{1,2}_{r,0}.$$

This is a Hilbert space when endowed with the inner product

$$\langle \eta, \zeta \rangle := \eta_0 \zeta_0 + \langle \eta_1, \zeta_1 \rangle_{W^{1,2}_{r,0}}.$$

which induces the norm

$$\|\eta\|^2 = |\eta_0|^2 + \int_{-r}^{0} |\eta_1'(\xi)|^2 d\xi.$$

This is the Hilbert space where our infinite dimensional system will be embedded.

3 The optimal control problem

In this section we give the precise formulation of the optimal control problem that we are going to study.

Given $y_0 \in (0, +\infty)$ and $\delta \in L^2_r$, we consider the optimal control of the following differential equation with delay in the control variable

$$\begin{cases} y'(t) = ay(t) + b_0 u(t) + \int_{-r}^{0} b_1(\xi) u(t+\xi) d\xi; \\ y(0) = y_0; \quad u(s) = \delta(s), \; s \in [-r, 0); \end{cases} \tag{1}$$

with state constraint $y(\cdot) > 0$ and control constraint $u(\cdot) \in U \subset \mathbb{R}$. The value $y_0 \in (0, +\infty)$ in the state equation (1) represents the initial state of the system, while the function δ represents the past of the control, which is considered as given.

Concerning the control set U we assume the following, that will be a standing assumption throughout the paper.

Hypothesis 3.1. *$U = [0, \bar{u}]$, where $\bar{u} \in [0, +\infty]$. When $\bar{u} = +\infty$, the set U is intended as $U = [0, +\infty)$.*

Concerning the parameters appearing in (1) we assume the following assumptions that will be standing throughout the paper as well.

Hypothesis 3.2.

(i) $a, b_0 \in \mathbb{R}$;

(ii) $b_1 \in W^{1,2}_{r,0}$, *and* $b_1 \neq 0$.

The fact that $b_1 \neq 0$ means that the delay really appears in the state equation.

Given $u(\cdot) \in L^2_{loc}([0, +\infty); \mathbb{R})$, there exists a unique locally absolutely continuous solution $y : [0, +\infty] \to \mathbb{R}$ of (1), provided explicitly by the variation of constants formula

$$y(t) = y_0 e^{at} + \int_0^t e^{a(t-s)} f(s) ds, \tag{2}$$

where

$$f(s) = b_0 u(s) + \int_{-r}^{0} b_1(\xi) u(s+\xi) d\xi; \qquad u(s) = \delta(s), \ s \in [-r, 0].$$

We notice that f is well defined, as b_1 is bounded and $u(\cdot) \in L^2_{loc}([0,+\infty),\mathbb{R})$. We denote by $y(t; y_0, \delta(\cdot), u(\cdot))$ the solution to (2) with initial datum $(y_0, \delta(\cdot))$ and under the control $u(\cdot) \in L^2_{loc}([0,+\infty);\mathbb{R})$. We notice that $y(t; y_0, \delta(\cdot), u(\cdot))$ solves the delay differential equation (1) only for almost every $t \geq 0$.

We define the class of the admissible controls for the problem with state constraint $y(\cdot) > 0$ as

$$\mathcal{U}(y_0, \delta(\cdot)) := \{u(\cdot) \in L^2_{loc}([0,+\infty); U) \mid y(\cdot, y_0, \delta; u(\cdot)) > 0\}.$$

Setting $y(t) := y(t; y_0, \delta(\cdot), u(\cdot))$, we define the objective functional

$$J_0(y_0, \delta(\cdot); u(\cdot)) = \int_0^{+\infty} e^{-\rho t} \big(g_0(y(t)) - h_0(u(t))\big) dt, \tag{3}$$

where $\rho > 0$ and $g_0 : [0,+\infty) \to \mathbb{R}$, $h_0 : U \to \mathbb{R}$ are measurable functions satisfying the following, that will be standing assumptions throughout the paper.

Hypothesis 3.3.

(i) $g_0 \in C([0,+\infty);\mathbb{R})$, *it is concave, nondecreasing and bounded from above. Without loss of generality we assume that* $g(0) = 0$ *and set*

$$\bar{g}_0 := \lim_{y \to +\infty} g_0(y). \tag{4}$$

(ii) $h_0 \in C(U) \cap C^1(U^\circ)$, *where* U° *denotes the interior part of* U*. Moreover it is nondecreasing, convex and not constant. Without loss of generality we assume* $h_0(0) = 0$*.*

The optimization problem consists in the maximization of the objective functional J_0 over the set of all admissible strategies $\mathcal{U}(y_0, \delta(\cdot))$, i.e.

$$\max_{u(\cdot) \in \mathcal{U}(y_0, \delta(\cdot))} J_0(y_0, \delta(\cdot); u(\cdot)). \tag{5}$$

Remark 3.4.

(i) *The assumption that* g_0 *is bounded from above (Hypothesis 3.3-(i)) is quite unpleasant, if we think about the applications. However we stress that this assumption is taken here just for convenience and can be replaced with a suitable assumption on the growth of* g_0*, relating it to the requirement of a large enough discount factor* ρ*.*

(ii) *We consider a delay* r *belonging to* $(-\infty, 0]$*. However one can obtain the same results even allowing* $r = -\infty$*, suitably redefining the boundary conditions as limits. In the definition of the Sobolev spaces* $W^{k,2}_{-\infty,0}$*, the boundary conditions required become*

$$W^{k,2}_{-\infty,0} := \left\{ f \in W^{k,2} \mid \lim_{r \to -\infty} f^{(i)}(r) = 0, \ \forall i = 0, 1, ..., k-1 \right\} \subset W^{k,2}_{-\infty}.$$

4 Representation in infinite dimension

In this section we restate the delay differential equation (1) as an abstract evolution equation in a suitable infinite dimensional space. The infinite dimensional setting is represented by the Hilbert space $H = \mathbb{R} \times W^{1,2}_{r,0}$. The following argument is just a suitable rewriting in $\mathbb{R} \times W^{1,2}_{r,0}$ of the method illustrated in [11] in the framework of the product space $\mathbb{R} \times L^2$. We will use some basic concepts from the Semigroups Theory, for which we refer to [22].

Let A be the unbounded linear operator

$$A : \mathcal{D}(A) \subset H \to H, \quad (\eta_0, \eta_1(\cdot)) \mapsto (a\eta_0 + \eta_1(0), -\eta_1'(\cdot)), \tag{6}$$

where a is the constant appearing in (1), defined on

$$\mathcal{D}(A) = \mathbb{R} \times W^{2,2}_{r,0}.$$

It is possible to show by direct computations that A is a (densely defined) closed operator and generates a C_0-semigroup $(S_A(t))_{t\geq 0}$ in H. However, we provide the proof of that in the Appendix by using some known facts from the Semigroups Theory. The explicit expression of $S_A(t)$ is (see the Appendix)

$$S_A(t)\eta = \left(\eta_0 e^{at} + \int_{(-t)\vee(-r)}^{0} \eta_1(\xi) e^{a(\xi+t)} d\xi, \ \eta_1(\cdot - t)\mathbf{1}_{[-r,0]}(\cdot - t) \right), \quad \eta = (\eta_0, \eta_1(\cdot)) \in H. \tag{7}$$

By [37, Ch. 2, Prop. 4.7], there exist $M > 0$, $\omega \in \mathbb{R}$ such that

$$\|S_A(t)\| \leq M e^{\omega t}, \quad t \geq 0.$$

More precisely (see the Appendix) we have

$$\|S_A(t)\| \leq (2 + r^3)^{\frac{1}{2}} e^{at}, \quad t \geq 0. \tag{8}$$

In the space H we set $b := (b_0, b_1(\cdot))$ and define the bounded linear operator

$$B : U \to H, \quad u \mapsto bu.$$

Often we will identify the operator B with b.

Given $u(\cdot) \in L^2_{loc}([0, +\infty), \mathbb{R})$, $\eta \in H$, we can consider the abstract equation in H

$$\begin{cases} Y'(t) = AY(t) + Bu(t), \\ Y(0) = \eta. \end{cases} \tag{9}$$

We will use two concepts of solution to (9), that in our case coincide each other. For details we refer to [37, Ch. 2, Sec. 5]. In the definitions below the integral in dt is intended as Bochner integral in the Hilbert space H.

Definition 4.1.

(i) *We call mild solution of* (9) *the function* $Y \in C([0, +\infty), H)$ *defined as*

$$Y(t) = S_A(t)\eta + \int_0^t S_A(t - \tau) Bu(\tau) d\tau, \quad t \geq 0. \tag{10}$$

(ii) *We call weak solution of* (9) *a function* $Y \in C([0,+\infty), H)$ *such that, for any* $\phi \in \mathcal{D}(A^*)$,

$$\langle Y(t), \phi\rangle = \langle \eta, \phi\rangle + \int_0^t \langle Y(\tau), A^*\phi\rangle d\tau + \int_0^t \langle Bu(\tau), \phi\rangle d\tau, \quad \forall t \geq 0.$$

From now on we denote by $Y(\cdot;\eta,u(\cdot))$ the mild solution of (9). We notice that the definition of mild solution is the infinite dimensional version of the variation of constants formula. By a well-known result (see [37, Ch. 2, Prop. 5.2]), the mild solution is also the (unique) weak solution.

4.1 Equivalence with the original problem

In order to state equivalence between the DDE (1) and the abstract evolution equation (9), we need to link the canonical $\mathbb{R}$-valued integration with the $W^{1,2}_{r,0}$-valued integration. This is provided by the following lemma whose proof is standard. We omit it for brevity.

Lemma 4.2. *Let* $0 \leq a < b$ *and* $f \in L^2\left([a,b]; W_0^{1,2}\right)$. *Then*

$$\left(\int_a^b f(t)dt\right)(\xi) = \int_a^b f(t)(\xi)dt, \qquad \forall \xi \in [-r,0],$$

where the integral in dt *in the left handside is intended as Bochner integral in the space* $W_0^{1,2}$.

We need to study the adjoint operator A^* in order to use the concept of weak solution of (9).

Proposition 4.3. *We have*

$$\mathcal{D}(A^*) = \left\{\phi = (\phi_0, \phi_1(\cdot)) \in H \mid \phi_1 \in W_r^{2,2}, \quad \phi_1'(0) = 0\right\}$$

and

$$A^*\phi = \left(a\phi_0,\ \xi \mapsto \phi_1'(\xi) + \phi_0(\xi + r) - \phi_1'(-r)\right), \quad \phi \in \mathcal{D}(A^*). \tag{11}$$

Proof. Let

$$\mathcal{D} := \left\{\phi = (\phi_0, \phi_1(\cdot)) \in H \mid \phi_1 \in W_r^{2,2}, \quad \phi_1'(0) = 0\right\}.$$

First of all we notice that, defining $A^*\phi$ on $\mathcal{D}$ as in (11), we have $A^*\phi \in H$. Now notice that

$$\psi_1'(-r) = 0, \quad \psi_1(0) = \int_{-r}^0 \psi_1'(\xi)d\xi, \qquad \forall \psi \in \mathcal{D}(A). \tag{12}$$

Therefore, taking into account (12), we have for every $\psi \in \mathcal{D}(A)$ and every $\phi \in \mathcal{D}$

$$\begin{aligned}
\langle A\psi, \phi\rangle &= a\psi_0\phi_0 + \psi_1(0)\phi_0 - \int_{-r}^0 \psi_1''(\xi)\phi_1'(\xi)d\xi \\
&= a\psi_0\phi_0 + \left(\int_{-r}^0 \psi_1'(\xi)d\xi\right)\phi_0 - \psi_1'(0)\phi_1'(0) + \psi_1'(-r)\phi_1'(-r) + \int_{-r}^0 \psi_1'(\xi)\phi_1''(\xi)d\xi \\
&= a\psi_0\phi_0 + \int_{-r}^0 \psi_1'(\xi)\left(\phi_0 + \phi_1''(\xi)\right)d\xi = \langle \psi, A^*\phi\rangle
\end{aligned} \tag{13}$$

where

$$\begin{cases} (A^*\phi)_0 = a\phi_0, \\ (A^*\phi)_1(\xi) = \int_{-r}^{\xi}(\phi_0 + \phi_1''(\xi))d\xi = \phi_0 \cdot (\xi + r) + \phi_1'(\xi) - \phi_1'(-r). \end{cases}$$

The equality above shows that $\mathcal{D} \subset \mathcal{D}(A^*)$ and that A^* acts as claimed in (11) on the elements of $\mathcal{D}$.

Now we have to show that $\mathcal{D} = \mathcal{D}(A^*)$. For sake of brevity here we only sketch the proof of this fact[3], as a complete proof would require a study of the adjoint semigroup $S_{A^*}(t)$ in the space H. We observe that $\mathcal{D}$ is dense in H. Moreover an explicit computation of the adjoint semigroup would show that $S_{A^*}(t)\mathcal{D} \subset \mathcal{D}$ for any $t \geq 0$. Hence, by [18, Th. 1.9, p. 8], $\mathcal{D}$ is dense in $\mathcal{D}(A^*)$ endowed with the graph norm. Finally, using (11) it is easy to show that $\mathcal{D}$ is closed in the graph norm of A^* and therefore $\mathcal{D}(A^*) = \mathcal{D}$. $\square$

Let $v \in L^2_r$ and consider the function

$$(v * b_1)(\xi) = \int_{-r}^{\xi} b_1(\tau)v(\tau - \xi)d\tau, \quad \xi \in [-r, 0].$$

First of all we notice that $(v * b_1)(-r) = 0$. Extend b_1 to a $W^{1,2}_r(\mathbb{R})$ function on $\mathbb{R}$ equal to 0 in $(-\infty, -r)$ (recall that $b_1(-r) = 0$) and extend v to an $L^2(\mathbb{R};\mathbb{R})$ function simply defining it equal to 0 out of $[-r, 0]$. Then the function above can be rewritten as

$$(v * b_1)(\xi) = \int_{\mathbb{R}} b_1(\tau)v(\tau - \xi)\mathbf{1}_{(-\infty,0]}(\tau - \xi)d\tau, \quad \xi \in [-r, 0].$$

Since $v\mathbf{1}_{(-\infty,0]} \in L^2(\mathbb{R};\mathbb{R})$ and $b_1 \in W^{1,2}(\mathbb{R};\mathbb{R})$, [13, Lemma 8.4] yields $v * b_1 \in W^{1,2}_{r,0}$ and

$$(v * b_1)'(\xi) = \int_{-r}^{\xi} b_1'(\tau)v(\tau - \xi)d\tau. \tag{14}$$

Consider still v extended to 0 out of $[-r, 0]$ and set $v_\xi(\tau) := v(\tau - \xi)$, $\tau \in [-r, 0]$ for $\xi \in [-r, 0]$. Of course $v_\xi \in L^2_r$ and $\|v_\xi\|_{L^2_r} \leq \|v\|_{L^2_r}$ for every $\xi \in [-r, 0]$. Then, due to (14) and by Holder's inequality we have

$$\begin{aligned}\|v * b_1\|^2_{W^{1,2}_{r,0}} &= \int_{-r}^{0} \left| \int_{-r}^{\xi} b_1'(\tau)v(\tau - \xi)d\tau \right|^2 d\xi = \int_{-r}^{0} \left| \int_{-r}^{\xi} b_1'(\tau)v_\xi(\tau)d\tau \right|^2 d\xi \\ &\leq \int_{-r}^{0} \left(\int_{-r}^{0} |b_1'(\tau)v_\xi(\tau)|d\tau \right)^2 d\xi \ \leq \int_{-r}^{0} \left(\|b_1'\|^2_{L^2_r} \|v_\xi\|^2_{L^2_r} \right) d\xi \leq r\|b_1'\|^2_{L^2_r}\|v\|^2_{L^2_r}. \end{aligned} \tag{15}$$

Let us introduce the continuous linear map M:

$$\begin{array}{rccl} M: & \mathbb{R} \times L^2([-r,0];\mathbb{R}) & \longrightarrow & H \\ & (z, v) & \longmapsto & (z, v * b_1) = \left(z, \int_{-r}^{\cdot} b_1(\tau)v(\tau - \cdot)d\tau\right). \end{array} \tag{16}$$

Due to (15), M is bounded. Call

$$\mathcal{M} := \mathrm{Im}(M).$$

Remark 4.4. *Of course $\mathcal{M}$ is a linear subspace of H. It should be possible using [10] that is not closed.* $\square$

[3] To this regard we observe that we will use in the following only the fact $\mathcal{D} \subset \mathcal{D}(A^*)$ and that (11) holds true on the elements of $\mathcal{D}$, which has been proven rigorously. More precisely we will use the fact that $(1, 0) \in \mathcal{D} \subset \mathcal{D}(A^*)$ in the proof of Theorem 4.5.

Theorem 4.5. *Let $y_0 \in \mathbb{R}$, $\delta \in L^2_r$, $u(\cdot) \in L^2_{loc}([0,+\infty),\mathbb{R})$. Set*

$$\eta := M(y_0, \delta(\cdot)) \in \mathcal{M}; \quad Y(t) := Y(t;\eta,u(\cdot)), \quad t \geq 0. \tag{17}$$

Then

$$Y(t) = M(Y_0(t), u(t+\cdot)), \qquad \forall t \geq 0. \tag{18}$$

Moreover, let $y(\cdot) := y(\cdot; y_0, \delta, u(\cdot))$ be the unique solution to (1)*. Then*

$$y(t) = Y_0(t), \qquad \forall t \geq 0. \tag{19}$$

Proof. Let Y be the mild solution defined by (10) with initial condition η given by (17). On the second component it reads

$$\begin{aligned} Y_1(t) &= T(t)\eta_1 + \int_0^t [T(t-s)b_1]u(s)ds \\ &= \mathbf{1}_{[-r,0]}(\cdot - t)\eta_1(\cdot - t) + \int_0^t \mathbf{1}_{[-r,0]}(\cdot - t + s)b_1(\cdot - t + s)u(s)ds \end{aligned} \tag{20}$$

where $(T(t))_{t\geq 0}$ is the semigroup of truncated right shifts on $W^{1,2}_0$ that is

$$[T(t)\phi](\xi) = \mathbf{1}_{[-r,0]}(\xi - t)\phi(\xi - t), \quad \xi \in [-r, 0].$$

We recall that by hypothesis $\eta = M(y_0, \delta(\cdot))$, so we write the second component of the initial datum

$$\eta_1(\xi) = \int_{-r}^{\xi} b_1(\alpha)\delta(\alpha - \xi)d\alpha.$$

Then, by (20) and due to Lemma 4.2, the second component evaluated at ξ is

$$Y_1(t)(\xi) = \mathbf{1}_{[-r,0]}(\xi - t)\int_{-r}^{\xi - t} b_1(\alpha)u(\alpha - \xi + t)d\alpha + \int_0^t \mathbf{1}_{[-r,0]}(\xi - t + s)b_1(\xi - t + s)u(s)ds. \tag{21}$$

Taking into account that $0 \leq s \leq t$, we have $\xi - t \leq \xi - t + s \leq \xi$, so that, setting $\alpha = \xi - t + s$ in the second part of the right handside of (21), it becomes

$$\begin{aligned} Y_1(t)(\xi) &= \mathbf{1}_{[-r,0]}(\xi - t)\int_{-r}^{\xi - t} b_1(\alpha)u(\alpha - \xi + t)d\alpha + \int_{\xi - t}^{\xi} \mathbf{1}_{[-r,0]}(\alpha)b_1(\alpha)u(\alpha - \xi + t)d\alpha \\ &= \int_{-r}^{(\xi - t)\vee(-r)} b_1(\alpha)u(\alpha - \xi + t)d\alpha + \int_{(\xi - t)\vee(-r)}^{\xi} b_1(\alpha)u(\alpha - \xi + t)d\alpha. \\ &= \int_{-r}^{\xi} b_1(\alpha)u(\alpha + t - \xi)d\alpha. \end{aligned} \tag{22}$$

Therefore, due to (16), the identity (18) is proved.

Let us show now (19). Setting $\xi = 0$ in (22) we get

$$Y_1(t)(0) = \int_{-r}^{0} b_1(\alpha)u(t+\alpha)d\alpha. \tag{23}$$

Now we use the fact that $Y(\cdot)$ is also a weak solution of (9). From Proposition 4.3 we know that

$$(1,0) \in \mathcal{D}(A^*), \quad A^*(1,0) = (a,\ \xi \mapsto \xi + r)\,. \tag{24}$$

Therefore taking into account (24) and (23) and Definition 4.1-(ii), we have for almost every $t \geq 0$

$$\begin{aligned} Y_0'(t) = \frac{d}{dt}\langle Y(t),(1,0)\rangle &= \langle Y(t), A^*(1,0)\rangle + \langle Bu(t),(1,0)\rangle \\ &= aY_0(t) + \int_{-r}^{0} Y_1(t)'(\xi)d\xi + b_0 u(t) \\ &= aY_0(t) + Y_1(t)(0) - Y_1(t)(-r) + b_0 u(t) \\ &= aY_0(t) + \int_{-r}^{0} b_1(\xi)u(t+\xi)d\xi + b_0 u(t). \end{aligned}$$

Therefore $Y_0(t)$ solves (1) with initial data $(y_0, \delta(\cdot))$, so it must coincide with $y(t)$. □

We can use the above result to reformulate the optimization problem (3) in the space H. Let

$$H_+ := (0,+\infty) \times W^{1,2}_{r,0}.$$

Let $\eta \in H$ and define the (possibly empty) set

$$\begin{aligned} \mathcal{U}(\eta) &:= \{u(\cdot) \in L^2_{loc}([0,+\infty);U) \mid Y_0(t;\eta,u(\cdot)) > 0, \ \forall t \geq 0\} \\ &= \{u(\cdot) \in L^2_{loc}([0,+\infty);U) \mid Y(t;\eta,u(\cdot)) \in H_+, \ \forall t \geq 0\}. \end{aligned}$$

Given $u(\cdot) \in \mathcal{U}(\eta)$ define

$$J(\eta; u(\cdot)) = \int_0^{+\infty} e^{-\rho t}\Big(g(Y(t;\eta,u(\cdot))) \ - \ h_0(u(t))\Big)dt. \tag{25}$$

where

$$g: H_+ \longrightarrow \mathbb{R}, \quad g(\eta) := g_0(\eta_0). \tag{26}$$

Due to (19), if $\eta = M(y_0, \delta(\cdot))$ then

$$\mathcal{U}(\eta) = \mathcal{U}(y_0, \delta(\cdot))$$

and

$$J(\eta; u(\cdot)) = J_0(y_0, \delta(\cdot); u(\cdot)),$$

where J_0 is the objective functional defined in (3). Therefore, we have reduced the original problem (5) to

$$\max_{u(\cdot)\in\mathcal{U}(y_0,\delta(\cdot))} J(\eta; u(\cdot)), \ \ \eta = M(y_0,\delta(\cdot)) \in \mathcal{M}.$$

Although we are interested to solve the problem for initial data $\eta \in \mathcal{M}$, as these are the initial data coming from the real original problem, we enlarge the problem to data $\eta \in H$ and consider the functional (25) defined also for these data. So the problem is

$$\max_{u(\cdot)\in\mathcal{U}(\eta)} J(\eta; u(\cdot)), \ \ \eta \in H. \tag{27}$$

5 The value function in the Hilbert space

In this section we study some qualitative properties of the value function V associated to the optimization problem (27) in the space H. For $\eta \in H$ the value function of our problem is the function

$$V : H \longrightarrow \mathbb{R}, \quad V(\eta) := \sup_{u(\cdot)\in\mathcal{U}(\eta)} J(\eta, u(\cdot))$$

with the convention

$$\sup \emptyset = -\infty.$$

We notice that V is bounded from above due to the Hypotheses 3.3. More precisely

$$V(\eta) \leq \int_0^{+\infty} e^{-\rho t}\bar{g}_0 dt = \frac{1}{\rho}\bar{g}_0, \qquad \forall\, \eta \in \mathcal{D}(V).$$

The domain of the value function V is defined as

$$\mathcal{D}(V) := \{\eta \in H \mid V(\eta) > -\infty\}.$$

Of course

$$\mathcal{D}(V) \subset \{\eta \in H \mid \mathcal{U}(\eta) \neq \emptyset\}.$$

Before to proceed, we introduce a weaker norm in H, which is the natural one to deal with the unbounded linear term in the study of the HJB equation.

5.1 The norm $\|\cdot\|_{-1}$

We are going to define a norm weaker than $\|\cdot\|$ we will deal with. To this aim, we introduce the following assumption, that will be a standing assumption in the rest of the paper and will not be repeated.

Hypothesis 5.1. $a \neq 0$.

Remark 5.2. *First of all we notice that Hypothesis 5.1 is not very restrictive for the applications, as the coefficient a in the model often represents a depreciation factor (so $a < 0$) or a growth rate (so $a > 0$). However, the case $a = 0$ can be treated translating the problem as follows. Take $a = 0$. The state equation in infinite dimension is* (9) *with*

$$A : (\phi_0, \phi_1(\cdot)) \mapsto \Big(\phi_1(0), -\phi_1'(\cdot)\Big).$$

We can rewrite it as

$$Y'(t) = \tilde{A}Y(t) - P_0 Y(t) + Bu(t),$$

where

$$P_0 : H \mapsto H, \quad P_0\phi = (\phi_0, 0); \quad \tilde{A} = A + P_0.$$

Then everything we will do can be suitably replaced dealing with this translated equation. □

Just for convenience from now on we assume that Hypothesis 5.1 holds (Remark 5.2 shows that it is not strictly necessary). Due to Hypothesis 5.1, the inverse operator of A is well defined. This is a bounded operator $H \to D(A)$ whose explicit expression is

$$\begin{aligned} A^{-1}: (H, \|\cdot\|) \quad &\longrightarrow \quad (\mathcal{D}(A), \|\cdot\|), \\ \eta \quad &\longmapsto \quad \left(\frac{\eta_0 + \int_{-r}^0 \eta_1(s)ds}{-a}, - \int_{-r}^{\xi} \eta_1(s)ds \right). \end{aligned}$$

We define in H the norm $\|\cdot\|_{-1}$ as

$$\|\eta\|_{-1} := \|A^{-1}\eta\|,$$

so

$$\|\eta\|_{-1}^2 = \left| \frac{\eta_0 + \int_{-r}^0 \eta_1(s)ds}{a} \right|^2 + \int_{-r}^0 |\eta_1(s)|^2 ds. \tag{28}$$

Lemma 5.3. *The norms $\|\cdot\|_{-1}$ and $\|\cdot\|_X$ are equivalent in H.*

Proof. Let $\eta = (\eta_0, \eta_1) \in H$. Taking into account (28) and by Hölder's inequality, we have

$$\begin{aligned} \|\eta\|_X^2 &= |\eta_0|^2 + \int_{-r}^0 |\eta_1(\xi)|^2 d\xi \\ &= \left| \eta_0 + \int_{-r}^0 \eta_1(\xi)d\xi - \int_{-r}^0 \eta_1(\xi)d\xi \right|^2 + \int_{-r}^0 |\eta_1(\xi)|^2 d\xi \\ &\le 2\left| \eta_0 + \int_{-r}^0 \eta_1(\xi)d\xi \right|^2 + 2\left| \int_{-r}^0 \eta_1(\xi)d\xi \right|^2 + \int_{-r}^0 |\eta_1(\xi)|^2 d\xi \\ &\le 2\left| \eta_0 + \int_{-r}^0 \eta_1(\xi)d\xi \right|^2 + 2\left(\int_{-r}^0 |\eta_1(\xi)|d\xi \right)^2 + \int_{-r}^0 |\eta_1(\xi)|^2 d\xi \\ &\le 2a^2 \left| \frac{\eta_0 + \int_{-r}^0 \eta_1(\xi)d\xi}{a} \right|^2 + 2r^2 \int_{-r}^0 |\eta_1(\xi)|^2 d\xi + \int_{-r}^0 |\eta_1(\xi)|^2 d\xi \\ &\le C\|\eta\|_{-1}^2, \end{aligned}$$

where $C = \max\{2a^2, 2r^2 + 1\}$.

On the other hand, still using (28) and Hölder's inequality, we have

$$\|\eta\|_{-1}^2 = \left| \frac{\eta_0 + \int_{-r}^0 \eta_1(s)ds}{a} \right|^2 + \int_{-r}^0 |\eta_1(s)|^2 ds \le \frac{2}{a^2}|\eta_0|^2 + \int_{-r}^0 |\eta_1(s)|^2 ds \le C'\|\eta\|_X^2,$$

where $C' = \max\left\{ \frac{2}{a^2}, 1 \right\}$. So the claim is proved. □

From Lemma 5.3 we get the following

Corollary 5.4. *There exists a constant $C_{a,r} > 0$ such that*

$$|\eta_0| \le C_{a,r}\|\eta\|_{-1}, \quad \forall \eta \in H. \tag{29}$$

Remark 5.5. *Corollary 5.4 represents a crucial issue and motivates our choice of working in the product space* $\mathbb{R} \times W^{1,2}_{r,0}$ *in place of the more usual product space* $\mathbb{R} \times L^2_r$. *Indeed, embedding the problem in* $\mathbb{R} \times L^2_r$ *and defining everything in the same way in this bigger space, we would not be able to have an estimate of type* (29) *controlling* $|\eta_0|$ *by* $\|\eta\|_{-1}$. *But this estimate is necessary to prove the continuity of the value function with respect to* $\|\cdot\|_{-1}$, *since in this way* g *is continuous in* $(H_+, \|\cdot\|_{-1})$. *On the other hand, the continuity of* V *with respect to* $\|\cdot\|_{-1}$ *is necessary to have a suitable property for the superdifferential of* V *(see Proposition 5.14), allowing to handle the unbounded linear term in the HJB equation.*

We show with an example that an estimate like (29) *cannot be obtained in the* $\mathbb{R} \times L^2_r$ *setting. Consider in* $\mathbb{R} \times L^2_r$ *the sequence*

$$\eta^n = (\eta^n_0, \eta^n_1(\cdot)), \quad \eta^n_0 := 1, \;\; \eta^n_1(\cdot) = -n\mathbf{1}_{[-1/n,0]}(\cdot), \quad n \geq 1.$$

Let $r > 1/n$. *In this setting, we have*

$$\begin{aligned}\|\eta^n\|^2_{-1} &= \left|\frac{\eta^n_0 + \int_{-r}^0 \eta^n_1(s)ds}{a}\right|^2 + \int_{-r}^0 \left|-\int_{-r}^{\xi} \eta^n_1(s)ds\right|^2 d\xi = 0 + \int_{-\frac{1}{n}}^0 \left|\int_{-\frac{1}{n}}^{\xi} n ds\right|^2 d\xi \\ &= \int_{-\frac{1}{n}}^0 n^2 \left(\xi + \frac{1}{n}\right)^2 d\xi = \frac{1}{3n} \longrightarrow 0.\end{aligned}$$

Therefore, we have $|\eta^n_0| = 1$ *and* $\|\eta^n\|_{-1} \to 0$. *This shows that an estimate like* (29) *does not hold in this setting.* □

5.2 Concavity and $\|\cdot\|_{-1}$-continuity of the value function

We are going to prove that V is concave and continuous with respect to $\|\cdot\|_{-1}$. First of all, recall that we have defined

$$H_+ := (0, +\infty) \times W^{1,2}_{r,0}.$$

We introduce also the spaces

$$\mathcal{G} := \{\eta \in H_+ \mid 0 \in \mathcal{U}(\eta)\},$$

$$\mathcal{F} := \left\{\eta \in H_+ \;\middle|\; \eta_0 + \int_{-\xi}^0 \eta_1(s)e^{as}ds > 0, \quad \forall \xi \in [-r, 0]\right\},$$

$$H_{++} := (0, +\infty) \times \{\eta_1(\cdot) \in W^{1,2}_{r,0} \mid \eta_1(\cdot) \geq 0, \;\forall\, \xi \in [-r, 0]\}.$$

Proposition 5.6.

(i) $H_{++} \subset \mathcal{F} = \mathcal{G} \subset \mathcal{D}(V) \subset H_+$ *and* $V(\eta) \geq 0$ *for every* $\eta \in \mathcal{F} = \mathcal{G}$.

(ii) $\mathcal{F}$ *is open with respect to* $\|\cdot\|_{-1}$.

Proof. (i) The inclusions

$$H_{++} \subset \mathcal{F} \subset H_+$$

are obvious. Let $\eta \in \mathcal{F}$ and set $Y(\cdot) := Y(\cdot;\eta,0)$. Due to Definition 4.1-(i) and to the definition of the set $\mathcal{F}$, we have

$$
\begin{aligned}
Y_0(t) = [S_A(t)\eta]_0 &= \eta_0 e^{at} + \int_{(-t)\vee(-r)}^{0} e^{a(t+\xi)}\eta_1(\xi)d\xi, \\
&= e^{at}\left(\eta_0 + \int_{(-t)\vee(-r)}^{0} e^{a\xi}\eta_1(\xi)d\xi\right) > 0, \ \forall\, t \geq 0
\end{aligned}
$$

so this claim is proved. This shows that $\mathcal{F} = \mathcal{G}$. Now let $\eta \in \mathcal{G}$. Since g is nondecreasing and $g(0) = 0$, $h_0(0) = 0$, we have

$$
V(\eta) \geq J(\eta,0) = \int_0^{+\infty} e^{-\rho t}\left(g(Y(t;\eta,0)) - h_0(0)\right)dt \geq \int_0^{+\infty} e^{-\rho t}\left(g(0) - h_0(0)\right)dt = 0.
$$

As a byproduct this shows that $V(\eta) \geq 0$ on $\mathcal{G}$ and that $\mathcal{G} \subset \mathcal{D}(V)$, so the proof of item (i) is complete.

(ii) Let $\bar{\eta} \in \mathcal{F}$. We have to prove that

$$
\exists\, \varepsilon \text{ such that } B_{\|\cdot\|_{-1}}(\bar{\eta},\varepsilon) \subset \mathcal{F}. \tag{30}
$$

Due to Lemma 5.3, (30) is equivalent to

$$
\exists\, \varepsilon \text{ such that } \ B_{\|\cdot\|_X}(\bar{\eta},\varepsilon) \subset \mathcal{F}. \tag{31}
$$

Let $\varepsilon > 0$ and $\eta \in B_{\|\cdot\|_X}(\bar{\eta},\varepsilon)$. Then we have

$$
\begin{cases}
|\eta_0 - \bar{\eta}_0| < \varepsilon^{\frac{1}{2}}, \\
\\
\|\eta_1 - \bar{\eta}_1\|_{L^2_r} < \varepsilon^{\frac{1}{2}}.
\end{cases}
$$

Therefore

$$
\begin{aligned}
&\left|\left(\eta_0 + \int_{-\xi}^{0} e^{as}\eta_1(s)ds\right) - \left(\bar{\eta}_0 + \int_{-\xi}^{0} e^{as}\bar{\eta}_1(s)ds\right)\right| \\
&\leq |\eta_0 - \bar{\eta}_0| + \left|\int_{-\xi}^{0} e^{as}\left(\eta_1(s) - \bar{\eta}_1(s)\right)ds\right| \\
&\leq |\eta_0 - \bar{\eta}_0| + r^{1/2}e^{|a|r}\|\eta_1 - \bar{\eta}\|_{L^2} < \varepsilon^{\frac{1}{2}} + r^{1/2}e^{|a|r}\varepsilon^{1/2}.
\end{aligned} \tag{32}
$$

where the second inequality follows from Holder's inequality. Then (31) straightly follows from (32) taking a sufficiently small $\varepsilon > 0$, so that the proof is complete. □

Definition 5.7. *Let $\eta \in \mathcal{D}(V)$, $\varepsilon > 0$. An admissible control $u^\varepsilon(\cdot) \in \mathcal{U}(\eta)$ is said ε-optimal for the initial state η if $J(\eta;u^\varepsilon(\cdot)) > V(\eta) - \varepsilon$.*

Proposition 5.8. *The set $\mathcal{D}(V)$ is convex and the value function V is concave on $\mathcal{D}(V)$.*

Proof. Let η, $\bar{\eta} \in \mathcal{D}(V)$ and set, for $\lambda \in [0,1]$, $\eta_\lambda := \lambda\eta + (1-\lambda)\bar{\eta}$. For $\varepsilon > 0$, let $u^\varepsilon(\cdot) \in \mathcal{U}(\eta)$ and $\bar{u}^\varepsilon(\cdot) \in \mathcal{U}(\bar{\eta})$ be two controls ε-optimal for the initial states $\eta, \bar{\eta}$ respectively. Set

$$
y(\cdot) := y(\cdot;\eta,u^\varepsilon(\cdot)),\ \bar{y}(\cdot) := \bar{y}(\cdot;\bar{\eta},\bar{u}^\varepsilon(\cdot)),\ u^\lambda(\cdot) := \lambda u^\varepsilon(\cdot) + (1-\lambda)\bar{u}^\varepsilon(\cdot).
$$

Finally set $y_\lambda(\cdot) := \lambda y(\cdot) + (1-\lambda)\bar{y}(\cdot)$. The function h_0 is convex so one has

$$h_0(u^\lambda(t)) \leq \lambda h_0(u^\varepsilon(t)) + (1-\lambda) h_0(\bar{u}^\varepsilon(t)), \; t \geq 0.$$

Moreover, by linearity of the state equation, we have

$$Y(t;\eta_\lambda, u_\lambda(\cdot)) = \lambda Y(y;\eta, u^\varepsilon(\cdot)) + (1-\lambda) Y(t;\bar{\eta}, \bar{u}^\varepsilon(\cdot)).$$

Hence, by concavity of g we have

$$g(Y(t;\eta_\lambda, u_\lambda(\cdot))) \geq \lambda g(Y(t;\eta, u^\varepsilon(\cdot))) + (1-\lambda) g(Y(t;\bar{\eta}, \bar{u}^\varepsilon(\cdot))), \; t \geq 0.$$

So, we have

$$\begin{aligned} V(\eta_\lambda) \geq J(\eta_\lambda, u^\lambda(\cdot)) &= \int_0^{+\infty} e^{-\rho t} \Big(g(Y(t;\eta_\lambda, u_\lambda)) - h_0(u^\lambda(t)) \Big) dt \\ &\geq \int_0^{+\infty} e^{-\rho t} \left(\lambda g(Y(t;\eta, u^\varepsilon)) + (1-\lambda) g(Y(t;\bar{\eta}, \bar{u}^\varepsilon)) - \lambda h_0(u^\varepsilon(t)) - (1-\lambda) h_0(\bar{u}^\varepsilon(t)) \right) dt \\ &= \lambda J(\eta, u^\varepsilon) + (1-\lambda) J(\bar{\eta}, \bar{u}^\varepsilon) > \lambda V(\eta) + (1-\lambda) V(\bar{\eta}) - \varepsilon \end{aligned}$$

Since ε is arbitrary, this shows both the claims. □

Corollary 5.9. *The value function V is continuous with respect to $\|\cdot\|_{-1}$ in $\mathcal{F}$.*

Proof. The function V is concave and bounded from below in the $\|\cdot\|_{-1}$ open set $\mathcal{F}$. Therefore the claim follows by a result of Convex Analysis (see e.g. [21, Ch. 1, Cor. 2.4]). □

Remark 5.10. *$\mathcal{F}$ is open also with respect to $\|\cdot\|$.* □

5.3 Monotonicity of the value function

Let us introduce the following partial order relation in H. Given η, $\zeta \in H$, we say

$$\eta \geq \zeta \iff \eta_0 \geq \zeta_0; \;\; \eta_1(\xi) \geq \zeta_1(\xi), \; \forall \xi \in [-r, 0]. \tag{33}$$

Analogously, denoting by m the Lebesgue measure in $[-r, 0]$, we say that

$$\eta > \zeta \iff \eta \geq \zeta \;\text{ and }\; \eta_0 > \zeta_0 \text{ or } m\{\eta_1(\xi) > \zeta_1(\xi)\} > 0, \; \forall \xi \in [-r, 0]. \tag{34}$$

Proposition 5.11. *The value function V is nondecreasing with respect to the order relation defined above. Moreover, for all $\eta \in \mathcal{D}(V)$, $h > 0$ (in the sense of* (34)*), we have*

$$\lim_{s \to +\infty} V(\eta + sh) = \frac{1}{\rho} \bar{g}_0. \tag{35}$$

Proof. It is straightforward to check from (7) that $S_A(t)$ is positive preserving, which means that

$$\eta \geq 0 \implies S_A(t)\eta \geq 0.$$

Let η, $\zeta \in \mathcal{D}(V)$ with $\eta \geq \zeta$. Let $u(\cdot) \in \mathcal{U}(\eta)$ and consider $Y(\cdot;\eta, u(\cdot))$. We have

$$Y(\cdot;\eta, u(\cdot)) - Y(\cdot;\zeta, u(\cdot)) = S_A(t)\,(\eta - \zeta) \geq 0. \tag{36}$$

Therefore

$$Y_0(t;\eta,u(\cdot)) \geq Y_0(t;\zeta,u(\cdot)).$$

This shows that $u(\cdot) \in \mathcal{U}(\eta)$. Hypothesis 5.12 implies that g is nondecreasing with respect to the order relation defined above. Set

$$\beta(t) := \int_0^t S_A(t-\tau)Bu(\tau)d\tau.$$

Then, also taking into account (36),

$$\begin{aligned} & J(\eta;u(\cdot)) - J(\zeta;u(\cdot)) \\ &= \int_0^{+\infty} e^{-\rho t}\left(g(Y(t;\eta,u(\cdot)) - g(Y(t;\zeta,u(\cdot)))\right)dt \\ &= \int_0^{+\infty} e^{-\rho t}\left(g(S_A(t)(\eta) + \beta(t)) - g(S_A(t)(\zeta) + \beta(t))\right)dt. \end{aligned}$$

So also $V(\eta) \geq V(\zeta)$ and the first part of the claim is proved.

Let us show the second part. Since $h(0) = 0$, we have

$$V(\eta+sh) \geq J(\eta+sh;0) = \int_0^{+\infty} e^{-\rho t} g\left(S_A(t)(\eta+sh)\right)dt = \int_0^{+\infty} e^{-\rho t} g_0\left([S_A(t)(\eta+sh)]_0\right)dt.$$

By (7) we have

$$\lim_{s\uparrow+\infty} [S_A(t)(\eta+sh)]_0 = +\infty, \quad \forall\, t \geq 0.$$

So, since g_0 is nondecreasing, by monotone convergence we get

$$\lim_{s\to+\infty} V(\eta+sh) = \int_0^{+\infty} e^{-\rho t}\bar{g}_0 dt = \frac{1}{\rho}\bar{g}_0,$$

the claim. □

Hypothesis 5.12. *g_0 is strictly increasing.*

Proposition 5.13. *Let Hypothesis 5.12 hold. We have the following statements:*

(i) *$V(\eta) < \frac{1}{\rho}\bar{g}_0$ for every $\eta \in \mathcal{D}(V)$.*

(ii) *For every $\eta \in \mathcal{D}(V)$ and $h \in H$ with $h > 0$ in the sense of (34), the function*

$$[0,+\infty) \to \mathbb{R},\ s \mapsto V(\eta+sh) \tag{37}$$

is strictly increasing.

Proof. (i) Let $u(\cdot) \in \mathcal{U}(\eta)$. Set $C := (2+r^3)^{\frac{1}{2}}$. By (8) we have

$$\|Y(t)\| \leq Ce^{at}\|\eta\| + \int_0^t Ce^{a(t-\tau)}\|b\||u(\tau)|d\tau \leq Ce^{at}\left(\|\eta\| + \|b\|\int_0^t |u(\tau)|d\tau\right). \tag{38}$$

With regard to the structure of U, we distinguish the two cases $\bar{u} < +\infty$, $\bar{u} = +\infty$.

Case $\bar{u} < +\infty$. In this case (38) yields

$$|Y_0(t)| \leq \|Y(t)\| \leq Ce^{\omega t}\left(\|\eta\| + \frac{1}{a}\|b\|\bar{u}t\right), \quad \forall\, t \geq 0. \tag{39}$$

Let δ be such that

$$g_0\left(Ce^{\omega}\left(\|\eta\| + \frac{1}{a}\|b\|\bar{u}\right)\right) = \bar{g}_0 - \delta. \tag{40}$$

Since g_0 is strictly increasing we have $\delta > 0$. Then, for every $u(\cdot) \in \mathcal{U}(\eta)$, we have thanks to (39) and (40) and since $h_0 \geq 0$

$$\begin{aligned}
J(\eta; u(\cdot)) = \int_0^{+\infty} & e^{-\rho t}\left(g(Y(t)) - h_0(u(t))\right) dt \leq \int_0^{+\infty} e^{-\rho t} g(Y_0(t)) dt \\
&\leq \int_0^1 e^{-\rho t} g_0\left(Ce^{\omega}\left(\|\eta\| + \frac{1}{a}\|b\|\bar{u}\right)\right) dt + \int_1^{+\infty} e^{-\rho t}\bar{g}_0 dt \\
&\leq \frac{1}{\rho}(\bar{g}_0 - \delta)\left(1 - e^{-\rho}\right) + \frac{1}{\rho}e^{-\rho}\bar{g}_0.
\end{aligned}$$

Taking the supremum over $u(\cdot) \in \mathcal{U}(\eta)$, in the inequality above we get

$$V(\eta) = \sup_{u(\cdot)\in\mathcal{U}(\eta)} J(\eta; u(\cdot)) \leq \frac{1}{\rho}(\bar{g}_0 - \delta)\left(1 - e^{-\rho}\right) + \frac{1}{\rho}e^{-\rho}\bar{g}_0 < \frac{1}{\rho}\bar{g}_0,$$

getting the claim in this case.

Case $\bar{u} = \infty$. By Hypothesis 3.3-(ii) there exist C_0, C_1 constant such that

$$h_0(u) \geq C_0 u - C_1, \quad \forall u \in U. \tag{41}$$

Given (38) we want to find an upper bound for $|Y_0(t)|$ as (39), in order to argue as before and get the claim. In this case, since $\bar{u} = +\infty$, we do not have directly this upper bound over all $u(\cdot) \in \mathcal{U}(\eta)$, but only on good controls (ε-optimal, which still suffices). Let $\varepsilon > 0$ and let $u(\cdot) \in \mathcal{U}(\eta)$ be an ε-optimal control for η. Then by (41)

$$\begin{aligned}
V(\eta) - \varepsilon < J(\eta; u(\cdot)) = \int_0^{+\infty} & e^{-\rho t}\left(g(Y(t)) - h_0(u(t))\right) dt \\
&\leq \int_0^{+\infty} e^{-\rho t}\left(g(Y(t)) - C_0 u(t) + C_1\right) dt.
\end{aligned}$$

So we have

$$\begin{aligned}
C_0 \int_0^1 e^{-\rho t}|u(t)|dt \leq C_0 \int_0^{+\infty} & e^{-\rho t}|u(t)|dt < \int_0^{+\infty} e^{-\rho t}\left(g(Y(t)) + C_1\right) dt - V(\eta) + \varepsilon \\
&\leq \int_0^{+\infty} e^{-\rho t}\left(\bar{g}_0 + C_1\right) dt - V(\eta) + \varepsilon = \frac{1}{\rho}(\bar{g}_0 + C_1) - V(\eta) + \varepsilon < M.
\end{aligned}$$

This means that there exists some $M' > 0$ such that

$$\int_0^1 |u(t)|dt \leq M', \qquad \forall\, u(\cdot) \in \mathcal{U}(\eta)\ \varepsilon - \text{optimal}.$$

Therefore an upper bound as (39) holds true for $t \in [0,1]$ for the controls $u(\cdot) \in \mathcal{U}(\eta)$ which are ε-optimal. This allows to conclude as before.

(ii) By Propositions 5.8 and 5.11, we know that the real function (37) is concave and nondecreasing. Then, assuming by contradiction that it is not strictly increasing, it must exist $\bar{s} \geq 0$ such that $V(\eta + \bar{s}h)$ is constant on the half line $[\bar{s}; +\infty)$. This fact would contradict claim (i) and (35), so we conclude. $\square$

5.4 Superdifferential of concave $\|\cdot\|_{-1}$-continuous function

Motivated by Proposition 5.8 and Corollary 5.9, in this subsection we focus on the properties of the superdifferential of concave and $\|\cdot\|_{-1}$-continuous functions. This will be useful to prove the regularity result in the next section. We recall first some definitions and basic results from non-smooth analysis concerning the generalized differentials. For the details we refer to [40].

Let v be a concave continuous function defined on some open set $\mathcal{A}$ of H. Given $\eta \in \mathcal{A}$ the superdifferential of v at η is the convex and closed set

$$D^+v(\eta) = \{p \in H \mid v(\zeta) - v(\eta) \leq \langle \zeta - \eta, p\rangle, \ \ \forall \zeta \in \mathcal{A}\}.$$

The set of the *reachable gradients* at $\eta \in \mathcal{A}$ is defined as

$$D^*v(\eta) := \{p \in H \mid \exists (\eta_n) \subset \mathcal{A}, \ \eta_n \to \eta, \text{ such that } \exists \nabla v(\eta_n) \text{ and } \nabla v(\eta_n) \rightharpoonup p\}.$$

As known (see [40, Ch. 1, Prop. 1.11]) $D^+v(\eta)$ is a closed convex not empty subset of H. Moreover the set-valued map $\mathcal{A} \longrightarrow \mathcal{P}(H)$, $\eta \mapsto D^+v(\eta)$ is locally bounded (see [40, Ch. 1, Prop. 1.11]). Also we have the representation (see [14, Cor. 4.7])

$$D^+v(\eta) = \overline{co}(D^*v(\eta)), \quad \eta \in \mathcal{A}. \tag{42}$$

Given $p, h \in H$, with $\|h\| = 1$, we denote

$$p_h := \langle p, h\rangle.$$

We introduce the directional superdifferential of v at η along the direction h

$$D_h^+v(\eta) := \{\alpha \in \mathbb{R} \mid v(\eta + \gamma h) - v(\eta) \leq \gamma\alpha, \ \forall \gamma \in \mathbb{R}\}.$$

This set is a nonempty closed and bounded interval $[a, c] \subset \mathbb{R}$. More precisely, since $v(\eta)$ is concave, we have

$$a = v_h^+(\eta), \quad c = v_h^-(\eta),$$

where $v_h^+(\eta), v_h^-(\eta)$ denote respectively the right and the left derivatives of the real function $s \mapsto v(\eta + sh)$ at $s = 0$. By definition of $D^+v(\eta)$, the projection of $D^+v(\eta)$ on b must be contained in $D_h^+v(\eta)$, that is

$$D_h^+v(\eta) \supset \left\{p_h \mid p \in D^+v(\eta)\right\}. \tag{43}$$

On the other hand, Proposition 2.24 in [40, Ch. 1] states that

$$a = \inf\{\langle p, h\rangle \mid p \in D^+v(\eta)\} \quad c = \sup\{\langle q, h\rangle, \mid q \in D^+v(\eta)\},$$

and the sup and inf above are attained. This means that there exist $p, q \in D^+v(\eta)$ such that

$$a = \langle p, h\rangle, \quad c = \langle q, h\rangle.$$

Since $D^+v(\eta)$ is convex, we see that also the converse inclusion of (43) is true. Therefore

$$D_h^+v(\eta) = \left\{p_h \mid p \in D^+v(\eta)\right\}. \tag{44}$$

Proposition 5.14. *Let $v : \mathcal{F} \longrightarrow \mathbb{R}$ be a concave function continuous with respect to $\|\cdot\|_{-1}$ and let $\eta \in \mathcal{F}$, $p \in D^*v(\eta)$. Then*

(i) $p \in \mathcal{D}(A^*)$*;*

(ii) there exists a sequence $\eta_n \to \eta$ such that for each $n \in \mathbb{N}$ there exists $\nabla v(\eta_n)$ and $\nabla v(\eta_n) \in \mathcal{D}(A^*)$;

(iii) $\nabla v(\eta_n) \rightharpoonup p$ *and* $A^*\nabla v(\eta_n) \rightharpoonup A^*p$.

Proof. See [27, Prop. 3.12-(4)] and [28, Rem. 2.11]. □

6 Dynamic Programming and HJB equation

We are ready to approach the problem by the Dynamic Programming. From now on, just for convenience, we assume without loss of generality that $\|b\| = 1$.

Theorem 6.1 (Dynamic Programming Principle)**.** *For any $\eta \in \mathcal{D}(V)$ and for any $\tau \geq 0$,*

$$V(\eta) = \sup_{u(\cdot)\in\mathcal{U}(\eta)} \left[\int_0^\tau e^{-\rho t} \left(g\left(Y(t;\eta,u(\cdot))\right) \,-\, h_0(u(t)) \right) dt + e^{-\rho\tau} V\left(Y(\tau;\eta,u(\cdot))\right) \right].$$

Proof. See e.g. [37, Th. 1.1, Ch. 6]. □

The differential version of the Dynamic Programming Principle is the HJB equation. We consider this equation in the set $\mathcal{F}$. It is

$$\rho v(\eta) = \langle A\eta, \nabla v(\eta)\rangle \,+\, g(\eta) \,+\, \sup_{u\in U} \left\{ \langle Bu, \nabla v(\eta)\rangle - h_0(u) \right\}, \qquad \eta \in \mathcal{F}. \tag{45}$$

We introduce the following Inada's type assumption on h_0.

Hypothesis 6.2. $\lim_{u\downarrow 0} h_0'(u) = 0, \quad \lim_{u\uparrow \bar{u}} h_0'(u) = +\infty.$

Defining the Legendre transform of h_0

$$\mathcal{H}(p_0) := \sup_{u\in U}\{up_0 - h_0(u)\}, \tag{46}$$

due to Hypothesis 6.2, it is easily checked that

$$\begin{cases} \mathcal{H}(p_0) = 0, \text{ if } p_0 \leq 0, \\ \mathcal{H}(p_0) > 0, \text{ if } p_0 > 0. \end{cases}$$

Since

$$\sup_{u\in U} \left\{ \langle Bu, p\rangle \,-\, h_0(u) \right\} = \sup_{u\in U} \left\{ \langle u, B^*p\rangle \,-\, h_0(u) \right\},$$

taking into account that $B^*p = \langle b, p\rangle$, (45) can be rewritten as

$$\rho v(\eta) = \langle \eta, A^*\nabla v(\eta)\rangle \,+\, g(\eta) \,+\, \mathcal{H}(\langle \nabla v(\eta), b\rangle), \qquad \eta \in \mathcal{F}. \tag{47}$$

We note that the nonlinear term in (47) can be defined without requiring the full regularity of v, but only the C^1-smoothness of v with respect to the direction b. Indeed, denoting coherently with (44) by v_b the directional derivative of v with respect to b, we can intend the nonlinear term in (47) as $\mathcal{H}(v_b(\eta))$. So we can write (47) as

$$\rho v(\eta) = \langle \eta, A^*\nabla v(\eta)\rangle \,+\, g(\eta) \,+\, \mathcal{H}(v_b(\eta)), \qquad \eta \in \mathcal{F}. \tag{48}$$

Proposition 6.3. *Let Hypothesis 6.2 hold true. Then the function* $\mathcal{H}$ *is finite and strictly convex in* $(0, +\infty)$.

Proof. Let $\tilde{U} := [-\bar{u}, \bar{u}]$. If $\bar{u} = +\infty$, the set $\tilde{U}$ is intended as $\mathbb{R}$. Let

$$\tilde{h}_0(u) := \begin{cases} h_0(u) & \text{if } u \in [0, \bar{u}], \\ h_0(-u) & \text{if } u \in [-\bar{u}, 0]. \end{cases}$$

The Legendre transform of $\tilde{h}_0$ is

$$\tilde{\mathcal{H}}(p_0) := \sup_{u \in \tilde{U}} \{up - \tilde{h}_0(u)\}.$$

Due to Hypothesis 3.3-(ii) and to [41, Cor. 26.4.1], $\tilde{\mathcal{H}}(p)$ is finite and strictly convex in $\mathbb{R}$. In order to get the claim, we need just to prove that that for $p_0 > 0$ we have $\tilde{\mathcal{H}}(p_0) = \mathcal{H}(p_0)$. Indeed if $p_0 > 0$

$$\tilde{\mathcal{H}}(p_0) = \sup_{u \in \tilde{U}} \{up_0 - \tilde{h}_0(u)\} = \sup_{u \in U} \{up_0 - \tilde{h}_0(u)\} = \sup_{u \in U} \{up_0 - h_0(u)\} = \mathcal{H}(p_0),$$

where the second equality follows from Hypothesis 6.2. □

6.1 The HJB equation: viscosity solutions

In this subsection we are going to prove that the value function V is a viscosity solution of the HJB equation (48). To this aim, we need to define a suitable set of regular test functions. This is the set

$$\mathcal{T} := \Big\{ \varphi \in C^1(H) \mid \nabla\varphi(\cdot) \in \mathcal{D}(A^*), \ A^*\nabla\varphi : H \to H \text{ is continuous} \Big\}.$$

Let us define, for $u \geq 0$, the differential operator $\mathcal{L}^u$ on $\mathcal{T}$ by

$$[\mathcal{L}^u\varphi](\eta) := -\rho\varphi(\eta) + \langle \eta, A^*\nabla\varphi(\eta) \rangle + u\langle \nabla\varphi(\eta), b \rangle.$$

The proof of the following chain's rule can be found in [37, Ch. 2, Prop. 5.5].

Lemma 6.4. *Let* $\eta \in H$, $\varphi \in \mathcal{T}$, $u(\cdot) \in L^2_{loc}([0, +\infty); \mathbb{R})$ *and set* $Y(t) := Y(t; \eta, u(\cdot))$. *Then the following chain's rule holds:*

$$e^{-\rho t}\varphi(Y(t)) - \varphi(\eta) = \int_0^t e^{-\rho s}[\mathcal{L}^{u(s)}\varphi](Y(s))ds, \quad \forall\, t \geq 0.$$ □

Definition 6.5.

(i) *A continuous function* $v : \mathcal{F} \to \mathbb{R}$ *is called a viscosity subsolution of* (48) *if, for each couple* $(\eta_M, \varphi) \in \mathcal{F} \times \mathcal{T}$ *such that* $v - \varphi$ *has a local maximum at* η_M, *we have*

$$\rho v(\eta_M) \leq \langle \eta_M, A^*\nabla\varphi(\eta_M) \rangle + g(\eta_M) + \mathcal{H}\left(\varphi_b(\eta_M)\right).$$

(ii) *A continuous function* $v : \mathcal{F} \to \mathbb{R}$ *is called a viscosity supersolution of* (48) *if, for each couple* $(\eta_m, \varphi) \in \mathcal{F} \times \mathcal{T}$ *such that* $v - \varphi$ *has a local minimum at* η_m, *we have*

$$\rho v(\eta_m) \geq \langle \eta_m, A^*\nabla\varphi(\eta_m) \rangle + g(\eta_m) + \mathcal{H}\left(\varphi_b(\eta_M)\right).$$

(iii) *A continuous function $v : \mathcal{F} \to \mathbb{R}$ is called a viscosity solution of* (48) *if it is both a viscosity sub and supersolution of* (48).

We introduce the following assumption on U.

Hypothesis 6.6. *Either* (i) $\bar{u} < +\infty$ *or*

$$\text{(ii)} \quad \bar{u} = +\infty \quad \textit{and} \quad \exists\, \alpha > 0 \;\; \textit{such that} \; \liminf_{u \to +\infty} \frac{h_0(u)}{u^{1+\alpha}} > 0. \tag{49}$$

Remark 6.7. *We notice that Hypothesis 6.6-(ii) is just slightly stronger than the assumption* $\lim_{u\to+\infty} h_0'(u) = +\infty$ *in Hypothesis 6.2.*

Lemma 6.8. *Let Hypotheses 6.6 hold. Then, for every $\eta \in \mathcal{F}$, $\varepsilon > 0$, there exists M_ε such that*

$$\int_0^{+\infty} e^{-\rho t}|u(t)|^{1+\alpha} dt \leq M_\varepsilon \quad \forall\, u^\varepsilon(\cdot) \in \mathcal{U}(\eta)\ \varepsilon - \textit{optimal for}\, \eta,$$

where α is the constant appearing in (49).

Proof. If Hypothesis 6.6-(i) holds, the proof is trivial. So, let Hypothesis 6.6-(ii) hold true. By such assumption, there exist constants $M_0, M_1 > 0$ such that

$$h_0(u) \geq M_0 u^{1+\alpha}(t) - M_1.$$

Let $u^\varepsilon(\cdot) \in \mathcal{U}(\eta)$ an ε-optimal control for η. Then

$$\begin{aligned} V(\eta) - \varepsilon < J(\eta; u^\varepsilon(\cdot)) &= \int_0^{+\infty} e^{-\rho t} \left(g(Y(t)) - h_0(u(t)) \right) dt \\ &\leq \int_0^{+\infty} e^{-\rho t} \left(g(Y(t)) - M_0 |u(t)|^{1+\alpha} + M_1 \right) dt. \end{aligned} \tag{50}$$

From (50) we get

$$\begin{aligned} M_0 \int_0^{+\infty} e^{-\rho t}|u(t)|^{1+\alpha} dt &\leq \int_0^{+\infty} e^{-\rho t} \left(g(Y(t)) + M_1 \right) dt - V(\eta) + \varepsilon \\ &\leq \int_0^{+\infty} e^{-\rho t} \left(\bar{g}_0 + M_1 \right) dt - V(\eta) + \varepsilon < \frac{\bar{g}_0 + M_1}{\rho} - V(\eta) + \varepsilon =: M_\varepsilon. \end{aligned}$$

So the claim is proved. □

Theorem 6.9. *Let Hypotheses 6.6 hold. The value function V is a viscosity solution of* (48) *on $\mathcal{F}$.*

Proof. *Subsolution property.* Let $(\eta_M, \varphi) \in \mathcal{F} \times \mathcal{T}$ be such that $V - \varphi$ has a local maximum at η_M. Without loss of generality we can suppose $V(\eta_M) = \varphi(\eta_M)$. Let us suppose, by contradiction, that there exists $\nu > 0$ such that

$$2\nu \leq \rho V(\eta_M) - \left(\langle \eta_M, A^* \nabla \varphi(\eta_M) \rangle + g(\eta_M) + \mathcal{H}\left(\varphi_b(\eta_M) \right) \right).$$

Let us define the function

$$\tilde{\varphi}(\eta) := V(\eta_M) + \langle \nabla \varphi(\eta_M), \eta - \eta_M \rangle, \quad \eta \in H. \tag{51}$$

We have

$$\nabla\tilde{\varphi}(\eta) = \nabla\varphi(\eta_M), \quad \forall\, \eta \in H.$$

Thus $\tilde{\varphi} \in \mathcal{T}$ and we have as well

$$2\nu \le \rho V(\eta_M) - \left(\langle \eta_M, A^*\nabla\tilde{\varphi}(\eta_M)\rangle + g(\eta) + \mathcal{H}\left(\tilde{\varphi}_b(\eta_M)\right)\right).$$

Now, we know that V is a concave function and that $V-\varphi$ has a local maximum at η_M, so that

$$V(\eta) \le V(\eta_M) + \langle \nabla\varphi(\eta_M), \eta - \eta_M\rangle. \tag{52}$$

Thus, by (51) and (52)

$$\tilde{\varphi}(\eta_M) = V(\eta_M) \quad \tilde{\varphi}(\eta) \ge V(\eta), \quad \forall\, \eta \in \mathcal{F}. \tag{53}$$

Let $B_\varepsilon := B_{(H,\|\cdot\|)}(\eta_M, \varepsilon)$. Due to the properties of the functions belonging to $\mathcal{T}$, we can find $\varepsilon > 0$ such that

$$\nu \le \rho V(\eta) - \left(\langle \eta, A^*\nabla\tilde{\varphi}(\eta)\rangle + g(\eta) + \mathcal{H}\left(\tilde{\varphi}_b(\eta_M)\right)\right), \quad \forall \eta \in B_\varepsilon.$$

Take a sequence $\delta_n > 0$ such that $\delta_n \to 0$. For each $n \in \mathbb{N}$, take a δ_n-optimal control $u_n(\cdot) \in \mathcal{U}(\eta)$ and set $Y^n(\cdot) := Y(\cdot;\eta_M, u_n(\cdot))$. Define

$$t_n := \inf\{t \ge 0 \mid \|Y^n(t) - \eta_M\| = \varepsilon\} \wedge 1$$

with the agreement that $\inf \emptyset = +\infty$. Of course t_n is well defined and belongs to $(0,1]$. Moreover, by continuity of $t \mapsto Y^n(t)$, we have $Y^n(t) \in B_\varepsilon$, for $t \in [0, t_n)$. By definition of δ_n-optimal control, we have as consequence of the Dynamic Programming Principle

$$\delta_n \ge -\int_0^{t_n} e^{-\rho t}\left[g(Y^n(t)) - h_0(u_n(t))\right] dt - \left(e^{-\rho t_n} V(Y(t_n)) - V(\eta_M)\right). \tag{54}$$

Therefore, by (53) and (54),

$$\begin{aligned}
\delta_n &\ge -\int_0^{t_n} \left[g(Y^n(t)) - h_0(u_n(t))\right] dt - \left(e^{-\rho t_n}\left(\tilde{\varphi}(Y^n(t_n))\right) - \tilde{\varphi}(\eta_M)\right) \\
&= \int_0^{t_n} e^{-\rho t}\left[g(Y^n(t)) - h_0(u_n(t)) + [\mathcal{L}^{u_n(t)}\tilde{\varphi}](Y^n(t))\right] dt \\
&\ge -\int_0^{t_n} e^{-\rho t}\left[g(Y^n(t)) - \rho\tilde{\varphi}(Y^n(t)) + \langle A^*\nabla\tilde{\varphi}(Y^n(t)), Y^n(t)\rangle + \mathcal{H}(\tilde{\varphi}_b(Y^n(t)))\right] dt \ge t_n\nu.
\end{aligned}$$

Therefore, since $\delta_n \to 0$ we also have $t_n \to 0$. We claim that $t_n \to 0$ implies

$$\|Y^n(t_n) - \eta_M\| \longrightarrow 0. \tag{55}$$

This would be a contradiction of the definition of t_n, concluding the proof. Let us prove (55). Using the definition of mild solution (4.1) of $Y^n(t_n)$, we have

$$\begin{aligned}
\|Y^n(t_n) - \eta_M\| &= \left\| S_A(t_n)\eta_M + \int_0^{t_n} S_A(t_n - \tau)Bu_n(\tau)d\tau \; - \; \eta_M \right\| \\
&\le \|\left(S_A(t_n) - I\right)\eta_M\| + \left\| \int_0^{t_n} S_A(t_n - \tau)Bu_n(\tau)d\tau \right\| \\
&\le \|\left(S_A(t_n) - I\right)\eta_M\| + \int_0^{t_n} \|S_A(t_n - \tau)\|_{\mathcal{L}(H)}\|B\||u_n(\tau)|d\tau.
\end{aligned}$$

By the estimate (8) of the $S_A(\cdot)$, to prove that the right side of above inequality converges to 0, it suffices to prove that

$$\int_0^{t_n} |u_n(s)|ds \to 0. \tag{56}$$

We have to distinguish the two case. If Hypothesis 6.6-(i) holds true, since $t_n \to 0$ we have directly (56). If Hypothesis 6.6-(ii) holds true, set $\beta > 1$ and $1/\beta + 1/\alpha = 1$. Then $\alpha = \beta/(\beta-1)$ and by Hölder's inequality

$$\int_0^{t_n} |u_n(s)|ds \leq \left(\int_0^{t_n} |u_n(\tau)|^\beta d\tau\right)^{\frac{1}{\beta}} t_n^{\frac{\beta-1}{\beta}}.$$

Since by Lemma 6.8 we know that $\left(\int_0^{t_n} |u_n(\tau)|^\beta d\tau\right)^{\frac{1}{\beta}}$ is bounded and since $t_n \to 0$, we have again (56). So the proof of this part is complete.

Supersolution property. The proof that V is a viscosity supersolution is more standard. We refer to [37, Ch. 6, Th. 3.2]. □

6.2 Smoothness of viscosity solutions

In this subsection we are going to show our main result that is the proof of a C^1 directional regularity result for viscosity solutions to the HJB (45). In particular, this result applies to the value function when $b > 0$ (in the sense of (34)) and all the hypotheses collected so far hold true.

Lemma 6.10. *Let $v : \mathcal{F} \longrightarrow \mathbb{R}$ be a concave $\|\cdot\|_{-1}$-continuous function and suppose that $\eta \in \mathcal{F}$ is a differentiability point for v and that $\nabla v(\eta) = \xi$. Then*

1. *There exists a $\varphi \in \mathcal{T}$ such that $v - \varphi$ has a local maximum at η and $\nabla\varphi(\eta) = \xi$.*
2. *There exists a $\varphi \in \mathcal{T}$ such that $v - \varphi$ has a local minimum at η and $\nabla\varphi(\eta) = \xi$.*

Proof. See [27, Lemma 4.5]. □

Theorem 6.11. *Let v be a concave $\|\cdot\|_{-1}$-continuous viscosity solution of (48) on $\mathcal{F}$, which is strictly increasing along the direction b. Then v is differentiable along b at each $\eta \in \mathcal{F}$, $v_b(\eta) \in (0, +\infty)$ and the function $\eta \mapsto v_b(\eta)$ is continuous in $\mathcal{F}$.*

Proof. Let $\eta \in \mathcal{F}$ and $p, q \in D^*v(\eta)$. Due to Proposition 5.14, there exist sequences (η_n), $(\tilde{\eta}_n) \subset \mathcal{F}$ such that:

- $\eta_n \to \eta$, $\tilde{\eta}_n \to \eta$;
- $\nabla v(\eta_n)$ and $\nabla v(\tilde{\eta}_n)$ exist for all $n \in \mathbb{N}$ and $\nabla v(\eta_n) \rightharpoonup p$, $\nabla v(\tilde{\eta}_n) \rightharpoonup q$;
- $A^*\nabla v(\eta_n) \rightharpoonup A^*p$ and $A^*\nabla v(\tilde{\eta}_n) \rightharpoonup A^*q$.

Recall that, given $\eta \in H$, we have defined

$$\eta_b := \langle \eta, b \rangle.$$

Due to Lemma 6.10 and Theorem 6.9 we can write, for each $n \in \mathbb{N}$,

$$\rho v(\eta_n) = \langle \eta_n, A^*\nabla v(\eta_n)\rangle + g(\eta_n) + \mathcal{H}(v_b(\eta_n))$$

$$\rho v(\tilde{\eta}_n) = \langle \eta_n, A^*\nabla v(\tilde{\eta}_n)\rangle + g(\tilde{\eta}_n) + \mathcal{H}(v_b(\tilde{\eta}_n)).$$

So, letting $n \to \infty$ we get

$$\langle \eta, A^* p\rangle + g(\eta) + \mathcal{H}(p_b) = \rho v(\eta) = \langle \eta, A^* q\rangle + g(\eta) + \mathcal{H}(q_b). \tag{57}$$

On the other hand $\lambda p + (1-\lambda) q \in D^+ v(\eta)$ for any $\lambda \in (0,1)$, so that we have the subsolution inequality

$$\rho v(\eta) \le \langle \eta, A^*[\lambda p + (1-\lambda)q]\rangle + g(\eta) + \mathcal{H}(\lambda p_b + (1-\lambda) q_b),\ \forall\, \lambda \in (0,1). \tag{58}$$

Combining (57) and (58) we get

$$\mathcal{H}(\lambda p_b + (1-\lambda) q_b) \ge \lambda \mathcal{H}(p_b) + (1-\lambda)\mathcal{H}(q_b). \tag{59}$$

Notice that, since $p, q \in D^* v(\eta)$, we have also $p, q \in D^+ v(\eta)$. Since v is strictly increasing along b we must have $p_b, q_b \in (0, +\infty)$. Since $\mathcal{H}$ is strictly convex on $(0,+\infty)$, (59) yields $p_b = q_b$. Due to (44) we have that $p_b,\ q_b \in D_b^+ v(\eta)$. With this argument we have shown that the projection of $D^* v(\eta)$ onto b is a singleton. Due to (42), this implies that also the projection of $D^+ v(\eta)$ onto b is a singleton. Due to (44) we have that $D_b^+ v(\eta)$ is a singleton too. Since v is concave, this is enough to conclude that it is differentiable along the direction b at η and $v_b(\eta) \in (0,+\infty)$.

Now we prove the second claim of the Theorem, that is that the map $\eta \mapsto v_b(\eta)$ is continuous in $\mathcal{F}$. To this aim we take $\eta \in \mathcal{F}$ and a sequence $(\eta^n) \subset \mathcal{F}$ such that $\eta^n \to \eta$. We have to prove that $v_b(\eta^n) \to v_b(\eta)$. Being v concave, by definition of superdifferential (44) for every $n \in \mathbb{N}$, there exists $p_n \in D^+ v(\eta_n)$ such that $\langle p_n, b\rangle = v_b(\eta_n) \in D_b^+(\eta_n)$. Since v is concave, it is also locally Lipschitz continuous, so that the super-differential is a locally bounded multi-function (see [40, Ch. 1, Prop. 2.5]). Therefore, from each subsequence (p_{n_k}) we can extract a sub-subsequence $(p_{n_{k_h}})$ such that

$$p_{n_{k_h}} \longrightarrow p \in \mathcal{F}$$

for some limit point p. Due to concavity of v, this limit point must belong to $D^+ v(\eta)$. We have shown in the first part of the proof that the projection of $D^+ v(\eta)$ onto b is the singleton $v_b(\eta)$, so that it must be

$$\langle p, b\rangle = v_b(\eta).$$

With this argument we have shown that, from each subsequence $(v_b(\eta^{n_k}))$, we can extract a sub-subsequence $(v_b(\eta^{n_{k_h}}))$ such that

$$v_b(\eta^{n_{k_h}}) = \langle p_{n_{k_h}}, b\rangle \to \langle p, b\rangle = v_b(\eta).$$

The claim follows by the usual argument on subsequences. □

Remark 6.12. *Notice that in the assumption of Theorem 6.11 we do not require that v is the value function, but only that it is a concave $\|\cdot\|_{-1}$-continuous viscosity solution of* (48) *strictly increasing along b.*

Corollary 6.13. *Let $b > 0$ (in the sense of* (34)*) and Hypotheses 5.12, 6.2, 6.6 hold. Then V is differentiable along b at any point $\eta \in \mathcal{F}$, $V_b(\eta) \in (0,+\infty)$ and the function $\eta \mapsto V_b(\eta)$ is continuous in $\mathcal{F}$.*

Proof. Due to Proposition 5.8 the function V is concave in $\mathcal{F}$ and due to Corollary 5.9 it is $\|\cdot\|_{-1}$-continuous therein. Moreover, since $b > 0$, due to Proposition 5.13, it is strictly increasing along b . Finally, by Theorem 6.9 it is a viscosity solution of the HJB equation (48). Therefore Theorem 6.11 applies to V and we have the claim. □

Corollary 6.13 allows to define a feedback map in classical form when the assumptions of this corollary are fulfilled. Indeed we can define the map

$$\mathcal{P}(\eta) := \operatorname{argmax}_{u \in U} \{uV_b(\eta) - h_0(u)\}, \quad \eta \in \mathcal{F}. \tag{60}$$

Existence, uniqueness and continuity in $\mathcal{F}$ of the argmax follow from (46). With this map at hand, we can try to study the associated closed loop equation and prove a Verification Theorem stating the optimality of feedback controls, as it is done in [28]. The main problem is represented by the study of the closed loop equation, which, differently form [27], has to be approached directly in infinite dimension. These topics are the object of further research.

7 Application to optimal advertising

In this section we show how our setting covers a model for optimal advertising considered in the classical economic literature. The first model for the optimal advertising problem goes back to Nerlove and Arrow [38], which has paved the way for the development of a number of more complex models on the subject[4]. Here we describe the model proposed by Pawels in [39], which fits our setting.

In this model a monopolistic firm is considered. The rate of sales of this firm at time t, indicated by $q(t)$, is given by

$$q(t) = f(p(t), K(t)), \tag{61}$$

where $p(\cdot)$ is the price at time t of the commodity produced and sold by the monopolist; $K(t)$ is the monopolist's stock of goodwill at time t; f is a demand function, concave and increasing on K, convex and decreasing on p. The rate of advertising expenditure at time t is denoted by $s(t)$[5]. This expenditure is transformed by a production function of goodwill G: in the absence of any time lags, the value of $G(s(t))$ gives the gross increase in the stock of goodwill resulting from a rate of advertising $s(t)$. It is assumed that

$$G(0) = 0, \quad G' > 0, \quad G'' < 0. \tag{62}$$

However time lags due to the memory of the past advertising are considered in the model and the resulting state equation is

$$\begin{cases} \dot{K}(t) = -\delta K(t) + \int_{-\infty}^{t} w(t-\sigma)G(s(\sigma))d\sigma, \\ K(0) = K_0, \quad s(\sigma) = s_0(\sigma), \ \sigma \in (-\infty, 0), \end{cases} \tag{63}$$

where $\delta > 0$ is a forgetting factor and $w : [0, +\infty) \to \mathbb{R}^+$ is a function weighting of the past of the control with the property that

$$\int_{-\infty}^{t} w(t-\sigma)d\sigma = 1. \tag{64}$$

[4]We refer to [29], a survey on this subject that also explains the importance of the introduction of memory effects in the state equation.

[5]Goodwill can be interpreted as "image" in the consumer's mind, with this image being revived through advertising or, more in general, through an expenditure improving this image (for example, we can suppose that also the expenditure is in R&D to innovate brings an increase of the consumer's belief).

Total costs at time t are $C(q(t)) + s(t)$, where C represents the cost function of production and is assumed to be increasing and convex. Finally the total revenue at time t is

$$R(p(t), K(t)) = p(t)f(p(t), K(t)) - C\left(f(p(t), K(t))\right). \tag{65}$$

The optimal control problem consists in maximize, over the controls $s(\cdot), p(\cdot) : [0, +\infty) \to \mathbb{R}$, the present value of future profits:

$$\int_0^{+\infty} e^{-\rho t}\left(R(p(t), K(t)) - s(t)\right) dt. \tag{66}$$

The problem is studied in [39] by means of a Maximum Principle, using the theoretical background provided by [42]. We stress that no constraints on the state and the control variables are imposed in [39], while our approach, as we have shown, allows to treat the constrained case.

First of all, we notice that the optimization with respect to $p(t)$ can be performed pointwise inside the integral (66), as this variable does not appear in the state equation (64). Indeed the optimality condition for $p(t)$ is

$$\frac{\partial R}{\partial p}(p(t), K(t)) = 0,$$

i.e.

$$p(t) = \left[\frac{\partial R}{\partial p}(\cdot, K(t))\right]^{-1}(0).$$

If we suppose that f is separable in p, K, i.e. in the form $f(p, K) = \alpha(p)\beta(K)$ and that C is linear, i.e. in the form $C(q) = \gamma q$, $\gamma > 0$, then the optimum $p(t)$ is constant. More precisely, supposing suitable assumptions[6] on α, it is the unique solution $\bar{p} > \gamma$ to the equation

$$(p - \gamma)\alpha'(p) + \alpha(p) = 0.$$

Suppose that this is the case and that $w \equiv 0$ on $(r, +\infty)$ for some $r > 0$. Then the model above is covered by our setting through suitable transformations as follows. Setting $b_1(\xi) := w(-\xi)$, $\xi \in [-r, 0]$, and $u(\sigma) := G(s(\sigma))$, the state equation becomes like ours

$$\dot{K}(t) = -\delta K(t) + \int_{-r}^{0} b_1(\xi)u(t + \xi)d\xi, \tag{67}$$

and the space of control becomes $U := [0, G(+\infty)]$. On the other hand, with these transformations, the functional (66) becomes

$$\int_0^{+\infty} e^{-\rho t}\big(R(\bar{p}, K(t)) - G^{-1}(u(t))\big)dt. \tag{68}$$

Therefore, setting $h_0 := G^{-1}$ and $g_0(\cdot) := R(\bar{p}, \cdot)$, the functional (68) looks like (3)[7]. Now we can apply Theorem 6.11: the value function V of this transformed problem is smooth along the

[6] For instance, we may take as in the example of [39], $\alpha(p) = Ap^{-\eta}$, $A > 0$, $\eta > 1$.

[7] Hypotehsis 3.3-(ii) is fulfilled by $h_0 = G^{-1}$ due to (62). Instead, Hypotheses 3.3-(i), 6.2 and 6.6 on g_0, h_0 will correspond to suitable ones on β, G:

- Hypothesis 3.3-(i) corresponds just to require β fulfilling Hypothesis 3.3-(i);
- Hypothesis 6.2 becomes $G'(0) = +\infty$, $G'(+\infty) = 0$;
- Hypothesis 6.6 becomes either (i) $G(+\infty) < +\infty$ or (ii) $\exists\ \alpha > 0$ such that $\limsup_{\zeta \to +\infty} \frac{G(\zeta)}{\zeta^{\frac{1}{1+\alpha}}} = 0$.

direction $b = (0, b_1)$ in the space H. Therefore, formally the optimal feedback control for the original problem (66) is

$$s^*(t) = G^{-1}\left(\mathcal{P}(Y^*(t))\right),$$

where $\mathcal{P}$ is the feedback map of the transformed problem defined in (60) and $Y^*(t)$ is the solution of the infinite dimensional closed loop equation

$$\begin{cases} Y'(t) = AY(t) + B\mathcal{P}(Y(t)), \\ Y(0) = \eta = \left(K_0, \int_{-r}^{\cdot} b_1(\tau)G(s_0(\tau - \cdot))d\tau\right). \end{cases} \tag{69}$$

As observed at the end of the previous section, the problem is now to study the closed loop equation (69) (existence and uniqueness), the admissibility of the associated trajectory and prove by verification that the solution found really defines an optimal control for the problem. If this is true, then the optimal control can be expressed in closed loop form by

$$s(t) = G^{-1}\left(\mathcal{P}\left(K(t), \int_{-r}^{\cdot} b_1(\xi)s(t+\tau-\cdot)d\tau\right)\right).$$

8 Explicit solution in a special case

In this section we provide the explicit solution to the HJB equation (48) in a special case. Consider the case

$$U = [0, +\infty); \qquad g(y_0) = \gamma y_0, \quad \gamma > 0; \qquad h_0(u) = \delta u^2, \quad \delta > 0.$$

We notice that this case is actually out of our assumptions, as the function g_0 above is not bounded from above as required in Hypothesis 3.3. Nevertheless, as observed in Remark 3.4-(ii), the assumption of boundedness from above of g_0 can be replaced by a suitable assumption on ρ. Indeed the right assumption in this case is that the discount factor ρ is able to kill the uncontrolled part of the growth rate of $y(\cdot)$, i.e. it amounts to require $\rho > a$.

The Legendre transform of h is

$$h^*(p) := \sup_{u\geq 0}\{pu - h_0(u)\} = \sup_{u\geq 0}\{pu - \delta u^2\} = \frac{p^2}{4\delta}.$$

where the maximum above is obtained at $u^*(p) = \frac{p}{2\delta}$. Assuming as in Section 6 that $\|b\| = 1$, the HJB equation is

$$\rho v(\eta) = \langle \eta, A^*\nabla v(\eta)\rangle + \gamma y_0 + \frac{1}{4\delta}v_b(\eta)^2, \qquad \eta \in \mathcal{F}. \tag{70}$$

We look for a solution in the affine form

$$v(y) = \langle y, \alpha\rangle + \beta, \qquad \alpha \in \mathcal{D}(A^*),\ \beta \in \mathbb{R}. \tag{71}$$

Plugging (71) into (70) we see that it must be

$$\rho\left(\langle y, \alpha\rangle + \beta\right) = \langle y, A^*\alpha\rangle + \gamma y_0 + \frac{1}{4\delta}|\langle b, \alpha\rangle|^2. \tag{72}$$

This means that we have to solve the following system

$$\begin{cases} \rho\alpha = A^*\alpha + (\gamma, 0), & \alpha \in \mathcal{D}(A^*), \\ \rho\beta = \dfrac{1}{4\delta}|\langle b, \alpha\rangle|^2, & \beta \in \mathbb{R}. \end{cases} \tag{73}$$

We recall that the operator A^* is

$$A^*\alpha = \left(a\alpha_0,\ \xi \mapsto \alpha_1'(\xi) + \alpha_0 \cdot (\xi + r) - \alpha_1'(-r)\right), \quad \alpha \in \mathcal{D}(A^*),$$

and

$$\mathcal{D}(A^*) = \left\{\alpha = (\alpha_0, \alpha_1(\cdot)) \in H \mid \alpha_1 \in W_r^{2,2}, \quad \alpha_1'(0) = 0\right\}.$$

Thus (73) can be written as

$$\rho\alpha_0 = a\alpha_0 + \gamma,\ \beta = \frac{|\langle b, \alpha\rangle|^2}{4\delta\rho}, \tag{74}$$

$$\begin{cases} \rho\alpha_1(\xi) = \alpha_1'(\xi) + \alpha_0 \cdot (\xi + r) - \alpha_1'(-r) \\ \alpha_1(-r) = 0,\ \alpha_1'(0) = 0. \end{cases} \tag{75}$$

By (74) it follows

$$\alpha_0 = \frac{\gamma}{(\rho - a)}. \tag{76}$$

To find a solution to (75), consider the equation

$$\begin{cases} \rho\alpha_1(\xi) = \alpha_1'(\xi) + \alpha_0 \cdot (\xi + r) - \sigma, \\ \\ \alpha(-r) = 0. \end{cases} \tag{77}$$

This equation has solution $\alpha_1(\xi; \sigma)$ for all $\sigma \in \mathbb{R}$ given by

$$\begin{aligned} \alpha_1(\xi; \sigma) &= -\int_{-r}^{\xi} e^{\rho(\xi - s)} (\sigma + \alpha_0 \cdot (s + r))\, ds \\ &= \frac{\sigma}{\rho}\left(1 - e^{\rho(\xi + r)}\right) + \frac{\alpha_0}{\rho}(\xi + r) + \frac{\alpha_0}{\rho^2}\left(1 - e^{\rho(\xi + r)}\right). \end{aligned} \tag{78}$$

We note that, for any given $\sigma \in \mathbb{R}$, we have

$$\alpha_1'(-r; \sigma) = \frac{\sigma}{\rho}(-\rho) + \frac{\alpha_0}{\rho} - \frac{\alpha_0}{\rho} = -\sigma,$$

so that the equation

$$\begin{cases} \rho\alpha_1(\xi) = \alpha_1'(\xi) + \alpha_0 \cdot (\xi + r) - \alpha_1'(-r) \\ \alpha_1(-r) = 0,\ \alpha_1'(0) = 0, \end{cases}$$

admits a family of solutions $\alpha_1(\cdot; \alpha_1'(-r))$. We have the freedom to choose $\alpha_1'(-r)$ to match the last condition of (75), imposing $\alpha_1'(0; \alpha_1'(-r)) = 0$. From (78) we get the equation

$$-\alpha_1'(-r)e^{\rho r} + \frac{\alpha_0}{\rho} - \frac{\alpha_0}{\rho}e^{\rho r} = 0,$$

so that

$$\alpha_1'(-r) = \frac{\alpha_0}{\rho}\left(e^{-\rho r} - 1\right). \tag{79}$$

Finally, by (78) and (79) the solution of (77) is

$$\alpha_1(\xi) = \frac{\alpha_0}{\rho^2}\left(e^{-\rho r} - e^{\rho\xi}\right) + \frac{\alpha_0}{\rho}(\xi + r). \tag{80}$$

By (76) and (80), we have that

$$\alpha = (\alpha_0, \alpha_1(\xi)) = \left(\frac{\gamma}{\rho - a}, \frac{\alpha_0}{\rho^2}\left(e^{-\rho r} - e^{\rho\xi}\right) + \frac{\alpha_0}{\rho}(\xi + r)\right), \tag{81}$$

and we can get the expression of β by (74) and (81). Taking into account α and β, due to (71) we have

$$v_b(\eta) = \langle b, \alpha\rangle.$$

By (60), the feedback map associated to this solution is the constant map

$$G(\eta) = \frac{\langle b, \alpha\rangle}{2\delta}. \tag{82}$$

The state equation associated to the constant control $\frac{\langle b, \alpha\rangle}{2\delta}$ is

$$\begin{cases} Y'(t) = AY(t) + B\dfrac{\langle b, \alpha\rangle}{2\delta}, \\ Y(0) = \eta, \end{cases} \tag{83}$$

which admits the unique mild solution

$$Y^*(t) = S_A(t)\eta + \int_0^t S_A(t-\tau)B\frac{\langle b, \alpha\rangle}{2\delta}d\tau.$$

On the first component the expression above yields

$$Y_0^*(t) = e^{at}\left(\eta_0 + \int_{(-t)\vee(-r)}^0 \eta_1(\xi)e^{a\xi}d\xi\right) + \frac{\langle b, \alpha\rangle}{2\delta}\int_0^t \left(e^{a(t-\tau)}b_0 + \int_{(-t)\vee(-r)}^0 e^{a(t+\xi-\tau)}b_1(\xi)d\xi\right)d\tau.$$

The admissibility of this trajectory is related to the sign of $\langle b, \alpha\rangle$. Actually we have

$$\begin{aligned} \langle b, \alpha\rangle &= b_0\alpha_0 + \int_{-r}^0 b_1'(\xi)\alpha_1'(\xi)d\xi = b_0\frac{\gamma}{\rho - a} + \int_{-r}^0 b_1'(\xi)\left(-\frac{\alpha_0}{\rho}e^{\rho\xi} + \frac{\alpha_0}{\rho}\right)d\xi \\ &= b_0\frac{\gamma}{\rho - a} - \frac{\alpha_0}{\rho}\left(b_1(0) - \rho\int_{-r}^0 b_1(\xi)e^{\rho\xi}d\xi\right) + \frac{\alpha_0}{\rho}b_1(0) \\ &= b_0\frac{\gamma}{\rho - a} + \frac{\gamma}{\rho - a}\int_{-r}^0 b_1(\xi)e^{\rho\xi}d\xi = \frac{\gamma}{\rho - a}b_0 + \int_{-r}^0 b_1(\xi)e^{\rho\xi}d\xi. \end{aligned} \tag{84}$$

If $b \geq 0$ (in the sense of (33)), we have $\langle b, \alpha\rangle \geq 0$. Therefore, since $\eta \in \mathcal{F}$, if $b \geq 0$ we have $Y_0^*(t) \geq 0$ for every $t \geq 0$. So, by standard arguments we can prove the following verification theorem.

Theorem 8.1. *Let $\eta \in \mathcal{F}$ and let $b \geq 0$ (in the sense of* (33)*). Then $v(\eta) = V(\eta)$ and the constant control*

$$c^*(t) = \frac{\langle b, \alpha\rangle}{2\delta}$$

is the unique optimal control starting from η.

Appendix: the semigroup S_A in the space H

Hereafter, given $f \in L^2$, with a slight abuse of notation we shall intend it extended on $[-r,+\infty)$ setting $f \equiv 0$ on $(0,+\infty)$. Consider the space $X = \mathbb{R} \times L^2$ endowed with the inner product

$$\langle \cdot,\cdot\rangle_X = \langle \cdot,\cdot\rangle_{\mathbb{R}} + \langle \cdot,\cdot\rangle_{L^2},$$

which makes it a Hilbert space. On this space we consider the unbounded linear operator

$$\bar{A}^* : \mathcal{D}(\bar{A}^*) \subset X \longrightarrow X, \quad (\eta_0, \eta_1(\cdot)) \longmapsto (a\eta_0, \eta_1'(\cdot)), \tag{85}$$

defined on the domain

$$\mathcal{D}(\bar{A}^*) = \big\{\eta = (\eta_0, \eta_1(\cdot)) \,|\, \eta_1 \in W^{1,2},\ \eta_1(0) = \eta_0\big\}.$$

It is well known (see [22]) that $\bar{A}^*$ is a closed operator which generates a C_0-semigroup $(S_{\bar{A}^*}(t))_{t\geq 0}$ on X. More precisely the explicit expression of $S_{\bar{A}^*}(t)$ acting on $\psi = (\psi_0, \psi_1(\cdot)) \in X$ is

$$S_{\bar{A}^*}(t)\psi = \Big(e^{at}\psi_0, \mathbf{1}_{[-r,0]}(t+\xi)\psi_1(t+\xi) + \mathbf{1}_{[0,+\infty)}(t+\xi)e^{a(t+\xi)}\psi_0\big|_{\xi\in[-r,0]}\Big). \tag{86}$$

On the other hand it is possible to show (see e.g. [27]) that $\bar{A}^*$ is the adjoint in X of

$$\begin{array}{rccc} \bar{A}: & \mathcal{D}(\bar{A}) \subset X & \longrightarrow & X \\ & (\eta_0, \eta_1(\cdot)) & \longmapsto & (a\eta_0 + \eta_1(0), -\eta_1'(\cdot)), \end{array} \tag{87}$$

where

$$\mathcal{D}(\bar{A}) = \mathbb{R} \times W_0^{1,2} = H.$$

It follows (see [22]) that $\bar{A}$ generates on X the C_0-semigroup $(S_{\bar{A}}(t))_{t\geq 0}$ where

$$S_{\bar{A}}(t) = S_{\bar{A}^*}(t)^*, \quad \forall t \geq 0$$

and the adjoint is taken in X. We can compute the explicit expression of the semigroup $S_{\bar{A}}(t)$ through the relation

$$\langle S_{\bar{A}}(t)\phi, \psi\rangle = \langle \phi, S_{\bar{A}^*}(t)\psi\rangle, \quad \forall\, \phi = (\phi_0, \phi_1(\cdot)) \in X,\ \forall\, \psi = (\psi_0, \psi_1(\cdot)) \in X.$$

By (86), we calculate

$$\begin{aligned} \langle \bar{S}_{\bar{A}}(t)\phi, \psi\rangle = {} & \phi_0 e^{at}\psi_0 + \int_{-r}^{(-t)\vee(-r)} \phi_1(\xi)\psi_1(t+\xi)d\xi \\ & + \int_{-(t)\vee(-r)}^0 \phi_1(\xi)\psi_0 e^{a(t+\xi)}d\xi = \phi_0 e^{at}\psi_0 + \int_{(-r+t)\wedge 0}^0 \phi_1(\xi - t)\psi_1(\xi)d\xi \\ & + \int_{(-t)\vee(-r)}^0 \phi_1(\xi)e^{a(\xi+t)}\psi_0 d\xi. \end{aligned} \tag{88}$$

So we can write the explicit form of the semigroup $\bar{S}(t)$ as

$$S_{\bar{A}}(t)\phi = \left(\phi_0 e^{at} + \int_{(-t)\vee(-r)}^0 \phi_1(\xi)e^{a(\xi+t)}d\xi,\, T(t)\phi_1\right), \quad \phi = (\phi_0, \phi_1(\cdot)) \in X, \tag{89}$$

where $(T(t))_{t\geq 0}$ is the semigroup of truncated right shifts on L^2 defined as

$$[T(t)f](\xi) = \begin{cases} f(\xi - t), & -r \leq \xi - t, \\ 0, & \text{otherwise}, \end{cases} \tag{90}$$

for $f \in L^2$. So, we may rewrite the above expression as

$$S_{\bar{A}}(t)\phi = \left(\phi_0 e^{at} + \int_{(-t)\vee(-r)}^{0} \phi_1(\xi) e^{a(\xi+t)} d\xi,\ \phi_1(\cdot - t)\mathbf{1}_{[-r,0]}(\cdot - t) \right), \quad (\phi_0, \phi_1(\cdot)) \in X. \tag{91}$$

Equation (91) yields the explicit expression of the semigroup $(\bar{S}(t))_{t\geq 0}$.

We have defined the semigroup $S_{\bar{A}}(t)$ and its infinitesimal generator $(\bar{A}, \mathcal{D}(\bar{A}))$ in the space X. Therefore, by well-known results (see [22, chapter II, pag 124]), we get that $\bar{A}|_{\mathcal{D}(\bar{A}^2)}$ is the generator of a C_0-semigroup on $(\mathcal{D}(\bar{A}), \|\cdot\|_{\mathcal{D}(\bar{A})})$, which is nothing but the restriction of $S_{\bar{A}}$ to this subspace. Now we notice that

$$\mathcal{D}(\bar{A}) = H, \quad \|\cdot\|_{\mathcal{D}(\bar{A})} \sim \|\cdot\|, \quad \mathcal{D}(\bar{A}^2) = W_0^{2,2} = \mathcal{D}(A), \quad \bar{A}|_{W_0^{2,2}} = A,$$

where A is the operator defined in (6). Hence, we conclude that A generates a C_0-semigroup on H, whose expression is the same given in (89). We denote such semigroup by S_A. We recall (see e.g. [37, Ch. 2, Prop. 4.7]) that if $S(t)$ is a C_0 semigroup on a Banach space H, then there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$, such that

$$\|S(t)\| \leq Me^{\omega t}, \qquad t \geq 0. \tag{92}$$

In this case, using Holder's inequality and taking into account that $\phi_1(-r) = 0$ we compute for every $t \geq 0$

$$\begin{aligned}
\left| \phi_0 e^{at} + \int_{(-t)\vee(-r)}^{0} \phi_1(\xi) e^{a(\xi+t)} d\xi \right|^2 &\leq 2e^{2at}|\phi_0|^2 + 2e^{2at} \left(\int_{-r}^{0} |\phi_1(\xi)| d\xi \right)^2 \\
&\leq 2e^{-at}|\phi_0|^2 + 2e^{2at} r \left(\int_{-r}^{0} |\phi_1(\xi)|^2 d\xi \right) \\
&\leq 2e^{2at}|\phi_0|^2 + 2e^{2at} r \left(\int_{-r}^{0} \left| \int_{-r}^{\xi} \phi_1'(s) ds \right|^2 d\xi \right) \\
&\leq 2e^{2at}|\phi_0|^2 + 2e^{2at} r \left(\int_{-r}^{0} (r+\xi) \left(\int_{-r}^{\xi} |\phi_1'(s)|^2 ds \right) d\xi \right) \\
&\leq 2e^{2at}|\phi_0|^2 + e^{2at} r^3 \|\phi_1\|^2_{W^{1,2}_{r,0}}.
\end{aligned}$$

Moreover

$$\|T(t)\|_{\mathcal{L}(W^{1,2}_{r,0})} \leq 1,\ \forall t \in [0, r]; \quad \|T(t)\|_{\mathcal{L}(W^{1,2}_{r,0})} = 0,\ \forall t > r.$$

The computations above show that

$$\|e^{tA}\|_{\mathcal{L}(H)} \leq (2 + r^3)^{1/2} e^{at}, \quad \forall t \geq 0. \tag{93}$$

So, setting

$$\omega = a, \qquad M = (2 + r^3)^{1/2}, \tag{94}$$

(92) is verified.

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