On the set of imputations induced by the k-additive core

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Abstract

An extension to the classical notion of core is the notion of k-additive core, that is, the set of k-additive games which dominate a given game, where a k-additive game has its Möbius transform (or Harsanyi dividends) vanishing for subsets of more than k elements. Therefore, the 1-additive core coincides with the classical core. The advantages of the k-additive core is that it is never empty once $k \geq 2$, and that it preserves the idea of coalitional rationality. However, it produces k-imputations, that is, imputations on individuals and coalitions of at most k individuals, instead of a classical imputation. Therefore one needs to derive a classical imputation from a k-order imputation by a so-called sharing rule. The paper investigates what set of imputations the k-additive core can produce from a given sharing rule.

Keywords: game theory, core, k-additive game, selectope

1 Introduction

A central problem in cooperative game theory is to define a rational way to share the total worth of a game. Specifically, let N be the set of players, and v a game in characteristic function form, assigning to each coalition $S \subseteq N$ a worth $v(S)$, which in the case of profit game, represents the benefit arising from the cooperation among members of S. Suppose that the best way to generate profit is to form the grand coalition N. An important question is how to share the total benefit $v(N)$ among the players. Any systematic way of sharing $v(N)$ is called a solution of the game.

The core [7, 19] is one of the most popular concepts of solution, and has been largely studied (see, e.g., [4, 13, 15]). It is defined as the set of preimputations which are coalitionally rational, i.e., there is no coalition S such that the value $v(S)$ that S can achieve

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by itself is strictly greater than the payoff $x(S)$ given to S. This rationality condition ensures that no coalition has interest to leave the grand coalition N.

The main drawback of the core is that it is often empty, so that other concepts of solution have to be sought for. The literature abunds on this topic, and many new solution concepts have been proposed, for example the kernel [2], the selectope [11, 3], the nucleolus [17], the Shapley value [18] and so on.

Although all these propositions have their own merits, they depart from the fundamental idea of coalitional rationality of the core[1]. To keep as much as possible this idea, Grabisch and Miranda have proposed the notion of k-additive core [10, 14]. Roughly speaking, the condition of coalitional rationality $x(S) \geq v(S)$ for all $S \subseteq N$ is preserved, but the notion of imputation/payoff is enlarged: $x(S)$ is no more the sum of payoffs to individuals in S, i.e., $x(S) = \sum_{i \in S} x_i$, but it is a sum of payoffs to individuals and possibly to coalitions of size at most k in S. Such general imputations are called k-order imputations. It is proved in [14] that as soon as $k = 2$, the k-additive core is never empty. The drawback is that eventually each player should receive an individual payoff. Therefore, once a k-order imputation has been selected, it remains in a second step to compute from it a classical imputation.

The aim of this paper is precisely to study what kind of imputation one can find through the k-additive core. We will show that this question is closely related to the selectope, and that surprisingly, any preimputation can be attained through the 2-additive core of a game. The paper is a continuation of [14], published in the present journal. In addition to its interest for game theory, we believe it can bring a significant contribution to the problem of cost or benefit sharing. On the mathematical point of view, it uses mainly polyhedral techniques, and gives interesting insights on the structure of commonly used set of solutions for games.

The paper is organized as follows. Section 2 introduces the basic material on the k-additive core and related notions. Section 3 gives some basic results about the convex polytopes of the monotonic k-additive core and the convex part of the k-additive core. Then, Sections 4 and 5 give the main results of the paper, i.e., the set of imputations induced by some classes of sharing values on the k-additive core and the monotonic k-additive core.

Throughout the paper, we will often omit braces for singletons and sets. Also we will write sets in capital italic, collections of sets in capital calligraphic, and mappings either in small italic or sans serif. For any vector $x \in \mathbb{R}^n$ and $S \subseteq \{1, \ldots, n\}$, we use the shorthand $x(S) := \sum_{i \in S} x_i$.

2 Notations and definitions

A *game* is a pair (N, v) where $N := \{1, \ldots, n\}$ is the set of players, and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. If there is no fear of ambiguity, we will call a game simply v.

We denote by $\mathcal{P}(N) = 2^N$ the set of all subsets of N, while the set of all subsets of N

[1] As remarked by one the referees, it is true that the nucleolus is nevertheless close to the idea of coalitional rationality, through the notion of excess. However, we may say that the nucleolus aims at minimizing the excesses, without paying attention if these excesses are positive (in which case coalitional rationality is violated) or negative.

of cardinality smaller or equal to k is denoted by $\mathcal{P}^k(N)$.

A game is *monotone* if $S \subseteq T \subseteq N$ implies $v(S) \leq v(T)$. It is *additive* if $v(S \cup T) = v(S) + v(T)$ for every pair of disjoint coalitions S, T.

For any game v, its *Möbius transform* [16] (or Harsanyi dividend [12]) is a set function $m : 2^N \to \mathbb{R}$ defined by

$$m(S) := \sum_{T \subseteq S} (-1)^{|S \setminus T|} v(T), \quad \forall S \subseteq N.$$

If m is given, it is possible to recover v by $v(S) = \sum_{T \subseteq S} m(T)$. A game v is *k-additive* [8] if its Möbius transform vanishes for sets of more than k players: $m(S) = 0$ if $|S| > k$, and there exists at least one $S \subseteq N$ of k players such that $m(S) \neq 0$. Note that a 1-additive game is an additive game.

We introduce various sets:

(i) The set of all games with player set N: $\mathcal{G}(N) = \mathbb{R}^{2^n - 1}$;

(ii) The set of all monotonic games with player set N: $\mathcal{MG}(N)$;

(iii) The set of all at most k-additive games $\mathcal{G}^k(N) = \mathbb{R}^{\eta(k)}$, and at most k-additive monotonic games $\mathcal{MG}^k(N)$, with $\eta(k) := \sum_{\ell=1}^{k} \binom{n}{\ell}$. Note that $\mathcal{G}^1(N)$ denotes the set of additive games.

(iv) The set of *selectors* on N:

$$\mathcal{A}(N) := \{\alpha : 2^N \setminus \{\emptyset\} \to N, \quad S \mapsto \alpha(S) \in S\};$$

(v) The set of *sharing functions* on N:

$$\mathcal{Q}(N) = \Big\{ q : 2^N \setminus \{\emptyset\} \times N \to [0,1] \mid q(K,i) = 0 \text{ if } i \notin K, \ \sum_{i \in K} q(K,i) = 1, \quad \emptyset \neq K \subseteq N \Big\}.$$

If q is such that for all K there exists $i \in K$ such that $q(K,i) = 1$, then q is a selector. Conversely, any selector can be viewed as a sharing function. Moreover, $\mathcal{Q}(N)$ is a convex polyhedron whose vertices are the selectors.

Next, we introduce various mappings defined on $\mathcal{G}(N)$ (or any subset like $\mathcal{MG}(N)$, $\mathcal{G}^k(N)$, etc.). Most of the following notions are usually not considered as mappings, but it is very convenient here to do so.

(i) The *preimputation set* $\mathsf{PI} : \mathcal{G}(N) \to 2^{\mathbb{R}^n}$

$$\mathsf{PI}(v) := \{x \in \mathbb{R}^N \mid x(N) = v(N)\};$$

(ii) The *imputation set* $\mathsf{I} : \mathcal{G}(N) \to 2^{\mathbb{R}^n}$

$$\mathsf{I}(v) := \{x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N\};$$

(iii) The *core* $\mathsf{C} : \mathcal{G}(N) \to 2^{\mathbb{R}^n}$ [19]

$$\begin{aligned}\mathsf{C}(v) &:= \{x \in \mathbb{R}^n \mid x(S) \geq v(S), \ \ \forall S \subset N, \text{ and } x(N) = v(N)\} \\ &= \{\phi \in \mathcal{G}^1(N) \mid \phi(S) \geq v(S), \ \ \forall S \subset N, \text{ and } \phi(N) = v(N)\};\end{aligned}$$

(iv) The *monotonic core* $\mathsf{MC} : \mathcal{G}(N) \to 2^{\mathbb{R}^n}$

$$\mathsf{MC}(v) := \{\phi \in \mathcal{MG}^1(N) \mid \phi(S) \geq v(S), \ \ \forall S \subset N, \text{ and } \phi(N) = v(N)\};$$

(v) The *positive core* $\mathsf{C}_+ : \mathcal{G}(N) \to 2^{\mathbb{R}^n_+}$ [5]

$$\mathsf{C}_+(v) := \{x \in \mathbb{R}^n_+ \mid x(S) \geq v(S), \ \ \forall S \subset N, \text{ and } x(N) = v(N)\};$$

(vi) The *k-additive core* $\mathsf{C}^k : \mathcal{G}(N) \to 2^{\mathcal{G}^k(N)}$ [14]

$$\mathsf{C}^k(v) := \{\phi \in \mathcal{G}^k(N) \mid \phi(S) \geq v(S), \ \ \forall S \subset N, \text{ and } \phi(N) = v(N)\},$$

and similarly the k-additive monotonic core MC^k and the k-additive positive core C^k_+;

(vii) The *selector value* $x^\alpha : \mathcal{G}(N) \to \mathbb{R}^n$ for any selector $\alpha \in \mathcal{A}(N)$ [3]

$$x^\alpha_i(v) := \sum_{S \mid \alpha(S) = i} m^v(S), \quad i \in N$$

where m^v is the Möbius transform of v;

(viii) The *sharing value* $x^q : \mathcal{G}(N) \to \mathbb{R}^n$ for any sharing function $q \in \mathcal{Q}(N)$ [3, 1]:

$$x^q_i(v) := \sum_{S \ni i} q(S, i) m^v(S), \quad i \in N;$$

(ix) The *selectope* $\mathsf{S} : \mathcal{G}(N) \to 2^{\mathbb{R}^n}$ [11]

$$\mathsf{S}(v) := \text{conv}\{x^\alpha(v) \mid \alpha \in \mathcal{A}(N)\} = \{x^q(v) \mid q \in \mathcal{Q}(N)\}.$$

We may write [3]

$$\mathsf{S} = \bigcup_{q \in \mathcal{Q}(N)} x^q.$$

(x) The *marginal value* $p^\sigma : \mathcal{G}(N) \to \mathbb{R}^n$, with $\sigma \in \mathfrak{S}(N)$, the set of permutations on N

$$p^\sigma_{\sigma(i)}(v) := v(S_i) - v(S_{i-1}), \quad i \in N,$$

where $S_i := \{\sigma(1), \ldots, \sigma(i)\}$. Each marginal value is a selector value (and hence a sharing value): the selector corresponding to p^σ is α which selects in S the player of maximal rank[2] [3].

[2]We adopt the convention: i is the rank, and $\sigma(i)$ the player of rank i.

(xi) The *Weber set* $\mathsf{W} : \mathcal{G}(N) \to 2^{\mathbb{R}^n}$ [21]

$$\mathsf{W}(v) = \mathrm{conv}\{p^\sigma(v) \mid \sigma \in \mathfrak{S}(N)\}.$$

From the above remark, we have $\mathsf{W}(v) \subseteq \mathsf{S}(v)$ for any game v.

(xii) The Shapley value $\mathsf{sh} : \mathcal{G}(N) \to \mathbb{R}^n$ [18]. Since $\mathsf{sh}(v) \in \mathsf{S}(v)$, we write $\mathsf{sh} \in \mathsf{S}$ (particular sharing value).

We make several noteworthy remarks on the k-additive core.

- The k-additive core of v is the set of *at most* k-additive games dominating v. Therefore, it contains the core of v when the latter is nonempty.
- An element ϕ of the k-additive core induces by its Möbius transform m^ϕ a preimputation on all coalitions of at most k players, since by definition $m^\phi(S) = 0$ for all $S \subseteq N$ such that $|S| > k$, and $\sum_{S \in \mathcal{P}^k(N)} m^\phi(S) = v(N)$. We call such a (generalized) imputation a *k-order preimputation*. Note that in general, m^ϕ need not be positive everywhere.
- The idea of coalitional rationality is kept in the following sense: given an element ϕ of the k-additive core, for any coalition S, the sum of all k-imputations received by S (that is, all quantities $m^\phi(T)$ for any subcoalition T of S of at most k members) is equal or exceeds $v(S)$. So the members of S, on the level of k-imputations, have no incentive to leave the game.

It remains to derive from a given k-order preimputation m^ϕ a classical preimputation x, by sharing for every coalition $S \in \mathcal{P}^k(N)$ the amount $m^\phi(S)$ among players in S, i.e., by using a sharing function $q \in \mathcal{Q}$ applied on m^ϕ. In other words, any preimputation obtained from m^ϕ is a sharing value $x^q(\phi)$ for some $q \in \mathcal{Q}$, and vice-versa. It follows that the set of preimputations derived from an element ϕ of the k-additive core is the selectope $\mathsf{S}(\phi)$. In short, the set of preimputations which can be derived from the k-additive core is $\mathsf{S}(\mathsf{C}^k(v))$.

3 Basic results and facts on the k-additive core

The k-additive core is a polyhedron of dimension $\sum_{i=1}^{k} \binom{n}{i} - 1$, possibly unbounded. A study of its vertices has been done in [10], with results similar to the Shapley-Ichiishi result for convex games. By contrast, the monotonic k-additive core is always bounded, but has many more vertices than the k-additive core, and it seems quite difficult to study them.

A noticeable fact shown in [14] is that $\mathsf{C}^k(v) \neq \emptyset$ for any game in $\mathcal{G}(N)$, as soon as $k \geq 2$. However, this property does not hold for the monotonic k-additive core. Exact conditions for nonemptiness are given in [14]; we call *k-balanced-monotone* a game v such that $\mathsf{MC}^k(v) \neq \emptyset$.

We prove some elementary facts concerning the polytope $\mathsf{MC}^k(v)$ and the convex part of the k-additive core.

Proposition 1. For every k-balanced-monotone game v on N, any $2 \leq k \leq n$, any sharing value x^q we have:

$$\text{ext}(x^q(\mathsf{MC}^k(v))) \subseteq x^q(\text{ext}(\mathsf{MC}^k(v)))$$
$$x^q(\mathsf{MC}^k(v)) = \text{conv}(x^q(\text{ext}(\mathsf{MC}^k(v)))).$$

The result holds also if MC^k is replaced by $\text{conv}(\text{ext}(\mathsf{C}^k))$.

Proof. It is well known from the theory of polyhedra that if P is a polytope in $\mathbb{R}^m$ and ρ is a linear mapping from $\mathbb{R}^m$ to $\mathbb{R}^p$, $m \geq p$, then $\rho(P)$ is a polytope. Moreover, a vertex y of $\rho(P)$ is necessarily the image from a vertex of P (indeed, suppose that no x such that $\rho(x) = y$ is a vertex. Then it exists $x_1, x_2 \in P$, $\alpha \in]0,1[$ such that $x = \alpha x_1 + (1-\alpha)x_2$. By linearity, we get $y = \rho(x) = \alpha\rho(x_1) + (1-\alpha)\rho(x_2)$, contradicting the fact that y is a vertex of $\rho(P)$), but the converse does not hold in general.

Since $\mathsf{MC}^k(v)$ is a polytope and x^q is linear, the first relation holds by the above fact. Now, since $\text{conv}(\text{ext}(P)) = P$, we get

$$x^q(\mathsf{MC}^k(v)) = \text{conv}(\text{ext}(x^q(\mathsf{MC}^k(v)))) = \text{conv}(x^q(\text{ext}(\mathsf{MC}^k(v)))),$$

the second equality coming from the inclusion relation. □

By Proposition 1 and the fact that $\mathsf{S} = \bigcup_{q \in \mathcal{Q}(N)} x^q$, we get:

Corollary 1. For every k-balanced-monotone game v on N and any $2 \leq k \leq n$, we have:

$$\mathsf{S}(\mathsf{MC}^k(v)) \subseteq \text{conv}(\mathsf{S}(\text{ext}(\mathsf{MC}^k(v)))).$$

The result holds also if MC^k is replaced by $\text{conv}(\text{ext}(\mathsf{C}^k))$.

Equality holds if $\mathsf{S}(\mathsf{MC}^k(v))$ is a convex set, which does not seem to be true in general.

Proposition 2. Suppose that $\mathsf{C}(v) \neq \emptyset$. Then for any $2 \leq k \leq n$ and any sharing value x^q

$$x^q(\mathsf{C}^k(v)) \supseteq \mathsf{C}(v).$$

The same result holds with the monotonic core.

Proof. Clear from the fact that $\mathsf{C}^k(v) \supseteq \mathsf{C}(v)$, and that $\phi \in \mathsf{C}(v)$ (considered as an element of $\mathcal{G}^1(N)$) implies $x^q(\phi) = \phi$. □

Remark 1. (i) We have $\mathsf{MC} = \mathsf{C}_+$. Indeed, an element of the positive core is monotone. Conversely, take x in the monotonic core. It implies that for any $S \subset N$ and $i \notin S$, we have $x(S \cup i) - x(S) = x(i) \geq 0$. Obviously, this is no more true for the k-additive core.

(ii) It is easy to find that $v(\{i\}) \geq 0$ for all $i \in N$ is a sufficient condition for ensuring $\mathsf{C}_+ = \mathsf{C}$ (and therefore $\mathsf{MC} = \mathsf{C}$). But a similar condition for the k-additive case seems to be hard to find. Clearly, if v^* dominates v, the monotonicity of v does not imply the monotonicity of v^* in general.

4 Preimputations induced by the k-additive core

We begin by recalling the following result on systems of inequalities.

Lemma 1. Consider the system of n linear inequalities

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, \quad (i \in I)$$
$$\sum_{j=1}^{n} a'_{ij}x_j = b'_i, \quad (i \in E),$$

where I, E are index sets forming a partition of $\{1, \ldots, n\}$. Consider a given $j_0 \in \{1, \ldots, n\}$ such that $a'_{ij_0} = 0$ for all $i \in E$, and define $I_0 := \{i \in I \mid a_{ij_0} = 0\}$ (possibly empty). If all a_{ij_0}, $i \in I \setminus I_0$, have the same sign, then the above system is equivalent to

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, \quad (i \in I_0)$$
$$\sum_{j=1}^{n} a'_{ij}x_j = b'_i, \quad (i \in E).$$

Proof. This result may be deduced from the Fourier-Motzkin elimination. Otherwise, simply remark the following: suppose w.l.o.g. that $a_{ij_0} > 0$ for all $i \in I \setminus I_0$. Then, any inequality $\sum_{j=1}^{n} a_{ij}x_j \leq b_i, \quad (i \in I \setminus I_0)$ will be satisfied for any $x_1, \ldots, x_{j_0-1}, x_{j_0+1}, \ldots, x_n$, for a sufficiently negatively large value of x_{j_0}. □

This observation is the key for the next theorem.

Theorem 1. For any x^q, $q \in \mathcal{Q}(N)$ such that $q(K, i) > 0$ for all $K \subseteq N$ and $i \in K$, for any $v \in \mathcal{G}(N)$, for any $2 \leq k \leq n$, we have

$$x^q(\mathsf{C}^k(v)) = \mathsf{PI}(v).$$

Therefore, $\mathsf{S} \circ \mathsf{C}^k = \mathsf{PI}$.

Proof. Take any $v \in \mathcal{G}(N)$, and any $k \geq 2$. Since $\mathsf{C}^k(v) \neq \emptyset$, for any $v^* \in \mathsf{C}^k(v)$ with Möbius transform m^*, we have:

$$m^*(i) \geq v(i), \quad i \in N$$
$$\sum_{\substack{K \subseteq S \\ |K| \leq k}} m^*(K) \geq v(S), \quad S \subset N, |S| > 1$$
$$\sum_{\substack{K \subseteq N \\ |K| \leq k}} m^*(K) = v(N).$$

Take any x^q with sharing system $q \in \mathcal{Q}(N)$ and write for simplicity $x := x^q(v^*)$. We have by definition

$$x_i = m^*(i) + \sum_{\substack{K \ni i \\ 2 \leq |K| \leq k}} q(K, i)m^*(K).$$

Then

$$m^*(i) = x_i - \sum_{\substack{K \ni i \\ 2 \le |K| \le k}} q(K,i) m^*(K).$$

Replacing in the above system, we get:

$$\begin{aligned}
x_i - \sum_{\substack{K \ni i \\ 2 \le |K| \le k}} q(K,i) m^*(K) &\ge v(i), \quad i \in N \\
\sum_{i \in S} x_i - \sum_{i \in S} \sum_{\substack{K \ni i \\ 2 \le |K| \le k}} q(K,i) m^*(K) + \sum_{\substack{K \subseteq S \\ 2 \le |K| \le k}} m^*(K) &\ge v(S), \quad S \subset N, |S| > 1 \\
\sum_{i \in N} x_i - \sum_{i \in N} \sum_{\substack{K \ni i \\ 2 \le |K| \le k}} q(K,i) m^*(K) + \sum_{\substack{K \subseteq N \\ 2 \le |K| \le k}} m^*(K) &= v(N).
\end{aligned}$$

The second line becomes

$$\sum_{i \in S} x_i - \sum_{\substack{K \subseteq S \\ 2 \le |K| \le k}} m^*(K) \underbrace{\sum_{i \in K} q(K,i)}_{=1} - \sum_{\substack{K \cap S \neq \emptyset \\ K \not\subseteq S \\ 2 \le |K| \le k}} m^*(K) \sum_{i \in K \cap S} q(K,i) + \sum_{\substack{K \subseteq S \\ 2 \le |K| \le k}} m^*(K) \ge v(S)$$

or

$$\sum_{i \in S} x_i - \sum_{\substack{K \cap S \neq \emptyset \\ K \not\subseteq S \\ 2 \le |K| \le k}} m^*(K) \sum_{i \in K \cap S} q(K,i) \ge v(S).$$

Therefore the system becomes:

$$\begin{aligned}
x_i - \sum_{\substack{K \ni i \\ 2 \le |K| \le k}} q(K,i) m^*(K) &\ge v(i), \quad i \in N \\
\sum_{i \in S} x_i - \sum_{\substack{K \cap S \neq \emptyset \\ K \not\subseteq S \\ 2 \le |K| \le k}} m^*(K) \sum_{i \in K \cap S} q(K,i) &\ge v(S), \quad S \subset N, |S| > 1 \\
\sum_{i \in N} x_i &= v(N).
\end{aligned}$$

Now, by positivity of q, applying Lemma 1 successively to all $m^*(K)$, it remains only

$$\sum_{i \in N} x_i = v(N).$$

□

Remark 2. $x^q = \mathsf{sh}$ fulfills the condition, hence $\mathsf{sh}(\mathsf{C}^k(v)) = \mathsf{PI}(v)$. It can be interpreted by saying that any preimputation can be seen as the Shapley value of some element (in general not unique) of the 2-additive core.

We turn now to the case of selector values.

Theorem 2. Let $\alpha \in \mathcal{A}(N)$ be a selector and x^α the corresponding selector value. Then, for any $2 \leq k \leq n$ we have

$$x^\alpha(\mathsf{C}^k(v)) = \{x \in \mathsf{PI}(v) \mid x(S) \geq v(S), \quad \forall S \subset N, S \notin \mathcal{C}(\alpha)\}$$

where $\mathcal{C}(\alpha) := \{S \subset N \mid \exists K \subseteq N, 2 \leq |K| \leq n, K \cap S \neq \emptyset, K \not\subseteq S, \alpha(K) \in S\}$.

Proof. From proof of Theorem 1, we know that, for any $v^* \in \mathsf{C}^k(v)$ with Möbius transform m^*, the system of inequalities is:

$$\begin{aligned}
x_i - \sum_{\substack{K \ni i \\ 2 \leq |K| \leq k}} q(K,i) m^*(K) &\geq v(i), \quad i \in N \\
\sum_{i \in S} x_i - \sum_{\substack{K \cap S \neq \emptyset \\ K \not\subseteq S \\ 2 \leq |K| \leq k}} m^*(K) \sum_{i \in K \cap S} q(K,i) &\geq v(S), \quad S \subset N, |S| > 1 \\
\sum_{i \in N} x_i &= v(N),
\end{aligned}$$

with q corresponding to x^α, that is, $q(K,i) = 1$ if and only if $\alpha(K) = i$, and 0 otherwise. Therefore, the system becomes:

$$\begin{aligned}
x_i - \sum_{\substack{K \mid \alpha(K) = i \\ 2 \leq |K| \leq k}} m^*(K) &\geq v(i), \quad i \in N \text{ s.t. } \alpha(K) = i \text{ for some } K, 2 \leq |K| \leq k \\
x_i &\geq v(i), \quad i \in N \text{ otherwise} \\
\sum_{i \in S} x_i - \sum_{\substack{K \cap S \neq \emptyset \\ K \not\subseteq S \\ \alpha(K) \in S \\ 2 \leq |K| \leq k}} m^*(K) &\geq v(S), \quad S \subset N, |S| > 1 \text{ s.t. } S \in \mathcal{C}(\alpha) \\
\sum_{i \in S} x_i &\geq v(S), \quad S \subset N, |S| > 1 \text{ s.t. } S \notin \mathcal{C}(\alpha) \\
\sum_{i \in N} x_i &= v(N).
\end{aligned}$$

Applying Lemma 1 successively to all $m^*(K)$, we get the result. □

Therefore, $x^\alpha(\mathsf{C}^k(v))$ is a superset of $\mathsf{C}(v)$, whose structure depends on which coalitions appear in the inequalities. We can be more specific by taking marginal values, which are particular selector values.

Theorem 3. For any permutation $\sigma \in \mathfrak{S}$, for any $v \in \mathcal{G}(N)$, for any $2 \leq k \leq n$, we have

$$p^\sigma(\mathsf{C}^k(v)) = \Big\{x \in \mathsf{PI}(v) \mid \sum_{j=1}^{i} x_{\sigma(j)} \geq v(\{\sigma(1), \ldots, \sigma(i)\}), \quad i = 1, \ldots, n-1\Big\}. \quad (1)$$

Proof. This time q corresponds to p^σ, that is:

$$q(K,i) = 1 \text{ if and only if } \ell_\sigma(K) = i$$

and $q(K,i) = 0$ otherwise, where $\ell_\sigma(K)$ is the last element of K in the order σ. Therefore, proceeding as for Theorem 2, we get:

$$\begin{aligned}
x_{\sigma(1)} &\geq v(\{\sigma(1)\}) \\
x_i - \sum_{\substack{K|\ell_\sigma(K)=i \\ 2\leq|K|\leq k}} m^*(K) &\geq v(i), \quad i = 2,\dots,n \\
x_{\sigma(1)} + \cdots + x_{\sigma(i)} &\geq v(\{\sigma(1),\dots,\sigma(i)\}), \quad i = 2,\dots,n-1 \\
\sum_{i\in S} x_i - \sum_{\substack{K\cap S\neq\emptyset \\ K\not\subseteq S \\ \ell_\sigma(K)\in S \\ 2\leq|K|\leq k}} m^*(K) &\geq v(S), \quad S \subset N, S \neq \{\sigma(1),\dots,\sigma(i)\} \text{ for some } i \in N \\
\sum_{i\in N} x_i &= v(N).
\end{aligned}$$

(note that in the 4th inequality, the set of K such that $K \cap S \neq \emptyset, K \not\subseteq S, \ell_\sigma(K) \in S, 2 \leq |K| \leq k$ is never empty due to the assumption that S is not one of the sets in the chain induced by σ) Applying Lemma 1 successively to all $m^*(K)$, we get the result. □

Let us study the structure of $p^\sigma(\mathsf{C}^k(v))$. We denote by C_σ the maximal chain associated to σ, that is, $C_\sigma = \{S_1,\dots,S_n\}$, with $S_i := \{\sigma(1),\dots,\sigma(i)\}$. We introduce $v_{|C_\sigma}$ the restriction of v to the maximal chain C_σ (game with restricted cooperation). Therefore, we have

$$p^\sigma(\mathsf{C}^k(v)) = \mathsf{C}(v_{|C_\sigma}).$$

The core of games with restricted cooperation has been largely studied (see a survey in [9]). We can derive directly from known results the structure of $p^\sigma(\mathsf{C}^k(v))$. A first observation is that this set is nonempty, for it contains the marginal value $p^\sigma(v)$, obviously solution of the set of inequalities (1). Moreover, $p^\sigma(v)$ is the unique vertex of $p^\sigma(\mathsf{C}^k(v))$. Indeed, there are $n-1$ inequalities and one equality in (1), hence from it we can derive only one set of n equalities, which precisely defines $p^\sigma(v)$.

It remains to find the extremal rays of $p^\sigma(\mathsf{C}^k(v))$, which can be obtained by known results about the core of games with restricted cooperation. The collection C_σ being a chain, it is a distributive lattice of height n, whose join-irreducible elements are $\{\sigma(1)\}$, $\{\sigma(1),\sigma(2)\},\dots,\{\sigma(1),\dots,\sigma(n)\}$. We cite the following result due to Tomizawa.

Proposition 3. (Tomizawa [20], also cited in Fujishige [6, Th. 3.26]) Let $\mathcal{F} \subseteq 2^N$ be a distributive lattice of height n, with joint irreducible elements $J_1,\dots,J_n$. The extremal rays of $\mathsf{C}(0)$, the recession cone of $\mathsf{C}(v)$, are of the form $(1_j,-1_i)$, with $i \in N$ such that the smallest join-irreducible element J containing i satisfies $|J| > 1$, and $j \in J$, such that the smallest join-irreducible element J' containing j is the predecessor of J in the poset of joint-irreducible elements.

Applying this result to our case, we find the following.

Proposition 4. For any permutation $\sigma \in \mathfrak{S}$, for any $v \in \mathcal{G}(N)$, for any $2 \leq k \leq n$, $p^\sigma(\mathsf{C}^k(v))$ is a pointed unbounded polyhedron, with unique vertex $p^\sigma(v)$, and extreme rays given by $\mathbf{1}_{\sigma(i-1)} - \mathbf{1}_{\sigma(i)}$, $i = 2, \ldots, n$.

As a consequence of Theorem 2, Theorem 3 and Proposition 2, we have immediately

Theorem 4. For any $2 \leq k \leq n$, for any $v \in \mathcal{G}(N)$, we have

$$\bigcap_{\sigma \in \mathfrak{S}} p^\sigma(\mathsf{C}^k(v)) = \bigcap_{\alpha \in \mathcal{A}(N)} x^\alpha(\mathsf{C}^k(v)) = \bigcap_{x \in \mathsf{S}} x(\mathsf{C}^k(v)) = \mathsf{C}(v).$$

5 Preimputations induced by the monotonic k-additive core

The case of the monotonic core is much more tricky to study, because monotonicity induces supplementary inequalities which make Lemma 1 inapplicable. Nevertheless, a result can be derived for the case of selector values.

A first simple but important observation is the following.

Lemma 2. If P_1, P_2 are two polyhedra in $\mathbb{R}^m$, and ρ is a linear mapping from $\mathbb{R}^m$ to $\mathbb{R}^p$, then $\rho(P_1 \cap P_2) = \rho(P_1) \cap \rho(P_2)$.

Proof. Let P_1, P_2 be defined by sets of k and ℓ linear inequalities respectively, in variables $x_1, \ldots, x_m$. Then $P_1 \cap P_2$ is defined by the union of these two sets of inequalities. Now, $\rho(P_1), \rho(P_2)$ are polyhedra defined by sets of k and ℓ linear inequalities in variables $y_1, \ldots, y_p$, and the union of these two systems defines $\rho(P_1) \cap \rho(P_2)$. But this system is also the transform by ρ of the system defining $P_1 \cap P_2$, hence it represents $\rho(P_1 \cap P_2)$ as well. □

We apply this result with $P_1 = \mathsf{C}^k(v)$ and P_2 the polyhedron of monotone k-additive games. Clearly $\mathsf{MC}^k(v) = P_1 \cap P_2$, and it suffices to study the transform of P_2 by sharing values. Polyhedron P_2 reads, in the space of the Möbius transform (see [1]):

$$P_2 = \big\{ m \in \mathbb{R}^{\eta(k)} \mid \sum_{\substack{K \ni i, K \subseteq S \\ 1 \leq |K| \leq k}} m(K) \geq 0, \quad S \subseteq N, S \neq \emptyset, i \in S \big\},$$

with $\eta(k) := \sum_{\ell=1}^{k} \binom{n}{\ell}$.

Theorem 5. Let $\alpha \in \mathcal{A}(N)$ be a selector satisfying the following property: if K, K' are such that $2 \leq |K|, |K'| \leq k$ and $q(K, i) = q(K', i) = 1$ for some $i \in N$, then any $K'' \subseteq K \cup K'$ such that $2 \leq |K''| \leq k$ satisfies $q(K'', i) = 1$. Let x^α be the corresponding selector value. Then, for any $2 \leq k \leq n$, for any k-balanced-monotone game v we have

$$x^\alpha(\mathsf{MC}^k(v)) = \{x \in \mathsf{PI}(v) \cap \mathbb{R}^n_+ \mid x(S) \geq v(S), \quad \forall S \subset N, S \notin \mathcal{C}(\alpha)\}$$

where $\mathcal{C}(\alpha) := \{S \subset N \mid \exists K \subseteq N, 2 \leq |K| \leq n, K \cap S \neq \emptyset, K \not\subseteq S, \alpha(K) \in S\}$.

Proof. From Lemma 2, it suffices to compute $p^\sigma(P_2)$. For ease, we express $x^q(P_2)$ for any sharing value $q \in \mathcal{Q}(N)$. For any game v in P_2 with Möbius transform m we have

$$m(i) \geq 0, \quad i \in N$$
$$\sum_{\substack{K \ni i, K \subseteq S \\ 1 \leq |K| \leq k}} m(K) \geq 0, \quad S \subseteq N, |S| > 1, i \in S.$$

Writing for simplicity $x := x^q(v)$, we have by definition

$$x_i = m(i) + \sum_{\substack{K \ni i \\ 2 \leq |K| \leq k}} q(K,i) m(K),$$

so that

$$m(i) = x_i - \sum_{\substack{K \ni i \\ 2 \leq |K| \leq k}} q(K,i) m(K),$$

for all $i \in N$. Replacing in the above system, we get:

$$x_i - \sum_{\substack{K \ni i \\ 2 \leq |K| \leq k}} q(K,i) m(K) \geq 0, \quad i \in N \tag{2}$$

$$x_i + \sum_{\substack{K \ni i \\ K \subseteq S \\ 2 \leq |K| \leq k}} (1 - q(K,i)) m(K) - \sum_{\substack{K \ni i \\ K \not\subseteq S \\ 2 \leq |K| \leq k}} q(K,i) m(K) \geq 0, \quad S \subseteq N, |S| > 1, i \in S. \tag{3}$$

Let us consider the selector value x^α for some $\alpha \in \mathcal{A}(N)$. Then $q(K,i) = 1$ or 0, for every $K \subseteq N$, $2 \leq |K| \leq k$, and every $i \in N$. Consider i to be fixed, and denote by $\mathcal{K}$ the collection of sets K, $2 \leq |K| \leq k$, such that $q(K,i) = 1$. If $\mathcal{K}$ is empty, then (2) reduces to

$$x_i \geq 0.$$

Suppose then that $\mathcal{K}$ is not the empty collection, and consider $S = \bigcup \mathcal{K}$, i.e., the union of all sets in $\mathcal{K}$. Observe that $|S| > 1$ and $S \ni i$. Equation (3) for this S gives:

$$x_i \geq 0,$$

since by hypothesis any $K \subseteq S$ such that $K \ni i$ and $2 \leq |K| \leq k$ is a member of $\mathcal{K}$, and any $K \not\subseteq S$ satisfies $q(K,i) = 0$ by definition of $\mathcal{K}$. Hence in any case, we get the inequality $x_i \geq 0$. This reasoning can be done for any $i \in N$, therefore $x_1, \ldots, x_n \geq 0$.

The remaining inequalities have the form

$$x_i + \sum_{\substack{K \ni i \\ 2 \leq |K| \leq k}} \epsilon m(K) \geq 0,$$

with $\epsilon = 0$, 1 or -1. Let us eliminate all variables $m(K)$ by Fourier-Motzkin elimination from this system of inequalities. Observe that elimination amounts to add pairs of inequalities, possibly multiplied by some positive constants. Therefore, as a result of elimination, it will remain only inequalities of the form

$$a_1 x_1 + \ldots + a_n x_n \geq 0,$$

with $a_1, \ldots, a_n \geq 0$, so that they are all redundant with $x_1, \ldots, x_n \geq 0$. □

Observe that any marginal value p^σ is a sharing value having the property requested in Theorem 5. Therefore, we have the following result.

Corollary 2. For any permutation $\sigma \in \mathfrak{S}$, for any $v \in \mathcal{G}(N)$, for any $2 \leq k \leq n$, for any k-balanced-monotone game v, we have

$$p^\sigma(\mathsf{MC}^k(v)) = \Big\{ x \in \mathsf{PI}(v) \cap \mathbb{R}^n_+ \mid \sum_{j=1}^{i} x_{\sigma(j)} \geq v(\{\sigma(1), \ldots, \sigma(i)\}), \quad i = 1, \ldots, n-1, \Big\}. \quad (4)$$

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