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TILINGS OF THE PLANE AND THURSTON SEMI-NORM

JEAN-RENÉ CHAZOTTES, JEAN-MARC GAMBAUDO, AND FRANÇOIS GAUTERO

Abstract. We show that the problem of tiling the Euclidean plane with a finite set of polygons (up to translation) boils down to prove the existence of zeros of a non-negative convex function defined on a finite-dimensional simplex. This function is a generalisation, in the framework of branched surfaces, of the Thurston semi-norm originally defined for compact 3-manifolds.

1. Introduction

Consider a finite collection $\mathscr{P} = \{p_1, \dots, p_n\}$, where for $j = 1, \dots, n$, p_j is a polygon in the Euclidean plane $\mathbb{R}^2$, indexed by j and with colored edges. These decorated polygons are called *prototiles*. In the sequel a *rational prototile* (resp. *integral prototile*) is a prototile whose vertices have rational (resp. integer) coordinates. A tiling of $\mathbb{R}^2$ made with $\mathscr{P}$ is a collection $(t_i)_{i\geqslant 0}$ of polygons called *tiles* indexed by a symbol $k(i)$ in $\{1, \dots, n\}$, such that:

- the tiles cover the plane: $\cup_{i\geqslant 0} t_i = \mathbb{R}^2$;
- the tiles have disjoint interiors: $\mathrm{int}(t_i) \cap \mathrm{int}(t_j) = \varnothing$ whenever $i \neq j$;
- whenever two distinct tiles intersect, they do it along a common edge and the colors match;
- for each $i \geqslant 0$, t_i is a translated copy of $p_{k(i)}$.

We denote by $\Omega_{\mathscr{P}}$ the set of all tilings made with $\mathscr{P}$. Clearly this set may be empty. When $\Omega_{\mathscr{P}} \neq \varnothing$ the group of translation acts on $\Omega_{\mathscr{P}}$ as follows:

$$\Omega_{\mathscr{P}} \times \mathbb{R}^2 \ni (T, u) \mapsto T + u \in \Omega_{\mathscr{P}}$$

where $T + u = (t_i + u)_{i\geqslant 0}$ whenever $T = (t_i)_{i\geqslant 0}$. A tiling $T \in \Omega_{\mathscr{P}}$ is *periodic* if there exist two independent vectors u_1 and u_2 in $\mathbb{R}^2$ such that $T = T + u_1 = T + u_2$. R. Berger [4] proved that the problem to know whether or not $\Omega_{\mathscr{P}}$ is empty (*i.e.* if one can or cannot tile the plane with $\mathscr{P}$) is not decidable. More precisely he showed that there is no algorithm that takes as input any family of prototiles $\mathscr{P}$ and gives as output in a finite time one of the following two results: $\mathscr{P}$ can tile the plane or $\mathscr{P}$ cannot tile the plane. Berger also showed that this undecidability is strongly related to the fact that there exist collections of prototiles $\mathscr{P}$ which tile the plane ($\Omega_{\mathscr{P}} \neq \varnothing$) but which cannot tile it periodically. Before we state our main result,

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we need some notations and definitions.

The Anderson-Putnam CW-complex associated with a collection $\mathscr{P}$ of prototiles is the cell complex $AP_{\mathscr{P}}$ (see [1]) [1] made with 2-cells, 1-cells and 0-cells constructed as follows. There is one 2-cell for each prototile and these 2-cells are glued along their colored edges by translation [1]. An edge e_{k_0} of p_{k_0} is glued to an edge e_{k_1} of p_{k_1} if and only if:

- they have the same color;
- there exists a vector v_{k_0,k_1} in $\mathbb{R}^2$ such that $e_{k_1} = e_{k_0} + v_{k_0,k_1}$.

For $i = 1, 2$, the vector space of linear combinations with real coefficients of the oriented i-cells is denoted by $C_i(AP_{\mathscr{P}}, \mathbb{R})$, its elements are called i-chains and the coefficients are called coordinates. For any chain c in $C_i(AP_{\mathscr{P}}, \mathbb{R})$, we denote by $|c|$ its ℓ_1-norm. By convention, for each i-chain c , $-c$ is the chain which corresponds to an inversion of the orientation. Notice that there is a natural orientation of the 2-cells induced by the orientation of $\mathbb{R}^2$, but there is no natural orientation of the 1-cells. Given an arbitrary orientation on each 1-cell of $AP_{\mathscr{P}}$, the 2-cells that contain this edge are split in two parts: the positive ones for which the orientation on the edge coincides with the one induced by the orientation of the 2-cell and the negative ones for which both orientations disagree. Notice that this splitting is independent, up to reversing, on the arbitrary choice of the orientation of the 1-cells.

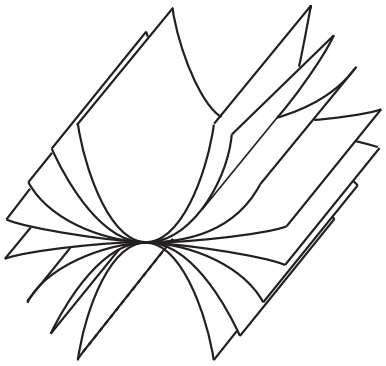


FIGURE 1. Local view of the Anderson-Putnam complex.

We define the linear boundary operator

$$\partial : C_2(AP_{\mathscr{P}}, \mathbb{R}) \to C_1(AP_{\mathscr{P}}, \mathbb{R})$$

which assigns to any face the sum of the edges at its boundary, weighted with a positive (resp. negative) sign if the induced orientation fits (resp. does not fit) with the orientation chosen for these edges. The kernel of the operator ∂ is the vector space of 2-cycles which we denote $H_2(AP_{\mathscr{P}}, \mathbb{R})$. It is well known that (up to isomorphism) the vector space $H_2(AP_{\mathscr{P}}, \mathbb{R})$ is a topological invariant of $AP_{\mathscr{P}}$ that coincides with the second singular homology group of the branched surface $AP_{\mathscr{P}}$,

[1] Actually the construction given by Anderson and Putnam is made in a particular case that suppose that $\mathscr{P}$ tiles the plane. They get a cell complex which is smaller than the one we defined here, however the basic ideas of both constructions are the same.

see *e.g.* [19]. The canonical orientation of the faces allows us to characterize the vector space $H_2(AP_{\mathscr{P}},\mathbb{R}) \subset C_2(AP_{\mathscr{P}},\mathbb{R})$ as follows. A 2-chain is a 2-cycle if and only if for each edge e the sum of the coordinates of the positive faces containing e is equal to the sum of the coordinates of the negative faces. This gives a set of m linear equations with integer coefficients for n variables (where m is the dimension of $C_1(AP_{\mathscr{P}},\mathbb{R})$ and n the dimension of $C_2(AP_{\mathscr{P}},\mathbb{R})$). These equations are called the *switching rules*. Let us say that a 2-cycle is *non-negative* if its coordinates are greater than or equal to zero and denote by $H_2^+(AP_{\mathscr{P}},\mathbb{R})$, the closed cone of non-negative cycles, *i.e.* the closed cone of cycles with non-negative coordinates and by $S_2(AP_{\mathscr{P}},\mathbb{R})$ the simplex made of all non-negative cycles whose ℓ_1-norm is 1. Finally, let us say that a 2-cycle is integral (resp. rational) if its coordinates are integers (resp. rational numbers).

When $\Omega_{\mathscr{P}} \neq \varnothing$, *i.e.* when one can tile $\mathbb{R}^2$ with $\mathscr{P}$, $\Omega_{\mathscr{P}}$ inherits a natural metrizable topology. A metric δ defining this topology can be chosen as follows. Let $B_\epsilon(0)$ stand for the open ball with radius ϵ centered at 0 in $\mathbb{R}^d$ and $B_\epsilon[T] := \{t_j \in T : t_j \cap \bar{B}_\epsilon(0) \neq \varnothing\}$ be the collection of tiles in T which intersect $\bar{B}_\epsilon(0)$. Consider two tilings T and T' in $\Omega_{\mathscr{P}}$ and let A be the set of ϵ in $(0,1)$ such that there exist u and u' in $\mathbb{R}^d$, with $|u|, |u'| \leqslant \epsilon/2$, so that $B_{1/\epsilon}[T+u] = B_{1/\epsilon}[T'+u']$. Then

$$\delta(T,T') = \begin{cases} \inf A & \text{if } A \neq \varnothing \\ 1 & \text{if not.} \end{cases}$$

In words, T and T' are close if, up to a small translation, they agree exactly in a large neighborhood of the origin. Equipped with such a metric, $\Omega_{\mathscr{P}}$ is a compact metric space and $\mathbb{R}^2$ acts by translation in a continuous way. Let $\mathcal{M}(\Omega_{\mathscr{P}})$ be the set of finite translation-invariant measures on $\Omega_{\mathscr{P}}$. It is non-empty because the group $\mathbb{R}^2$ is amenable. The subset of translation-invariant probability measures is denoted by $\Theta(\Omega_{\mathscr{P}})$. There exists a natural morphism:

$$\begin{array}{rccl} \mathrm{Ev}: & \mathcal{M}(\Omega_{\mathscr{P}}) & \to & C_2(AP_{\mathscr{P}},\mathbb{R}) \\ & \mu & \mapsto & \mathrm{Ev}(\mu) = \displaystyle\sum_{j=1}^{j=n} \frac{\mu(\pi^{-1}(p_j))}{\lambda(p_j)}\, p_j \end{array}$$

where λ stands for the Lebesgue measure in $\mathbb{R}^2$ and $\pi : \Omega_{\mathscr{P}} \to AP_{\mathscr{P}}$ is the natural projection which associates to each tiling the location of the origin of $\mathbb{R}^2$ in the (translated copy of the) prototile it belongs to. The switching rules are related to translation invariance:

Proposition 1.1. [3]

$$\mathrm{Ev}(\mathcal{M}(\Omega_{\mathscr{P}})) \subset H_2^+(AP_{\mathscr{P}},\mathbb{R}) \quad \text{and} \quad \mathrm{Ev}(\Theta(\Omega_{\mathscr{P}})) \subset S_2(AP_{\mathscr{P}},\mathbb{R}).$$

It turns out that the previous inclusions may not be onto: one can find a set of prototiles and non-negative coefficients associated with these prototiles satisfying the switching rules which are not the weights of some finite translation-invariant measure (see [7]). Clearly the set $\mathrm{Ev}(\Theta(\Omega_{\mathscr{P}}))$ is a closed convex subset of $S_2(AP_{\mathscr{P}},\mathbb{R})$, and the aim of this paper is to characterize this subset, that is to give geometric conditions on the non-negative 2-cycles ensuring that they are images of finite measures. We shall construct a non-negative continuous convex function $S_2(AP_{\mathscr{P}},\mathbb{R}) \ni c \mapsto |||c||| \in \mathbb{R}$ which we call the *asymptotic Thurston semi-norm*. Our main result is the following theorem:

Theorem 1.1.
A collection of prototiles $\mathscr{P}$ tiles the plane if and only if $H_2^+(AP_{\mathscr{P}}, \mathbb{R}) \neq \{0\}$ and the asymptotic Thurston semi-norm has a zero in $S_2(AP_{\mathscr{P}}, \mathbb{R})$. In this case

$$c \in \mathrm{Ev}(\Theta(\Omega_{\mathscr{P}})) \Longleftrightarrow |||c||| = 0.$$

2. The asymptotic Thurston semi-norm

• **The Thurston semi-norm:** We start with a brief historical account on the Thurston semi-norm, in order to shed some light on the ideas at the origin of this work. If M is a closed (*i.e.* compact, without boundary) 3-manifold, any closed surface $\mathcal{F}$ embedded in M defines an integer 2-cycle in $C_2(M;\mathbb{Z})$, and thus a homology class $[\mathcal{F}] \in H_2(M;\mathbb{Z})$, the second homology group of M. Relying upon this observation, W.P. Thurston introduced in [20] a semi-norm $\|\cdot\| : H_2(M;\mathbb{Z}) \to \mathbb{N}$ defined by:

$$\|u\| = \min\{|\chi(\mathcal{F})|; [\mathcal{F}] = u\},$$

where $\chi(.)$ stands for the Euler characteristic[2]. This semi-norm vanishes on the classes which can be represented by tori, and only on these classes. Thurston (semi)-norm has been intensively studied, see for instance [12, 13, 8] among many others. D. Gabai [9] (see also [14]) proved that this norm does not change if one considers singular surfaces (*i.e.* just "mapped in" surfaces, instead of embedded ones) in M. On another hand, U. Oertel in [13] used *branched surfaces* to describe the unit-ball of this norm: non-negative integer solutions to the so-called "switch-equations" define integer 2-cycles which in turn give surfaces embedded in the 3-manifold M, and carried by the branched surface W (non-negative real solutions yield measured laminations). When W *is* the ambient space, *i.e.* is no more assumed to be embedded in a 3-manifold, an integer 2-cycle of a branched surface W still defines a surface mapped in W, and carried by W provided that the cycle is non-negative. It is thus natural to try to extend the definition of the Thurston norm when dealing with "abstract" branched surfaces. Unfortunately, it happens that the multiplicativity fails in this setting. This leads us to adapt the definition by considering an asymptotic version $|||u||| = \lim_{n\to+\infty} \dfrac{\|nu\|}{n}$ (compare with [5]).

• **The asymptotic Thurston semi-norm:**

Consider a topological oriented surface $\mathcal{F}$ made with finitely many polygons of the Euclidean plane $t_1, \ldots, t_s$ glued along their edges, full edge to full edge. We say that $\mathcal{F}$ is carried by $AP_{\mathscr{P}}$ if there exists a continuous cellular map $\pi : \mathcal{F} \to AP_{\mathscr{P}}$ such that:

- for each $i \in \{1, \ldots s\}$, there exists $k(i) \in \{1, \ldots, n\}$ such that π is an isometry between $\mathrm{int}(t_i)$ and $\mathrm{int}(p_{k(i)})$;
- π is a local embedding.

The orientation of $\mathcal{F}$ induces an orientation of each polygon t_i in $\mathcal{F}$ which may fit or not with the canonical orientation of the polygon $p_{k(i)}$. We set

$$\epsilon_i = \begin{cases} 1 & \text{if the orientation is preserved} \\ -1 & \text{if not} \end{cases}$$

[2]We recall that the Euler characteristic of a surface is the sum of the Euler characteristics of its connected components.

and define for each $j \in \{1, \dots, n\}$:

$$c_j = \sum_{i \mid k(i)=j} \epsilon_i.$$

Clearly the vector $c(\mathcal{F}) \equiv \sum_{j=1}^{j=n} c_j \cdot p_j$, is a cycle in $H_2(AP_{\mathscr{P}}; \mathbb{Z})$.

For each cycle $c \neq 0$ in $H_2(AP_{\mathscr{P}}; \mathbb{Z})$, we denote by $\hat{c}$ the set of all compact oriented surfaces $\mathcal{F}$ that are carried by the flat branched surface $AP_{\mathscr{P}}$ and whose associated 2-cycles $c_{\mathcal{F}}$ is equal to c.

Lemma 2.1. *For any cycle $c = \sum_{j=1}^{j=n} c_j \cdot p_j \neq 0$ in $H_2(AP_{\mathscr{P}}; \mathbb{Z})$, there exists a surface $\mathcal{F}$ in $\hat{c}$ made with exactly $|c_j|$ copies of p_j, for each $j = 1, \dots, n$.*

Proof. A proof of this lemma is already given in [10] for non-negative cycles, and the same arguments work here. For each $j \in \{1, \dots, n\}$ consider $|c_j|$ copies of p_j with orientation given by the sign of c_j. Let p_{k_0} and p_{k_1} be two distinct polygons chosen in the collection of $\sum_{j=1}^{j=n} |c_j|$ polygons.

An edge e_{k_0} of p_{k_0} can be glued to an edge e_{k_1} of p_{k_1} if and only if:

- they have the same color;
- they are translated copies one of the other;
- their orientations (induced by the orientations of the polygons) are different.

If both polygons have the same orientation, the gluing is made by the translation that maps e_{k_0} to e_{k_1}. If the two polygons have different orientations, the gluing is made by post-composing the translation that maps e_{k_0} to e_{k_1} with the orthogonal symmetry with axis e_{k_1}. We perform this series of gluing as long as possible until exhaustion of the polygons in the collection. We have thus built a compact oriented (not necessarily connected) surface $\mathcal{F}$ such that $c(\mathcal{F}) = c$. □

For any integral cycle c, define

$$\begin{aligned} \|c\| &\equiv -\max_{\mathcal{F} \in \hat{c}} (\chi(\mathcal{F})) \quad \text{if } c \neq 0 \\ \text{and} \quad \|0\| &\equiv 0. \end{aligned}$$

Remark 2.2. *For any integral cycle c:*

$$\|c\| = 0 \Longleftrightarrow \hat{c} \text{ contains a torus.}$$

For any integer $m > 0$ and any pair of integral cycles c, c', we have:

$$\|mc\| \leqslant m\|c\| \quad \text{and} \quad \|c + c'\| \leqslant \|c\| + \|c'\|.$$

It follows that for any integral cycle c, the limit of the sequence $((1/m)\|mc\|)_m$ exists. Hence we define:

$$|||c||| \equiv \lim_{m \to +\infty} \frac{1}{m} \|mc\|.$$

We observe that for each integer $m > 0$, and each pair of integral cycles c, c', we have:

$$|||mc||| = m|||c||| \quad \text{and} \quad |||c + c'||| \leqslant |||c||| + |||c'|||.$$

Thus, we can extend the definition of the asymptotic Thurston semi-norm $|||\cdot|||$ to the cycles in $H_2(AP_{\mathscr{P}};\mathbb{Q})$ by setting:

$$|||c||| \equiv \frac{1}{m}|||mc|||$$

for all integer m such that mc is an integral cycle.

Remark 2.3. *For any cycle $c \in H_2(AP_{\mathscr{P}};\mathbb{Q})$, $|||c||| = 0$ does not imply that $\hat{c}$ contains a torus.*

Lemma 2.4. *There exists a constant $C_{\mathscr{P}} > 0$ such that for any cycle in $H_2(AP_{\mathscr{P}},\mathbb{Q})$, we have*

$$|||c||| \leqslant C_{\mathscr{P}}|c|.$$

Proof. Let $c' = mc$ be a cycle in $H_2(AP_{\mathscr{P}},\mathbb{Z})$ different from 0, that reads $c' = \sum_{j=1}^{j=n} c_j \cdot p_j$. Following Lemma 2.1, one can find a surface $\mathcal{F}$ in $\hat{c}$ made with exactly $|c_j|$ copies of p_j, for each $j = 1,\dots,n$. Let s_{p_j} be the number of vertices of the polygons in $\mathscr{P}$ and set $s_{\mathscr{P}} = \max_{j=1,\dots,n} s_{p_j}$. We clearly have:

$$|\chi(\mathcal{F})| \leqslant s_{\mathscr{P}}|c'|$$

whence

$$\|c'\| \leqslant s_{\mathscr{P}}|c|.$$

It follows that:

$$|||c'||| \leqslant s_{\mathscr{P}}|c|. \qquad \square$$

Corollary 2.1. *The asymptotic Thurston semi-norm $|||\cdot|||$ is Lipschitz continuous on $H_2(AP_{\mathscr{P}};\mathbb{Q})$.*

It follows that the function $c \mapsto |||c|||$ can be extended to a non-negative continuous function defined on $H_2(AP_{\mathscr{P}};\mathbb{R})$, and thus in particular on the whole simplex $S_2(AP_{\mathscr{P}},\mathbb{R})$. We call it the *asymptotic Thurston map*. From the subadditivity of the map $c \mapsto |||c|||$ we easily get:

Lemma 2.5. *When restricted to $S_2(AP_{\mathscr{P}},\mathbb{R})$, the asymptotic Thurston map is a bounded convex continuous non-negative map.*

Corollary 2.2. *When restricted to $S_2(AP_{\mathscr{P}},\mathbb{R})$, the set of zeros of the asymptotic Thurston map is a (possibly empty) convex subset of the simplex $S_2(AP_{\mathscr{P}},\mathbb{R})$.*

3. Wang tilings

We start with some basic definitions. Let $\mathcal{W} = \{w_1,\dots,w_n\}$ be a finite collection of unit squares whose vertices have integer coordinates in $\mathbb{R}^2$, and with colored edges. We say that $\mathcal{W}$ is a collection of *Wang prototiles*. A Wang tiling is a tiling made with $\mathcal{W}$ such that abutting edges of adjacent tiles have the same color. In 1966, R. Berger [4] gave a first example of a set of Wang prototiles which can tile the plane but cannot tile it periodically. This example was made with a collection of 20426 Wang prototiles. Since then, similar examples with a smaller set of Wang prototiles have been found. The state of the art is the example found by K. Culik [6] (see also [11]) made with 13 Wang prototiles and shown in Figure 2.

Let us introduce two notions concerning Wang tilings that we will use in the proof of the main theorem.

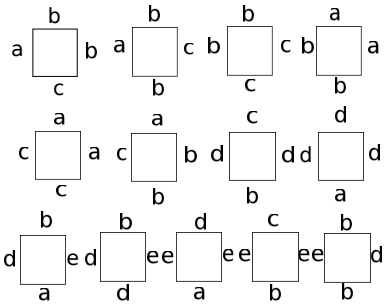


FIGURE 2. A collection of Wang prototiles.

Forgetting colors:
For a given collection of Wang prototiles $\mathcal{W} = \{w_1, \dots, w_n\}$, we consider the set $\widehat{\mathcal{W}} = \{\widehat{w}_1, \dots, \widehat{w}_n\}$, where, for $j = 1, \dots, n$, $\widehat{w}_j$ is deduced from w_j by forgetting the colors on its sides and keeping its index j. It follows that $AP_{\widehat{\mathcal{W}}}$ is a collection of n unit squares (indexed by j in $\{1, \dots, n\}$) glued respectively along their horizontal edges and their vertical edges.

Remark 3.1. *$\Omega_{\widehat{\mathcal{W}}}$ is non empty and periodic orbits are dense in $\widehat{\Omega}_{\mathcal{W}}$. Whenever $\Omega_{\mathcal{W}}$ is non empty, there is an isometric embedding of $\Omega_{\mathcal{W}}$ into $\Omega_{\widehat{\mathcal{W}}}$ which commutes with translations.*

Enforcing colors:
Consider a collection of Wang prototiles $\mathcal{W} = \{w_1, \dots, w_n\}$ and fix $p > 0$. For each j in $\{1, \dots, n\}$ we consider the collection of tilings of the square $[-p-1/2, p+1/2]^2$ made with translated copies of prototiles in $\mathcal{W}$ so that colors of common edges of adjacent tiles coincide and such that the central tile which covers $[-1/2, +1/2]^2$ is a copy of w_j. We denote by $\{T^1_{w_j,p}, \dots, T^{l(w_j,\mathcal{W},p)}_{w_j,p}\}$ this collection of tilings.

For each tiling $T^l_{w_j,p}$ of $[-p-1/2, p+1/2]^2$ we consider the 4 following 'colors':

- $\mathrm{Up}(T^l_{w_j,p})$ which is the restriction of $T^l_{w_j,p}$ to $[-p-1/2, p+1/2] \times [1/2, p+1/2]$,
- $\mathrm{Down}(T^l_{w_j,p})$ which is the restriction of $T^l_{w_j,p}$ to $[-p-1/2, p+1/2] \times [-p-1/2, -1/2]$,
- $\mathrm{Left}(T^l_{w_j,p})$ which is the restriction of $T^l_{w_j,p}$ to $[-p-1/2, -1/2] \times [-p-1/2, p+1/2]$,
- $\mathrm{Right}(T^l_{w_j,p})$ which is the restriction of $T^l_{w_j,p}$ to $[1/2, p+1/2] \times [-p-1/2, p+1/2]$,

and associate the Wang prototile $w_{j,l}$ whose index is the pair (j, l) and whose edges inherit the color:

- $\mathrm{Up}(T^l_{w_j,p})$ for the top edge;
- $\mathrm{Down}(T^l_{w_j,p})$ for the bottom edge;
- $\mathrm{Left}(T^l_{w_j,p})$ for the left edge;
- and $\mathrm{Right}(T^l_{w_j,p})$ for the right edge.

We denote by $\mathcal{W}^p$ the collection of Wang prototiles $w_{j,l}$ when j runs from 1 to n and l from 1 to $l(w_j, \mathcal{W}, p)$.

Remark 3.2. *$\mathcal{W}$ tiles the plane if and only if, for each $p > 1$, $\mathcal{W}^p$ (and thus $AP_{\mathcal{W}^p}$) is well defined.*

The importance of Wang tilings stems from the following result proved by L. Sadun and R. Williams (which is in fact valid in any dimension).

Theorem 3.3. [17] *For any finite collection of prototiles $\mathscr{P}$ which tiles the plane, there exists a finite collection of Wang prototiles $\mathcal{W}$ such that the dynamical systems $(\Omega_{\mathscr{P}}, \mathbb{R}^2)$ and $(\Omega_{\mathcal{W}}, \mathbb{R}^2)$ are orbit equivalent.*

Actually the homeomorphism that realizes the orbit equivalence of Theorem 3.3 has some important rigidity properties that will be detailed in the next section.

4. Reduction to Wang tilings

Lemma 4.1. *Theorem 1.1 is true if it is true for any finite collection of Wang prototiles.*

Proof. The proof splits in the proof of 3 claims.
Claim 1:
Theorem 1.1 is true if it is true for any finite collection of rational prototiles.
Proof of Claim 1: In order to prove this claim we need, as announced above, to go deeper in the proof of Sadun and Williams of Theorem 3.3 in [17] and, for the sake of convenience, we sketch the construction given therein in full details. Consider a finite collection of prototiles $\mathscr{P} = \{p_1, \dots, p_n\}$ that tiles the plane. For any $\epsilon > 0$, one can construct a finite collection of rational prototiles $\mathscr{P}' = \{p'_1, \dots, p'_n\}$ such that:

- Each p'_i has the same number of edges as p_i, and the p'_i's are ϵ-close to the p_i's for the Haussdorff distance and the corresponding edges have the same colors;
- The Anderson-Putnam complex $AP_{\mathscr{P}'}$ is homeomorphic to $AP_{\mathscr{P}}$.

This amounts to solve finitely many equations with integral coefficients and to use the fact that for such systems of equations, rational solutions are dense in the set of solutions. The one-to-one correspondence of the 2-cells of $AP_{\mathscr{P}'}$ and $AP_{\mathscr{P}}$ yields the natural identifications:

$$H_2^+(AP_{\mathscr{P}'}, \mathbb{R}) = H_2^+(AP_{\mathscr{P}}, \mathbb{R}) \quad \text{and} \quad S_2(AP_{\mathscr{P}'}, \mathbb{R}) = S_2(AP_{\mathscr{P}}, \mathbb{R}).$$

On the one hand, the construction of the asymptotic Thurston norm on $S_2(AP_{\mathscr{P}'}, \mathbb{R})$ coincides with the similar construction on $S_2(AP_{\mathscr{P}}, \mathbb{R})$. On the other hand, there is a natural cone isomorphism $\mathcal{I}$ between $\Theta(\Omega_{\mathscr{P}'})$ and $\Theta(\Omega_{\mathscr{P}})$ which is defined by:

$$\mathcal{I}(\mu)(A) = \sum_{i=1}^{i=n} \mu(\pi^{-1}(p_i) \cap A)) \cdot \frac{\lambda(p_i)}{\lambda(p'_i)},$$

for any mesurable set A in $\Omega_{\mathscr{P}}$ and any measure μ in $\Theta(\Omega_{\mathscr{P}})$. It follows easily that

$$\mathrm{Ev}(\mathcal{I}(\mu)) = \mathrm{Ev}(\mu), \quad \forall \mu \in \Theta(\Omega_{\mathscr{P}}).$$

This proves Claim 1.
Let us illustrate the above construction on the classical example of Penrose tilings. Consider the 'thin' and 'fat' triangles displayed in Figure 3 [3]. Together with their

[3]It is customary to use arrowheads to indicate adjacency rules. Each triangle can be represented as a polyhedron by replacing the arrowheads by appropriate dents and bumps to fit the general definition of tilings given above.

rotation by multiples of $2\pi/10$, they generate a set of prototiles $\mathscr{P}$ with 40 elements which in turn, generates the Penrose dynamical system $(\Omega_{\mathscr{P}}, \mathbb{R}^2)$.

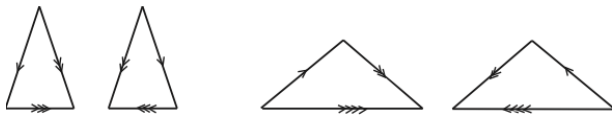


FIGURE 3. The tiles of the Penrose tiling.

Figure 4 shows a patch in $\mathbb{R}^2$ tiled by Penrose prototiles.

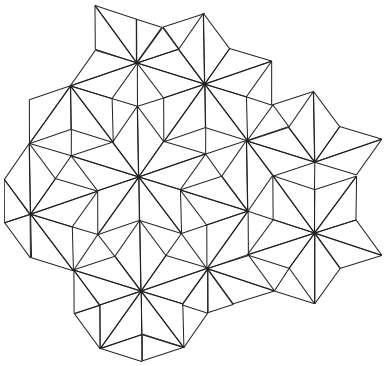


FIGURE 4. A patch of a Penrose tiling.

Figure 5 shows now a patch tiled with rational prototiles.

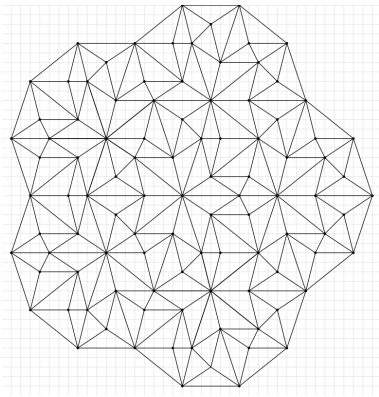


FIGURE 5. A patch of the rational Penrose tiling.

Claim 2:
Theorem 1.1 is true for any finite collection of rational prototiles if it is true for any finite collection of integral prototiles.
Proof of Claim 2: The next step in [17] is to transform a finite collection of rational prototiles $\mathscr{P}'$ into a finite collection of integral prototiles. Using a homothety with

an integral dilatation factor, one can transform the collection $\mathscr{P}' = \{p'_1, \ldots, p'_n\}$ in a family of integer prototiles $\mathscr{P}'' = \{p''_1, \ldots, p''_n\}$. Clearly both dynamical systems $(\Omega_{\mathscr{P}'}, \mathbb{R}^2)$ and $\Omega_{\mathscr{P}''}, \mathbb{R}^2)$ are orbit equivalent, the homeomorphism that realizes this equivalence maps translation invariant measures onto translation invariant measures and the two Anderson-Putnam complex $AP_{\mathscr{P}'}$ and $AP_{\mathscr{P}''}$ are homothetic. This proves Claim 2.

Claim 3:
Theorem 1.1 is true for any finite collection of integral prototiles if it is true for any finite collection of Wang prototiles.
Proof of Claim 3: One proceeds in 2 steps:

1 One replaces the straight edges of the prototiles in $\mathscr{P}''$ with zig-zags, that is with sequences of unit displacements in the coordinates directions. We denote by $\widehat{\mathscr{P}} = \{\hat{p}_1, \ldots, \hat{p}_n\}$ the new collection of prototiles obtained this way. Figure 6 shows how, in the particular case of the Penrose collection of rational prototiles, the patch described in Figure 5 is transformed.

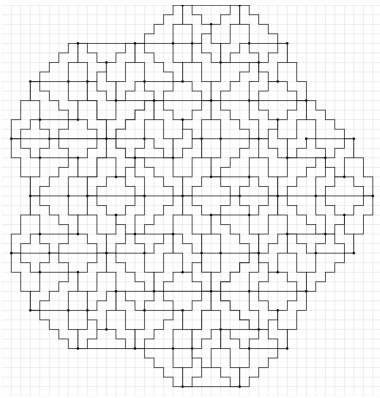


FIGURE 6. A patch of a square Penrose tiling.

2 It remains to put a label and appropriate colors on the edges of each square in each prototile in $\widehat{\mathscr{P}}$ to obtain a Wang tiling. The encoding is made in such a way that each edge of a square which is in the interior of a prototile of $\widehat{\mathscr{P}}$ forces as neighbors only the square which is its neighbor in the prototile and that any edge which meets the boundary of a prototile in $\widehat{\mathscr{P}}$ has its color given by the one of the prototile it belongs to. We denote by $\mathcal{W}$ the finite collection of Wang prototiles obtained with this construction.

It follows that the dynamical systems $(\Omega_{\mathcal{W}}, \mathbb{R}^2)$ and $(\Omega_{\widehat{\mathscr{P}}}, \mathbb{R}^2)$ are conjugate and that both Anderson-Putnam complexes $AP_{\mathcal{W}}$ and $AP_{\widehat{\mathscr{P}}}$ are homeomorphic. This allows us to identify:

$$H_2^+(AP_{\widehat{\mathscr{P}}}, \mathbb{R}) = H_2^+(AP_{\mathcal{W}}, \mathbb{R}) \quad \text{and} \quad S_2(AP_{\widehat{\mathscr{P}}}, \mathbb{R}) = S_2(AP_{\mathcal{W}}, \mathbb{R})$$

and proves Claim 3. □

5. Proof of Theorem 1.1

Thanks to Lemma 4.1, we only need to prove Theorem 1.1 for finite collections of Wang prototiles $\mathcal{W} = \{w_1, \dots, w_n\}$.

• **Assume first that $\Omega_{\mathcal{W}}$ is non empty.**
This implies that the set of translation-invariant probability measures $\Theta(\Omega_{\mathcal{W}})$ is non empty. Consider an ergodic measure $\mu \in \Theta(\Omega_{\mathcal{W}})$. From the Birkhoff Ergodic Theorem, we know that for μ-almost every tiling T in $\Omega_{\mathcal{W}}$ and for every prototile w_j in $\mathcal{W}$

$$\lim_{p\to+\infty} \frac{1}{(2p+1)^2}\, \mathcal{N}(w_j, p) = \mu(\pi^{-1}(w_j))$$

where $\mathcal{N}(w_i, p)$ stands for the number of copies of w_i that appear in T in the square $[-1/2-p, p+1/2]^2$. Fix $p > 0$ and consider the periodic tiling $\widehat{T}_p$ in $\Omega_{\widehat{\mathcal{W}}}$ obtained from T by repeating the pattern of T in $[-1/2-p, p+1/2]^2$. More precisely for any (q, r) in $\mathbb{Z}^2$, the tile of $\widehat{T}_p$ centered at (q, r) corresponds to the same prototile as the tile centered at $(q - u(2p+1), r - v(2p+1))$ where (u, v) is the pair of integers chosen so that $(q - u(2p+1), r - v(2p+1)) \in [-1/2-p, p+1/2]^2$. Consider the probability measure $\hat{\mu}_p$ which is equidistributed along the $\mathbb{R}^2$-orbit of the tiling $\widehat{T}_p$. On the one hand, notice that $\mathrm{Ev}(\hat{\mu}_p)$ is a cycle in $S_2(AP_{\widehat{\mathcal{W}}}, \mathbb{R})$ which is given by

$$\mathrm{Ev}(\hat{\mu}_p) = \sum_{j=1}^{j=n} \hat{\mu}_p(\pi^{-1}(\hat{w}_j))\, \hat{w}_j = \frac{1}{(2p+1)^2} \sum_{j=1}^{j=n} \mathcal{N}(w_i, p)\, \hat{w}_j.$$

It follows that $$\lim_{p\to+\infty} \mathrm{Ev}(\hat{\mu}_p) = \sum_{j=1}^{j=n} \mu(\pi^{-1}(w_j))\, \hat{w}_j.$$

On the other hand, the natural inclusion $\Omega_{\mathcal{W}} \subset \Omega_{\widehat{\mathcal{W}}}$ allows one to consider the measure μ as a measure $\hat{\mu}$ in $\Theta(\Omega_{\widehat{\mathcal{W}}})$ and the cycle $\mathrm{Ev}(\mu) = \sum_{j=1}^{j=n} \mu(\pi^{-1}(w_j))\, w_j$ in $S_2(AP_{\mathcal{W}}, \mathbb{R})$ can be identified (through the above inclusion) with the cycle $\mathrm{Ev}(\hat{\mu}) = \sum_{j=1}^{j=n} \mu(\pi^{-1}(w_j))\hat{w}_j$ in $S_2(AP_{\widehat{\mathcal{W}}}, \mathbb{R})$. Therefore

$$\lim_{p\to+\infty} \mathrm{Ev}(\hat{\mu}_p) = \mathrm{Ev}(\hat{\mu}).$$

Since the $\mathbb{R}^2$-orbit of the tiling $\widehat{T}_p$ is a 2-torus embedded in $\Omega_{\widehat{\mathcal{W}}}$, it follows directly that $\|\!|\, \mathrm{Ev}(\hat{\mu}_p) \,|\!\| = 0$. The continuity of the Thurston semi-norm (Lemma 2.5) implies that $\|\!|\, \mathrm{Ev}(\hat{\mu}) \,|\!\| = 0$ and thus $\|\!|\, \mathrm{Ev}(\mu) \,|\!\| = 0$. We conclude that $H_2^+(AP_{\mathcal{W}}, \mathbb{R}) \neq \{0\}$ and the set of zeros of the asymptotic Thurston semi-norm on $S_2(AP_{\mathcal{W}}, \mathbb{R})$ is non empty and contains $\mathrm{Ev}(\Theta(\Omega_{\mathcal{W}}))$.

• **Assume now $H_2^+(AP_{\mathcal{W}}, \mathbb{R}) \neq \{0\}$ and that the Thurston semi-norm has a zero in $S_2(AP_{\mathcal{W}}, \mathbb{R})$.**
Let $c \in S_2(AP_{\mathcal{W}}, \mathbb{R})$ be such that $\|\!|c|\!\| = 0$. The continuity of the Thurston semi-norm and the density of rational cycles in $S_2(AP_{\mathcal{W}}, \mathbb{R})$, imply that there exist a sequence of cycles $(c_\ell)_{\ell\geqslant 0}$ in $S_2(AP_{\mathcal{W}}, \mathbb{Q})$ such that $\lim_{\ell\to+\infty} c_\ell = c$, a sequence of integers $(n_\ell)_{\ell\geqslant 0}$ such that $n_\ell c_\ell$ is a non-negative integral cycle, and a sequence of

surfaces $(\mathcal{F}_\ell)_{\ell\geqslant 0}$ such that for each $\ell \geqslant 0$:

$$\mathcal{F}_\ell \in \widehat{n_\ell c_\ell} \quad \text{and} \quad \lim_{\ell\to+\infty} \frac{|\chi(\mathcal{F}_\ell)|}{n_\ell} = 0.$$

Fix now $p > 0$ and, for each ℓ big enough, consider the surface (with boundary) $\mathcal{F}_{\ell,p}$ which is made of all the Wang tiles of $\mathcal{F}_\ell$ that are at the center of the square $[-1/2-p, p+1/2]^2$ immersed in $\mathcal{F}_\ell$. These are the tiles at a distance larger than p from a singular point, *i.e.* the tiles which cannot be connected to a singularity with a path contained in less than p Wang tiles. We observe that the number of Wang tiles which are at distance 0 of a singularity is smaller than $4\,|\chi(\mathcal{F}_{l,p})|$. On the other hand, for any $p > 0$, tiles which are at distance p from a singularity must share an edge or a vertex with tiles which are at a distance smaller than $p-1$ from the singularity and conversely, each tile which is at distance $p-1$ from a singularity is in contact with at most eight tiles which are at distance p from this singularity. It follows that the number of Wang tiles which are at a distance smaller than p from a singularity is smaller than $K(p)\,|\chi(\mathcal{F}_{\ell,p})|$, where $K(p) = 4(1+\cdots+8^p)$. Let $c_{\ell,p}$ be the chain associated to $\mathcal{F}_{\ell,p}$ in $C_2(AP_{\mathcal{W}}, \mathbb{R})$.

We have:
$$|c_{\ell,p} - n_\ell c_\ell| \leqslant K(p)\,|\chi(\mathcal{F}_{\ell,p})|$$

and thus
$$\lim_{\ell\to+\infty} \frac{c_{\ell,p}}{|c_{\ell,p}|} = c.$$

Clearly $\mathcal{W}^p$ is non empty since, when ℓ is big enough, $\mathcal{F}_{\ell,p}$ is non empty. From the very construction of $\mathcal{F}_{\ell,p}$ we get that the chain $c_{\ell,p}$ is the image of a 2-chain $c^{(p)}_{\ell,p}$ in $C_2(AP_{\mathcal{W}^p}, \mathbb{R})$ through the canonical projection $\pi^{(p)} : C_2(AP_{\mathcal{W}^p}, \mathbb{R}) \to C_2(AP_{\mathcal{W}}, \mathbb{R})$. Let $c^{(p)}$ be an accumulation point in $C_2(AP_{\mathcal{W}^p}, \mathbb{R})$ of the sequence of normalized chains $\left(\frac{c^{(p)}_{\ell,p}}{|c^{(p)}_{\ell,p}|}\right)_{\ell>0}$. We easily check that

$$|\partial c^{(p)}_{\ell,p}| \leqslant K(p)\,|\chi(\mathcal{F}_{\ell,p})|$$

which implies that $c^{(p)}$ is a non-negative 2-cycle in $H_2^+(AP_{\mathcal{W}^p}, \mathbb{R})$ and that $\pi^p(c^{(p)}) = c$. It follows that

$$c \in \pi^{(p)}(H_2^+(AP_{\mathcal{W}^p}, \mathbb{R})),\ \forall p > 0.$$

Since it is well known (see [3] or [2]) that

$$\Theta(\Omega_{\mathcal{W}}) = \bigcap_{p>0} H_2^+(AP_{\mathcal{W}^p}, \mathbb{R})$$

we deduce that $\mathcal{W}$ tiles the plane and that the zeros of the Thurston semi-norm on $S_2(AP_{\mathcal{W}}, \mathbb{R})$ are contained in $\Theta(\Omega_{\mathcal{W}})$.

This ends the proof of our main Theorem.

6. Discussion and examples

Let us examine the different situations that may occur according to the family $\mathcal{W}$ of Wang prototiles we consider. One extreme situation is when $\mathcal{W}$ does not tile the plane, in this case either $H_2^+(AP_{\mathcal{W}}, \mathbb{R}) = \{0\}$ or $H_2^+(AP_{\mathcal{W}}, \mathbb{R}) \neq \{0\}$ and the Thurston semi-norm remains strictly positive in $S_2(AP_{\mathcal{W}}, \mathbb{R})$. The other extreme situation is when the colors of the edges are forgotten. In this situation, periodic orbits are dense in $\Omega_{\mathcal{W}}$, $\mathrm{Ev}(\Theta(\Omega_{\mathcal{W}}))$ is the whole simplex $S_2(AP_{\mathcal{W}}, \mathbb{R})$ and the Thurston semi-norm vanishes on the whole simplex.

Let us now concentrate on the case when $H_2^+(AP_{\mathcal{W}}, \mathbb{R}) \neq \varnothing$ and the (convex) set of zeros of the asymptotic Thurston semi-norm (which coincides with $\mathrm{Ev}(\Theta(\Omega_{\mathcal{W}}))$) is non empty. Different cases may occur:

- $\mathrm{Ev}(\Theta(\Omega_{\mathcal{W}}))$ is reduced to a single cycle c.
  - If c is not rational, then $\mathcal{W}$ cannot tile the plane periodically. This is exactly the situation we studied earlier for the Penrose tiling.
  - If c is rational,
    - either $\|c\| = 0$ which means that $\mathcal{W}$ can tile the plane periodically;
    - or $\|c\| \neq 0$ and $\mathcal{W}$ cannot tile the plane periodically: this is exactly what happens for the Robinson set of Wang prototiles [15].
- $\mathrm{Ev}(\Theta(\Omega_{\mathcal{W}}))$ is not reduced to a single cycle. In this case we are left with a series of questions, for instance:
  - **Question 1**: Can we find $\mathcal{W}$ such that rational cycles are not dense in $\mathrm{Ev}(\Theta(\Omega_{\mathcal{W}}))$?
  - **Question 2**: When does $\mathrm{Ev}(\Theta(\Omega_{\mathcal{W}}))$ contain a ball in $S_2(AP_{\mathcal{W}}, \mathbb{R})$?

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