Rigidity results with applications to best constants and symmetry of Caffarelli-Kohn-Nirenberg and logarithmic Hardy inequalities

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Abstract We take advantage of a rigidity result for the equation satisfied by an extremal function associated with a special case of the Caffarelli-Kohn-Nirenberg inequalities to get a symmetry result for a larger set of inequalities. The main ingredient is a reparametrization of the solutions to the Euler-Lagrange equations and estimates based on the rigidity result. The symmetry results cover a range of parameters which go well beyond the one that can be achieved by symmetrization methods or comparison techniques so far.

Keywords Caffarelli-Kohn-Nirenberg inequalities; Hardy-Sobolev inequality; extremal functions; ground state; bifurcation; branches of solutions; Emden-Fowler transformation; radial symmetry; symmetry breaking; rigidity; Keller-Lieb-Thirring inequalities

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1 Introduction and main results

Let $2^* := \infty$ if $d = 1$, 2, and $2^* := 2\,d/(d-2)$ if $d \ge 3$. Define

$$\vartheta(p,d) := \frac{d\,(p-2)}{2\,p}, \qquad a_c := \frac{d-2}{2}\ ,$$

and consider the space $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$ obtained by completion of $\mathcal{D}(\mathbb{R}^d \setminus \{0\})$ with respect to the norm $v \mapsto \| \, |x|^{-a}\, \nabla v \, \|^2_{\mathrm{L}^2(\mathbb{R}^d)}$. We will be concerned with the following two families of inequalities

Caffarelli-Kohn-Nirenberg Inequalities (CKN) [2] Let $d \ge 1$. For any $p \in [2, 2^*]$ if $d \ge 3$ or $p \in [2, 2^*)$ if $d = 1$, 2, for any $\theta \in [\vartheta(p,d), 1]$ with $\theta > 1/2$ if $d = 1$, there exists a positive constant $\mathsf{C}_{\mathrm{CKN}}(\theta, p, a)$ such that

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx \right)^{\frac{2}{p}} \le \mathsf{C}_{\mathrm{CKN}}(\theta,p,a) \left(\int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \right)^{\theta} \left(\int_{\mathbb{R}^d} \frac{|v|^2}{|x|^{2\,(a+1)}}\,dx \right)^{1-\theta} \tag{1}$$

holds true for any $v \in \mathcal{D}^{1,2}_a(\mathbb{R}^d)$. Here a, b and p are related by $b = a - a_c + d/p$, with the restrictions $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$ and $a + 1/2 < b \le a+1$ if $d = 1$. Moreover, the constants $\mathsf{C}_{\mathrm{CKN}}(\theta,p,a)$ are uniformly bounded outside a neighborhood of $a = a_c$.

In [4], a new class of inequalities, called *weighted logarithmic Hardy inequalities*, was considered. These inequalities can be obtained from (1) by taking $\theta = \gamma\,(p-2)$ and passing to the limit as $p \to 2_+$.

Weighted Logarithmic Hardy Inequalities (WLH)[4] Let $d \ge 1$, $a < a_c$, $\gamma \ge d/4$ and $\gamma > 1/2$ if $d = 2$. Then there exists a positive constant $\mathsf{C}_{\mathrm{WLH}}(\gamma, a)$ such that, for any $v \in \mathcal{D}^{1,2}_a(\mathbb{R}^d)$ normalized by

$$\int_{\mathbb{R}^d} |x|^{-2\,(a+1)}\,|v|^2\,dx = 1\ ,$$

we have

$$\int_{\mathbb{R}^d} \frac{|v|^2}{|x|^{2\,(a+1)}}\,\log\left(|x|^{2\,(a_c-a)}\,|v|^2\right)dx \le 2\,\gamma\,\log\left[\mathsf{C}_{\mathrm{WLH}}(\gamma,a)\int_{\mathbb{R}^d}\frac{|\nabla v|^2}{|x|^{2\,a}}\,dx\right]. \tag{2}$$

Moreover, the constants $\mathsf{C}_{\mathrm{WLH}}(\gamma,a)$ are uniformly bounded outside a neighborhood of $a = a_c$.

It is very convenient to reformulate the Caffarelli-Kohn-Nirenberg inequality in cylindrical variables as in [3]. By means of the Emden-Fowler transformation

$$s = \log|x| \in \mathbb{R}\ , \quad \omega = \frac{x}{|x|} \in \mathbb{S}^{d-1}\ , \quad y = (s,\omega)\ , \quad u(y) = |x|^{a_c-a}\,v(x)\ ,$$

Inequality (1) for v is equivalent to a Gagliardo-Nirenberg-Sobolev inequality for the function u on the cylinder $\mathcal{C} := \mathbb{R} \times \mathbb{S}^{d-1}$:

$$\mathsf{K}_{\rm CKN}(\theta,p,\Lambda)\,\|u\|^2_{\mathrm{L}^p(\mathcal{C})} \le \Big(\|\nabla u\|^2_{\mathrm{L}^2(\mathcal{C})}+\Lambda\,\|u\|^2_{\mathrm{L}^2(\mathcal{C})}\Big)^{\theta}\,\|u\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathcal{C})} \quad \forall\, u\in \mathrm{H}^1(\mathcal{C})\ . \tag{3}$$

Here and throughout the rest of the work we set

$$\Lambda := (a_c-a)^2\ .$$

Similarly, with $u(y) = |x|^{a_c-a}\,v(x)$, Inequality (2) is equivalent to

$$\int_{\mathcal{C}} |u|^2\,\log|u|^2\,dy \le 2\,\gamma\,\log\left[\frac{1}{\mathsf{K}_{\rm WLH}(\gamma,\Lambda)}\Big(\|\nabla u\|^2_{\mathrm{L}^2(\mathcal{C})}+\Lambda\Big)\right]\ , \tag{4}$$

for any $u\in \mathrm{H}^1(\mathcal{C})$ such that $\|u\|_{\mathrm{L}^2(\mathcal{C})}=1$. In both cases, we consider on $\mathcal{C}$ the measure $d\mu = |\mathbb{S}^{d-1}|^{-1} d\omega\,ds$ obtained by normalizing the surface of $\mathbb{S}^{d-1}$ to 1 (that is, the uniform probability measure), tensorized with the usual Lebesgue measure on the axis of the cylinder.

We are interested in *symmetry* and *symmetry breaking* issues: when do we know that equality in (1) and (2) is achieved by radial functions or, alternatively, by functions depending only on s in (3) and (4)? Related with inequality (3) is the Rayleigh quotient:

$$\mathcal{Q}^\theta_\Lambda[u] := \frac{\big(\|\nabla u\|_2^2+\Lambda\,\|u\|_2^2\big)^{\theta}\,\|u\|_2^{2\,(1-\theta)}}{\|u\|_p^2}\ .$$

Here $\|u\|_q := \big(\int_{\mathcal{C}}|u|^q\,d\mu\big)^{1/q}$. Then (3) and (4) are equivalent to state that

$$\mathsf{K}_{\rm CKN}(\theta,p,\Lambda) = \inf_{u\in \mathrm{H}^1(\mathcal{C})\setminus\{0\}} \mathcal{Q}^\theta_\Lambda[u]\ ,$$

$$\mathsf{K}_{\rm WLH}(\gamma,\Lambda) = \inf_{\substack{u\in \mathrm{H}^1(\mathcal{C})\setminus\{0\}\\ \|u\|_2=1}} \big(\|\nabla u\|_2^2+\Lambda\big)\ e^{-\frac{1}{2\,\gamma}\int_{\mathcal{C}}|u|^2\,\log|u|^2\,d\mu}\ .$$

Let $\mathsf{K}^*_{\rm CKN}(\theta,p,\Lambda)$ and $\mathsf{K}^*_{\rm WLH}(\gamma,\Lambda)$ be the corresponding values of the infimum when the set of minimization is restricted to functions depending only on s. The main interest of introducing the measure $d\mu$ is that $\mathsf{K}^*_{\rm CKN}(\theta,p,\Lambda)$ and $\mathsf{K}^*_{\rm WLH}(\gamma,\Lambda)$ are independent of the dimension and can be computed for $d=1$ by solving the problem on the real line $\mathbb{R}$.

Radial symmetry of $v=v(x)$ means that $u=u(s,\omega)$ is independent of ω. Up to translations in s and a multiplication by a constant, the optimal functions in the class of functions depending only on $s\in\mathbb{R}$ solve the equation

$$-u_*'' + \Lambda\,u_* = u_*^{p-1} \quad \text{in} \quad \mathbb{R}$$

if $\theta=1$. See Section 2 if $\theta<1$. Up to translations in s, non-negative solutions of this equation are all equal to the function

$$u_*(s) := \frac{A}{\big[\cosh(B\,s)\big]^{\frac{2}{p-2}}} \quad \forall\, s\in\mathbb{R}\ , \tag{5}$$

with $A^{p-2}=\frac{p}{2}\,\Lambda$ and $B=\frac12\,\sqrt{\Lambda}\,(p-2)$. The uniqueness up to translations is a standard result (see for instance [11, Proposition B.2] for a proof).

The symmetry breaking issue is now reduced to the question of knowing whether the inequalities

$$\mathsf{K}_{\rm CKN}(\theta,p,\Lambda)\le\mathsf{K}^*_{\rm CKN}(\theta,p,\Lambda)\quad\text{and}\quad\mathsf{K}_{\rm WLH}(\gamma,\Lambda)\le\mathsf{K}^*_{\rm WLH}(\gamma,\Lambda)\tag{6}$$

are strict or not, when $d\ge2$. Symmetry breaking occurs if the inequality is strict and then optimal functions *are not* symmetric (symmetric means: depending only on s in the setting of the cylinder, or on $|x|$ in the case of the Euclidean space). In [4, pp. 2048 and 2057], the values of the symmetric constants have been computed. They are given by

$$\mathsf{K}^*_{\rm CKN}(\theta,p,\Lambda):=\left[\tfrac{2p\theta+2-p}{(p-2)^2}\right]^{\frac{p-2}{2\,p}}\left[\tfrac{2p\theta}{2p\theta+2-p}\right]^{\theta}\left[\tfrac{p+2}{4}\right]^{\frac{6-p}{2\,p}}\left[\frac{\sqrt{\pi}\;\Gamma\left(\frac{2}{p-2}\right)}{\Gamma\left(\frac{2}{p-2}+\frac12\right)}\right]^{\frac{p-2}{p}}\Lambda^{\theta-\frac{p-2}{2\,p}}\tag{7}$$

and

$$\mathsf{K}^*_{\rm WLH}(\gamma,\Lambda)=\frac{\gamma\left(8\pi^{d+1}\,e\right)^{\frac{1}{4\gamma}}}{\Gamma\left(\frac d2\right)^{\frac{1}{2\gamma}}}\left(\frac{4\Lambda}{4\gamma-1}\right)^{\frac{4\gamma-1}{4\gamma}}\quad\text{if }\gamma>\tfrac14\ ,$$

$$\mathsf{K}^*_{\rm WLH}(\gamma,\Lambda)=\frac{2\pi^{d+1}\,e}{\Gamma\left(\frac d2\right)^2}\qquad\text{if}\quad\gamma=\tfrac14\ .$$

Let

$$\Lambda_{\rm FS}(\theta,p,d):=4\,\tfrac{d-1}{p^2-4}\,\tfrac{(2\,\theta-1)\,p+2}{p+2}\quad\text{and}\quad\Lambda_\star(1,p,d):=\tfrac14\,(d-1)\,\tfrac{6-p}{p-2}\ .\tag{8}$$

We will define $\Lambda_\star(\theta,p,d)$ for $\theta<1$ later in the Introduction. Symmetry breaking occurs for any $\Lambda>\Lambda_{\rm FS}$ according to a result of V. Felli and M. Schneider in [15] for $\theta=1$ and in [4] for $\theta<1$ (also see [3] for previous results and [14] if $d=2$ and $\theta=1$). This symmetry breaking is a straightforward consequence of the fact that for $\Lambda>\Lambda_{\rm FS}$, the symmetric *optimals* are saddle points of an energy functional, and thus cannot be even local minima. As a consequence, we know that $\mathsf{K}_{\rm CKN}(\theta,p,\Lambda)<\mathsf{K}^*_{\rm CKN}(\theta,p,\Lambda)$ if $\Lambda>\Lambda_{\rm FS}(\theta,p,d)$.

Concerning the log Hardy inequality, it was shown in [4] that symmetry breaking occurs, that is, $\mathsf{K}_{\rm WLH}(\gamma,\Lambda)<\mathsf{K}^*_{\rm WLH}(\gamma,\Lambda)$, when either $d=2$ and $\gamma>1/2$ or $d\ge3$ and $\gamma\ge d/4$ provided that

$$\Lambda>(d-1)\left(\gamma-\tfrac14\right)\ .$$

Concerning symmetry, if $\theta=1$, from [12], we know that symmetry holds for CKN for any $\Lambda\le\Lambda_\star(1,p,d)$. The precise statement goes as follows.

Theorem 1 [12] *Let $d\ge2$. For any $p\in[2,2^*]$ if $d\ge3$ or $p\in[2,\infty)$ if $d=2$, under the conditions*

$$0<\mu\le\Lambda_\star(1,p,d)\qquad\text{and}\qquad\mathcal{Q}^1_\mu[u]\le\mathsf{K}^*_{\rm CKN}(1,p,\mu)\ ,$$

the solution of

$$-\,\Delta u+\mu\,u=u^{p-1}\quad\text{on}\quad\mathcal{C}\tag{9}$$

is given by the one-dimensional equation, written on $\mathbb{R}$. It is unique, up to translations.

Theorem 1 is a *rigidity result*. In [12], the proof is given for a minimizer of $\mathcal{Q}^1_\mu$, which therefore satisfies $\mathcal{Q}^1_\mu[u] \le \mathsf{K}^*_{\rm CKN}(1,p,\mu)$, but the reader is invited to check that only the latter condition is used in the proof. The proof is based on a chain of estimates which involve optimal interpolation inequalities on the sphere and the Keller-Lieb-Thirring inequality. These inequalities turn out to be equalities, and equality in each of the inequalities is shown to imply that the solution only depends on s (no angular dependence). The result of Theorem 1 gives a sufficient condition for symmetry when $\theta = 1$. We shall say that *any minimizer is symmetric* if it is given by (5), up to multiplications by constants and translations.

Theorem 2 [12] *Let $d \ge 2$. For any $p \in [2, 2^*]$ if $d \ge 3$ or any $p \in [2,\infty)$ if $d = 2$, if $0 < \Lambda \le \Lambda_\star(1,p,d)$, then $\mathsf{K}_{\rm CKN}(1,p,\Lambda) = \mathsf{K}^*_{\rm CKN}(1,p,\Lambda)$ and any minimizer is symmetric.*

In [12], the case $\theta < 1$ is also considered. According to [12, Theorem 9], for any $d \ge 3$, any $p \in (2, 2^*)$ and any $\theta \in [\vartheta(p,d), 1)$, we have the estimate

$$\mathfrak{C}(\theta,p)^{-\frac{2\,\theta}{q+2}}\,\mathsf{K}^*_{\rm CKN}(\theta,\Lambda,p) \le \mathsf{K}_{\rm CKN}(\theta,\Lambda,p) \le \mathsf{K}^*_{\rm CKN}(\theta,\Lambda,p) \tag{10}$$

where $q := \frac{2\,(p-2)}{(2\,\theta-1)\,p+2}$ and

$$\mathfrak{C}(\theta,p) := \frac{(p+2)^{\frac{p+2}{(2\,\theta-1)\,p+2}}}{(2\,\theta-1)\,p+2}\,\big(2-\tfrac{p}{2}\,(1-\theta)\big)^{1-\frac{q}{2}}\cdot\left(\frac{\Gamma(\frac{p}{p-2})}{\Gamma(\frac{\theta\,p}{p-2})}\right)^{2\,q}\left(\frac{\Gamma(\frac{2\,\theta\,p}{p-2})}{\Gamma(\frac{2\,p}{p-2})}\right)^{q}$$

under the condition $a_c^2 < \Lambda \le \frac{(d-1)}{\mathfrak{C}(\theta,p)}\,\frac{(2\,\theta-3)\,p+6}{4\,(p-2)}$. If $\theta = 1$, the equality case in the last inequality characterizes $\Lambda_\star(1,p,d)$ as defined in (8). However (10) does not give a range for symmetry unless $\theta = 1$.

Much more is known. According to [13,5], there is a continuous curve $p \mapsto \Lambda_{\rm s}(\theta,p,d)$ with $\lim_{p\to 2_+}\Lambda_{\rm s}(\theta,p,d) = \infty$ and $\Lambda_{\rm s}(\theta,p,d) > a_c^2$ for any $p \in (2,2^*)$ such that symmetry holds for any $\Lambda \le \Lambda_{\rm s}(1,p,d)$ and there is symmetry breaking if $\Lambda > \Lambda_{\rm s}(1,p,d)$, for any $\theta \in [\vartheta(p,d),1)$. Additionally, we have that $\lim_{p\to 2^*}\Lambda_{\rm s}(1,p,d) = a_c^2$ if $d \ge 3$ and, if $d = 2$, $\lim_{p\to\infty}\Lambda_{\rm s}(1,p,d) = 0$ and $\lim_{p\to\infty} p^2\Lambda_{\rm s}(1,p,d) = 4$. The existence of this function $\Lambda_{\rm s}$ has been proven in an indirect way, and it is not explicitly known. It has been a long-standing question to decide whether the curves $p \to \Lambda_{\rm s}(\theta,p,d)$ and the curve $p \to \Lambda_{\rm FS}(\theta,p,d)$ coincide or not. This is still an open question, at least for $\theta = 1$. For $\theta < 1$, and for some specific values of p, it has been shown that, in some cases, $\Lambda_{\rm s}(\theta,p,d) < \Lambda_{\rm FS}(\theta,p,d)$; see [5] for more details, as well as some symmetry results based on symmetrization techniques. A scenario based on numerical computations and asymptotic expansions at the point where non-symmetric positive solutions bifurcate from the symmetric ones has been proposed; see [7,9,10] for details.

Our interest in this work is to establish symmetry of the minimizers of CKN for $\theta < 1$ as well as of the log Hardy inequalities, thus identifying the corresponding sharp constants.

Our first result is an extension of Theorem 2 to the case $\theta < 1$. Our goal is to give explicit estimates of the range for which symmetry holds. This requires some notations and a preliminary result. We set

$$\Pi^*(\theta,p,q) := \left(\frac{\mathsf{K}^*_{\rm CKN}(\theta,p,1)}{\mathsf{K}^*_{\rm CKN}(1,q,1)^{\frac{q\,(p-2)}{p\,(q-2)}}}\right)^{\frac{1}{\theta-\frac{q\,(p-2)}{p\,(q-2)}}} \,. \tag{11}$$

Next we define

$$q^* = q^*(\theta,p) := \frac{2\,p\,\theta}{2-p\,(1-\theta)} \,. \tag{12}$$

The condition $\theta > \frac{q\,(p-2)}{p\,(q-2)}$ is equivalent to $q > q^*(\theta,p)$ and we can notice that $p < q^*(\theta,p) < 2^*$ for any $\theta \in (\vartheta(p,d),1)$. For $d \ge 3$ we define

$$\Lambda_1(\theta,p,d) := \max_{q\in(q^*,2^*)} \min\left\{\Lambda_\star(1,q,d), \frac{\theta\,\Lambda_\star(1,p,d)}{(1-\theta)\,\Pi^*(\theta,p,q)+\theta}\right\} \,,$$

whereas for $d = 2$

$$\Lambda_1(\theta,p,2) := \max_{q\in(q^*,6)} \min\left\{\Lambda_\star(1,q,2), \frac{\theta\,\Lambda_\star(1,p,2)}{(1-\theta)\,\Pi^*(\theta,p,q)+\theta}\right\} \,.$$

Next, we can also define

$$\mathsf{N}(\theta,p) := \frac{\big(\mathsf{K}^*_{\rm CKN}(\theta,p,1)\big)^{1/\theta}}{\mathsf{K}^*_{\rm CKN}\big(1,q^*(\theta,p),1\big)} \,. \tag{13}$$

We refer to Section 3 for an explicit expression of $\mathsf{N}(\theta,p)$. We introduce the exponent

$$\beta = \beta(\theta,p) := 1 - \frac{p-2}{2\,p\,\theta} \,. \tag{14}$$

For $2 < p < 6$ and $\theta \in (\vartheta(p,3),1)$ we denote by $\mathsf{x}^* = \mathsf{x}^*(\theta,p)$ the unique root of the equation

$$\theta\,(6-p)\big(x^\beta - \mathsf{N}\big)\,x - \big(2\,p\,\theta - 3\,(p-2)\big)\,\big(\theta\,\big(x^\beta-\mathsf{N}\big) + (1-\theta)\,(x-1)\,\mathsf{N}\big) = 0 \,,$$

in the interval $(\mathsf{N}^{1/\beta},\infty)$ for $\mathsf{N} = \mathsf{N}(\theta,p)$, see Lemma 2 in Section 3. Next we define

$$\Lambda_2(\theta,p,d) := \frac{\Lambda_\star(1,q^*,d)}{\mathsf{x}^*(\theta,p)} = \frac14\,(d-1)\,\frac{2\,p\,\theta-3\,(p-2)}{(p-2)\,\mathsf{x}^*(\theta,p)} \,,$$

and

$$\Lambda_\star(\theta,p,d) := \max\Big\{\Lambda_1(\theta,p,d), \Lambda_2(\theta,p,d)\Big\} \,.$$

Theorem 3 *Suppose that either $d = 2$ and $p \in (2,6)$ or else $d \ge 3$ and $p \in (2,2^*)$. Then*

$$\mathsf{K}_{\rm CKN}(\theta,p,\Lambda) = \mathsf{K}^*_{\rm CKN}(\theta,p,\Lambda) \,,$$

and any minimizer of CKN (3) is symmetric *provided that one of the following conditions is satisfied:*

(i) $d = 2$, $\theta \in (\vartheta(p,2),1)$ *and* $0 < \Lambda \le \Lambda_1(\theta,p,2)$.
(ii) $d = 2$, $\theta \in (\vartheta(p,3),1)$ *and* $0 < \Lambda \le \Lambda_\star(\theta,p,2)$,
(iii) $d \ge 3$, $\theta = \vartheta(p,d)$ *and* $0 < \Lambda \le \Lambda_2(\theta,p,d)$,
(iv) $d \ge 3$, $\theta \in (\vartheta(p,d),1)$ *and* $0 < \Lambda \le \Lambda_\star(\theta,p,d)$.

Our definition of $\Lambda_\star(\theta,p,d)$ for $\theta < 1$ is consistent with the definition of $\Lambda_\star(1,p,d)$ given in (8) because

$$\lim_{\theta\to 1} \Lambda_1(\theta,p,d) = \lim_{\theta\to 1} \Lambda_2(\theta,p,d) = \Lambda_\star(1,p,d) \ .$$

One of the drawbacks in the definition of $\Lambda_2(\theta,p,d)$ is that $\mathsf{x}^*(\theta,p)$ given by Lemma 2 is not explicit. For an explicit estimate of $\Lambda_2(\theta,p,d)$ see Proposition 2 in Section 5.

By passing to the limit as $p \to 2_+$ in the criterion $\Lambda \le \Lambda_2(\theta,p,d)$, we also obtain an explicit condition for symmetry in the weighted logarithmic Hardy inequalities. For any $\mathsf{N}_0 > 1$, consider the smallest root $x > \mathsf{N}_0^{1/\beta_0}$ of

$$4\,\gamma\,x^{\beta_0+1} - (8\,\gamma-3)\,\mathsf{N}_0\,x + (4\,\gamma-3)\,\mathsf{N}_0 = 0 \quad \text{with} \quad \beta_0 = 1 - \frac{1}{4\,\gamma}$$

and denote it by $\mathsf{x}_0^*(\gamma)$ if $\mathsf{N}_0 = \mathsf{N}_0(\gamma) := \lim_{p\to 2_+} \mathsf{N}(\gamma\,(p-2),p)$. An elementary but tedious computation shows that

$$\mathsf{N}_0(\gamma) = 2^{1-\frac{3}{4\,\gamma}}\,e^{\frac{1}{4\,\gamma}}\,\frac{(2\,\gamma-1)^{1-\frac{1}{\gamma}}}{(4\,\gamma-1)^{1-\frac{3}{4\,\gamma}}}\left(\frac{\Gamma\left(2\,\gamma-\frac12\right)}{\Gamma\left(2\,\gamma-1\right)}\right)^{\frac{1}{2\,\gamma}} . \tag{15}$$

Let us define

$$\Lambda_0(\gamma,d) := \frac{(d-1)\ (\gamma-3/4)}{\mathsf{x}_0^*(\gamma)} \ . \tag{16}$$

We then have

Theorem 4 *Assume that either* $d = 2$ *or* 3 *and* $\gamma > 3/4$, *or* $d \ge 4$ *and* $\gamma \ge d/4$. *Then*

$$\mathsf{K}_{\rm WLH}(\gamma,\Lambda) = \mathsf{K}^*_{\rm WLH}(\gamma,\Lambda) \ ,$$

and any minimizer of (4) *is* symmetric *provided that*

$$0 < \Lambda \le \Lambda_0(\gamma,d) \ .$$

For an explicit estimate of $\Lambda_0(\gamma,d)$ see Proposition 3 in Section 5.

Theorem 3 provides us with a *rigidity result*, which is stronger than a simple symmetry result. As a consequence, our estimates of Theorem 3 for the symmetry region cannot be optimal.

Theorem 5 *Suppose that either* $d = 2$ *and* $p \in (2,6)$ *or else* $d \ge 3$ *and* $p \in (2,2^*)$. *If* $\theta > \vartheta(p,\min\{3,d\})$, *then*

$$\Lambda_\star(\theta,p,d) < \Lambda_{\rm s}(\theta,p,d) \le \Lambda_{\rm FS}(\theta,p,d) \ .$$

If either $d = 3$ *and* $\theta = \vartheta(p,3)$, *or* $d = 2$ *and* $\theta > 0$, *then*

$$\Lambda_2(\theta,p,d) < \Lambda_{\rm s}(\theta,p,d) \le \Lambda_{\rm FS}(\theta,p,d) \ .$$

It can be conjectured that $\Lambda_{\rm s}(\theta,p,d) = \Lambda_{\rm FS}(\theta,p,d)$ holds in the limit case $\theta = 1$, and probably also for θ close enough to 1, on the basis of the numerical results of [9] and the formal computations of [10]. On the other hand, it is known from [5] that $\Lambda_{\rm s}(\theta,p,d) < \Lambda_{\rm FS}(\theta,p,d)$ when $\theta - \vartheta(p,d)$ is small enough, at least for some values of p and d.

The expressions involved in the statement of Theorem 3 look quite technical, but they are interesting for two reasons:

- Theorem 3 determines a range for symmetry which goes well beyond what can be achieved using standard methods and is somewhat unexpected in view of the estimate of [12, Theorem 9]. It is a striking observation that the reparametrization method which has been extensively used in [9,10] allows us to extend to $\theta < 1$ results which were known only for $\theta = 1$.
- Even if they cannot be optimal as shown in Theorem 5, the estimates of Theorem 3 are rather accurate from the numerical point of view, as will be illustrated in Section 5.

This paper is organized as follows. Section 2 is devoted to the reparametrization and the proof of symmetry when $\Lambda \le \Lambda_1(\theta,p,d)$ in the subcritical case $\vartheta(p,d) < \theta < 1$. To the price of some additional technicalities, the range $\Lambda \le \Lambda_2(\theta,p,d)$ and $\vartheta(p,\min\{3,d\}) \le \theta < 1$ is covered in Section 3. The proofs of Theorems 3 and 5 are established in Section 4. The last section is devoted to an explicit approximation of Λ_0 and Λ_2, and some numerical results which illustrate Theorems 3 and 5. The reader interested in the strategy of the proofs as well as the origin of the expressions of $\Lambda_1(\theta,p,d)$ and $\Lambda_2(\theta,p,d)$ is invited to read first Section 2 and the proof of Lemma 5 in Section 3.

2 Reparametrization and a first symmetry result

We begin by a reparametrization of the branches of the solutions which allows us to reduce the case corresponding to $\theta < 1$ and Λ to the case corresponding to $\theta = 1$ and some related μ, as in Theorem 1. Consider an optimal function u for (3), which therefore satisfies

$$\mathsf{K}_{\rm CKN}(\theta,p,\Lambda) = \mathcal{Q}^\theta_\Lambda[u] = (t+\Lambda)^\theta\,\frac{\|u\|_2^2}{\|u\|_p^2} \quad \text{with} \quad t := \frac{\|\nabla u\|_2^2}{\|u\|_2^2}\ .$$

According to [5, Theorem1], such a function u exists for any $\theta > \vartheta(p,d)$. As a critical point of $\mathcal{Q}^\theta_\Lambda$, u solves (9) with

$$\theta\,\mu = (1-\theta)\,t + \Lambda$$

if it has been normalized by the condition

$$\|\nabla u\|_2^2 + \Lambda\,\|u\|_2^2 = \theta\,\|u\|_p^p\ .$$

Because of the zero-homogeneity of $\mathcal{Q}_\Lambda^\theta$, such a condition can be imposed without restriction and is equivalent to

$$\|u\|_2^2 = \frac{\theta}{t+\Lambda}\,\|u\|_p^p\ . \tag{17}$$

Proposition 1 *Let us assume that u is a solution of (9), satisfying $\mathcal{Q}_\Lambda^\theta[u] = \mathsf{K}_{\rm CKN}(\theta,p,\Lambda)$ and (17), with $\theta\,\mu = (1-\theta)\,t+\Lambda$. Then we have*

$$\mathcal{Q}_\mu^1[u] \le \mathsf{K}^*_{\rm CKN}(1,p,\mu)\ . \tag{18}$$

Proof From (6) we know that

$$(t+\Lambda)^\theta\,\frac{\|u\|_2^2}{\|u\|_p^2} \le \mathsf{K}^*_{\rm CKN}(\theta,p,\Lambda)\ .$$

Using (17), we rewrite this estimate as

$$\theta\,(t+\Lambda)^{\theta-1}\,\|u\|_p^{p-2} \le \mathsf{K}^*_{\rm CKN}(\theta,p,\Lambda)\ .$$

Using (17) again and the expression of μ, we obtain

$$Q_\mu^1[u] = \frac{\theta\,(t+\mu)}{t+\Lambda}\,\|u\|_p^{p-2} = \|u\|_p^{p-2} \le f(t,\theta,\Lambda,p)\,\mathsf{K}^*_{\rm CKN}(1,p,\mu)$$

with

$$f(t,\theta,\Lambda,p) := \frac{1}{\theta\,(t+\Lambda)^{\theta-1}}\,\frac{\mathsf{K}^*_{\rm CKN}(\theta,p,\Lambda)}{\mathsf{K}^*_{\rm CKN}(1,p,\mu)}\ .$$

Using the expression of μ and (7), we find that

$$f(t,\theta,\Lambda,p) = \tfrac{(p+2)^{\frac{p+2}{2\,p}}}{(2\,p)^{1-\theta}}\,\big(\tfrac{\Lambda\,\theta}{2+(2\,\theta-1)\,p}\big)^{\theta-\frac{p-2}{2\,p}}\,(t+\Lambda)^{1-\theta}\,\big((1-\theta)\,t+\Lambda\big)^{-\frac{p+2}{2\,p}}$$

achieves its maximum at $t_0 := \Lambda\,\big(\frac{2\,p\,\theta}{p-2}-1\big)^{-1} > 0$. Hence $f(t) \le f(t_0) = 1$, which concludes the proof.

Using the notations (11) and (12), we obtain our first symmetry result, which goes as follows.

Lemma 1 *Suppose that either $d=2$ and $p\in(2,6)$ or else $d\ge 3$, $p\in(2,2^*)$. If $\theta\in(\vartheta(p,d),1)$ and*

$$\Lambda \le \min\left\{\Lambda_\star(1,q,d),\,\frac{\theta\,\Lambda_\star(1,p,d)}{(1-\theta)\,\Pi^*(\theta,p,q)+\theta}\right\}$$

for some $q\in\big(q^*(\theta,p),6\big)$ when $d=2$, or for some $q\in\big(q^*(\theta,p),2^*\big)$ when $d\ge 3$, then any optimal function for (3) is symmetric.

Proof Let u be a solution as in Proposition 1. From (6), we know that

$$\mathsf{K}^*_{\rm CKN}(\theta,p,\Lambda)\ge(t+\Lambda)^\theta\,\frac{\|u\|_2^2}{\|u\|_p^2}\ .$$

For $p<q<\min\{6,2^*\}$ we have by Hölder's inequality, $\|u\|_p\le\|u\|_2^\delta\,\|u\|_q^{1-\delta}$ provided $\delta=\frac{2}{p}\,\frac{q-p}{q-2}$, and thus $1-\delta=\frac{q}{p}\,\frac{p-2}{q-2}$. Hence

$$\mathsf{K}^*_{\rm CKN}(\theta,p,\Lambda)\ge(t+\Lambda)^\theta\,\left(\frac{\|u\|_2^2}{\|u\|_q^2}\right)^{1-\delta}\ .$$

Now, for any $\lambda\in(0,\Lambda_\star(1,q,d)]$, we know from Theorem 2 that

$$\|u\|_q^2\le\frac{\|\nabla u\|_2^2+\lambda\,\|u\|_2^2}{\mathsf{K}^*_{\rm CKN}(1,q,\lambda)}\ ,$$

which shows that

$$\mathsf{K}^*_{\rm CKN}(\theta,p,\Lambda)\ge(t+\Lambda)^\theta\,\left(\frac{\mathsf{K}^*_{\rm CKN}(1,q,\lambda)}{t+\lambda}\right)^{1-\delta}\ .$$

Summarizing, we have found that

$$\frac{(t+\Lambda)^\theta}{(t+\lambda)^{1-\delta}}\le\frac{\mathsf{K}^*_{\rm CKN}(\theta,p,\Lambda)}{(\mathsf{K}^*_{\rm CKN}(1,q,\lambda))^{1-\delta}}\quad\text{if}\quad\lambda\le\Lambda_\star(1,q,d)\ . \tag{19}$$

Next we can make the ansatz $\lambda=\Lambda$. Provided $\Lambda\le\Lambda_\star(1,q,d)$, we get that

$$(t+\Lambda)^{\theta+\delta-1}\le\frac{\mathsf{K}^*_{\rm CKN}(\theta,p,\Lambda)}{(\mathsf{K}^*_{\rm CKN}(1,q,\Lambda))^{1-\delta}}=\big(\Pi^*(\theta,p,q)\,\Lambda\big)^{\theta+\delta-1}\ ,$$

so that $t\le(\Pi^*(\theta,p,q)-1)\,\Lambda$. According to Theorem 1, u is symmetric if

$$\frac{1}{\theta}\,\big((1-\theta)\,t+\Lambda\big)=\mu\le\Lambda_\star(1,p,d)\ , \tag{20}$$

because (18) holds by Proposition 1. This completes the proof.

In the next section we shall consider an alternative ansatz for which $\lambda\neq\Lambda$.

3 Another symmetry result

In this section we establish an estimate similar to the one of Lemma 1 but based on a different ansatz, which moreover covers the critical case $\theta=\vartheta(p,d)$. We recall that $\beta=\beta(\theta,p)=1-\frac{p-2}{2\,p\,\theta}$ has been defined in (14). The proof is slightly more technical than the one of Lemma 1. We start with an auxiliary result.

Lemma 2 *For any* $\mathsf{N} > 1$, $p < 6$ *and* $\theta \in (\vartheta(p,3),1)$, *if* $\beta = \beta(\theta,p)$ *is given by* (14), *the equation*

$$\theta\,(6-p)\,\big(x^\beta - \mathsf{N}\big)\,x - \big(2\,p\,\theta - 3\,(p-2)\big)\,\big(\theta\,\big(x^\beta - \mathsf{N}\big) + (1-\theta)\,(x-1)\,\mathsf{N}\big) = 0\ ,$$

has a unique root in the interval $(\mathsf{N}^{1/\beta},\infty)$.

When $\mathsf{N} = \mathsf{N}(\theta,p) > 1$ is given by (13), we denote this root by $\mathsf{x}^* = \mathsf{x}^*(\theta,p)$.

Proof Consider the function

$$f(x) := \theta\,(6-p)\big(x^\beta - \mathsf{N}\big)\,x - \big(2\,p\,\theta - 3\,(p-2)\big)\big[\theta\big(x^\beta - \mathsf{N}\big) + (1-\theta)(x-1)\mathsf{N}\big]\,,$$

and notice first that $f(\mathsf{N}^{1/\beta}) < 0$ because $\theta > \vartheta(p,3)$ and $\mathsf{N}^{1/\beta} > 1$. Next we observe that $\alpha := 2\,p\,\theta - 3\,(p-2) = 2\,p\,\big(\theta - \vartheta(p,3)\big) = 6 - p - 2\,p\,(1-\theta)$ and compute

$$f'(x) = (6-p)\,\theta\,\left[(1+\beta)\,x^\beta - \mathsf{N}\right] - 2\,p\,\big(\theta - \vartheta(p,3)\big)\left[\beta\,\theta\,x^{\beta-1} + (1-\theta)\,\mathsf{N}\right]$$

and

$$f''(x) = \beta\,\theta\,x^{\beta-2}\,\left[(6-p)\,(1+\beta)\,x - (\beta-1)\big(6-p-2\,p\,(1-\theta)\big)\right] > 0$$

for any $x > 1$. Using the fact that $\mathsf{N} > 1$, we find that

$$f'(\mathsf{N}^{1/\beta}) \ge 2\,(p-2)\,(1-\theta)\,\mathsf{N} > 0\ .$$

It follows that the function $f(x)$ is increasing and convex for $x > \mathsf{N}^{1/\beta}$. Since $f(\mathsf{N}^{1/\beta}) < 0$ we conclude that $f(x)$ has a unique root for $x > \mathsf{N}^{1/\beta}$.

When $\mathsf{N} = \mathsf{N}(\theta,p)$ we only need to check that $\mathsf{N}(\theta,p) > 1$. This is shown in Lemma 4. Before, we need a preliminary estimate. Consider the *Digamma function* $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

Lemma 3 *For all* $z > 0$, *we have*

$$\frac{1}{2\,z} < \psi\left(z + \tfrac12\right) - \psi(z) < \ln\left(1 + \tfrac{1}{2\,z}\right) + \frac{1}{z} - \frac{2}{2\,z+1}\ .$$

Proof We use the following representation formula (*cf.* [1, § 6.3.21, p. 259]):

$$\psi(z) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-z\,t}}{1-e^{-t}}\right)\,dt$$

and elementary manipulations to get the lower bound

$$\psi\left(z + \tfrac12\right) - \psi(z) = \int_0^\infty \frac{e^{-z\,t}}{1+e^{-t/2}}\,dt > \frac12\int_0^\infty e^{-z\,t}\,dt = \frac{1}{2\,z}\ .$$

As for the upper bound, we have the equivalences

$$\ln\left(1+\tfrac{1}{2\,z}\right)+\frac{1}{z}-\frac{2}{2\,z+1}-\int_0^\infty\frac{e^{-z\,t}}{1+e^{-t/2}}\,dt>0$$
$$\Longleftrightarrow\ln\left(1+\tfrac{1}{2\,z}\right)+\int_0^\infty e^{-z\,t}\,dt-\frac{2}{2\,z+1}-\int_0^\infty\frac{e^{-z\,t}}{1+e^{-t/2}}\,dt>0$$
$$\Longleftrightarrow\ln\left(1+\tfrac{1}{2\,z}\right)+\int_0^\infty\frac{e^{-t/2}\,e^{-z\,t}}{1+e^{-t/2}}\,dt-\frac{2}{2\,z+1}>0\ .$$

The result follows from

$$\int_0^\infty\frac{e^{-(z+\frac12)\,t}}{1+e^{-t/2}}\,dt>\frac12\int_0^\infty e^{-(z+\frac12)\,t}\,dt=\frac{1}{2\,z+1}$$

and, by monotonicity of the function $z\mapsto\ln\left(1+\frac{1}{2\,z}\right)-\frac{1}{2\,z+1}$,

$$\ln\left(1+\tfrac{1}{2\,z}\right)>\frac{1}{2\,z+1}\ .$$

Lemma 4 *Assume that $2<p<6$ and $\vartheta(p,2)<\theta\le1$. Then the function $\theta\mapsto\mathsf{N}(\theta,p)$ is decreasing and $\mathsf{N}(1,p)=1$.*

Proof $\mathsf{N}(1,p)=1$ is a consequence of the definition of N. Using the precise value pf $\mathsf{K}^*_{\rm CKN}(\theta,p,\Lambda)$, we obtain the following explicit expression of the function $\mathsf{N}(\theta,p)$, namely

$$\left(\tfrac{2}{2-p\,(1-\theta)}\right)^{\frac{p-2}{2\,p\,\theta}}\left(\tfrac{p+2}{4}\right)^{\frac{6-p}{2\,p\,\theta}}\left(\tfrac{2\,(2-p\,(1-\theta))}{(2\,\theta-1)\,p+2}\right)^{\frac{2\,p\,\theta-3\,(p-2)}{2\,p\,\theta}}\left[\frac{\Gamma\left(\frac{2}{p-2}\right)\Gamma\left(\frac{2-p\,(1-\theta)}{p-2}+\frac12\right)}{\Gamma\left(\frac{2}{p-2}+\frac12\right)\Gamma\left(\frac{2-p\,(1-\theta)}{p-2}\right)}\right]^{\frac{p-2}{p\,\theta}}.$$

Let us define $G:=\mathsf{N}^\theta$ and compute

$$\frac{1}{G}\,\frac{\partial G}{\partial\theta}=\frac{p\,\theta-2\,(p-2)}{2-p\,(1-\theta)}-\frac{2\,p\,\theta-3\,(p-2)}{(2\,\theta-1)\,p+2}+\ln\left(\frac{2\,(2-p\,(1-\theta))}{(2\,\theta-1)\,p+2}\right)$$
$$+\frac{\Gamma'\left(\frac{2-p\,(1-\theta)}{p-2}+\frac12\right)}{\Gamma\left(\frac{2-p\,(1-\theta)}{p-2}+\frac12\right)}-\frac{\Gamma'\left(\frac{2-p\,(1-\theta)}{p-2}\right)}{\Gamma\left(\frac{2-p\,(1-\theta)}{p-2}\right)}\ .$$

By Lemma 3 we get that

$$\frac{1}{G}\,\frac{\partial G}{\partial\theta}<\frac{p\,\theta-2\,(p-2)}{2-p\,(1-\theta)}-\frac{2\,p\,\theta-3\,(p-2)}{(2\,\theta-1)\,p+2}+\ln\left(\frac{2\,(2-p\,(1-\theta))}{(2\,\theta-1)\,p+2}\right)$$
$$+\ln\left(\frac{(2\,\theta-1)\,p+2}{2\,(2-p\,(1-\theta))}\right)+\frac{p-2}{2-p\,(1-\theta)}-\frac{2\,(p-2)}{(2\,\theta-1)\,p+2}=0\ .$$

Since

$$\frac{1}{G}\,\frac{\partial G}{\partial\theta}=\ln\mathsf{N}+\frac{\theta}{\mathsf{N}}\,\frac{\partial\mathsf{N}}{\partial\theta}<0\ ,$$

for $\theta\in(\vartheta(p,2),1]$ and $\mathsf{N}(1,p)=1$, it follows that $\frac{\partial}{\partial\theta}\mathsf{N}<0$.

After these preliminaries, we can now state the main result of this section.

Lemma 5 *Assume that*

$$2 < p < 6 \quad \text{and} \quad \vartheta(p,3) < \theta < 1 \quad \text{if} \quad d = 2 \quad \text{or} \quad 3 \ ,$$
$$2 < p < 2^* \quad \text{and} \quad \vartheta(p,d) \le \theta < 1 \quad \text{if} \quad d \ge 4 \ .$$

Then any optimal function for (3) *is symmetric if* $\Lambda < \Lambda_2(\theta,p,d)$. *Moreover, we have* $\lim_{\theta\to 1} \Lambda_2(\theta,p,d) = \Lambda_\star(1,p,d)$.

Proof As in the proof of Lemma 1, the starting point of our estimate is inequality (19), which becomes

$$\frac{t+\Lambda}{t+\lambda} \le \mathsf{N}(\theta,p) \left(\frac{\Lambda}{\lambda}\right)^\beta \quad \text{if} \quad \lambda < \Lambda_\star(1,q,d) \ ,$$

under the restriction that we choose $q = q^*(\theta,p)$ given by (12), that is $1-\delta = \theta$ with δ as in (11). Remarkably, we observe that, for this specific value of q, we have

$$\theta - \frac{p-2}{2\,p} = \left(1 - \frac{q-2}{2\,q}\right)(1-\delta)$$

and, as a consequence,

$$\frac{t+\Lambda}{t+\lambda} \le \mathsf{N} \left(\frac{\Lambda}{\lambda}\right)^\beta$$

where $\beta := 1 - \frac{p-2}{2\,p\,\theta}$ and $\mathsf{N} = \mathsf{N}(\theta,p)$. Hence we get that

$$t \le \frac{\mathsf{N}\,\Lambda^\beta\,\lambda - \Lambda\,\lambda^\beta}{\lambda^\beta - \mathsf{N}\,\Lambda^\beta} =: \bar t \ .$$

As in the proof of Lemma 1, we can apply Theorem 1 if

- Condition (20) holds and a sufficient condition is therefore given by the condition

  $$(1-\theta)\,\bar t + \Lambda \le \theta\,\Lambda_\star(1,p,d) \ ,$$

  that is,

  $$\big(\theta\,\Lambda_\star(1,p,d) - \Lambda\big)\,\big(\lambda^\beta - \mathsf{N}\,\Lambda^\beta\big) \ge (1-\theta)\,\big(\mathsf{N}\,\Lambda^\beta\,\lambda - \Lambda\,\lambda^\beta\big) \ .$$

- Condition $\lambda < \Lambda_\star(1,q,d)$, which is required to get (19), holds, *i.e.*,

  $$\lambda < \Lambda_\star(1,q,d) = \frac{1}{4}\,(d-1)\,\frac{6-q}{q-2} = \frac{1}{4}\,(d-1)\,\frac{2\,p\,\theta - 3\,(p-2)}{p-2} \ .$$

For a suitable $x = \lambda/\Lambda > \mathsf{N}^{1/\beta}$, to be chosen, these two conditions amount to

$$\Lambda \le \phi(x) := \frac{\theta\,\Lambda_\star(1,p,d)\,\big(x^\beta - \mathsf{N}\big)}{\theta\,(x^\beta - \mathsf{N}) + (1-\theta)\,(x-1)\,\mathsf{N}}\ ,$$
$$\Lambda < \chi(x) := \frac{1}{4}\,(d-1)\,\frac{2\,p\,\theta - 3\,(p-2)}{p-2}\,\frac{1}{x}\ .$$

After replacing $\Lambda_\star(1,p,d)$ by its value according to (8), we get that $\phi(x)-\chi(x)$ has the sign of $f(x)$ as defined in the proof of Lemma 2. By Corollary 4, we know that $\mathsf{N} \ge 1$ and conclude henceforth that any minimizer is *symmetric* if $\Lambda < \chi(\mathsf{x}^*(\theta,p)) = \Lambda_2(\theta,p,d)$.

In the limiting regime corresponding to as $\theta \to 1_-$, we observe that $\phi(x) = \Lambda_\star(1,p,d)$ and $\chi(x) = \Lambda_\star(1,p,d)/x$, so that $\lim_{\theta\to1}\Lambda_2(\theta,p,d) = \chi(1) = \Lambda_\star(1,p,d)$.

4 Proof of the main results

Proof (Theorem 3) It is a straightforward consequence of Lemma 1 and Lemma 5. Notice that $\lim_{\theta\to1}\Lambda_1(\theta,p,d) = \Lambda_\star(1,p,d)$ because

$$\lim_{\theta\to1}\frac{\theta\,\Lambda_\star(1,p,d)}{(1-\theta)\,\Pi^*(\theta,p,q)+\theta} = \Lambda_\star(1,p,d)\ .$$

Proof (Theorem 5) The function $q \mapsto \Lambda_1(1,q,d)$ is monotone decreasing and

$$q^*(\theta,p) - p = \frac{p\,(p-2)\,(1-\theta)}{2-p\,(1-\theta)} \ge 0$$

so that, for $i = 1,\ 2$,

$$\Lambda_i(\theta,p,d) \le \Lambda_\star(1,q^*(\theta,p),d) \le \Lambda_\star(1,p,d) < \Lambda_{\rm FS}(\theta,p,d)\ .$$

By definition of $\Lambda_{\rm s}(\theta,p,d)$, we know that $\Lambda_\star(\theta,p,d) \le \Lambda_{\rm s}(\theta,p,d)$. By Theorem 3, if $\Lambda = \Lambda_\star(\theta,p,d)$ any minimizer for $\mathsf{K}_{\rm CKN}(\theta,p,\Lambda)$ is symmetric. On the other hand, by continuity, we know that

$$\mathsf{K}_{\rm CKN}\big(\theta,p,\Lambda_{\rm s}(\theta,p,d)\big) = \mathsf{K}^*_{\rm CKN}\big(\theta,p,\Lambda_{\rm s}(\theta,p,d)\big)\ .$$

Let us assume that $\Lambda_{\rm s}(\theta,p,d) < \Lambda_{\rm FS}(\theta,p,d)$ and consider a sequence $(\lambda_n)_{n\in\mathbb{N}}$ converging to $\Lambda_{\rm s}(\theta,p,d)$ with $\lambda_n > \Lambda_{\rm s}(\theta,p,d)$. If u_n is a non-symmetric minimizer of $\mathsf{K}_{\rm CKN}(\theta,p,\lambda_n)$, we can pass to the limit: up to the extraction of a subsequence, $(u_n)_{n\in\mathbb{N}}$ converges in $\mathrm{H}^1(\mathcal{C})$ towards a minimizer u for $\mathsf{K}_{\rm CKN}(\theta,p,\Lambda_{\rm s}(\theta,p,d))$. The function u cannot only depend on s, because any symmetric minimizer for $\mathsf{K}^*_{\rm CKN}(\theta,p,\Lambda)$ is a strict local minimum in $\mathrm{H}^1(\mathcal{C})$ due to the fact that $\Lambda_{\rm s}(\theta,p,d) < \Lambda_{\rm FS}(\theta,p,d)$. Hence, for $\Lambda = \Lambda_{\rm s}(\theta,p,d)$ there are two distinct minimizers for $\mathsf{K}_{\rm CKN}(\theta,p,\Lambda)$: one is symmetric and the other one *is not* symmetric. This proves that $\Lambda_\star(\theta,p,d) < \Lambda_{\rm s}(\theta,p,d)$ if $\theta > \vartheta(p,\min\{3,d\})$.

In the other cases, that is, if either $d = 3$ and $\theta = \vartheta(p,3)$, or $d = 2$ and $\theta > 0$, the same method applies if we replace $\Lambda_\star(\theta,p,d)$ by $\Lambda_2(\theta,p,d)$.

Proof (Theorem 4) Let us consider $f(x)$ as in the proof of Lemma 2 and assume that $\theta = \gamma\,(p-2)$. As $p \to 2_+$, $f(x)/(p-2)$ converges towards

$$f_0(x) := 4\,\gamma\,x^{\beta_0+1} - (8\,\gamma-3)\,\mathsf{N}_0\,x + (4\,\gamma-3)\,\mathsf{N}_0 \quad \text{with} \quad \beta_0 = 1 - \tfrac{1}{4\,\gamma}\ .$$

We easily check that the function $f_0(x)$ is convex for $x > 0$, $f_0(\mathsf{N}_0^{1/\beta_0}) < 0$ and $f_0'(\mathsf{N}_0^{1/\beta_0}) = 2\,\mathsf{N}_0 > 0$. We conclude that $f_0(x)$ has a unique root for $x > \mathsf{N}_0^{1/\beta_0}$. We denote this unique root by $\mathsf{x}_0^* = \mathsf{x}_0^*(\gamma)$. It follows that $\mathsf{x}^*(\gamma\,(p-2),p)$ converges to $\mathsf{x}_0^*(\gamma)$ as $p \to 2_+$. Symmetry then is established by passing to the limit for any $\Lambda \in \big(0, \Lambda_0(\gamma,d)\big)$ with $\Lambda_0(\gamma,d)$ given by (16).

5 An approximation and some numerical results

The functions $\mathsf{x}^*(\theta,p)$ and $\mathsf{x}_0^*(\gamma)$ which enter in the results of Theorem 3 and Theorem 4 are not explicit but easy to estimate, which in turn gives explicit estimates of $\Lambda_2(\theta,p,d)$ and $\Lambda_0(\gamma,d)$. Let

$$\alpha = 2\,p\,\big(\theta - \vartheta(p,3)\big) = 2\,p\,\theta - 3\,(p-2)\ ,$$
$$\beta = \beta(\theta,p) = 1 - \frac{p-2}{2\,p\,\theta}\ ,$$

and

$$\Lambda_{2,\mathrm{approx}}(\theta,p,d) := \frac{(d-1)\,\alpha}{4\,(p-2)}\,\frac{\beta\,\theta\,(6-p) - \alpha\,(1-\theta+\beta\,\theta\,\mathsf{N}^{-1/\beta})}{\beta\,\theta\,(6-p)\,\mathsf{N}^{1/\beta} - \alpha\,(\beta\,\theta+1-\theta)}\ .$$

Proposition 2 *Suppose that either $d = 2$ and $p \in (2,6)$ or else $d \ge 3$ and $p \in (2,2^*)$. Then for any $\theta \in (\vartheta(p,3),1)$, we have the estimate*

$$\Lambda_2(\theta,p,d) > \Lambda_{2,\mathrm{approx}}(\theta,p,d)\ .$$

Proof Let us consider the function f defined in the proof of Lemma 2 and recall that $f''(x)$ is positive for any $x \ge \mathsf{N}^{1/\beta} > 1$. Moreover we verify that

$$f(\mathsf{N}^{1/\beta}) = -\,(1-\theta)\,\alpha\,\mathsf{N}\,(\mathsf{N}^{1/\beta}-1) < 0\ ,$$
$$f'(\mathsf{N}^{1/\beta}) = \mathsf{N}\,\Big[\beta\,\theta\,(6-p) - \alpha\,\big(1-\theta+\beta\,\theta\,\mathsf{N}^{-1/\beta}\big)\Big] > 0.$$

which provides the estimate

$$\mathsf{x}^*(\theta,p) < \mathsf{N}^{1/\beta} - \frac{f(\mathsf{N}^{1/\beta})}{f'(\mathsf{N}^{1/\beta})} = \frac{\beta\,\theta\,(6-p)\,\mathsf{N}^{1/\beta} - \alpha\,(\beta\,\theta+1-\theta)}{\beta\,\theta\,(6-p) - \alpha\,\big(1-\theta+\beta\,\theta\,\mathsf{N}^{-1/\beta}\big)}\ ,$$

and the result follows.

Next we give an estimate of $\Lambda_0(\gamma,d)$ in Theorem 4. Let

$$\Lambda_{0,\mathrm{approx}}(\gamma,d):=\frac{(d-1)\,(\gamma-\frac34)}{2\,(\gamma-\frac14)\,\mathsf{N}_0^{\frac{4\gamma}{4\gamma-1}}-2\,(\gamma-\frac34)}\ ,$$

with $\mathsf{N}_0(\gamma)$ as defined by (15).

Proposition 3 *Assume that $d\ge2$ and $\gamma>3/4$. Then*

$$\Lambda_0(\gamma,d)>\Lambda_{0,\mathrm{approx}}(\gamma,d)\ .$$

Proof Recall that $\beta_0=1-\frac{1}{4\,\gamma}$. Let us consider the function f_0 defined in the proof of Theorem 4. We note that $f_0''(x)$ is positive for $x>0$. Moreover we verify that $f_0'(\mathsf{N}_0^{1/\beta_0})=2\,\mathsf{N}_0>0$ and $f_0(\mathsf{N}_0^{1/\beta_0})=-(4\,\gamma-3)\,\mathsf{N}_0\,(\mathsf{N}_0^{1/\beta_0}-1)<0$, which provides the estimates

$$\mathsf{x}_0^*(\gamma)<\mathsf{N}_0^{1/\beta_0}-\frac{f(\mathsf{N}_0^{1/\beta_0})}{f'(\mathsf{N}_0^{1/\beta})}=2\,(\gamma-\tfrac14)\,\mathsf{N}_0^{\frac{4\gamma}{4\gamma-1}}-2\,(\gamma-\tfrac34)\ ,$$

and the result follows.

To conclude this paper, let us illustrate Theorems 3 and 5 with some numerical results. First we address the case of subcritical $\theta\in(\vartheta(p,d),1)$ and compare $\Lambda_\star$ with Λ_{FS}: Fig. 1 corresponds to the particular case $d=5$ and $\theta=0.5$.

The expression of $\Lambda_\star(\theta,p,d)$ is not explicit but easy to compute numerically. We recall that $\Lambda_\star$ is the maximum of Λ_1 and Λ_2, both of them being non-explicit. In practice, for low values of the dimension d, the relative difference of Λ_1 and Λ_2 is in the range of a fraction of a percent to a few percents, depending on θ and on the exponent p. Moreover, we numerically observe that $\Lambda_1\le\Lambda_2$, at least for the values of the parameters considered in Fig. 1. The estimate $\Lambda_{2,\mathrm{approx}}(\theta,p,d)$ of Proposition 2 is remarkably good.

In Fig. 2, we consider the critical case $\theta=\vartheta(p,d)$. The plot corresponds to $d=5$ and all p in the interval $(2,10/3)$. The exponent $\vartheta(p,d)$ is the one which enters in the Gagliardo-Nirenberg inequality

$$\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)}\le\mathsf{C}_{\mathrm{GN}}(p,d)\,\|\nabla u\|^{2\,\vartheta(p,d)}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|u\|^{2\,(1-\vartheta(p,d))}_{\mathrm{L}^2(\mathbb{R}^d)}\quad\forall\,u\in\mathrm{H}^1(\mathbb{R}^d)$$

on the Euclidean space $\mathbb{R}^d$, *without weights*. Here $\mathsf{C}_{\mathrm{GN}}(p,d)$ denotes the optimal constant and $p\in(2,\infty)$ if $d=1$ or 2, $p\in(2,2^*]$ if $d\ge3$. The optimizers are radially symmetry but not known explicitly.

It has been shown in [8, Theorem 1.4] that optimal functions for (1) exist if $\mathsf{C}_{\mathrm{GN}}(p,d)<\mathsf{C}_{\mathrm{CKN}}(\theta,p,a)$. On the other hand, optimal functions cannot be symmetric $\mathsf{C}_{\mathrm{GN}}(p,d)>\mathsf{C}^*_{\mathrm{CKN}}(\theta,p,a)$: see [5, Section 5] for further details and consequences. This symmetry breaking condition determines a curve $p\mapsto\Lambda_{\mathrm{GN}}(p,d)$ which has been computed numerically in [6,7]: there are values of p and d for which the condition $\Lambda>\Lambda_{\mathrm{GN}}(p,d)$, which guarantees symmetry breaking (but not existence), is weaker than the condition $\Lambda>\Lambda_{\mathrm{FS}}(\theta,p,d)$,

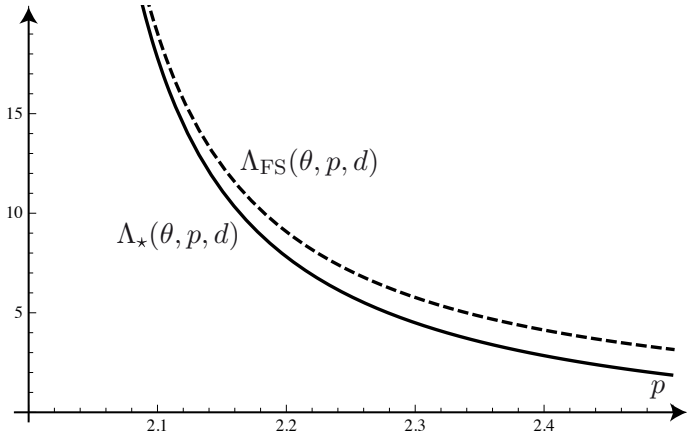


Fig. 1 Curves $p \mapsto \Lambda_\star(\theta,p,d)$ and $\Lambda \mapsto \Lambda_{\rm FS}(\theta,p,d)$ with $\theta = 0.5$ and $d = 5$. Symmetry holds for $\Lambda \le \Lambda_\star(\theta,p,d)$, while symmetry is broken for $\Lambda \ge \Lambda_{\rm FS}(\theta,p,d)$. The relative difference of Λ_1 and Λ_2, *i.e.*, $\Lambda_2(\theta,p,d)/\Lambda_1(\theta,p,d) - 1$, is below 4%. The estimate of Proposition 2 is such that $1 - \Lambda_{2,\rm approx}(\theta,p,d)/\Lambda_2(\theta,p,d)$ is of the order of 5×10^{-3}.

that is $\Lambda_{\rm GN}(p,d) < \Lambda_{\rm FS}(\theta,p,d)$. See Fig. 2. A rather complete scenario of explanations, based on numerical computations and some formal expansions, has been established in [9,10]. As it had to be expected, we numerically observe that $\Lambda_\star(\theta,p,d) \le \min\{\Lambda_{\rm FS}(\theta,p,d), \Lambda_{\rm GN}(p,d)\}$ when $\theta = \vartheta(p,d)$, for any $p \in (2,2^*)$.

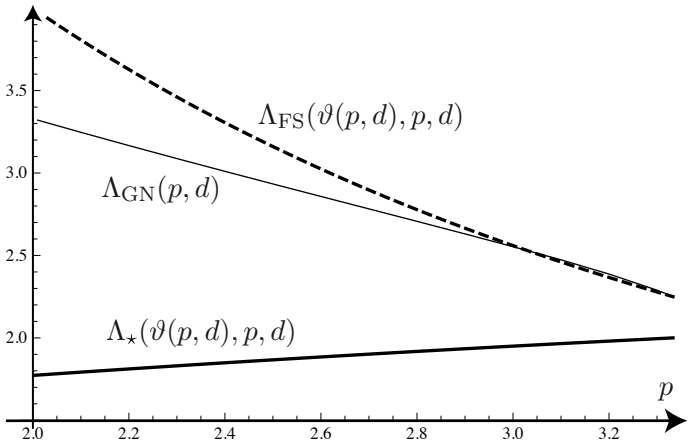


Fig. 2 With $\theta = \vartheta(p,d)$, the curve $p \mapsto \Lambda_\star(\theta,p,d)$ is always below the curves $p \mapsto \Lambda_{\rm FS}(\theta,p,d)$ and $p \mapsto \Lambda_{\rm GN}(p,d)$ for any $p \in (2,2^*)$, although $\Lambda_{\rm FS}$ and $\Lambda_{\rm GN}$ are not ordered. The plot corresponds to $d = 5$ and we may notice that $\Lambda_{\rm GN}(p,d) < \Lambda_{\rm FS}$ if p is small enough.

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