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FRACTIONAL SOBOLEV SPACES AND FUNCTIONS OF BOUNDED VARIATION OF ONE VARIABLE

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Abstract

We investigate the 1D Riemann-Liouville fractional derivative focusing on the connections with fractional Sobolev spaces, the space BV of functions of bounded variation, whose derivatives are not functions but measures and the space SBV, say the space of bounded variation functions whose derivative has no Cantor part. We prove that SBV is included in $W^{s,1}$ for every $s \in (0,1)$ while the result remains open for BV. We study examples and address open questions.

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Key Words and Phrases: Fractional calculus, Bounded variation functions, Riemann-Liouville derivative, Marchaud derivative, Sobolev spaces.

1. Introduction

The aim of this work is to investigate the *fractional derivative* concepts and make the connection between the related (so called fractional) Sobolev spaces and the spaces of functions of bounded variation whose derivatives are not functions but measures. Here, we only deal with the 1D case and we hope to extend results to higher dimensions via slicing theorems. Our main concern to investigate this connection is to consider variational models in the context of image processing. Indeed, variational models including many kinds of first order derivative have been studied from the seminal work of Rudin et al. [29]. We have used generalized second order derivatives in different segmentation and image analysis context [6, 7, 12, 13]. The use of derivatives of order in (0,1) is not standard in image analysis though it could be a quite useful tool for texture analysis to our opinion. However, pioneer works have been done (see [26] for example) using a finite dimensional setting. From a mathematical point of view, the natural underlying space is the space of functions of bounded variation. However, beyond the use of the total variation, it appears that more accurate penalization tools are needed. This is the case, for example, for texture analysis where the structure involves fractal dimension. To our opinion, the fractional derivative could be the suitable tool to describe images textures by providing quantitative information via the differentiation order.

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There are two main classes of definitions for fractional differentiation whose connections are not fully explored to our knowledge. The fractional derivative in the sense of Gagliardo is not explicitely defined (not even almost everywhere) but it is implicitly assumed by the setting of fractional Sobolev Spaces and the underlying norm (see [10] for example). It is, in some sense, a global definition which can be easily handled via the Fourier transform in the Hilbertian case. It is *a priori* well suited for variational analysis, especially in view of the Riesz distributional gradient. The second approach is based on the Riemann-Liouville fractional derivative (in short RL) and may be pointwise defined. We choose to focus on the RL derivative : there are many variants of the fractional derivatives/integrals definition as the Grunwald-Letnikov, Caputo, Weyl ones [9, 15, 23] but the RL derivative can be considered as a generic one. One can refer to[16] for a review of the different definitions. For a complete study of these derivatives one can refer to the book by Samko and al. [30] that contains an extensive bibliography in particular with respect to the pioneer work of Hardy-Littlewood. Moreover, in [33], the connection is made with metric and measure spaces, in particular the Hausdorff measure. We decided to use this derivative concept because it seems more adapted to applications and numerical computations than the Gagliardo one. The RL derivative is widely used by physicists [36, 37], in automatics, control theory and image processing as well, especially to deal with image enhancement and texture analysis [26]; in [22] calculus of variations problems where the cost functional involves fractional derivative are investigated. Nevertheless, the context is often a discrete one and there is not much analysis (to our knowledge) in the infinite dimensional setting. In particular, the link between the classical spaces of bounded variation functions and the fractional Sobolev spaces is not clear. To our knowledge, there is no paper that compare the BV space and the fractional Sobolev spaces in the RL sense. Indeed, the concept of fractional Sobolev spaces is not much developed for the RL derivative, though this fractional derivative concept is commonly used in engineering. One can refer to [8, 20, 21] however.
The limitation of the present paper is twofold. Here we consider only the 1D case: indeed we extensively exploit the notion of good representative of BV functions of one variable, which is not available for several variables. However, 1D results are necessary to consider higher dimension cases via slicing theorems for example ([3] section 3.11). In addition, since we are interested in image processing applications, we consider only bounded intervals. Of course, it is still possible to extend the present analysis to unbounded intervals. However, in that case the different notions of fractional derivatives induce significant differences [14, 9, 35]. This implies more investigation to compare the effects of the different approaches and this does not fit our prior concern.

The paper is organized as follows. Section 2 is devoted to the presentation of the two main approaches with a special focus on the Riemann-Liouville fractional derivatives: the main tools are recalled. In Section 3, we define RL-fractional Sobolev spaces $W^{s,1}_{RL,a+}$ and give basic properties. In the last section, we perform a comparison between these fractional Sobolev spaces, the classical BV space and the space SBV of functions whose distributional derivative is a special measure in the sense of De Giorgi (see [3]). In particular we prove

that

$$SBV \subset \bigcap_{s\in(0,1)} W^{s,1} \qquad \text{and} \qquad \bigcap_{s\in(0,1)} W^{s,1} \setminus BV \neq \emptyset \,.$$

2. **Fractional Calculus and Fractional Sobolev Spaces**

In this section we present the two main (different) definitions of fractional Sobolev spaces that we can find in the literature. We are in particular interested to the case where the differentiation order is $s \in [0,1)$ in order to study the fractional spaces between L^1 and $W^{1,1}$ and their relationship with BV. In the sequel, we consider the 1D framework. We recall that the space $AC(a,b)$ of absolutely continuous functions coincides with the Sobolev space $W^{1,1}(a,b)$ defined by

$$W^{1,1}(a,b) := \left\{ u \in L^1(a,b) \mid u' \in L^1(a,b) \right\},$$

endowed with the norm

$$\|u\|_{W^{1,1}} = \|u\|_{L^1} + \|u'\|_{L^1}.$$

Here and in the sequel $\frac{d}{dx}u$ denotes the classical derivative of u, u' denotes the distributional derivative of u, and $\dot{u}$ stands for the absolutely continuous part of u' ([3]). Recall that

$$W^{1,1}(a,b) \subset \mathcal{C}^0([a,b]),$$

where $\mathcal{C}^0([a,b])$ is the space of continuous functions on $[a,b]$ (see [1, 5, 24] for example) and

$$\forall u \in W^{1,1}(a,b), \forall y \in [a,b] \qquad \|u\|_{L^\infty} \leq |u(y)| + \|u'\|_{L^1} \,.$$

2.1. **Gagliardo's fractional Sobolev Spaces.**

This section is devoted to recalling the classical definition of fractional Sobolev spaces in the sense of Gagliardo:

Definition 2.1 (**Gagliardo's spaces**). *Let $s \in (0,1)$. For any $p \in [1,+\infty)$ we define the following space:*

$$W_G^{s,p}(a,b) = \left\{ u \in L^p(a,b) \;:\; \frac{|u(x)-u(y)|}{|x-y|^{\frac{1}{p}+s}} \in L^p([a,b]\times[a,b]) \right\}. \tag{2.1}$$

This is a Banach space endowed with the norm

$$\|u\|_{W_G^{s,p}(a,b)} = \left[\int_{[a,b]} |u(x)|^p dx \;+\; \int_{[a,b]}\int_{[a,b]} \frac{|u(x)-u(y)|^p}{|x-y|^{1+sp}} dx\, dy \right]^{\frac{1}{p}}.$$

$W_G^{s,p}(a,b)$ is the interpolated space between $L^p(a,b)$ and $\mathrm{W}^{1,p}(a,b)$ and the term

$$[u]_{W_G^{s,p}(a,b)} = \left[\int_{[a,b]}\int_{[a,b]} \frac{|u(x)-u(y)|^p}{|x-y|^{1+sp}} dx\, dy \right]^{\frac{1}{p}} \tag{2.2}$$

is the so-called Gagliardo semi-norm of u. We have in particular [18] that

$$W_G^{s,p}(a,b) \subseteq \mathrm{W}_G^{r,p}(a,b) \qquad \forall \quad 0 < r < s \leq 1 \,.$$

If $s = m + \tau > 1$ with $m \in \mathbb{N}$, $\tau \in [0, 1[$, such a definition can be generalized to higher orders by setting

$$W^{s,p}_G(a,b) = \{u \in \mathrm{W}^{m,p}(a,b) \ : \ D^m u \in \mathrm{W}^{\tau,p}(a,b)\}.$$

This point of view is related to interpolation theory (see [1, 4, 17, 19, 32, 34] for example). There is a huge literature concerning these fractional differentiation methods that we cannot mention here. However, let us quote a recent work by Shieh and Spector [31] who define the *distributional Riesz fractional gradient* for every $\mathcal{C}^\infty_c(\mathbb{R})$ function (with compact support) as

$$D^s u(x) = \frac{\Gamma(s/2)}{2^s\sqrt{\pi}\,\Gamma((1-s)/2)} \int_{\mathbb{R}} \frac{u'(t)}{(x-t)^s}\,dt$$

and define the fractional Sobolev spaces $X^{s,p}$ $(s \in (0,1), 1 < p < \infty)$ as the closure of $\mathcal{C}^\infty_c(\mathbb{R})$ with respect to the norm $\|u\| := \left(\|u\|_{L^p(\mathbb{R})} + \|D^s u\|_{L^p(\mathbb{R})}\right)^{1/p}$. Here Γ stands for the classical Gamma function [25]. Nevertheless, this definition is to be considered in a reflexive framework and will not be useful for our purpose since we need to choose $p = 1$ (see Section 3.) Moreover, we deal with bounded intervals and this definition should be applied only to functions with *prescribed* compact support (for example).

2.2. **Fractional integration and differentiation theory.** Another point of view to deal with fractional derivatives is the one we describe in the sequel: the archetypal definition is the one known as Riemann-Liouville, though there are many variants [16] that we do not consider. The point of view is different from the Gagliardo one. This viewpoint aims to a pointwise definition of derivatives by using fractional integrals while the Gagliardo's fractional Sobolev Spaces are defined by interpolation and global approach. As we already mentioned it, we decided to focus on this second type which seems more suitable with respect to applications.

2.2.1. **Fractional integrals.** From now on $[a,b]$ is a nonempty bounded interval of $\mathbb{R}$. We start by defining the fractional integral for L^1-functions:

Definition 2.2. *Let $u \in L^1([a,b])$. For every $s \in (0,1]$ we define the left-side and right-side Riemann-Liouville fractional integrals, by setting respectively*

$$\forall x \in [a,b] \qquad I^s_{a+}[u](x) = \frac{1}{\Gamma(s)} \int_a^x \frac{u(t)}{(x-t)^{1-s}} dt,$$

$$\forall x \in [a,b] \qquad I^s_{b-}[u](x) = \frac{1}{\Gamma(s)} \int_x^b \frac{u(t)}{(t-x)^{1-s}} dt.$$

The properties of left-side and right-side integrals are similar. In the sequel we list only the main results for the left-side integral I^s_{a+}.

The fractional integration theory has been extensively studied in [30]. Next proposition recall the main properties of the fractional integral

Proposition 2.1. *For any $s \in (0,1)$, the fractional integral I^s_{a+} is a continuous operator from*

(i) *$L^p(a,b)$ into $L^p(a,b)$ for every $p \geq 1$,*

(ii) $L^p(a,b)$ *into* $L^r(a,b)$ *for every* $p \in [1, 1/s)$ *and* $r \in [1, p/(1-sp))$,
(iii) $L^p(a,b)$ *into* $\mathcal{C}^{0,s-\frac{1}{p}}(a,b)$, *for every* $p > 1/s$
(iv) $L^p(a,b)$ *into* $L^r(a,b)$ *with* $r \in [1,\infty)$, *for* $p = 1/s$,
(v) $L^\infty(a,b)$ *into* $\mathcal{C}^{0,s}(a,b)$.

Moreover, with $p \geq 1$, *we have*

$$\forall u \in L^p(a,b) \qquad \lim_{s\to 0_+} \|I^s_{a+}u - u\|_{L^p(a,b)} = 0\,. \tag{2.3}$$

Here $\mathcal{C}^{0,s}(a,b)$ denotes the space of Hölder (continuous) functions of order s.

Remark 2.1 (**Fractional integral of BV-functions**). *We point out that to ensure the Hölder-regularity of the fractional integral we need to work with* L^p*-functions with* $p > 1$. *The case* $p = 1$ *is not covered from the previous proposition.*
However, the point (v) *guarantees such a regularity for bounded functions, which helps to study an important subset of* $L^1(a,b)$, *namely* $BV([a,b])$ *(see section 4). Indeed, in dimension one, every function of bounded variation is bounded ([5] chapter 10), so we get*

$$I^s_{a+}(BV([a,b])) \subset \mathcal{C}^{0,s}(a,b) \qquad \forall\, s \in (0,1)\,. \tag{2.4}$$

2.2.2. **Fractional derivatives and representability.** There are several different definitions of fractional derivatives. We recall next the definition of Riemann-Liouville and Marchaud derivatives and refer to [27, 30] for a deeper analysis of the fractional differentiation theory.

Definition 2.3 (**Riemann-Liouville fractional derivative**). *Let* $u \in L^1(a,b)$ *and* $n-1 \leq s < n$ *(*n *integer). The left Riemann-Liouville derivative of* u *at* $x \in [a,b]$ *is defined by*

$$D^s_{a+}u(x) = \frac{d^n}{dx^n} I^{n-s}_{a+}[u](x) = \frac{1}{\Gamma(n-s)} \frac{d^n}{dx^n} \int_a^x \frac{u(t)}{(x-t)^{s-n+1}} dt \tag{2.5}$$

at points where the classical derivative d^n/dx^n *exists.*
If such a derivative exists at x *for* $s = 0$, $n = 1$, *then it coincides with the function* u *at* x.
Similarly, we may define the right Riemann-Liouville derivative of u *at* $x \in [a,b]$ *as*

$$D^s_{b-}u(x) = \frac{d^n}{dx^n} I^{n-s}_{b-}[u](x) = \frac{1}{\Gamma(n-s)} \frac{d^n}{dx^n} \int_x^b \frac{u(t)}{(t-x)^{s-n+1}} dt \tag{2.6}$$

if the last term exists.

Example 2.1 (**Power function**). *We consider the function* $u(x) = x^k$ *(*$k \geq 0$*) on* $[0,1]$, *say* $a = 0$ *and* $b = 1$. *Then for every* $s \in [n-1, n)$ *and any* $x \in (0,1)$ *the fractional*

derivative at x is defined as

$$\begin{aligned} D^s_{0+}x^k &= \frac{1}{\Gamma(n-s)}\frac{d^n}{dx^n}\int_0^x t^k(x-t)^{n-s-1}dt \\ &= \frac{1}{\Gamma(n-s)}\frac{d^n}{dx^n}\left[x^{n+k-s}\int_0^1 (1-v)^{n-s-1}v^k dv\right] \\ &= \frac{1}{\Gamma(n-s)}\frac{d^n}{dx^n}\left[x^{n+k-s}B(n-s,k+1)\right] \\ &= \frac{1}{\Gamma(n-s)}\frac{\Gamma(k+1)\Gamma(n-s)}{\Gamma(k+1+n-s)}\frac{d^n}{dx^n}x^{n+k-s} = \frac{\Gamma(k+1)}{\Gamma(k-s+1)}x^{k-s} \end{aligned}$$

where we exploited the Beta Euler function $B(\nu,\mu)=\int_0^1 t^{\nu-1}(1-t)^{\mu-1}=\frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\nu+\mu)}$ and

$$\frac{d^n}{dx^n}x^\tau = \frac{\Gamma(\tau+1)}{\Gamma(\tau-n+1)}x^{\tau-n} \quad \forall\,\tau\geq 0\,.$$

If s is a positive integer number, then the fractional derivative $D^s_{0+}x^k$ coincides with the classical one.
If $k=0$, then the left fractional derivative of the constant function (say D^s_{0+}) is null if and only if s is a strictly positive integer number.
Remark also that

$$D^s_{0+}x^{s-k}=0 \qquad \forall\, s>0\,, k=1,...,1+[s]$$

where $[s]$ denotes the integer part of the real number s.

Now we focus on the case $n=1$.

Definition 2.4. (Representability) *A function $f\in L^1(a,b)$ is said to be representable, and it is represented by a fractional integral if $f\in I^s_{a+}(L^1(a,b))$ for some $s\in(0,1)$.*

Next Theorem [30] gives a representability criterion:

Theorem 2.1. (L^1-representability) *Assume $f\in L^1(a,b)$.*
Then $f\in I^s_{a+}(L^1(a,b))$ for a given $s\in(0,1]$ if and only if

$$I^{1-s}_{a+}[f]\in W^{1,1}(a,b) \quad \textit{and} \quad I^{1-s}_{a+}[f](a)=0\,.$$

Moreover, if $u\in L^1(a,b)$ is such that $f=I^s_{a+}[u]$ then $u=D^s_{a+}f$ a.e. on (a,b).

As an immediate consequence, we have the following result:

Corollary 2.1. *Assume $s\in(0,1)$;*

$$\forall u\in L^1(a,b) \qquad D^s_{a+}I^s_{a+}[u]=u, \tag{2.7}$$

and

$$\forall u\in I^s_{a+}(L^1(a,b)) \qquad I^s_{a+}D^s_{a+}u=u. \tag{2.8}$$

In the previous Corollary, (2.7) proves that fractional differentiation can be seen as the inverse operator of the fractional integration.
The converse is not true in general: a counterexample is given by the power function x^{s-k} ($k=1,...,1+[s]$) whose s-fractional derivative is null if and only if $k=0$ and $s>0$

(Example 2.1). This is similar to the classical integration and differentiation theories where the integral of u' differs from u for a constant. However, according to (2.8), for every function that can be represented as a fractional integral, the fractional integration acts as the reciprocal operator of the fractional differentiation.

Using Theorem 2.1, it is easy to verify that the power function x^{s-1} is not represented by a fractional integral. In fact $I_{a+}^{1-s}[x^{s-1}] \equiv \Gamma(s-1)$ so that it belongs to $W^{1,1}(a,b)$ but it does not verify $I_{a+}^{1-s}[x^{s-1}](0) = 0$.

Let us give a comment about the relationship between D_{a+}^s and I_{a+}^s. There are two kind of results on the fractional integral that can be very useful to study the properties of the fractional derivative :

- The first-one is a representability result (for instance Theorem 2.1) that gives conditions for a function f to be represented as the fractional integral of an other function u. This is quite important because it allows to easily prove that f admits a fractional derivative. However, the representability of a function u (i.e., $u = I_{a+}^s \varphi$ with $\varphi \in L^1$) is only a sufficient condition to get the existence of the derivative. The power function and the Heaviside function give two examples of functions that are not representable and whose fractional derivative exists (see previous discussion and Example 4.1).
- The second kind of result are embedding results, as (2.4), that give some informations on the regularity of the fractional integral to get the Riemann-Liouville fractional derivative existence.

2.2.3. **Marchaud derivative and representability for $p \in (1,\infty)$.** The representability result given by Theorem 2.1 can be improved by characterizing the set of the functions $u \in L^p(a,b)$ $(p>1)$ represented by another L^p-function

$$u = I_{a+}^s[f] \quad f \in L^p(a,b) \;\; s \in (0,1)\,.$$

In order to address this issue we need to introduce a slightly different definition/notion of fractional derivative.

According to [30] p. 110, we note that, for $\mathcal{C}^1$-functions and every $s \in (0,1)$, the use of integration by parts gives

$$D_{a+}^s u(x) = \frac{u(x)}{\Gamma(1-s)(x-a)^s} + \frac{s}{\Gamma(1-s)} \int_a^x \frac{u(x)-u(t)}{(x-t)^{1+s}}\, dt \quad \forall\, x \in (a,b]\,. \tag{2.9}$$

The *Marchaud fractional derivative* is defined as the second summand in the right-hand side term of (2.9). To extend this setting to non-smooth functions we need to define the integral by a limit, which leads to the following definition:

Definition 2.5 (**Marchaud fractional derivative**). *Let $u \in L^p(a,b)$ and $s \in (0,1)$. The left-side Marchaud derivative of u at $x \in (a,b]$ is defined by*

$$\mathbf{D}_{a+}^s u = \lim_{\varepsilon \to 0} \mathbf{D}_{a+,\varepsilon}^s u$$

in L^p with respect to the strong topology with

$$\mathbf{D}_{a+,\varepsilon}^s u(x) = \frac{u(x)}{\Gamma(1-s)(x-a)^s} + \frac{s}{\Gamma(1-s)}\, \psi_\varepsilon(x)$$

and

$$\psi_\varepsilon(x) = \begin{cases} \displaystyle\int_a^{x-\varepsilon} \frac{u(x)-u(t)}{(x-t)^{1+s}}\,dt & \text{if } x \geq a+\varepsilon\,, \\ \displaystyle\int_a^{x-\varepsilon} \frac{u(x)}{(x-t)^{1+s}}\,dt & \text{if } a \leq x \leq a+\varepsilon\,. \end{cases} \tag{2.10}$$

Note that the definition of ψ_ε for $a \leq x \leq a+\varepsilon$ is obtained by continuing the function u by zero beyond the interval $[a,b]$. The passage to the limit depends on the functional space we are working with. We remark that such a derivative is not defined at $x=a$ and that a necessary condition for the derivative to exist is $u(a)=0$

The right-side derivative can be defined similarly by using the integral between x and b. In the following we state the main results about Marchaud differentiation for the left-side derivative, but similar results can be obtained for the right-side one.

As expected, if u is a $\mathcal{C}^1$-function, the Marchaud and Riemann-Liouville derivative coincide for every $s \in (0,1)$, and their expression is given by (2.9):

$$\forall\, u \in \mathcal{C}^1([a,b]) \qquad D^s_{a+}u(x) = \mathbf{D}^s_{a+}u(x) \quad \forall\, x \in (a,b].$$

Next result generalizes Theorem 2.1 and Corollary 2.1 :

Theorem 2.2 ([30]-Theorems 13.1 -13.2)**.** *Let be $s \in (0,1)$.*

1. *For every $f = I^s_{a+}[u]$ where $u \in L^p(a,b)$ with $p \geq 1$, we get $\mathbf{D}^s_{a+}f = u$.*
2. *Let be $p \in (1,\infty)$. For any $f \in L^p(a,b)$, we get $f = I^s_{a+}[u]$ with $u \in L^p(a,b)$ if and only if the limit of the family $\{\psi_\varepsilon\}$ as $\varepsilon \to 0$, where ψ_ε is defined in (2.10) exists (for the L^p norm topology).*

Remark 2.2 (**Marchaud vs Riemann-Liouville derivative**)**.** *We point out that, for every $s \in (0,1)$, we have*

$$\forall\, u \in I^s_{a+}(L^1(a,b)), \quad \text{for a.e. } x \in (a,b] \quad D^s_{a+}u(x) = \mathbf{D}^s_{a+}u(x),$$

because of Theorems 2.1 and 2.2.This implies in particular that

$$\forall\, u \in \mathcal{C}^{0,s+\alpha}(a,b), \quad \text{for a.e. } x \in (a,b] \qquad D^s_{a+}u(x) = \mathbf{D}^s_{a+}u(x),$$

if $s+\alpha < 1$. This is a useful result in order to study the fractional derivative because the Marchaud derivative it is easier to handle.

3. Riemann-Liouville Fractional Sobolev space ($p=1$)

In this section, we define the Sobolev spaces associated to the Riemann-Liouville fractional derivative for $p=1$. The case $p=1$ is of particular interest since we aim to study the relationship between these spaces and the spaces of functions of bounded variation.

A first possible definition could be given by the following set for $s \in (0,1)$:

$$\left\{u \in L^1(a,b) \mid D^s_{a+}u \in L^1(a,b)\right\}\,,$$

which contains all the L^1-functions such that the Riemann-Liouville fractional derivative or order s for a given $s \in (0,1)$ belongs to L^1. We noticed that if the Riemann-Liouville fractional derivative of u exists for some s, then, referring to the same s, $I^s_{a+}[u]$ is differentiable almost everywhere. However, we have no information on the differential properties of the fractional integral. These differential properties are not completely described by

the pointwise derivative though it exists a.e. This shows that the previous definition is not suitable to obtain a generalized integration by parts formula.

Therefore, to develop a satisfactory theory of fractional Sobolev spaces we use a more suitable definition in the next section.

3.1. **Riemann-Liouville Fractional Sobolev spaces.** Following [8, 21] where these spaces are denoted $AC^{s,1}_{a^+}$, we may define the Riemann-Liouville Fractional Sobolev spaces as follows:

Definition 3.1. *Let $s \in [0,1)$. We denote by*

$$W^{s,1}_{RL,a+}(a,b) := \{u \in L^1(a,b) \mid I^{1-s}_{a+}[u] \in W^{1,1}(a,b)\ \}.$$

A similar space $W^{s,1}_{RL,b-}(a,b)$ can be defined by using the right-side fractional integral :

$$W^{s,1}_{RL,b-}(a,b) = \{u \in L^1(a,b) \mid I^{1-s}_{b-}[u] \in W^{1,1}(a,b)\ \}.$$

Short notations $W^{s,1}_{RL,a+}$, $W^{s,1}_{RL,b-}$ will be used whenever there is no risk of confusion. For any $s \in (0,1)$, the space $W^{s,1}_{RL,a+}$ is a Banach space endowed with the norm

$$\|u\|_{W^{s,1}_{RL,a+}} := \|u\|_{L^1(a,b)} + \|I^{1-s}_{a+}u\|_{W^{1,1}(a,b)}.$$

Note that this definition does not mean that u is representable but its fractional integral $f = I^{1-s}_{a+}u$ is representable. We shall describe the representable functions of $W^{s,1}_{RL,a+}$ next (it is related to their trace).

Remark 3.1. *If $u \in L^1(a,b)$ and $I^{s-1}_{a+}[u]$ is differentiable at every point with a derivative as an L^1-function, then $u \in W^{s,1}_{RL,a+}$. Indeed, in such case $I^{s-1}_{a+}[u] \in W^{1,1}(a,b)$ (see [28]).*

3.2. **Embedding properties.** Before performing comparisons between these fractional Sobolev spaces and the spaces of bounded variation functions, we investigate some embedding relations.

Theorem 3.1. *Assume $0 < s < r < 1$ and consider $u \in L^1(a,b) \cap I^r_{a+}(L^1(a,b))$. Then $u \in W^{s,1}_{RL,a+}(a,b)$ and*

$$\|u\|_{W^{s,1}_{RL,a+}(a,b)} \leq C_{s,r}\|u\|_{W^{r,1}_{RL,a+}(a,b)}\,.$$

Proof. Let be $u \in L^1(a,b) \cap I^r_{a+}(L^1(a,b))$. Then u is representable and Theorem 2.1 yields that $u \in W^{r,1}_{RL,a+}$. As a direct consequence of Theorem 31 in [21], we get that for every $0 < s < r \leq 1$ the embedding

$$W^{r,1}_{RL,a+} \subset W^{s,1}_{RL,a+}$$

is compact. Therefore $u \in W^{s,1}_{RL,a+}$ and (by continuity of the embedding) we get

$$\|u\|_{W^{s,1}_{RL,a+}} \leq C_{s,r}\|u\|_{W^{r,1}_{RL,a+}}\,.$$

Next theorem gives a relationship between Riemann-Liouville Sobolev spaces and Gagliardo Sobolev spaces:

Theorem 3.2. *Let be $s, r \in (0,1)$ such that $r > s$. Then*

$$W^{r,1}_G(a,b) \cap I^s_{a+}(L^1(a,b)) \subset W^{s,1}_{RL,a+}(a,b)$$

with continuous injection. More precisely,

$$\forall u \in W^{r,1}_G(a,b) \cap I^s_{a+}(L^1(a,b)) \qquad \|u\|_{W^{s,1}_{RL,a+}(a,b)} \leq C\|u\|_{W^{r,1}_G(a,b)}\,.$$

Proof. Let us choose $s \in (0,1)$ and $r > s$ (in $(0,1)$). Let be $u \in I^s_{a+}(L^1(a,b))$. It is represented by a fractional integral of a L^1-function, so its Riemann-Liouville and Marchaud derivative coincide. Due to Remark 2.2, the RL fractional derivative coincide with the Marchaud derivative for functions in the RL fractional Sobolev space of the same order. Therefore to achieve the claim, it will be enough showing that the Gagliardo norm of a fractional Sobolev space keeps under control the the L^1 norm of the Marchaud derivative of the same order. Recall that

$$\mathbf{D}^s_{a+,\varepsilon}u(x) = \frac{u(x)}{\varepsilon^s\Gamma(1-s)} \qquad \text{if } a \leq x \leq a+\varepsilon$$

and

$$\mathbf{D}^s_{a+,\varepsilon}u(x) = \frac{1}{\Gamma(1-s)}\frac{u(x)}{(x-a)^s} + \frac{s}{\Gamma(1-s)}\int_a^{x-\varepsilon}\frac{u(x)-u(t)}{(x-t)^{1+s}}\,dt \qquad \text{if } x \geq a+\varepsilon.$$

$$\begin{aligned}\|\mathbf{D}^s_{a+,\varepsilon}u\|_{L^1(a+\varepsilon,b)} \leq& \frac{1}{\Gamma(1-s)}\int_{a+\varepsilon}^b\frac{|u(x)|}{(x-a)^s}dx + \frac{s}{\Gamma(1-s)}\int_{a+\varepsilon}^b\int_a^{x-\varepsilon}\frac{|u(x)-u(t)|}{|x-t|^{1+s}}\,dt\,dx\\ \leq& \frac{1}{\Gamma(1-s)}\int_{a+\varepsilon}^b\frac{|u(x)|}{(x-a)^s}dx + \frac{s}{\Gamma(1-s)}[u]_{W^{s,1}_G(a,b)}.\end{aligned}$$

where the Gagliardo semi-norm $[u]_{W^{s,1}_G(a,b)}$ is given by (2.2). Moreover

$$\|\mathbf{D}^s_{a+,\varepsilon}u\|_{L^1(a,a+\varepsilon)} \leq \frac{1}{\varepsilon^s\Gamma(1-s)}\int_a^{a+\varepsilon}|u(x)|dx;$$

finally

$$\|\mathbf{D}^s_{a+,\varepsilon}u\|_{L^1(a,b)} \leq \frac{1}{\Gamma(1-s)}\left(\int_{a+\varepsilon}^b\frac{|u(x)|}{(x-a)^s}dx + \varepsilon^{-s}\int_a^{a+\varepsilon}|u(x)|ds + s[u]_{W^{s,1}_G(a,b)}\right). \quad (3.1)$$

Now, we know ([18] - section 6. for example) that

$$W^{r,1}_G(a,b) \subset L^p(a,b)$$

with continuous embedding for $p \in [1, \frac{1}{1-r}]$ and compact embedding if $p \in [1, \frac{1}{1-r})$.As $u \in W^{r,1}_G(a,b)$, then $u \in L^p(a,b)$ where p can be chosen such as $\frac{1}{1-s} < p < \frac{1}{1-r}$ since that $r > s$. Let us call $p' = p/(p-1)$ the conjugate exponent and apply Hölder inequalities to relation 3.1. Note that p' satisfies $1 - sp' > 0$. We get

$$\begin{aligned}\|\mathbf{D}^s_{a+,\varepsilon}u\|_{L^1(a,b)} \leq & \frac{\|u\|_{L^p(a+\varepsilon,b)}}{\Gamma(1-s)}\left[\frac{(b-a)^{1-sp'}-\varepsilon^{1-sp'}}{1-sp'}\right]^{1/p'} \\ & + \frac{\|u\|_{L^p(a,a+\varepsilon)}}{\Gamma(1-s)}\varepsilon^{(1-sp')/p'} \\ & + \frac{s}{\Gamma(1-s)}[u]_{W^{s,1}_G(a,b)} \\ \leq & \frac{\|u\|_{L^p}}{\Gamma(1-s)}\left(\left[\frac{(b-a)^{1-sp'}-\varepsilon^{1-sp'}}{1-sp'}\right]^{1/p'} + \varepsilon^{(1-sp')/p'}\right) \\ & + \frac{s}{\Gamma(1-s)}[u]_{W^{s,1}_G(a,b)}\ .\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ gives

$$\|\mathbf{D}^s_{a+}u\|_{L^1(a,b)} \leq \frac{\|u\|_{L^p}}{\Gamma(1-s)}\left(\frac{(b-a)^{(1-sp')}}{1-sp'}\right)^{1/p'} + \frac{s}{\Gamma(1-s)}[u]_{W^{s,1}_G(a,b)}\ .$$

As $\|u\|_{L^p} \leq C\|u\|_{W^{r,1}_G(a,b)}$ and

$$[u]_{W^{s,1}_G(a,b)} \leq [u]_{W^{r,1}_G(a,b)} \leq \|u\|_{W^{r,1}_G(a,b)}$$

we finally get

$$\|\mathbf{D}^s_{a+}u\|_{L^1(a,b)} \leq C(s,r,a,b)\|u\|_{W^{r,1}_G(a,b)}\ .$$

This ends the proof.

Remark 3.2. (1) *Note that for every $s,r \in (0,1)$ with $s<r$, the function $u: x \mapsto \frac{(x-a)^{s+1}}{\Gamma(s+2)}$ belongs to $W^{r,1}_G(a,b) \cap I^s_{a+}(L^1(a,b))$ so that this space is not reduced to $\{0\}$. Indeed u is a $W^{1,1}(a,b)$ function and lso a $W^{r,1}_G(a,b)$ function since $W^{1,1}(a,b) \subset W^{r,1}_G(a,b)$. Moreover a short computation similar to the one of example 2.1 gives that $u = I^s_{a+}(x-a)$ which proves that $u \in I^s_{a+}(L^1(a,b))$.*

(2) *Note that $s=r$ is the critical case in the above theorem. We cannot handle this case with the same arguments.*

(3) *The equality issue is an open question. In other terms, can we find a function in $W^{s,1}_{RL,a+}$ and not in $W^{s,1}_G(a,b) \cap I^s_a(L^1(a,b))$?*

Next, we investigate the relationship between $W^{1,1}(a,b)$ and the fractional Sobolev spaces $W^{s,1}_{RL,a+}$. A crucial tool is the following integration by parts formula.

Proposition 3.1 ([8]). *If $0 \leq \frac{1}{p} < s < 1$ and $0 \leq \frac{1}{r} < s < 1$, then for every $u \in W^{s,1}_{RL,a+}$ and $v \in W^{r,1}_{RL,b-}$ we get*

$$\int_a^b (D^s_{a+}u)(t)v(t)\,dt = \int_a^b (D^s_{b-}v)(t)u(t)\,dt + u(b)\,I^{1-s}_{b-}[v](b) - I^{1-s}_{a+}[u](a)\,v(a)\ . \tag{3.2}$$

Example 3.1 (Smooth functions). *Set $[a,b]=[0,1]$. For every $u\in C^\infty([0,1],\mathbb{R})$ and for every $s\in[0,1)$ we get for any $x\neq 0$, using integration by parts twice:*

$$\begin{aligned} D^s_{0+}u(x) &= \frac{1}{\Gamma(1-s)}\frac{d}{dx}\int_0^x u(t)(x-t)^{-s}dt \\ &= \frac{1}{(1-s)\Gamma(1-s)}\frac{d}{dx}\left[u(0)x^{1-s}+\int_0^x u'(t)(x-t)^{1-s}dt\right] \\ &= \frac{1}{\Gamma(1-s)}\left[u(0)x^{-s}+\int_0^x u'(t)(x-t)^{-s}dt\right] \\ &= \frac{1}{\Gamma(1-s)}u(0)x^{-s}+\frac{1}{\Gamma(2-s)}u'(0)x^{1-s}+\frac{1}{\Gamma(2-s)}\int_0^x u''(t)(x-t)^{1-s}dt \end{aligned}$$

since $\Gamma(z+1)=z\Gamma(z)$ for every $z>0$. As $x\mapsto x^{-s}$ belongs to $L^1(0,1)$ and u'' is bounded and $s\in[0,1)$, we get that $D^s_{0+}u$ belongs to $L^1(0,1)$. Moreover, we have

$$\begin{aligned} \|u'-D^s_{0+}u-u(0)\|_{L^1([0,1],\mathbb{R})} &= \int_0^1\left|u'(x)-\frac{u(0)x^{-s}}{\Gamma(1-s)}-\frac{u'(0)x^{1-s}}{\Gamma(2-s)}-\int_0^x\frac{u''(t)(x-t)^{1-s}}{\Gamma(2-s)}dt-u(0)\right|dx \\ &\le |u(0)|\int_0^1\left|1-\frac{x^{-s}}{\Gamma(1-s)}\right|dx + \\ &\quad \int_0^1\left|u'(x)-\frac{u'(0)x^{1-s}}{\Gamma(2-s)}-\int_0^x\frac{u''(t)(x-t)^{1-s}}{\Gamma(2-s)}dt\right|dx \\ &\le |u(0)|\left|1-\frac{1}{\Gamma(2-s)}\right| + \\ &\quad \int_0^1\left|u'(x)-\frac{u'(0)x^{1-s}}{\Gamma(2-s)}-\int_0^x\frac{u''(t)(x-t)^{1-s}}{\Gamma(2-s)}dt\right|dx. \end{aligned}$$

Indeed, we observe that $x\mapsto 1-\frac{x^{-s}}{\Gamma(1-s)}$ does not vanish on $(0,1)$ so that

$$\int_0^1\left|1-\frac{x^{-s}}{\Gamma(1-s)}\right|dx=\left|\int_0^1(1-\frac{x^{-s}}{\Gamma(1-s)}\,dx\right|=\left|1-\frac{1}{\Gamma(2-s)}\right|.$$

Since $\lim_{s\to 1}\Gamma(2-s)=\Gamma(1)=1$, by the Lebesgue convergence theorem, we get

$$\|u'-D^s_{0+}u-u(0)\|_{L^1([0,1],\mathbb{R})}\overset{s\to 1}{\longrightarrow}\int_0^1\left|u'(x)-u'(0)-\int_0^x u''(t)dt\right|dx=0.$$

Moreover, if $u(0)=0$ then $D^s_{0+}u$ is well defined at $x=0$ and

$$\forall x\in[0,1]\qquad \lim_{s\to 1}D^s_{0+}(u)(x)=u'(x),$$

and not only almost everywhere.

Now, we compare the usual Sobolev space $W^{1,1}(a,b)$ and the Riemann-Liouville fractional Sobolev spaces. We already know that $W^{1,1}(a,b)$ is compactly embedded in $W^{s,1}_{RL,a+}$ for any $s\in[0,1)$ thanks to Theorem 31 of [21] again. Now, we investigate the behavior of $D^s_{a+}u$ with respect to s as $s\to 1$.
In the sequel $\mathcal{M}_{a,b}$ is the space of (finite) Radon measures on $\mathbb{R}$ with support contained

in $[a,b]$, that can be identified to the dual of $\mathcal{C}([a,b])$ the space of continuous functions with support in $[a,b]$ The notation $\langle\cdot,\cdot\rangle$ stands for the duality product between $\mathcal{M}_{a,b}$ and $\mathcal{C}([a,b])$. We recall that a family of Radon measures μ_ε weakly converges to μ in $\mathcal{M}_{a,b}$ (notation $\mu_\varepsilon \rightharpoonup \mu$) as $\varepsilon \to 0$ if

$$\forall \varphi \in \mathcal{C}([a,b]) \qquad \lim_{\varepsilon\to 0} \mu_\varepsilon(\varphi) = \mu(\varphi)\ .$$

Next $\mathcal{L}^1$ denotes the 1D Lebesgue measure and δ_a the Dirac measure at a say

$$\forall \varphi \in \mathcal{C}(\mathbb{R}) \qquad \langle \delta_a, \varphi\rangle = \varphi(a)\ .$$

We recall that any $L^p(a,b)$ function f may be considered as a measure by the identification $f \mapsto f\mathcal{L}^1$ with

$$\forall \varphi \in \mathcal{C}([a,b]) \qquad \left\langle f\mathcal{L}^1, \varphi\right\rangle = \int_{\mathbb{R}} \varphi(x) f(x)\, dx\ .$$

Theorem 3.3. *Assume that $u \in W^{1,1}(a,b)$. Then, $\dot{u} = u'$ and*

$$D_a^s u \overset{\mathcal{M}_{a,b}}{\rightharpoonup} u'\mathcal{L}^1 + u(a)\delta_a \quad as \quad s \to 1.$$

In addition, if $u(a) = 0$, then we have

$$D_{a+}^s u \overset{L^1(a,b)}{\longrightarrow} u' \quad as \quad s \to 1.$$

Proof. Let be $u \in W^{1,1}(a,b)$. Then $u \in W^{s,1}_{RL,a+}$ for any $s \in [0,1)$. Moreover u is defined everywhere since $W^{1,1}(a,b)$ is continuously embedded in $\mathcal{C}([a,b])$(see [1]). Integration by parts provide the next representation for fractional integral and derivative:

$$I_{a+}^{1-s}[u](x) = \frac{1}{\Gamma(2-s)}\left[u(a)\,(x-a)^{1-s} + \int_a^x u'(t)(x-t)^{1-s}dt\right] \tag{3.3}$$

and

$$\begin{aligned} D_{a+}^s u(x) &= \frac{1}{\Gamma(1-s)}\left[u(a)\,(x-a)^{-s} + \int_a^x u'(t)(x-t)^{-s}dt\right] \\ &= \frac{u(a)(x-a)^{-s}}{\Gamma(1-s)} + I_{a+}^{1-s}[u'](x)\,. \end{aligned} \tag{3.4}$$

As $s \in (0,1)$ we get that the fractional derivative is defined almost everywhere on $(a,b]$ and belongs to $L^1(a,b)$ with

$$\|D_{a+}^s u\|_{L^1(a,b)} \leq \frac{|u(a)|(b-a)^{1-s}}{\Gamma(2-s)} + \|I_{a+}^{1-s}[u']\|_{L^1}(a,b) \leq C\left(|u(a)| + \|u'\|_{L^1}(a,b)\right). \tag{3.5}$$

Now, as $u' \in L^1(a,b)$ then by relation (2.3) we get the strong convergence of $I_a^{1-s}u'$ to u' in $L^1(a,b)$ as $s \to 1$. Therefore we get the result in the case where $u(a) = 0$ using (3.4).

Moreover, for every $\varphi \in \mathcal{C}^1(a,b)$

$$\begin{aligned} \frac{u(a)}{\Gamma(1-s)}\int_a^b (x-a)^{-s}\varphi(x)dx = \ & \frac{u(a)}{\Gamma(2-s)}\left[\varphi(b)(b-a)^{1-s} - \int_a^b (x-a)^{1-s}\varphi'(x)dx\right] \\ & \overset{s\to 1}{\to} u(a)\varphi(a). \end{aligned}$$

So

$$\left\langle D^s_{a+}u,\varphi\right\rangle = \frac{u(a)}{\Gamma(1-s)}\int_a^b (x-a)^{-s}\varphi(x)dx + \int_a^b I_a^{1-s}[u'](x)\varphi(x)\,dx$$
$$\to u(a)\varphi(a) + \int_a^b u'(x)\varphi(x)\,dx = \left\langle u(a)\delta_a + u'\mathcal{L}^1,\varphi\right\rangle .$$

We conclude with the density of the $\mathcal{C}^1$ functions in the space of continuous functions: let be $\varphi\in\mathcal{C}(a,b)$ and $\varphi_n\in\mathcal{C}^1$ such that $\|\varphi_n-\varphi\|_\infty\to 0$. With (3.5) we get for every $s\in(0,1)$

$$|\left\langle D^s_{a+}u,\varphi-\varphi_n\right\rangle| \le \|D^s_{a+}u\|_{L^1}\|\varphi_n-\varphi\|_\infty \le C\left(|u(a)|+\|u'\|_{L^1}(a,b)\right)\|\varphi_n-\varphi\|_\infty \overset{n\to+\infty}{\to} 0.$$

Similarly

$$|\left\langle u(a)\delta_a + u'\mathcal{L}^1,\varphi-\varphi_n\right\rangle| \le \left(|u(a)|+\|u'\|_{L^1}(a,b)\right)\|\varphi_n-\varphi\|_\infty \overset{n\to+\infty}{\to} 0\ .$$

Let be $\varepsilon>0$. There exists $n_0\in\mathbb{N}$ such that

$$\forall n\ge n_0,\ \forall s\in(0,1)\qquad |\left\langle D^s_{a+}u,\varphi-\varphi_n\right\rangle| + |\left\langle u(a)\delta_a+u'\mathcal{L}^1,\varphi-\varphi_n\right\rangle| \le \frac{\varepsilon}{2}\ .$$

Fix $n^*\ge n_0$. As $\varphi_{n^*}\in\mathcal{C}^1(a,b)$, there exists $\eta^*>0$ such that

$$\forall s \text{ such that } 1-s\le\eta^*\qquad |\left\langle D^s_{a+}u-u(a)\delta_a+u'\mathcal{L}^1,\varphi_{n^*}\right\rangle| \le \frac{\varepsilon}{2}.$$

Finally, as

$$|\left\langle D^s_{a+}u-u(a)\delta_a-u'\mathcal{L}^1,\varphi\right\rangle| \le |\left\langle D^s_{a+}u,\varphi-\varphi_{n^*}\right\rangle| + |\left\langle u(a)\delta_a+u'\mathcal{L}^1,\varphi-\varphi_{n^*}\right\rangle|$$
$$+\left\langle D^s_{a+}u-u(a)\delta_a-u'\mathcal{L}^1,\varphi_{n^*}\right\rangle .$$

we get for every s such that $1-s\le\eta^*$

$$|\left\langle D^s_{a+}u-u(a)\delta_a+u'\mathcal{L}^1,\varphi\right\rangle| \le \varepsilon\ .$$

This proves that

$$\forall\varphi\in\mathcal{C}(a,b)\qquad \left\langle D^s_{a+}u,\varphi\right\rangle \to \left\langle u(a)\delta_a+u'\mathcal{L}^1,\varphi\right\rangle .$$

Remark 3.3. *Previous Theorem shows that, even if u is a smooth function,its fractional derivative cannot converge in L^1 to u' whenever $u(a)\neq 0$, since some of the its mass must concentrates at $x=a$. The simplest example is provided by the constant function $u\equiv 1$ in $(0,1)$. Then*

$$u'\equiv 0\ ,\qquad D^s_{a+}u(x) = \frac{(x-a)^{-s}}{\Gamma(1-s)}$$

and

$$\|D^s_{a+}u\|_{L^1(a,b)} = \frac{(b-a)^{1-s}}{\Gamma(2-s)} \overset{s\to 1}{\longrightarrow} 1 \neq \|u'\|_{L^1(a,b)}\ .$$

4. **Comparison with BV and SBV**

Let us recall the definition and the main properties of the space of functions of bounded variation (see [3, 5] for example), defined by

$$BV(a,b) = \{u \in L^1(a,b) \mid TV(u) < +\infty\},$$

where (a,b) is a bounded, open subset of $\mathbb{R}$,

$$TV(u) := \sup\left\{\int_a^b u(x)\,\xi'(x)\,dx \mid \xi \in \mathcal{C}_c^1(a,b),\ \|\xi\|_\infty \le 1\right\}. \tag{4.1}$$

and $\mathcal{C}_c^1(a,b)$ denotes the space of functions of class $\mathcal{C}^1$ with compact support in (a,b). The space $BV(a,b)$, endowed with the norm $\|u\|_{BV(a,b)} = \|u\|_{L^1} + TV(u)$, is a Banach space. The derivative in the sense of distributions of every $u \in BV(a,b)$ is a bounded Radon measure, denoted u', and $TV(u)$ is the total variation of u. For more details, one can refer to [3].

Proposition 4.1 ([3])**.** *Let (a,b) be a bounded open subset of $\mathbb{R}$*

(1) *For any $u \in BV(a,b)$, the total variation coincides with the essential pointwise variation. Precisely,*

$$TV(u) = \inf_{v=u \ a.e.} \left\{\sup_{\mathcal{T}} \left(\sum_i |v(t_{i+1}) - v(t_i)|\right)\right\}$$

where $\mathcal{T}$ stands for any subdivision $\mathcal{T} = \{t_0 = a < t_1 < \cdots < t_n = b\}$ of (a,b) .

(2) *For every $u \in BV(a,b)$, the Radon measure u' can be decomposed into $u' = \dot{u}\,dx + (\dot{u})^\perp$, where $\dot{u}\,dx$ is the absolutely continuous part of u' with respect of the Lebesgue measure and $(\dot{u})^\perp$ is the singular part.*

(3) *The mapping $u \mapsto TV(u)$ is lower semi-continuous from $BV(a,b)$ to $\mathbb{R}^+$ for the $L^1(a,b)$ topology.*

(4) *$BV(a,b) \subset L^\sigma(a,b)$ with continuous embedding, for $\sigma \in [1,\infty]$.*

(5) *$BV(a,b) \subset L^\sigma(a,b)$ with compact embedding, for $\sigma \in [1,\infty)$.*

The singular part $(\dot{u})^\perp$ of the derivative has a jump part and a Cantor component. The $SBV(a,b)$ space (see [3] for example) is the space of functions in $BV(a,b)$ whose derivative has no singular Cantor component. The functions of $SBV(a,b)$ have two components: one is regular and belongs to $W^{1,1}(a,b)$ and the other one is a countable summation of characteristic functions. More precisely, any increasing function in $SBV(a,b)$ can be written as

$$u(x) = u(a) + \int_a^x \dot{u}(t)\,dt + \sum_{x_k \in J_u} p_k \chi_{[x_k,1]}(x) \qquad x \in [a,b]$$

where J_u denotes the (at most countable) set of jump points of u and $p_k = u^+(x_k) - u^-(x_k)$ denotes the positive jump of u at x_k. This describes all the functions of $SBV(a,b)$ since any SBV-function can be written as the difference of two SBV increasing functions. Indeed, any SBV-function can be written as the difference of two BV increasing functions. These functions are obviously in SBV thanks to the unique decomposition (up to additive constant) of the derivative ([3]-Corollary 3.33).

Next example shows that there exists a SBV-function that belongs to $W^{s,1}_{RL,a+}$ for any $s \in (0,1)$. This confirms the regularizing behavior of the fractional integral operator and represents a preliminary result in order to prove the relationship between SBV function and fractional Sobolev space.

Example 4.1 (Step functions). *Let $u : [0,1] \to \mathbb{R}$ $(a = 0)$, $u(x) = \chi_{[\alpha,1]}(x)$ with $\alpha \in (0,1)$. We consider $s \in [0,1)$. For every $x \in [\alpha,1]$ we get*

$$I^{1-s}_{0+}[\chi_{[\alpha,1]}](x) = \begin{cases} 0 & \text{if} \quad x \in [0,\alpha) \\ \dfrac{(x-\alpha)^{1-s}}{\Gamma(2-s)} & \text{if} \quad x \in [\alpha,1] \end{cases} \tag{4.2}$$

which proves that $I^{1-s}_{0+}[\chi_{[\alpha,1]}] \in W^{1,1}(0,1)$ so that $\chi_{[\alpha,1]} \in W^{s,1}_{RL,0+}(0,1)$.

The fractional derivative is given by

$$D^s_{0+}\chi_{[\alpha,1]}(x) = \begin{cases} 0 & \text{if} \quad x \in [0,\alpha] \\ \dfrac{(x-\alpha)^{-s}}{\Gamma(1-s)} & \text{if} \quad x \in (\alpha,1] \end{cases} \tag{4.3}$$

and, for every $s \in (0,1)$, we have that

$$\int_\alpha^1 \frac{(x-\alpha)^{-s}}{\Gamma(1-s)} = \left[\frac{(x-\alpha)^{1-s}}{\Gamma(2-s)}\right]_\alpha^1 = \frac{(1-\alpha)^{1-s}}{\Gamma(2-s)}$$

which implies that

$$\|D^s_{0+}u\|_{L^1([0,1])} \overset{s\to 1}{\longrightarrow} 1 = TV(u) \tag{4.4}$$

where $TV(u)$ denotes the total variation of u on $(0,1)$.

Next result is useful to prove Theorem 4.1:

Lemma 4.1. *Let $\{f_k\} \subset W^{1,1}(a,b)$ a sequence of non-negative functions with $f_k(a) = 0$ and non-negative derivative. We suppose also that*

$$\sum_k f_k\,,\ \sum_k f_k' \in L^1(a,b)\,. \tag{4.5}$$

Then

$$\left(\sum_k f_k(x)\right)' = \sum_k f_k'(x) \qquad a.e.\ x \in [a,b]\,. \tag{4.6}$$

Proof. The result follows from the monotone convergence theorem and the hypothesis on f_k. In fact, for every $x \in [a,b]$ we get

$$\sum_k f_k(x) = \sum_k \int_a^x f_k'(t)\,dt = \int_a^x \sum_k f_k'(t)\,dt\,,$$

which proves that $\sum_k f_k \in W^{1,1}(a,b)$ and (4.6) follows.

Theorem 4.1. *For every $s \in (0,1)$, it holds that $SBV(a,b) \subset W^{s,1}_{RL,a+}(a,b)$ and*

$$D^s_{a+}u \overset{\mathcal{M}_{a,b}}{\rightharpoonup} \dot{u}\,\mathcal{L}^1 + u(a^+)\delta_a + \sum_{x_k \in J_u} p_k \delta_{x_k} \quad as \quad s \to 1\,.$$

Moreover, if $u(a^+) = 0$ *then*

$$\|D^s_{a+}u\|_{L^1(a,b)} \to TV(u) \quad as \quad s \to 1\,.$$

Here $u(a^+)$ *denotes the right limit of* u *at zero and* $J_u \subset (a,b)$ *is the jump set of* u*.*

Proof. With a simple change of variables, we can assume that $[a,b] = [0,1]$. This will make the proof easier to read. As every SBV-function can be written as the difference of two increasing SBV functions, we prove the result for a SBV-increasing function. Such a function u can be written as

$$u(x) = u(0^+) + \int_0^x \dot{u}(t)\,dt + \sum_{x_k \in J_u} p_k \chi_{[x_k,1]}(x) \qquad a.e.\, x \in [0,1]$$

where $u(0^+)$ denotes the right limit of u at zero, J_u denotes the (at most countable) set of jump points of u and $p_k = u^+(x_k) - u^-(x_k)$ denotes the positive jump of u at x_k. In particular

$$TV(u) = \int_0^1 |\dot{u}(t)|\,dt + \sum_{x_k \in J_u} p_k\,.$$

The function u can be written as the sum of two functions $u = u_a + u_j$ where u_a is the absolutely continuous part ($u_a \in W^{1,1}(0,1)$) and u_j is the jump part. They are given by:

$$u_a(x) = u(0^+) + \int_0^x \dot{u}(t)\,dt \text{ a.e. } x \in [0,1]\,,$$

$$u_j = \sum_{x_k \in J_u} p_k \chi_{[x_k,1]}\,.$$

• We first consider u_a. Since u_a belongs to $W^{1,1}(a,b)$, for every $s \in (0,1)$ it belongs to $W^{s,1}_{RL,a+}$ and its fractional derivative is given by (3.4):

$$D^s_{0+}(x) = \frac{u(0^+)x^{-s}}{\Gamma(1-s)} + I^{1-s}_{0+}[\dot{u}](x) \text{ a.e. } x \in [0,1].$$

Moreover with Theorem 3.3 we have

$$\begin{array}{lll} \text{if } u(0^+) = 0 & D^s_{0+}u \overset{L^1(0,1)}{\longrightarrow} \dot{u} & \text{as} \quad s \to 1\,, \\ \text{otherwise} & D^s_{0+}u \overset{\mathcal{M}_{0,1}}{\rightharpoonup} \dot{u}\,\mathcal{L}^1 + u(0^+)\delta_0 & \text{as} \quad s \to 1\,. \end{array} \tag{4.7}$$

• We consider now u_j with $s \in (0,1)$. We use Example 4.1 and set

$$f_k = I^{1-s}_{0+}[p_k \chi_{[x_k,1]}] = \begin{cases} 0 & \text{if} \quad x \in [0,x_k) \\ p_k \dfrac{(x-x_k)^{1-s}}{\Gamma(2-s)} & \text{if} \quad x \in [x_k,1] \end{cases}$$

and

$$f_k'(x) = \begin{cases} 0 & \text{if} \quad x \in [0,x_k) \\ p_k \dfrac{(x-x_k)^{-s}}{\Gamma(1-s)} & \text{if} \quad x \in [x_k,1]. \end{cases}$$

Functions f_k and f_k' are L^1 for every k and $f_k(0)=0$. Moreover, we get

$$\sup_n \left(\int_0^1 \sum_{k\le n} f_k(t)\,dt \right) \le \sup_n \sum_{k\le n} \frac{p_k}{\Gamma(3-s)} \le \frac{TV(u)}{\Gamma(3-s)} \ ,$$

and

$$\sup_n \left(\int_0^1 \sum_{k\le n} f_k'(t)\,dt \right) \le \sup_n \sum_{k\le n} \frac{p_k}{\Gamma(2-s)} \le \frac{TV(u)}{\Gamma(2-s)} \ .$$

So, with the monotone convergence theorem, we have that the series

$$I_{0+}^{1-s}u_j = \sum_{x_k\in J_u} f_k \text{ and } \sum_{x_k\in J_u} f_k'$$

are normally convergent in $L^1(0,1)$. Using Lemma 4.1 we get

$$(I_{0+}^{1-s}u_j)'(x) = \sum_{x_k\in J_u} p_k \frac{(x-x_k)^{-s}}{\Gamma(1-s)}\chi_{[x_k,1]} \ a.e.\, x\in(0,1).$$

This means that $I_{0+}^{1-s}u_j \in W^{1,1}(0,1)$ and consequently, $u_j \in W^{s,1}_{RL,a+}$. So we may conclude that $SBV(a,b)\subset W^{s,1}_{RL,a+}$.
It remains to prove the weak* convergence of $D^s_{0+}u_j$ towards $\sum_{x_k\in J_u} p_k\delta_{x_k}$. Using again the monotone convergence theorem one gets

$$D^s_{0+}u_j(x) = \sum_{x_k\in J_u} p_k \frac{(x-x_k)^{-s}}{\Gamma(1-s)}\chi_{[x_k,1]} \ \text{ a.e. } x\in(0,1).$$

Let $\varphi\in\mathcal{C}^1(0,1)$ with compact support.

$$\left\langle D^s_{0+}u_j,\varphi\right\rangle = \int_0^1 D^s_{0+}u_j(x)\varphi(x)\,dx = \sum_{x_k\in J_u}\frac{p_k}{\Gamma(1-s)}\int_0^1 (x-x_k)^{-s}\chi_{[x_k,1]}(x)\varphi(x)\,dx \ .$$

Here, we used again that the series is normally convergent. An integration by parts gives

$$\frac{1}{\Gamma(1-s)}\int_0^1 (x-x_k)^{-s}\chi_{[x_k,1]}(x)\varphi(x)\,dx = -\frac{1}{\Gamma(2-s)}\int_{x_k}^1 (x-x_k)^{1-s}\varphi'(x)\,dx.$$

So the Lebesgue dominating theorem gives that

$$\lim_{s\to1}\frac{1}{\Gamma(1-s)}\int_0^1 (x-x_k)^{-s}\chi_{[x_k,1]}(x)\varphi(x)\,dx = -\int_{x_k}^1 \varphi'(x)\,dx = \varphi(x_k) \ .$$

We finally obtain, always with the same tools, that

$$\lim_{s\to1}\left\langle D^s_{0+}u_j,\varphi\right\rangle = \sum_{x_k\in J_u} p_k\varphi(x_k) = \left\langle \sum_{x_k\in J_u} p_k\delta_k,\varphi\right\rangle.$$

We conclude by density as in the proof of Theorem 3.3.

Remark 4.1. *We may note that if we extend the functions u with support in $[a,b]$ by 0 below a and denote them similarly, we may consider $(-\infty,b]$ instead of $[a,b]$. Then the appearance of the Dirac measure at a is consistent with the distributional derivative of u on $(-\infty,b)$ since there is a jump at a.*

The subsequent remarks point out that the fractional Sobolev spaces are larger than SBV and give some relationship between BV and $W^{s,1}$.

Remark 4.2 (**Cantor-Vitali function**). *The Cantor-Vitali function is an example of increasing continuous function on $[0,1]$ whose classical derivative is defined and null at a.e. point. It is well known that such a function is of bounded variation but is not a SBV-function. Precisely such a function is Hölder-continuous with exponent $\alpha = \ln 2/\ln 3$ (i.e., the Hausdorff dimension of the Cantor set). Then, the Riemann-Liouville derivative is well defined at every point for every $s \in (0,\alpha)$. In fact, it is surely possible to prove the result for any $s \in (0,1)$ using an adapted Cantor-like function whose Hölder exponent would be s. Moreover the Riemann-Liouville derivative of order s belongs to $\mathcal{C}^{0,\alpha-s}([0,1])$, and we get that the Cantor-Vitali function belongs to $W^{s,1}_{RL,0+}([0,1])$ for every $s \in (0,\alpha)$. This proves in particular that in general*

$$(BV \setminus SBV) \cap W^{s,1}_{RL,0+} \neq \emptyset\,,$$

since the function we exhibit belongs to BV and $W^{s,1}_{RL,0+}$ and not to SBV.

Remark 4.3 (**A continuous but non Hölder-continuous function in RL spaces**). *Set $u(x) = \big(\ln(x/2)\big)^{-1}$ if $x \in (0,1)$ and $u(0) = 0$. This u provides an example of monotone, continuous function which is not α Hölder-continuous for any $\alpha \in (0,1)$, but u belongs to $\bigcap_{s\in(0,1)} W^{s,1}$.*

Remark 4.4 (**Relationship between BV and $W^{s,1}_{RL,0+}$**). *In [27] the authors investigate the relationship between usual a.e. differentiation and the fractional Riemann-Liouville derivative definition. Several interesting examples are given. One of them is given by the Weierstrass function defined as*

$$W(x) = \sum_{n=0}^{\infty} q^{-n}(e^{iq^n x} - e^{iq^n a}) \qquad x \in [a,b]$$

where $q > 1$. It is proved that W has continuous and bounded fractional Riemann-Liouville derivatives of all orders $s < 1$. However, since W is nowhere differentiable it cannot be of bounded variation.

This implies that Riemann-Liouville fractional Sobolev spaces are not contained in $BV([a,b])$. Then we can state

$$SBV \subset \bigcap_{s\in(0,1)} W^{s,1}\,, \qquad \bigcap_{s\in(0,1)} W^{s,1} \setminus BV \neq \emptyset\,.$$

The question to be addressed now is the relation between BV and $W^{s,1}_{RL,0+}$. Indeed, we have either $BV \subset W^{s,1}_{RL,0+}$ or they are completely different spaces whose intersection contains SBV (the case $W^{s,1}_{RL,0+} \subset BV$ is excluded).

5. Conclusion

In this paper we try to make connections between the two main definitions of fractional derivatives : the *pointwise* one whose typical representation is the RL derivative and the *global* one which is typically the Gagliardo one. In view of a more precise description of the derivative of order $s \in (0,1)$ with respect to BV functions we have also proved preliminary results to compare $SBV(a,b)$ and $W^{s,1}_{RL,a+}$. Open problems are numerous. In particular, it remains to strongly connect the Riemann-Liouville theory with the Gagliardo one. This would allow to perform comparison between $W^{s,1}_{RL,a+}$ and Besov-spaces for example. In addition, we have to understand precisely how $W^{s,1}_{RL,a+}$ behaves with respect to $BV(a,b)$ to get some information about the $BV \backslash W^{1,1}$ functions. In particular, we proved that $SBV(a,b) \subset W^{s,1}_{RL,a+}$ and exhibit a function in $W^{s,1}_{RL,a+}$, that does not belong to $BV(a,b)$; however we still don't know if $BV(a,b) W^{s,1}_{RL,a+}$. In addition, it remains to prove density results and continuity/compactness results in view of variational models involving the RL derivative.

Another important issue is also to address engineering and/or imaging problems in a rigorous mathematical framework. From that point of view, the fractional derivative concept widely used in engineering is the Caputo one :

Definition 5.1 (**Caputo fractional derivative**). [2, 11] *Let $u \in L^1(a,b)$ and $n-1 \leq s < n$ (where n integer).*
The left Caputo fractional derivatives of u at $x \in [a,b]$ is defined by

$$ {}^{C}D^{s}_{a+}u(x) \;=\; I^{n-s}_{a+}\left[\frac{d^n}{dx^n}u\right](x) = \frac{1}{\Gamma(n-s)}\int_a^x \frac{u^{(n)}(t)}{(x-t)^{s-n+1}}dt \tag{5.1} $$

when the right-hand side is defined. The right Caputo fractional derivatives of u at $x \in [a,b]$ is defined in a similar way.

The main advantage of Caputo derivatives, which makes them the preferred ones in many engineering applications, is the fact that the initial conditions for fractional differential equations with Caputo derivatives are expressed by integer derivatives at time 0, say quantities with a straightforward physical interpretation. This relies on the Laplace transform of Caputo fractional derivative, formally identical to the classical formula for integer derivatives (in contrast to the formula for RL fractional derivatives):

$$ \mathcal{L}\left\{ {}^{C}D^{\alpha}_{a+}u \right\}(s) \;=\; s^{\alpha}\,\mathcal{L}\{u\} \;-\; \sum_{j=0}^{n-1} s^{\alpha-j-1}u^{j}(0) \qquad n-1 < \alpha \leq n\,. $$

The connection between Riemann-Liouville and Caputo derivatives, when they both exist, is given by the relationship (see[2] for example):

$$ {}^{C}D^{\alpha}_{a+}u(x) \;=\; D^{\alpha}_{a+}u(x) \;-\; \sum_{j=0}^{n-1} \frac{u^{j}(a)}{\Gamma(j-\alpha+1)}\,(x-a)^{j-\alpha}\,. $$

In particular:

$$ {}^{C}D^{\alpha}_{a+}u(x) \;=\; D^{\alpha}_{a+}u(x) \qquad \text{if } u(a) = u'(a) = \cdots = u^{n-1}(a) = 0\,. $$

Therefore all the results and comments stated in the present paper concerning Riemann-Liouville derivatives can be easily transferred to Caputo derivatives. This will be precisely addressed in a forthcoming work.

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