



Higher-order expansion of misfit functional for defect identification in elastic solids

Marc Bonnet, Rémi Cornaggia

► To cite this version:

Marc Bonnet, Rémi Cornaggia. Higher-order expansion of misfit functional for defect identification in elastic solids. WAVES 2015 The 12th International Conference on Mathematical and Numerical Aspects of Wave Propagation, Jul 2015, Karlsruhe, Germany. hal-01388731

HAL Id: hal-01388731

<https://hal.science/hal-01388731>

Submitted on 27 Oct 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Higher-order expansion of misfit functional for defect identification in elastic solids

Marc Bonnet¹, Rémi Cornaggia^{1,*}

¹Poems (UMR 7231 CNRS-INRIA-ENSTA), Palaiseau, France

*Email: remi.cornaggia@ensta-paristech.fr

Abstract. In this work, least-squares functionals commonly used for defect identification are expanded in powers of the small radius of a trial inclusion, in the context of time-harmonic elastodynamics, generalizing to higher orders the concept of topological derivative. Such expansion, whose derivation and evaluation are facilitated by using an adjoint state, provides a basis for the quantitative estimation of flaws whereby a region of interest may be exhaustively probed at reasonable computational cost.

Keywords: Topological derivative, identification, elastodynamics, asymptotic analysis

Problem statement. We consider a reference (i.e defect-free) 3D elastic solid Ω characterized by Hooke's tensor \mathcal{C} and mass density ρ . The time-harmonic background displacement \mathbf{u} then solves

$$\langle \mathbf{u}, \mathbf{w} \rangle_{\Omega}^{\mathcal{C}} - \omega^2 \langle \mathbf{u}, \mathbf{w} \rangle_{\Omega}^{\rho} = \mathcal{F}(\mathbf{w}) \quad \forall \mathbf{w} \in \mathcal{W}, \quad (1)$$

where $\langle \cdot, \cdot \rangle_D^{\mathcal{C}}$ and $\langle \cdot, \cdot \rangle_D^{\rho}$ denote the stiffness and mass bilinear forms associated to a given domain D characterized by (\mathcal{C}, ρ) , $\mathcal{W} \subset H^1(\Omega)$ is the function space incorporating the relevant essential boundary conditions (if any), the linear form $\mathcal{F} \in \mathcal{W}'$ defines the applied time-harmonic loading and ω is the angular frequency.

Assuming the presence of a defect inside Ω , and that we can measure the resulting displacement \mathbf{u}^{ex} on a surface Γ , we define the least-squares cost functional $J(\mathbf{w})$, with the elastodynamic displacement \mathbf{w} associated to the given excitation and a known trial defect, by:

$$J(\mathbf{w}) = \frac{1}{2} \int_{\Gamma} |\mathbf{w}(\mathbf{x}) - \mathbf{u}^{\text{ex}}(\mathbf{x})|^2 dS_{\mathbf{x}} \quad (2)$$

We now consider a specific trial defect $B_a = \mathbf{z} + a\mathcal{B}$ (Fig 1), centered at $\mathbf{z} \in \Omega$, of small size a and reference shape \mathcal{B} . It is a perfectly bonded inclusion filled with a material characterized by its Hooke tensor $\mathcal{C}^* = \mathcal{C} + \Delta\mathcal{C}$ and mass density $\rho^* = \rho + \Delta\rho$. We denote \mathbf{u}_a the displacement in the perturbed domain, and $\mathbf{v}_a = \mathbf{u}_a - \mathbf{u}$ the displacement perturbation. $J(\mathbf{u}_a)$ admits the

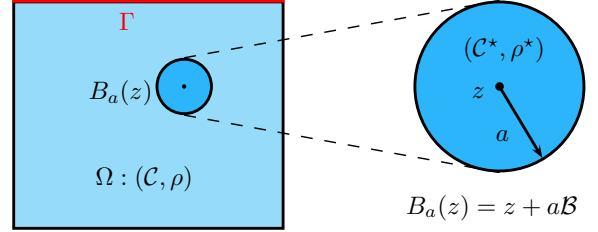


Figure 1: Computational domain and inclusion.

exact expansion about \mathbf{u} :

$$\begin{aligned} J(\mathbf{u}_a) &= J(\mathbf{u}) + J'(\mathbf{u}; \mathbf{v}_a) + J''(\mathbf{u}; \mathbf{v}_a, \mathbf{v}_a) \\ &= J(\mathbf{u}) + \Re \int_{\Gamma} \overline{(\mathbf{u} - \mathbf{u}^{\text{ex}})} \cdot \mathbf{v}_a + \frac{1}{2} \int_{\Gamma} |\mathbf{v}_a|^2 \end{aligned} \quad (3)$$

The goal is now to expand $J(\mathbf{u}_a)$ in powers of a . Similar expansions have been studied in e.g. [2] for rigid obstacles in 3D acoustic media and [5] for holes in 2D elastic bodies.

Define the adjoint field \mathbf{p} as the solution of

$$\langle \mathbf{p}, \mathbf{w} \rangle_{\Omega}^{\mathcal{C}} - \omega^2 \langle \mathbf{p}, \mathbf{w} \rangle_{\Omega}^{\rho} = J'(\mathbf{u}; \mathbf{w}) \quad \forall \mathbf{w} \in \mathcal{W}. \quad (4)$$

We can then compute $J'(\mathbf{u}; \mathbf{v}_a)$ as

$$J'(\mathbf{u}; \mathbf{v}_a) = -\langle \mathbf{p}, \mathbf{u}_a \rangle_{B_a}^{\Delta\mathcal{C}} + \omega^2 \langle \mathbf{p}, \mathbf{u}_a \rangle_{B_a}^{\Delta\rho} \quad (5)$$

Expanding \mathbf{v}_a in powers of a is now needed. As we will see, $J''(\mathbf{u}; \mathbf{v}_a, \mathbf{v}_a)$ requires only the leading contribution of $\mathbf{v}_a|_{\Gamma}$ whereas a higher-order expansion of $\mathbf{v}_a|_{B_a}$ is needed for evaluating (5).

Expansion of the solution perturbation.

Following e.g. [3], \mathbf{v}_a solves the integro-differential Lippmann-Schwinger equation:

$$\mathcal{L}_a[\mathbf{v}_a](\mathbf{x}) = -\langle \mathbf{u}, \mathbf{G} \rangle_{B_a}^{\Delta\mathcal{C}} + \omega^2 \langle \mathbf{u}, \mathbf{G} \rangle_{B_a}^{\Delta\rho} \quad (6)$$

with $\mathcal{L}_a[\mathbf{v}](\mathbf{x}) := \mathbf{v}(\mathbf{x}) + \langle \mathbf{v}, \mathbf{G} \rangle_{B_a}^{\Delta\mathcal{C}} - \omega^2 \langle \mathbf{v}, \mathbf{G} \rangle_{B_a}^{\Delta\rho}$ and $\mathbf{G} = \mathbf{G}(\cdot, \mathbf{x})$ is the elastodynamic Green's tensor for a unit point force applied at \mathbf{x} and satisfying homogeneous boundary conditions consistent with problem (1) on $\partial\Omega$. Substituting the ansatz

$$\begin{aligned} \mathbf{v}_a(\mathbf{x}) &= a\mathbf{V}_1(\bar{\mathbf{x}}) + a^2\mathbf{V}_2(\bar{\mathbf{x}}) + \frac{1}{2}a^3\mathbf{V}_3(\bar{\mathbf{x}}) \\ &\quad + \frac{1}{6}a^4\mathbf{V}_4(\bar{\mathbf{x}}) + \delta_a(\mathbf{x}), \quad \mathbf{x} \in B_a \end{aligned} \quad (7)$$

(with $\bar{x} := (\mathbf{x} - \mathbf{z})/a \in \mathcal{B}$) into (6) and expanding the resulting equation in powers of a (in particular using that $\mathbf{G}(\boldsymbol{\xi}, \mathbf{x}) = a^{-1}\mathbf{G}_\infty(\bar{\boldsymbol{\xi}} - \bar{\mathbf{x}}) + O(1)$, with \mathbf{G}_∞ denoting the *static* full-space Kelvin fundamental solution) yields a sequence of integral equations for the \mathbf{V}_j . These equations correspond to *elastostatic* problems for the normalized inclusion \mathcal{B} embedded in an unbounded reference medium, and are solved with the help of Eshelby's equivalent inclusion method [4].

The remainder $\boldsymbol{\delta}_a$ in (7) solves an integro-differential equation of the form $\mathcal{L}_a[\boldsymbol{\delta}_a] = \boldsymbol{\gamma}_a$. The operator $\mathcal{L}_a : H^1(B_a) \rightarrow H^1(B_a)$ is shown to be invertible with bounded inverse, while $\boldsymbol{\gamma}_a$ can be estimated as $\|\boldsymbol{\gamma}_a\|_{H^1(B_a)} = O(a^{11/2})$. Consequently, there exists a constant $C > 0$ independent of a such that

$$\|\boldsymbol{\delta}_a\|_{H^1(B_a)} \leq C a^{11/2}. \quad (8)$$

For $\mathbf{x} \notin B_a$, plugging (7) in the form $\mathbf{v}_a(\mathbf{x}) \approx a\mathbf{V}_1(\bar{\mathbf{x}})$ into (6) yields the outer expansion

$$\begin{aligned} \mathbf{v}_a(\mathbf{x}) = & -a^3 [\boldsymbol{\nabla} \mathbf{u}(\mathbf{z}) : \boldsymbol{\mathcal{A}} : \boldsymbol{\nabla} \mathbf{G}(\mathbf{z}, \mathbf{x}) \\ & - \omega^2 \Delta \rho |\mathcal{B}| \mathbf{u}(\mathbf{z}) \cdot \mathbf{G}(\mathbf{z}, \mathbf{x})] + o(a^3), \end{aligned} \quad (9)$$

$\boldsymbol{\mathcal{A}}$ being the elastic moment tensor associated to \mathcal{B} , \mathcal{C} and $\Delta \mathcal{C}$ [1, 3].

Cost functional expansion. Substituting (7) into (5) and (9) into (3), $J(\mathbf{u}_a)$ is finally found to have an expansion of the form:

$$J(\mathbf{u}_a) = J_6(a, \mathbf{z}) + o(a^6) \quad (10)$$

$$\begin{aligned} \text{with } J_6(a, \mathbf{z}) = & J(\mathbf{u}) + a^3 \mathcal{T}_3(\mathbf{z}) + a^4 \mathcal{T}_4(\mathbf{z}) \\ & + a^5 \mathcal{T}_5(\mathbf{z}) + a^6 \mathcal{T}_6(\mathbf{z}), \end{aligned}$$

the $o(a^6)$ estimate resulting from (8) and (9).

The $\mathcal{T}_j(\mathbf{z})$ are found to be given in terms of (i) the background field \mathbf{u} and its derivatives at \mathbf{z} , (ii) the adjoint field \mathbf{p} and its derivatives at \mathbf{z} , (iii) $\boldsymbol{\mathcal{A}}$ and other elastic moment tensors that involve the material parameters, the shape \mathcal{B} and the angular frequency ω , and (iv) the complementary part of \mathbf{G} , i.e. $\mathbf{G} - \mathbf{G}_\infty = (\mathbf{G}_{\infty, \omega} - \mathbf{G}_\infty) + \mathbf{G}_C$, where $\mathbf{G}_{\infty, \omega}$ is the elastodynamic full-space fundamental solution and \mathbf{G}_C accounts for the boundedness of Ω . In particular, $\mathcal{T}_3(\mathbf{z})$ is the well-known topological derivative:

$$\mathcal{T}_3(\mathbf{z}) = -[\boldsymbol{\nabla} \mathbf{u} : \boldsymbol{\mathcal{A}} : \boldsymbol{\nabla} \mathbf{p} - \omega^2 \Delta \rho |\mathcal{B}| \mathbf{u} \cdot \mathbf{p}](\mathbf{z}).$$

Moreover, $\mathcal{T}_4(\mathbf{z}) = 0$ for any centrally-symmetric shape \mathcal{B} . The complementary part $\mathbf{G}_{\infty, \omega} - \mathbf{G}_\infty$ (known analytically) is involved in $\mathcal{T}_5(\mathbf{z})$ and

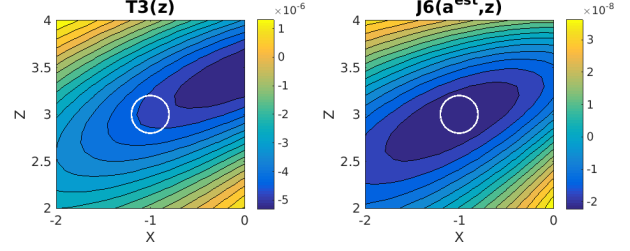


Figure 2: $\mathcal{T}_3(\mathbf{z})$ and $J_6(a^{\text{est}}, \mathbf{z})$ plotted in the (XZ) plane around the obstacle (in white).

$\mathcal{T}_6(\mathbf{z})$, while \mathbf{G}_C appears in $\mathcal{T}_6(\mathbf{z})$ only. Since the exact computation of \mathbf{G}_C would require solving an elastodynamic problem on Ω for each trial location \mathbf{z} , we plan to use an approximation method to save computational time.

Closed-form formulae for the \mathcal{T}_j can be obtained when \mathcal{B} is spherical (for which case we provide explicit expressions) or ellipsoidal.

Identification. Following [2], estimates of the location \mathbf{z}^{est} and size a^{est} of the real defect can then be sought as minimizers of $J_6(a, \mathbf{z})$, with \mathbf{z} spanning a predefined sampling grid. This entails computing the $\mathcal{T}_j(\mathbf{z})$ over the sampling grid and minimizing $a \mapsto J_6(a, \mathbf{z})$ for each \mathbf{z} , the latter step being very fast and straightforward.

A preliminary example is set in free space (so that $\mathbf{G}_C = 0$) for a spherical scatterer of radius $0.1\lambda_S$ illuminated by a plane P-wave travelling along the positive x -direction, with a discrete array of displacement sensors lying behind the scatterer. The above procedure yields the size estimate $a^{\text{est}} \approx 0.105\lambda_S$; moreover, the estimated location \mathbf{z}^{est} is found to be very close to the true center of the scatterer. The contour plot of $J_6(a^{\text{est}}, \mathbf{z})$ (Fig 2) shows improved localisation (relative to the topological derivative $\mathcal{T}_3(\mathbf{z})$) for this partial-aperture configuration.

References

- [1] H. Ammari and H. Kang. *Polarization and moment tensors*. Springer, 2007.
- [2] M. Bonnet. Inverse acoustic scattering by small-obstacle expansion of a misfit function. *Inverse Problems*, 24(3):035022, 2008.
- [3] M. Bonnet and G. Delgado. The topological derivative in anisotropic elasticity. *Quart. J. Mech. Appl. Math.*, 66:557–586, 2013.
- [4] T. Mura. *Micromechanics of defects in solids*. M. Nijhoff Publishers, 1982.
- [5] M. Silva, M. Matalon, and D. A. Tortorelli. Higher order topological derivatives in elasticity. *Int J Solids Struct.*, 47:3053 – 3066, 2010.