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A Method for 3D Direction of Arrival Estimation for General Arrays Using Multiple Frequencies

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***Abstract*—We develop a novel high-resolution method for the estimation of the direction of incidence of energy emitted by sources in 3D space from wideband measurements collected by a planar array of sensors. We make use of recent generalizations of Kronecker's theorem and formulate the direction of arrival estimation problem as an optimization problem in the space of sequences generating so called "general domain Hankel matrices" of fixed rank. The unequal sampling at different wavelengths is handled by using appropriate interpolation operators. The algorithm is operational for general array geometries (i.e., not restricted to, e.g., rectangular arrays) and for equidistantly as well as unequally spaced receivers. Numerical simulations for different sensor arrays and various signal-to-noise ratios are provided and demonstrate its excellent performance.**

***Index Terms*—3D DOA estimation, Kronecker's theorem, general domain Hankel matrix, irregular sampling, ADMM**

I. Introduction

Direction of arrival (DOA) estimation, i.e., the localization of sources in space emitting energy from measurements obtained by a collection of passive sensors, is an important problem that arises in a large number of applications in radar, sonar, telecommunications and astrophysics, to name but a few [1], [2]. Let X be a planar grid of (not necessarily equally spaced) receivers, located near the origin in the (x_1, x_2)-plane of $\mathbb{R}^3$ and consider one distant point source located in direction of the unit vector $(\theta_1, \theta_2, \theta_3)$ (where $\theta_3 = \pm\sqrt{1-\theta_1^2-\theta_2^2}$) that emits a signal. We write $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{\theta} = (\theta_1, \theta_2)$ and denote the data measured by the receivers by $s(\boldsymbol{x}, t)$, whereas its Fourier transform with respect to time is denoted by $\hat{s}(\boldsymbol{x}, \lambda)$. By geometrical considerations, the signature of one single source can be written

$$\hat{s}(\boldsymbol{x}, \lambda) = \hat{s}((0,0), \lambda) e^{2\pi i \frac{x_1\theta_1 + x_2\theta_2}{\lambda}} = c_\lambda e^{2\pi i \frac{\boldsymbol{x}\cdot\boldsymbol{\theta}}{\lambda}}. \quad (1)$$

Consequently, K distinct sources will yield data of the form

$$\hat{s}(\boldsymbol{x}, \lambda) = \sum_{k=1}^{K} c_{\lambda,k} e^{2\pi i \frac{\boldsymbol{x}\cdot\boldsymbol{\theta}_k}{\lambda}} + \epsilon_\lambda(\boldsymbol{x}), \quad (2)$$

where ϵ_λ represents noise. The 3D DOA estimation problem for one fixed wavelength λ is hence equivalent to estimating the 2D (spatial) frequency parameters $\boldsymbol{\theta}_k$ [3]. A significant number of methods have been proposed in the literature for this problem, often extending methods for 1D spectral line estimation (such as the ESPRIT method) to the 2D case, see, e.g., [4]–[7]. While they have been successfully used in applications, most methods are affected by at least one of the following conceptual limitations. First, they do require specific sensor arrangements (such as rectangular or circular arrays) and/or equally spaced sensors, i.e., $x_n = k\Delta x$ for $n \in \{1, 2\}$ in (2), where Δx is the spacing and k takes integer values. Second, they can only be applied for one individual wavelength λ, while the data are often observed at multiple wavelengths $\{\lambda_j\}_{j=1}^J$. Note that for the 2D DOA estimation problem, a number of methods have been proposed that enable wide-band sources and/or general array geometries to be considered, cf., e.g., [8]–[11].

The goal of the present contribution is to develop a novel high-resolution 3D DOA joint estimation procedure for *multiple wavelengths*, without any assumptions on the source spectra, that can be used for planar arrays with *general geometry* and equidistant as well as *unequally spaced receivers*. The method relies on a low-rank matrix approximation formulation based on recent generalizations of Kronecker's theorem [12]–[14] and makes use of the following original key ingredients. First, in the case of a rectangular (but not necessarily equally spaced) grid $\boldsymbol{X}$ and one single wavelength λ, it is explained in [13, Section VI] how to retrieve the parameters $\boldsymbol{\theta}_k$ from $\hat{s}$ (see also [15] for a related approach). The theoretical ground behind this approach (extension of Kronecker's theorem to several variables) is established in [16]. Second, it has recently been shown in [14] how this result may be used in a discretized form including the case of non-rectangular domains through the use of "general domain Hankel matrices". Third, in the one dimensional setting, a strategy for utilizing several wavelengths $\{\lambda_j\}_j$ jointly to determine the DOA, based on an auxiliary equidistant grid and interpolation operators, was proposed in [17]. Here, we generalize this idea to 2 dimensions and non-equidistant sensor spacing. Finally, we develop an efficient alternating directions method of multipliers (ADMM) [18] based algorithm that combines these three ingredients to a robust method for the estimation of 3D DOA of K distant transmitters. Throughout, we assume that the number of sources K is known (see, e.g., [13, Section III.D] for a model order selection method for unequally spaced receivers).

II. Formulation using Hankel matrices

A. Problem formulation

We denote by $\boldsymbol{X}$ the location of the (not necessarily equally spaced) receivers. To formulate the problem for multiple wave-

lengths, for each λ_j, $j = 1, \dots J$, we let f_j denote the function

$$f_j(\boldsymbol{x}) = \hat{s}(\lambda_j \boldsymbol{x}, \lambda_j) = \sum_{k=1}^{K} c_{j,k} e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{\theta}_k} + \epsilon_j(\boldsymbol{x}), \qquad (3)$$

where we have redefined $c_{j,k} = c_{\lambda_j,k}$ and $\epsilon_j(x) = \epsilon_{\lambda_j}(x)$ for easier reading. Since s is measured at the grid $\boldsymbol{X}$, each f_j lives on a dilated grid $\boldsymbol{X}_j = \{\lambda_j \boldsymbol{x} : \ \boldsymbol{x} \in \boldsymbol{X}\}$. We use an auxiliary equidistant grid $\boldsymbol{\Omega}$ on a convex domain that covers $\cup_j \boldsymbol{X}_j$. More precisely, we let Ω be a convex domain such that

$$\Omega \supset \cup_j \boldsymbol{X}_j, \qquad (4)$$

and we let $\boldsymbol{\Omega}$ be an equidistant grid that covers Ω. Set

$$g_j(\boldsymbol{x}) = \sum_{k=1}^{K} c_{j,k} e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{\theta}_k}, \quad \boldsymbol{x} \in \boldsymbol{\Omega}, \qquad (5)$$

i.e., g_j ideally denotes the noiseless model on the larger grid $\boldsymbol{\Omega}$. Let I_j be an operator that interpolates between the grid $\boldsymbol{\Omega}$ and the grid $\boldsymbol{X}_j$. The problem of retrieving the unknown g_j's from the measured data f_j can thus be formulated as

$$\underset{\{c_{j,k}\}_{j,k},\ \{\boldsymbol{\theta}_k\}_k}{\operatorname{argmin}} \sum_j \|I_j g_j - f_j\|^2, \qquad (6)$$

with the constraint that

$$g_j(x_1, x_2) = \sum_{k=1}^{K} c_{j,k} e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{\theta}_k}, \qquad (7)$$

i.e. the functions g_j share the K frequencies but have different coefficients. The above problem formulation will force the method to find one set $\{\boldsymbol{\theta}_k\}$ that fits with all the data $\{f_j\}_j$.

B. On generalized multidimensional Hankel matrices

We now review the necessary ingredients needed to recast Problem (6-7) into a structured low rank matrix approximation problem. Consider one single function g (e.g., one element of the set $(g_j)_j$). Let $\boldsymbol{\Xi}$ and $\boldsymbol{\Upsilon}$ be equally spaced grids such that $\boldsymbol{\Xi} + \boldsymbol{\Upsilon} = \boldsymbol{\Omega}$, where $\boldsymbol{\Xi} + \boldsymbol{\Upsilon} = \{\boldsymbol{x} + \boldsymbol{y} : \ \boldsymbol{x} \in \boldsymbol{\Xi}, \ \boldsymbol{y} \in \boldsymbol{\Upsilon}\}$. The general domain Hankel operator $\boldsymbol{\Gamma}_g$ transforms functions on $\boldsymbol{\Upsilon}$ to functions on $\boldsymbol{\Xi}$ via the formula

$$\boldsymbol{\Gamma}_g h(\boldsymbol{x}) = \sum_{\boldsymbol{y} \in \boldsymbol{\Upsilon}} g(\boldsymbol{x} + \boldsymbol{y}) h(\boldsymbol{y}), \quad \boldsymbol{x} \in \boldsymbol{\Xi}. \qquad (8)$$

By identifying functions on $\boldsymbol{\Upsilon}$ and $\boldsymbol{\Xi}$ with vectors (e.g., by using the lexicographical order), the operator $\boldsymbol{\Gamma}_g$ can be represented by a structured matrix H_g, as explained in detail in [14]. Note in particular that H_g becomes a block Hankel matrix when $\boldsymbol{\Xi} = \boldsymbol{\Upsilon}$ is a square. Moreover, $\boldsymbol{\Gamma}_g$ has rank K if g is of the form (7), cf., [16]. Thus, for $\boldsymbol{\Gamma}_f$ with f of the form (3), it is reasonable to assume that if $\boldsymbol{\Gamma}_g$ is the closest rank K operator in this class, then g is of the form

$$g(x_1, x_2) = \sum_{k=1}^{K} c_k e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{\theta}_k} \qquad (9)$$

and the parameters $\{\boldsymbol{\theta}_k\}_k$ can be extracted from g as described in [13]. To clarify this statement, note that there are (many) rank K operators $\boldsymbol{\Gamma}_g$ whose symbols are exponential polynomials [16], i.e., in the representation (9), c_k would be polynomials and the sum would contain fewer terms. In the discrete setting there are also other exceptional cases which do not fit with the model (9), as described in [14]. However, for an f of the form (3), it is not likely that the *closest* rank K operator $\boldsymbol{\Gamma}_g$ to $\boldsymbol{\Gamma}_f$ would exhibit this exceptional structure.

Finally, note that the range of $\boldsymbol{\Gamma}_g$ is the space of all linear combinations of the functions $e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{\theta}_k}$ on $\boldsymbol{\Xi}$ [16, Lemma 4.2].

C. Operator formulation of (6-7)

We now move back to the setting of several functions g_j. Since the rank of an operator is the dimension of its range, it follows from the above that the concatenation

$$H((g_j)_j) = [H_{g_1}, H_{g_2}, \dots, H_{g_J}] \qquad (10)$$

will have rank K if and only if the g_j's are of the form (9) and share the *same parameters* $\boldsymbol{\theta_k}$, but not necessarily the same coefficients $c_{j,k}$. The problem (6) can now be reformulated as

$$\underset{H \text{ is of form (10)}}{\operatorname{argmin}} \mathcal{R}_K(H) + \frac{1}{2}\|I_j g_j - f_j\|^2, \qquad (11)$$

where $\mathcal{R}_K$ is ∞ whenever $\operatorname{rank}(H) > K$ and 0 else. With $\check{H}$ denoting the solution to (11), the nodes $\{\boldsymbol{\theta}_k\}_{k=1}^K$ can be readily obtained by determining the roots of a system of two polynomials in two variables which is generated by any two vectors in the kernel of $\check{H}\check{H}^*$ following the approach of [19].

III. An ADMM based solution for DOA estimation

Following [13], we propose to solve (11) using ADMM, cf., e.g., [18]. The interpolation operators I_j in (11) can be realized as the tensor-product of one-dimensional interpolation functions φ [20], [21]. In the numerical experiments reported below, we use the Lanczos function with a lobes, $\varphi(x) = \operatorname{sinc}(x)\operatorname{sinc}(x/a)$ for $|x| \geq a$ and 0 otherwise.

We introduce the augmented Lagrangian for (11)

$$\begin{aligned} L(A, (g_j)_j, \Lambda) \ = \ & \mathcal{R}_K(A) + \frac{1}{2}\sum_j \|I_j g_j - f_j\|^2 \\ & + \langle \Lambda, A - H((g_j)_j) \rangle_{\mathsf{Re}} + \frac{\rho}{2}\|A - H((g_j)_j)\|_2^2, \end{aligned} \qquad (12)$$

where $\|\cdot\|_2$ is the Frobenius norm, ρ the penalty parameter and H is defined via (10). The ADMM routine consists in iterating updates for A^q, $(g_j^q)_j$ and Λ^q, where $q = 0, 1, 2, \dots$

A. Updating A

It is easy to see (cp. [13, Section III.B]) that A^{q+1} is the rank K matrix that is closest to $B^q = H((g_j^q)_j) - \Lambda/\rho$, which can be computed using the Schmidt-Eckart-Young theorem. This requires the computation of the SVD of B^q, which is costly since B^q has many columns. Yet, the cost can be significantly reduced as follows. Let $B^q = USV^*$ where U and S are taken to be square matrices. Then $B^q B^{q*} = US^2U^*$ and

$$B^q B^{q*} = \sum_j (H_{g_j} - \Lambda_j/\rho)(H_{g_j} - \Lambda_j/\rho)^*, \qquad (13)$$

where Λ_j denotes the submatrix of Λ corresponding to the position of H_{g_j} in $H((g_j)_j)$. Therefore, U and S can be extracted by an SVD of the smaller square matrix $B^q B^{q*}$. As long as it is not rank deficient, we then have $V^* = S^{-1}U^*B^q$. Now let S_K denote a copy of S in which all except the first K singular values are set to zero and $\bar{U}_K = U(:, 1:K)$ the first

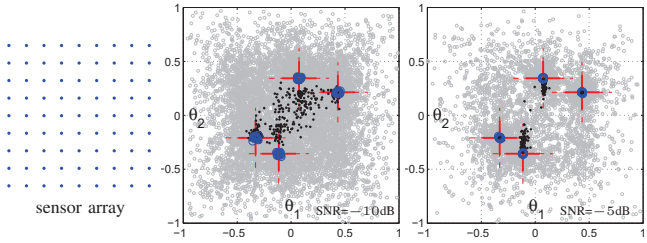


Fig. 1: Square array of 81 equidistant sensors. Sensor arrangement (left), results for SNR=−10dB (center) and SNR=−5dB (right): theory (red crosses); proposed joint wavelength $\text{DOA}_{\text{joint}}$ (blue circles), individual wavelength based SW (grey circles) and SW_{med} (black dots).

K columns of U. Then $A^{q+1} = US_KV^* = US_KS^{-1}U^*B^q$ and, moreover, $US_KS^{-1}U^* = \bar{U}_K\bar{U}_K^* =: P_{U,K}$ is the projection onto the first K singular vectors of U. The update step for A^{q+1} hence becomes

$$A^{q+1} = P_{U,K}B^q. \tag{14}$$

B. Updating $(g_j)_j$

Again, the following steps are suitable modifications of those in [13, Section III.B], adapted here to multiple dimensions and general domain $\mathbf{\Omega}$. We introduce the weight-function

$$w(\boldsymbol{z}) = \sum_{\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{z}} \chi_{\mathbf{\Xi}}(\boldsymbol{x})\chi_{\mathbf{\Upsilon}}(\boldsymbol{y}),$$

where $\chi_{\mathbf{\Xi}}$ denotes the characteristic function of $\mathbf{\Xi}$. For fixed j and $\boldsymbol{z} \in \mathbf{\Omega}$, let $S_{j,\boldsymbol{z}}$ be the matrix with ones at all positions where the value $g_j(\boldsymbol{z})$ shows up in the matrix $H((g_j)_j)$, and zeroes elsewhere. Formally, we can define $S_{j,\boldsymbol{z}}$ as $S_{j,\boldsymbol{z}} = H((h_l)_l)$ where $h_l = \chi_{\{\boldsymbol{z}\}}$ if $l = j$ and the 0-function on $\mathbf{\Omega}$ if $l \neq j$. Therefore, the operation

$$b_j^{q+1}(\boldsymbol{z}) = \left\langle A^{q+1} + \frac{\Lambda_j^q}{\rho}, S_{j,\boldsymbol{z}} \right\rangle \tag{15}$$

produces a function on $\mathbf{\Omega}$ for each j. A short calculation yields

$$g_j^{q+1} = \operatorname*{argmin}_g \|I_jg - f_j\|^2 + \rho \sum_{\boldsymbol{z}\in\mathbf{\Omega}} w(\boldsymbol{z}) \left| g(\boldsymbol{z}) - \frac{b_j^{q+1}(\boldsymbol{z})}{w(\boldsymbol{z})} \right|^2 \tag{16}$$

which can be solved by least squares (note that this can be done for each j independently). The operator I_j, acting on functions on $\mathbf{\Omega}$, clearly has a matrix realization, given, e.g., the lexicographical ordering on the grid $\mathbf{\Omega}$. We use the same notation for this matrix. Similarly, we define M_w to be the matrix corresponding to the operator that multiplies functions on $\mathbf{\Omega}$ pointwise with w. With this notation, the solution to (16) becomes (cp. [13, Eq. (17)])

$$g_j^{q+1} = \left(I_j^*I_j + \rho M_w\right)^{-1}\left(I_j^*f_j + \rho b_j^{q+1}\right). \tag{17}$$

C. Updating Λ

It follows straightforwardly from the ADMM scheme that

$$\Lambda^{q+1} = \Lambda^q + \rho(A^{q+1} - H((g_j^{q+1})_j)). \tag{18}$$

Upon convergence, the parameters $\boldsymbol{\theta}_k$ can be retrieved from $(g_j)_j$ as outlined in Section II-C.

Method \ SNR (dB)	-10	-5	0	5	10
$\text{DOA}_{\text{joint}}$	0.7304	0.2206	0.1106	0.0626	0.0380
SW	26.8226	15.2271	6.6841	2.7814	1.9434
SW_{mean}	13.7797	8.7245	4.2333	1.8462	1.3235
SW_{med}	9.5548	2.9494	0.3072	0.1028	0.0499

TABLE I: Average error $\varepsilon_\angle$ in estimated direction (in degrees) using the metric described in Section IV.

IV. Results

We illustrate the performance of the proposed method by means of numerical simulations with $K = 4$ sources in circular white Gaussian noise at different levels of signal-to-noise ratio (SNR), using $J = 32$ wavelengths between $\lambda_1 = 1$ and $\lambda_J = 2$ (w.r.t. normalized units $\Delta x = 1$ and wave speed $c = 1$). The parameters $\{\boldsymbol{\theta}_k\}_{k=1}^K$ are chosen as pseudo-random numbers on the square $[-\frac{1}{2}, \frac{1}{2}]^2$ and indicated by red crosses in Figs. 1 and 2. The data $\hat{s}(\boldsymbol{x}, \lambda)$ is generated using coefficients $c_{\lambda,k}$ whose real and imaginary parts are independent uniformly distributed random variables.

We compare the proposed multiple wavelength estimation procedure, denoted $\text{DOA}_{\text{joint}}$, with estimates obtained for each wavelength λ_j individually using the 2D frequency estimation algorithm proposed in [13], denoted SW (for Single Wavelength). This method was shown to attain the Cramér-Rao bound for reasonable SNR values and readily extends to irregular spacing and non-rectangular domains using the above developments. It is thus used here as a reference method for individual wavelength based estimation. Moreover, we construct from SW estimates for all wavelengths, defined as the average (median) over wavelengths of the ordered estimates obtained by SW and denoted SW_{mean} (SW_{med}), respectively.

Square array. We use an equidistant square array of $S = 9 \times 9 = 81$ sensors, as plotted in Fig. 1 (left). The results for 100 independent noise realizations are illustrated in Fig. 1 for SNR values of -10dB (center) and -5dB (right): the correct values of $\boldsymbol{\theta}$ are indicated by red crosses, the estimates obtained with the proposed method $\text{DOA}_{\text{joint}}$ are given by blue circles, those obtained by SW by small grey circles (for all individual wavelengths), and the median over ordered single-wavelength estimates SW_{med} by black dots. The results clearly indicate that the proposed DOA estimation procedure provides excellent estimates, even when the data are severely corrupted by noise. In contrast, the estimates obtained for each wavelength individually, and even those of the robust statistic over wavelengths SW_{med}, can be far off the true values.

Table I reports quantitative results for the performance of the proposed method in terms of the error in estimated direction defined as $\varepsilon_\angle = \arcsin\left(||(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) - (\theta_1, \theta_2, \theta_3)||\right)$ (averaged over noise realizations and sources). The proposed multiple wavelength based DOA estimates are of significantly better quality than those based on individual frequencies, in particular for low SNR: Indeed, the errors in estimated directions of $\text{DOA}_{\text{joint}}$ are up to one order of magnitude smaller than those of SW_{med}, and up to two orders of magnitude smaller than those of SW.

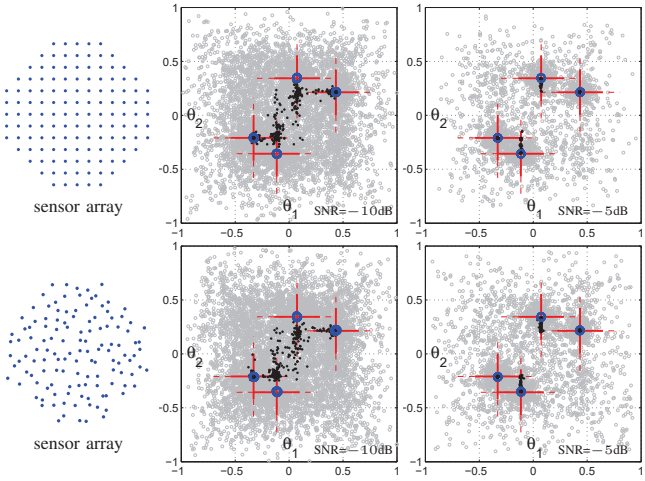


Fig. 2: 129 sensors on a non-square domain using regular (top row) and irregular (bottom row) spacing. Sensor arrangement (left), results for SNR=−10dB (center) and SNR=−5dB (right): theory (red crosses); proposed joint wavelength $\text{DOA}_{\text{joint}}$ (blue circles), median over wavelengths SW_{med} (black dots), individual wavelengths SW (grey circles).

Non-square array and irregularly spaced sensors. We illustrate the performance of the proposed method for sensor arrangements on non-square domains and with unequal spacing (using 129 sensors). The domains considered are depicted in Fig. 2 (left) for equidistant (top) and and irregular (bottom) spacing, respectively. Results obtained for 100 independent noise realizations and SNR values of -10dB and -5dB are plotted in Fig. 2 (center and right column, respectively; same color code as in Fig. 1). Comparison with Fig. 1 yields the following conclusions. First and foremost, the proposed method is clearly effective for general sensor arrangements on non-square domains as well as for the case of irregularly spaced sensors, with performance comparable to those for the square regularly spaced sensor array. Second, the proposed joint wavelength based procedure $\text{DOA}_{\text{joint}}$ dramatically outperforms the individual wavelength based approaches, with conclusions similar to those obtained in the previous paragraph. Finally, note that the performance overall is slightly above the ones reported in Fig. 1 (estimates are slightly more concentrated around the true values for SW and SW_{med}).

V. Conclusions

We have developed a novel high-resolution procedure for 3D DOA estimation using multiple wavelengths. The method makes use of a recent extension of Kronecker's theorem and general domain Hankel matrices to cast the 3D DOA estimation problem into an optimization problem that can be resolved efficiently using ADMM. The method can be used for general planar sensor array geometry and for equidistant as well as unequally spaced sensors. To our knowledge, this is the first and only operational method that achieves this flexibility. It can hence be employed in applications where regular sensor grids can not be achieved (e.g., in sensor networks, seismology, remote sensing applications). Numerical simulations indicate the excellent performance of the method, both for rectangular, nonrectangular, equidistant and unequally spaced receivers.

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