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The Gamma Power Half-Logistic Distribution: Theory and Applications

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Abstract

In this paper, we propose a new three-parameter distribution on the positive real line, called the gamma power half-logistic distribution. This new distribution is derived to the gamma-G family of distributions pioneered by Zografos and Balakrishnan (2009) and the power half-logistic distribution introduced by Krishnarani (2016). Among its advantages, the corresponding hazard rate function has various kinds of shapes, which constitutes a positive point in the context of statistical modelling. A part of the paper is devoted to some of its main mathematical features, including quantiles, skewness, kurtosis, moments, incomplete moments, mean deviations, Bonferroni and Lorenz curves, stochastic ordering, reliability parameter and distribution of order statistics. Then, the gamma power half-logistic model is investigated in view of data analysis. We use the maximum likelihood method for estimating the model parameters, with a simulation study attesting the good performance of the method. The practical aspect is discussed with the help of two real life data sets.

Keywords: half-logistic distribution; power half-logistic distribution; gamma-G family of distributions; hazard rate; lifetime data; maximum likelihood method.

AMS Subject: 60E05, 62E15, 62F10.

1 Introduction

By their intrinsic definitions, the distributions defined on the positive real line, i.e., $(0, +\infty)$, are appropriate to model the duration of time until a certain phenomenon happens (death times of patients, time to mechanical failure...). The notorious distributions on $(0, +\infty)$ includes the chi, Dagum, exponential, Fréchet, gamma, Gompertz, Lomax, Pareto, Rayleigh and Weibull distributions, and those defined as a fold at zero of symmetric (around zero) distributions as the half-normal, half-Student, half-Cauchy and half-logistic distributions. All of them are widely used as models to analyze data sets in many applied fields, as computer science, engineering, economics, biological studies, medical sciences and hydrology. Thanks to its simplicity, tractable mathematical properties and ability to fit correctly survival data, the half-logistic (HL) distribution has been the object of all the attentions. We refer to (Balakrishnan, 1985; Balakrishnan and Puthenpura, 1986;

Balakrishnan, 1992) and Olapade (2003), and the references therein. Let us just mention that it is characterized by the survival function (sf) given by

$$\bar{\Pi}(x) = \frac{2}{1 + e^{\alpha x}}, \quad \alpha, x > 0. \quad (1)$$

In recent years, aiming to extend or exploit some of its features, several generalizations and extensions of the HL distribution have been proposed. Among them, there are the generalized half-logistic (GHL) distribution by Torabi and Bagheri (2010), the type 1 generalized half logistic (Type 1 GHL) distribution by Olapade (2014), the exponentiated HL-G family of distributions (EHL-G) by Cordeiro et al. (2014) type I half-logistic-G family of distributions (Type 1 GHL-G) by Cordeiro et al. (2016), the power half-logistic (PHL) distribution by Krishnarani (2016), the half-logistic generalized Weibull (HLGW) distribution by Anwar and Bibi (2018), the Marshall Olkin half-logistic (MOHL) distribution by Yegen and Ozel (2018) and the Kumaraswamy type I half-logistic (KHL) distribution by El-Sherpieny and Elsehetry (2019). Let us now focus on the PHL distribution by Krishnarani (2016). First of all, we can presented it as a simple two parameter distribution extending the half logistic distribution by the use of the power transformation method, in the same way that the Weibull distribution extends the exponential distribution. The corresponding sf is given by

$$\bar{G}(x) = \bar{\Pi}(x^\beta) = \frac{2}{1 + e^{\alpha x^\beta}}, \quad \alpha, \beta, x > 0. \quad (2)$$

It is shown in Krishnarani (2016) that β has an important role on the curvatures of the related probability density function (pdf) and on the nature of the tails of the PHL distribution, demonstrating more flexible properties in comparison to the former HL distribution. Thus, the related statistical model is adequate to model data sets having tail probability less or greater than the basic HL model. This is illustrated in Krishnarani (2016) with the help of three real-life data sets. There is however a room for improvement in terms of model flexibility as suggested by (Krishnarani, 2016, Figures 3 and 4), where a lack of bathtub or reversed J shapes for the hazard rate function (hrf) can be observed.

In this paper, we explore a natural extension of the PHL distribution. Following the spirit of Castellares et al. (2015) for the logistic distribution (on the whole real line), we consider the gamma-G family of distributions established by Zografos and Balakrishnan (2009) with the PHL distribution for baseline distribution. Let us now present the gamma-G family of distributions, in full generality. From a baseline sf denoted by $\bar{G}(x)$, the cumulative distribution function (cdf) of the gamma-G family of distributions is given by

$$F(x) = \gamma_1(\delta, -\log[\bar{G}(x)]), \quad \delta, x > 0, \quad (3)$$

where $\gamma_1(\delta, z)$ denotes the so-called regularized lower incomplete gamma function defined by $\gamma_1(\delta, z) = \gamma(\delta, z)/\Gamma(\delta)$, $\gamma(\delta, z) = \int_0^z t^{\delta-1} e^{-t} dt$ and $\Gamma(\delta) = \int_0^{+\infty} t^{\delta-1} e^{-t} dt$. As established in numerous works in the field, the new parameter δ aims to increase flexibility to the baseline distribution, including it as a special case by taking $\delta = 1$. The usefulness of the gamma-G family of distributions is discussed in detail in (Nadarajah and Rocha, 2016, gamma-G I distributions, page 11), with a wide broad of baseline distributions and applications. We thus introduce the gamma power half-logistic (GPHL) distribution characterized by the cdf defined by the cdf of the gamma-G family of distributions given by (3) with the sf of the PHL distribution (2). In this study, we defend the merits and advantages of using the GPHL distribution in a statistical setting.

The rest of the paper is unfolded in the following manner. Section 2 describes the GPHL distribution by their main probabilistic functions, with the analytical study of their shapes. Some structural properties of the GPHL distribution are exhibited in Section 3, with natural ideas of extensions. Estimation of the unknown GPHL model parameters are explored in Section 4 with the

maximum likelihood method. In particular, a Monte Carlo simulation study examines the precision of the obtained maximum likelihood estimates. Applications of the GPLH model are performed in Section 5 by the consideration of two real data sets. A brief conclusion is given in Section 6.

2 The GPLH distribution

2.1 Crucial functions

As described in the introduction, the GPLH distribution is characterized by the cdf obtained by substituting the PHL sf given by (2) in the definition of the gamma-G cdf given by (3). Hence, the cdf of the GPLH distribution is given by

$$F(x) = \gamma_1 \left[\delta, -\log \left(\frac{2}{1 + e^{\alpha x^\beta}} \right) \right], \quad \delta, \alpha, \beta, x > 0. \quad (4)$$

By differentiation, after some algebra, the pdf corresponding to (4) is given by

$$f(x) = \frac{2\alpha\beta x^{\beta-1} e^{\alpha x^\beta}}{\Gamma(\delta) (1 + e^{\alpha x^\beta})^2} \left[-\log \left(\frac{2}{1 + e^{\alpha x^\beta}} \right) \right]^{\delta-1}. \quad (5)$$

The sf and hrf of the GPLH distribution are, respectively, given by

$$S(x) = 1 - F(x) = 1 - \gamma_1 \left[\delta, -\log \left(\frac{2}{1 + e^{\alpha x^\beta}} \right) \right]$$

and

$$h(x) = \frac{f(x)}{S(x)} = \frac{2\alpha\beta x^{\beta-1} e^{\alpha x^\beta}}{\Gamma(\delta) (1 + e^{\alpha x^\beta})^2} \left[-\log \left(\frac{2}{1 + e^{\alpha x^\beta}} \right) \right]^{\delta-1} \left\{ 1 - \gamma_1 \left[\delta, -\log \left(\frac{2}{1 + e^{\alpha x^\beta}} \right) \right] \right\}^{-1}.$$

2.2 Illustrations

Figure 1 displays some plots of $f(x)$ and $h(x)$ for different values of δ , α and β . The plots in Figure 1 (a) reveal that $f(x)$ can have reversed-J, right skewed shapes, left-skewed and approximately symmetric. The plots in Figure 1 (b) indicate that $h(x)$ can have IFR (increasing failure rate), DFR (decreasing failure rate) and BT (bathtub) shapes. These different kinds of shapes show the high degree of flexibility of the proposed GPLH distribution in comparison to the former PHL distribution, as mentioned in Introduction.

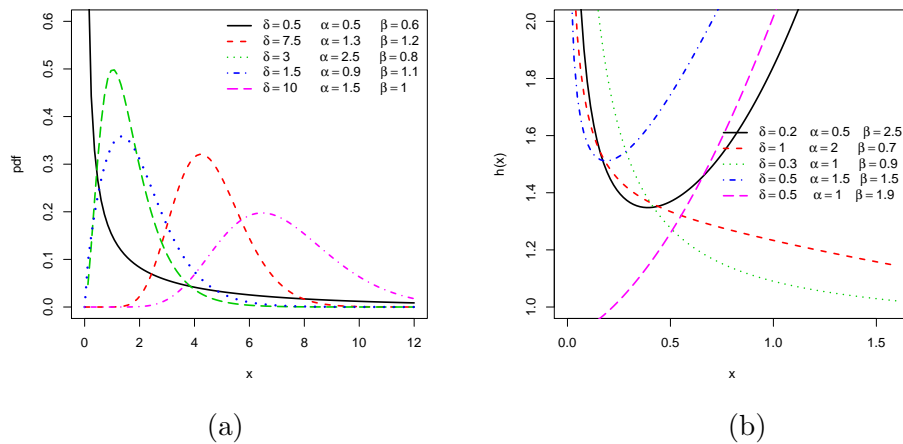


Figure 1: Plots of the (a) GPLH pdf (b) GPLH hrf for certain parameter values.

The rest of the section is devoted to some analytical study of these shapes.

2.3 Asymptotes

The following result presents the asymptotes for $f(x)$ and $h(x)$.

Proposition 1 *We have*

$$\lim_{x \rightarrow 0} f(x) = \begin{cases} +\infty & \text{if } \delta\beta < 1, \\ \frac{\alpha^\delta}{\delta\Gamma(\delta)2^\delta} & \text{if } \delta\beta = 1, \\ 0 & \text{if } \delta\beta > 1, \end{cases} \quad \lim_{x \rightarrow +\infty} f(x) = 0$$

and

$$\lim_{x \rightarrow 0} h(x) = \begin{cases} +\infty & \text{if } \delta\beta < 1, \\ \frac{\alpha^\delta}{\delta\Gamma(\delta)2^\delta} & \text{if } \delta\beta = 1, \\ 0 & \text{if } \delta\beta > 1, \end{cases} \quad \lim_{x \rightarrow +\infty} h(x) = \begin{cases} 0 & \text{if } \beta < 1, \\ 2\alpha & \text{if } \beta = 1, \\ +\infty & \text{if } \beta > 1. \end{cases}$$

Proof. We proceed by using equivalences of the involved functions.

As $x \rightarrow 0$, since $-\log(2/(1 + e^{\alpha x^\beta})) \sim 1 - 2/(1 + e^{\alpha x^\beta}) \sim (1/2)\alpha x^\beta$, we have

$$f(x) \sim \frac{\alpha^\delta \beta}{\Gamma(\delta)2^\delta} x^{\delta\beta-1}.$$

Therefore, as $x \rightarrow 0$, if $\delta\beta < 1$, $f(x) \rightarrow +\infty$, if $\delta\beta = 1$, $f(x) \rightarrow \alpha^\delta/(\delta\Gamma(\delta)2^\delta)$ and if $\delta\beta > 1$, $f(x) \rightarrow 0$.

As $x \rightarrow +\infty$, since $-\log(2/(1 + e^{\alpha x^\beta})) \sim \alpha x^\beta$, we have

$$f(x) \sim \frac{2\alpha^\delta \beta}{\Gamma(\delta)} x^{\delta\beta-1} e^{-\alpha x^\beta}.$$

Hence, for any values of the parameters, as $x \rightarrow +\infty$, we have $f(x) \rightarrow 0$.

As $x \rightarrow 0$, we have

$$h(x) \sim f(x) \sim \frac{\alpha^\delta \beta}{\Gamma(\delta)2^\delta} x^{\delta\beta-1}.$$

Therefore, as $x \rightarrow 0$, if $\delta\beta < 1$, $h(x) \rightarrow +\infty$, if $\delta\beta = 1$, $h(x) \rightarrow \alpha^\delta/(\delta\Gamma(\delta)2^\delta)$ and if $\delta\beta > 1$, $h(x) \rightarrow 0$.

As $x \rightarrow +\infty$, since $\gamma_1(\delta, x) \sim 1 - x^{\delta-1}e^{-x}/\Gamma(\delta)$ and $-\log(2/(1 + e^{\alpha x^\beta})) \sim \alpha x^\beta$, we have

$$S(x) \sim \frac{1}{\Gamma(\delta)} \alpha^{\delta-1} x^{\beta(\delta-1)} e^{-\alpha x^\beta}.$$

Therefore,

$$h(x) = \frac{f(x)}{S(x)} \sim 2\alpha\beta x^{\beta-1}.$$

Hence, as $x \rightarrow +\infty$, if $\beta < 1$, $h(x) \rightarrow 0$, if $\beta = 1$, $h(x) \rightarrow 2\alpha$ and if $\beta > 1$, $h(x) \rightarrow +\infty$. This ends the proof of Proposition 1. \square

2.4 Shapes

The shapes of $f(x)$ and $h(x)$ can be described analytically. The critical points of $f(x)$ are the roots of the equation given by $\partial \log[f(x)]/\partial x = 0$, i.e.,

$$\left[\beta - \alpha\beta x^\beta e^{\alpha x^\beta} + \alpha\beta x^\beta - e^{\alpha x^\beta} + \beta e^{\alpha x^\beta} - 1 \right] \log \left(\frac{2}{1 + e^{\alpha x^\beta}} \right) = (1 - \delta)\alpha\beta x^{\beta-1} e^{\alpha x^\beta}.$$

As usual, if $x = x_0$ is a root, then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether $\lambda(x_0) < 0$, $\lambda(x_0) > 0$, or $\lambda(x_0) = 0$, where $\lambda(x) = \partial^2 \log[f(x)]/\partial x^2$, i.e.,

$$\lambda(x) = \frac{(\delta - 1)\alpha^2 \beta^2 x^{2\beta-2} e^{2\alpha x^\beta} \left\{ 2 - \delta + \log \left(\frac{2}{1 + e^{\alpha x^\beta}} \right) \right\}}{\left[\log \left(\frac{2}{1 + e^{\alpha x^\beta}} \right) \right]^2 (1 + e^{\alpha x^\beta})^2}.$$

Similarly, the critical points of $h(x)$ are the roots of the equation given by $\partial \log[h(x)]/\partial x = 0$, i.e.,

$$\frac{\beta - \alpha\beta x^\beta e^{\alpha x^\beta} + \alpha\beta x^\beta - e^{\alpha x^\beta} + \beta e^{\alpha x^\beta} - 1}{x(1 + e^{\alpha x^\beta})} = \frac{(1 - \delta)\alpha\beta x^{\beta-1} e^{\alpha x^\beta}}{\log \left(\frac{2}{1 + e^{\alpha x^\beta}} \right) (1 + e^{\alpha x^\beta})} - \frac{2\alpha\beta x^{\beta-1} e^{\alpha x^\beta}}{\Gamma(\delta) (1 + e^{\alpha x^\beta})^2} \left[-\log \left(\frac{2}{1 + e^{\alpha x^\beta}} \right) \right]^{\delta-1} \left\{ 1 - \gamma_1 \left[\delta, -\log \left(\frac{2}{1 + e^{\alpha x^\beta}} \right) \right] \right\}^{-1}.$$

Here again, if $x = x_\bullet$ is a root, then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether $\theta(x_\bullet) < 0$, $\theta(x_\bullet) > 0$, or $\theta(x_\bullet) = 0$, where $\theta(x) = \partial^2 \log[h(x)]/\partial x^2$. To save place, we omit the analytical expression of $\theta(x)$.

These critical points, as well as their nature, can be determined numerically using any mathematical softwares (R, Matlab, Mathematica...).

3 Some properties of the GPHL distribution

3.1 Quantile function

The quantile function of the GPHL distribution, denoted by $Q(u)$, satisfies the equation $F(Q(u)) = Q(F(u)) = u$, for any $u \in (0, 1)$. After some algebra, it is given by

$$Q(u) = \left\{ \frac{1}{\alpha} \log \left(2e^{\gamma_1^{-1}(\delta, u)} - 1 \right) \right\}^{1/\beta}, \quad u \in (0, 1), \quad (6)$$

where $\gamma_1^{-1}(\delta, u)$ denotes the inverse function of $\gamma_1(\delta, u)$ (the so-called inverse of the regularized lower incomplete gamma function, see DiDonato and Morris (1986)).

The median is given by $M_{ed} = Q(1/2)$. In a similar way, we can also determine the quartiles and octiles.

The quantile function is also useful to examine the skewness and kurtosis of the GPHL distribution. One can evaluate the Bowley skewness and the Moors kurtosis, for instance. The Bowley skewness is given by

$$B = \frac{Q(3/4) + Q(1/4) - 2Q(2/4)}{Q(3/4) - Q(1/4)}$$

and the Moors kurtosis is given by

$$M = \frac{Q(3/8) - Q(1/8) + Q(7/8) - Q(5/8)}{Q(3/4) - Q(1/4)}.$$

Further details on the Bowley skewness and Moors kurtosis can be found in Kenney and Keeping (1962) and Moors (1998), respectively. These measures have the advantages to be less sensitive to outliers and they exist even for distributions without moments.

A remarkable function related to $Q(u)$, with of statistical importance, is the quantile density function given by

$$q(u) = \frac{1}{f(Q(u))} = \frac{2\Gamma(\delta)e^{\gamma_1^{-1}(\delta,u)} \left\{ \frac{1}{\alpha} \log \left(2e^{\gamma_1^{-1}(\delta,u)} - 1 \right) \right\}^{1/\beta-1}}{\alpha\beta(2e^{\gamma_1^{-1}(\delta,u)} - 1) \left\{ \gamma_1^{-1}(\delta,u) \right\}^{\delta-1} e^{-\gamma_1^{-1}(\delta,u)}}, \quad u \in (0, 1).$$

The implication of the quantile density function in statistics is discussed in Parzen (1979).

3.2 Some distributional results

Here we develop some results in distribution involving the GPLH distribution.

3.2.1 Simple connexions

There exist connexions between the GPLH distribution and standard distributions. Some of these connexions are presented below.

Let U be a random variable following the uniform distribution over $(0, 1)$. Then, using the quantile function $Q(u)$ given by (6), the random variable X defined by

$$X = Q(U) = \left\{ \frac{1}{\alpha} \log \left(2e^{\gamma_1^{-1}(\delta,U)} - 1 \right) \right\}^{1/\beta} \quad (7)$$

follows the GPLH distribution.

Now, we say that a random variable follows the gamma distribution $\mathcal{G}_{am}(1, \delta)$ if it has the cdf given by $K(x) = \gamma_1(\delta, x)$, $x > 0$. If X is a random variable following the GPLH distribution, then the random variable Y defined by

$$Y = -\log \left(\frac{2}{1 + e^{\alpha X^\beta}} \right)$$

follows the gamma distribution $\mathcal{G}_{am}(1, \delta)$.

Also, if Y is a random variable following the gamma distribution $\mathcal{G}_{am}(1, \delta)$, since $\gamma(\delta, Y)$ follows the uniform distribution on $(0, 1)$, (7) implies that the random variable X defined by

$$X = Q(\gamma(\delta, Y)) = \left\{ \frac{1}{\alpha} \log (2e^Y - 1) \right\}^{1/\beta}$$

follows the GPLH distribution.

3.2.2 Some log GPLH distributions

Some log transformations of the GPLH distributions are now investigated, extending those in (Krishnarani, 2016, Subsection 5.1).

Let X be a random variable following the GPLH distribution.

- Let $Y = e^X$. Then, the corresponding pdf is given by

$$f(y) = \frac{2\alpha\beta\{\log(y)\}^{\beta-1}e^{\alpha\{\log(y)\}^\beta}}{\Gamma(\delta)y(1 + e^{\alpha\{\log(y)\}^\beta})^2} \left[-\log \left(\frac{2}{1 + e^{\alpha\{\log(y)\}^\beta}} \right) \right]^{\delta-1}, \quad y > 1.$$

The distribution of Y is called the log positive GPLH distribution.

- Let $Z = e^{-X}$. Then, the corresponding pdf is given by

$$f(z) = \frac{2\alpha\beta\{-\log(z)\}^{\beta-1}e^{\alpha\{-\log(z)\}^\beta}}{\Gamma(\delta)z(1+e^{\alpha\{-\log(z)\}^\beta})^2} \left[-\log\left(\frac{2}{1+e^{\alpha\{-\log(z)\}^\beta}}\right) \right]^{\delta-1}, \quad z \in (0, 1).$$

The distribution of Z is called the log negative GPHL distribution.

Simple distributions derived to the GPHL distribution, with one parameter and different supports, are presented below.

- Let $Y = e^{\alpha X^\beta}$. Then, the pdf of Y is given by

$$f(y) = \frac{1}{\Gamma(\delta)(1+y)^2} \left[-\log\left(\frac{2}{1+y}\right) \right]^{\delta-1}, \quad y > 1.$$

Let $Z = e^{-\alpha X^\beta}$. Then, the pdf of Z is given by

$$f(z) = \frac{1}{\Gamma(\delta)(1+z)^2} \left[-\log\left(\frac{2z}{1+z}\right) \right]^{\delta-1}, \quad z \in (0, 1).$$

3.3 Linear expansions for the pdf and cdf

The following result investigates useful expansions for $F(x)$ and $f(x)$ in terms of exponential function of the form $e^{-s\alpha x^\beta}$, where s denotes an integer.

Proposition 2 *The cdf $F(x)$ given by (4) can be expressed as*

$$F(x) = \sum_{\ell, m=0}^{+\infty} c_{\ell, m} e^{-(\ell+m)\alpha x^\beta}, \quad x > 0, \quad (8)$$

where

$$c_{\ell, m} = \frac{1}{\Gamma(\delta-1)} \sum_{k=0}^{+\infty} \sum_{j=0}^k \binom{k+1-\delta}{k} \binom{k}{j} \binom{\delta+k}{\ell} \binom{-\ell}{m} \frac{(-1)^{j+k+\ell} 2^\ell p_{j, k}}{(\delta-1-j)(\delta+k)},$$

$\binom{b}{a}$ denotes the (generalized) binomial coefficient, i.e., $\binom{b}{a} = b(b-1)\dots(b-a+1)/a!$ and $p_{j, k}$ is calculated recursively by using $p_{j, 0} = 1$ and, for strictly positive integer k ,

$$p_{j, k} = \frac{1}{k} \sum_{i=1}^k [k-i(j+1)] \frac{(-1)^{i+1}}{i+1} p_{j, k-i}.$$

Also, the pdf $f(x)$ given by (5) can be expressed as

$$f(x) = \sum_{\ell, m=0 (\ell+m>0)}^{+\infty} c_{\ell, m}^* \left[(\ell+m)\alpha\beta x^{\beta-1} e^{-(\ell+m)\alpha x^\beta} \right], \quad (9)$$

where $c_{\ell, m}^* = -c_{\ell, m}$.

Proof. It follows from a general result in (Nadarajah et al., 2015, Section 3) on the gamma-G family of distributions, with $\bar{G}(x)$ as baseline sf, that

$$F(x) = \sum_{k=0}^{+\infty} b_k [1 - \bar{G}(x)]^{\delta+k},$$

where

$$b_k = \frac{1}{(\delta + k)\Gamma(\delta - 1)} \binom{k+1-\delta}{k} \sum_{j=0}^k \frac{(-1)^{j+k}}{\delta - 1 - j} \binom{k}{j} p_{j,k}.$$

On the other hand, by the generalized binomial formula and $\bar{G}(x) = 2/(1 + e^{\alpha x^\beta}) = 2e^{-\alpha x^\beta}/(1 + e^{-\alpha x^\beta})$, we have

$$\begin{aligned} [1 - \bar{G}(x)]^{\delta+k} &= \sum_{\ell=0}^{+\infty} \binom{\delta+k}{\ell} (-1)^\ell [\bar{G}(x)]^\ell = \sum_{\ell=0}^{+\infty} \binom{\delta+k}{\ell} (-1)^\ell 2^\ell e^{-\ell\alpha x^\beta} [1 + e^{-\alpha x^\beta}]^{-\ell} \\ &= \sum_{\ell,m=0}^{+\infty} \binom{\delta+k}{\ell} \binom{-\ell}{m} (-1)^\ell 2^\ell e^{-(\ell+m)\alpha x^\beta}. \end{aligned}$$

By combining the equalities above, we obtain the desired expansion for $F(x)$. The expansion for $f(x)$ follows by differentiation. This ends the proof of Proposition 2. \square

One can remark that the expression of the GPHL pdf given by (9) is sums of pdfs of the Weibull distribution with parameters $(\ell + m)\alpha$ and β , i.e., with pdf $\kappa_{\ell,m}(x) = (\ell + m)\alpha\beta x^{\beta-1} e^{-(\ell+m)\alpha x^\beta}$, $x > 0$. This expression is useful to have expression of several probabilistic measures. Some of them are presented in the next subsections.

3.4 Moments and related measures

Hereafter, X denotes a random variable following the GPHL distribution, i.e., having the cdf given by (4).

Let r be an integer. Then, the (raw) r -th moment of X is given by

$$\mu'_r = \mathbb{E}(X^r) = \int_{-\infty}^{+\infty} x^r f(x) dx = \int_0^{+\infty} x^r \frac{2\alpha\beta x^{\beta-1} e^{\alpha x^\beta}}{\Gamma(\delta) (1 + e^{\alpha x^\beta})^2} \left[-\log \left(\frac{2}{1 + e^{\alpha x^\beta}} \right) \right]^{\delta-1} dx.$$

Since, as $x \rightarrow 0$, we have $x^r f(x) \sim [1/(2^\delta \Gamma(\delta))] \alpha^\delta \beta x^{r+\delta\beta-1}$ and, as $x \rightarrow +\infty$, we have $x^r f(x) \sim (1/\Gamma(\delta)) 2\alpha^\delta \beta x^{r+\delta\beta-1} e^{-\alpha x^\beta}$, μ'_r always exists by using the criteria of the Riemann integrals.

Alternative expressions of this integral are possible via some change of variables, as the following ones:

$$\mu'_r = \int_0^{+\infty} \left\{ \frac{1}{\alpha} \log(2e^y - 1) \right\}^{r/\beta} \frac{1}{\Gamma(\delta)} y^{\delta-1} e^{-y} dy = \int_0^1 \left\{ \frac{1}{\alpha} \log(2e^{\gamma_1^{-1}(\delta,u)} - 1) \right\}^{r/\beta} du.$$

This integral can be computed numerically by using any mathematical softwares.

Alternatively, we can use the expression of $f(x)$ given by (9). Hence, we have

$$\begin{aligned} \mu'_r &= \sum_{\ell,m=0}^{+\infty} c_{\ell,m}^* \int_0^{+\infty} x^r [(\ell + m)\alpha\beta x^{\beta-1} e^{-(\ell+m)\alpha x^\beta}] dx \\ &= \Gamma\left(\frac{r}{\beta} + 1\right) \sum_{\ell,m=0}^{+\infty} c_{\ell,m}^* [(\ell + m)\alpha]^{-r/\beta}. \end{aligned}$$

Several quantities can be derived to μ'_r . The most important of them are described below. The mean of X is given by μ'_1 . Furthermore, the r -th central moment of X is given by

$$\mu_r = \mathbb{E}([X - \mu'_1]^r) = \sum_{k=0}^r \binom{r}{k} (-1)^k (\mu'_1)^k \mu'_{r-k}. \quad (10)$$

The variance of X is given by $\mu_2 = \mu'_2 - (\mu'_1)^2$. The r -th descending factorial moment of X is given by

$$\mu'_{(r)} = \mathbb{E}[X(X-1)\dots(X-r+1)] = \sum_{k=0}^r s_{sti}(r, k) \mu'_k,$$

where $s_{sti}(r, k)$ denotes the Stirling number of the first kind defined by $s_{sti}(r, k) = (1/k!) [x(x-1)\dots(x-r+1)]^{(k)}|_{x=0}$.

Also, the cumulants of X can be calculated by the recursive formula given by

$$\kappa_r = \mu'_r - \sum_{k=1}^{r-1} \binom{r-1}{k-1} \mu'_{r-k} \kappa_k,$$

where, as initial value, $\kappa_1 = \mu'_1$.

In particular, we have $\kappa_2 = \mu'_2 - (\mu'_1)^2$, $\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3$ and $\kappa_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3(\mu'_2)^2 + 12\mu'_2(\mu'_1)^2 - 6(\mu'_1)^4$. The skewness γ_1 is given by $\gamma_1 = \kappa_3/\kappa_2^{3/2}$, the normalized kurtosis γ_2 is given by κ_4/κ_2^2 and the non normalized kurtosis is given by $\beta_2 = \gamma_2 + 3$.

The moment generating function of X is given by

$$M(t) = \int_{-\infty}^{+\infty} e^{tx} f(x) dx = \int_0^{+\infty} e^{tx} \frac{2\alpha\beta x^{\beta-1} e^{\alpha x^\beta}}{\Gamma(\delta) (1 + e^{\alpha x^\beta})^2} \left[-\log \left(\frac{2}{1 + e^{\alpha x^\beta}} \right) \right]^{\delta-1} dx.$$

Since, as $x \rightarrow 0$, we have $e^{tx} f(x) \sim (1/[2^\delta \Gamma(\delta)]) \alpha^\delta \beta x^{\delta\beta-1}$ and, as $x \rightarrow +\infty$, we have $e^{tx} f(x) \sim (1/\Gamma(\delta)) 2\alpha^\delta \beta x^{\delta\beta-1} e^{tx-\alpha x^\beta}$, $M(t)$ exists if $t \leq 0$ if $\beta > 0$ (without restriction, a priori), or $t \leq \alpha$ if $\beta = 1$, or $t \in \mathbb{R}$ if $\beta > 1$, by using criteria of the Riemann integrals. As for the raw moments, we can investigate several changes of variables to have a more tractable expression of the integral. Also, if the raw moments are available, we have

$$M(t) = \sum_{r=0}^{+\infty} \frac{t^r}{r!} \mu'_r.$$

An alternative expression follows from the expansion of $f(x)$ given by (9). Thus, we have

$$\begin{aligned} M(t) &= \sum_{\ell, m=0}^{+\infty} c_{\ell, m}^* \int_0^{+\infty} e^{tx} \left[(\ell + m) \alpha \beta x^{\beta-1} e^{-(\ell+m)\alpha x^\beta} \right] dx \\ &= \sum_{r, \ell, m=0}^{+\infty} c_{\ell, m}^* \frac{t^r}{r!} [(\ell + m) \alpha]^{-r/\beta} \Gamma \left(\frac{r}{\beta} + 1 \right). \end{aligned}$$

3.5 Incomplete moments

Let $\mathbf{1}_A$ be the indicator function of over an event A , i.e., $\mathbf{1}_A = 1$ if A is satisfied and $\mathbf{1}_A = 0$ elsewhere. Then, the r -th incomplete moment of X is defined by

$$\mu_r^*(t) = \mathbb{E}(X^r \mathbf{1}_{\{X \leq t\}}) = \int_{-\infty}^t x^r f(x) dx = \int_0^t x^r \frac{2\alpha\beta x^{\beta-1} e^{\alpha x^\beta}}{\Gamma(\delta) (1 + e^{\alpha x^\beta})^2} \left[-\log \left(\frac{2}{1 + e^{\alpha x^\beta}} \right) \right]^{\delta-1} dx.$$

Other integral expressions can be given via some change of variables. As for the moments, we can compute this integral numerically.

On the other side, by virtue of the linear expansion of $f(x)$ given by (9), we have

$$\begin{aligned}\mu_r^*(t) &= \sum_{\ell, m=0}^{+\infty} c_{\ell, m}^* \int_0^t x^r [(\ell + m)\alpha\beta x^{\beta-1} e^{-(\ell+m)\alpha x^\beta}] dx \\ &= \sum_{\ell, m=0}^{+\infty} c_{\ell, m}^* [(\ell + m)\alpha]^{-r/\beta} \gamma\left(\frac{r}{\beta} + 1, (\ell + m)\alpha t^\beta\right).\end{aligned}$$

Some important quantities involving $\mu_r^*(t)$ with $r = 1$ are described below. The mean deviation of X about the mean μ_1' is given by

$$\xi_1 = \mathbb{E}(|X - \mu_1'|) = 2\mu_1' F(\mu_1') - 2\mu_1^*(\mu_1').$$

Similarly, the mean deviation of X about the median M_{ed} is given by

$$\xi_2 = \mathbb{E}(|X - M_{ed}|) = \mu_1' - 2\mu_1^*(M_{ed}).$$

The Bonferroni curve is given by

$$B(u) = \frac{1}{u\mu_1'} \mu_1^*(Q(u)) = \frac{1}{u\mu_1'} \mu_1^* \left(\left\{ \frac{1}{\alpha} \log \left(2e^{\gamma_1^{-1}(\delta, u)} - 1 \right) \right\}^{1/\beta} \right), \quad u \in (0, 1).$$

The Lorenz curve is given by $L(u) = uB(u)$, $u \in (0, 1)$. These curves are useful in many areas as economics, insurance, reliability, demography and medicine.

3.6 Stochastic ordering

A result on the stochastic ordering involving the GPLH distribution with fixed parameters α and β is presented below.

Proposition 3 *Let X be a random variable having the pdf $f_1(x)$ given by (5) with parameters δ_1 , α and β and Y be a random variable having the pdf $f_2(x)$ given by (5) with parameters δ_2 , α and β . Then, if $\delta_1 \leq \delta_2$, we have $X \leq_{lr} Y$.*

Proof. We have

$$\frac{f_1(x)}{f_2(x)} = \frac{\Gamma(\delta_2)}{\Gamma(\delta_1)} \left[-\log \left(\frac{2}{1 + e^{\alpha x^\beta}} \right) \right]^{\delta_1 - \delta_2}.$$

Hence, by differentiation, since $\delta_1 \leq \delta_2$, we have

$$\frac{\partial}{\partial x} \frac{f_1(x)}{f_2(x)} = \frac{\Gamma(\delta_2)}{\Gamma(\delta_1)} (\delta_1 - \delta_2) \left[-\log \left(\frac{2}{1 + e^{\alpha x^\beta}} \right) \right]^{\delta_1 - \delta_2 - 1} \frac{\alpha \beta x^{\beta-1} e^{\alpha x^\beta}}{1 + e^{\alpha x^\beta}} \leq 0.$$

Therefore, the ratio function $f_1(x)/f_2(x)$ is decreasing, implying that $X \leq_{lr} Y$. This ends the proof of Proposition 3. \square

The complete theory on stochastic ordering can be found in Shaked (1994).

3.7 On a reliability parameter

Here, we investigate a reliability parameter related to the GPLH distribution with fixed parameter β . This parameter is defined as follows. Let X be a random variable having the pdf $f_1(x)$ given by (5) with parameters δ_1 , α_1 and β and Y be a random variable having the cdf $F_2(x)$ given by (4) with parameters δ_2 , α_2 and β , independent of X . Then, we consider the reliability parameter of the

GPHL distribution defined by $R = \mathbb{P}(Y < X)$. We refer to Kotz et al. (2003) for the implication of this parameter in the setting of the reliability theory. In terms of integrals, R can be expressed as

$$\begin{aligned} R &= \int_{-\infty}^{+\infty} f_1(x) F_2(x) dx \\ &= \int_0^{+\infty} \frac{2\alpha_1 \beta x^{\beta-1} e^{\alpha_1 x^\beta}}{\Gamma(\delta_1) (1 + e^{\alpha_1 x^\beta})^2} \left[-\log \left(\frac{2}{1 + e^{\alpha_1 x^\beta}} \right) \right]^{\delta_1-1} \gamma_1 \left[\delta_2, -\log \left(\frac{2}{1 + e^{\alpha_2 x^\beta}} \right) \right] dx \\ &= \int_0^{+\infty} \frac{2\alpha_1 e^{\alpha_1 y}}{\Gamma(\delta_1) (1 + e^{\alpha_1 y})^2} \left[-\log \left(\frac{2}{1 + e^{\alpha_1 y}} \right) \right]^{\delta_1-1} \gamma_1 \left[\delta_2, -\log \left(\frac{2}{1 + e^{\alpha_2 y}} \right) \right] dy. \end{aligned}$$

This integral can be computed numerically. Also, remark that it is independent of β . A linear expression can be given by using the linear expressions of $F_2(x)$ and $f_1(x)$, respectively given by (8) and (9) with the appropriate notations for the coefficients c_s and c_s^* according to the definitions of the parameters δ_2 , α_2 and δ_1 , α_1 , i.e.,

$$F_2(x) = \sum_{\ell, m=0}^{+\infty} c_{\ell, m}(\delta_2, \alpha_2) e^{-(\ell+m)\alpha_2 x^\beta}, \quad f_1(x) = \sum_{s, t=0}^{+\infty} c_{s, t}^*(\delta_1, \alpha_1) \left[(s+t)\alpha_1 \beta x^{\beta-1} e^{-(s+t)\alpha_1 x^\beta} \right].$$

Hence,

$$\begin{aligned} R &= \sum_{\ell, m, s, t=0}^{+\infty} c_{\ell, m}(\delta_2, \alpha_2) c_{s, t}^*(\delta_1, \alpha_1) \int_0^{+\infty} (s+t)\alpha_1 \beta x^{\beta-1} e^{-(\ell+m)\alpha_2 x^\beta - (s+t)\alpha_1 x^\beta} dx \\ &= \sum_{\ell, m, s, t=0}^{+\infty} c_{\ell, m}(\delta_2, \alpha_2) c_{s, t}^*(\delta_1, \alpha_1) \frac{(s+t)\alpha_1}{(\ell+m)\alpha_2 + (s+t)\alpha_1}. \end{aligned}$$

Furthermore, if $\delta_1 = \delta_2$ and $\alpha_1 = \alpha_2$, X and Y becomes identically distributed and we rediscover the value $R = 1/2$.

3.8 Order statistics

The order statistics are widely used in many statistical applications. Most of them are described in David and Nagaraja (2003), for instance. This subsection is devoted to the order statistics of the GPHL distribution. Let X_1, \dots, X_n be a random sample from the GPHL distribution and $X_{i:n}$ be the i -th order statistic, i.e., the i -th random variable such that, by arranging X_1, \dots, X_n in increasing order, we have $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$. In particular, the first order statistic is given by $X_{1:n} = \inf(X_1, X_2, \dots, X_n)$ and the last order statistic is given by $X_{n:n} = \sup(X_1, X_2, \dots, X_n)$. Then, the cdf of $X_{i:n}$ is given by

$$\begin{aligned} F_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} \frac{(-1)^k}{k+i} \binom{n-i}{k} [F(x)]^{k+i} \\ &= \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} \frac{(-1)^k}{k+i} \binom{n-i}{k} \left\{ \gamma_1 \left[\delta, -\log \left(\frac{2}{1 + e^{\alpha x^\beta}} \right) \right] \right\}^{k+i}, \quad x > 0. \end{aligned}$$

The corresponding pdf is given by

$$\begin{aligned}
f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} f(x) [F(x)]^{i-1} [1-F(x)]^{n-i} \\
&= \frac{n!}{(i-1)!(n-i)!} \frac{2\alpha\beta x^{\beta-1} e^{\alpha x^\beta}}{\Gamma(\delta) (1+e^{\alpha x^\beta})^2} \left[-\log \left(\frac{2}{1+e^{\alpha x^\beta}} \right) \right]^{\delta-1} \left\{ \gamma_1 \left[\delta, -\log \left(\frac{2}{1+e^{\alpha x^\beta}} \right) \right] \right\}^{i-1} \\
&\quad \times \left\{ 1 - \gamma_1 \left[\delta, -\log \left(\frac{2}{1+e^{\alpha x^\beta}} \right) \right] \right\}^{n-i}.
\end{aligned}$$

In particular, the pdf corresponding to $X_{1:n}$ is given by

$$f_{1:n}(x) = n \frac{2\alpha\beta x^{\beta-1} e^{\alpha x^\beta}}{\Gamma(\delta) (1+e^{\alpha x^\beta})^2} \left[-\log \left(\frac{2}{1+e^{\alpha x^\beta}} \right) \right]^{\delta-1} \left\{ 1 - \gamma_1 \left[\delta, -\log \left(\frac{2}{1+e^{\alpha x^\beta}} \right) \right] \right\}^{n-1}$$

and the pdf corresponding to $X_{n:n}$ is given by

$$f_{n:n}(x) = n \frac{2\alpha\beta x^{\beta-1} e^{\alpha x^\beta}}{\Gamma(\delta) (1+e^{\alpha x^\beta})^2} \left[-\log \left(\frac{2}{1+e^{\alpha x^\beta}} \right) \right]^{\delta-1} \left\{ \gamma_1 \left[\delta, -\log \left(\frac{2}{1+e^{\alpha x^\beta}} \right) \right] \right\}^{n-1}.$$

The r -th moment of $X_{i:n}$ is given by

$$\begin{aligned}
\mu_r^o &= \mathbb{E}(X_{i:n}^r) = \int_{-\infty}^{+\infty} x^r f_{i:n}(x) dx \\
&= \int_0^{+\infty} x^r \frac{n!}{(i-1)!(n-i)!} \frac{2\alpha\beta x^{\beta-1} e^{\alpha x^\beta}}{\Gamma(\delta) (1+e^{\alpha x^\beta})^2} \left[-\log \left(\frac{2}{1+e^{\alpha x^\beta}} \right) \right]^{\delta-1} \\
&\quad \times \left\{ \gamma_1 \left[\delta, -\log \left(\frac{2}{1+e^{\alpha x^\beta}} \right) \right] \right\}^{i-1} \left\{ 1 - \gamma_1 \left[\delta, -\log \left(\frac{2}{1+e^{\alpha x^\beta}} \right) \right] \right\}^{n-i} dx \\
&= \frac{n!}{(i-1)!(n-i)!} \int_0^{+\infty} \left\{ \frac{1}{\alpha} \log(2e^y - 1) \right\}^{r/\beta} \frac{1}{\Gamma(\delta)} y^{\delta-1} e^{-y} [\gamma_1(\delta, y)]^{i-1} [1 - \gamma_1(\delta, y)]^{n-i} dy \\
&= \frac{n!}{(i-1)!(n-i)!} \int_0^1 \left\{ \frac{1}{\alpha} \log(2e^{\gamma_1^{-1}(\delta, u)} - 1) \right\}^{r/\beta} u^{i-1} (1-u)^{n-i} du.
\end{aligned}$$

This integral can be evaluated numerically.

3.9 Ideas of extensions

Following the spirit of (Krishnarani, 2016, Subsection 5.2), a prime idea of extension of the GPHL distribution is to introduce a new parameter τ by considering the cdf

$$F(x) = \gamma_1 \left[\delta, -\log \left(\frac{1+\tau}{1+\tau e^{\alpha x^\beta}} \right) \right], \quad \delta, \alpha, \beta, \tau, x > 0.$$

The role of τ is to skew the tail of the distribution, by modulating the influence of the exponential function $e^{\alpha x^\beta}$. The related distribution is called the extended GPHL (EGPHL) distribution.

Another idea of extension is to define some general GPHL-G families of distributions. Two of them are described below. Let $G(x)$ be a cdf of an univariate continuous distribution. Then we can define the two following GPHL-G families of distributions:

- the odd GPHL-G family of distributions characterized by the cdf given by

$$F(x) = \gamma_1 \left[\delta, -\log \left(\frac{2}{1+e^{\alpha[G(x)]^\beta/(1-G(x))^\beta}} \right) \right], \quad \delta, \alpha, \beta > 0, \quad x \in \mathbb{R}.$$

- the log GPHL-G family of distributions characterized by the cdf given by

$$F(x) = \gamma_1 \left[\delta, -\log \left(\frac{2}{1 + e^{\alpha \{-\log[1-G(x)]\}^\beta}} \right) \right], \quad \delta, \alpha, \beta > 0, \quad x \in \mathbb{R}.$$

New flexible statistical models based on these ideas can be elaborated. This needs however further developments and validations, that we leave for a future work.

4 Estimation of parameters

4.1 Maximum likelihood estimation

We consider the estimation of the unknown parameters of the GPHL distribution by the method of maximum likelihood. Let x_1, \dots, x_n be a sample of size n from the GPHL distribution (characterized by the cdf given by (4) and with pdf (5)). The log-likelihood function for the vector of parameters $\Theta = (\delta, \alpha, \beta)^\top$ can be expressed as

$$\begin{aligned} \ell(\Theta) = & n \log \left(\frac{2\alpha\beta}{\Gamma(\delta)} \right) + (\beta - 1) \sum_{i=1}^n \log(x_i) + \alpha \sum_{i=1}^n x_i^\beta - 2 \sum_{i=1}^n \log \left(1 + e^{\alpha x_i^\beta} \right) \\ & + (\delta - 1) \sum_{i=1}^n \log \left[-\log \left(\frac{2}{1 + e^{\alpha x_i^\beta}} \right) \right]. \end{aligned}$$

The components of the score vector $U(\Theta)$ are given by

$$\begin{aligned} U_\delta(\Theta) &= \frac{\partial \ell(\Theta)}{\partial \delta} = -n \frac{\Gamma'(\delta)}{\Gamma(\delta)} + \sum_{i=1}^n \log \left[-\log \left(\frac{2}{1 + e^{\alpha x_i^\beta}} \right) \right], \\ U_\alpha(\Theta) &= \frac{\partial \ell(\Theta)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n x_i^\beta - \sum_{i=1}^n \frac{x_i^\beta e^{\alpha x_i^\beta}}{1 + e^{\alpha x_i^\beta}} + (\delta - 1) \sum_{i=1}^n \frac{x_i^\beta e^{\alpha x_i^\beta}}{\left[-\left(1 + e^{\alpha x_i^\beta} \right) \log \left(\frac{2}{1 + e^{\alpha x_i^\beta}} \right) \right]}, \\ U_\beta(\Theta) &= \frac{\partial \ell(\Theta)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^n \log(x_i) + \alpha \sum_{i=1}^n x_i^\beta \log(x_i) - 2\alpha \sum_{i=1}^n \frac{x_i^\beta \log(x_i) e^{\alpha x_i^\beta}}{1 + e^{\alpha x_i^\beta}} \\ &+ \alpha(\delta - 1) \sum_{i=1}^n \frac{x_i^\beta \log(x_i) e^{\alpha x_i^\beta}}{\left[-\left(1 + e^{\alpha x_i^\beta} \right) \log \left(\frac{2}{1 + e^{\alpha x_i^\beta}} \right) \right]}. \end{aligned}$$

Setting these equations equal to zero and solving them simultaneously also yields the maximum likelihood estimates (MLEs) of the model parameters. They cannot be solved analytically but some Newton-Raphson type algorithms can be applied to obtain numerical evaluations of these estimates. Under some regularity conditions, it is well-known that the maximum likelihood estimators are asymptotically unbiased, convergent and normal. These properties allow the construction of crucial statistical objects as confidence intervals and statistical tests. All the related theory can be found in Cox and Hinkley (1974). Some properties of the MLEs are illustrated in the next subsection via a simulation study.

4.2 Monte Carlo simulation study

Here, we evaluate the precision of the MLEs in the estimation of the GPHL parameters by using Monte Carlo simulations. The simulation study is repeated 5000 times each with sample sizes $n = 50, 100, 300$. The following parameter scenarios are considered: I: $\delta = 0.5, \alpha = 0.5$ and $\beta = 1$,

II: $\delta = 0.5$, $\alpha = 1.5$ $\beta = 1$, III: $\delta = 1.5$, $\alpha = 0.5$ and $\beta = 1$, IV: $\delta = 1.5$, $\alpha = 1.5$ and $\beta = 1$, V: $\delta = 1.5$, $\alpha = 2.5$ and $\beta = 0.5$, VI: $\delta = 0.7$, $\alpha = 0.3$ and $\beta = 2.5$.

Table 1 gives the average biases (Bias) of the MLEs, mean square errors (MSE) and model-based coverage probabilities (CPs) for the parameters δ , α and β under these scenarios and the different sample sizes. Based on the simulation results, we conclude that the MLEs perform well in estimating the parameters of the GPLH distribution. Furthermore, the CPs of the confidence intervals are quite close to the 95% nominal levels. Therefore, the MLEs and their asymptotic results can be adopted for estimating and constructing confidence intervals for the model parameters.

Table 1: Monte Carlo simulation results: Biases, MSEs and CPs.

Parameter	n	I			II			III		
		Bias	MSE	CP	Bias	MSE	CP	Bias	MSE	CP
δ	50	0.010	0.037	0.91	0.052	0.072	0.98	0.033	0.407	0.89
	100	0.013	0.023	0.94	0.010	0.026	0.98	0.063	0.214	0.94
	300	0.007	0.008	0.95	0.000	0.006	0.95	0.040	0.075	0.96
α	50	0.187	0.491	1.00	0.570	4.077	0.88	0.209	0.465	0.99
	100	0.062	0.058	0.98	0.305	1.390	0.92	0.062	0.093	0.98
	300	0.013	0.013	0.96	0.086	0.208	0.95	0.011	0.014	0.96
β	50	0.343	1.228	0.99	0.495	3.206	0.86	0.374	1.287	0.98
	100	0.131	0.275	0.97	0.269	1.115	0.91	0.115	0.326	0.97
	300	0.027	0.071	0.96	0.077	0.179	0.95	0.020	0.076	0.96
Parameter	n	IV			V			VI		
		Bias	MSE	CP	Bias	MSE	CP	Bias	MSE	CP
δ	50	0.271	0.938	0.99	0.358	1.035	1.00	-0.090	0.057	0.87
	100	0.122	0.294	0.97	0.067	0.273	1.00	-0.075	0.032	0.90
	300	0.022	0.059	0.95	0.002	0.056	0.99	-0.051	0.013	0.93
α	50	0.597	4.007	0.86	0.511	5.886	0.82	0.163	0.103	1.00
	100	0.256	1.511	0.90	0.638	4.383	0.90	0.093	0.029	1.00
	300	0.064	0.166	0.94	0.225	1.097	0.94	0.044	0.006	0.99
β	50	0.513	3.132	0.84	0.299	1.170	0.74	0.588	1.074	0.96
	100	0.218	1.193	0.89	0.303	0.853	0.86	0.341	0.426	0.94
	300	0.056	0.146	0.94	0.097	0.195	0.92	0.146	0.106	0.95

5 Illustrations with real data sets

In this section, the GPLH model is applied to model two real life data sets. We compare the fits of the GPLH model with the exponentiated Nadarajah-Haghighi (ENH) model defined by Lemonte (2013), the beta-exponential (BE) model introduced by Nadarajah and Kotz (2006) and the PHL model proposed by Krishnarani (2016). We estimate the model parameters by using the maximum likelihood method. We compare the goodness-of-fit of the models using Cramér-von Mises (W^*) and Anderson-Darling (A^*) statistics. Their mathematical definitions can be found in Chen and Balakrishnan (1995). In addition, we consider the Kolmogorov-Smirnov (K-S) statistic. In general, the smaller the values of these statistics, the better the fit to the data. The cdfs of the GPLH, ENH, BE and PHL models are given by:

Table 2: MLEs, their SEs (in parentheses) and goodness-of-fit measures for the strength data.

Distribution	Estimates			A^*	W^*	K-S
GPHL(δ, α, β)	0.8514 (0.2858)	5.5887 (1.3054)	0.0813 (0.0810)	0.9246	0.16780	0.1396
ENH(β, α, λ)	9.3842 (1.8948)	21.2951 (16.5962)	0.0421 (0.0342)	2.7858	0.5312	0.2107
BE(a, b, λ)	16.7598 (3.5990)	6.8285 (8.4755)	0.8587 (0.6948)	3.4713	0.6643	0.2265
PHL(α, β)	5.0492 (0.5193)	0.1252 (0.0385)		0.9590	0.1974	0.1890

$$F_{GPHL}(x, \delta, \alpha, \beta) = \gamma_1 \left[\delta, -\log \left(\frac{2}{1 + e^{\alpha x \beta}} \right) \right], \quad \delta, \alpha, \beta, x > 0,$$

$$F_{ENH}(x, \beta, \alpha, \lambda) = \left(1 - e^{1-(1+\lambda x)^\alpha} \right)^\beta, \quad \beta, \alpha, \lambda, x > 0,$$

$$F_{BE}(x, a, b, \lambda) = \frac{1}{\int_0^1 t^{a-1}(1-t)^{b-1} dt} \int_0^{1-e^{-\lambda x}} t^{a-1}(1-t)^{b-1} dt, \quad a, b, \lambda, x > 0,$$

$$F_{PHL}(x, \alpha, \beta) = \frac{2}{1 + e^{\alpha x \beta}}, \quad \alpha, \beta, x > 0.$$

Strength data

The first data set is taken from Smith and Naylor (1987) which represents the strength of 1.5 cm glass fibers, measured at National physical laboratory, England. The data are: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89. The MLEs (with SEs in parenthesis), A^* , W^* and K-S statistics are listed in Table 2. All three goodness-of-fit statistics indicate that the GPHL model provides the best fit. For a visual comparison, the histogram of the data estimated pdf and cdf, P-P plot and Q-Q plot are displayed in Figure 2. Clearly, the GPHL model fits the data more closely.

Breaking stress of carbon fibers (GPa)

The second data are taken from Nichols and Padgett (2006) on the breaking stress of carbon fibers (in Gba). The data are: 0.39, 3.70, 4.42, 2.67, 1.25, 1.89, 2.35, 2.74, 2.41, 2.93, 4.38, 2.88, 2.55, 2.73, 3.19, 3.22, 1.84, 2.82, 2.59, 2.50, 3.22, 3.39, 2.05, 2.03, 3.60, 1.69, 2.81, 3.68, 3.65, 1.61, 3.11, 3.28, 4.20, 2.48, 3.75, 2.12, 3.27, 3.09, 3.33, 0.85, 2.43, 3.15, 2.87, 1.87, 2.55, 1.61, 2.95, 1.08, 1.47, 3.15, 3.31, 2.79, 2.97, 2.56, 3.11, 4.90, 3.31, 4.70, 3.39, 1.80, 3.56, 1.57, 2.85, 2.03, 2.96, 2.53. The MLEs (with SEs in parenthesis), A^* , W^* and K-S statistics are listed in Table 3. Again, all three goodness-of-fit statistics indicate that the GPHL model provides the best fit. For a visual comparison, the histogram of the data estimated pdf and cdf, P-P plot and Q-Q plot are displayed in Figure 3. We observe that the GPHL model fits well the data.

Table 3: MLEs, their SEs (in parentheses) and goodness-of-fit measures for the breaking stress of carbon fibers.

Distribution	Estimates			A^*	W^*	K-S
GPHL(δ, α, β)	1.0482 (0.3983)	2.8713 (0.7405)	0.0621 (0.0752)	0.4029	0.0663	0.0743
ENH(β, α, λ)	4.5816 (0.8390)	6.0911 (2.0535)	0.0727 (0.0280)	0.9009	0.1722	0.1143
BE(a, b, λ)	7.4229 (1.3408)	6.0563 (4.0672)	0.3060 (0.1577)	1.4260	0.2657	0.1371
PHL(α, β)	2.9579 (0.2949)	0.0537 (0.0204)		1.4074	0.3665	0.1746

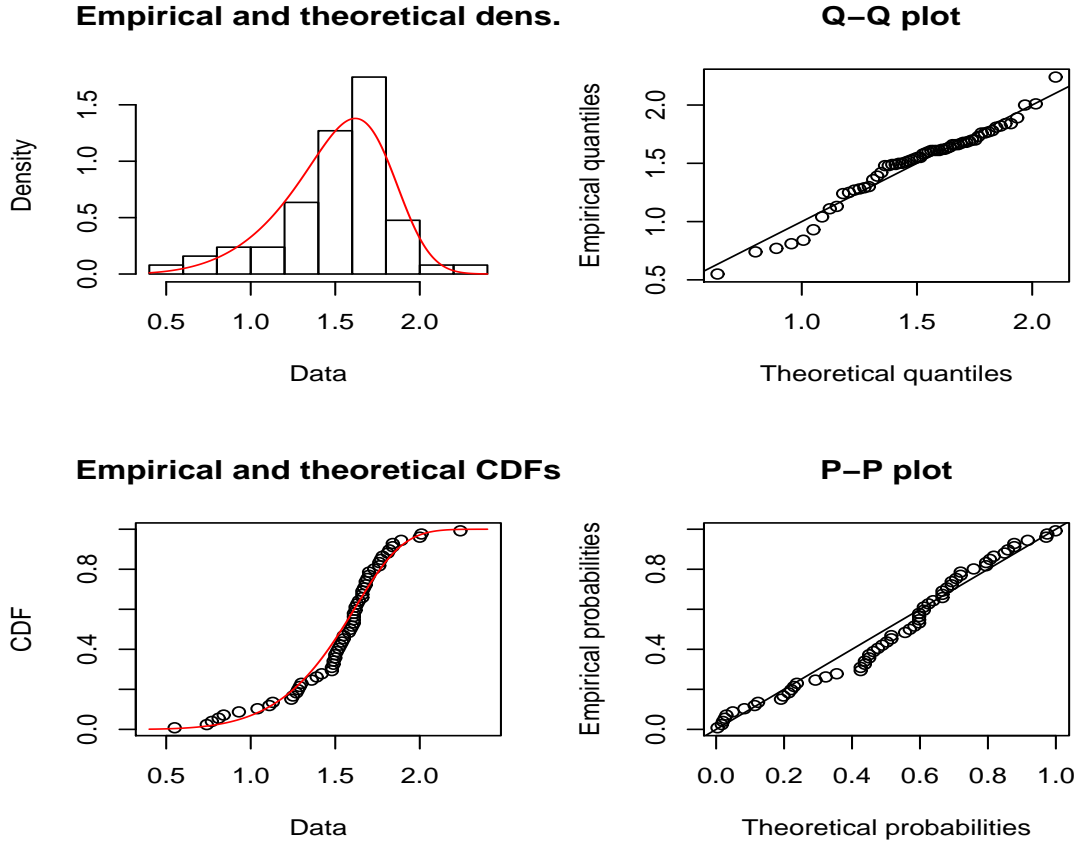


Figure 2: The fitted pdfs, cdfs, Q-Q and P-P plots for the strength data.

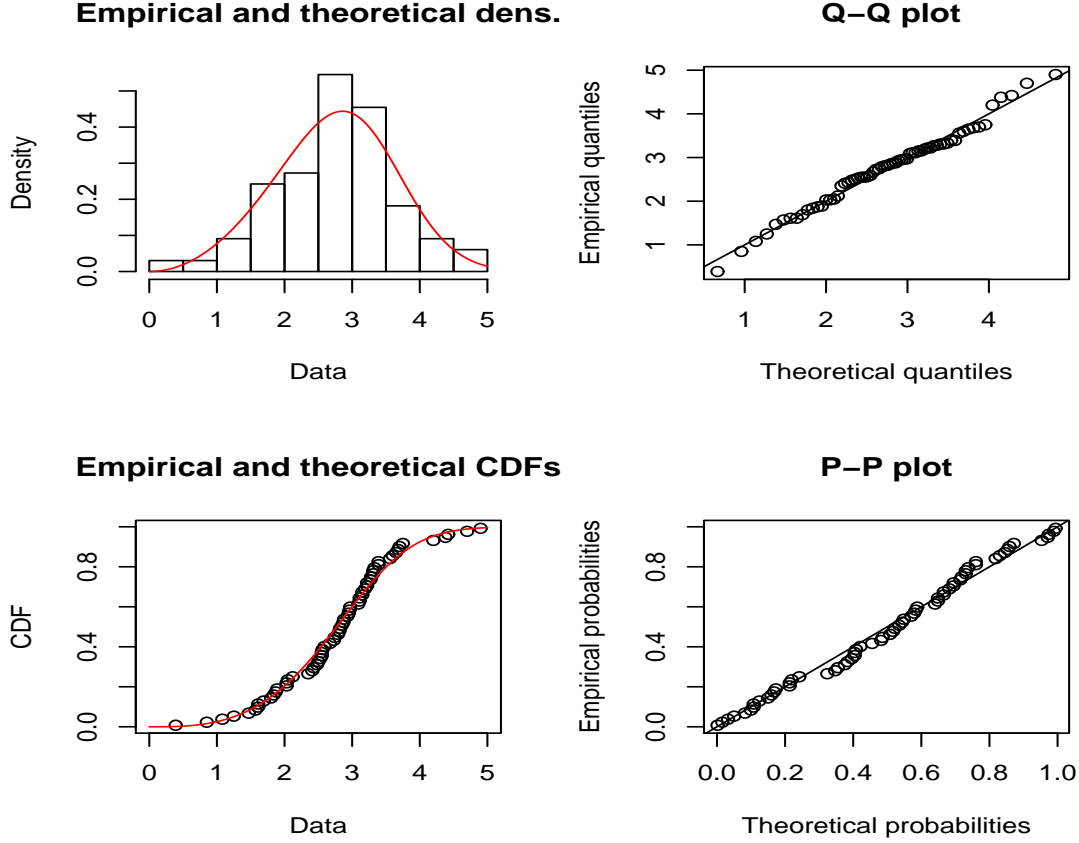


Figure 3: The fitted pdfs, cdfs, Q-Q and P-P plots for the breaking stress of carbon fibers.

6 Concluding remarks

A new distribution on the positive real-line is constructed using the gamma-G family of distributions introduced by Zografos and Balakrishnan (2009) and the power half-logistic distribution introduced by Krishnarani (2016). It is called the gamma power half-logistic distribution (GPHL for short). Among its advantages, the GPHL distribution possesses a hazard rate function with very flexible behavior. We also investigate several of its analytical properties as quantiles, skewness, kurtosis, moments, incomplete moments, mean deviations, Bonferroni and Lorenz curves, stochastic ordering, reliability parameter and distribution of order statistics. Then, the estimation of the GPHL model parameters are done by using the maximum likelihood function. Finally, the GPHL model is used to analyze two real data sets in order to illustrate its usefulness. Also, some possible extensions of the GPHL model are introduced, investigations on their applicability will be conducted in a future work.

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