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Coupled heat equation in thermoelasticity with temperature dependent moduli

Djaffar Boussaa
boussaa@lma.cnrs-mrs.fr

Aix Marseille Univ, CNRS, Centrale Marseille, LMA UMR 7031, Marseille, France

Abstract

New and consistent expressions for the coupled heat equation are developed within the framework of small-strain thermoelasticity for both the Fourier and Cattaneo–Vernotte conduction models. These expressions place no restrictions on the changes in temperature, allow for the temperature dependence of the thermoelastic moduli, and include all the coupling terms as functions of the thermoelastic moduli and their derivatives. As applications, (i) an extended Lord–Shulman-type model is derived that takes into account the temperature dependence of the thermoelastic moduli, and (ii) the equations underpinning the experimental technique of thermoelastic stress analysis are revisited.

Keywords: generalized thermoelasticity; temperature dependent properties; Cattaneo–Vernotte heat conduction equation; Lord–Shulman model; thermoelastic stress analysis

1 Introduction

The thermoelastic coupling terms occurring in the heat conduction equation are usually small and neglected in practice [1–3]. Taking them into account, however, has been found important in many applications, such as the modeling of the vibration of resonant microelectromechanical systems (MEMS) [4–6], dynamic crack propagation [7–9], thermal shocks [10–14], ultrafast laser heating in thermal processing of materials [15–18], and wave propagation [19–22]. In some of these applications, using the Cattaneo–Vernotte conduction model instead of Fourier’s law and/or accounting for the temperature dependence of the material moduli have resulted in better predictions [14, 23, 24].

The interest in retaining thermoelastic coupling terms is not limited to extreme applications. For instance, these terms are the foundation of the experimental technique known as thermoelastic stress analysis (TSA), which does not involve severe conditions. In a typical TSA test, temperature changes of an order of magnitude of one thousandth of a Kelvin are induced by sinusoidal reversible straining at a frequency of the order of magnitude of ten Hertz [25, 26]. Interestingly, despite the smallness of the temperature changes in TSA tests, models that (i) take into account the temperature dependence of the material moduli and (ii) involve their temperature-derivatives have been found to explain experimental observations more satisfactorily than linear thermoelasticity [27–29], in which the elasticities and the coefficients of thermal expansion are assumed temperature-independent.

In the case of small strains, small temperature changes, and temperature-independent thermoelastic moduli, there is a consensus on the explicit form of the coupled heat equation for both the Fourier and Cattaneo–Vernotte conduction models [30]. In the case of large temperature changes and temperature-dependent thermoelastic moduli, there currently seems to be no consensus on the explicit form of the coupled heat equation, even for the Fourier conduction model. So far, several distinct expressions have been used that (i) involve different, often implicit, assumptions, and (ii) generally lack some of the coupling terms. Thus, the expressions obtained by letting the material moduli depend on temperature in the coupled heat equation obtained in linear thermoelasticity [14, 31–35] implicitly neglect, among other things, the coupling terms involving the temperature derivatives of the elasticity tensor. Similarly, the expressions stemming from polynomial expansions of the Helmholtz energy to an order higher than second in the temperature change [14, 36–38] also lack some of the coupling terms because of the truncation. Moreover, such expansions result in forms for the thermoelastic moduli, as functions of temperature, that do not fit well the actual temperature dependence of these moduli in many instances, for example, between room and cryogenic temperatures [39].

The purpose of this study is to provide expressions for the coupled heat equation that include all the coupling terms for both the Fourier and Cattaneo–Vernotte conduction models while taking into account large temperature changes, defined as temperature changes large enough to require consideration of the temperature dependence of the thermoelastic moduli. The thermoelastic

framework used here assumes the strain to be small but does not place the usual restrictions imposed in linear thermoelasticity on the temperature changes [40–42]. The approach is based on the expressions of the thermodynamic potentials obtained in [43].

The rest of the paper is organized as follows. Section 2 reviews some basic thermodynamic results of thermoelasticity with temperature-dependent material moduli. Section 3 introduces thermoelastic moduli that enter the expressions of the coupled heat equations to be developed in later sections. The presentation emphasizes the dependence of these moduli on the mechanical variables in addition to temperature. Section 4 develops the coupled heat equation for both the Fourier and the Cattaneo–Vernotte conduction models, with strain and temperature as independent variables. Section 5 does the same with stress and temperature as independent variables. Section 6 applies the results of Sections 4 and 5 to obtain (i) approximate expressions for the coupled heat equation, including a Lord–Shulman-type model that accounts for the temperature dependence of the thermoelastic moduli and (ii) a specialization of the coupled heat equations to the adiabatic case, which is the framework underpinning the TSA. Section 7 offers conclusions.

2 Framework, energy balance, potentials and heat conduction laws

2.1 Framework

The framework of this study is small-strain thermoelasticity with allowance made for large changes in temperature, as defined in the introduction, and temperature dependence of the material moduli [40, 41, 43]. The basic assumption of the framework is that the stress-strain relation is an affine function with temperature-dependent coefficients. This amounts to assuming that the elasticity tensor and, as to be expected from a thermoelasticity model, the thermal strain tensor are temperature dependent. This assumption fixes the form of the thermodynamic potentials as functions of temperature and either stress or strain [44], and therefore that of the remaining thermoelastic moduli, i.e., the heat capacities, the coefficient of thermal expansion (CTE) tensor, and the stress-temperature tensor.

2.2 Local energy balance equation

In small deformation, the local form of the energy balance equation reads [45, p. 42]

$$\dot{u} = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\epsilon}} - \operatorname{div} \boldsymbol{q} + r, \quad (1)$$

where the central dot denotes the inner product between two second-order tensors, the overdot denotes differentiation with respect to time, u is the internal energy per unit reference volume, $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{\epsilon}$ is the (small) strain

tensor, \mathbf{q} is the heat flux vector, and r is the heat supply per unit reference volume.

Let h denote the enthalpy per unit reference volume. The identity

$$u = h + \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} \quad (2)$$

allows rewriting the energy balance equation (1) in terms of the enthalpy as

$$\dot{h} = -\boldsymbol{\epsilon} \cdot \dot{\boldsymbol{\sigma}} - \operatorname{div} \mathbf{q} + r. \quad (3)$$

2.3 Helmholtz potential in the case of temperature-dependent moduli

The Helmholtz potential per unit reference volume, expressed in terms of its natural variables, is given by [44]

$$\begin{aligned} f(\boldsymbol{\epsilon}, T) &= \frac{1}{2} \boldsymbol{\epsilon} \cdot \mathbf{L}(T) \boldsymbol{\epsilon} + \mathbf{l}(T) \cdot \boldsymbol{\epsilon} \\ &\quad - \int_{T_0}^T \left(\int_{T_0}^{\xi} \frac{C_{\boldsymbol{\epsilon}}(\mathbf{0}, \nu)}{\nu} d\nu \right) d\xi - \eta_0 (T - T_0) + f_0. \end{aligned} \quad (4)$$

where T is the temperature; T_0 is the reference temperature; \mathbf{L} is the isothermal elasticity tensor; \mathbf{l} is the thermal stress, i.e., the stress that develops in the material when the temperature is made to vary under zero-strain conditions; $C_{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, T)$ is the heat capacity at constant strain per unit reference volume at a state $(\boldsymbol{\epsilon}, T)$ and $C_{\boldsymbol{\epsilon}}(\mathbf{0}, T)$ its value at the state $(\mathbf{0}, T)$; ν and ξ are dummy variables; and η_0 and f_0 are the values of the entropy and the Helmholtz potential at the reference state, respectively. The elasticity tensor \mathbf{L} is assumed to be symmetric, and the thermal stress \mathbf{l} is assumed to vanish at the reference state (i.e., $\mathbf{l}(T_0) = \mathbf{0}$).

From

$$\boldsymbol{\sigma} = \left(\frac{\partial f}{\partial \boldsymbol{\epsilon}} \right)_T, \quad (5)$$

$$\eta = - \left(\frac{\partial f}{\partial T} \right)_{\boldsymbol{\epsilon}}, \quad (6)$$

it follows that the state equations associated with the Helmholtz potential (4) are

$$\boldsymbol{\sigma} = \mathbf{L}(T) \boldsymbol{\epsilon} + \mathbf{l}(T), \quad (7)$$

$$\eta = -\frac{1}{2} \boldsymbol{\epsilon} \cdot \frac{d\mathbf{L}}{dT} \boldsymbol{\epsilon} - \frac{d\mathbf{l}}{dT} \cdot \boldsymbol{\epsilon} + \int_{T_0}^T \frac{C_{\boldsymbol{\epsilon}}(\mathbf{0}, \nu)}{\nu} d\nu + \eta_0, \quad (8)$$

where η is the entropy.

Combining the Gibbs–Helmholtz equation

$$u = f - T \left(\frac{\partial f}{\partial T} \right)_{\boldsymbol{\epsilon}} \quad (9)$$

with the Helmholtz potential (4) yields the following expression for the internal energy:

$$u(\boldsymbol{\epsilon}, T) = \frac{1}{2} \boldsymbol{\epsilon} \cdot \left(\mathbf{L} - T \frac{d\mathbf{L}}{dT} \right) \boldsymbol{\epsilon} + \left(\mathbf{l} - T \frac{d\mathbf{l}}{dT} \right) \cdot \boldsymbol{\epsilon} + \int_{T_0}^T C_{\boldsymbol{\epsilon}}(\mathbf{0}, \nu) d\nu + f_0 + \eta_0 T_0. \quad (10)$$

2.4 Gibbs potential in the case of temperature-dependent moduli

The Gibbs potential per unit reference volume, expressed in terms of its natural variables, is given by [44]

$$g(\boldsymbol{\sigma}, T) = -\frac{1}{2} \boldsymbol{\sigma} \cdot \mathbf{M}(T) \boldsymbol{\sigma} - \mathbf{m}(T) \cdot \boldsymbol{\sigma} - \int_{T_0}^T \left(\int_{T_0}^{\xi} \frac{C_{\boldsymbol{\sigma}}(\mathbf{0}, \nu)}{\nu} d\nu \right) d\xi + g_0 - \eta_0 (T - T_0). \quad (11)$$

where $\mathbf{M} = \mathbf{L}^{-1}$ is the compliance tensor; \mathbf{m} is the thermal strain, i.e., the strain, measured from the reference state, that develops in the material when the temperature is made to vary under zero-stress conditions; $C_{\boldsymbol{\sigma}}(\boldsymbol{\sigma}, T)$ is the heat capacity at constant stress per unit reference volume at a state $(\boldsymbol{\sigma}, T)$ and $C_{\boldsymbol{\sigma}}(\mathbf{0}, T)$ is its value at the state $(\mathbf{0}, T)$; and g_0 is the value of the Gibbs potential at the reference state. By definition, $\mathbf{m}(T_0) = \mathbf{0}$.

From

$$\boldsymbol{\epsilon} = - \left(\frac{\partial g}{\partial \boldsymbol{\sigma}} \right)_{T}, \quad (12)$$

$$\eta = - \left(\frac{\partial g}{\partial T} \right)_{\boldsymbol{\sigma}}, \quad (13)$$

it follows that the state equations associated with the Gibbs potential (11) are

$$\boldsymbol{\epsilon} = \mathbf{M}(T) \boldsymbol{\sigma} + \mathbf{m}(T), \quad (14)$$

$$\eta = \frac{1}{2} \boldsymbol{\sigma} \cdot \frac{d\mathbf{M}}{dT} \boldsymbol{\sigma} + \frac{d\mathbf{m}}{dT} \cdot \boldsymbol{\sigma} + \int_{T_0}^T \frac{C_{\boldsymbol{\sigma}}(\mathbf{0}, \nu)}{\nu} d\nu + \eta_0. \quad (15)$$

Combining the Gibbs–Helmholtz equation

$$h = g - T \left(\frac{\partial g}{\partial T} \right)_{\boldsymbol{\sigma}} \quad (16)$$

with the Gibbs potential (11) gives

$$h(\boldsymbol{\sigma}, T) = -\frac{1}{2}\boldsymbol{\sigma} \cdot \left(\mathbf{M} - T \frac{d\mathbf{M}}{dT} \right) \boldsymbol{\sigma} - \left(\mathbf{m} - T \frac{d\mathbf{m}}{dT} \right) \cdot \boldsymbol{\sigma} + \int_{T_0}^T C_{\boldsymbol{\sigma}}(\mathbf{0}, \nu) d\nu + g_0 + \eta_0 T_0. \quad (17)$$

2.5 Heat conduction laws

Two heat conduction laws are used below. The first is Fourier's law, i.e.,

$$\mathbf{q} = -\mathbf{k} \text{grad } T, \quad (18)$$

where \mathbf{k} is the heat conductivity tensor, which, in general, is a function of the thermodynamic state.

The second is the Cattaneo–Vernotte heat conduction law, which is characterized by the following equation:

$$\mathbf{q} + \tau \dot{\mathbf{q}} = -\mathbf{k} \text{grad } T, \quad (19)$$

where τ is the relaxation time, assumed to be a nonnegative constant. This equation is postulated to hold as given for general anisotropy [46, 47].

3 Thermoelastic moduli

3.1 Coefficient of thermal expansion tensor

The CTE tensor, $\boldsymbol{\alpha}$, is defined by

$$\boldsymbol{\alpha} = \left(\frac{\partial \boldsymbol{\epsilon}}{\partial T} \right)_{\boldsymbol{\sigma}}. \quad (20)$$

From (14) it follows that

$$\boldsymbol{\alpha}(\boldsymbol{\sigma}, T) = \frac{d\mathbf{M}}{dT} \boldsymbol{\sigma} + \frac{d\mathbf{m}}{dT}. \quad (21)$$

Unlike in linear thermoelasticity, where \mathbf{M} is independent of T and \mathbf{m} is linearly dependent on T so that $\boldsymbol{\alpha}$ is independent of the state, equation (21) shows that the CTE tensor depends on $\boldsymbol{\sigma}$ in addition to T .

The free thermal expansion coefficient (CFTE) tensor, $\boldsymbol{\alpha}_0(T)$, is defined as $\boldsymbol{\alpha}(\mathbf{0}, T)$ in (21), i.e.,

$$\boldsymbol{\alpha}_0(T) = \frac{d\mathbf{m}}{dT}, \quad (22)$$

whence

$$\mathbf{m}(T) = \int_{T_0}^T \boldsymbol{\alpha}_0(\nu) d\nu, \quad (23)$$

where $\mathbf{m}(T_0) = \mathbf{0}$ was used.

The CFTE tensor defined by (22) is referred to as the tangent or instantaneous CFTE tensor. Another CFTE tensor is used in the literature, namely, the secant CFTE tensor, which is defined as

$$\boldsymbol{\alpha}_0^{\text{sec}}(\nu) = \frac{1}{T - T_0} \mathbf{m}(T) = \frac{1}{T - T_0} \int_{T_0}^T \boldsymbol{\alpha}_0(\nu) \, d\nu. \quad (24)$$

In terms of this quantity, the stress-strain relation reads

$$\boldsymbol{\epsilon} = \mathbf{M}(T) \boldsymbol{\sigma} + \boldsymbol{\alpha}_0^{\text{sec}}(T) (T - T_0). \quad (25)$$

In linear thermoelasticity, $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}_0$ and $\boldsymbol{\alpha}_0^{\text{sec}}$ coincide and the fact that they are not distinguished from each other has no consequences. This is not the case if the temperature dependence of the elasticities is taken into consideration.

3.2 Stress-temperature tensor

The stress-temperature tensor $\boldsymbol{\beta}$ is defined by

$$\boldsymbol{\beta} = \left(\frac{\partial \boldsymbol{\sigma}}{\partial T} \right)_{\boldsymbol{\epsilon}}. \quad (26)$$

From (7) it follows that

$$\boldsymbol{\beta}(\boldsymbol{\epsilon}, T) = \frac{d\mathbf{L}}{dT} \boldsymbol{\epsilon} + \frac{d\mathbf{l}}{dT}. \quad (27)$$

As with $\boldsymbol{\alpha}$, the dependence of the elasticities on T causes a dependence of $\boldsymbol{\beta}$ on the mechanical variable, here $\boldsymbol{\epsilon}$.

The discussion in Subsection 3.1 on $\boldsymbol{\alpha}$ can be repeated for $\boldsymbol{\beta}$ in an obvious manner, yielding

$$\boldsymbol{\beta}_0(T) = \boldsymbol{\beta}(\mathbf{0}, T), \quad (28)$$

$$\mathbf{l}(T) = \int_{T_0}^T \boldsymbol{\beta}_0(\nu) \, d\nu, \quad (29)$$

and

$$\boldsymbol{\beta}_0^{\text{sec}}(\nu) = \frac{1}{T - T_0} \mathbf{l}(T) = \frac{1}{T - T_0} \int_{T_0}^T \boldsymbol{\beta}_0(\nu) \, d\nu, \quad (30)$$

as well as

$$\boldsymbol{\sigma} = \mathbf{L}(T) \boldsymbol{\sigma} + \boldsymbol{\beta}_0^{\text{sec}}(T) (T - T_0). \quad (31)$$

3.3 Heat capacities

The heat capacity per unit reference volume at constant strain, $C_{\boldsymbol{\epsilon}}$, is defined by

$$C_{\boldsymbol{\epsilon}} = T \left(\frac{\partial \eta}{\partial T} \right)_{\boldsymbol{\epsilon}}. \quad (32)$$

It follows from (8) that

$$C_{\epsilon}(\epsilon, T) = C_{\epsilon}(\mathbf{0}, T) - T \left(\frac{1}{2} \epsilon \cdot \frac{d^2 \mathbf{L}}{dT^2} \epsilon + \frac{d^2 \mathbf{l}}{dT^2} \cdot \epsilon \right). \quad (33)$$

In view of (10), it is possible to check that, as was to be expected,

$$\left(\frac{\partial u}{\partial T} \right)_{\epsilon} = C_{\epsilon}(\epsilon, T). \quad (34)$$

The heat capacity per unit reference volume at constant stress, C_{σ} , is defined by

$$C_{\sigma} = T \left(\frac{\partial \eta}{\partial T} \right)_{\sigma}, \quad (35)$$

which, combined with (8), gives

$$C_{\sigma}(\sigma, T) = C_{\sigma}(\mathbf{0}, T) + T \left(\frac{1}{2} \sigma \cdot \frac{d^2 \mathbf{M}}{dT^2} \sigma + \frac{d^2 \mathbf{m}}{dT^2} \cdot \sigma \right). \quad (36)$$

In view of (17), one can check that, as was to be expected,

$$\left(\frac{\partial h}{\partial T} \right)_{\sigma} = C_{\sigma}(\sigma, T). \quad (37)$$

Within the thermoelasticity framework used here, the thermoelastic moduli discussed in this subsection depend not only on the temperature but also on the mechanical variable, which will increase the number of coupling terms.

3.4 Remark

If expressions in terms of the thermal strain, \mathbf{m} , its first T -derivative, α_0 , and its second T -derivative, $d\alpha_0/dT$, are preferred to those involving the thermal stress, \mathbf{l} , then the following equations can be used:

$$\mathbf{l} = -\mathbf{L}\mathbf{m}, \quad (38)$$

$$\frac{d\mathbf{l}}{dT} = -\frac{d\mathbf{L}}{dT}\mathbf{m} - \mathbf{L}\alpha_0, \quad (39)$$

$$\frac{d^2 \mathbf{l}}{dT^2} = -\frac{d^2 \mathbf{L}}{dT^2} \mathbf{m} - 2 \frac{d\mathbf{L}}{dT} \alpha_0 - \mathbf{L} \frac{d\alpha_0}{dT}. \quad (40)$$

Equation (38) can be derived from (7) by noting that during a free thermal expansion $\sigma = \mathbf{0}$ and $\epsilon = \mathbf{m}$.

4 Coupled heat equation with strain and temperature as independent variables

4.1 General form

From (1), (7), (10), (26), (27), (33) and (34) it follows that

$$C_{\epsilon}(\epsilon, T) \dot{T} = -\operatorname{div} \mathbf{q} + T \boldsymbol{\beta} \cdot \dot{\epsilon} + r. \quad (41)$$

4.2 Case of Fourier's law

Inserting (18) into (41) gives

$$\operatorname{div}(\mathbf{k} \operatorname{grad} T) + r = C_\epsilon(\boldsymbol{\epsilon}, T) \dot{T} - T \boldsymbol{\beta}(\boldsymbol{\epsilon}, T) \cdot \dot{\boldsymbol{\epsilon}}. \quad (42)$$

The thermoelastic coupling occurs through not only $\dot{\boldsymbol{\epsilon}}$ terms as in linear thermoelasticity but also through $\boldsymbol{\epsilon}$ terms, in view of the expressions of $\boldsymbol{\beta}(\boldsymbol{\epsilon}, T)$ and $C_\epsilon(\boldsymbol{\epsilon}, T)$ as given by (27) and (33), respectively. Another potential source of coupling is the possible dependence of \mathbf{k} on $\boldsymbol{\epsilon}$. A thermoelastic model where the two-way coupling is caused by the dependence of \mathbf{k} on $\boldsymbol{\epsilon}$ is described in [48].

The coupling in (42) is now discussed according to whether or not the conductivity depends on the strain.

Case of $\boldsymbol{\epsilon}$ -dependent conductivity If the conductivity depends on the strain, then the two-way coupling is operative whether or not the problem depends on time.

Case of $\boldsymbol{\epsilon}$ -independent conductivity If the conductivity is independent of the strain, then, in general, the two-way coupling is inoperative in time-independent problems and operative in time-dependent problems. For the two-way coupling to be inoperative and the heat equation (42) uncoupled from the mechanical fields in the time-dependent case, it is necessary that the stress-temperature tensor $\boldsymbol{\beta}$ be zero. The vanishing of $\boldsymbol{\beta}$ implies that \mathbf{L} is independent of T and \mathbf{l} is zero. In turn, this implies that C_ϵ is independent of the strain. The vanishing of the stress-temperature tensor $\boldsymbol{\beta}$ is therefore also a sufficient condition for the heat equation to be uncoupled from the mechanical fields.

4.3 Case of the Cattaneo–Vernotte law

Combining (41) and its time derivative with (19) gives

$$\begin{aligned} \operatorname{div}(\mathbf{k} \operatorname{grad} T) + r + \tau \dot{r} &= \tau C_\epsilon \ddot{T} \\ &+ \tau \left(\frac{\partial C_\epsilon}{\partial T} \right)_\epsilon \dot{T}^2 \\ &+ \left[C_\epsilon + \tau \left(\left(\frac{\partial C_\epsilon}{\partial \boldsymbol{\epsilon}} \right)_T - \boldsymbol{\beta} - T \left(\frac{\partial \boldsymbol{\beta}}{\partial T} \right)_\epsilon \right) \cdot \dot{\boldsymbol{\epsilon}} \right] \dot{T} \\ &- \left[\boldsymbol{\beta} \cdot \dot{\boldsymbol{\epsilon}} + \tau \left(\boldsymbol{\beta} \cdot \ddot{\boldsymbol{\epsilon}} + \dot{\boldsymbol{\epsilon}} \cdot \left(\frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\epsilon}} \right)_T \dot{\boldsymbol{\epsilon}} \right) \right] T, \end{aligned} \quad (43)$$

where

$$\begin{aligned}\left(\frac{\partial C_\epsilon}{\partial \epsilon}\right)_T &= -T \left(\frac{d^2 \mathbf{L}}{dT^2} \epsilon + \frac{d^2 \mathbf{l}}{dT^2} \right), \\ \left(\frac{\partial C_\epsilon}{\partial T}\right)_\epsilon &= \frac{\partial C_\epsilon}{\partial T}(\mathbf{0}, T) - \frac{1}{2} \epsilon \cdot \frac{d^2 \mathbf{L}}{dT^2} \epsilon - \frac{d^2 \mathbf{l}}{dT^2} \cdot \epsilon \\ &\quad - T \left(\frac{1}{2} \epsilon \cdot \frac{d^3 \mathbf{L}}{dT^3} \epsilon + \frac{d^3 \mathbf{l}}{dT^3} \cdot \epsilon \right),\end{aligned}$$

and

$$\begin{aligned}\left(\frac{\partial \beta}{\partial \epsilon}\right)_T &= \frac{d\mathbf{L}}{dT}, \\ \left(\frac{\partial \beta}{\partial T}\right)_\epsilon &= \frac{d^2 \mathbf{L}}{dT^2} \epsilon + \frac{d^2 \mathbf{l}}{dT^2}.\end{aligned}$$

The following higher order Maxwell relation can be used in (43)

$$\left(\frac{\partial C_\epsilon}{\partial \epsilon}\right)_T = -T \left(\frac{\partial \beta}{\partial T}\right)_\epsilon. \quad (44)$$

Similar conclusions to those reached in subsection 4.2 on the thermoelastic coupling hold true in this subsection. In particular, for the thermoelastic coupling to be inoperative, it suffices that \mathbf{k} be independent of ϵ and β be zero.

Equation (43) represents the general coupled heat equation in the case of the Cattaneo–Vernotte law. It contains all the coupling terms in explicit form while being more convenient and more accurate than the equations based on the expansion of the Helmholtz potential in power series of the temperature change.

5 Coupled heat equation with stress and temperature as independent variables

5.1 General form

From (3), (14), (17), (20), (21), (36) and (37) it follows that

$$-\operatorname{div} \mathbf{q} + r = C_\sigma \dot{T} + T \boldsymbol{\alpha} \cdot \dot{\boldsymbol{\sigma}}. \quad (45)$$

5.2 Case of Fourier's law

Combining (18) and (45) gives

$$\operatorname{div} (\mathbf{k}(\boldsymbol{\sigma}, T) \operatorname{grad} T) + r = C_\sigma(\boldsymbol{\sigma}, T) \dot{T} + T \boldsymbol{\alpha}(\boldsymbol{\sigma}, T) \cdot \dot{\boldsymbol{\sigma}}, \quad (46)$$

where $\boldsymbol{\alpha}(\boldsymbol{\sigma}, T)$ and $C_\sigma(\boldsymbol{\sigma}, T)$ are given by (21) and (36), respectively.

It can be shown, as in Subsection 4.2, that in the case of σ -independent conductivity the vanishing of the CTE tensor α is a necessary and sufficient condition for the heat equation to be uncoupled from the mechanical fields. The vanishing of only the CFTE tensor α_0 does not suffice to uncouple the heat equation from the mechanical fields.

5.3 Case of the Cattaneo–Vernotte law

Combining (45) and its time derivative with (19) gives

$$\begin{aligned}
\operatorname{div}(\mathbf{k} \operatorname{grad} T) + r + \tau \dot{r} &= \tau C_\sigma \ddot{T} \\
&+ \tau \left(\frac{\partial C_\sigma}{\partial T} \right)_\sigma \dot{T}^2 \\
&+ \left[C_\sigma + \tau \left(\left(\frac{\partial C_\sigma}{\partial \sigma} \right)_T + \alpha + T \left(\frac{\partial \alpha}{\partial T} \right)_\sigma \right) \cdot \dot{\sigma} \right] \dot{T} \\
&+ \left[\alpha \cdot \dot{\sigma} + \tau \left(\alpha \cdot \ddot{\sigma} + \dot{\sigma} \cdot \left(\frac{\partial \alpha}{\partial \sigma} \right)_T \dot{\sigma} \right) \right] T,
\end{aligned} \tag{47}$$

where

$$\begin{aligned}
\left(\frac{\partial C_\sigma}{\partial \sigma} \right)_T &= T \left(\frac{d^2 M}{dT^2} \sigma + \frac{d^2 m}{dT^2} \right), \\
\left(\frac{\partial C_\sigma}{\partial T} \right)_\sigma &= \frac{\partial C_\sigma}{\partial T}(\mathbf{0}, T) + \frac{1}{2} \sigma \cdot \frac{d^2 M}{dT^2} \sigma \\
&+ \frac{d^2 m}{dT^2} \cdot \sigma + T \left(\frac{1}{2} \sigma \cdot \frac{d^3 M}{dT^3} \sigma + \frac{d^3 m}{dT^3} \cdot \sigma \right),
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{\partial \alpha}{\partial \sigma} \right)_T &= \frac{dM}{dT}, \\
\left(\frac{\partial \alpha}{\partial T} \right)_\sigma &= \frac{d^2 M}{dT^2} \sigma + \frac{d^2 m}{dT^2}.
\end{aligned}$$

Note that in (47)

$$\left(\frac{\partial C_\sigma}{\partial \sigma} \right)_T = T \left(\frac{\partial \alpha}{\partial T} \right)_\sigma. \tag{48}$$

Equations (43) and (47) are of course equivalent. It is keeping all the coupling terms that makes it possible to pass from one equation to the other without unnecessary additional assumptions. The appendix gives relations that can be used to move from (43) to (47).

The conclusions reached in Subsection 5.2 on the thermoelastic coupling hold true in this subsection. In particular, for the thermoelastic coupling to be inoperative, it suffices that \mathbf{k} be independent of σ and α (and not α_0) be zero.

6 Applications

6.1 Some simplified expressions

Equations (42), (43), (46), and (47) are numerically challenging and are not meant to be used as they are in all circumstances. They are rather intended to build justified approximate models. As they do not involve any approximations other than those defining their theoretical framework (small strains and linear stress-strain relation with temperature-dependent coefficients) and contain all the corresponding coupling terms, they make it possible to estimate on a case-by-case basis the order of magnitude of each of the coupling terms and decide which ones to keep. This subsection discusses some such approximate models.

6.1.1 An approximate heat equation within Fourier conduction

If the temperature gradient, temperature rate, strain rate, heat supply, together with strain, are assumed to be small of the same order, then, up to first-order, the heat equation for Fourier's law (42) reduces to

$$\operatorname{div}(\mathbf{k}(\mathbf{0}, T) \operatorname{grad} T) + r = C_\epsilon(\mathbf{0}, T) \dot{T} - T \frac{d\mathbf{l}}{dT}(T) \cdot \dot{\epsilon}. \quad (49)$$

This equation is valid even for large temperature changes.

If in addition to the above assumptions the relative change in temperature, $(T - T_0)/T_0$, is also assumed to be small, then the foregoing equation reduces to

$$\operatorname{div}(\mathbf{k}(\mathbf{0}, T_0) \operatorname{grad} T) + r = C_\epsilon(\mathbf{0}, T_0) \dot{T} - T_0 \frac{d\mathbf{l}}{dT}(T_0) \cdot \dot{\epsilon}. \quad (50)$$

Since $d\mathbf{l}/dT(T_0) = -\mathbf{L}(T_0) \boldsymbol{\alpha}_0(T_0)$, (50) can be recognized as the heat conduction equation used in linear thermoelasticity, namely,

$$\operatorname{div}(\mathbf{k}(\mathbf{0}, T_0) \operatorname{grad} T) + r = C_\epsilon(\mathbf{0}, T_0) \dot{T} + T_0 \mathbf{L}(T_0) \boldsymbol{\alpha}_0(T_0) \cdot \dot{\epsilon}. \quad (51)$$

It is worth noting that it does not suffice to replace T_0 by T in the foregoing equation to retrieve the coupled heat equation in the case of temperature-dependent material moduli (42), or even the approximate version (49).

6.1.2 An extended Lord–Shulman model

In the case of the Cattaneo–Vernotte conduction also, if the temperature gradient, temperature rates, strain rates, heat supply, and heat supply rate, together with strain, are assumed to be small, then (43) reduces to

$$\operatorname{div}(\mathbf{k}(\mathbf{0}, T) \operatorname{grad} T) + r + \tau \dot{r} = C_\epsilon(\mathbf{0}, T) (\dot{T} + \tau \ddot{T}) - T \frac{d\mathbf{l}}{dT}(T) \cdot (\dot{\epsilon} + \tau \ddot{\epsilon}). \quad (52)$$

Equation (52) takes into account large temperature changes and the temperature dependence of the moduli. It therefore represents an extended version of the classical Lord–Shulman equation.

Again, if in addition to the above assumptions the temperature change is also assumed to be small, then the above equation reduces to

$$\operatorname{div}(\mathbf{k}(\mathbf{0}, T_0) \operatorname{grad} T) + r = C_\epsilon(\mathbf{0}, T_0) \left(\dot{T} + \tau \ddot{T} \right) + T_0 \mathbf{L}(T_0) \boldsymbol{\alpha}_0(T_0) \cdot (\dot{\epsilon} + \tau \ddot{\epsilon}). \quad (53)$$

This is the anisotropic version of the coupled heat equation proposed by Lord and Shulman [37]. Replacing T_0 by T in this equation gives back neither (43) nor (52).

6.2 Application to TSA

The independent variables commonly used in TSA are the temperature and stress and the conduction model is Fourier's law. This subsection will therefore focus on (46). This equation is first particularized to the adiabatic case, which is the usual case considered in TSA studies. The results obtained are then compared with those given in [28], which are presently the most widely used [29, 49–53] if there is a need to take into account the experimentally observed dependence of the thermoelastic parameter on both temperature and stress [27, 54].

In the adiabatic case, i.e., $\mathbf{q} = \mathbf{0}$ and $r = 0$, (46) reduces to

$$C_\sigma(\boldsymbol{\sigma}, T) \frac{\dot{T}}{T} = -\boldsymbol{\alpha}(\boldsymbol{\sigma}, T) \cdot \dot{\boldsymbol{\sigma}}, \quad (54)$$

or equivalently, by (21), to

$$C_\sigma(\boldsymbol{\sigma}, T) \frac{\dot{T}}{T} = -\left(\frac{d\mathbf{M}}{dT} \boldsymbol{\sigma} + \boldsymbol{\alpha}_0 \right) \cdot \dot{\boldsymbol{\sigma}}. \quad (55)$$

Specializing equation (55) to the isotropic case gives

$$\begin{aligned} C_\sigma(\boldsymbol{\sigma}, T) \frac{\dot{T}}{T} = & - \left[\alpha_0 + \left(\frac{\nu}{E^2} \frac{dE}{dT} - \frac{1}{E} \frac{d\nu}{dT} \right) \operatorname{tr} \boldsymbol{\sigma} \right] \operatorname{tr} \dot{\boldsymbol{\sigma}} \\ & + \left(\frac{1 + \nu}{E^2} \frac{dE}{dT} - \frac{1}{E} \frac{d\nu}{dT} \right) \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\sigma}}, \end{aligned} \quad (56)$$

where α_0 is the coefficient of free thermal expansion ($\boldsymbol{\alpha}_0 = \alpha_0 \mathbf{i}$, with \mathbf{i} being the second-order identity tensor), E is Young's modulus and ν is Poisson's ratio.

Equation (56) is now compared with equation (2.23) given in [28], which is often referred to as “revised higher order theory equation.” The general form of the two equations is the same. The terms involving the mechanical moduli are identical in the two equations. However, those involving the heat capacities and those involving the coefficients of free thermal expansion differ. First, the heat capacity at constant strain is used in [28, equation (2.23)], whereas the heat capacity at constant stress is used in (56). Second, the secant CFTE is used in [28, equation (2.23)], whereas the tangent CFTE is used in (56). The two equations are therefore different for general adiabatic transformations.

The difference can be explained by approximations used in [28] but not used here. The derivation of the stress-based equations [28, equation (2.22)] and [28, equation (2.23)] from the strain-based equation [28, equation (2.20)] and [28, equation (2.21)] has required approximations, such as “omitting higher order terms”. By contrast, (56) is free of such approximations.

Now, as far as TSA is concerned, the new equations should not a priori question most existing results, as C_σ and C_ϵ are close to each other, as are, even more so, the tangent and secant CFTE tensors in view of the smallness of the change in temperature in typical TSA tests.

7 Conclusions

This study has developed expressions for the coupled heat equation within the framework of small-strain thermoelasticity with temperature-dependent thermoelastic moduli and without the restrictions on temperature changes imposed in linear thermoelasticity. The expressions, developed for the Fourier and Cattaneo–Vernotte conduction laws, contain all the coupling terms, which involve not only the thermoelastic moduli but also their derivatives with respect to the independent state variables. Retaining all the coupling terms has made it possible to present a coherent framework without unnecessary assumptions and to obtain equations that are more accurate and more convenient than those obtained by polynomial expansion. As applications, the expressions obtained were used (i) to derive a new Lord–Shulman-type model that accounts for the temperature dependence of the thermoelastic moduli and large changes in temperature, and (2) to revisit the theory underpinning TSA.

Appendix

The heat equations (43) and (47) are of course equivalent. One can use the following relations to derive (47) from (43):

- Terms in C_ϵ

$$\begin{aligned}
C_\epsilon &= C_\sigma - T\alpha \cdot M^{-1}\alpha \\
\left(\frac{\partial C_\epsilon}{\partial T}\right)_\epsilon &= \left(\frac{\partial}{\partial T}(C_\sigma - T\alpha \cdot M^{-1}\alpha)\right)_\sigma \\
&\quad - \left(\frac{\partial}{\partial \sigma}(C_\sigma - T\alpha \cdot M^{-1}\alpha)\right)_T \cdot M^{-1}\alpha \\
&= \left(\frac{\partial C_\sigma}{\partial T}\right)_\sigma - \left(\frac{\partial C_\sigma}{\partial \sigma}\right)_T \cdot M^{-1}\alpha - \alpha \cdot M^{-1}\alpha \\
&\quad - 2T\alpha \cdot M^{-1}\left(\frac{\partial \alpha}{\partial T}\right)_\sigma + 3T\alpha \cdot M^{-1}\frac{dM}{dT}M^{-1}\alpha \\
\left(\frac{\partial C_\epsilon}{\partial \epsilon}\right)_T &= T\left(-2M^{-1}\frac{dM}{dT}M^{-1}\alpha + M^{-1}\left(\frac{\partial \alpha}{\partial T}\right)_\sigma\right)
\end{aligned}$$

- Terms in β

$$\begin{aligned}
\beta &= -M^{-1}\alpha \\
\left(\frac{\partial \beta}{\partial T}\right)_\epsilon &= 2M^{-1}\frac{dM}{dT}M^{-1}\alpha - M^{-1}\left(\frac{\partial \alpha}{\partial T}\right)_\sigma \\
\left(\frac{\partial \beta}{\partial \epsilon}\right)_T &= -M^{-1}\frac{dM}{dT}M^{-1}
\end{aligned}$$

- Terms in ϵ

$$\begin{aligned}
\epsilon &= M\sigma + m \\
\dot{\epsilon} &= \alpha\dot{T} + M\dot{\sigma} \\
\ddot{\epsilon} &= M\ddot{\sigma} + \alpha\ddot{T} + 2\frac{dM}{dT}\dot{\sigma}\dot{T} + \left(\frac{\partial \alpha}{\partial T}\right)_\sigma (\dot{T})^2
\end{aligned}$$

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