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# Modal correspondence theory in the class of all Euclidean frames

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## Abstract

The core of this article is the modal correspondence theory in the class of all Euclidean frames. It shows that with respect to the class of all Euclidean frames, every modal formula is first-order definable and the problem of deciding the modal definability of sentences is undecidable.

*Keywords:* Modal logic, first-order logic, correspondence theory, first-order definability, modal definability, Euclidean frames.

## 1 Introduction

An important and classical topic in modal logic is the correspondence between modal syntax and semantics and first-order syntax and semantics. Its study began in the 1970s with the works of van Benthem [4, 5], Goldblatt and Thomason [14] and Sahlqvist [20]. A number of questions since then have been explored, in particular, with respect to a fixed class of frames, the decidability of the first-order definability of modal formulas and the decidability of the modal definability of sentences.

Over the class of all frames, the two problems (first-order definability of modal formulas and modal definability of sentences) have been proved to be undecidable by Chagrov and Chagrova [7–9] and Chagrova [10, 11]. See also [3] for a number of other undecidability results. Over the class of all *S5*-frames (based on equivalence relations) and over the class of all *KD45*-frames (based on serial, transitive and Euclidean relations), the two problems are decidable, as proved by Balbiani and Tinchev [1, 2] and Georgiev [12, 13].

In this article, we answer the following question left open in [13]: is there a natural class of frames for which one of the two problems is undecidable but not the other? We have discovered such a class, namely the class of all *K5*-frames (based on Euclidean relations). It turns out that over this class of frames, every modal formula is first-order definable and the problem of deciding the modal definability of sentences is undecidable. Section 2 introduces different classes of frames and studies their relationships. In Section 3, the modal language we will work with is presented. Section 4 presents the first-order language we will work with. In Section 5, the definitions of abstract types of frames and concrete types of frames are given. Section 6 contains the proof that every modal formula is first-order definable with respect to the class of all Euclidean frames. In Section 7, the proof that the problem of deciding the modal definability of sentences with respect to the class of all Euclidean frames is undecidable is presented.

## 2 Euclidean frames

A *frame* is a couple  $\mathcal{F} = (W, R)$  where  $W$  is a non-empty set of ‘states’ and  $R$  is a binary relation of ‘accessibility’ on  $W$ . Let  $\mathcal{F} = (W, R)$  be a frame. For all frames  $\mathcal{F}' = (W', R')$ , we say  $\mathcal{F}'$  is a *subframe* of  $\mathcal{F}$  iff  $W' \subseteq W$  and  $R' = R \cap (W' \times W')$ . For all  $s \in W$ , let  $R(s) = \{t \in W : sRt\}$  and  $R^{-1}(s) = \{t \in W : tRs\}$ . Let  $\text{dom}(R) = \{s \in W : R(s) \neq \emptyset\}$  and  $\text{ran}(R) = \{s \in W : R^{-1}(s) \neq \emptyset\}$ . For all  $A \subseteq W$ , let  $|A|$  be the cardinality of  $A$ . Moreover, let  $R(A) = \{t \in W : sRt \text{ for some } s \in A\}$  and  $R^{-1}(A) = \{t \in W : tRs \text{ for some } s \in A\}$ . For all  $s \in W$ , let  $R^*(s) = \bigcup \{R^n(s) : n \in \mathbb{N}\}$  where  $R^0(s) = \{s\}$  and for all  $n \geq 1$ ,  $R^n(s) = R(R^{n-1}(s))$ . For all  $s \in W$ , we say  $s$  is a *root* of  $\mathcal{F}$  iff  $R^*(s) = W$ . We say  $\mathcal{F}$  is *universal* iff  $R = W \times W$ . We say  $\mathcal{F}$  is *reflexive* iff for all  $s \in W$ ,  $sRs$ . We say  $\mathcal{F}$  is *symmetric* iff for all  $s, t \in W$ , if  $sRt$  then  $tRs$ . We say  $\mathcal{F}$  is *connected* iff for all  $s, t, u \in W$ , if  $sRt$  and  $sRu$  then either  $tRu$ , or  $uRt$ . We say  $\mathcal{F}$  is *Euclidean* iff for all  $s, t, u \in W$ , if  $sRt$  and  $sRu$  then  $tRu$ .

LEMMA 1

If  $\mathcal{F}$  is Euclidean then  $\mathcal{F}$  is connected. Moreover, for all  $s \in W$ ,  $R(s) \subseteq R(R(s))$  and  $R^*(s) = \{s\} \cup R(R(s))$ .

We say  $\mathcal{F}$  is *simple* iff either  $R = \emptyset$ , or  $\text{dom}(R) = W$  and  $\text{ran}(R) \times \text{ran}(R) \subseteq R$ . In the latter case, obviously,  $\text{dom}(R) \setminus \text{ran}(R)$  and  $\text{ran}(R)$  constitute a partition of  $W$  and we also say that  $\mathcal{F}$  is *strongly simple*.

LEMMA 2

If  $\mathcal{F}$  is simple then  $\mathcal{F}$  is Euclidean.

We say  $\mathcal{F}$  is *non-universal* iff  $R \neq W \times W$ . We say  $\mathcal{F}$  is *non-reflexive* iff there exists  $s \in W$  such that not  $sRs$ . We say  $\mathcal{F}$  is a *flower* iff  $\mathcal{F}$  is non-universal and strongly simple.

LEMMA 3

If  $\mathcal{F}$  is a flower then  $\mathcal{F}$  is non-reflexive.

We say  $\mathcal{F}$  is *3-connected* iff for all  $s, t \in W$ , there exists  $u, v \in W$  such that  $sRu$ ,  $tRv$  and  $uRv$ .

LEMMA 4

If  $\mathcal{F}$  is Euclidean then  $\mathcal{F}$  is a flower iff  $\mathcal{F}$  is non-reflexive and 3-connected.

Let  $s \in W$ . The *connected component* of  $\mathcal{F}$  determined by  $s$  is the frame  $\mathcal{F}_s = (W_s, R_s)$  such that

- $W_s$  is the least subset of  $W$  such that
  - $s \in W_s$ ,
  - for all  $t \in W_s$ ,  $R(t) \subseteq W_s$ ,
  - for all  $t \in W_s$ ,  $R^{-1}(t) \subseteq W_s$ ,
- $R_s$  is the restriction of  $R$  to  $W_s$ .

LEMMA 5

If  $\mathcal{F}$  is Euclidean then for all  $s \in W$ , the connected component of  $\mathcal{F}$  determined by  $s$  is simple.

We say  $\mathcal{F}$  is a *singleton* iff  $|W| = 1$  and  $R = \emptyset$ . Obviously, if  $\mathcal{F}$  is a singleton then the unique state in  $W$  is a root of  $\mathcal{F}$ . Moreover, concerning subframes,  $\mathcal{F}$  has exactly one subframe:  $\mathcal{F}$  itself.

LEMMA 6

If  $\mathcal{F}$  is a singleton then  $\mathcal{F}$  is simple.

We say  $\mathcal{F}$  is *universal* iff  $R = W \times W$ . Obviously, if  $\mathcal{F}$  is universal then every state in  $W$  is a root of  $\mathcal{F}$ . Moreover, as for the subframes of  $\mathcal{F}$ , they are universal too.

LEMMA 7

If  $\mathcal{F}$  is universal then  $\mathcal{F}$  is strongly simple.

We say  $\mathcal{F}$  is a *cul-de-sac* iff there exists  $s \in \text{dom}(R) \setminus \text{ran}(R)$  such that  $(W \setminus \{s\}) \times (W \setminus \{s\}) \subseteq R$ . Obviously, if  $\mathcal{F}$  is a cul-de-sac then  $s$  is the unique root of  $\mathcal{F}$  and  $|\text{dom}(R) \setminus \text{ran}(R)| = 1$ . Moreover, about the subframes of  $\mathcal{F}$  containing  $s$  and intersecting  $R(s)$ , they are cul-de-sac too and  $s$  is their root.

LEMMA 8

If  $\mathcal{F}$  is a cul-de-sac then  $\mathcal{F}$  is non-universal and strongly simple.

We say  $\mathcal{F}$  is *clusterizable* iff either  $\mathcal{F}$  is a singleton, or  $\mathcal{F}$  is universal, or  $\mathcal{F}$  is a cul-de-sac.

LEMMA 9

If  $\mathcal{F}$  is clusterizable then  $\mathcal{F}$  is simple.

Let  $s \in W$ . The *generated subframe* of  $\mathcal{F}$  determined by  $s$  is the frame  $\mathcal{F}_s = (W_s, R_s)$  such that

- $W_s$  is the least subset of  $W$  such that
  - $s \in W_s$ ,
  - for all  $t \in W_s$ ,  $R(t) \subseteq W_s$ ,
- $R_s$  is the restriction of  $R$  to  $W_s$ .

LEMMA 10

If  $\mathcal{F}$  is Euclidean then for all  $s \in W$ , the generated subframe of  $\mathcal{F}$  determined by  $s$  is clusterizable.

Let  $\mathcal{C}_{rs}$  be the class of all reflexive symmetric frames,  $\mathcal{C}_{Euc}$  be the class of all Euclidean frames and  $\mathcal{C}_{flo}$  be the class of all flowers.

### 3 Modal language and satisfaction

It is now time to meet the modal language we will work with.

**Modal language.** Let us consider a countable set  $VAR$  of *propositional variables* (with typical members denoted  $p, q, \dots$ ). The set  $MOF$  of all *modal formulas* (with typical members denoted  $\varphi, \psi, \dots$ ) is inductively defined as follows:

- $\varphi ::= p \mid \perp \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$ .

We define the other Boolean constructs as usual. The modal formula  $\Diamond\varphi$  is obtained as the well-known abbreviation:  $\Diamond\varphi ::= \neg\Box\neg\varphi$ . We adopt the standard rules for omission of the parentheses. We write  $\varphi(p_1, \dots, p_n)$  to denote a modal formula whose propositional variables form a subset of  $\{p_1, \dots, p_n\}$ . Let  $\text{sf}$  be the function assigning to each modal formula  $\varphi$  the set  $\text{sf}(\varphi)$  of all its subformulas. It is defined as usual by induction on  $\varphi$ . For all modal formulas  $\varphi$ , the number of  $\varphi$ 's subformulas is denoted as  $|\text{sf}(\varphi)|$ .

**Satisfaction.** A *model* based on a frame  $\mathcal{F} = (W, R)$  is a triple  $\mathcal{M} = (W, R, V)$  where  $V$  is a function assigning to each propositional variable  $p$ , a subset  $V(p)$  of  $W$ . Given a model  $\mathcal{M} = (W, R, V)$ , the *satisfiability* of a modal formula  $\varphi$  at  $s \in W$ , in symbols  $\mathcal{M}, s \models \varphi$ , is inductively defined as follows:

- $\mathcal{M}, s \models p$  iff  $s \in V(p)$ ,
- $\mathcal{M}, s \not\models \perp$ ,
- $\mathcal{M}, s \models \neg\varphi$  iff  $\mathcal{M}, s \not\models \varphi$ ,
- $\mathcal{M}, s \models \varphi \vee \psi$  iff either  $\mathcal{M}, s \models \varphi$ , or  $\mathcal{M}, s \models \psi$ ,
- $\mathcal{M}, s \models \Box\varphi$  iff for all  $t \in W$ , if  $sRt$  then  $\mathcal{M}, t \models \varphi$ .

Obviously,  $\mathcal{M}, s \models \Diamond\varphi$  iff there exists  $t \in W$  such that  $sRt$  and  $\mathcal{M}, t \models \varphi$ . We say a modal formula  $\varphi$  is *true* in a model  $\mathcal{M}$ , in symbols  $\mathcal{M} \models \varphi$ , iff  $\varphi$  is satisfied at all states in  $\mathcal{M}$ . We say modal formula  $\varphi$  is *valid* in a frame  $\mathcal{F}$ , in symbols  $\mathcal{F} \models \varphi$ , iff  $\varphi$  is true in all models based on  $\mathcal{F}$ . We say a frame  $\mathcal{F}$  is *weaker* than a frame  $\mathcal{F}'$ , in symbols  $\mathcal{F} \preceq \mathcal{F}'$ , iff for all modal formulas  $\varphi$ , if  $\mathcal{F} \models \varphi$  then  $\mathcal{F}' \models \varphi$ . We say a modal formula  $\varphi$  is *valid* in a class  $\mathcal{C}$  of frames, in symbols  $\mathcal{C} \models \varphi$ , iff  $\varphi$  is valid in all frames in  $\mathcal{C}$ .

**Bounded morphic images.** Let  $\mathcal{F} = (W, R)$ ,  $\mathcal{F}' = (W', R')$  be frames. We say a function  $f$  assigning to each state  $s$  in  $\mathcal{F}$  a state  $f(s)$  in  $\mathcal{F}'$  is a *bounded morphism* from  $\mathcal{F}$  to  $\mathcal{F}'$  iff the following conditions are satisfied:

- for all states  $s, t$  in  $\mathcal{F}$ , if  $sRt$  then  $f(s)R'f(t)$ ,
- for all states  $s$  in  $\mathcal{F}$  and for all states  $t'$  in  $\mathcal{F}'$ , if  $f(s)R't'$  then there exists a state  $t$  in  $\mathcal{F}$  such that  $sRt$  and  $f(t) = t'$ .

We say  $\mathcal{F}'$  is a *bounded morphic image* of  $\mathcal{F}$  iff there exists a surjective bounded morphism from  $\mathcal{F}$  to  $\mathcal{F}'$ . For all states  $s$  in  $\mathcal{F}$  and for all states  $s'$  in  $\mathcal{F}'$ , we say  $(\mathcal{F}', s')$  is a *bounded morphic image* of  $(\mathcal{F}, s)$  iff there exists a surjective bounded morphism  $f$  from  $\mathcal{F}$  to  $\mathcal{F}'$ , such that  $f(s) = s'$ . See [6, Chapter 2] for more about bounded morphic images.

## 4 First-order language and satisfaction

It is now time to meet the first-order language we will work with.

**First-order language.** Let us consider a countable set of *individual variables* (with typical members denoted  $x, y, \dots$ ). A list  $x_1, \dots, x_m$  of individual variables will sometimes be written as  $\bar{x}$ . We leave it to the context to determine the length of such a list. The set *FOF* of all *first-order formulas* (denoted  $A, B, \dots$ ) is inductively defined as follows:

- $A ::= R_{\Box}(x, y) \mid x = y \mid \neg A \mid (A \vee B) \mid \forall x A$ .

We define the other Boolean constructs as usual. The first-order formula  $\exists x A$  is obtained as the well-known abbreviation:  $\exists x A ::= \neg \forall x \neg A$ . We adopt the standard rules for omission of the parentheses. For all first-order formulas  $A$ , let  $\text{fiv}(A)$  be the set of all free individual variables occurring in  $A$ . When  $\bar{x}$  is a list of pairwise distinct individual variables, we write  $A(\bar{x})$  to denote a first-order formula  $A$  whose free individual variables belongs to  $\bar{x}$ . A first-order formula  $A$  is called a *sentence* if  $\text{fiv}(A) = \emptyset$ .

Let  $\tau: \text{FOF} \rightarrow \text{FOF}$  be the function inductively defined as follows:

- $\tau(R_{\Box}(x, y))$  is  $\exists z (R_{\Box}(x, z) \wedge R_{\Box}(y, z))$ ,
- $\tau(x = y)$  is  $x = y$ ,
- $\tau(\perp)$  is  $\perp$ ,
- $\tau(\neg A)$  is  $\neg \tau(A)$ ,

- $\tau(A \vee B)$  is  $\tau(A) \vee \tau(B)$ ,
- $\tau(\forall x A)$  is  $\forall x (R_{\square}(x, x) \vee \tau(A))$ .

The reader may easily verify that for all first-order formulas  $A$ ,  $\text{fiv}(\tau(A)) = \text{fiv}(A)$ .

**Satisfaction.** Given a frame  $\mathcal{F} = (W, R)$ , the *satisfiability* of a first-order formula  $A(\bar{x})$  in  $\mathcal{F}$  with respect to a list  $\bar{s}$  of states in  $\mathcal{F}$ , in symbols  $\mathcal{F} \models A(\bar{x}) [\bar{s}]$ , is inductively defined as follows:

- $\mathcal{F} \models R_{\square}(x_i, x_j) [\bar{s}]$  iff  $s_i R s_j$ ,
- $\mathcal{F} \models x_i = x_j [\bar{s}]$  iff  $s_i = s_j$ ,
- $\mathcal{F} \models \neg A [\bar{s}]$  iff  $\mathcal{F} \not\models A [\bar{s}]$ ,
- $\mathcal{F} \models A \vee B [\bar{s}]$  iff either  $\mathcal{F} \models A [\bar{s}]$ , or  $\mathcal{F} \models B [\bar{s}]$ ,
- $\mathcal{F} \models \forall x A(\bar{x}, x) [\bar{s}]$  iff for all states  $s$  in  $\mathcal{F}$ ,  $\mathcal{F} \models A(\bar{x}, x) [\bar{s}, s]$ .

Obviously,  $\mathcal{F} \models \exists x A(\bar{x}, x) [\bar{s}]$  iff there exists a state  $s$  in  $\mathcal{F}$  such that  $\mathcal{F} \models A(\bar{x}, x) [\bar{s}, s]$ . We say a first-order formula  $A(\bar{x})$  is *valid* in a frame  $\mathcal{F}$ , in symbols  $\mathcal{F} \models A(\bar{x})$ , iff  $A(\bar{x})$  is satisfied in  $\mathcal{F}$  with respect to all lists  $\bar{s}$  of states in  $\mathcal{F}$ . We say a first-order formula  $A$  is *valid* in a class  $\mathcal{C}$  of frames, in symbols  $\mathcal{C} \models A$ , iff  $A$  is valid in all frames in  $\mathcal{C}$ .

**Relativized reducts and stable classes of frames.** Let  $\mathcal{F}, \mathcal{F}'$  be the frames. We say  $\mathcal{F}'$  is a *relativized reduct* of  $\mathcal{F}$  iff there exists a first-order formula  $A(\bar{x}, x)$  and there exists a list  $\bar{s}$  of states in  $\mathcal{F}$  such that  $\mathcal{F}'$  is the restriction of  $\mathcal{F}$  to the set of all states  $s$  in  $\mathcal{F}$  such that  $\mathcal{F} \models A(\bar{x}, x) [\bar{s}, s]$ . In that case,  $\mathcal{F}'$  is called the *relativized reduct* of  $\mathcal{F}$  with respect to  $A(\bar{x}, x)$  and  $\bar{s}$ . See [17, Chapter 5] for more about relativized reducts. Let  $\mathcal{C}$  be a class of frames. We say  $\mathcal{C}$  is *stable* iff there exists a first-order formula  $A(\bar{x}, x)$  and there exists a sentence  $B$  such that

- for all frames  $\mathcal{F}$  in  $\mathcal{C}$ , for all lists  $\bar{s}$  of states in  $\mathcal{F}$  and for all frames  $\mathcal{F}'$ , if  $\mathcal{F}'$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(\bar{x}, x)$  and  $\bar{s}$  then  $\mathcal{F}'$  is in  $\mathcal{C}$ ,
- for all frames  $\mathcal{F}_0$  in  $\mathcal{C}$ , there exists frames  $\mathcal{F}, \mathcal{F}'$  in  $\mathcal{C}$  and there exists a list  $\bar{s}$  of states in  $\mathcal{F}$  such that  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(\bar{x}, x)$  and  $\bar{s}$ ,  $\mathcal{F} \models B$ ,  $\mathcal{F}' \not\models B$  and  $\mathcal{F} \preceq \mathcal{F}'$ .

See [3] for more about stable classes of frames.

## 5 Types

An *abstract type* is a triple  $(\epsilon, \sigma, \mu) \in \{0, 1\} \times \mathbb{N}^2$  such that either  $\epsilon = 1, \sigma = 0$  and  $\mu = 0$ , or  $\epsilon = 0, \sigma \geq 1$  and  $\mu = \sigma$ , or  $\epsilon = 1, \sigma \geq 1$  and  $\mu \geq \sigma$ . Let  $\Delta_{\omega}$  be the set of all abstract types. For all positive integers  $n$ , let  $\Delta_n$  be the set of all abstract types  $(\epsilon, \sigma, \mu)$  such that  $\epsilon + \mu \leq n$ .

LEMMA 11

For all positive integers  $n$ ,  $\Delta_n$  is finite.

Let  $\mathcal{F} = (W, R)$  be a finite clusterizable frame. Hence, either  $\mathcal{F}$  is a singleton, or  $\mathcal{F}$  is universal, or  $\mathcal{F}$  is a cul-de-sac. Let  $s$  be a root of  $\mathcal{F}$ . The *concrete type* of  $\mathcal{F}$  is the triple  $(\epsilon, \sigma, \mu) \in \{0, 1\} \times \mathbb{N}^2$  where

- if  $sRs$  then  $\epsilon = 0$  else  $\epsilon = 1$ ,
- $\sigma = |R(s)|$ ,
- $\mu = |R(R(s))|$ ,

A clusterizable frame has several roots only if it is universal. As a result, the concrete type of  $\mathcal{F}$  as defined above does not depend on the choice of  $\mathcal{F}$ 's root when this choice is possible. Remark that if  $\mathcal{F}$  is a singleton then note  $sRs$ ,  $|R(s)| = 0$ ,  $|R(R(s))| = 0$  and  $\mathcal{F}$  is of type  $(1, 0, 0)$ , if  $\mathcal{F}$  is universal

then  $sRs$ ,  $|R(s)| \geq 1$ ,  $|R(R(s))| = |R(s)|$  and  $\mathcal{F}$  is of type  $(0, |W|, |W|)$  and if  $\mathcal{F}$  is a cul-de-sac then note  $sRs$ ,  $|R(s)| \geq 1$ ,  $|R(R(s))| \geq |R(s)|$  and  $\mathcal{F}$  is of type  $(1, |R(s)|, |W| - 1)$ .

LEMMA 12

Let  $\mathcal{F} = (W, R)$  be a finite clusterizable frame. If the concrete type of  $\mathcal{F}$  is the triple  $(\epsilon, \sigma, \mu) \in \{0, 1\} \times \mathbb{N}^2$  then  $\epsilon + \mu = |W|$ .

LEMMA 13

For all  $(\epsilon, \sigma, \mu) \in \{0, 1\} \times \mathbb{N}^2$ , if  $(\epsilon, \sigma, \mu)$  is the concrete type of some finite clusterizable frame then  $(\epsilon, \sigma, \mu)$  is an abstract type.

For all abstract types  $(\epsilon, \sigma, \mu)$ ,

- if  $\epsilon = 1, \sigma = 0$  and  $\mu = 0$  then let  $\mathcal{F}_{(\epsilon, \sigma, \mu)} = (W_{(\epsilon, \sigma, \mu)}, R_{(\epsilon, \sigma, \mu)})$  be the frame where  $W_{(\epsilon, \sigma, \mu)} = \{0\}$  and  $R_{(\epsilon, \sigma, \mu)} = \emptyset$ ,
- if  $\epsilon = 0, \sigma \geq 1$  and  $\mu = \sigma$  then let  $\mathcal{F}_{(\epsilon, \sigma, \mu)} = (W_{(\epsilon, \sigma, \mu)}, R_{(\epsilon, \sigma, \mu)})$  be the frame where  $W_{(\epsilon, \sigma, \mu)} = \{0, \dots, \sigma - 1\}$  and  $R_{(\epsilon, \sigma, \mu)} = \{0, \dots, \sigma - 1\} \times \{0, \dots, \sigma - 1\}$ ,
- if  $\epsilon = 1, \sigma \geq 1$  and  $\mu \geq \sigma$  then let  $\mathcal{F}_{(\epsilon, \sigma, \mu)} = (W_{(\epsilon, \sigma, \mu)}, R_{(\epsilon, \sigma, \mu)})$  be the frame where  $W_{(\epsilon, \sigma, \mu)} = \{0, 1, \dots, \sigma, \sigma + 1, \dots, \mu\}$  and  $R_{(\epsilon, \sigma, \mu)} = (\{0\} \times \{1, \dots, \sigma\}) \cup (\{1, \dots, \sigma, \sigma + 1, \dots, \mu\} \times \{1, \dots, \sigma, \sigma + 1, \dots, \mu\})$ .

Remark that for all abstract types  $(\epsilon, \sigma, \mu)$ , 0 is a root of  $\mathcal{F}_{(\epsilon, \sigma, \mu)}$ .

LEMMA 14

For all abstract types  $(\epsilon, \sigma, \mu)$ ,  $\mathcal{F}_{(\epsilon, \sigma, \mu)}$  is a finite clusterizable frame of type  $(\epsilon, \sigma, \mu)$ .

For all abstract types  $(\epsilon, \sigma, \mu)$ ,

- if  $\epsilon = 1, \sigma = 0$  and  $\mu = 0$  then let  $A_{(\epsilon, \sigma, \mu)}(x)$  be the first-order formula

$$\forall y \neg R_{\square}(x, y),$$

- if  $\epsilon = 0, \sigma \geq 1$  and  $\mu = \sigma$  then let  $A_{(\epsilon, \sigma, \mu)}(x)$  be the conjunction of the first-order formulas

$$R_{\square}(x, x),$$

$$\exists y_1 \dots \exists y_{\sigma} \left( \bigwedge_{1 \leq i \leq \sigma} R_{\square}(x, y_i) \wedge \bigwedge_{1 \leq i < j \leq \sigma} y_i \neq y_j \right),$$

- if  $\epsilon = 1, \sigma \geq 1$  and  $\mu \geq \sigma$  then let  $A_{(\epsilon, \sigma, \mu)}(x)$  be the conjunction of the first-order formulas

$$\neg R_{\square}(x, x),$$

$$\exists y_1 \dots \exists y_{\sigma} \left( \bigwedge_{1 \leq i \leq \sigma} R_{\square}(x, y_i) \wedge \bigwedge_{1 \leq i < j \leq \sigma} y_i \neq y_j \right),$$

$$\exists y (R_{\square}(x, y) \wedge \exists z_{\sigma+1} \dots \exists z_{\mu} \left( \bigwedge_{\sigma+1 \leq i \leq \mu} \neg R_{\square}(x, z_i) \wedge R_{\square}(y, z_i) \wedge$$

$$\bigwedge_{\sigma+1 \leq i < j \leq \mu} z_i \neq z_j \right)).$$

Remark that for all abstract types  $(\epsilon, \sigma, \mu)$ ,  $x$  is the only free individual variable occurring in  $A_{(\epsilon, \sigma, \mu)}(x)$ .

LEMMA 15

For all abstract types  $(\epsilon, \sigma, \mu)$ ,  $\mathcal{F}_{(\epsilon, \sigma, \mu)} \models A_{(\epsilon, \sigma, \mu)}(x) [0]$ .

Let  $\varphi$  be a modal formula. Let  $\Delta(\varphi) = \{(\epsilon, \sigma, \mu) \in \Delta_\omega : \mathcal{F}_{(\epsilon, \sigma, \mu)} \not\models \varphi \text{ and } \epsilon + \mu \leq |\text{sf}(\varphi)|\}$ . Obviously,  $\Delta(\varphi) \subseteq \Delta_{|\text{sf}(\varphi)|}$ . Hence, by Lemma 11,  $\Delta(\varphi)$  is finite. Let  $A(\varphi)$  be the first-order formula  $\forall x \bigwedge \{\neg A_{(\epsilon, \sigma, \mu)}(x) : (\epsilon, \sigma, \mu) \in \Delta(\varphi)\}$ . Remark that  $A(\varphi)$  is a sentence.

## 6 First-order definability

Let  $\mathcal{C}$  be a class of frames. We say a modal formula  $\varphi$  is *first-order definable* with respect to  $\mathcal{C}$  iff there exists a sentence  $A$  such that for all frames  $\mathcal{F}$  in  $\mathcal{C}$ ,  $\mathcal{F} \models \varphi$  iff  $\mathcal{F} \models A$ . In that case, we say that  $A$  is a *first-order definition* of  $\varphi$  with respect to  $\mathcal{C}$ . The goal of this section is to prove that every modal formula is first-order definable with respect to  $\mathcal{C}_{Euc}$ .

LEMMA 16

Let  $\mathcal{F} = (W, R)$  be an Euclidean frame and  $s$  be a state in  $\mathcal{F}$ . Let  $\mathcal{F}_s = (W_s, R_s)$  be the generated subframe of  $\mathcal{F}$  determined by  $s$ . Let  $(\epsilon, \sigma, \mu)$  be an abstract type. If  $\mathcal{F} \models A_{(\epsilon, \sigma, \mu)}(x) [s]$  then  $(\mathcal{F}_{(\epsilon, \sigma, \mu)}, 0)$  is a bounded morphic image of  $(\mathcal{F}_s, s)$ .

PROOF. Suppose  $\mathcal{F} \models A_{(\epsilon, \sigma, \mu)}(x) [s]$ . We have to consider the following cases.

Case ‘ $\epsilon = 1, \sigma = 0$  and  $\mu = 0$ ’. Hence,  $\mathcal{F}_{(\epsilon, \sigma, \mu)}$  is a singleton. Moreover,  $A_{(\epsilon, \sigma, \mu)}(x)$  is the first-order formula  $\forall y \neg R_\square(x, y)$ . Since  $\mathcal{F}$  is Euclidean and  $\mathcal{F} \models A_{(\epsilon, \sigma, \mu)}(x) [s]$ , therefore  $\mathcal{F}_s$  is a singleton. Since  $\mathcal{F}_{(\epsilon, \sigma, \mu)}$  is a singleton, therefore  $(\mathcal{F}_{(\epsilon, \sigma, \mu)}, 0)$  is a bounded morphic image of  $(\mathcal{F}_s, s)$ .

Case ‘ $\epsilon = 0, \sigma \geq 1$  and  $\mu = \sigma$ ’. Hence,  $\mathcal{F}_{(\epsilon, \sigma, \mu)}$  is universal and contains exactly  $\sigma$  states. Moreover,  $A_{(\epsilon, \sigma, \mu)}(x)$  is the conjunction of the first-order formulas  $R_\square(x, x)$  and  $\exists y_1 \dots \exists y_\sigma (\bigwedge \{R_\square(x, y_i) : 1 \leq i \leq \sigma\} \wedge \bigwedge \{y_i \neq y_j : 1 \leq i < j \leq \sigma\})$ . Since  $\mathcal{F}$  is Euclidean and  $\mathcal{F} \models A_{(\epsilon, \sigma, \mu)}(x) [s]$ , therefore  $\mathcal{F}_s$  is universal and contains at least  $\sigma$  states. Since  $\mathcal{F}_{(\epsilon, \sigma, \mu)}$  is universal and contains exactly  $\sigma$  states, therefore  $(\mathcal{F}_{(\epsilon, \sigma, \mu)}, 0)$  is a bounded morphic image of  $(\mathcal{F}_s, s)$ .

Case ‘ $\epsilon = 1, \sigma \geq 1$  and  $\mu \geq \sigma$ ’. Hence,  $\mathcal{F}_{(\epsilon, \sigma, \mu)}$  is a cul-de-sac, contains exactly  $\sigma$  states accessible from its root in 1 step and contains exactly  $\mu - \sigma$  states not accessible from its root in 1 step and accessible from its root in 2 steps. Moreover,  $A_{(\epsilon, \sigma, \mu)}(x)$  is the conjunction of the first-order formulas  $\neg R_\square(x, x)$ ,  $\exists y_1 \dots \exists y_\sigma (\bigwedge \{R_\square(x, y_i) : 1 \leq i \leq \sigma\} \wedge \bigwedge \{y_i \neq y_j : 1 \leq i < j \leq \sigma\})$  and  $\exists y (R_\square(x, y) \wedge \exists z_{\sigma+1} \dots \exists z_\mu (\bigwedge \{\neg R_\square(x, z_i) \wedge R_\square(y, z_i) : \sigma + 1 \leq i \leq \mu\} \wedge \bigwedge \{z_i \neq z_j : \sigma + 1 \leq i < j \leq \mu\}))$ . Since  $\mathcal{F}$  is Euclidean and  $\mathcal{F} \models A_{(\epsilon, \sigma, \mu)}(x) [s]$ , therefore  $\mathcal{F}_s$  is a cul-de-sac, contains at least  $\sigma$  states accessible from its root in 1 step and contains at least  $\mu - \sigma$  states not accessible from its root in 1 step and accessible from its root in 2 steps. Since  $\mathcal{F}_{(\epsilon, \sigma, \mu)}$  is a cul-de-sac, contains exactly  $\sigma$  states accessible from its root in 1 step and contains exactly  $\mu - \sigma$  states not accessible from its root in 1 step and accessible from its root in 2 steps, therefore  $(\mathcal{F}_{(\epsilon, \sigma, \mu)}, 0)$  is a bounded morphic image of  $(\mathcal{F}_s, s)$ . ■

LEMMA 17

Let  $\mathcal{F} = (W, R)$  be an Euclidean frame. Let  $\varphi$  be a modal formula. If  $\mathcal{F} \not\models \varphi$  then there exists  $(\epsilon, \sigma, \mu) \in \Delta_\omega$  such that  $\mathcal{F}_{(\epsilon, \sigma, \mu)} \not\models \varphi$ ,  $\epsilon + \mu \leq |\text{sf}(\varphi)|$  and  $\mathcal{F} \models \exists x A_{(\epsilon, \sigma, \mu)}(x)$ .

PROOF. Suppose  $\mathcal{F} \not\models \varphi$ . Let  $\mathcal{M} = (W, R, V)$  be a model based on  $\mathcal{F}$  and  $s$  be a state in  $\mathcal{M}$  such that  $\mathcal{M}, s \not\models \varphi$ . Let  $\mathcal{F}_s = (W_s, R_s)$  be the generated subframe of  $\mathcal{F}$  determined by  $s$ . Since  $\mathcal{F}$  is Euclidean, therefore by Lemma 10,  $\mathcal{F}_s$  is clusterizable. Let  $V_s$  be the function assigning to each propositional variable  $p$  the subset  $V_s(p) = V(p) \cap W_s$  of  $W_s$ . Let  $\mathcal{M}_s = (W_s, R_s, V_s)$ . Since  $\mathcal{M}, s \not\models \varphi$ , therefore by the Generated Submodel Lemma [6, Proposition 2.6 (iii)],  $\mathcal{M}_s, s \not\models \varphi$ . Since  $\mathcal{F}_s$  is clusterizable, therefore we have to consider the following cases.

Case ‘ $\mathcal{F}_s$  is a singleton’. Let  $\mathcal{F}' = (W', R')$  where  $W' = \{s\}$  and  $R' = \emptyset$ . Remark that  $\mathcal{F}'$  is a singleton. Let  $V'$  be  $V_s$ . Let  $\mathcal{M}' = (W', R', V')$ . Since  $\mathcal{M}_s, s \not\models \varphi$ , therefore  $\mathcal{M}', s \not\models \varphi$ . Hence,  $\mathcal{F}' \not\models \varphi$ . Moreover, obviously,  $|W'| \leq |\text{sf}(\varphi)|$ .

Case ‘ $\mathcal{F}_s$  is universal’. Let  $\Box\psi_1, \dots, \Box\psi_k$  be an enumeration of the set of all  $\varphi$ 's subformulas of the form  $\Box\psi$  such that  $\mathcal{M}_s, s \not\models \Box\psi$ . Obviously,  $k < |\text{sf}(\varphi)|$ . For all positive integers  $i$ , if  $i \leq k$  then let  $t_i \in R_s(s)$  be such that  $\mathcal{M}_s, t_i \not\models \psi_i$ . Let  $\mathcal{F}' = (W', R')$  where  $W' = \{s\} \cup \{t_1, \dots, t_k\}$  and  $R' = R_s \cap (W' \times W')$ . Remark that  $\mathcal{F}'$  is universal. Let  $V'$  be the restriction of  $V_s$  to  $W'$ . Let  $\mathcal{M}' = (W', R', V')$ . The reader may easily prove by induction on  $\theta \in \text{sf}(\varphi)$  that for all  $u \in W'$ ,  $\mathcal{M}_s, u \models \theta$  iff  $\mathcal{M}', u \models \theta$ . Since  $\mathcal{M}_s, s \not\models \varphi$ , therefore  $\mathcal{M}', s \not\models \varphi$ . Hence,  $\mathcal{F}' \not\models \varphi$ . Moreover, obviously,  $|W'| \leq |\text{sf}(\varphi)|$ .

Case ‘ $\mathcal{F}_s$  is a cul-de-sac’. Let  $t \in R_s(s)$ . Let  $\Box\psi_1, \dots, \Box\psi_k$  be an enumeration of the set of all  $\varphi$ 's subformulas of the form  $\Box\psi$  such that  $\mathcal{M}_s, s \not\models \Box\psi$  and  $\Box\chi_1, \dots, \Box\chi_l$  be an enumeration of the set of all  $\varphi$ 's subformulas of the form  $\Box\chi$  such that  $\mathcal{M}_s, s \models \Box\chi$  and  $\mathcal{M}_s, t \not\models \Box\chi$ . Obviously,  $k + l < |\text{sf}(\varphi)|$ . We consider the cases ‘ $k = 0$ ’ and  $k \geq 1$ . In the former case, let  $\mathcal{F}' = (W', R')$  where  $W' = \{s\}$  and  $R' = \emptyset$ . Remark that  $\mathcal{F}'$  is a singleton. In the latter case, for all positive integers  $i$ , if  $i \leq k$  then let  $u_i \in R_s(s)$  be such that  $\mathcal{M}_s, u_i \not\models \psi_i$  and for all positive integers  $j$ , if  $j \leq l$  then let  $v_j \in R_s(t) \setminus R_s(s)$  be such that  $\mathcal{M}_s, v_j \not\models \chi_j$ . Let  $\mathcal{F}' = (W', R')$  where  $W' = \{s\} \cup \{u_1, \dots, u_k\} \cup \{v_1, \dots, v_l\}$  and  $R' = R_s \cap (W' \times W')$ . Remark that  $\mathcal{F}'$  is a cul-de-sac. In both cases, let  $V'$  be the restriction of  $V_s$  to  $W'$ . Let  $\mathcal{M}' = (W', R', V')$ . The reader may easily prove by induction on  $\theta \in \text{sf}(\varphi)$  that for all  $w \in W'$ ,  $\mathcal{M}_s, w \models \theta$  iff  $\mathcal{M}', w \models \theta$ . Since  $\mathcal{M}_s, s \not\models \varphi$ , therefore  $\mathcal{M}', s \not\models \varphi$ . Hence,  $\mathcal{F}' \not\models \varphi$ . Moreover, obviously,  $|W'| \leq |\text{sf}(\varphi)|$ .

In all cases, let  $(\epsilon, \sigma, \mu)$  be the type of  $\mathcal{F}'$ . The reader may easily verify that  $(\mathcal{F}', s)$  and  $(\mathcal{F}_{(\epsilon, \sigma, \mu)}, 0)$  are isomorphic. Since  $\mathcal{F}' \not\models \varphi$ , therefore  $\mathcal{F}_{(\epsilon, \sigma, \mu)} \not\models \varphi$ . Moreover, since  $|W'| \leq |\text{sf}(\varphi)|$ , therefore by Lemma 12,  $\epsilon + \mu \leq |\text{sf}(\varphi)|$ . In other respect, obviously,  $(\mathcal{F}', s)$  can be isomorphically embedded into  $(\mathcal{F}_s, s)$ . Since  $(\mathcal{F}', s)$  and  $(\mathcal{F}_{(\epsilon, \sigma, \mu)}, 0)$  are isomorphic, therefore  $(\mathcal{F}_{(\epsilon, \sigma, \mu)}, 0)$  can be isomorphically embedded into  $(\mathcal{F}_s, s)$ . By Lemma 15,  $\mathcal{F}_{(\epsilon, \sigma, \mu)} \models A_{(\epsilon, \sigma, \mu)}(x)$  [0]. Since  $A_{(\epsilon, \sigma, \mu)}(x)$  is an existential first-order formula and  $(\mathcal{F}_{(\epsilon, \sigma, \mu)}, 0)$  can be isomorphically embedded into  $(\mathcal{F}_s, s)$ , therefore  $\mathcal{F}_s \models A_{(\epsilon, \sigma, \mu)}(x)$  [s]. Hence,  $\mathcal{F} \models A_{(\epsilon, \sigma, \mu)}(x)$  [s]. Thus,  $\mathcal{F} \models \exists x A_{(\epsilon, \sigma, \mu)}(x)$ . ■

LEMMA 18

Let  $\mathcal{F} = (W, R)$  be an Euclidean frame. Let  $\varphi$  be a modal formula. The following conditions are equivalent:

- $\mathcal{F} \models \varphi$ ,
- $\mathcal{F} \models A(\varphi)$ .

PROOF. Suppose  $\mathcal{F} \models \varphi$  and  $\mathcal{F} \not\models A(\varphi)$ . Let  $s \in W$  be such that  $\mathcal{F} \not\models \bigwedge \{ \neg A_{(\epsilon, \sigma, \mu)}(x) : (\epsilon, \sigma, \mu) \in \Delta(\varphi) \}$  [s]. Let  $(\epsilon, \sigma, \mu) \in \Delta(\varphi)$  be such that  $\mathcal{F} \models A_{(\epsilon, \sigma, \mu)}(x)$  [s]. Hence, by definition of  $\Delta(\varphi)$ ,  $\mathcal{F}_{(\epsilon, \sigma, \mu)} \not\models \varphi$ . Moreover, by Lemma 16,  $\mathcal{F}_{(\epsilon, \sigma, \mu)}$  is a bounded morphic image of  $\mathcal{F}_s$ , the generated subframe of  $\mathcal{F}$  determined by  $s$ . Since  $\mathcal{F} \models \varphi$ , therefore by the Generated Subframe Lemma [6, Theorem 3.14 (ii)] and the Bounded Morphism Lemma [6, Theorem 3.14 (iii)],  $\mathcal{F}_{(\epsilon, \sigma, \mu)} \models \varphi$ : a contradiction.

Suppose  $\mathcal{F} \models A(\varphi)$  and  $\mathcal{F} \not\models \varphi$ . By Lemma 17, let  $(\epsilon, \sigma, \mu)$  be an abstract type such that  $\mathcal{F}_{(\epsilon, \sigma, \mu)} \not\models \varphi$ ,  $\epsilon + \mu \leq |\text{sf}(\varphi)|$  and  $\mathcal{F} \models \exists x A_{(\epsilon, \sigma, \mu)}(x)$ . Hence, by definition of  $\Delta(\varphi)$ ,  $(\epsilon, \sigma, \mu) \in \Delta(\varphi)$ . Moreover, let  $s \in W$  be such that  $\mathcal{F} \models A_{(\epsilon, \sigma, \mu)}(x) [s]$ . Since  $\mathcal{F} \models A(\varphi)$  and  $(\epsilon, \sigma, \mu) \in \Delta(\varphi)$ , therefore  $\mathcal{F} \models \neg A_{(\epsilon, \sigma, \mu)}(x) [s]$ : a contradiction. ■

THEOREM 1

Every modal formula is first-order definable with respect to  $\mathcal{C}_{Euc}$ . Moreover, for all modal formulas  $\varphi$ ,  $A(\varphi)$  is a first-order definition of  $\varphi$  with respect to  $\mathcal{C}_{Euc}$ .

PROOF. By Lemma 18. ■

## 7 Modal definability

Let  $\mathcal{C}$  be a class of frames. We say a sentence  $A$  is *modally definable* with respect to  $\mathcal{C}$  iff there exists a modal formula  $\varphi$  such that for all frames  $\mathcal{F}$  in  $\mathcal{C}$ ,  $\mathcal{F} \models A$  iff  $\mathcal{F} \models \varphi$ . In that case, we say that  $\varphi$  is a *modal definition* of  $A$  with respect to  $\mathcal{C}$ . The goal of this section is to prove that the problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{Euc}$  is undecidable.

LEMMA 19

$\mathcal{C}_{flo}$  is stable.

PROOF. Let  $A(x_1, x)$  be the first-order formula  $\neg R_{\square}(x_1, x_1) \wedge \exists y (x_1 \neq y \wedge \neg R_{\square}(y, y)) \wedge x_1 \neq x$ . Let  $B$  be the sentence  $\exists z_1 \exists z_2 (z_1 \neq z_2 \wedge \neg R_{\square}(z_1, z_1) \wedge \neg R_{\square}(z_2, z_2) \wedge \forall y (R_{\square}(z_1, y) \leftrightarrow R_{\square}(z_2, y)))$ . Obviously,  $\mathcal{C}_{flo}$  and  $A(x_1, x)$  satisfy the first condition defining stable classes of frames (condition (a)). As for the second condition defining stable classes of frames (condition (b)), let  $\mathcal{F}_0 = (W_0, R_0)$  be a frame in  $\mathcal{C}_{flo}$  and  $s_0 \in W_0$  be such that not  $s_0 R_0 s_0$ . Consider the frame  $\mathcal{F} = (W, R)$  in  $\mathcal{C}_{flo}$  defined as follows:

- $W = W_0 \cup \{s_1\}$ ,
- $R$  is  $R_0 \cup (\{s_1\} \times R_0(s_0))$ .

Obviously,  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(x_1, x)$  and  $s_1$  and  $\mathcal{F} \models B$ . Let  $E$  be the set of all  $s \in W$  such that not  $s R s$ . Let  $\simeq$  be the equivalence relation on  $E$  such that  $s \simeq t$  iff  $R(s) = R(t)$ . Let  $E'$  be a selector set for  $E/\simeq$ , i.e.  $E'$  possesses exactly one representative state from any equivalence class modulo  $\simeq$ . Consider the frame  $\mathcal{F}' = (W', R')$  in  $\mathcal{C}_{flo}$  defined as follows:

- $W' = (W \setminus E) \cup E'$ ,
- $R'$  is  $R \cap (W' \times W')$ .

Obviously,  $\mathcal{F}' \not\models B$ . Let  $f$  be the function assigning to each state  $s$  in  $\mathcal{F}$  a state  $f(s)$  in  $\mathcal{F}'$ , such that if  $s \in E$  then  $f(s)$  is the unique representative state of  $s$  in  $E'$ , else  $f(s)$  is the state  $s$ . Obviously,  $\mathcal{F}'$  is a bounded morphic image of  $\mathcal{F}$ . Hence, by the Bounded Morphism Lemma [6, Theorem 3.14 (iii)],  $\mathcal{F} \preceq \mathcal{F}'$ . ■

LEMMA 20

$\mathcal{C}_{Euc}$  is stable.

PROOF. Let  $A(x_1, x_2, x)$  be the first-order formula  $x_1 \neq x \wedge x_2 \neq x$ . Let  $B$  be the sentence  $\exists z_1 \exists z_2 (z_1 \neq z_2 \wedge \neg \exists y (R_{\square}(y, z_1) \vee R_{\square}(y, z_2) \vee R_{\square}(z_1, y) \vee R_{\square}(z_2, y)))$ . Obviously,  $\mathcal{C}_{Euc}$  and  $A(x_1, x_2, x)$  satisfy the first condition defining stable classes of frames (condition (a)). As for the second condition defining stable classes of frames (condition (b)), let  $\mathcal{F}_0 = (W_0, R_0)$  be a frame in  $\mathcal{C}_{Euc}$ . Consider the frames

$\mathcal{F} = (W, R)$ ,  $\mathcal{F}' = (W', R')$  in  $\mathcal{C}_{Euc}$  defined as follows:

- $W = W_0 \cup \{s_1, s_2\}$ ,
- $R$  is  $R_0$ ,
- $W' = \text{dom}(R_0) \cup \text{ran}(R_0) \cup \{s'\}$ ,
- $R'$  is  $R_0$ .

Obviously,  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(x_1, x_2, x)$  and  $s_1, s_2$ ,  $\mathcal{F} \models B$ ,  $\mathcal{F}' \not\models B$  and  $\mathcal{F}'$  is a bounded morphic image of  $\mathcal{F}$ . Hence, by the Bounded Morphism Lemma [6, Theorem 3.14 (iii)],  $\mathcal{F} \preceq \mathcal{F}'$ . ■

LEMMA 21

Let  $\mathcal{F} = (W, R)$  be a reflexive and symmetric frame. Let  $\mathcal{F}' = (W', R')$  be the frame where

- $W' = W \cup \{\{s, t\} : s, t \in W \text{ and } sRt\}$ ,
- $R' = \{(s, \{t, u\}) : s, t, u \in W, tRu \text{ and } s \in \{t, u\}\} \cup \{(\{s, t\}, \{u, v\}) : s, t, u, v \in W, sRt \text{ and } uRv\}$ .

$\mathcal{F}'$  is a flower. Moreover, for all first-order formulas  $A(\bar{x})$  and for all lists  $\bar{s}$  of states in  $\mathcal{F}$ , the following conditions are equivalent:

- $\mathcal{F} \models A(\bar{x}) [\bar{s}]$ ,
- $\mathcal{F}' \models \tau(A(\bar{x})) [\bar{s}]$ .

PROOF. The proof of the first part of the lemma saying that  $\mathcal{F}'$  is a flower is left to the reader. We prove the second part of the lemma by induction on  $A(\bar{x})$ . We only consider the cases ‘ $A(\bar{x})$  is of the form  $R_{\square}(x, y)$ ’ and ‘ $A(\bar{x})$  is of the form  $\forall z B(\bar{x}, z)$ ’.

Case ‘ $A(\bar{x})$  is of the form  $R_{\square}(x, y)$ ’. Let  $s, t$  be states in  $\mathcal{F}$ . Suppose  $\mathcal{F} \models R_{\square}(x, y) [s, t]$ . Hence,  $sRt$ . Thus,  $\{s, t\}$  is a state in  $\mathcal{F}'$  such that  $sR'\{s, t\}$  and  $tR'\{s, t\}$ . Consequently,  $\mathcal{F}' \models R_{\square}(x, z) \wedge R_{\square}(y, z) [s, t, \{s, t\}]$ . Hence,  $\mathcal{F}' \models \tau(R_{\square}(x, y)) [s, t]$ . Reciprocally, suppose  $\mathcal{F}' \models \tau(R_{\square}(x, y)) [s, t]$ . Let  $w'$  be a state in  $\mathcal{F}'$  such that  $\mathcal{F}' \models R_{\square}(x, z) \wedge R_{\square}(y, z) [s, t, w']$ . Thus,  $sR'w'$  and  $tR'w'$ . Let  $u, v \in W$  be such that  $uRv$  and  $w' = \{u, v\}$ . Since  $sR'w'$  and  $tR'w'$ , therefore  $s \in \{u, v\}$  and  $t \in \{u, v\}$ . Consequently, either  $s = u$ , or  $s = v$  and either  $t = u$ , or  $t = v$ . Since  $\mathcal{F}$  is reflexive and symmetric and  $uRv$ , therefore  $sRt$ . Hence,  $\mathcal{F} \models R_{\square}(x, y) [s, t]$ .

Case ‘ $A(\bar{x})$  is of the form  $\forall z B(\bar{x}, z)$ ’. Let  $\bar{s}$  be a list of states in  $\mathcal{F}$ . Suppose  $\mathcal{F}' \not\models \tau(\forall z B(\bar{x}, z)) [\bar{s}]$ . Let  $w'$  be a state in  $\mathcal{F}'$  such that  $\mathcal{F}' \not\models R_{\square}(z, z) \vee \tau(B(\bar{x}, z)) [\bar{s}, w']$ . Hence,  $\mathcal{F}' \not\models R_{\square}(z, z) [\bar{s}, w']$  and  $\mathcal{F}' \not\models \tau(B(\bar{x}, z)) [\bar{s}, w']$ . Thus, not  $w'R'w'$ . Consequently,  $w'$  is a state in  $\mathcal{F}$ . Since  $\mathcal{F}' \not\models \tau(B(\bar{x}, z)) [\bar{s}, w']$ , therefore by induction hypothesis,  $\mathcal{F} \not\models B(\bar{x}, z) [\bar{s}, w']$ . Hence,  $\mathcal{F} \not\models \forall z B(\bar{x}, z) [\bar{s}]$ . Reciprocally, suppose  $\mathcal{F} \not\models \forall z B(\bar{x}, z) [\bar{s}]$ . Let  $w$  be a state in  $\mathcal{F}$  such that  $\mathcal{F} \not\models B(\bar{x}, z) [\bar{s}, w]$ . Hence, not  $wR'w$  and by induction hypothesis,  $\mathcal{F}' \not\models \tau(B(\bar{x}, z)) [\bar{s}, w]$ . Thus,  $\mathcal{F}' \not\models R_{\square}(z, z) \vee \tau(B(\bar{x}, z)) [\bar{s}, w]$ . Consequently,  $\mathcal{F}' \not\models \tau(\forall z B(\bar{x}, z)) [\bar{s}]$ . ■

LEMMA 22

Let  $\mathcal{F} = (W, R)$  be a flower. Let  $\mathcal{F}' = (W', R')$  be the frame where

- $W' = \{s \in W : \text{not } sRs\}$ ,
- $R' = \{(s, t) : s, t \in W, \text{not } sRs, \text{not } tRt \text{ and there exists } u \in W \text{ such that } sRu \text{ and } tRu\}$ .

$\mathcal{F}'$  is reflexive and symmetric. Moreover, for all first-order formulas  $A(\bar{x})$  and for all lists  $\bar{s}'$  of states in  $\mathcal{F}'$ , the following conditions are equivalent:

- $\mathcal{F}' \models A(\bar{x}) [\bar{s}']$ ,
- $\mathcal{F} \models \tau(A(\bar{x})) [\bar{s}']$ .

PROOF. The proof of the first part of the lemma saying that  $\mathcal{F}'$  is reflexive and symmetric is left to the reader. We prove the second part of the lemma by induction on  $A(\bar{x})$ . We only consider the cases ‘ $A(\bar{x})$  is of the form  $R_{\square}(x, y)$ ’ and ‘ $A(\bar{x})$  is of the form  $\forall z B(\bar{x}, z)$ ’.

Case ‘ $A(\bar{x})$  is of the form  $R_{\square}(x, y)$ ’. Let  $s', t'$  be states in  $\mathcal{F}'$ . Suppose  $\mathcal{F}' \models R_{\square}(x, y) [s', t']$ . Hence,  $s'R't'$ . Let  $u \in W$  be such that  $s'Ru$  and  $t'Ru$ . Thus,  $\mathcal{F} \models R_{\square}(x, z) \wedge R_{\square}(y, z) [s', t', u]$ . Consequently,  $\mathcal{F} \models \tau(R_{\square}(x, y)) [s', t']$ . Reciprocally, suppose  $\mathcal{F} \models \tau(R_{\square}(x, y)) [s', t']$ . Let  $w$  be a state in  $\mathcal{F}$  such that  $\mathcal{F} \models R_{\square}(x, z) \wedge R_{\square}(y, z) [s', t', w]$ . Hence,  $s'Rw$  and  $t'Rw$ . Thus,  $s'R't'$ . Consequently,  $\mathcal{F}' \models R_{\square}(x, y) [s', t']$ .

Case ‘ $A(\bar{x})$  is of the form  $\forall z B(\bar{x}, z)$ ’. Let  $\bar{s}'$  be a list of states in  $\mathcal{F}'$ . Suppose  $\mathcal{F} \not\models \tau(\forall z B(\bar{x}, z)) [\bar{s}']$ . Let  $w$  be a state in  $\mathcal{F}$  such that  $\mathcal{F} \not\models R_{\square}(z, z) \vee \tau(B(\bar{x}, z)) [\bar{s}', w]$ . Hence,  $\mathcal{F} \not\models R_{\square}(z, z) [\bar{s}', w]$  and  $\mathcal{F} \not\models \tau(B(\bar{x}, z)) [\bar{s}', w]$ . Thus, not  $wRw$ . Consequently,  $w$  is a state in  $\mathcal{F}'$ . Since  $\mathcal{F} \not\models \tau(B(\bar{x}, z)) [\bar{s}', w]$ , therefore by induction hypothesis,  $\mathcal{F}' \not\models B(\bar{x}, z) [\bar{s}', w]$ . Hence,  $\mathcal{F}' \not\models \forall z B(\bar{x}, z) [\bar{s}']$ . Reciprocally, suppose  $\mathcal{F}' \not\models \forall z B(\bar{x}, z) [\bar{s}']$ . Let  $w'$  be a state in  $\mathcal{F}'$  such that  $\mathcal{F}' \not\models B(\bar{x}, z) [\bar{s}', w']$ . Hence, not  $w'Rw'$  and by induction hypothesis,  $\mathcal{F} \not\models \tau(B(\bar{x}, z)) [\bar{s}', w']$ . Thus,  $\mathcal{F} \not\models R_{\square}(z, z) \vee \tau(B(\bar{x}, z)) [\bar{s}', w']$ . Consequently,  $\mathcal{F} \not\models \tau(\forall z B(\bar{x}, z)) [\bar{s}']$ . ■

LEMMA 23

For all sentences  $A$ , the following conditions are equivalent:

- $\mathcal{C}_{rs} \models A$ ,
- $\mathcal{C}_{flo} \models \tau(A)$ .

PROOF. By Lemmas 21 and 22. ■

LEMMA 24

Let  $B_{flo}$  be the conjunction of the following sentences:  $\exists x \neg R_{\square}(x, x)$  and  $\forall x \forall y \exists z \exists t (R_{\square}(x, z) \wedge R_{\square}(y, t) \wedge R_{\square}(z, t))$ . For all sentences  $A$ , the following conditions are equivalent:

- $\mathcal{C}_{flo} \models A$ ,
- $\mathcal{C}_{Euc} \models B_{flo} \rightarrow A$ .

PROOF. It suffices to remark that by Lemmas 2 and 3, every flower is Euclidean and by Lemma 4, an Euclidean frame is a flower iff it validates  $B_{flo}$ . ■

LEMMA 25

The problem of deciding the validity of sentences in  $\mathcal{C}_{flo}$  is undecidable.

PROOF. By Lemma 23 and a result of Rogers [19] saying that deciding the validity of sentences in  $\mathcal{C}_{rs}$  is undecidable, the problem of deciding the validity of sentences in  $\mathcal{C}_{flo}$  is undecidable. ■

LEMMA 26

The problem of deciding the validity of sentences in  $\mathcal{C}_{Euc}$  is undecidable.

PROOF. By Lemmas 24 and 25, the problem of deciding the validity of sentences in  $\mathcal{C}_{Euc}$  is undecidable. ■

THEOREM 2

The problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{flo}$  is undecidable.

PROOF. By Lemma 19,  $\mathcal{C}_{flo}$  is stable. Hence, by Theorem 1 in [3], the problem of deciding the validity of sentences in  $\mathcal{C}_{flo}$  is reducible to the problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{flo}$ . Since by Lemma 25, the problem of deciding the validity of sentences in  $\mathcal{C}_{flo}$  is

undecidable, therefore the problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{flo}$  is undecidable. ■

### THEOREM 3

The problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{Euc}$  is undecidable.

PROOF. By Lemma 20,  $\mathcal{C}_{Euc}$  is stable. Hence, by Theorem 1 in [3], the problem of deciding the validity of sentences in  $\mathcal{C}_{Euc}$  is reducible to the problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{Euc}$ . Since by Lemma 26, the problem of deciding the validity of sentences in  $\mathcal{C}_{Euc}$  is undecidable, therefore the problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{Euc}$  is undecidable. ■

## 8 Conclusion and open problems

We have proved that over the class of all Euclidean frames, every modal formula is first-order definable and the problem of deciding the modal definability of sentences is undecidable. Much remains to be done.

By enriching the ordinary modal language with the universal modality [15] or the difference modality [18], one generally increases its expressivity. Hence, in the class of all  $K5$ -frames studied in this article, it would be interesting to see how things change when the language is enriched with these modal constructs.

The computability of first-order definability and modal definability in the class of all  $S5$ -frames and in the class of all  $KD45$ -frames have been considered in [1, 2, 12, 13]. Thus, we would like to determine the computability of first-order definability and modal definability in the class of all  $K45$ -frames (based on transitive and Euclidean relations).

For what concerns the computability of satisfiability in different classes of Euclidean frames, Halpern and Rêgo [16] have suggested to explore a somewhat larger class of frames: the class of all connected frames. Consequently, it seems interesting to determine the computability of first-order definability and modal definability in the class of all connected frames.

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