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ON THE BEILINSON–BLOCH–KATO CONJECTURE FOR RANKIN–SELBERG MOTIVES

YIFENG LIU, YICHAO TIAN, LIANG XIAO, WEI ZHANG, AND XINWEN ZHU

ABSTRACT. In this article, we study the Beilinson–Bloch–Kato conjecture for motives corresponding to the Rankin–Selberg product of conjugate self-dual automorphic representations, within the framework of the Gan–Gross–Prasad conjecture. We show that if the central critical value of the Rankin–Selberg L -function does not vanish, then the Bloch–Kato Selmer group with coefficients in a favorable field of the corresponding motive vanishes. We also show that if the class in the Bloch–Kato Selmer group constructed from certain diagonal cycle does not vanish, which is conjecturally equivalent to the nonvanishing of the central critical first derivative of the Rankin–Selberg L -function, then the Bloch–Kato Selmer group is of rank one.

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1. INTRODUCTION

In this article, we study the Beilinson–Bloch–Kato conjecture for motives corresponding to the Rankin–Selberg product of conjugate self-dual automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F) \times \mathrm{GL}_{n+1}(\mathbb{A}_F)$ for a CM number field F , within the framework of the Gan–Gross–Prasad conjecture [GGP12] for the pair of unitary groups $\mathrm{U}(n) \times \mathrm{U}(n+1)$. For the background on the Beilinson–Bloch–Kato conjecture, which is a generalization of the famous Birch and Swinnerton-Dyer conjecture from elliptic curves to higher dimensional algebraic varieties, we refer to the introduction of [Liu16].

1.1. Main results. Let F/F^+ be a totally imaginary quadratic extension of a totally real number field. We first state one of our main results that is least technical to understand.

Theorem 1.1.1 (Corollary 8.2.5). *Let $n \geq 2$ be an integer. Let A and A' be two modular elliptic curves over F^+ such that $\mathrm{End}(A_{\overline{F}}) = \mathrm{End}(A'_{\overline{F}}) = \mathbb{Z}$. Suppose that*

- (a) $A_{\overline{F}}$ and $A'_{\overline{F}}$ are not isogenous to each other;
- (b) both $\mathrm{Sym}^{n-1} A$ and $\mathrm{Sym}^n A'$ are modular; and
- (c) $[F^+ : \mathbb{Q}] > 1$ if $n \geq 3$.

If the (central critical) L -value $L(n, \mathrm{Sym}^{n-1} A_F \times \mathrm{Sym}^n A'_F)$ does not vanish, then the Bloch–Kato Selmer group

$$H_f^1(F, \mathrm{Sym}^{n-1} H_{\mathrm{et}}^1(A_{\overline{F}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathrm{Sym}^n H_{\mathrm{et}}^1(A'_{\overline{F}}, \mathbb{Q}_\ell)(n))$$

vanishes for all but finitely many rational primes ℓ .

Remark 1.1.2. The finite set of rational primes ℓ that are excluded in Theorem 1.1.1 can be effectively bounded. We now explain the three conditions in Theorem 1.1.1.

- (a) is necessary for ℓ to satisfy (L3) and (L5) in Definition 8.1.1. Otherwise, there might be no rational primes ℓ satisfying (L3) and (L5).
- (b) is necessary since our approach only applies to Galois representations arising from automorphic representations. We summarise the current knowledge on the modularity of symmetric powers of elliptic curves in Remark 8.2.6.
- (c) is necessary only for technical reasons. First, we do not know Hypothesis 3.2.9, which concerns cohomology of unitary Shimura varieties, yet for $N \geq 4$ if $F^+ = \mathbb{Q}$. Second, we do not have (an appropriate replacement for) Theorem D.1.3, a result generalizing [CS17], when $F^+ = \mathbb{Q}$ since the corresponding Shimura variety is not proper. Indeed, as long as we have these results as expected, (c) can be lifted.

Theorem 1.1.1 is a special case of a more general result concerning the Bloch–Kato Selmer groups of Galois representations associated to conjugate self-dual automorphic representations. To reduce the burden of long and technical terminology in the future, we first introduce the following definition, which will serve for the entire article.

Definition 1.1.3. We say that a complex representation Π of $\mathrm{GL}_N(\mathbb{A}_F)$ with $N \geq 1$ is *relevant* if

- (1) Π is an irreducible cuspidal automorphic representation;
- (2) $\Pi \circ \mathfrak{c} \simeq \Pi^\vee$, where $\mathfrak{c} \in \mathrm{Gal}(F/F^+)$ is the complex conjugation;
- (3) for every archimedean place τ of F , Π_τ is isomorphic to the (irreducible) principal series representation induced by the characters $(\arg^{N-1}, \arg^{N-3}, \dots, \arg^{3-N}, \arg^{1-N})$, where $\arg: \mathbb{C}^\times \rightarrow \mathbb{C}^\times$ is the *argument character* defined by the formula $\arg(z) := z/\sqrt{z\bar{z}}$.

Now we can state our main result in the context of automorphic representations, of which Theorem 1.1.1 is a special case. Till the end of the next subsection, we will take an integer $n \geq 2$, and denote by n_0 and n_1 the unique even and odd numbers in $\{n, n+1\}$, respectively.

Theorem 1.1.4 (Theorem 8.2.1). *Let Π_0 and Π_1 be relevant representations of $\mathrm{GL}_{n_0}(\mathbb{A}_F)$ and $\mathrm{GL}_{n_1}(\mathbb{A}_F)$, respectively. Let $E \subseteq \mathbb{C}$ be a strong coefficient field of both Π_0 and Π_1 (Definition 3.2.5). Suppose $[F^+ : \mathbb{Q}] > 1$ if $n \geq 3$. If $L(\frac{1}{2}, \Pi_0 \times \Pi_1) \neq 0$, then for all admissible primes λ of E with respect to (Π_0, Π_1) , the Bloch–Kato Selmer group $H_f^1(F, \rho_{\Pi_0, \lambda} \otimes_{E_\lambda} \rho_{\Pi_1, \lambda}(n))$ vanishes. Here, $\rho_{\Pi_\alpha, \lambda}$ is the Galois representation of F with coefficients in E_λ associated to Π_α for $\alpha = 0, 1$, as described in Proposition 3.2.4 and Definition 3.2.5.*

In fact, Theorem 8.2.1 is slightly stronger than the one stated here.

Remark 1.1.5. The notion of admissible primes appeared in Theorem 1.1.4 is introduced in Definition 8.1.1, which consists of a long list of assumptions, some of which are rather technical. Here, we would like to comment on the essence of these assumptions.

- (L1) is elementary and excludes only finitely many primes λ .
- (L2) is elementary and excludes only finitely many primes λ .
- (L3) is expected to hold for every prime λ if and only if the (conjectural) automorphic product $\Pi_0 \boxtimes \Pi_1$, which is an irreducible admissible representation of $\mathrm{GL}_{n(n+1)}(\mathbb{A}_F)$, remains cuspidal.
- (L4) is expected to hold for all but finitely many primes λ .
- (L5) is basically saying that, under (L4), the image of the pair of residual Galois representations $(\bar{\rho}_{\Pi_0, \lambda}, \bar{\rho}_{\Pi_1, \lambda})$ contains an element of a particular form. It is expected to hold for all but finitely many primes λ if the two automorphic representations Π_0 and Π_1 are not correlated in some manner. For example, when $n = 2$, we expect that as long as Π_1 is not an automorphic twist of $\mathrm{Sym}^2 \Pi_0$ after any base change, then (L5) holds for all but finitely many primes λ .
- (L6) is a technical assumption that is only used in the argument of an R=T theorem concerning Galois deformations in Appendix E. It is expected to hold for all but finitely many primes λ (see Conjecture E.8.1 and Theorem E.8.4).
- (L7) is a technical assumption for the vanishing of certain Hecke localized cohomology of unitary Shimura varieties off middle degree. In fact, when $[F^+ : \mathbb{Q}] > 1$, (L7) holds for all but finitely many primes λ by Corollary D.1.4.

In fact, we have dedicated ourselves to obtaining the following family of abstract examples in which all but finitely many primes are admissible. Note that neither the following theorem nor Theorem 1.1.1 implies the other.

Theorem 1.1.6 (Corollary 8.2.3). *Let Π_0 , Π_1 , and E be as in Theorem 1.1.4. Suppose that*

- (a) *there exists a very special inert prime \mathfrak{p} of F^+ (Definition 3.3.4) such that $\Pi_{0, \mathfrak{p}}$ is Steinberg, and $\Pi_{1, \mathfrak{p}}$ is unramified whose Satake parameter contains 1 exactly once¹;*
- (b) *for $\alpha = 0, 1$, there exists a nonarchimedean place w_α of F such that Π_{α, w_α} is supercuspidal;*
- (c) *$[F^+ : \mathbb{Q}] > 1$ if $n \geq 3$.*

If $L(\frac{1}{2}, \Pi_0 \times \Pi_1) \neq 0$, then for all but finitely many primes λ of E , the Bloch–Kato Selmer group $H_f^1(F, \rho_{\Pi_0, \lambda} \otimes_{E_\lambda} \rho_{\Pi_1, \lambda}(n))$ vanishes.

Remark 1.1.7. In (a) of Theorem 1.1.6, if the CM field F is Galois or contains an imaginary quadratic field, then a very special inert prime of F^+ is simply a prime of F^+ that is inert in F , of degree 1 over \mathbb{Q} , whose underlying rational prime is odd and unramified in F .

Now we state our result in the (Selmer) rank 1 case. Let Π_0 and Π_1 be relevant representations of $\mathrm{GL}_{n_0}(\mathbb{A}_F)$ and $\mathrm{GL}_{n_1}(\mathbb{A}_F)$, respectively. Let $E \subseteq \mathbb{C}$ be a strong coefficient field of both Π_0 and Π_1

¹Note that the Satake parameter of $\Pi_{1, \mathfrak{p}}$ has to contain 1 at least once by Definition 1.1.3(2).

(Definition 3.2.5). Suppose that the global epsilon factor of $\Pi_0 \times \Pi_1$ is -1 . Then the Beilinson–Bloch–Kato conjecture predicts that if $L'(\frac{1}{2}, \Pi_0 \times \Pi_1) \neq 0$, then the Bloch–Kato Selmer group $H_f^1(F, \rho_{\Pi_0, \lambda} \otimes_{E_\lambda} \rho_{\Pi_1, \lambda}(n))$ has rank 1. However, what we can prove now is half of this implication. Namely, for every prime λ of E , we will construct explicitly an element Δ_λ in (the direct sum of finitely many copies of) $H_f^1(F, \rho_{\Pi_0, \lambda} \otimes_{E_\lambda} \rho_{\Pi_1, \lambda}(n))$ in Subsection 8.3 (more precisely, Δ_λ is the class in (8.9)). In fact, by Conjecture 8.3.1 and Beilinson’s conjecture on the injectivity of the ℓ -adic Abel–Jacobi map, the nonvanishing of Δ_λ is equivalent to the nonvanishing of $L'(\frac{1}{2}, \Pi_0 \times \Pi_1)$. Then our theorem in the rank 1 case reads as follows.

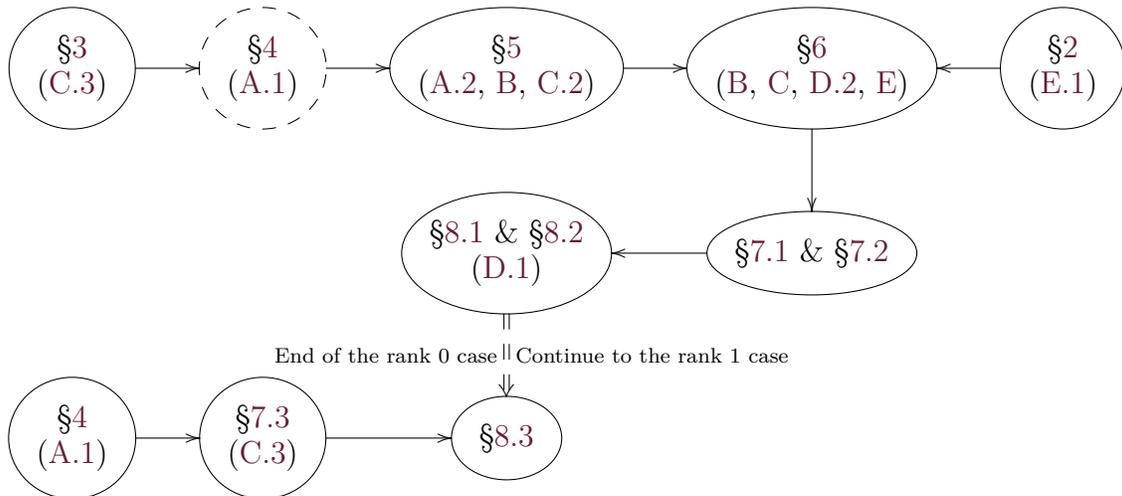
Theorem 1.1.8 (Theorem 8.3.2). *Let Π_0 and Π_1 be relevant representations of $\mathrm{GL}_{n_0}(\mathbb{A}_F)$ and $\mathrm{GL}_{n_1}(\mathbb{A}_F)$, respectively. Let $E \subseteq \mathbb{C}$ be a strong coefficient field of both Π_0 and Π_1 (Definition 3.2.5). Suppose $[F^+ : \mathbb{Q}] > 1$ if $n \geq 3$. For all admissible primes λ of E with respect to (Π_0, Π_1) , if $\Delta_\lambda \neq 0$, then the Bloch–Kato Selmer group $H_f^1(F, \rho_{\Pi_0, \lambda} \otimes_{E_\lambda} \rho_{\Pi_1, \lambda}(n))$ is of dimension 1 over E_λ .*

In fact, Theorem 8.3.2 is slightly stronger than the one stated here. We also have an analogue of Theorem 1.1.6 in the rank 1 case, whose statement we omit.

Remark 1.1.9. In both Theorem 1.1.4 and Theorem 1.1.8, the assumption that $[F^+ : \mathbb{Q}] > 1$ if $n \geq 3$ can be lifted once Hypothesis 3.2.9 is known for $N \geq 4$ when $F^+ = \mathbb{Q}$.

1.2. Road map for the article. The very basic idea of bounding Selmer groups as in our main theorems follows from Kolyvagin [Kol90], namely, we construct a system of torsion Galois cohomology classes serving as annihilators of (reduction of) Selmer groups. However, our system is not a generalization of the Euler–Kolyvagin system originally constructed by Kolyvagin. Instead, our system is constructed via level-raising congruences, which was first introduced by Bertolini and Darmon in the case of Heegner points in the study of certain Iwasawa main conjecture of elliptic curves [BD05]. The first example where such level-raising system was used to bound Selmer groups beyond the Heegner point case was performed by one of us in [Liu16], for the so-called twisted triple product automorphic motives. In the sequels [Liu19] and [LT], the case of the so-called cubic triple product automorphic motives was also studied. From this point of view, our current article is a vast generalization of the previous results mentioned above. We have to point out that, although the fundamental ideas do not vary too much, the level of difficulty of realizing all the steps in our current work is tremendously higher than all of the past ones. In fact, in order to study the arithmetic level-raising for unitary groups of even ranks at least 4, we have to use the theory of Galois deformations, which seems to be a new application of the latter.

The following is a road map for reading the main part of the article, where we indicate the need from the five appendices in the parentheses.



The proof of Theorem 1.1.8 is based on the proof of Theorem 1.1.4. We may regard the transition from the rank 0 case to the rank 1 case as an induction step. As seen from the road map, for the rank 0 case alone, Section 4, Subsection A.1, Subsection 7.3, and, of course, Subsection 8.3 are not needed. However, we strongly recommend the readers to go through Section 4 even if they are only interested in the rank 0 case, as Section 4 is an appropriate warm-up for reading Section 5, which is parallel but much more complicated.

In what follows, we explain the main steps in the proof of Theorem 1.1.4. Some of the notations in the rest of this subsection are *ad hoc*, only for the purpose of explaining ideas, hence will be obsolete or differ from the main text.

The initial step (which although will not appear until Subsection 8.2) is to translate the condition that $L(\frac{1}{2}, \Pi_0 \times \Pi_1) \neq 0$ into a more straightforward statement. This is exactly the content of the global Gan–Gross–Prasad conjecture [GGP12]. In fact, as stated in Lemma 8.2.2, we may construct a pair of hermitian spaces $(V_n^\circ, V_{n+1}^\circ)$ over F (with respect to F/F^+) in which V_n° is totally positive definite of rank n , and $V_{n+1}^\circ = V_n^\circ \oplus F \cdot 1$ where 1 has norm 1. For $\alpha = 0, 1$, put $\text{Sh}(V_{n_\alpha}^\circ) := \text{U}(V_{n_\alpha}^\circ)(F^+) \backslash \text{U}(V_{n_\alpha}^\circ)(\mathbb{A}_{F^+}^\infty)$ as a Shimura (pro-)set. We may further find cuspidal automorphic representations π_0 and π_1 contained in the space of locally constant functions on $\text{Sh}(V_{n_0}^\circ)$ and $\text{Sh}(V_{n_1}^\circ)$ satisfying $\text{BC}(\pi_0) \simeq \Pi_0$ and $\text{BC}(\pi_1) \simeq \Pi_1$, respectively, such that

$$(1.1) \quad \mathcal{P}(f_0, f_1) := \int_{\text{Sh}(V_{n_0}^\circ)} f_0(h) f_1(h) dh \neq 0$$

for some $f_0 \in \pi_0$ and $f_1 \in \pi_1$ valued in O_E . Such result was first obtained by one of us [Zha14] under some local restrictions. Those restrictions are all lifted till very recently through some new techniques in the study of trace formulae [BPLZZ]. In what follows, we will fix open compact subgroups of $\text{U}(V_{n_0}^\circ)(\mathbb{A}_{F^+}^\infty)$ and $\text{U}(V_{n_1}^\circ)(\mathbb{A}_{F^+}^\infty)$ that fix f_0 and f_1 , respectively, and will carry them implicitly in the notation.

The next step is to bring the set $\text{Sh}(V_{n_\alpha}^\circ)$ into arithmetic geometry so that the period (1.1) can be related to certain Galois cohomology classes. Now we choose a special inert prime \mathfrak{p} of F^+ (see Definition 3.3.4) with sufficiently large underlying rational prime p , so that all data appeared so far are unramified above p . For $\alpha = 0, 1$, we attach to $V_{n_\alpha}^\circ$ *canonically* a strictly semistable scheme $\mathbf{M}_{\mathfrak{p}}(V_{n_\alpha}^\circ)$ over $\text{Spec } \mathbb{Z}_{p^2}$ of relative dimension $n_\alpha - 1$, whose complex generic fiber is *non-canonically* isomorphic to the disjoint union of finitely many Shimura varieties attached to the *nearby* hermitian space of $V_{n_\alpha}^\circ$ by changing local components at \mathfrak{p} and one archimedean place. Moreover, we can write its special fiber $\mathbf{M}_{\mathfrak{p}}(V_{n_\alpha}^\circ)$ over $\text{Spec } \mathbb{F}_{p^2}$ as a union of $\mathbf{M}_{\mathfrak{p}}^\circ(V_{n_\alpha}^\circ)$ and $\mathbf{M}_{\mathfrak{p}}^\bullet(V_{n_\alpha}^\circ)$, in which $\mathbf{M}_{\mathfrak{p}}^\circ(V_{n_\alpha}^\circ)$ is geometrically a $\mathbb{P}^{n_\alpha-1}$ -fibration over the Shimura set $\text{Sh}(V_{n_\alpha}^\circ)$. However, the reality is much more intricate, as the geometry of the other stratum $\mathbf{M}_{\mathfrak{p}}^\bullet(V_{n_\alpha}^\circ)$, which is rather mysterious, will also involve in the later computation. In fact, one key effort we pay is to show that only the basic locus of the stratum $\mathbf{M}_{\mathfrak{p}}^\bullet(V_{n_\alpha}^\circ)$ will play a role in the computation. For the basic locus, we show that its normalization is geometrically a fibration over the Shimura set $\text{Sh}(V_{n_\alpha}^\circ)$ (but with a slightly different level structure at \mathfrak{p}) by certain Deligne–Lusztig varieties of dimension $r_\alpha := \lfloor \frac{n_\alpha}{2} \rfloor$, introduced in Subsection A.2. The study of various geometric aspects of the scheme $\mathbf{M}_{\mathfrak{p}}(V_{n_\alpha}^\circ)$, including its associated Rapoport–Zink spectral sequence and its functorial behavior from n to $n + 1$, will be carried out in Section 5.

The automorphic input will be thrown into the scheme $\mathbf{M}_{\mathfrak{p}}(V_{n_\alpha}^\circ)$ from the third step, in Section 6, where we study the local Galois cohomology of certain cohomology of $\mathbf{M}_{\mathfrak{p}}(V_{n_\alpha}^\circ)$ localized at some Hecke ideals. More precisely, we fix an admissible prime λ of E with respect to (Π_0, Π_1) , and denote by O_λ and k_λ the ring of integers and the residue field of E_λ , respectively. For $\alpha = 0, 1$, the Satake parameters of Π_α induce a homomorphism $\phi_\alpha: \mathbb{T}_{n_\alpha} \rightarrow k_\lambda$ with kernel \mathfrak{m}_α , where \mathbb{T}_{n_α} is certain abstract spherical Hecke algebra for unitary groups of rank n_α . When $\alpha = 0$ (resp. $\alpha = 1$),

we need to study the singular (resp. unramified) part of the local Galois cohomology

$$(1.2) \quad H^1(\mathbb{Q}_{p^2}, H_{\mathfrak{I}}^{n_\alpha-1}(\overline{M}_{\mathfrak{p}}(V_{n_\alpha}^\circ), R\Psi O_\lambda(r_\alpha))_{\mathfrak{m}_\alpha}),$$

where $\overline{M}_{\mathfrak{p}}(V_{n_\alpha}^\circ) := M_{\mathfrak{p}}(V_{n_\alpha}^\circ) \otimes_{\mathbb{F}_{p^2}} \overline{\mathbb{F}}_p$, and $H_{\mathfrak{I}}$ denotes certain invariant part of the étale cohomology (a subtlety that can be ignored at this moment). The question boils down to the arithmetic level-raising phenomenon (resp. existence of Tate cycles) when $\alpha = 0$ (resp. $\alpha = 1$). However, in both cases, we have to rely on the recent progress on the Tate conjecture for Shimura varieties achieved by some of us [XZ]. Now we would like to continue the discussion on the case where $\alpha = 0$, since it is more interesting and more involved, and omit the case where $\alpha = 1$. The first key point is to figure out the correct condition so that the level-raising phenomenon (namely, from unramified to mildly ramified at the place \mathfrak{p}) happens on the cohomology (1.2) in a way that can be understood: we say that \mathfrak{p} is a *level-raising prime* with respect to λ if $\ell \nmid p(p^2 - 1)$, and the mod λ Satake parameter of $\Pi_{0,\mathfrak{p}}$ contains the pair $\{p, p^{-1}\}$ exactly once and does not contain the pair $\{-1, -1\}$. Suppose that \mathfrak{p} is such a prime, we show that there is a canonical isomorphism

$$(1.3) \quad H_{\text{sing}}^1(\mathbb{Q}_{p^2}, H_{\mathfrak{I}}^{n_0-1}(\overline{M}_{\mathfrak{p}}(V_{n_0}^\circ), R\Psi O_\lambda(r_0))/\mathfrak{m}_0) \simeq O_\lambda[\text{Sh}(V_{n_0}^\circ)]/\mathfrak{m}_0$$

of O_λ -modules of finite length. Note that by our condition on \mathfrak{p} , the right-hand side of (1.3) is nonvanishing, which implies that the left-hand side is also nonvanishing; in other words, we see the level-raising phenomenon in $H_{\mathfrak{I}}^{n_0-1}(\overline{M}_{\mathfrak{p}}(V_{n_0}^\circ), R\Psi O_\lambda(r_0))$. The proof of (1.3) is the technical heart of this article (for example, it uses materials from all of the five appendices). Through studying the geometry and intersection theory on the special fiber $M_{\mathfrak{p}}(V_{n_0}^\circ)$ in Section 5 and some of the appendices, we can conclude that $O_\lambda[\text{Sh}(V_{n_0}^\circ)]/\mathfrak{m}_0$ is canonically a subquotient of $H_{\text{sing}}^1(\mathbb{Q}_{p^2}, H_{\mathfrak{I}}^{n_0-1}(\overline{M}_{\mathfrak{p}}(V_{n_0}^\circ), R\Psi O_\lambda(r_0))/\mathfrak{m}_0)$. Thus, it remains to show that the two sides of (1.3) have the same cardinality. For this, we use the theory of Galois deformations. We construct a global Galois deformation O_λ -algebra R^{mix} with two quotient algebras R^{unr} and R^{ram} , together with a natural R^{unr} -module H^{unr} and a natural R^{ram} -module H^{ram} . They satisfy the following relation: if we put $R^{\text{cong}} := R^{\text{unr}} \otimes_{R^{\text{mix}}} R^{\text{ram}}$, which is an Artinian O_λ -algebra, then we have natural isomorphisms

$$\begin{aligned} H^{\text{unr}} \otimes_{R^{\text{unr}}} R^{\text{cong}} &\simeq O_\lambda[\text{Sh}(V_{n_0}^\circ)]/\mathfrak{m}_0, \\ H^{\text{ram}} \otimes_{R^{\text{ram}}} R^{\text{cong}} &\simeq H_{\text{sing}}^1(\mathbb{Q}_{p^2}, H_{\mathfrak{I}}^{n_0-1}(\overline{M}_{\mathfrak{p}}(V_{n_0}^\circ), R\Psi O_\lambda(r_0))/\mathfrak{m}_0). \end{aligned}$$

Thus, we only need to show that H^{unr} and H^{ram} are both finite free over R^{unr} and R^{ram} , respectively, of the same rank. The finite-freeness follows from an $R=T$ theorem, proved in Appendix E. It is worth pointing out that our $R=T$ theorem is over the initial base field F , that is, we do not take a favourable CM extension of F as people usually do like in [CHT08, Tho12], for example; in particular, we have to deal with certain ramification at nonsplit places of F^+ . The comparison of ranks can be performed over E_λ , which turns out to be an automorphic problem and is solved in Subsection 6.4 based on Subsection D.2. Therefore, we obtain (1.3). In practice, we also need a mod λ^m version of (1.3).

The fourth step is to merge the study of (1.2) for n_0 and n_1 together, to obtain the so-called *first explicit reciprocity law* for the Rankin–Selberg product of Galois representations. As an application, we construct a system of torsion Galois cohomology classes whose image in the singular part of the local Galois cohomology at \mathfrak{p} of the product Galois representation is controlled by the period integral (1.1). This step is sort of routine, once we have enough knowledge on (1.2); it is completed in Subsection 7.2.

The final step of the proof of Theorem 1.1.4 will be performed in Subsection 8.2, where we use the system of torsion Galois cohomology classes constructed in the previous step, together with

some Galois theoretical facts from Section 2, to bound the Selmer group, which is possible due to the nonvanishing of (1.1).

1.3. Notations and conventions. In this subsection, we setup some common notations and conventions for the entire article, including appendices, unless otherwise specified. The notations in the previous two subsections will not be relied on from this moment, and should not be kept for further reading.

Generalities:

- Denote by $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ the monoid of nonnegative integers.
- We only apply the operation $\sqrt{}$ to positive real numbers, which takes values in positive real numbers as well.
- For a set S , we denote by $\mathbb{1}_S$ the characteristic function of S .
- The eigenvalues or generalized eigenvalues of a matrix over a field k are counted with multiplicity (namely, dimension of the corresponding eigenspace or generalized eigenspace); in other words, they form a multi-subset of an algebraic extension of k .
- For every rational prime p , we fix an algebraic closure $\overline{\mathbb{Q}}_p$ of \mathbb{Q}_p with the residue field $\overline{\mathbb{F}}_p$. For every integer $r \geq 1$, we denote by \mathbb{Q}_{p^r} the subfield of $\overline{\mathbb{Q}}_p$ that is an unramified extension of \mathbb{Q}_p of degree r , by \mathbb{Z}_{p^r} its ring of integers, and by \mathbb{F}_{p^r} its residue field.
- For a nonarchimedean place v of a number field K , we write $\|v\|$ for the cardinality of the residue field of K_v .
- We use standard notations from the category theory. The category of sets is denoted by \mathbf{Set} . For a category \mathcal{C} , we denote by \mathcal{C}^{op} its opposite category, and denote by $\mathcal{C}/_A$ the category of morphisms to A for an object A of \mathcal{C} . For another category \mathcal{D} , we denote by $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ the category of functors from \mathcal{C} to \mathcal{D} . In particular, we denote by $\mathbf{PC} := \mathbf{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$ the category of presheaves on \mathcal{C} , which contains \mathcal{C} as a full subcategory by the Yoneda embedding. Isomorphisms in a category will be indicated by \simeq .
- All rings (but not algebras) are commutative and unital. For a ring L and an L -algebra L' , we denote by $\mathbf{Mod}(L')$ the category of left L' -modules.
- If a base ring is not specified in the tensor operation \otimes , then it is \mathbb{Z} .
- For a ring L and a set S , denote by $L[S]$ the L -module of L -valued functions on S of finite support.

Definition 1.3.1. Let K be a subfield of \mathbb{C} . We say that an intermediate extension $K \subseteq K' \subseteq \mathbb{C}$ is K -normal if K'/K is finite, and every automorphism in $\text{Aut}(\mathbb{C}/K)$ stabilizes K' .

Algebraic geometry:

- We denote by the category of schemes by \mathbf{Sch} and its full subcategory of locally Noetherian schemes by \mathbf{Sch}' . For a scheme S (resp. Noetherian scheme S), we denote by $\mathbf{Sch}/_S$ (resp. $\mathbf{Sch}'/_S$) the category of S -schemes (resp. locally Noetherian S -schemes). If $S = \text{Spec } R$ is affine, we also write $\mathbf{Sch}/_R$ (resp. $\mathbf{Sch}'/_R$) for $\mathbf{Sch}/_S$ (resp. $\mathbf{Sch}'/_S$).
- For a scheme X over an affine scheme $\text{Spec } R$ and a commutative R -algebra S , we write $X \otimes_R S$ or even X_S for $X \times_{\text{Spec } R} \text{Spec } S$.
- For a scheme S in characteristic p for some rational prime p , we denote by $\sigma: S \rightarrow S$ the absolute p -power Frobenius morphism. For a perfect field κ of characteristic p , we denote by $W(\kappa)$ its Witt ring, and by abuse of notation, $\sigma: W(\kappa) \rightarrow W(\kappa)$ the canonical lifting of the p -power Frobenius map.
- For a scheme S and a locally free \mathcal{O}_S -module \mathcal{V} of finite rank, we denote by $\mathbb{P}(\mathcal{V}) \rightarrow S$ the moduli scheme of quotient line bundles of \mathcal{V} over S , known as the *projective fibration* associated to \mathcal{V} .

- For a scheme S and (sheaves of) \mathcal{O}_S -modules \mathcal{F} and \mathcal{G} , we denote by $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ the quasi-coherent sheaf of \mathcal{O}_S -linear homomorphisms from \mathcal{F} to \mathcal{G} .
- For two positive integers r, s , we denote by $M_{r,s}$ the scheme of r -by- s matrices, and put $M_r := M_{r,r}$ for short; we also denote by $GL_r \subseteq M_r$ the subscheme of invertible r -by- r matrices. Then GL_1 is simply the multiplicative group $\mathbf{G}_m := \mathbb{Z}[T, T^{-1}]$; but we will distinguish between GL_1 and \mathbf{G}_m according to the context.
- For a number field K , a commutative group scheme $G \rightarrow S$ equipped with an action by O_K over some base scheme S , and an ideal $\mathfrak{a} \subset O_K$, we denote by $G[\mathfrak{a}]$ the maximal closed subgroup scheme of G annihilated by all elements in \mathfrak{a} .
- By a *coefficient ring* for étale cohomology, we mean either a finite ring, or a finite extension of \mathbb{Q}_ℓ , or the ring of integers of a finite extension of \mathbb{Q}_ℓ . In the latter two cases, we regard the étale cohomology via suitable ℓ -adic formalism. We say that a coefficient ring L is *n-coprime* for a positive integer n if n is invertible in L in the first case, and $\ell \nmid n$ in the latter two cases.

Group theory: Let G and $\tilde{\Gamma}$ be groups, and Γ a subgroup of $\tilde{\Gamma}$. Let L be a ring.

- Denote by Γ^{ab} the maximal abelian quotient of Γ .
- For a homomorphism $\rho: \Gamma \rightarrow GL_r(L)$ for some $r \geq 1$, we denote by $\rho^\vee: \Gamma \rightarrow GL_r(L)$ the contragredient homomorphism, which is defined by the formula $\rho^\vee(x) = {}^t\rho(x)^{-1}$ for every $x \in \Gamma$.
- For a homomorphism $\rho: \Gamma \rightarrow G$ and an element $\gamma \in \tilde{\Gamma}$ that normalizes Γ , we let $\rho^\gamma: \Gamma \rightarrow G$ be the homomorphism defined by $\rho^\gamma(x) = \rho(\gamma x \gamma^{-1})$ for every $x \in \Gamma$.
- We say that two homomorphisms $\rho_1, \rho_2: \Gamma \rightarrow G$ are conjugate if there exists an element $g \in G$ such that $\rho_1 = g \circ \rho_2 \circ g^{-1}$, that is, $\rho_1(x) = g\rho_2(x)g^{-1}$ for every $x \in \Gamma$.
- The L -module $L[G]$ is naturally an L -algebra, namely, the group algebra of G with coefficients in L .
- Suppose that G is a locally compact and totally disconnected topological group. For an open compact subgroup K of G , the L -module $L[K \backslash G / K]$ (of bi- K -invariant compactly supported L -valued functions on G) is naturally an L -algebra, where the algebra structure is given by the composition of cosets. In particular, the unit element of $L[K \backslash G / K]$ is always $\mathbb{1}_K$.
- For every integer $r \geq 1$, we denote by J_r the standard upper triangular nilpotent Jordan block

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

or size r -by- r .

Combinatorics:

Notation 1.3.2. We recall the λ -analogue of binomial coefficients:

$$[0]_\lambda = 1, \quad [n]_\lambda = \frac{\lambda^n - 1}{\lambda - 1}, \quad [n]_\lambda! = [n]_\lambda \cdot [n-1]_\lambda \cdots [1]_\lambda, \quad \begin{bmatrix} n \\ m \end{bmatrix}_\lambda = \frac{[n]_\lambda!}{[n-m]_\lambda! \cdot [m]_\lambda!}$$

for integers $0 \leq m \leq n$. For $r \geq 0$ and $q \in \mathbb{N}$, we put

$$\begin{cases} \mathbf{d}_{r,q} := \sum_{\delta=0}^r (-1)^\delta (2\delta+1) q^{\delta(\delta+1)} \begin{bmatrix} 2r+1 \\ r-\delta \end{bmatrix}_{-q}, \\ \mathbf{d}_{r,q}^\bullet := \frac{1}{q+1} \left(\mathbf{d}_{r,q} + \frac{(-q)^{r+1} - 1}{q+1} (q+1)(q^3+1) \cdots (q^{2r-1}+1) \right). \end{cases}$$

Ground fields:

- Let $\mathbf{c} \in \text{Aut}(\mathbb{C}/\mathbb{Q})$ be the complex conjugation.
- Throughout the article, we fix a *subfield* $F \subseteq \mathbb{C}$ that is a number field and is stable under \mathbf{c} ; it is assumed to be a CM field except in Section 2.
- Let $F^+ \subseteq F$ be the maximal subfield on which \mathbf{c} acts by the identity.
- Let \overline{F} be the Galois closure of F in \mathbb{C} . Put $\Gamma_F := \text{Gal}(\overline{F}/F)$ and $\Gamma_{F^+} := \text{Gal}(\overline{F}/F^+)$.
- Denote by Σ_∞ (resp. Σ_∞^+) the set of complex embeddings of F (resp. F^+) with $\tau_\infty \in \Sigma_\infty$ (resp. $\tau_\infty \in \Sigma_\infty^+$) the default one. For $\tau \in \Sigma_\infty$, we denote by $\tau^\mathbf{c}$ the its complex conjugation.
- For every rational prime p , denote by Σ_p^+ the set of all p -adic places of F^+ .
- Denote by Σ_{ram}^+ the union of Σ_p^+ for all p ramified in F .
- Denote by $\eta_{F/F^+} : \Gamma_{F^+} \rightarrow \{\pm 1\}$ the character associated to the extension F/F^+ .
- For every prime ℓ , denote by $\epsilon_\ell : \Gamma_{F^+} \rightarrow \mathbb{Z}_\ell^\times$ the ℓ -adic cyclotomic character.

For every place v of F^+ , we

- put $F_v := F \otimes_{F^+} F_v^+$; and define $\delta(v)$ to be 1 (resp. 2) if v splits (resp. does not split) in F ;
- fix an algebraic closure \overline{F}_v^+ of F_v^+ containing \overline{F} ; and put $\Gamma_{F_v^+} := \text{Gal}(\overline{F}_v^+/F_v^+)$ as a subgroup of Γ_{F^+} ;
- for a homomorphism r from Γ_{F^+} to another group, denote by r_v the restriction of r to the subgroup $\Gamma_{F_v^+}$.

For every nonarchimedean place w of F , we

- identify the Galois group Γ_{F_w} with $\Gamma_{F_v^+} \cap \Gamma_F$ (resp. $\mathbf{c}(\Gamma_{F_v^+} \cap \Gamma_F)\mathbf{c}$), where v is the underlying place of F^+ , if the embedding $F \hookrightarrow \overline{F}_v^+$ induces (resp. does not induce) the place w ;
- let $\mathbf{I}_{F_w} \subseteq \Gamma_{F_w}$ be the inertia subgroup;
- let κ_w be the residue field of F_w , and identify its Galois group Γ_{κ_w} with $\Gamma_{F_w}/\mathbf{I}_{F_w}$;
- denote by $\phi_w \in \Gamma_{F_w}$ a lifting of the *arithmetic* Frobenius element in Γ_{κ_w} .

Definition 1.3.3. We say that two subsets Σ_1^+ and Σ_2^+ of nonarchimedean places of F^+ are *strongly disjoint* if there is no rational prime underlying places from both sets.

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2. GALOIS COHOMOLOGY AND SELMER GROUPS

In this section, we make the Galois theoretical preparation for the proof of the main theorems. Most discussions in this section are generalizations from [Liu16, Liu19]. The material of this section will not be used until Section 6. In Subsection 2.1, we collect some lemmas on ℓ -adic modules with certain group actions. In Subsection 2.2, we study local Galois cohomology. In Subsection 2.3, we perform the discussion that is typical for Kolyvagin’s type of argument. The Selmer group and its variant will be introduced in Subsection 2.4. In Subsection 2.5, we study localization of Selmer groups. In Subsection 2.6, we study an example related to the Rankin–Selberg product.

We will start from a more general setup in order to make the discussion applicable to the orthogonal case as well, which may be studied in the future. Thus, we fix a *subfield* $F \subseteq \mathbb{C}$ that is a number field, *not* necessarily CM.

We fix an odd rational prime ℓ that is unramified in F , and consider a finite extension $E_\lambda/\mathbb{Q}_\ell$, with the ring of integers O_λ and the maximal ideal λ of O_λ . Recall that $\epsilon_\ell: \Gamma_{F^+} \rightarrow \mathbb{Z}_\ell^\times$ is the ℓ -adic cyclotomic character.

2.1. Preliminaries on adic modules. Let Γ be a profinite group and L a commutative topological \mathbb{Z}_ℓ -algebra.

Notation 2.1.1. We denote by $\mathbf{Mod}(\Gamma, L)$ the category of finitely generated L -modules equipped with a continuous action of Γ , and by $\mathbf{Mod}(\Gamma, L)_{\text{tor}}$ (resp. $\mathbf{Mod}(\Gamma, L)_{\text{fr}}$) the full subcategory of $\mathbf{Mod}(\Gamma, L)$ consisting of those whose underlying L -module is torsion (resp. free).

For the rest of this subsection, we further assume that Γ is a topologically finitely generated abelian group and L is Noetherian.

Definition 2.1.2. We say that an $L[\Gamma]$ -module M is *weakly semisimple* if

- (1) M is an object of $\mathbf{Mod}(\Gamma, L)$; and
- (2) the natural map $M^\Gamma \rightarrow M_\Gamma$ is an isomorphism.

Lemma 2.1.3. *Let M be an $L[\Gamma]$ -module that is an object of $\mathbf{Mod}(\Gamma, L)$. Then*

- (1) $M_\Gamma = 0$ implies $M^\Gamma = 0$;
- (2) M is weakly semisimple if and only if $(M/M^\Gamma)_\Gamma = 0$.

Proof. For (1), for every maximal ideal \mathfrak{m} of L , let \mathfrak{m}' be the maximal ideal of $L[[\Gamma]]$ topologically generated by \mathfrak{m} and Γ . As $M_\Gamma = 0$, we have $M \otimes_{L[[\Gamma]]} (L[[\Gamma]]/\mathfrak{m}') = 0$ hence $M_{\mathfrak{m}'} = 0$ by Nakayama’s lemma since $L[[\Gamma]]$ is Noetherian. Thus, we have $(M^\Gamma)_{\mathfrak{m}} = (M^\Gamma)_{\mathfrak{m}'} = 0$, which implies $M^\Gamma = 0$.

For (2), consider the short exact sequence $0 \rightarrow M^\Gamma \rightarrow M \rightarrow M/M^\Gamma \rightarrow 0$. Suppose that M is weakly semisimple. Then the natural map $(M^\Gamma)_\Gamma \rightarrow M_\Gamma$ is an isomorphism. Thus, we have $(M/M^\Gamma)_\Gamma = 0$. Conversely, suppose $(M/M^\Gamma)_\Gamma = 0$. Then we have $H^1(\Gamma, M/M^\Gamma) = 0$, and $H^0(\Gamma, M/M^\Gamma) = 0$ by (1). From the exact sequence

$$H^0(\Gamma, M/M^\Gamma) \rightarrow H^1(\Gamma, M^\Gamma) \rightarrow H^1(\Gamma, M) \rightarrow H^1(\Gamma, M/M^\Gamma),$$

we know that natural map $M^\Gamma \rightarrow M_\Gamma$, which coincides with $H^1(\Gamma, M^\Gamma) \rightarrow H^1(\Gamma, M)$ is an isomorphism.

The lemma is proved. □

Lemma 2.1.4. *We have*

- (1) *A finite direct sum of weakly semisimple $L[\Gamma]$ -modules is weakly semisimple.*
- (2) *A subquotient $L[\Gamma]$ -module of a weakly semisimple $L[\Gamma]$ -module is weakly semisimple.*

Proof. Part (1) is obvious.

For (2), we take a weakly semisimple $L[\Gamma]$ -module M . By Lemma 2.1.3(2), we have $(M/M^\Gamma)_\Gamma = 0$.

Let N be an $L[\Gamma]$ -submodule of M . Since L is Noetherian, N is an object of $\text{Mod}(\Gamma, L)$. As $N^\Gamma = N \cap M^\Gamma$, we have an inclusion $N/N^\Gamma \hookrightarrow M/M^\Gamma$. Since $(M/NM^\Gamma)_\Gamma = 0$, we have $H^0(\Gamma, M/NM^\Gamma) = 0$ by Lemma 2.1.3(1), which implies $(N/N^\Gamma)_\Gamma = 0$. By Lemma 2.1.3(2) again, we know that N is weakly semisimple.

Let N be a quotient $L[\Gamma]$ -module of M . Then we have a quotient map $M/M^\Gamma \rightarrow N/N^\Gamma$. Thus, we have $(N/N^\Gamma)_\Gamma = 0$. By Lemma 2.1.3(2) again, we know that N is weakly semisimple.

Part (2) is proved. \square

Lemma 2.1.5. *Let M be an $O_\lambda[\Gamma]$ -module that is an object of $\text{Mod}(\Gamma, O_\lambda)_{\text{fr}}$. Suppose that $M \otimes_{O_\lambda} O_\lambda/\lambda$ is weakly semisimple, and $\dim_{E_\lambda}(M \otimes_{O_\lambda} E_\lambda)^\Gamma \geq \dim_{O_\lambda/\lambda}(M \otimes_{O_\lambda} O_\lambda/\lambda)^\Gamma$. Then M is weakly semisimple as well, and $\dim_{E_\lambda}(M \otimes_{O_\lambda} E_\lambda)^\Gamma = \dim_{O_\lambda/\lambda}(M \otimes_{O_\lambda} O_\lambda/\lambda)^\Gamma$.*

Proof. Since M is a finitely generated free O_λ -module, both M^Γ and M/M^Γ are finitely generated free O_λ -modules. In particular, the map $M^\Gamma \otimes_{O_\lambda} O_\lambda/\lambda \rightarrow (M \otimes_{O_\lambda} O_\lambda/\lambda)^\Gamma$ is injective. As we have

$$\dim_{O_\lambda/\lambda} M^\Gamma \otimes_{O_\lambda} O_\lambda/\lambda = \text{rank}_{O_\lambda} M^\Gamma = \dim_{E_\lambda}(M \otimes_{O_\lambda} E_\lambda)^\Gamma,$$

the map $M^\Gamma \otimes_{O_\lambda} O_\lambda/\lambda \rightarrow (M \otimes_{O_\lambda} O_\lambda/\lambda)^\Gamma$ is an isomorphism. Thus, we have

$$\dim_{E_\lambda}(M \otimes_{O_\lambda} E_\lambda)^\Gamma = \dim_{O_\lambda/\lambda}(M \otimes_{O_\lambda} O_\lambda/\lambda)^\Gamma$$

and

$$(M/M^\Gamma) \otimes_{O_\lambda} O_\lambda/\lambda \simeq (M \otimes_{O_\lambda} O_\lambda/\lambda)/(M \otimes_{O_\lambda} O_\lambda/\lambda)^\Gamma.$$

As $M \otimes_{O_\lambda} O_\lambda/\lambda$ is weakly semisimple, we have $((M/M^\Gamma) \otimes_{O_\lambda} O_\lambda/\lambda)_\Gamma = 0$ by Lemma 2.1.3(2). By Nakayama's lemma, we have $(M/M^\Gamma)_\Gamma = 0$, which implies that M is weakly semisimple by Lemma 2.1.3(2). The lemma is proved. \square

To end this subsection, we record the following definition which slightly generalizes [Liu16, Definition 5.1], which will be used in later sections.

Definition 2.1.6. Consider an O_λ -module M and an element $x \in M$. We define the *exponent* and the *order* of x to be

$$\begin{aligned} \exp_\lambda(x, M) &:= \min\{d \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \mid \lambda^d x = 0\}, \\ \text{ord}_\lambda(x, M) &:= \sup\{d \in \mathbb{Z}_{\geq 0} \mid x \in \lambda^d M\} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}, \end{aligned}$$

respectively.

2.2. Local Galois cohomology. In this subsection, we study Galois cohomology locally at nonarchimedean places of F . Let w be a nonarchimedean place of F .

Notation 2.2.1. For a commutative topological \mathbb{Z}_ℓ -algebra L and $? \in \{, \text{tor}, \text{fr}\}$, we

- (1) put $\text{Mod}(F_w, L)_? := \text{Mod}(\Gamma_{F_w}, L)_?$;
- (2) denote by $\mathbf{-}(j): \text{Mod}(F_w, L)_? \rightarrow \text{Mod}(F_w, L)_?$ the functor of j -th Tate twist for $j \in \mathbb{Z}$; and
- (3) denote by $\mathbf{-}^\vee: \text{Mod}(F_w, L)_?^{\text{op}} \rightarrow \text{Mod}(F_w, L)_?$ the functor sending M to $\text{Hom}_L(M, L)$.

We also denote

$$\mathbf{-}_\mathbb{Q}: \text{Mod}(F_w, O_\lambda) \rightarrow \text{Mod}(F_w, E_\lambda)$$

the base change functor sending M to $M \otimes_{O_\lambda} E_\lambda$, and

$$\mathbf{-}^*: \text{Mod}(F_w, O_\lambda)^{\text{op}} \rightarrow \text{Mod}(F_w, O_\lambda)$$

the E_λ -Pontryagin duality functor sending M to $\mathrm{Hom}_{O_\lambda}(M, E_\lambda/O_\lambda)$. For every pair $m, m' \in \{1, 2, \dots, \infty\}$ with $m' \geq m$, we have a “reduction modulo $\lambda^{m'}$ ” functor

$$\bar{\cdot}^{(m)} := - \otimes_{O_\lambda} O_\lambda/\lambda^m : \mathrm{Mod}(F_w, O_\lambda/\lambda^{m'}) \rightarrow \mathrm{Mod}(F_w, O_\lambda/\lambda^m).$$

We usually write $\bar{\cdot}$ for $\bar{\cdot}^{(1)}$.

Recall that for every object $R \in \mathrm{Mod}(F_w, O_\lambda)$, we have a local Tate pairing

$$(2.1) \quad \langle \cdot, \cdot \rangle_w : H^1(F_w, R) \times H^1(F_w, R^*(1)) \xrightarrow{\cup} H^2(F_w, E_\lambda/O_\lambda) \simeq E_\lambda/O_\lambda,$$

which we will study in the following.

First, we study the case where ℓ is invertible in κ_w .

Definition 2.2.2. For every object R in either $\mathrm{Mod}(F_w, E_\lambda)$ or $\mathrm{Mod}(F_w, O_\lambda)$, we put

$$H_{\mathrm{sing}}^1(F_w, R) := H^1(I_{F_w}, R)^{\Gamma_{\kappa_w}};$$

and denote by $H_f^1(F_w, R)$ the kernel of the canonical map

$$\partial_w : H^1(F_w, R) \rightarrow H_{\mathrm{sing}}^1(F_w, R),$$

called the *finite part* of $H^1(F_w, R)$.

By the inflation-restriction exact sequence (see, for example, [Liu19, Lemma 2.6]), we know that ∂_w is surjective, and $H_f^1(F_w, R)$ is canonically isomorphic to $H^1(\kappa_w, R^{1_{F_w}})$.

Lemma 2.2.3. For $R \in \mathrm{Mod}(F_w, O_\lambda)_{\mathrm{tor}}$, the finite parts $H_f^1(F_w, R)$ and $H_f^1(F_w, R^*(1))$ are the exact annihilators of each other under the local Tate pairing $\langle \cdot, \cdot \rangle_w$ (2.1).

Proof. This is well-known. In fact, the cup product of $H_f^1(F_w, R)$ and $H_f^1(F_w, R^*(1))$ factors through $H^2(\kappa_w, R^{1_{F_w}} \otimes R^*(1)^{1_{F_w}})$, which is the zero group. The lemma then follows from an easy computation of length and the fact that the pairing $\langle \cdot, \cdot \rangle_w$ is perfect. \square

Second, we study the case that κ_w has characteristic ℓ . In particular, F_w is a finite unramified extension of \mathbb{Q}_ℓ . Denote by $\mathbf{-}_0 : \mathrm{Mod}(F_w, O_\lambda) \rightarrow \mathrm{Mod}(F_w, \mathbb{Z}_\ell)$ the obvious forgetful functor.

Definition 2.2.4. Let $a \leq b$ be two integers.

- (1) For an object $R \in \mathrm{Mod}(F_w, \mathbb{Z}_\ell)_{\mathrm{tor}}$, we say that R is *crystalline with Hodge–Tate weights in $[a, b]$* if $R = R''/R'$ where $R' \subseteq R''$ are two Γ_{F_w} -stable \mathbb{Z}_ℓ -lattices in a crystalline \mathbb{Q}_ℓ -representation of Γ_{F_w} with Hodge–Tate weights in $[a, b]$.
- (2) For an object $R \in \mathrm{Mod}(F_w, \mathbb{Z}_\ell)$, we say that R is *crystalline with Hodge–Tate weights in $[a, b]$* if $R/\ell^m R$ is a torsion crystalline module with Hodge–Tate weights in $[a, b]$ for every integer $m \geq 1$.
- (3) For an object $R \in \mathrm{Mod}(F_w, O_\lambda)$, we say that R is *crystalline with Hodge–Tate weights in $[a, b]$* if R_0 is.

Definition 2.2.5. For an object $R \in \mathrm{Mod}(F_w, O_\lambda)$ that is crystalline with Hodge–Tate weights in $[a, b]$, we define $H_f^1(F_w, R)$ to be the subset of $H^1(F_w, R) = H^1(F_w, R_0)$ consisting of elements s represented by an extension

$$0 \rightarrow R_0 \rightarrow R_s \rightarrow \mathbb{Z}_\ell \rightarrow 0$$

in the category $\mathrm{Mod}(F_w, \mathbb{Z}_\ell)$ such that R_s is crystalline with Hodge–Tate weights in $[a, b]$.

It follows that $H_f^1(F_w, R)$ is an O_λ -submodule of $H^1(F_w, R)$.

Lemma 2.2.6. *Suppose that the integers a, b satisfy $a \leq -1$, $b \geq 0$, and $b - a \leq \frac{\ell-2}{2}$. Then for every $R \in \text{Mod}(F_w, O_\lambda)_{\text{tor}}$ that is crystalline with Hodge–Tate weights in $[a, b]$, the restriction of the local Tate pairing $\langle \cdot, \cdot \rangle_w$ (2.1) to $H_f^1(F_w, R) \times H_f^1(F_w, R^*(1))$ takes values in $\mathfrak{d}_\lambda^{-1}/O_\lambda$, where $\mathfrak{d}_\lambda \subseteq O_\lambda$ is the different ideal of $E_\lambda/\mathbb{Q}_\ell$.*

Proof. We have a canonical map $\text{Tr}: (R^*)_0 \rightarrow (R_0)^*$ in the category $\text{Mod}(F_w, \mathbb{Z}_\ell)$ induced by the trace map $\text{Tr}_{E_\lambda/\mathbb{Q}_\ell}$, which induces a map $H^1(F_w, R^*(1)) \rightarrow H^1(F_w, (R_0)^*(1))$ under which the image of $H_f^1(F_w, R^*(1))$ is contained in $H_f^1(F_w, (R_0)^*(1))$. Take elements $x \in H_f^1(F_w, R)$ and $y \in H_f^1(F_w, R^*(1))$. Then we have for every $a \in O_\lambda$,

$$\text{Tr}_{E_\lambda/\mathbb{Q}_\ell}(a\langle x, y \rangle_w) = \text{Tr}_{E_\lambda/\mathbb{Q}_\ell}\langle ax, y \rangle_w = \langle ax, \text{Tr}(y) \rangle_w \in \mathbb{Q}_\ell/\mathbb{Z}_\ell.$$

However, $\langle ax, \text{Tr}(y) \rangle_w = 0$ by [Niz93, Proposition 6.2]. The lemma follows. \square

2.3. Some Galois-theoretical lemmas. In this subsection, we generalize some lemmas from [Liu16]. For a finite set Σ of places of F , we denote by $\Gamma_{F, \Sigma}$ the Galois group of the maximal subextension of \overline{F}/F that is unramified outside Σ .

Notation 2.3.1. For a commutative topological \mathbb{Z}_ℓ -algebra L and $? \in \{, \text{tor}, \text{fr}\}$, we put

$$\text{Mod}(F, L)_? := \varinjlim_{\Sigma} \text{Mod}(\Gamma_{F, \Sigma}, L)_?,$$

where the colimit is taken over all finite sets Σ of places of F with inflation as transition functors. We have functors $\text{---}(j)$, ---^\vee , $\text{---}_{\mathbb{Q}}$, ---^* , and $\text{---}^{(m)}$ similar to those in Notation 2.2.1. For an object $R \in \text{Mod}(F, L)$ and $i \in \mathbb{Z}$, we put

$$H^i(F, R) := \varinjlim_{\Sigma} H^i(\Gamma_{F, \Sigma}, R).$$

Moreover, for every place w of F , we have the restriction functor $\text{Mod}(F, L) \rightarrow \text{Mod}(F_w, L)$; and denote

$$\text{loc}_w: H^i(F, R) \rightarrow H^i(F_w, R)$$

the localization map.

Definition 2.3.2 ([Liu16, Definition 5.1]). Let G be a profinite group. For an object $R \in \text{Mod}(G, O_\lambda)_{\text{tor}}$, we define its *reducibility depth* to be the smallest integer $\mathfrak{r}_R \geq 0$ such that

- (1) if R' is a G -stable O_λ -submodule that is not contained in λR , then R' contains $\lambda^{\mathfrak{r}_R} R$;
- (2) for every positive integer m , the group $\text{End}_{O_\lambda[G]}(\overline{R}^{(m)})/O_\lambda \cdot \text{id}$ is annihilated by $\lambda^{\mathfrak{r}_R}$.

Note that if $R/\lambda R$ is absolutely irreducible, then $\mathfrak{r}_R = 0$.

Lemma 2.3.3. *Let $R \in \text{Mod}(F, O_\lambda)$ be an object such that $R_{\mathbb{Q}}$ is absolutely irreducible. Then there exists an integer \mathfrak{r}_R depending on R only, such that $\overline{R}^{(m)}$ has reducibility depth at most \mathfrak{r}_R for every positive integer m .*

Proof. The same argument in [Liu16, Lemma 5.2] apply to our case as well, with \mathbb{Z}/p^n replaced by O_λ/λ^m . \square

Now we fix a positive integer m . Consider an object $R \in \text{Mod}(F, O_\lambda/\lambda^m)_{\text{fr}}$. We denote by $\rho: \Gamma_F \rightarrow \text{GL}(R)$ the associated homomorphism. Let F_ρ/F be the Galois extension fixed by the kernel of ρ , and $G := \text{Gal}(F_\rho/F)$ the image of ρ . we have the restriction map

$$(2.2) \quad \text{Res}_\rho: H^1(F, R) \rightarrow H^1(F_\rho, R)^G = \text{Hom}_G(\Gamma_{F_\rho}^{\text{ab}}, R),$$

where $\Gamma_{F_\rho}^{\text{ab}} := \text{Gal}(F_\rho^{\text{ab}}/F_\rho)$ with $F_\rho^{\text{ab}} \subseteq \overline{F}$ the maximal abelian extension of F_ρ , which is equipped with the natural conjugation action by $G = \text{Gal}(F_\rho/F)$.

Lemma 2.3.4. *Suppose that either one of the following two assumptions holds:*

- (a) *the image of Γ_F in $\mathrm{GL}(\bar{\mathbb{R}})$ contains a nontrivial scalar element;*
- (b) *$\dim_{O_\lambda/\lambda} \bar{\mathbb{R}} \leq \min\{\frac{\ell+1}{2}, \ell - 3\}$, $\bar{\mathbb{R}}$ is a semisimple $O_\lambda/\lambda[\Gamma_F]$ -module, and moreover $\mathrm{Hom}_{O_\lambda/\lambda[\Gamma_F]}(\mathrm{End}(\bar{\mathbb{R}}), \bar{\mathbb{R}}) = 0$.*

Then the map Res_ρ (2.2) is injective.

Proof. By the inflation-restriction exact sequence, it suffices to show that $H^1(G, \mathbb{R}) = 0$.

In the situation (a), it follows that G contains a nontrivial scalar element of order coprime to ℓ . Then by the same argument in [Gro91, Proposition 9.1], we have $H^1(G, \mathbb{R}) = 0$.

Now we consider the situation (b). We prove by induction that $H^1(G, \bar{\mathbb{R}}^{(i)}) = 0$ for $1 \leq i \leq m$. Suppose $H^1(G, \bar{\mathbb{R}}^{(j)}) = 0$ for $1 \leq j \leq i < m$. By the short exact sequence

$$0 \rightarrow \bar{\mathbb{R}}^{(i+1)} \otimes_{O_\lambda/\lambda^{i+1}} \lambda^i/\lambda^{i+1} \rightarrow \bar{\mathbb{R}}^{(i+1)} \rightarrow \bar{\mathbb{R}}^{(i)} \rightarrow 0$$

of $O_\lambda[G]$ -modules, in which $\bar{\mathbb{R}}^{(i+1)} \otimes_{O_\lambda/\lambda^{i+1}} \lambda^i/\lambda^{i+1}$ is isomorphic to $\bar{\mathbb{R}}$, we know that $H^1(G, \bar{\mathbb{R}}^{(i+1)}) = 0$. Therefore, it remains to check the initial step that $H^1(G, \bar{\mathbb{R}}) = 0$.

Let $G^i \subseteq G$ be the kernel of the composite homomorphism $G \rightarrow \mathrm{GL}(\mathbb{R}) \rightarrow \mathrm{GL}(\bar{\mathbb{R}}^{(i)})$ for $1 \leq i \leq m$, so we obtain a filtration $0 = G^m \subseteq G^{m-1} \subseteq G^1 \subseteq G$ of normal subgroups of G . We prove by induction that $H^1(G/G^i, \bar{\mathbb{R}}) = 0$. For $i = 1$, since $\bar{\mathbb{R}}$ is a faithful semisimple $O_\lambda/\lambda[G/G^1]$ -module, G/G^1 has no nontrivial normal ℓ -subgroup. As $\dim_{O_\lambda/\lambda} \bar{\mathbb{R}} \leq \ell - 3$, we have $H^1(G/G^1, \bar{\mathbb{R}}) = 0$ by [Gur99, Theorem A]. Suppose $H^1(G/G^j, \bar{\mathbb{R}}) = 0$ for $1 \leq j \leq i < m$. By the inflation-restriction exact sequence

$$0 \rightarrow H^1(G/G^i, \bar{\mathbb{R}}) \rightarrow H^1(G/G^{i+1}, \bar{\mathbb{R}}) \rightarrow \mathrm{Hom}_G(G^i/G^{i+1}, \bar{\mathbb{R}}),$$

it suffices to show that $\mathrm{Hom}_G(G^i/G^{i+1}, \bar{\mathbb{R}}) = 0$, or equivalently, $\mathrm{Hom}_{O_\lambda/\lambda[G]}(G^i/G^{i+1} \otimes O_\lambda/\lambda, \bar{\mathbb{R}}) = 0$. Note that G^i/G^{i+1} is an $\mathbb{F}_\ell[G]$ -submodule of $\mathrm{End}(\bar{\mathbb{R}})$, hence $(G^i/G^{i+1}) \otimes O_\lambda/\lambda$ is an $O_\lambda/\lambda[G]$ -submodule of $\mathrm{End}(\bar{\mathbb{R}}) \otimes (O_\lambda/\lambda) \simeq \mathrm{End}(\bar{\mathbb{R}})^d$, where $d := [O_\lambda/\lambda : \mathbb{F}_\ell]$ is the degree. Since $\bar{\mathbb{R}}$ is a semisimple $O_\lambda/\lambda[G]$ -module and $2 \dim_{O_\lambda/\lambda} \bar{\mathbb{R}} < \ell + 2$, by [Ser94, Corollaire 1], we know that $\mathrm{End}(\bar{\mathbb{R}})$ is a semisimple $O_\lambda/\lambda[G]$ -module. In particular, we have $\mathrm{Hom}_{O_\lambda/\lambda[G]}(G^i/G^{i+1} \otimes O_\lambda/\lambda, \bar{\mathbb{R}}) = 0$ as $\mathrm{Hom}_G(\mathrm{End}(\bar{\mathbb{R}}), \bar{\mathbb{R}}) = 0$.

The lemma is proved. \square

Remark 2.3.5. In Lemma 2.3.4, assumption (a) is well-known to deduce the injectivity, and in fact the surjectivity as well, for the map Res_ρ ; this is the assumption in all previous works concerning Selmer groups of elliptic curves or their products and symmetric powers. However, for reduction of general automorphic Galois representations, assumption (a) is very hard to verify. Thus, we find an alternative, namely, assumption (b) for the injectivity of Res_ρ ; it looks much more complicated than (a), nevertheless can be achieved under certain mild conditions; see Corollary 8.2.3.

The map Res_ρ (2.2) induces an O_λ -linear pairing

$$[\ , \]: H^1(F, \mathbb{R}) \times \Gamma_{F_\rho}^{\mathrm{ab}} \rightarrow \mathbb{R},$$

such that the action of G on $\Gamma_{F_\rho}^{\mathrm{ab}}$ is compatible with that on \mathbb{R} . Let S be a finitely generated O_λ/λ^m -submodule of $H^1(F, \mathbb{R})$, and let F_S/F_ρ be the finite abelian extension such that $\mathrm{Gal}(F_\rho^{\mathrm{ab}}/F_S)$ is the subgroup of $\Gamma_{F_\rho}^{\mathrm{ab}}$ consisting of γ satisfying $[s, \gamma] = 0$ for every $s \in S$. As in [Liu16, Section 5.1], we introduce a sequence \mathfrak{f} that is given by $\mathfrak{f}(0) = 1$, $\mathfrak{f}(1) = 1$, $\mathfrak{f}(2) = 4$, $\mathfrak{f}(r+1) = 2(\mathfrak{f}(r) + 1)$ for $r \geq 2$.

Lemma 2.3.6. *Let the notation be as above. Suppose that either one of the two assumptions in Lemma 2.3.4 is satisfied. Then the induced pairing*

$$[\ , \]: S \times \text{Gal}(F_S/F_\rho) \rightarrow \mathbb{R}$$

induces an injective map $\theta_S: \text{Gal}(F_S/F_\rho) \rightarrow \text{Hom}_{O_\lambda}(S, \mathbb{R})$ of abelian groups that is compatible with G -actions. Moreover, if S is a free $O_\lambda/\lambda^{m-m'}$ -module of rank r_S for some integer $0 \leq m' \leq m$, then the O_λ -submodule of $\text{Hom}_{O_\lambda}(S, \mathbb{R})$ generated by the image of θ_S contains $\lambda^{i(r_S)\mathfrak{r}_R} \text{Hom}_{O_\lambda}(S, \mathbb{R})$, where $\mathfrak{r}_R \leq m$ is the reducibility depth of \mathbb{R} .

Proof. The same argument in [Liu16, Lemma 5.4] apply to our case as well, with \mathbb{Z}/p^n replaced by $O_\lambda/\lambda^{m-m'}$. Note that the proof only uses the injectivity, not the surjectivity, of the map Res_ρ (2.2). \square

2.4. Reduction of Selmer groups. We recall the following definition of the Bloch–Kato Selmer group from [BK90].

Definition 2.4.1 (Bloch–Kato Selmer group). For an object $\mathbb{R} \in \text{Mod}(F, E_\lambda)$, we define the *Bloch–Kato Selmer group* $H_f^1(F, \mathbb{R})$ of \mathbb{R} to be the E_λ -subspace of $H^1(F, \mathbb{R})$ consisting of elements s such that

- (1) $\text{loc}_w(s) \in H_f^1(F_w, \mathbb{R})$ (Definition 2.2.2) for every nonarchimedean place w of F ; and
- (2) $\text{loc}_w(s) \in H_f^1(F_w, \mathbb{R}) := \ker(H^1(F_w, \mathbb{R}) \rightarrow H^1(F_w, \mathbb{R} \otimes_{\mathbb{Q}_\ell} \mathbb{B}_{\text{cris}}))$ for every place w above ℓ , where \mathbb{B}_{cris} is Fontaine’s crystalline period ring for \mathbb{Q}_ℓ .

Definition 2.4.2. Consider an object $\mathbb{R} \in \text{Mod}(F, O_\lambda)_{\text{fr}}$.

- (1) We define the (*integral*) *Bloch–Kato Selmer group* $H_f^1(F, \mathbb{R})$ of \mathbb{R} to be inverse image of $H_f^1(F, \mathbb{R}_{\mathbb{Q}})$ under the obvious map $H^1(F, \mathbb{R}) \rightarrow H^1(F, \mathbb{R}_{\mathbb{Q}})$.
- (2) For $m \in \{1, 2, \dots, \infty\}$, we define $H_{f, \mathbb{R}}^1(F, \bar{\mathbb{R}}^{(m)})$ to be the image of $H_f^1(F, \mathbb{R})$ under the obvious map $H^1(F, \mathbb{R}) \rightarrow H^1(F, \bar{\mathbb{R}}^{(m)})$.

Lemma 2.4.3. *Consider an object $\mathbb{R} \in \text{Mod}(F, O_\lambda)_{\text{fr}}$. Suppose that we are in one of the two following cases*

- (1) *w is a nonarchimedean place of F not above ℓ at which \mathbb{R} is unramified.*
- (2) *w is a place of F above ℓ at which $\mathbb{R}_{\mathbb{Q}}$ is crystalline with Hodge–Tate weights in $[a, b]$ with $a \leq 0 \leq b$ and $b - a \leq \ell - 2$.*

Then for every positive integer m , we have

$$(2.3) \quad \text{loc}_w(H_{f, \mathbb{R}}^1(F, \bar{\mathbb{R}}^{(m)})) \subseteq H_f^1(F_w, \bar{\mathbb{R}}^{(m)}).$$

Proof. For (1), as \mathbb{R} is unramified at v , the natural map

$$H^1(I_{F_w}, \mathbb{R}) = \text{Hom}(I_{F_w}, \mathbb{R}) \rightarrow H^1(I_{F_w}, \mathbb{R}_{\mathbb{Q}}) = \text{Hom}(I_{F_w}, \mathbb{R}_{\mathbb{Q}})$$

is injective. We deduce from the following map of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_f^1(F_w, \mathbb{R}) & \longrightarrow & H^1(F_w, \mathbb{R}) & \longrightarrow & H_{\text{sing}}^1(F_w, \mathbb{R}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_f^1(F_w, \mathbb{R}_{\mathbb{Q}}) & \longrightarrow & H^1(F_w, \mathbb{R}_{\mathbb{Q}}) & \longrightarrow & H_{\text{sing}}^1(F_w, \mathbb{R}_{\mathbb{Q}}) \longrightarrow 0 \end{array}$$

that $H_f^1(F_w, \mathbb{R})$ consists of exactly the elements of $H^1(F_w, \mathbb{R})$ whose image in $H^1(F_w, \mathbb{R}_{\mathbb{Q}})$ belongs to $H_f^1(F_w, \mathbb{R}_{\mathbb{Q}})$. From this, we conclude that loc_w sends $H_f^1(F, \mathbb{R})$ into $H_f^1(F_w, \mathbb{R})$. Moreover, it is clear that the image of $H_f^1(F_w, \mathbb{R})$ under the reduction map $H^1(F_w, \mathbb{R}) \rightarrow H^1(F_w, \bar{\mathbb{R}}^{(m)})$ is contained

in $H_f^1(F_w, \bar{R}^{(m)})$. (In fact, the image is exactly equal to $H_f^1(F_w, \bar{R}^{(m)})$.) The desired inclusion (2.3) follows from this by the definition of $H_{f,R}^1(F, \bar{R}^{(m)})$.

For (2), by the main result of [Liu07], $H_f^1(F_w, R)$ is exactly the preimage of $H_f^1(F_w, R_{\mathbb{Q}})$ via the natural map $H^1(F_w, R) \rightarrow H^1(F_w, R_{\mathbb{Q}})$. It is clear then from the definition that the image of $H_f^1(F_w, R)$ in $H^1(F_w, \bar{R}^{(m)})$ lies in $H_f^1(F_w, \bar{R}^{(m)})$. \square

We recall the notion of purity for a local Galois representation.

Definition 2.4.4. Let w be a nonarchimedean place of F not above ℓ . Consider an object $R \in \text{Mod}(F_w, E_{\lambda})$. Let $\text{WD}(R)$ be the attached Weil–Deligne representation, and $\text{gr}_n \text{WD}(R)$ be the n -th graded piece of the monodromy filtration on $\text{WD}(R)$. For $\mu \in \mathbb{Z}$, we say that R is *pure of weight μ* if $\text{gr}_n \text{WD}(R)$ is pure of weight $\mu + n$ for each n , that is, the eigenvalues of ϕ_w on $\text{gr}_n \text{WD}(R)$ are Weil $\|w\|^{-(\mu+n)}$ -numbers.

From now to the end of this section, we suppose that the complex conjugation c restricts to an automorphism of F (of order at most two). We adopt the notation concerning ground fields in Subsection 1.3; in particular, we put $F^+ := F^{c=1}$. We also have a functor

$$-^c: \text{Mod}(F, L) \rightarrow \text{Mod}(F, L)$$

induced by the conjugation by c .

Lemma 2.4.5. *For every object $R \in \text{Mod}(F, E_{\lambda})$, the functor $-^c$ induces an isomorphism*

$$H_f^1(F, R) \simeq H_f^1(F, R^c)$$

of Selmer groups.

Proof. Regard elements in $H^1(F, -)$ as extensions. Then applying $-^c$ to extensions induces maps

$$H^1(F, R) \rightarrow H^1(F, R^c), \quad H^1(F, R^c) \rightarrow H^1(F, R)$$

which are inverse to each other. It is clear that conditions (1) and (2) in Definition 2.4.1 are preserved under such maps. The lemma follows. \square

Proposition 2.4.6. *Let R be an object in $\text{Mod}(F, O_{\lambda})_{\text{fr}}$ satisfying $R_{\mathbb{Q}}^c \simeq R_{\mathbb{Q}}^{\vee}(1)$ and such that $R_{\mathbb{Q}}$ is pure of weight -1 at every nonarchimedean place w of F not above ℓ . Take a finite set Σ of places of F . Then there exists a positive integer m_{Σ} , depending on R and Σ , such that for every free O_{λ} -submodule S of $H_f^1(F, R)$ that is saturated in $H_f^1(F, R)/H_f^1(F, R)_{\text{tor}}$ and every integer $m > m_{\Sigma}$, we have:*

- (1) $S^{(m)}$, the image of S in $H_{f,R}^1(F, \bar{R}^{(m)})$, is a free O_{λ}/λ^m -module of the same rank as S ;
- (2) $\text{loc}_w(\lambda^{m_{\Sigma}} S^{(m)}) = 0$ for every nonarchimedean place $w \in \Sigma$ not above ℓ .

Proof. Part (1) follows from the same argument for [Liu16, Lemma 5.9].

For (2), we look at the map

$$\text{loc}_{\Sigma}^{\infty \ell}: H_{f,R}^1(F, \bar{R}^{(m)}) \rightarrow \bigoplus_{w \in \Sigma, w \nmid \infty \ell} H^1(F_w, \bar{R}^{(m)}).$$

For every $w \nmid \infty \ell$, since $R_{\mathbb{Q}}$ is of pure weight of -1 at w , $R_{\mathbb{Q}}^c$ and $R_{\mathbb{Q}}^{\vee}(1)$ are of pure weight of -1 at w as well. Thus, we have $H^0(F_w, R_{\mathbb{Q}}) = 0$ and $H^2(F_w, R_{\mathbb{Q}}) \simeq H^0(F_w, R_{\mathbb{Q}}^{\vee}(1))^{\vee} = 0$, hence $H^1(F_w, R_{\mathbb{Q}}) = 0$ by the Euler characteristic formula (see also the proof of [Nek07, Proposition 4.2.2(1)]). Thus, $H^1(F_w, R)$ is annihilated by λ^{m_w} for some integer $m_w \geq 0$. We may enlarge m_w so that λ^{m_w} also annihilates $H^2(F_w, R)_{\text{tor}}$. Then it follows that $H^1(F_w, \bar{R}^{(m)})$ is annihilated by λ^{2m_w} . Now if we put $m_{\Sigma} := \max\{2m_w \mid w \in \Sigma, w \nmid \infty \ell\}$, then (2) follows. This completes the proof of the proposition. \square

To end this subsection, we recall the extension along j -polarization. This has been introduced in [CHT08, Section 1] when $[F : F^+] = 2$. We introduce the group scheme \mathcal{G}_N . When $[F : F^+] = 1$, it is just $\mathrm{GL}_N \times \mathrm{GL}_1$. When $[F : F^+] = 2$, it is the one in Notation E.1.1.

Definition 2.4.7. For a commutative topological \mathbb{Z}_ℓ -algebra L , an integer j , and an object R in $\mathrm{Mod}(F, L)$, a j -polarization of R is an isomorphism

$$\Xi: R^c \xrightarrow{\sim} R^\vee(j)$$

in $\mathrm{Mod}(F, L)$, such that $\Xi^{c^\vee}(j) = \mu_\Xi \cdot (-1)^{j+1} \cdot \Xi$ for some $\mu_\Xi \in \{\pm 1\}$. We say that R is j -polarizable if there exists a j -polarization.

Construction 2.4.8. Let R be a nonzero object in $\mathrm{Mod}(F, L)_{\mathrm{fr}}$ with the associated continuous homomorphism $\rho: \Gamma_F \rightarrow \mathrm{GL}(R)$, equipped with a j -polarization $\Xi: R^c \xrightarrow{\sim} R^\vee(j)$. Choose an isomorphism $R \simeq L^{\oplus N}$ of the underlying L -modules for a unique integer $N \geq 1$.

(1) When $[F : F^+] = 1$, we let

$$\rho_+: \Gamma_{F^+} \rightarrow \mathcal{G}_N(L)$$

be the continuous homomorphism sending $g \in \Gamma_{F^+} = \Gamma_F$ to $(\rho(g), \epsilon_\ell^j(g))$.

(2) When $[F : F^+] = 2$, the j -polarization Ξ gives rise to an element $B \in \mathrm{GL}_N(L)$ as in Lemma E.1.3(2) for the pair $(\rho, \eta_{F/F^+}^{\mu_\Xi} \epsilon_\ell^j)$. We let

$$\rho_+: \Gamma_F \rightarrow \mathcal{G}_N(L)$$

be the continuous homomorphism given in Lemma E.1.3(2).

In both cases, we call ρ_+ an *extension* of ρ , which depends on the choice of a basis of R .

2.5. Localization of Selmer groups. In this subsection, we study the behavior of Selmer groups under localization maps.

Notation 2.5.1. We take a nonzero object $R \in \mathrm{Mod}(F, O_\lambda)_{\mathrm{fr}}$ with the associated homomorphism $\rho: \Gamma_F \rightarrow \mathrm{GL}(R)$, together with a j -polarization $\Xi: R^c \xrightarrow{\sim} R^\vee(j)$. We fix an isomorphism $R \simeq O_\lambda^{\oplus N}$. Let

$$\rho_+: \Gamma_{F^+} \rightarrow \mathcal{G}_N(O_\lambda)$$

be the extension of ρ from Construction 2.4.8. For every integer $m \geq 1$, we have the induced homomorphisms

$$\begin{aligned} \bar{\rho}^{(m)}: \Gamma_F &\rightarrow \mathrm{GL}(\bar{R}^{(m)}) \simeq \mathrm{GL}_N(O_\lambda/\lambda^m), \\ \bar{\rho}_+^{(m)}: \Gamma_{F^+} &\rightarrow \mathcal{G}_N(O_\lambda/\lambda^m), \end{aligned}$$

and we omit the superscript (m) when $m = 1$.

We denote by $F^{(m)} := F_{\bar{\rho}^{(m)}}$ and $F_+^{(m)}$ the subfields of \bar{F} fixed by $\ker \bar{\rho}^{(m)}$ and $\ker \bar{\rho}_+^{(m)}$, respectively. In particular, we have $F \subseteq F^{(m)} \subseteq F_+^{(m)} \subseteq F^{(m)}(\zeta_{\ell^m})$.

Notation 2.5.2. For a positive integer m and an element $\gamma \in (\mathrm{GL}_N(O_\lambda/\lambda^m) \times (O_\lambda/\lambda^m)^\times) \mathfrak{c} \subseteq \mathcal{G}_N(O_\lambda/\lambda^m)$, we denote by $h_\gamma \in \mathrm{GL}_N(O_\lambda/\lambda^m)$ the first component of $\gamma^{[F:F^+]} \in \mathrm{GL}_N(O_\lambda/\lambda^m) \times (O_\lambda/\lambda^m)^\times$.

Now we fix a positive integer m and an element $\gamma \in (\mathrm{GL}_N(O_\lambda/\lambda^m) \times (O_\lambda/\lambda^m)^\times) \mathfrak{c} \subseteq \mathcal{G}_N(O_\lambda/\lambda^m)$. The following definition is essentially [Liu16, Definition 5.6].

Definition 2.5.3. We say that a place $w_+^{(m)}$ of $F_+^{(m)}$ is γ -associated if it is coprime to $\infty\ell$, unramified over F^+ , unramified in F_S , and such that its Frobenius substitution in $\mathrm{Gal}(F_+^{(m)}/F^+) \simeq \mathrm{im} \bar{\rho}_+^{(m)}$ coincides with γ .

Consider a finitely generated O_λ -submodule S of $H_{f,R}^1(F, \bar{R}^{(m)})$. We have the finite abelian extension $F_S/F^{(m)}$ from Subsection 2.3. Suppose that either one of the two assumptions in Lemma 2.3.4 is satisfied. Then by Lemma 2.3.6, we have an injective map

$$\theta_S: \text{Gal}(F_S/F^{(m)}) \rightarrow \text{Hom}_{O_\lambda}(S, \bar{R}^{(m)})$$

of abelian groups, equivariant under the action of $\text{Gal}(F^{(m)}/F)$. Take a γ -associated place $w_+^{(m)}$ of $F_+^{(m)}$, and denote by its underlying places of $F^{(m)}$ and F by $w^{(m)}$ and w , respectively. Since $F_S/F^{(m)}$ is abelian, $w^{(m)}$ has a well-defined Frobenius substitution $\Psi_{w^{(m)}} \in \text{Gal}(F_S/F^{(m)})$. Denote by $G_{S,\gamma}$ the subset of $\text{Gal}(F_S/F^{(m)})$ of elements $\Psi_{w^{(m)}}$ for all γ -associated places $w_+^{(m)}$.

On the other hand, as ϕ_w acts on $\bar{R}^{(m)}$ by h_γ , we have an isomorphism

$$(2.4) \quad H_f^1(F_w, \bar{R}^{(m)}) = H^1(\kappa_w, \bar{R}^{(m)}) \simeq \bar{R}^{(m)} / (h_\gamma - 1)\bar{R}^{(m)}$$

that sends a 1-cocycle on Γ_{κ_w} to its image of ϕ_w . By Lemma 2.4.3 and the fact that the underlying place of F^+ of $w_+^{(m)}$ is inert in F , we have the localization map

$$\text{loc}_w: H_{f,R}^1(F, \bar{R}^{(m)}) \rightarrow H_f^1(F_w, \bar{R}^{(m)}).$$

From (2.4), we know that $\text{loc}_w(s)(\Psi_{w^{(m)}})$ is a well-defined element in $(\bar{R}^{(m)})^{h_\gamma}$ for every $s \in S$.

Lemma 2.5.4. *Suppose that either one of the two assumptions in Lemma 2.3.4 is satisfied.*

- (1) *If $w_+^{(m)}$ is a γ -associated place of $F_+^{(m)}$, then we have $\theta_S(\Psi_{w^{(m)}})(s) = \text{loc}_w(s)(\Psi_{w^{(m)}})$ for every $s \in S$.*
- (2) *Suppose $\gamma \in \text{im } \bar{\rho}_+^{(m)}$, and that the order of h_γ is coprime to ℓ . Then we have*

$$G_{S,\gamma} = \theta_S^{-1} \text{Hom}(S, (\bar{R}^{(m)})^{h_\gamma}).$$

Proof. The same arguments in [Liu16, Lemma 5.7 and Lemma 5.8] apply to the current case as well with \mathbb{Q} replaced by F . \square

By Lemma 2.5.4, for every $r \in \mathbb{N}$, we have a map

$$\theta_{S,\gamma}^r: G_{S,\gamma}^r \rightarrow \text{Hom}_{O_\lambda}(S, ((\bar{R}^{(m)})^{h_\gamma})^{\oplus r})$$

of abelian groups induced by θ_S .

Proposition 2.5.5. *We make the following assumptions:*

- (1) $R_{\mathbb{Q}}$ is absolutely irreducible;
- (2) either one of the two assumptions in Lemma 2.3.4 is satisfied;
- (3) the order of h_γ is coprime to ℓ ; γ belongs to $\text{im } \bar{\rho}_+^{(m)}$;
- (4) $(\bar{R}^{(m)})^{h_\gamma}$ is free over O_λ/λ^m of rank r_γ for some $r_\gamma \in \mathbb{N}$; and
- (5) S is a free $O_\lambda/\lambda^{m-m_0}$ -module of rank r_S for some $m_0 \in \mathbb{N}$ and $r_S \in \mathbb{N}$.

Then there exists an element $(\Psi_1, \dots, \Psi_{r_\gamma r_S}) \in G_{S,\gamma}^{r_\gamma r_S}$ such that the image of the homomorphism $\theta_{S,\gamma}^{r_\gamma r_S}(\Psi_1, \dots, \Psi_{r_\gamma r_S})$ contains $\lambda^{m_0 + \mathfrak{f}(r_S)\mathfrak{r}_R}((\bar{R}^{(m)})^{h_\gamma})^{\oplus r_S}$, where \mathfrak{r}_R and $\mathfrak{f}(r_S)$ are the integers appearing in Lemma 2.3.3 and Lemma 2.3.6, respectively.

Proof. By Lemma 2.3.3 and Lemma 2.3.6, the O_λ -submodule generated by the image of θ_S contains $\lambda^{\mathfrak{f}(r_S)\mathfrak{r}_R} \text{Hom}_{O_\lambda}(S, \bar{R}^{(m)})$. Since h_γ has order coprime to ℓ , $\text{Hom}_{O_\lambda}(S, (\bar{R}^{(m)})^{h_\gamma})$ is a direct summand of $\text{Hom}_{O_\lambda}(S, \bar{R}^{(m)})$. It follows from Lemma 2.5.4(2) that the O_λ -submodule generated by $\theta_S(G_{S,\gamma})$ contains $\lambda^{\mathfrak{f}(r_S)\mathfrak{r}_R} \text{Hom}_{O_\lambda}(S, (\bar{R}^{(m)})^{h_\gamma})$. As $(\bar{R}^{(m)})^{h_\gamma}$ is free O_λ/λ^m -module of rank r_γ and S is a free $O_\lambda/\lambda^{m-m_0}$ -module of rank r_S , the proposition follows immediately. \square

Definition 2.5.6. Let the notation be as in Proposition 2.5.5. We say that an $r_\gamma r_S$ -tuple $(\Psi_1, \dots, \Psi_{r_\gamma r_S}) \in G_{S, \gamma}^{r_\gamma r_S}$ is (S, γ) -abundant if the image of the homomorphism $\theta_{S, \gamma}^{r_\gamma r_S}(\Psi_1, \dots, \Psi_{r_\gamma r_S})$ contains $\lambda^{m_0 + f(r_S) \mathfrak{v}_R}((\mathbb{R}^{(m)})^{h_\gamma})^{\oplus r_S}$.

Remark 2.5.7. In the applications later, we will use Proposition 2.5.5 with $r_\gamma = 1$ and $r_S \in \{1, 2\}$.

2.6. A Rankin–Selberg example. In this subsection, we discuss an example that is related to the Rankin–Selberg motives, which will be considered later. We take objects $R_\alpha \in \mathbf{Mod}(F, O_\lambda)_{\text{fr}}$ for $\alpha = 0, 1$ of rank $n_\alpha > 0$ with the associated homomorphism $\rho_\alpha: \Gamma_F \rightarrow \text{GL}(R_\alpha)$, together with a $(1 - \alpha)$ -polarization $\Xi_\alpha: R_\alpha^c \xrightarrow{\sim} R_\alpha^\vee(1 - \alpha)$. We fix isomorphisms $R_\alpha \simeq O_\lambda^{\oplus n_\alpha}$ for $\alpha = 0, 1$.

We assume that $n_0 = 2r_0$ is even and $n_1 = 2r_1 + 1$ is odd. Put $R := R_0 \otimes_{O_\lambda} R_1$; $\rho := \rho_0 \otimes \rho_1: \Gamma_F \rightarrow \text{GL}(R)$; and $\Xi := \Xi_0 \otimes \Xi_1: R^c \xrightarrow{\sim} R^\vee(1)$ which is a 1-polarization of R .

For a homomorphism ρ from Γ_F and a place w of F , we write ρ_w for the restriction of ρ to the subgroup Γ_{F_w} . Moreover, for clarity, we denote by $\bar{\epsilon}_\ell^{(m)}: \Gamma_{F^+} \rightarrow (O_\lambda/\lambda^m)^\times$ the reduction of ϵ_ℓ modulo λ^m for a positive integer m , and put $\bar{\epsilon}_\ell := \bar{\epsilon}_\ell^{(1)}$ for simplicity.

Lemma 2.6.1. *Let the notation be as above. Take a totally real F^+ -normal intermediate extension $F^+ \subseteq F' \subseteq \mathbb{C}$ (Definition 1.3.1) and a polynomial $\mathcal{P}(T) \in \mathbb{Z}[T]$. The following three statements are equivalent:*

- (1) *There exists a nonarchimedean place v of F^+ inert in F (under the unique place w of F) and splits completely in F' such that*
 - (a) *ℓ does not divide $\|v\| \mathcal{P}(\|v\|)$;*
 - (b) *both $\bar{\rho}_{0,w}$ and $\bar{\rho}_{1,w}$ are unramified and semisimple;*
 - (c) *the trivial character appears in each of $\bar{\rho}_{0,w}$, $\bar{\rho}_{1,w}$, and $\bar{\rho}_w$ with multiplicity one;*
 - (d) *if $[F : F^+] = 2$, then the unramified character sending ϕ_w to -1 does not appear in $\bar{\rho}_{0,w}$;*
 - (e) *if $[F : F^+] = 2$, then the unramified character sending ϕ_w to $-\|v\|$ does not appear in $\bar{\rho}_{1,w}$.*
- (2) *For every positive integer m , the image of the restriction of the homomorphism*

$$(\bar{\rho}_{0+}^{(m)}, \bar{\rho}_{1+}^{(m)}, \bar{\epsilon}_\ell^{(m)}): \Gamma_{F^+} \rightarrow \mathcal{G}_{n_0}(O_\lambda/\lambda^m) \times \mathcal{G}_{n_1}(O_\lambda/\lambda^m) \times (O_\lambda/\lambda^m)^\times$$

(see Notation 2.5.1 for the notation) to $\text{Gal}(\bar{F}/F')$ contains an element $(\gamma_0, \gamma_1, \xi)$ satisfying

- (a) *$\mathcal{P}(\xi)$ is invertible in O_λ/λ^m ;*
 - (b) *for $\alpha = 0, 1$, γ_α belongs to $(\text{GL}_{n_\alpha}(O_\lambda/\lambda^m) \times (O_\lambda/\lambda^m)^\times)_{\mathbf{c}}$ with order coprime to ℓ ;*
 - (c) *the kernels of $h_{\gamma_0} - 1$, $h_{\gamma_1} - 1$, and $h_{\gamma_0} \otimes h_{\gamma_1} - 1$ (Notation 2.5.2) are all free over O_λ/λ^m of rank 1;*
 - (d) *if $[F : F^+] = 2$, then h_{γ_0} does not have an eigenvalue that is equal to -1 in O_λ/λ ;*
 - (e) *if $[F : F^+] = 2$, then h_{γ_1} does not have an eigenvalue that is equal to $-\xi$ in O_λ/λ .*
- (3) *Part (2) holds for $m = 1$.*

Proof. For a nonarchimedean place v of F^+ , we choose an arithmetic Frobenius element ϕ_v in Γ_{F^+} .

First, we show that (1) implies (3). Let v be such a place in (1). Then (the conjugacy class of) ϕ_v belongs to (the normal subgroup) $\text{Gal}(\bar{F}/F')$ as v splits completely in F' . Let $(\gamma_0, \gamma_1, \xi)$ be the image of ϕ_v under the homomorphism $(\bar{\rho}_{0+}, \bar{\rho}_{1+}, \bar{\epsilon}_\ell)$, which is well-defined by (1b). Then (2a) follows from (1a); (2b) follows from the fact that v is inert in F and (1b); (2c) follows from (1c); (2d) follows from (1d); and (2e) follows from (1e).

Second, we show that (3) implies (2). Suppose that $(\gamma_0, \gamma_1, \xi)$ is an element in the image of $(\bar{\rho}_{0+}, \bar{\rho}_{1+}, \bar{\epsilon}_\ell)|_{\text{Gal}(\bar{F}/F')}$ satisfying (2a)–(2e). For every given positive integer m , take an element $(\gamma'_0, \gamma'_1, \xi')$ in the image of $(\bar{\rho}_{0+}^{(m)}, \bar{\rho}_{1+}^{(m)}, \bar{\epsilon}_\ell^{(m)})|_{\text{Gal}(\bar{F}/F')}$ whose reduction is $(\gamma_0, \gamma_1, \xi)$. Then after raising sufficiently large ℓ -power, $(\gamma'_0, \gamma'_1, \xi')$ will satisfy (2a)–(2e).

Last, we show that (2) implies (1). Take an element $(\gamma_0, \gamma_1, \xi)$ is an element in the image of $(\bar{\rho}_{0+}, \bar{\rho}_{1+}, \bar{\epsilon}_\ell)|_{\text{Gal}(\bar{F}/F')}$ satisfying (2a)–(2d). By the Chebotarev density theorem, we can find a nonarchimedean place v of F^+ coprime to ℓ , unramified in F , and splits completely in F' , such that $(\bar{\rho}_{0+}, \bar{\rho}_{1+}, \bar{\epsilon}_\ell)$ is unramified at v and sends ϕ_v to $(\gamma_0, \gamma_1, \xi)$. Then v is inert in F by (2b). Moreover, (1a) follows from (2a); (1b) follows from (2b); (1c) follows from (2c); (1d) follows from (2d); and (1e) follows from (2e).

The lemma is proved. \square

Lemma 2.6.2. *Take a positive integer m . Let v be a nonarchimedean place of F^+ coprime to ℓ and inert in F , such that the homomorphism $(\bar{\rho}_{0+}^{(m)}, \bar{\rho}_{1+}^{(m)}, \bar{\epsilon}_\ell^{(m)})$ is unramified at v and sends an arithmetic Frobenius element in $\Gamma_{F_v^+}$ to an element $(\gamma_0, \gamma_1, \xi)$ satisfying (2a)–(2c) in Lemma 2.6.1. Then all of $H_f^1(F_w, \bar{\rho}_0^{(m)})$, $H_f^1(F_w, \bar{\rho}_1^{(m)})$, and $H_f^1(F_w, \bar{\rho}^{(m)})$ are free O_λ/λ^m -modules of rank 1, where w is the place of F above v .*

Proof. We only consider $H_f^1(F_w, \bar{\rho}^{(m)})$; and the other two cases are similar. By (2b), we may write $\bar{\rho}_w^{(m)} \simeq \bigoplus_{i=1}^N \rho'_i$ where each ρ'_i is residually irreducible. By (2c), we know that the trivial representation appears in $\{\rho'_i\}_{i=1}^N$ exactly once, say ρ'_1 . Moreover, for $i > 1$, the residual representation of ρ'_i is not trivial, hence we have $H_f^1(F_w, \rho'_i) = 0$. Therefore, we have

$$H_f^1(F_w, \bar{\rho}^{(m)}) \simeq H_f^1(F_w, \rho'_1) = H_f^1(F_w, O_\lambda/\lambda^m)$$

which is a free O_λ/λ^m -module of rank 1. \square

In the remaining part, we discuss an example in the Rankin–Selberg case using elliptic curves. Let A_0 and A_1 be two elliptic curves over F^+ . For a rational prime ℓ (that is odd and unramified in F), we put

$$R_\alpha := (\text{Sym}_{\mathbb{Z}_\ell}^{n_\alpha-1} H_{\text{ét}}^1(A_{\alpha\bar{F}}, \mathbb{Z}_\ell))(r_\alpha)$$

as a $\mathbb{Z}_\ell[\Gamma_F]$ -module for $\alpha = 0, 1$. Then R_α is an object in $\text{Mod}(F, \mathbb{Z}_\ell)_{\text{fr}}$ of rank n_α with a canonical $(1 - \alpha)$ -polarization $\Xi_\alpha: R_\alpha \xrightarrow{\sim} R_\alpha^\vee(1 - \alpha)$. Put $R := R_0 \otimes_{\mathbb{Z}_\ell} R_1$ and $\Xi := \Xi_0 \otimes \Xi_1$ as above.

Proposition 2.6.3. *Suppose that $A_{0\bar{F}}$ and $A_{1\bar{F}}$ are not isogenous to each other and $\text{End}(A_{0\bar{F}}) = \text{End}(A_{1\bar{F}}) = \mathbb{Z}$. Take a totally real F^+ -normal intermediate extension $F^+ \subseteq F' \subseteq \mathbb{C}$ and a polynomial $\mathcal{P}(T) \in \mathbb{Z}[T]$. Then for sufficiently large ℓ , we have that*

- (1) *the image of $\bar{\rho}: \Gamma_F \rightarrow \text{GL}(R \otimes \mathbb{F}_\ell)$ contains a nontrivial scalar element;*
- (2) *all of $\bar{\rho}_0$, $\bar{\rho}_1$, and $\bar{\rho}_0 \otimes \bar{\rho}_1$ are absolutely irreducible; and*
- (3) *Lemma 2.6.1(3) holds for F' , $\mathcal{P}(T)$, and the coefficient field \mathbb{F}_ℓ .*

Proof. For $\alpha = 0, 1$ and every ℓ , we have the homomorphism

$$\bar{\rho}_{A_\alpha, \ell}: \Gamma_F \rightarrow \text{GL}(H_{\text{ét}}^1(A_{\alpha\bar{F}}, \mathbb{F}_\ell)) \simeq \text{GL}_2(\mathbb{F}_\ell).$$

Then we have $\bar{\rho}_\alpha = (\text{Sym}^{n_\alpha-1} \bar{\rho}_{A_\alpha, \ell})(r_\alpha)$ for $\alpha = 0, 1$. By our assumption on $A_{0\bar{F}}$ and $A_{1\bar{F}}$, and [Ser72, Théorème 6], for sufficiently large ℓ , the image of the homomorphism

$$(\bar{\rho}_{A_0, \ell}, \bar{\rho}_{A_1, \ell}, \bar{\epsilon}_\ell): \Gamma_F \rightarrow \text{GL}_2(\mathbb{F}_\ell) \times \text{GL}_2(\mathbb{F}_\ell) \times \mathbb{F}_\ell^\times$$

consists exactly of the elements (g_0, g_1, ξ) satisfying $\det g_0 = \det g_1 = \xi^{-1}$. Then both (1) and (2) follow immediately.

For (3), take an element $g \in \Gamma_F$ such that its image under $(\bar{\rho}_{A_0, \ell}, \bar{\rho}_{A_1, \ell}, \bar{\epsilon}_\ell)$ is in the conjugate class of

$$\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} ab & 0 \\ 0 & b^{-1} \end{pmatrix}, a^{-1} \right)$$

for $a, b \in \mathbb{F}_\ell^\times$ satisfying

- $\mathcal{P}(a^{-1}) \neq 0$,
- $(a^{2i}(ab^2)^{2j})^{[F':F^+]} \neq 1$ for $(i, j) \in \{r_0, r_0 - 1, \dots, 1 - r_0\} \times \{r_1, r_1 - 1, \dots, -r_1\}$ except for $(0, 0)$,
- $(a^{2i-1})^{[F':F^+]} \neq -1$ for $i \in \{r_0, r_0 - 1, \dots, 1 - r_0\}$, and
- $(a(ab^2)^{2j})^{[F':F^+]} \neq -1$ for $j \in \{r_1, r_1 - 1, \dots, -r_1\}$.

Such pair (a, b) always exists for sufficiently large ℓ . Then it is straightforward to check that the image $g^{[F':F^+]}c$ under $(\bar{\rho}_{0+}, \bar{\rho}_{1+}, \bar{\epsilon}_\ell)$ (under the notation of Lemma 2.6.1) satisfies (2a)–(2e) of Lemma 2.6.1. In particular, (3) follows. \square

3. PRELIMINARIES ON HERMITIAN STRUCTURES

In this section, we collect some constructions and results concerning objects with certain hermitian structure. In Subsection 3.1, we introduce hermitian spaces, their associated unitary groups and unitary Hecke algebras. In Subsection 3.2, we introduce unitary Shimura varieties and unitary Shimura sets. In Subsection 3.3, we review the notion of (generalized) CM types. In Subsection 3.4, we collect some facts about abelian schemes with hermitian structure, which will be parameterized by our unitary Shimura varieties. In Subsection 3.5, we introduce a moduli scheme parameterizing CM abelian varieties, which is an auxiliary moduli space in order to equip our unitary Shimura variety a moduli interpretation.

Let $N \geq 1$ be an integer.

3.1. Hermitian spaces and unitary Hecke algebras. We start by recalling the notion of the coefficient field for an automorphic representation of $\mathrm{GL}_N(\mathbb{A}_F)$. Let Π be an irreducible cuspidal automorphic (complex) representation of $\mathrm{GL}_N(\mathbb{A}_F)$.

Definition 3.1.1. The *coefficient field* of Π is defined to be the smallest subfield of \mathbb{C} , denoted by $\mathbb{Q}(\Pi)$, such that for every $\rho \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))$, Π and $\Pi \otimes_{\mathbb{C}, \rho} \mathbb{C}$ are isomorphic.

For a nonarchimedean place w of F such that Π_w is unramified, let

$$\alpha(\Pi_w) := \{\alpha(\Pi_w)_1, \dots, \alpha(\Pi_w)_N\} \subseteq \mathbb{C}$$

be the Satake parameter of Π_w and $\mathbb{Q}(\Pi_w) \subseteq \mathbb{C}$ be the subfield generated by the coefficients of the polynomial

$$\prod_{i=1}^N \left(T - \alpha(\Pi_w)_i \cdot \sqrt{\|\!|w\|\!|}^{N-1} \right) \in \mathbb{C}[T].$$

Lemma 3.1.2. *The coefficient field $\mathbb{Q}(\Pi)$ is the composition of $\mathbb{Q}(\Pi_w)$ for all nonarchimedean places w of F such that Π_w is unramified.*

Proof. Let $\mathbb{Q}(\Pi)'$ be the composition of $\mathbb{Q}(\Pi_w)$ for such w . By the construction of unramified principal series, it is clear that for every $\gamma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi)')$ and every w , Π_w and $\Pi_w \otimes_{\mathbb{C}, \gamma} \mathbb{C}$ have the same Satake parameter hence are isomorphic. By the strong multiplicity one property, we know that for $\gamma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi)')$, Π and $\Pi \otimes_{\mathbb{C}, \gamma} \mathbb{C}$ are isomorphic. Thus, $\mathbb{Q}(\Pi)$ is contained in $\mathbb{Q}(\Pi)'$. Conversely, for $\gamma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))$, Π_w and $\Pi_w \otimes_{\mathbb{C}, \gamma} \mathbb{C}$ must have the same Satake parameter for every w , which implies that γ fixes $\mathbb{Q}(\Pi_w)$ for every w . Thus, $\mathbb{Q}(\Pi_w)$ is contained in $\mathbb{Q}(\Pi)$ for every w , which implies $\mathbb{Q}(\Pi)' \subseteq \mathbb{Q}(\Pi)$. The lemma follows. \square

Definition 3.1.3 (Abstract Satake parameter). Let L be a ring. For a multi-subset $\alpha := \{\alpha_1, \dots, \alpha_N\} \subseteq L$, we put

$$P_\alpha(T) := \prod_{i=1}^N (T - \alpha_i) \in L[T].$$

Consider a nonarchimedean place v of F^+ not in Σ_{ram}^+ .

- (1) Suppose that v is inert in F . We define an (*abstract*) Satake parameter in L at v of rank N to be a multi-subset $\alpha \subseteq L$ of cardinality N . We say that α is *unitary* if $P_\alpha(T) = (-T)^N \cdot P_\alpha(T^{-1})$.
- (2) Suppose that v splits in F . We define an (*abstract*) Satake parameter in L at v of rank N to be a pair $\alpha := (\alpha_1; \alpha_2)$ of multi-subsets $\alpha_1, \alpha_2 \subseteq L$ of cardinality N , indexed by the two places w_1, w_2 of F above v . We say that α is *unitary* if $P_{\alpha_1}(T) = c \cdot T^N \cdot P_{\alpha_2}(T^{-1})$ for some constant $c \in L^\times$.

For two Satake parameters α_0 and α_1 in L at v of rank n_0 and n_1 , respectively, we may form their tensor product $\alpha_0 \otimes \alpha_1$ which is of rank $n_0 n_1$ in the obvious way. If α_0 and α_1 are both unitary, then so is $\alpha_0 \otimes \alpha_1$.

Notation 3.1.4. We denote by Σ_Π^+ the smallest (finite) set of nonarchimedean places of F^+ containing Σ_{ram}^+ such that Π_w is unramified for every nonarchimedean place w of F not above Σ_Π^+ . Take a nonarchimedean place v of F^+ not in Σ_Π^+ .

- (1) If v is inert in F , then we put $\alpha(\Pi_v) := \alpha(\Pi_w)$ for the unique place w of F above v .
- (2) If v splits in F into two places w_1 and w_2 , then we put $\alpha(\Pi_v) := (\alpha(\Pi_{w_1}); \alpha(\Pi_{w_2}))$.

Thus, $\alpha(\Pi_v)$ is a Satake parameter in \mathbb{C} at v of rank N .

Definition 3.1.5. Let v be a nonarchimedean place of F^+ inert in F , and L a ring in which $\|v\|$ is invertible. Let $P \in L[T]$ be a monic polynomial of degree N satisfying $P(T) = (-T)^N \cdot P(T^{-1})$.

- (1) When N is odd, we say that P is *Tate generic at v* if the constant term of $T^{-1}P(T+1)$ is invertible in L .
- (2) When N is odd, we say that P is *intertwining generic at v* if $P(-\|v\|)$ is invertible in L .
- (3) When N is even, we say that P is *level-raising special at v* if $P(\|v\|) = 0$ and the constant term of $T^{-1}P(T + \|v\|)$ is invertible in L .
- (4) When N is even, we say that P is *intertwining generic at v* if $P(-1)$ is invertible in L .

Remark 3.1.6. Suppose that L is a field in the above definition. Note that when N is odd, 1 appears in the Satake parameter and all other elements appear in pairs of the form $\{\alpha, \alpha^{-1}\}$; when N is even, elements in the Satake parameter appear in pairs of the form $\{\alpha, \alpha^{-1}\}$. Then

- (1) means that 1 appears exact once in the Satake parameter;
- (2) means that the pair $\{-\|v\|, -\|v\|^{-1}\}$ does not appear in the Satake parameter;
- (3) means that the pair $\{\|v\|, \|v\|^{-1}\}$ appears exactly once in the Satake parameter;
- (4) means that the pair $\{-1, -1\}$ does not appear in the Satake parameter.

We now introduce hermitian spaces.

Definition 3.1.7 (Hermitian space). Let R be a commutative $O_{F^+}[(\Sigma_{\text{ram}}^+)^{-1}]$ -algebra. A *hermitian space* over $O_F \otimes_{O_{F^+}} R$ of rank N is a free $O_F \otimes_{O_{F^+}} R$ -module V of rank N together with a perfect pairing

$$(\ , \)_V: V \times V \rightarrow O_F \otimes_{O_{F^+}} R$$

that is $O_F \otimes_{O_{F^+}} R$ -linear in the first variable and $(O_F \otimes_{O_{F^+}} R, c \otimes \text{id}_R)$ -linear in the second variable, and satisfies $(x, y)_V = (y, x)_V^\xi$ for $x, y \in V$. We denote by $U(V)$ the group of $O_F \otimes_{O_{F^+}} R$ -linear isometries of V , which is a reductive group over R .

Moreover, we denote by V_\sharp the hermitian space $V \oplus O_F \otimes_{O_{F^+}} R \cdot 1$ where 1 has norm 1. For an $O_F \otimes_{O_{F^+}} R$ -linear isometry $f: V \rightarrow V'$, we have the induced isometry $f_\sharp: V_\sharp \rightarrow V'_\sharp$.

Let v be a nonarchimedean place of F^+ not in Σ_{ram}^+ . Let $\Lambda_{N,v}$ be the unique up to isomorphism hermitian space over $O_{F_v} = O_F \otimes_{O_{F^+}} O_{F_v}^+$ of rank N , and $U_{N,v}$ its unitary group over $O_{F_v}^+$. Under

a suitable basis, the associated hermitian form of $\Lambda_{N,v}$ is given by the matrix

$$\begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

Consider the *local spherical Hecke algebra*

$$\mathbb{T}_{N,v} := \mathbb{Z}[\mathrm{U}_{N,v}(O_{F_v^+}) \backslash \mathrm{U}_{N,v}(F_v^+) / \mathrm{U}_{N,v}(O_{F_v^+})].$$

Note that according to our convention, the unit element is $\mathbb{1}_{\mathrm{U}_{N,v}(O_{F_v^+})}$. Let $\mathrm{A}_{N,v}$ be the maximal split diagonal subtorus of $\mathrm{U}_{N,v}$, and $\mathrm{X}_*(\mathrm{A}_{N,v})$ be its cocharacter group. Then there is a well-known Satake transform

$$(3.1) \quad \mathbb{T}_{N,v} \rightarrow \mathbb{Z}[\|v\|^{\pm\delta(v)/2}][\mathrm{A}_{N,v}(F_v^+) / \mathrm{A}_{N,v}(O_{F_v^+})] \simeq \mathbb{Z}[\|v\|^{\pm\delta(v)/2}][\mathrm{X}_*(\mathrm{A}_{N,v})]$$

as a homomorphism of algebras. Choose a uniformizer ϖ_v of F_v^+ .

Construction 3.1.8. Let L be a commutative $\mathbb{Z}[\|v\|^{\pm\delta(v)/2}]$ -algebra. Let α be a unitary Satake parameter in L at v of rank N . There are two cases.

- (1) Suppose that v is inert in F . Then a set of representatives of $\mathrm{A}_{N,v}(F_v^+) / \mathrm{A}_{N,v}(O_{F_v^+})$ can be taken as

$$\{(\varpi_v^{t_1}, \dots, \varpi_v^{t_N}) \mid t_1, \dots, t_N \in \mathbb{Z} \text{ satisfying } t_i + t_{N+1-i} = 0 \text{ for all } 1 \leq i \leq N\}.$$

Choose an order in α as $(\alpha_1, \dots, \alpha_N)$ satisfying $\alpha_i \alpha_{N+1-i} = 1$; we have a unique homomorphism

$$\mathbb{Z}[\|v\|^{\pm\delta(v)/2}][\mathrm{A}_{N,v}(F_v^+) / \mathrm{A}_{N,v}(O_{F_v^+})] \rightarrow L$$

of $\mathbb{Z}[\|v\|^{\pm\delta(v)/2}]$ -algebras sending the class of $(\varpi_v^{t_1}, \dots, \varpi_v^{t_N})$ to $\prod_{i=1}^{\lfloor \frac{N}{2} \rfloor} \alpha_i^{t_i}$. Composing with the Satake transform (3.1), we obtain a ring homomorphism

$$\phi_\alpha: \mathbb{T}_{N,v} \rightarrow L.$$

It is independent of the choices of the uniformizer ϖ_v and the order in α .

- (2) Suppose that v splits in F into two places w_1 and w_2 . Then a set of representatives of $\mathrm{A}_{N,v}(F_v^+) / \mathrm{A}_{N,v}(O_{F_v^+})$ can be taken as

$$\left\{ \left(\begin{pmatrix} \varpi_v^{t_1} & & & \\ & \ddots & & \\ & & \varpi_v^{t_N} & \\ & & & \varpi_v^{-t_1} \end{pmatrix}, \begin{pmatrix} \varpi_v^{-t_N} & & & \\ & \ddots & & \\ & & \varpi_v^{-t_1} & \\ & & & \varpi_v^{t_N} \end{pmatrix} \right) \mid t_1, \dots, t_N \in \mathbb{Z} \right\},$$

where the first diagonal matrix (resp. the second diagonal matrix) is regarded as an element in $\mathrm{A}_{N,v}(F_{w_1})$ (resp. $\mathrm{A}_{N,v}(F_{w_2})$). Choose orders in α_1 and α_2 as $(\alpha_{1,1}, \dots, \alpha_{1,N})$ and $(\alpha_{2,1}, \dots, \alpha_{2,N})$ satisfying $\alpha_{1,i} \alpha_{2,N+1-i} = 1$; we have a unique homomorphism

$$\mathbb{Z}[\|v\|^{\pm\delta(v)/2}][\mathrm{A}_{N,v}(F_v^+) / \mathrm{A}_{N,v}(O_{F_v^+})] \rightarrow L$$

of $\mathbb{Z}[\|v\|^{\pm\delta(v)/2}]$ -algebras sending the class of $(\varpi_v^{t_1}, \dots, \varpi_v^{t_N}; \varpi_v^{-t_N}, \dots, \varpi_v^{-t_1})$ to $\prod_{i=1}^N \alpha_{1,i}^{t_i}$. Composing with the Satake transform (3.1), we obtain a ring homomorphism

$$\phi_\alpha: \mathbb{T}_{N,v} \rightarrow L.$$

It is independent of the choices of the uniformizer ϖ_v , the order of the two places of F above v , and the orders in α_1 and α_2 .

Definition 3.1.9 (Abstract unitary Hecke algebra). For a finite set Σ^+ of nonarchimedean places of F^+ containing Σ_{ram}^+ , we define the *abstract unitary Hecke algebra away from Σ^+* to be the restricted tensor product

$$\mathbb{T}_N^{\Sigma^+} := \bigotimes'_v \mathbb{T}_{N,v}$$

over all $v \notin \Sigma_{\infty}^+ \cup \Sigma^+$ with respect to unit elements. It is a commutative \mathbb{Z} -algebra (that is, a ring under our convention).

Construction 3.1.10. Suppose that Π satisfies $\Pi \circ \mathfrak{c} \simeq \Pi^{\vee}$. For $v \notin \Sigma_{\Pi}^+$, the Satake parameter $\alpha(\Pi_v)$ is unitary. Thus by Construction 3.1.8, we have a homomorphism

$$\phi_{\Pi} := \bigotimes_{v \notin \Sigma_{\infty}^+ \cup \Sigma_{\Pi}^+} \phi_{\alpha(\Pi_v)}: \mathbb{T}_N^{\Sigma_{\Pi}^+} \rightarrow \mathbb{C}$$

which takes value in $\mathbb{Q}(\Pi)$ by Lemma 3.1.2. Here, we regard \mathbb{C} as a $\mathbb{Z}[\|v\|^{\pm\delta(v)/2}]$ -algebra by sending $\|v\|^{\pm\delta(v)/2}$ to $\sqrt{\|v\|^{\pm\delta(v)}}$. If moreover Π is regular algebraic, then it is cohomological [Clo90, Lemme 3.14]; hence $\mathbb{Q}(\Pi)$ is a number field and ϕ_{Π} takes value in $O_{\mathbb{Q}(\Pi)}$.

At last, we introduce some categories of open compact subgroups, which will be used later.

Definition 3.1.11. Let V be a hermitian space over F of rank N . Let \square be a finite set of nonarchimedean places of F^+ .

- (1) (Neat subgroups [Lan13]) We consider $U(V)(\mathbb{A}_{F^+}^{\infty, \square})$ as a subgroup of $GL_F(V)(\mathbb{A}_F^{\infty, \square})$ (by choosing an arbitrary basis). For a nonarchimedean place v of F^+ not in \square and an element $g_v \in U(V)(F_v^+)$, it makes sense to talk about the eigenvalues of g_v in \overline{F}_v^+ , which contains $\overline{\mathbb{Q}}$. Let $\Gamma(g_v)$ be the subgroup of $(\overline{F}_v^+)^{\times}$ generated by the eigenvalues of g_v . Note that the torsion subgroup $\Gamma(g_v)_{\text{tors}}$ lies in $\overline{\mathbb{Q}}^{\times}$. We say an element $g = (g_v) \in U(V)(\mathbb{A}_{F^+}^{\infty, \square})$ is *neat* if $\bigcap_{v \notin \square} \Gamma(g_v)_{\text{tors}} = \{1\}$, and a subgroup $K \subseteq U(V)(\mathbb{A}_{F^+}^{\infty, \square})$ is *neat* if all its elements are neat.
- (2) We define a category $\mathfrak{K}(V)^{\square}$ whose objects are neat open compact subgroups K of $U(V)(\mathbb{A}_{F^+}^{\infty, \square})$, and a morphism from K to K' is an element $g \in K \backslash U(V)(\mathbb{A}_{F^+}^{\infty, \square}) / K'$ satisfying $g^{-1}Kg \subseteq K'$. Denote by $\mathfrak{K}'(V)^{\square}$ the subcategory of $\mathfrak{K}(V)^{\square}$ that allows only identity double cosets as morphisms.
- (3) We define a category $\mathfrak{K}(V)_{\text{sp}}^{\square}$ whose objects are pairs $K = (K_b, K_{\sharp})$ where K_b is an object of $\mathfrak{K}(V)^{\square}$ and K_{\sharp} is an object of $\mathfrak{K}(V_{\sharp})^{\square}$ such that $K_b \subseteq K_{\sharp} \cap U(V)(\mathbb{A}_{F^+}^{\infty, \square})$, and a morphism from $K = (K_b, K_{\sharp})$ to $K' = (K'_b, K'_{\sharp})$ is an element $g \in K_b \backslash U(V)(\mathbb{A}_{F^+}^{\infty, \square}) / K'_b$ such that $g^{-1}K_b g \subseteq K'_b$ and $g^{-1}K_{\sharp} g \subseteq K'_{\sharp}$. We have the obvious functors

$$\mathfrak{K}_b: \mathfrak{K}(V)_{\text{sp}}^{\square} \rightarrow \mathfrak{K}(V)^{\square}, \quad \mathfrak{K}_{\sharp}: \mathfrak{K}(V)_{\text{sp}}^{\square} \rightarrow \mathfrak{K}(V_{\sharp})^{\square}$$

sending $K = (K_b, K_{\sharp})$ to K_b and K_{\sharp} , respectively. Note that $\mathfrak{K}(V)_{\text{sp}}^{\square}$ is a non-full subcategory of $\mathfrak{K}(V)^{\square} \times \mathfrak{K}(V_{\sharp})^{\square}$.

When \square is the empty set, we suppress it from all the notations above.

3.2. Unitary Shimura varieties and sets. We introduce hermitian spaces over F that will be used in this article.

Definition 3.2.1. Let V be a hermitian space over F of rank N .

- (1) We say that V is *standard definite* if it has signature $(N, 0)$ at every place in Σ_{∞}^+ .
- (2) We say that V is *standard indefinite* if it has signature $(N-1, 1)$ at τ_{∞} and $(N, 0)$ at other places in Σ_{∞}^+ .

First, we introduce unitary Shimura varieties. Take a standard indefinite hermitian space V over F of rank N . We have a functor

$$\begin{aligned} \text{Sh}(V, \mathbf{-}) : \mathfrak{K}(V) &\rightarrow \text{Sch}/F \\ K &\mapsto \text{Sh}(V, K) \end{aligned}$$

of Shimura varieties associated to the reductive group $\text{Res}_{F^+/\mathbb{Q}} \text{U}(V)$ and the Deligne homomorphism

$$\begin{aligned} h : \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{G}_m &\rightarrow (\text{Res}_{F^+/\mathbb{Q}} \text{U}(V)) \otimes_{\mathbb{Q}} \mathbb{R} \\ z &\mapsto \left(\begin{pmatrix} \bar{z}/z & & \\ & 1_{N-1} & \\ & & 1_N \end{pmatrix}, 1_N, \dots, 1_N \right) \in \text{U}(V)(F_{\tau_\infty}^+) \times \prod_{\tau \in \Sigma_\infty^+, \tau \neq \tau_\infty} \text{U}(V)(F_\tau^+), \end{aligned}$$

where we have identified $\text{U}(V)(F_{\tau_\infty}^+)$ with a subgroup of $\text{GL}_N(\mathbb{C})$ via the complex embedding τ_∞ of F .

Second, we introduce unitary Shimura sets. Take a standard definite hermitian space V over F of rank N . We have a functor

$$\begin{aligned} \text{Sh}(V, \mathbf{-}) : \mathfrak{K}(V) &\rightarrow \text{Set} \\ K &\mapsto \text{Sh}(V, K) := \text{U}(V)(F^+) \backslash \text{U}(V)(\mathbb{A}_{F^+}^\infty) / K. \end{aligned}$$

Remark 3.2.2. Whether the notion $\text{Sh}(V, \mathbf{-})$ stands for a scheme or a set depends on whether V is standard indefinite or standard definite; so there will be no confusion about notation. Of course, one can equip with $\text{Sh}(V, \mathbf{-})$ a natural scheme structure when V is standard definite; but we will not take this point of view in this article.

We now recall the notion of global base change.

Definition 3.2.3 (Global base change). Let V be a hermitian space over F of rank N , and π a discrete automorphic representation of $\text{U}(V)(\mathbb{A}_{F^+})$.

- (1) We define a *global base change* of π is an automorphic representation $\text{BC}(\pi)$ of $\text{GL}_N(\mathbb{A}_F)$ that is a finite isobaric sum of discrete automorphic representations such that $\text{BC}(\pi)_v \simeq \text{BC}(\pi_v)$ holds for all but finitely many nonarchimedean places v of F^+ such that π_v is unramified. By the strong multiplicity one property for GL_N [PS79], if $\text{BC}(\pi)$ exists, then it is unique up to isomorphism.
- (2) We say that π is *stable* if $\text{BC}(\pi)$ exists and is cuspidal.²

Proposition 3.2.4. *Let Π be a relevant representation of $\text{GL}_N(\mathbb{A}_F)$.*

- (1) *For every nonarchimedean place w of F , Π_w is tempered.*
- (2) *For every isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$, there is a semisimple continuous homomorphism*

$$\rho_{\Pi, \iota_\ell} : \Gamma_F \rightarrow \text{GL}_N(\overline{\mathbb{Q}}_\ell),$$

unique up to conjugation, satisfying that for every nonarchimedean place w of F , the Frobenius semisimplification of the associated Weil–Deligne representation of $\rho_{\Pi, \iota_\ell}|_{\Gamma_{F_w}}$ corresponds to the irreducible admissible representation $\iota_\ell \Pi_w | \det|_{w^2}^{\frac{1-N}{2}}$ of $\text{GL}_N(F_w)$ under the local Langlands correspondence. Moreover, ρ_{Π, ι_ℓ}^c and $\rho_{\Pi, \iota_\ell}^v(1-N)$ are conjugate.

Proof. Part (1) is [Car12, Theorem 1.2]. For (2), the Galois representation ρ_{Π, ι_ℓ} is constructed in [CH13, Theorem 3.2.3], and the local-global compatibility is obtained in [Car12, Theorem 1.1] and [Car14, Theorem 1.1]. The last property in (2) follows from the previous one and the Chebotarev density theorem. \square

²This is slightly more restrictive than the usual definition of stable representations by requiring $\text{BC}(\pi)$ discrete.

Definition 3.2.5. We say that a subfield $E \subseteq \mathbb{C}$ is a *strong coefficient field* of Π if E is a number field containing $\mathbb{Q}(\Pi)$ (Definition 3.1.1); and for every prime λ of E , there exists a continuous homomorphism

$$\rho_{\Pi, \lambda}: \Gamma_F \rightarrow \mathrm{GL}_N(E_\lambda),$$

necessarily unique up to conjugation, such that for every isomorphism $\iota_\ell: \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$ inducing the prime λ , $\rho_{\Pi, \lambda} \otimes_{E_\lambda} \overline{\mathbb{Q}}_\ell$ and ρ_{Π, ι_ℓ} are conjugate, where ρ_{Π, ι_ℓ} is the homomorphism from Proposition 3.2.4(2).

Remark 3.2.6. By [CH13, Proposition 3.2.5], strong coefficient field of Π exists. Moreover, under Hypothesis 3.2.9 below, $\mathbb{Q}(\Pi)$ is already a strong coefficient field of Π if Π is isomorphic to $\mathrm{BC}(\pi)$ for a relevant pair (V, π) (see Definition 3.2.7 below) in which V is standard *indefinite*.

Definition 3.2.7. Consider a pair (V, π) where V is a hermitian space over F , and π a discrete automorphic representation of $\mathrm{U}(V)(\mathbb{A}_{F^+})$. We say that (V, π) is *relevant* if either one of the following two situations happens:

- (1) V is standard definite, and π^∞ appears in

$$\varinjlim_{K \in \mathcal{R}'(V)} \mathbb{C}[\mathrm{Sh}(V, K)];$$

- (2) V is standard indefinite, and π^∞ appears in

$$\varinjlim_{K \in \mathcal{R}'(V)} \iota_\ell^{-1} H_{\mathrm{et}}^i(\mathrm{Sh}(V, K)_{\overline{F}}, \overline{\mathbb{Q}}_\ell)$$

for some isomorphism $\iota_\ell: \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$ and some $i \in \mathbb{Z}$.

Proposition 3.2.8. *Let (V, π) be a relevant pair. Then $\mathrm{BC}(\pi)$ exists.*

Proof. When V is standard definite, this is proved in [Lab, Corollaire 5.3]. When V is standard indefinite, this is proved in [Shi, Theorem 1.1].³ \square

Hypothesis 3.2.9. Consider an integer $N \geq 1$. For every standard indefinite hermitian space V over F of rank N , every discrete automorphic representation π of $\mathrm{U}(V)(\mathbb{A}_{F^+})$ such that $\mathrm{BC}(\pi)$ exists and is a relevant representation of $\mathrm{GL}_N(\mathbb{A}_F)$, and every isomorphism $\iota_\ell: \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$, the semisimplification of the $\overline{\mathbb{Q}}_\ell[\Gamma_F]$ -module

$$W^{N-1}(\pi) := \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell[\mathrm{U}(V)(\mathbb{A}_{F^+}^\infty)]} \left(\iota_\ell \pi^\infty, \varinjlim_{\mathcal{R}'(V)} H_{\mathrm{et}}^{N-1}(\mathrm{Sh}(V, K)_{\overline{F}}, \overline{\mathbb{Q}}_\ell) \right)$$

is isomorphic to the underlying $\overline{\mathbb{Q}}_\ell[\Gamma_F]$ -module of $\rho_{\mathrm{BC}(\pi), \iota_\ell}^\mathbb{C}$.

Proposition 3.2.10. *Hypothesis 3.2.9 holds for $N \leq 3$, and for $N > 3$ if $[F^+ : \mathbb{Q}] > 1$.*

Proof. By Proposition C.3.1, we know that $W^{N-1}(\pi)$ is of dimension N .

The case for $N = 1$ follows directly from the definition of the canonical model of Shimura varieties over reflex fields. The case for $N = 2$ is proved in [Liu, Theorem D.6(2)]. The case for $N = 3$ when $F^+ = \mathbb{Q}$ follows from the main result of [Rog92]. The case for $N \geq 3$ when $[F^+ : \mathbb{Q}] > 1$ will be proved in [KSZ]. \square

³In fact, in [Shi], the author considers the case for unitary similitude group and assumes that F contains an imaginary quadratic field. However, we can obtain the result in our setup by modifying the argument as in the proof of Theorem D.1.3.

3.3. Generalized CM type and reflexive closure. We denote by $\mathbb{N}[\Sigma_\infty]$ the commutative monoid freely generated by the set Σ_∞ , which admits an action of $\text{Aut}(\mathbb{C}/\mathbb{Q})$ via the set Σ_∞ .

Definition 3.3.1. A *generalized CM type of rank N* is an element

$$\Psi = \sum_{\tau \in \Sigma_\infty} r_\tau \tau \in \mathbb{N}[\Sigma_\infty]$$

satisfying $r_\tau + r_{\tau^c} = N$ for every $\tau \in \Sigma_\infty$. For such Ψ , we define its *reflex field* $F_\Psi \subseteq \mathbb{C}$ to be the fixed subfield of the stabilizer of Ψ in $\text{Aut}(\mathbb{C}/\mathbb{Q})$. A *CM type* is simply a generalized CM type of rank 1. For a CM type Φ , we say that Φ contains τ if its coefficient r_τ equals 1.

Definition 3.3.2. We define the *reflexive closure* of F , denoted by F_{rflx} , to be the subfield of \mathbb{C} generated by F and F_Φ for every CM type Φ of F . Put $F_{\text{rflx}}^+ := (F_{\text{rflx}})^{c=1}$.

Remark 3.3.3. It is clear that F_{rflx} is a CM field, and is F -normal (Definition 1.3.1); F_{rflx}^+ is the maximal totally real subfield of F_{rflx} , and is F^+ -normal. In many cases, we have $F_{\text{rflx}} = F$ hence $F_{\text{rflx}}^+ = F^+$; for example, when F is Galois or contains an imaginary quadratic field.

Definition 3.3.4. We say that a prime \mathfrak{p} of F^+ is *special inert* if the following are satisfied:

- (1) \mathfrak{p} is inert in F ;
- (2) the underlying rational prime p of \mathfrak{p} is odd and is unramified in F ;
- (3) \mathfrak{p} is of degree one over \mathbb{Q} , that is, $F_{\mathfrak{p}}^+ = \mathbb{Q}_p$.

By abuse of notation, we also denote by \mathfrak{p} for its induced prime of F .

We say that a special inert prime \mathfrak{p} of F^+ is *very special inert* if there exists a prime \mathfrak{p}' of F_{rflx}^+ above \mathfrak{p} satisfying $(F_{\text{rflx}}^+)_{\mathfrak{p}'} = F_{\mathfrak{p}}^+ (= \mathbb{Q}_p)$.

Remark 3.3.5. In Definition 3.3.4, (3) is proposed only for the purpose of simplifying computations on Dieudonné modules in Sections 4 and 5; it is not really necessary as results in these two sections should remain valid without (3). However, dropping (3) will vastly increase the burden of notations and computations in those two sections, where the technicality is already heavy.

In what follows in this article, we will often take a rational prime p that is unramified in F , and an isomorphism $\iota_p: \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_p}$. By composing with ι_p , we regard Σ_∞ also as the set of p -adic embeddings of F . We also regard \mathbb{Q}_p as a subfield of \mathbb{C} via ι_p^{-1} .

Notation 3.3.6. We introduce the following important notations.

- (1) In what follows, whenever we introduce some finite unramified extension \mathbb{Q}_p^\dagger of \mathbb{Q}_p , we denote by \mathbb{Z}_p^\dagger its ring of integers and put $\mathbb{F}_p^\dagger := \mathbb{Z}_p^\dagger/p\mathbb{Z}_p^\dagger$.
- (2) For every $\tau \in \Sigma_\infty$, we denote by $\mathbb{Q}_p^\tau \subseteq \mathbb{C}$ the composition of $\tau(F)$ and \mathbb{Q}_p , which is unramified over \mathbb{Q}_p . For a scheme $S \in \text{Sch}/\mathbb{Z}_p^\tau$ and an \mathcal{O}_S -module \mathcal{F} with an action $O_F \rightarrow \text{End}_{\mathcal{O}_S}(\mathcal{F})$, we denote by \mathcal{F}_τ the maximal \mathcal{O}_S -submodule of \mathcal{F} on which O_F acts via the homomorphism $\tau: O_F \rightarrow \mathbb{Z}_p^\tau \rightarrow \mathcal{O}_S$.
- (3) We denote by $\mathbb{Q}_p^\diamond \subseteq \mathbb{C}$ the composition of \mathbb{Q}_p^τ for all $\tau \in \Phi$, which is unramified over \mathbb{Q}_p . We can identify Σ_∞ with $\text{Hom}(O_F, \mathbb{Z}_p^\diamond) = \text{Hom}(O_F, \mathbb{F}_p^\diamond)$. In particular, the p -power Frobenius map σ acts on Σ_∞ .
- (4) For a generalized CM type Ψ of rank N , we denote by $\mathbb{Q}_p^\Psi \subseteq \mathbb{C}$ the composition of \mathbb{Q}_p , F , and F_Ψ , which is contained in \mathbb{Q}_p^\diamond .
- (5) For a (functor in) scheme over \mathbb{Z}_p^\dagger written like $\mathbf{X}_\dagger(\cdots)$, we put $\mathbf{X}_\dagger(\cdots) := \mathbf{X}_\dagger(\cdots) \otimes_{\mathbb{Z}_p^\dagger} \mathbb{F}_p^\dagger$ and $\mathbf{X}_\dagger^\eta(\cdots) := \mathbf{X}_\dagger(\cdots) \otimes_{\mathbb{Z}_p^\dagger} \mathbb{Q}_p^\dagger$. For a (functor in) scheme over \mathbb{F}_p^\dagger written like $\mathbf{X}_\dagger^\eta(\cdots)$, we put $\overline{\mathbf{X}}_\dagger^\eta(\cdots) := \mathbf{X}_\dagger^\eta(\cdots) \otimes_{\mathbb{F}_p^\dagger} \overline{\mathbb{F}_p}$. Similar conventions are applied to morphisms as well.

3.4. Unitary abelian schemes. We first introduce some general notations about abelian schemes.

Notation 3.4.1. Let A be an abelian scheme over a scheme S . We denote by A^\vee the *dual abelian variety* of A over S . We denote by $H_1^{\text{dR}}(A/S)$ (resp. $\text{Lie}_{A/S}$, and $\omega_{A/S}$) for the *relative de Rham homology* (resp. *Lie algebra*, and *dual Lie algebra*) of A/S , all regarded as locally free \mathcal{O}_S -modules. We have the following *Hodge exact sequence*

$$(3.2) \quad 0 \rightarrow \omega_{A^\vee/S} \rightarrow H_1^{\text{dR}}(A/S) \rightarrow \text{Lie}_{A/S} \rightarrow 0$$

of sheaves on S . When the base S is clear from the context, we sometimes suppress it from the notation.

Definition 3.4.2 (Unitary abelian scheme). We prescribe a subring $\mathbb{P} \subseteq \mathbb{Q}$. Let S be a scheme in Sch/\mathbb{P} .

- (1) An O_F -abelian scheme over S is a pair (A, i) in which A is an abelian scheme over S and $i: O_F \rightarrow \text{End}_S(A) \otimes \mathbb{P}$ is a homomorphism of algebras.
- (2) A *unitary O_F -abelian scheme* over S is a triple (A, i, λ) in which (A, i) is an O_F -abelian scheme over S , and $\lambda: A \rightarrow A^\vee$ is a quasi-polarization such that $i(a^\varepsilon)^\vee \circ \lambda = \lambda \circ i(a)$ for every $a \in O_F$, and there exists $c \in \mathbb{P}^\times$ making $c\lambda$ a polarization.
- (3) For two O_F -abelian schemes (A, i) and (A', i') over S , a (quasi-)homomorphism from (A, i) to (A', i') is a (quasi-)homomorphism $\varphi: A \rightarrow A'$ such that $\varphi \circ i(a) = i'(a) \circ \varphi$ for every $a \in O_F$. We will usually refer to such φ as an O_F -linear (quasi-)homomorphism.

Moreover, we will usually suppress the notion i if it is insensitive.

Definition 3.4.3 (Signature type). Let Ψ be a generalized CM type of rank N (Definition 3.3.1). Consider a scheme $S \in \text{Sch}/_{O_{F_\Psi} \otimes \mathbb{P}}$. We say that an O_F -abelian scheme (A, i) over S has *signature type* Ψ if for every $a \in O_F$, the characteristic polynomial of $i(a)$ on $\text{Lie}_{A/S}$ is given by

$$\prod_{\tau \in \Sigma_\infty} (T - \tau(a))^{r_\tau} \in \mathcal{O}_S[T].$$

Construction 3.4.4. Let K be an $O_{F_\Psi} \otimes \mathbb{P}$ -algebra that is an algebraically closed field. Suppose that we are given a unitary O_F -abelian scheme (A_0, i_0, λ_0) over K of signature type Φ that is a CM type, and a unitary O_F -abelian scheme (A, i, λ) over K of signature type Ψ . For every set \square of places of \mathbb{Q} containing ∞ and the characteristic of K , if not zero, we construct a hermitian space

$$\text{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}^\square}^{\lambda_0, \lambda} (H_1^{\text{ét}}(A_0, \mathbb{A}^\square), H_1^{\text{ét}}(A, \mathbb{A}^\square))$$

over $F \otimes_{\mathbb{Q}} \mathbb{A}^\square = F \otimes_{F^+} (F^+ \otimes_{\mathbb{Q}} \mathbb{A}^\square)$, with the underlying $F \otimes_{\mathbb{Q}} \mathbb{A}^\square$ -module

$$\text{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}^\square} (H_1^{\text{ét}}(A_0, \mathbb{A}^\square), H_1^{\text{ét}}(A, \mathbb{A}^\square))$$

equipped with the pairing

$$(x, y) := i_0^{-1} \left((\lambda_{0*})^{-1} \circ y^\vee \circ \lambda_* \circ x \right) \in i_0^{-1} \text{End}_{F \otimes_{\mathbb{Q}} \mathbb{A}^\square} (H_1^{\text{ét}}(A_0, \mathbb{A}^\square)) = F \otimes_{\mathbb{Q}} \mathbb{A}^\square.$$

Now we take a rational prime p that is unramified in F , and take the prescribed subring \mathbb{P} in Definition 3.4.2 to be $\mathbb{Z}_{(p)}$. We also choose an isomorphism $\iota_p: \mathbb{C} \simeq \overline{\mathbb{Q}}_p$, and adopt Notation 3.3.6.

Definition 3.4.5. Let A and B be two abelian schemes over a scheme $S \in \text{Sch}/_{\mathbb{Z}_{(p)}}$. We say that a quasi-homomorphism (resp. quasi-isogeny) $\varphi: A \rightarrow B$ is a *quasi- p -homomorphism* (resp. *quasi- p -isogeny*) if there exists some $c \in \mathbb{Z}_{(p)}^\times$ such that $c\varphi$ is a homomorphism (resp. isogeny). A quasi-isogeny φ is *prime-to- p* if both φ and φ^{-1} are quasi- p -isogenies. We say that a quasi-polarization λ of A is *p -principal* if λ is a prime-to- p quasi-isogeny.

Note that for a unitary O_F -abelian scheme (A, i, λ) , the quasi-polarization λ is a quasi- p -isogeny. To continue, take a generalized CM type $\Psi = \sum_{\tau \in \Sigma_\infty} r_\tau \tau$ of rank N .

Remark 3.4.6. Let A be an O_F -abelian scheme of signature type Ψ over a scheme $S \in \text{Sch}/\mathbb{Z}_p^\diamond$ for some $\tau \in \Sigma_\infty$. Then (3.2) induces a short exact sequence

$$0 \rightarrow \omega_{A^\vee/S, \tau} \rightarrow H_1^{\text{dR}}(A/S)_\tau \rightarrow \text{Lie}_{A/S, \tau} \rightarrow 0$$

of locally free \mathcal{O}_S -modules of ranks $N - r_\tau$, N , and r_τ , respectively. If S belongs to $\text{Sch}/\mathbb{Z}_p^\diamond$, then we have decompositions

$$\begin{aligned} H_1^{\text{dR}}(A/S) &= \bigoplus_{\tau \in \Sigma_\infty} H_1^{\text{dR}}(A/S)_\tau, \\ \text{Lie}_{A/S} &= \bigoplus_{\tau \in \Sigma_\infty} \text{Lie}_{A/S, \tau}, \\ \omega_{A/S} &= \bigoplus_{\tau \in \Sigma_\infty} \omega_{A/S, \tau} \end{aligned}$$

of locally free \mathcal{O}_S -modules.

Notation 3.4.7. Take $\tau \in \Sigma_\infty$. Let (A, λ) be a unitary O_F -abelian scheme of signature type Ψ over a scheme $S \in \text{Sch}/\mathbb{Z}_p^\diamond$. We denote

$$\langle \cdot, \cdot \rangle_{\lambda, \tau}: H_1^{\text{dR}}(A/S)_\tau \times H_1^{\text{dR}}(A/S)_{\tau^c} \rightarrow \mathcal{O}_S$$

the \mathcal{O}_S -bilinear pairing induced by the polarization λ , which might be degenerate. Moreover, for an \mathcal{O}_S -submodule $\mathcal{F} \subseteq H_1^{\text{dR}}(A/S)_\tau$, we denote by $\mathcal{F}^\perp \subseteq H_1^{\text{dR}}(A/S)_{\tau^c}$ its (right) orthogonal complement under the above pairing, if λ is clear from the context.

Next we review some facts from the Serre–Tate theory [Kat81] and the Grothendieck–Messing theory [Mes72], tailored to our application. Let Ψ be a generalized CM type of rank N such that $r_\tau r_{\tau^c} = 0$ for every τ not above $\underline{\tau}_\infty$. Consider a closed immersion $S \hookrightarrow \hat{S}$ in $\text{Sch}/\mathbb{Z}_p^\diamond$ with an ideal equipped with a locally nilpotent PD-structure, and a unitary O_F -abelian scheme (A, λ) of signature type Ψ over S . We let $H_1^{\text{cris}}(A/\hat{S})$ be the evaluation of the first relative crystalline homology of A/S at the PD-thickening $S \hookrightarrow \hat{S}$, which is a locally free $\mathcal{O}_{\hat{S}} \otimes O_F$ -module. The polarization λ induces a pairing

$$(3.3) \quad \langle \cdot, \cdot \rangle_{\lambda, \tau_\infty}^{\text{cris}}: H_1^{\text{cris}}(A/\hat{S})_{\tau_\infty} \times H_1^{\text{cris}}(A/\hat{S})_{\tau_\infty^c} \rightarrow \mathcal{O}_{\hat{S}}.$$

We define two groupoids

- $\text{Def}(S, \hat{S}; A, \lambda)$, whose objects are unitary O_F -abelian schemes $(\hat{A}, \hat{\lambda})$ of signature type Ψ over \hat{S} that lift (A, λ) ;
- $\text{Def}'(S, \hat{S}; A, \lambda)$, whose objects are pairs $(\hat{\omega}_{\tau_\infty}, \hat{\omega}_{\tau_\infty^c})$ where for each $\tau = \tau_\infty, \tau_\infty^c$, $\hat{\omega}_\tau \subseteq H_1^{\text{cris}}(A/\hat{S})_\tau$ is a subbundle that lifts $\omega_{A^\vee/S, \tau} \subseteq H_1^{\text{dR}}(A/S)_\tau$, such that $\langle \hat{\omega}_{\tau_\infty}, \hat{\omega}_{\tau_\infty^c} \rangle_{\lambda, \tau_\infty}^{\text{cris}} = 0$.

Proposition 3.4.8. *The functor from $\text{Def}(S, \hat{S}; A, \lambda)$ to $\text{Def}'(S, \hat{S}; A, \lambda)$ sending $(\hat{A}, \hat{\lambda})$ to $(\omega_{\hat{A}^\vee/\hat{S}, \tau_\infty}, \omega_{\hat{A}^\vee/\hat{S}, \tau_\infty^c})$ is a natural equivalence.*

Proof. By étale descent, we may replace $S \hookrightarrow \hat{S}$ by $S \otimes_{\mathbb{Z}_p^\diamond} \mathbb{Z}_p^\diamond \hookrightarrow \hat{S} \otimes_{\mathbb{Z}_p^\diamond} \mathbb{Z}_p^\diamond$. Then we have a decomposition

$$H_1^{\text{cris}}(A/\hat{S}) = \bigoplus_{\tau \in \Sigma_\infty} H_1^{\text{cris}}(A/\hat{S})_\tau$$

similar to the one in Notation 3.3.6. Note that for $\tau \notin \{\tau_\infty, \tau_\infty^c\}$, the subbundle $\omega_{A^\vee/S, \tau}$ has a unique lifting to either zero or the entire $H_1^{\text{cris}}(A/\hat{S})_\tau$. Thus, the proposition follows from the Serre–Tate and Grothendieck–Messing theories. \square

To end this subsection, we review some notions for abelian schemes in characteristic p .

Notation 3.4.9. Let A be an abelian scheme over a scheme $S \in \text{Sch}/\mathbb{F}_p$. Put

$$A^{(p)} := A \times_{S, \sigma} S,$$

where σ is the absolute Frobenius morphism of S . Then we have

- (1) a canonical isomorphism $H_1^{\text{dR}}(A^{(p)}/S) \simeq \sigma^* H_1^{\text{dR}}(A/S)$ of \mathcal{O}_S -modules;
- (2) the Frobenius homomorphism $\text{Fr}_A: A \rightarrow A^{(p)}$ which induces the *Verschiebung map*

$$\mathbf{V}_A := (\text{Fr}_A)_*: H_1^{\text{dR}}(A/S) \rightarrow H_1^{\text{dR}}(A^{(p)}/S)$$

of \mathcal{O}_S -modules;

- (3) the *Verschiebung* homomorphism $\text{Ver}_A: A^{(p)} \rightarrow A$ which induces the *Frobenius map*

$$\mathbf{F}_A := (\text{Ver}_A)_*: H_1^{\text{dR}}(A^{(p)}/S) \rightarrow H_1^{\text{dR}}(A/S)$$

of \mathcal{O}_S -modules.

For a subbundle H of $H_1^{\text{dR}}(A/S)$, we denote by $H^{(p)}$ the subbundle of $H_1^{\text{dR}}(A^{(p)}/S)$ that corresponds to $\sigma^* H$ under the isomorphism in (1). In what follows, we will suppress A in the notations \mathbf{F}_A and \mathbf{V}_A if the reference to A is clear.

Remark 3.4.10. In Notation 3.4.9, we have $\ker \mathbf{F} = \text{im } \mathbf{V} = \omega_{A^{(p)}/S}$ and $\ker \mathbf{V} = \text{im } \mathbf{F}$.

If $S = \text{Spec } \kappa$ for a field κ of characteristic p , then we have a canonical isomorphism $H_1^{\text{dR}}(A^{(p)}/\kappa) \simeq H_1^{\text{dR}}(A/\kappa) \otimes_{\kappa, \sigma} \kappa$. Thus, by abuse of notation, we have

- the (κ, σ) -linear Frobenius map $\mathbf{F}: H_1^{\text{dR}}(A/\kappa) \rightarrow H_1^{\text{dR}}(A/\kappa)$ and
- if κ is perfect, the (κ, σ^{-1}) -linear *Verschiebung map* $\mathbf{V}: H_1^{\text{dR}}(A/\kappa) \rightarrow H_1^{\text{dR}}(A/\kappa)$.

Suppose that κ is perfect. Recall that we have the *covariant* Dieudonné module $\mathcal{D}(A)$ associated to the p -divisible group $A[p^\infty]$, which is a free $W(\kappa)$ -module, such that $\mathcal{D}(A)/p\mathcal{D}(A)$ is canonically isomorphic to $H_1^{\text{dR}}(A/\kappa)$. Moreover, again by abuse of notation, we have

- the $(W(\kappa), \sigma)$ -linear Frobenius map $\mathbf{F}: \mathcal{D}(A) \rightarrow \mathcal{D}(A)$ lifting the one above, and
- the $(W(\kappa), \sigma^{-1})$ -linear *Verschiebung map* $\mathbf{V}: \mathcal{D}(A) \rightarrow \mathcal{D}(A)$ lifting the one above,

respectively, satisfying $\mathbf{F} \circ \mathbf{V} = \mathbf{V} \circ \mathbf{F} = p$.

Remark 3.4.11. Take $\tau \in \Sigma_\infty$. For a scheme $S \in \text{Sch}/\mathbb{F}_p^\tau$ and an O_F -abelian scheme A over S , we have $(H_1^{\text{dR}}(A/S)_\tau)^{(p)} = H_1^{\text{dR}}(A^{(p)}/S)_{\sigma\tau}$ under Notations 3.3.6 and 3.4.9.

If $S = \text{Spec } \kappa$ for a perfect field κ (containing \mathbb{F}_p^τ), then applying Notation 3.3.6 to the $W(\kappa)$ -module $\mathcal{D}(A)$, we obtain $W(\kappa)$ -submodules $\mathcal{D}(A)_{\sigma^i \tau} \subseteq \mathcal{D}(A)$ for every $i \in \mathbb{Z}$. From Remark 3.4.10, we obtain

- the $(W(\kappa), \sigma)$ -linear Frobenius map $\mathbf{F}: \mathcal{D}(A)_\tau \rightarrow \mathcal{D}(A)_{\sigma\tau}$ and
- the $(W(\kappa), \sigma^{-1})$ -linear *Verschiebung map* $\mathbf{V}: \mathcal{D}(A)_\tau \rightarrow \mathcal{D}(A)_{\sigma^{-1}\tau}$

by restriction. We have canonical isomorphisms and inclusions:

$$\mathbf{V}\mathcal{D}(A)_{\sigma\tau}/p\mathcal{D}(A)_\tau \simeq \omega_{A^\vee, \tau} \subseteq \mathcal{D}(A)_\tau/p\mathcal{D}(A)_\tau \simeq H_1^{\text{dR}}(A)_\tau.$$

Notation 3.4.12. Take $\tau \in \Sigma_\infty$. Let (A, λ) be a unitary O_F -abelian scheme of signature type Ψ over $\text{Spec } \kappa$ for a perfect field κ containing \mathbb{F}_p^τ . We have a pairing

$$\langle \cdot, \cdot \rangle_{\lambda, \tau}: \mathcal{D}(A)_\tau \times \mathcal{D}(A)_{\tau^c} \rightarrow W(\kappa)$$

lifting the one in Notation 3.4.7. We denote by $\mathcal{D}(A)_\tau^\vee$ the $W(\kappa)$ -dual of $\mathcal{D}(A)_\tau$, as a submodule of $\mathcal{D}(A)_{\tau^c} \otimes \mathbb{Q}$. In what follows, unless we specify, the dual is always with respect to the default quasi-polarization.

The following lemma will be repeatedly used in later discussion.

Lemma 3.4.13. *Suppose that F^+ is contained in \mathbb{Q}_p (via the embedding $\tau: F^+ \hookrightarrow \mathbb{C} \simeq \overline{\mathbb{Q}_p}$) with \mathfrak{p} the induced p -adic prime. Let $\varpi \in O_{F^+}$ be an element such that $\text{val}_{\mathfrak{p}}(\varpi) = 1$. Consider two O_F -abelian schemes A and B over a scheme $S \in \text{Sch}/_{\mathbb{F}_{p^2}}$. Let $\alpha: A \rightarrow B$ and $\beta: B \rightarrow A$ be two O_F -linear quasi- p -isogenies (Definition 3.4.5) such that $\beta \circ \alpha = \varpi \cdot \text{id}_A$ (hence $\alpha \circ \beta = \varpi \cdot \text{id}_B$). Then*

(1) *For $\tau \in \{\tau_\infty, \tau_\infty^c\}$, the induced maps*

$$\begin{aligned} \alpha_{*,\tau}: H_1^{\text{dR}}(A/S)_\tau &\rightarrow H_1^{\text{dR}}(B/S)_\tau, \\ \beta_{*,\tau}: H_1^{\text{dR}}(B/S)_\tau &\rightarrow H_1^{\text{dR}}(A/S)_\tau \end{aligned}$$

satisfy the relations $\ker \alpha_{,\tau} = \text{im } \beta_{*,\tau}$ and $\ker \beta_{*,\tau} = \text{im } \alpha_{*,\tau}$; and these kernels and images are locally free \mathcal{O}_S -modules.*

(2) *We have*

$$\text{rank}_{\mathcal{O}_S} \text{Lie}_{B/S, \tau_\infty} - \text{rank}_{\mathcal{O}_S} \text{Lie}_{A/S, \tau_\infty} = \text{rank}_{\mathcal{O}_S}(\ker \alpha_{*, \tau_\infty}) - \text{rank}_{\mathcal{O}_S}(\ker \alpha_{*, \tau_\infty^c}).$$

(3) *Let λ_A and λ_B be two quasi-polarizations on A and B , respectively, so that (A, λ_A) and (B, λ_B) become unitary O_F -abelian schemes of dimension $n[F^+ : \mathbb{Q}]$. Suppose that $\alpha^\vee \circ \lambda_B \circ \alpha = \varpi \lambda_A$.*

(a) *If both λ_A and λ_B are p -principal, then we have*

$$\text{rank}_{\mathcal{O}_S}(\ker \alpha_{*, \tau_\infty}) + \text{rank}_{\mathcal{O}_S}(\ker \alpha_{*, \tau_\infty^c}) = n.$$

(b) *If λ_A is p -principal and $\ker \lambda_B[\mathfrak{p}^\infty]$ is of rank p^2 , then we have*

$$\text{rank}_{\mathcal{O}_S}(\ker \alpha_{*, \tau_\infty}) + \text{rank}_{\mathcal{O}_S}(\ker \alpha_{*, \tau_\infty^c}) = n - 1.$$

(c) *If $\ker \lambda_A[\mathfrak{p}^\infty]$ is of rank p^2 and λ_B is p -principal, then we have*

$$\text{rank}_{\mathcal{O}_S}(\ker \alpha_{*, \tau_\infty}) + \text{rank}_{\mathcal{O}_S}(\ker \alpha_{*, \tau_\infty^c}) = n + 1.$$

(d) *If both $\ker \lambda_A[\mathfrak{p}^\infty]$ and $\ker \lambda_B[\mathfrak{p}^\infty]$ are of rank p^2 , respectively, then we have*

$$\text{rank}_{\mathcal{O}_S}(\ker \alpha_{*, \tau_\infty}) + \text{rank}_{\mathcal{O}_S}(\ker \alpha_{*, \tau_\infty^c}) = n.$$

(4) *Let λ_A and λ_B be two quasi-polarizations on A and B , respectively, so that (A, λ_A) and (B, λ_B) become unitary O_F -abelian schemes of dimension $n[F^+ : \mathbb{Q}]$. Suppose that $\alpha^\vee \circ \lambda_B \circ \alpha = \lambda_A$. If $\ker \lambda_A[\mathfrak{p}^\infty]$ is of rank p^2 and λ_B is p -principal, then we have*

$$\text{rank}_{\mathcal{O}_S}(\ker \alpha_{*, \tau_\infty}) + \text{rank}_{\mathcal{O}_S}(\ker \alpha_{*, \tau_\infty^c}) = 1.$$

Proof. We may assume S connected; and up to replacing α , β and ϖ by a common $\mathbb{Z}_{(p)}^\times$ -multiple, we may also that α and β are genuine isogenies.

For (1), it suffices to show that the induced maps

$$\begin{aligned} \alpha_*: H_1^{\text{dR}}(A/S) \otimes_{O_{F^+}} \mathbb{Z}_p &\rightarrow H_1^{\text{dR}}(B/S) \otimes_{O_{F^+}} \mathbb{Z}_p, \\ \beta_*: H_1^{\text{dR}}(B/S) \otimes_{O_{F^+}} \mathbb{Z}_p &\rightarrow H_1^{\text{dR}}(A/S) \otimes_{O_{F^+}} \mathbb{Z}_p \end{aligned}$$

satisfy the relations $\ker \alpha_* = \text{im } \beta_*$ and $\ker \beta_* = \text{im } \alpha_*$; and these kernels and images are locally free \mathcal{O}_S -modules.

Note that $A[\mathfrak{p}]$, $B[\mathfrak{p}]$, $\ker \alpha[\mathfrak{p}]$, and $\ker \beta[\mathfrak{p}]$ are all locally free finite group schemes over S with an action by $O_F/\mathfrak{p}O_F$. By the relation among α, β, ϖ , we may assume that $A[\mathfrak{p}]$ and $B[\mathfrak{p}]$ have degree p^{2d} ; $\ker \alpha[\mathfrak{p}]$ has degree p^r ; and $\ker \beta[\mathfrak{p}]$ has degree p^{2d-r} . As $\beta_* \circ \alpha_* = 0$ and $\alpha_* \circ \beta_* = 0$, it suffices to show that both $\ker \alpha_*$ and $\text{im } \beta_*$ (resp. both $\ker \beta_*$ and $\text{im } \alpha_*$) are locally direct factors of $H_1^{\text{dR}}(A/S) \otimes_{O_{F^+}} \mathbb{Z}_p$ (resp. $H_1^{\text{dR}}(B/S) \otimes_{O_{F^+}} \mathbb{Z}_p$) of rank r (resp. $2d - r$), which will follow if we can show that $\text{coker } \alpha_*$ and $\text{coker } \beta_*$ are locally free \mathcal{O}_S -modules of rank r and $2d - r$, respectively.

We now prove that $\text{coker } \alpha_*$ is a locally free \mathcal{O}_S -modules of rank r ; and the other case is similar. We follow the argument in [dJ93, Lemma 2.3]. Consider the big crystalline site $(S/\mathbb{Z}_p)_{\text{cris}}$ with the structural sheaf $\mathcal{O}_S^{\text{cris}}$. Let $\mathcal{D}(A[\mathfrak{p}^\infty])$ and $\mathcal{D}(B[\mathfrak{p}^\infty])$ denote by the covariant Dieudonné crystals on $(S/\mathbb{Z}_p)_{\text{cris}}$ of p -divisible groups $A[\mathfrak{p}^\infty]$ and $B[\mathfrak{p}^\infty]$, respectively. They are locally free $\mathcal{O}_S^{\text{cris}}$ -modules. We have a short exact sequence

$$(3.4) \quad 0 \rightarrow \alpha_* \mathcal{D}(A[\mathfrak{p}^\infty]) / \varpi \mathcal{D}(B[\mathfrak{p}^\infty]) \rightarrow \mathcal{D}(B[\mathfrak{p}^\infty]) / \varpi \mathcal{D}(B[\mathfrak{p}^\infty]) \rightarrow \mathcal{D}(B[\mathfrak{p}^\infty]) / \alpha_* \mathcal{D}(A[\mathfrak{p}^\infty]) \rightarrow 0$$

and a surjective map

$$(3.5) \quad \alpha_* : \mathcal{D}(A[\mathfrak{p}^\infty]) / \beta_* \mathcal{D}(B[\mathfrak{p}^\infty]) \rightarrow \alpha_* \mathcal{D}(A[\mathfrak{p}^\infty]) / \varpi \mathcal{D}(B[\mathfrak{p}^\infty])$$

of $\mathcal{O}_S^{\text{cris}}$ -modules. To show that $\text{coker } \alpha_*$ is a locally free \mathcal{O}_S -module of rank r , it suffices to show that $\mathcal{D}(B[\mathfrak{p}^\infty]) / \alpha_* \mathcal{D}(A[\mathfrak{p}^\infty])$ is a locally free $\mathcal{O}_S^{\text{cris}}/p\mathcal{O}_S^{\text{cris}}$ -module of rank r . By [BBM82, Proposition 4.3.1], $\mathcal{D}(B[\mathfrak{p}^\infty]) / \varpi \mathcal{D}(B[\mathfrak{p}^\infty])$ is a locally free $\mathcal{O}_S^{\text{cris}}/p\mathcal{O}_S^{\text{cris}}$ -module of rank $2d$. Thus, by (3.4) and (3.5), it suffices to show that the $\mathcal{O}_S^{\text{cris}}/p\mathcal{O}_S^{\text{cris}}$ -modules $\alpha_* \mathcal{D}(A[\mathfrak{p}^\infty]) / \varpi \mathcal{D}(B[\mathfrak{p}^\infty])$ and $\mathcal{D}(B[\mathfrak{p}^\infty]) / \alpha_* \mathcal{D}(A[\mathfrak{p}^\infty])$ are locally generated by r and $2d - r$ sections, respectively. However, this can be easily checked using classical Dieudonné modules after base change to geometric points of S . Thus, (1) is proved.

For (2), we know from (1) that both $\ker \alpha_{*,\tau_\infty}$ and $\ker \alpha_{*,\tau_\infty^c}$ are locally free \mathcal{O}_S -modules. We may assume that $S = \text{Spec } \kappa$ for a perfect field κ containing \mathbb{F}_{p^2} . Put $r := \dim_\kappa \text{Lie}_{A/\kappa,\tau_\infty}$ and $s := \dim_\kappa \text{Lie}_{B/\kappa,\tau_\infty^c}$. Then we have

$$s = \dim_\kappa (\omega_{B^\vee/\kappa,\tau_\infty^c}) = \dim_\kappa \frac{\text{VD}(B)_{\tau_\infty}}{p\mathcal{D}(B)_{\tau_\infty^c}}, \quad r = \dim_\kappa (\omega_{A^\vee/\kappa,\tau_\infty}) = \dim_\kappa \frac{\text{VD}(A)_{\tau_\infty}}{p\mathcal{D}(A)_{\tau_\infty^c}}.$$

Thus, we obtain

$$(3.6) \quad s - r = \dim_\kappa \frac{\text{VD}(B)_{\tau_\infty}}{p\mathcal{D}(B)_{\tau_\infty^c}} - \dim_\kappa \frac{\text{VD}(A)_{\tau_\infty}}{p\mathcal{D}(A)_{\tau_\infty^c}}.$$

Regarding $\mathcal{D}(A)$ as a submodule of $\mathcal{D}(B)$ via α_* , it follows that

$$(3.6) = \dim_\kappa \frac{\text{VD}(B)_{\tau_\infty}}{\text{VD}(A)_{\tau_\infty}} - \dim_\kappa \frac{p\mathcal{D}(B)_{\tau_\infty^c}}{p\mathcal{D}(A)_{\tau_\infty^c}} = \dim_\kappa \frac{\mathcal{D}(B)_{\tau_\infty}}{\mathcal{D}(A)_{\tau_\infty}} - \dim_\kappa \frac{\mathcal{D}(B)_{\tau_\infty^c}}{\mathcal{D}(A)_{\tau_\infty^c}} \\ = \dim_\kappa (\ker \alpha_{*,\tau_\infty}) - \dim_\kappa (\ker \alpha_{*,\tau_\infty^c}).$$

Thus, (2) is proved.

For (3), by assumption on λ_A , the alternating paring

$$\langle \cdot, \cdot \rangle_{\tau_\infty}^{\text{dR}} : H_1^{\text{dR}}(A/S)_{\tau_\infty} \times H_1^{\text{dR}}(A/S)_{\tau_\infty^c} \rightarrow \mathcal{O}_S$$

induced by λ_A is perfect.

In case (a), we have $\ker \alpha_{*,\tau_\infty} = (\ker \alpha_{*,\tau_\infty^c})^\perp$, the orthogonal complement of $\ker \alpha_{*,\tau_\infty^c} = \text{im } \beta_{*,\tau_\infty^c}$ under $\langle \cdot, \cdot \rangle_{\tau_\infty}^{\text{dR}}$, since λ_B is also p -principal.

In case (b), $\ker \alpha_{*,\tau_\infty}$ is a subbundle of $(\ker \alpha_{*,\tau_\infty^c})^\perp$ of corank 1. The identity follows immediately from the identity $\text{rank}_{\mathcal{O}_S}((\ker \alpha_{*,\tau_\infty^c})^\perp) + \text{rank}_{\mathcal{O}_S}(\ker \alpha_{*,\tau_\infty^c}) = n$.

In case (c), $(\ker \alpha_{*,\tau_\infty})^\perp$ is a subbundle of $\ker \alpha_{*,\tau_\infty^c}$ of corank 1. The identity follows immediately from the identity $\text{rank}_{\mathcal{O}_S}((\ker \alpha_{*,\tau_\infty})^\perp) + \text{rank}_{\mathcal{O}_S}(\ker \alpha_{*,\tau_\infty}) = n$.

In case (d), we have both situations in (b) and (c), and the identity follows by a similar reason.

The proof for (4) is similar to (3). We leave the detail to readers. \square

3.5. A CM moduli scheme. In this subsection, we introduce an auxiliary moduli scheme parameterizing certain CM abelian varieties, which will be used in Sections 4 and 5.

Definition 3.5.1. Let R be a commutative $\mathbb{Z}[(\text{disc } F)^{-1}]$ -algebra.

- (1) A *rational skew-hermitian space* over $O_F \otimes R$ of rank N is a free $O_F \otimes R$ -module W of rank N together with an R -bilinear skew-symmetric perfect pairing

$$\langle \cdot, \cdot \rangle_W : W \times W \rightarrow R$$

satisfying $\langle ax, y \rangle_W = \langle x, a^c y \rangle_W$ for every $a \in O_F \otimes R$ and $x, y \in W$.

- (2) Let W and W' be two rational skew-hermitian spaces over $O_F \otimes R$, a map $f: W \rightarrow W'$ is a *similitude* if f is an $O_F \otimes R$ -linear isomorphism such that there exists some $c(f) \in R^\times$ satisfying $\langle f(x), f(y) \rangle_{W'} = c(f) \langle x, y \rangle_W$ for every $x, y \in W$.
- (3) Two rational skew-hermitian spaces over $O_F \otimes R$ are *similar* if there exists a similitude between them.
- (4) For a rational skew-hermitian space W over $O_F \otimes R$, we denote by $\text{GU}(W)$ its *group of similitude* as a reductive group over R ; it satisfies that for every ring R' over R , $\text{GU}(W)(R')$ is the set of self-similitude of the rational skew-hermitian space $W \otimes_R R'$ over $O_F \otimes R'$.

We define a subtorus $T_0 \subseteq (\text{Res}_{O_F/\mathbb{Z}} \mathbf{G}_m) \otimes \mathbb{Z}[(\text{disc } F)^{-1}]$ such that for every commutative $\mathbb{Z}[(\text{disc } F)^{-1}]$ -algebra R ,

$$T_0(R) = \{a \in O_F \otimes R \mid \text{Nm}_{F/F^+} a \in R^\times\}.$$

Now we take a rational prime p that is unramified in F . We take the prescribed subring \mathbb{P} in Definition 3.4.2 to be $\mathbb{Z}_{(p)}$.

Remark 3.5.2. Let W_0 be a rational skew-hermitian space over $O_F \otimes \mathbb{Z}_{(p)}$ of rank 1. Then $\text{GU}(W_0)$ is canonically isomorphic to $T_0 \otimes_{\mathbb{Z}[(\text{disc } F)^{-1}]} \mathbb{Z}_{(p)}$. Moreover, the set of similarity classes of rational skew-hermitian spaces W'_0 over $O_F \otimes \mathbb{Z}_{(p)}$ of rank 1 such that $W'_0 \otimes_{\mathbb{Z}_{(p)}} \mathbb{A}$ is similar to $W_0 \otimes_{\mathbb{Z}_{(p)}} \mathbb{A}$ is canonically isomorphic to

$$\ker^1(T_0) := \ker \left(H^1(\mathbb{Q}, T_0) \rightarrow \prod_{v \leq \infty} H^1(\mathbb{Q}_v, T_0) \right),$$

which is a finite abelian group.

Definition 3.5.3. Let Φ be a CM type. We say that a rational skew-hermitian space W_0 over $O_F \otimes \mathbb{Z}_{(p)}$ of rank 1 has *type* Φ if for every $x \in W_0$ and every totally imaginary element $a \in F^\times$ satisfying $\text{Im } \tau(a) > 0$ for all $\tau \in \Phi$, we have $\langle ax, x \rangle_{W_0} \geq 0$.

Definition 3.5.4. For a rational skew-hermitian space W_0 over $O_F \otimes \mathbb{Z}_{(p)}$ of rank 1 and type Φ and an open compact subgroup $K_0^p \subseteq T_0(\mathbb{A}^{\infty, p})$, we define a presheaf $\mathbf{T}_p^1(W_0, K_0^p)$ on $\text{Sch}'_{/O_{F_\Phi} \otimes \mathbb{Z}_{(p)}}$ as follows: for every $S \in \text{Sch}'_{/O_{F_\Phi} \otimes \mathbb{Z}_{(p)}}$, we let $\mathbf{T}_p^1(W_0, K_0^p)(S)$ be the set of equivalence classes of triples $(A_0, \lambda_0, \eta_0^p)$ where

- (A_0, λ_0) is a unitary O_F -abelian scheme of signature type Φ over S such that λ_0 is p -principal;
- η_0^p is a K_0^p -level structure, that is, for a chosen geometric point s on every connected component of S , a $\pi_1(S, s)$ -invariant K_0^p -orbit of similitude

$$\eta_0^p : W_0 \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p} \rightarrow H_1^{\text{ét}}(A_{0s}, \mathbb{A}^{\infty, p})$$

of rational skew-hermitian spaces over $F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$, where $H_1^{\text{ét}}(A_{0s}, \mathbb{A}^{\infty, p})$ is equipped with the rational skew-hermitian form induced by λ_0 .

Two triples $(A_0, \lambda_0, \eta_0^p)$ and $(A'_0, \lambda'_0, \eta_0^{p'})$ are equivalent if there exists a prime-to- p O_F -linear quasi-isogeny $\varphi_0: A_0 \rightarrow A'_0$ carrying (λ_0, η_0^p) to $(c\lambda'_0, \eta_0^{p'})$ for some $c \in \mathbb{Z}_{(p)}^\times$.

For an object $(A_0, \lambda_0, \eta_0^p) \in \mathbf{T}_p^1(W_0, K_0^p)(\mathbb{C})$, its first homology $H_1(A_0(\mathbb{C}), \mathbb{Z}_{(p)})$ is a rational skew-hermitian space over $O_F \otimes \mathbb{Z}_{(p)}$ induced by λ_0 , which is of rank 1 and type Φ , and is everywhere locally similar to W_0 . Thus, by Remark 3.5.2, we obtain a map

$$\mathfrak{w}: \mathbf{T}_p^1(W_0, K_0^p)(\mathbb{C}) \rightarrow \ker^1(T_0)$$

sending $(A_0, \lambda_0, \eta_0^p) \in \mathbf{T}_p^1(W_0, K_0^p)(\mathbb{C})$ to the similarity class of $H_1(A_0(\mathbb{C}), \mathbb{Q})$.

It is known that when K_0^p is neat, $\mathbf{T}_p^1(W_0, K_0^p)$ is a scheme finite and étale over $O_{F_\Phi} \otimes \mathbb{Z}_{(p)}$. We define $\mathbf{T}_p(W_0, K_0^p)$ to be the minimal open and closed subscheme of $\mathbf{T}_p^1(W_0, K_0^p)$ containing $\mathfrak{w}^{-1}(W_0)$. The group $T_0(\mathbb{A}^{\infty, p})$ acts on $\mathbf{T}_p(W_0, K_0^p)$ via the formula

$$a.(A_0, \lambda_0, \eta_0^p) = (A_0, \lambda_0, \eta_0^p \circ a)$$

whose stabilizer is $T_0(\mathbb{Z}_{(p)})K_0^p$. In fact, $T_0(\mathbb{A}^{\infty, p})/T_0(\mathbb{Z}_{(p)})K_0^p$ is the Galois group of the Galois morphism

$$\mathbf{T}_p(W_0, K_0^p) \rightarrow \text{Spec}(O_{F_\Phi} \otimes \mathbb{Z}_{(p)}).$$

p

Definition 3.5.5. We denote by \mathfrak{T} the groupoid of $T_0(\mathbb{A}^{\infty, p})/T_0(\mathbb{Z}_{(p)})K_0^p$, that is, a category with a single object $*$ with $\text{Hom}(*, *) = T_0(\mathbb{A}^{\infty, p})/T_0(\mathbb{Z}_{(p)})K_0^p$.

Remark 3.5.6. As $\mathbf{T}_p(W_0, K_0^p)$ is object in $\text{Sch}/_{O_{F_\Phi} \otimes \mathbb{Z}_{(p)}}$ with an action by $T_0(\mathbb{A}^{\infty, p})/T_0(\mathbb{Z}_{(p)})K_0^p$, it induces a functor from \mathfrak{T} to $\text{Sch}/_{O_{F_\Phi} \otimes \mathbb{Z}_{(p)}}$, which we still denote by $\mathbf{T}_p(W_0, K_0^p)$. In what follows, we may often have another category \mathfrak{C} and will regard $\mathbf{T}_p(W_0, K_0^p)$ as a functor from $\mathfrak{C} \times \mathfrak{T}$ to $\text{Sch}/_{O_{F_\Phi} \otimes \mathbb{Z}_{(p)}}$ as the composition of the projection functor $\mathfrak{C} \times \mathfrak{T} \rightarrow \mathfrak{T}$ and the functor $\mathbf{T}_p(W_0, K_0^p): \mathfrak{T} \rightarrow \text{Sch}/_{O_{F_\Phi} \otimes \mathbb{Z}_{(p)}}$.

Notation 3.5.7. For a functor $X: \mathfrak{T} \rightarrow \text{Sch}$ and a coefficient ring L , we denote

$$H_{\mathfrak{T}}^i(X, L(j)) \subseteq H_{\text{ét}}^i(X(*), L(j)), \quad H_{\mathfrak{T}, c}^i(X, L(j)) \subseteq H_{\text{ét}, c}^i(X(*), L(j))$$

the maximal L -submodules, respectively, on which $T_0(\mathbb{A}^{\infty, p})/T_0(\mathbb{Z}_{(p)})K_0^p$ acts trivially.

Definition 3.5.8. Let κ be an algebraically closed field of characteristic p , and L a p -coprime coefficient ring. For a functor $X: \mathfrak{T} \rightarrow \text{Sch}/_{\kappa}$ such that $X(*)$ is smooth of finite type of dimension d , we define the \mathfrak{T} -trace map

$$\int_X^{\mathfrak{T}}: H_{\mathfrak{T}, c}^{2d}(X(*), L(d)) \rightarrow L$$

to be the composite map

$$H_{\mathfrak{T}, c}^{2d}(X(*), L(d)) \hookrightarrow H_c^{2d}(X(*), L(d)) \rightarrow \bigoplus_Y H_c^{2d}(Y, L(d)) \xrightarrow{\sum \text{tr}_Y} L,$$

where $\{Y\}$ is a set of representatives of \mathfrak{T} -orbits on the connected components of $X(*)$, and the second map is the natural projection. It is clear that the above composite map does not depend on the choice of $\{Y\}$.

4. UNITARY MODULI SCHEMES: SMOOTH CASE

In this section, we define and study certain smooth integral moduli scheme whose generic fiber is the product of a unitary Shimura variety and an auxiliary CM moduli. Since the materials in this section are strictly in the linear order, we will leave the summary of contents to each subsection.

We fix a special inert prime (Definition 3.3.4) \mathfrak{p} of F^+ (with the underlying rational prime p). We take the prescribed subring \mathbb{P} in Definition 3.4.2 to be $\mathbb{Z}_{(p)}$. We choose the following data

- a CM type Φ containing τ_∞ ;
- a rational skew-hermitian space W_0 over $O_F \otimes \mathbb{Z}_{(p)}$ of rank 1 and type Φ (Definition 3.5.3);
- a neat open compact subgroup $K_0^p \subseteq T_0(\mathbb{A}^{\infty,p})$;
- an isomorphism $\iota_p: \mathbb{C} \simeq \overline{\mathbb{Q}}_p$ such that $\iota_p \circ \tau_\infty: F^+ \hookrightarrow \overline{\mathbb{Q}}_p$ induces the place \mathfrak{p} of F^+ ;
- an element $\varpi \in O_{F^+}$ that is totally positive and satisfies $\text{val}_{\mathfrak{p}}(\varpi) = 1$, and $\text{val}_{\mathfrak{q}}(\varpi) = 0$ for every prime $\mathfrak{q} \neq \mathfrak{p}$ of F^+ above p .

We adopt Notation 3.3.6. In particular, \mathbb{F}_p^Φ contains \mathbb{F}_{p^2} . Since W_0 and K_0^p are insensitive and will never be changed in the remaining part of this section, we will not include them in all notations. However, we will keep the prime \mathfrak{p} in notations as, in later application, we need to choose different primes in a crucial step. Put $\mathbf{T}_{\mathfrak{p}} := \mathbf{T}_p(W_0, K_0^p) \otimes_{O_{F_\Phi} \otimes \mathbb{Z}_{(p)}} \mathbb{Z}_p^\Phi$.

4.1. Construction of moduli schemes. In this subsection, we construct our initial moduli schemes. We start from the datum $(V, \{\Lambda_{\mathfrak{q}}\}_{\mathfrak{q}|p})$ where

- V is a standard *indefinite* hermitian space (Definition 3.1.7) over F of rank $N \geq 1$, and
- $\Lambda_{\mathfrak{q}}$ is a self-dual $O_{F_{\mathfrak{q}}}$ -lattice in $V \otimes_F F_{\mathfrak{q}}$ for every prime \mathfrak{q} of F^+ above p .

Before defining the moduli functor, we need the following lemma to make sense of the later definition.

Lemma 4.1.1. *The field \mathbb{Q}_p^Φ contains F_Ψ with $\Psi = N\Phi - \tau_\infty + \tau_\infty^c$, which is a generalized CM type of rank N , for every $N \geq 1$.*

Proof. Take $\rho \in \text{Aut}(\mathbb{C}/\mathbb{Q}_p^\Phi) \subseteq \text{Aut}(\mathbb{C}/F)$. Then we have $\rho\Phi = \Phi$ and $\rho\tau_\infty = \tau_\infty$. Thus, we have $\rho(N\Phi - \tau_\infty + \tau_\infty^c) = N\Phi - \tau_\infty + \tau_\infty^c$ for every $N \geq 1$. The lemma follows. \square

Recall that we have the category $\text{Sch}'_{/\mathbb{Z}_p^\Phi}$ of locally Noetherian schemes over \mathbb{Z}_p^Φ , and $\text{PSch}'_{/\mathbb{Z}_p^\Phi}$ the category of presheaves on $\text{Sch}'_{/\mathbb{Z}_p^\Phi}$.

Definition 4.1.2. We define a functor

$$\begin{aligned} \mathbf{M}_{\mathfrak{p}}(V, -): \mathfrak{R}(V)^p \times \mathfrak{T} &\rightarrow \text{PSch}'_{/\mathbb{Z}_p^\Phi} \\ K^p &\mapsto \mathbf{M}_{\mathfrak{p}}(V, K^p) \end{aligned}$$

such that for every $S \in \text{Sch}'_{/\mathbb{Z}_p^\Phi}$, $\mathbf{M}_{\mathfrak{p}}(V, K^p)(S)$ is the set of equivalence classes of sextuples $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p)$ where

- $(A_0, \lambda_0, \eta_0^p)$ is an element in $\mathbf{T}_{\mathfrak{p}}(S)$;
- (A, λ) is a unitary O_F -abelian scheme of signature type $N\Phi - \tau_\infty + \tau_\infty^c$ over S (Definitions 3.4.2 and 3.4.3) such that λ is p -principal;
- η^p is a K^p -level structure, that is, for a chosen geometric point s on every connected component of S , a $\pi_1(S, s)$ -invariant K^p -orbit of isomorphisms

$$\eta^p: V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \rightarrow \text{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}}^{\lambda_0, \lambda}(\mathbf{H}_1^{\text{ét}}(A_{0s}, \mathbb{A}^{\infty,p}), \mathbf{H}_1^{\text{ét}}(A_s, \mathbb{A}^{\infty,p}))$$

of hermitian spaces over $F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} = F \otimes_{F^+} \mathbb{A}_{F^+}^{\infty,p}$. See Construction 3.4.4 (with $\square = \{\infty, p\}$) for the right-hand side.

Two sextuples $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p)$ and $(A'_0, \lambda'_0, \eta_0^{p'}; A', \lambda', \eta^{p'})$ are equivalent if there are prime-to- p O_F -linear quasi-isogenies $\varphi_0: A_0 \rightarrow A'_0$ and $\varphi: A \rightarrow A'$ such that

- φ_0 carries η_0^p to $\eta_0^{p'}$;
- there exists $c \in \mathbb{Z}_{(p)}^\times$ such that $\varphi_0^\vee \circ \lambda'_0 \circ \varphi_0 = c\lambda_0$ and $\varphi^\vee \circ \lambda' \circ \varphi = c\lambda$;
- the K^p -orbit of maps $v \mapsto \varphi_* \circ \eta^p(v) \circ (\varphi_{0*})^{-1}$ for $v \in V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ coincides with $\eta^{p'}$.

On the level of morphisms,

- a morphism $g \in K^p \backslash \mathbf{U}(V)(\mathbb{A}_F^{\infty,p})/K^{p'}$ of $\mathfrak{K}(V)^p$ maps $\mathbf{M}_p(V, K^p)(S)$ to $\mathbf{M}_p(V, K^{p'})(S)$ by changing η^p to $\eta^p \circ g$; and
- a morphism a of \mathfrak{T} acts on $\mathbf{M}_p(V, K^p)(S)$ by changing η_0^p to $\eta_0^p \circ a$.

We have apparently the forgetful morphism

$$(4.1) \quad \mathbf{M}_p(V, -) \rightarrow \mathbf{T}_p$$

in $\text{Fun}(\mathfrak{K}(V)^p \times \mathfrak{T}, \text{PSch}'_{/\mathbb{Z}_p^\Phi})$, the category of functors from $\mathfrak{K}(V)^p \times \mathfrak{T}$ to $\text{PSch}'_{/\mathbb{Z}_p^\Phi}$. Here, we regard \mathbf{T}_p as an object in $\text{Fun}(\mathfrak{K}(V)^p \times \mathfrak{T}, \text{Sch}'_{/\mathbb{Z}_p^\Phi})$ as in Remark 3.5.6. According to Notation 3.3.6, we shall denote by the base change of (4.1) to \mathbb{F}_p^Φ by $\mathbf{M}_p(V, -) \rightarrow \mathbf{T}_p$, which is a morphism in $\text{Fun}(\mathfrak{K}(V)^p \times \mathfrak{T}, \text{PSch}'_{/\mathbb{F}_p^\Phi})$.

Theorem 4.1.3. *The morphism (4.1) is represented by a quasi-projective smooth scheme over \mathbf{T}_p of relative dimension $N - 1$. Moreover, for every $K^p \in \mathfrak{K}(V)^p$, we have a canonical isomorphism for the relative tangent sheaf*

$$\mathcal{T}_{\mathbf{M}_p(V, K^p)/\mathbf{T}_p} \simeq \mathcal{H}om(\omega_{\mathcal{A}^\vee, \tau_\infty}, H_1^{\text{dR}}(\mathcal{A})_{\tau_\infty}/\omega_{\mathcal{A}^\vee, \tau_\infty})$$

where $(\mathcal{A}_0, \lambda_0, \eta_0^p; \mathcal{A}, \lambda, \eta^p)$ is the universal object over $\mathbf{M}_p(V, K^p)$. Moreover, (4.1) is projective if and only if its base change to \mathbb{Q}_p^Φ is.

Proof. This is well-known. We sketch the computation on the tangent sheaf hence the proof of the smoothness for readers' convenience. Take an object $K^p \in \mathfrak{K}(V)^p$. Since both K_0^p and K^p are neat, $\mathbf{M}_p(V, K^p)$ is an algebraic space. Thus, we have the universal object $(\mathcal{A}_0, \lambda_0, \eta_0^p; \mathcal{A}, \lambda, \eta^p)$ over $\mathbf{M}_p(V, K^p)$. By a standard argument in deformation theory, using Proposition 3.4.8, we know that the morphism $\mathbf{M}_p(V, K^p) \rightarrow \mathbf{T}_p$ is separated and smooth; and we have a canonical isomorphism for the tangent sheaf

$$\mathcal{T}_{\mathbf{M}_p(V, K^p)/\mathbf{T}_p} \simeq \mathcal{H}om(\omega_{\mathcal{A}^\vee, \tau_\infty}, H_1^{\text{dR}}(\mathcal{A})_{\tau_\infty}/\omega_{\mathcal{A}^\vee, \tau_\infty})$$

which is locally free of rank $N - 1$. Moreover, the canonical sheaf of $\mathbf{M}_p(V, K^p)$ is ample; hence $\mathbf{M}_p(V, K^p)$ is a quasi-projective scheme. The theorem is proved. \square

Let $K_{\mathfrak{q}}$ be the stabilizer of $\Lambda_{\mathfrak{q}}$ for every $\mathfrak{q} \mid p$; and put $K_p := \prod_{\mathfrak{q} \mid p} K_{\mathfrak{q}}$. As show in [RSZ, Section 3.2], there is a *canonical* “moduli interpretation” isomorphism of varieties over \mathbb{Q}_p^Φ

$$(4.2) \quad \mathbf{M}_p^\eta(V, -) \xrightarrow{\sim} \text{Sh}(V, -K_p) \times_{\text{Spec } F} \mathbf{T}_p^\eta$$

in $\text{Fun}(\mathfrak{K}(V)^p \times \mathfrak{T}, \text{Sch}_{/\mathbb{Q}_p^\Phi})_{/\mathbf{T}_p^\eta}$, where \mathfrak{T} acts on $\text{Sh}(V, -K_p) \times_{\text{Spec } F} \mathbf{T}_p^\eta$ through the second factor. See also Remark 4.1.5 below.

Lemma 4.1.4. *Let L be a p -coprime coefficient ring. The two specialization maps*

$$\begin{aligned} H_{\mathfrak{T}, c}^i(\mathbf{M}_p(V, -) \otimes_{\mathbb{Z}_p^\Phi} \overline{\mathbb{Q}}_p, L) &\rightarrow H_{\mathfrak{T}, c}^i(\overline{\mathbf{M}}_p(V, -), L), \\ H_{\mathfrak{T}}^i(\mathbf{M}_p(V, -) \otimes_{\mathbb{Z}_p^\Phi} \overline{\mathbb{Q}}_p, L) &\rightarrow H_{\mathfrak{T}}^i(\overline{\mathbf{M}}_p(V, -), L), \end{aligned}$$

are both isomorphisms. In particular, (4.2) induces isomorphisms

$$\begin{aligned} H_{\text{ét},c}^i(\text{Sh}(V, -\mathbf{K}_p)_{\overline{F}}, L) &\simeq H_{\mathfrak{X},c}^i(\overline{\mathbf{M}}_p(V, -), L), \\ H_{\text{ét}}^i(\text{Sh}(V, -\mathbf{K}_p)_{\overline{F}}, L) &\simeq H_{\mathfrak{X}}^i(\overline{\mathbf{M}}_p(V, -), L), \end{aligned}$$

in $\text{Fun}(\mathfrak{K}(V)^p, \text{Mod}(L[\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^\Phi)]))$ for every $i \in \mathbb{Z}$. Here, $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^\Phi)$ is regarded as a subgroup of $\text{Gal}(\overline{F}/F)$ under our fixed isomorphism $\iota_p: \mathbb{C} \simeq \overline{\mathbb{Q}}_p$.

Proof. Since $\mathbf{M}_p(V, -)$ is smooth over \mathbb{Z}_p^Φ , we have a canonical isomorphism $L \simeq R\Psi L$. When $\mathbf{M}_p(V, -)$ is proper, this is simply the proper base change. When $\mathbf{M}_p(V, -)$ is not proper, this follows from [LS18, Corollary 5.20]. \square

Remark 4.1.5. For readers' convenience, we describe (4.2) on complex points, which determines the isomorphism uniquely. It suffices to assign to every point

$$x = (A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p) \in \mathbf{M}_p(V, K^p)(\mathbb{C}).$$

a point in

$$\text{Sh}(V, K^p K_p)(\mathbb{C}) = \text{U}(V)(F^+) \backslash \left(V(\mathbb{C})_- / \mathbb{C}^\times \times \text{U}(V)(\mathbb{A}_{F^+}^\infty) / K^p K_p \right)$$

where $V(\mathbb{C})_- / \mathbb{C}^\times$ is the set of negative definite complex lines in $V \otimes_F \mathbb{C}$. Put

$$V_x := \text{Hom}_F(H_1(A_0(\mathbb{C}), \mathbb{Q}), H_1(A(\mathbb{C}), \mathbb{Q}))$$

equipped with a pairing in the way similar to Construction 3.4.4, which becomes a hermitian space over F of rank N . Moreover, it is standard indefinite. By the comparison between singular homology and étale homology, we have a canonical isometry of hermitian spaces

$$\rho: V_x \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \xrightarrow{\sim} \text{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}}^{\lambda_0, \lambda} (H_1^{\text{ét}}(A_0, \mathbb{A}^{\infty,p}), H_1^{\text{ét}}(A, \mathbb{A}^{\infty,p}))$$

which implies that $V_x \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \simeq V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ by the existence of the level structure η^p . On the other hand, we have a canonical decomposition

$$\text{Hom}_{O_F \otimes \mathbb{Z}_p} (H_1^{\text{ét}}(A_0, \mathbb{Z}_p), H_1^{\text{ét}}(A, \mathbb{Z}_p)) = \bigoplus_{\mathfrak{q}|p} \Lambda_{x,\mathfrak{q}}$$

of $O_F \otimes \mathbb{Z}_p$ -modules in which $\Lambda_{x,\mathfrak{q}}$ is a self-dual lattice in $V \otimes_F F_{\mathfrak{q}}$ for every prime \mathfrak{q} of F^+ above p . Thus, by the Hasse principle for hermitian spaces, this implies that hermitian spaces V_x and V are isomorphic. Choose an isometry $\eta_{\text{rat}}: V_x \rightarrow V$. Thus, we obtain an isometry

$$g^p := \eta_{\text{rat}} \circ \rho^{-1} \circ \eta^p: V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \rightarrow V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$$

as an element in $\text{U}(V)(\mathbb{A}_{F^+}^\infty)$. For every \mathfrak{q} above p , there exists an element $g_{\mathfrak{q}} \in \text{U}(V)(F_{\mathfrak{q}}^+)$ such that $g_{\mathfrak{q}} \Lambda_{\mathfrak{q}} = \eta_{\text{rat}} \Lambda_{x,\mathfrak{q}}$. Together, we obtain an element $g_x := (g^p, (g_{\mathfrak{q}})_{\mathfrak{q}|p}) \in \text{U}(V)(\mathbb{A}_{F^+}^\infty)$. Finally,

$$l_x := \{ \alpha \in \text{Hom}_F(H_1^{\text{dR}}(A_0/\mathbb{C}), H_1^{\text{dR}}(A/\mathbb{C})) \mid \alpha(\omega_{A_0^\vee, \tau_\infty}) \subseteq \omega_{A^\vee, \tau_\infty} \}$$

is a line in $V_x(\mathbb{C})$ such that $\eta_{\text{rat}}(l_x)$ is an element in $V(\mathbb{C})_- / \mathbb{C}^\times$. It is easy to check that the coset

$$\text{U}(V)(F^+)(\eta_{\text{rat}}(l_x), g_x K^p K_p)$$

does not depend on the choice of η_{rat} , hence gives rise an element in $\text{Sh}(V, K^p K_p)(\mathbb{C})$. It is clear that the action of a morphism a of \mathfrak{X} on x does not change the above coset.

4.2. Basic correspondence on special fiber. In this subsection, we construct and study the basic correspondence on the special fiber $M_{\mathfrak{p}}(V, -)$. Recall that we have chosen an element $\varpi \in O_{F^+}$ that is totally positive and satisfies $\text{val}_{\mathfrak{p}}(\varpi) = 1$, and $\text{val}_{\mathfrak{q}}(\varpi) = 0$ for every prime $\mathfrak{q} \neq \mathfrak{p}$ of F^+ above p .

Definition 4.2.1. We define a functor

$$\begin{aligned} S_{\mathfrak{p}}(V, -) : \mathfrak{K}(V)^p \times \mathfrak{T} &\rightarrow \text{PSch}'_{/\mathbb{F}_{\mathfrak{p}}} \\ K^p &\mapsto S_{\mathfrak{p}}(V, K^p) \end{aligned}$$

such that for every $S \in \text{Sch}'_{/\mathbb{F}_{\mathfrak{p}}}$, $S_{\mathfrak{p}}(V, K^p)(S)$ is the set of equivalence classes of sextuples $(A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p^*})$ where

- $(A_0, \lambda_0, \eta_0^p)$ is an element in $T_{\mathfrak{p}}(S)$;
- (A^*, λ^*) is a unitary O_F -abelian scheme of signature type $N\Phi$ over S such that $\ker \lambda^*[p^\infty]$ is trivial (resp. contained in $A^*[\mathfrak{p}]$ of rank p^2) if N is odd (resp. even);
- η^{p^*} is, for a chosen geometric point s on every connected component of S , a $\pi_1(S, s)$ -invariant K^p -orbit of isomorphisms

$$\eta^{p^*} : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p} \rightarrow \text{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}}^{\varpi \lambda_0, \lambda^*} (H_1^{\text{ét}}(A_{0s}, \mathbb{A}^{\infty, p}), H_1^{\text{ét}}(A_s^*, \mathbb{A}^{\infty, p}))$$

of hermitian spaces over $F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p} = F \otimes_{F^+} \mathbb{A}_{F^+}^{\infty, p}$. Note that here we are using $\varpi \lambda_0$ rather than λ_0 .

The equivalence relation and the action of morphisms in $\mathfrak{K}(V)^p \times \mathfrak{T}$ are defined similarly as in Definition 4.1.2.

We have apparently the forgetful morphism

$$S_{\mathfrak{p}}(V, -) \rightarrow T_{\mathfrak{p}}$$

in $\text{Fun}(\mathfrak{K}(V)^p \times \mathfrak{T}, \text{PSch}'_{/\mathbb{F}_{\mathfrak{p}}})$ which is represented by finite and étale schemes.

Now we take a point $s^* = (A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p^*}) \in S_{\mathfrak{p}}(V, K^p)(\kappa)$ where κ is a field containing $\mathbb{F}_{\mathfrak{p}}^{\Phi}$. By Remark 3.4.10, we have the (κ, σ) -linear Frobenius map

$$F : H_1^{\text{dR}}(A^*/\kappa)_{\tau_\infty} \rightarrow H_1^{\text{dR}}(A^*/\kappa)_{\sigma\tau_\infty} = H_1^{\text{dR}}(A^*/\kappa)_{\tau_\infty}.$$

We define a pairing

$$\{ , \}_{s^*} : H_1^{\text{dR}}(A^*/\kappa)_{\tau_\infty} \times H_1^{\text{dR}}(A^*/\kappa)_{\tau_\infty} \rightarrow \kappa$$

by the formula $\{x, y\}_{s^*} := \langle Fx, y \rangle_{\lambda^*, \tau_\infty}$ (Notation 3.4.7). To ease notation, we put

$$\mathcal{V}_{s^*} := H_1^{\text{dR}}(A^*/\kappa)_{\tau_\infty}.$$

Lemma 4.2.2. *The pair $(\mathcal{V}_{s^*}, \{ , \}_{s^*})$ is admissible of rank N (Definition A.1.1). In particular, the Deligne–Lusztig variety $\text{DL}_{s^*} := \text{DL}(\mathcal{V}_{s^*}, \{ , \}_{s^*}, \lceil \frac{N+1}{2} \rceil)$ (Definition A.1.2) is a geometrically irreducible projective smooth scheme in $\text{Sch}_{/\kappa}$ of dimension $\lfloor \frac{N-1}{2} \rfloor$ with a canonical isomorphism for its tangent sheaf*

$$\mathcal{T}_{\text{DL}_{s^*}/\kappa} \simeq \text{Hom}(\mathcal{H}/\mathcal{H}^\perp, (\mathcal{V}_{s^*})_{\text{DL}_{s^*}}/\mathcal{H})$$

where $\mathcal{H} \subseteq (\mathcal{V}_{s^*})_{\text{DL}_{s^*}}$ is the universal subbundle.

Proof. It follows from the construction that $\{ , \}_{s^*}$ is (κ, σ) -linear in the first variable and κ -linear in the second variable. By the signature condition Definition 4.2.1(2), the map $F : H_1^{\text{dR}}(A^*/\kappa)_{\tau_\infty} \rightarrow H_1^{\text{dR}}(A^*/\kappa)_{\tau_\infty}$ is an isomorphism, and the pairing $\langle F , \rangle_{\lambda^*, \tau_\infty}$ has kernel of rank 0 (resp. 1) if N is odd (resp. even). Thus, by Proposition A.1.3, it suffices to show that $(\mathcal{V}_{s^*}, \{ , \}_{s^*})$ is admissible.

Note that we have a canonical isomorphism $(\mathcal{V}_{s^*})_{\bar{\kappa}} = H_1^{\text{dR}}(A^*/\kappa)_{\tau_\infty} \otimes_{\kappa} \bar{\kappa} \simeq H_1^{\text{dR}}(A_{\bar{\kappa}}^*/\bar{\kappa})_{\tau_\infty}$, and that the $(\bar{\kappa}, \sigma)$ -linear Frobenius map $F : H_1^{\text{dR}}(A_{\bar{\kappa}}^*/\bar{\kappa})_{\tau_\infty} \rightarrow H_1^{\text{dR}}(A_{\bar{\kappa}}^*/\bar{\kappa})_{\tau_\infty}$ and the $(\bar{\kappa}, \sigma^{-1})$ -linear

Verschiebung map $V: H_1^{\text{dR}}(A_{\bar{\kappa}}^*/\bar{\kappa})_{\tau_\infty} \rightarrow H_1^{\text{dR}}(A_{\bar{\kappa}}^*/\bar{\kappa})_{\tau_\infty^\varepsilon}$ are both isomorphisms. Thus, we obtain a $(\bar{\kappa}, \sigma^2)$ -linear isomorphism $V^{-1}F: H_1^{\text{dR}}(A_{\bar{\kappa}}^*/\bar{\kappa})_{\tau_\infty} \rightarrow H_1^{\text{dR}}(A_{\bar{\kappa}}^*/\bar{\kappa})_{\tau_\infty}$. Denote by \mathcal{V}_0 the subset of $H_1^{\text{dR}}(A_{\bar{\kappa}}^*/\bar{\kappa})_{\tau_\infty}$ on which $V^{-1}F = \text{id}$, which is an \mathbb{F}_{p^2} -linear subspace. Since the p -divisible group $A[\mathfrak{p}^\infty]$ is supersingular, by Dieudonné's classification of crystals, the canonical map $\mathcal{V}_0 \otimes_{\mathbb{F}_{p^2}} \bar{\kappa} \rightarrow H_1^{\text{dR}}(A^*/\bar{\kappa})_{\tau_\infty} = (\mathcal{V}_{s^*})_{\bar{\kappa}}$ is an isomorphism. For $x, y \in \mathcal{V}_0$, we have

$$\{x, y\}_{s^*} = \langle Fx, y \rangle_{\lambda^*, \tau_\infty^\varepsilon} = \langle x, Vy \rangle_{\lambda^*, \tau_\infty}^\sigma = \langle x, Fy \rangle_{\lambda^*, \tau_\infty}^\sigma = -\langle Fy, x \rangle_{\lambda^*, \tau_\infty}^\sigma = -\{y, x\}_{s^*}.$$

Thus, $(\mathcal{V}_{s^*}, \{, \}_{s^*})$ is admissible. The lemma follows. \square

Definition 4.2.3. We define a functor

$$\begin{aligned} B_p(V, -): \mathfrak{K}(V)^p \times \mathfrak{T} &\rightarrow \text{PSch}'_{/\mathbb{F}_p^\Phi} \\ K^p &\mapsto B_p(V, K^p) \end{aligned}$$

such that for every $S \in \text{Sch}'_{/\mathbb{F}_p^\Phi}$, $B_p(V, K^p)(S)$ is the set of equivalence classes of decuples $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^*, \lambda^*, \eta^{p^*}; \alpha)$ where

- $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p)$ is an element of $M_p(V, K^p)(S)$;
- $(A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p^*})$ is an element of $S_p(V, K^p)(S)$;
- $\alpha: A \rightarrow A^*$ is an O_F -linear quasi- p -isogeny (Definition 3.4.5) such that
 - (a) $\ker \alpha[p^\infty]$ is contained in $A[\mathfrak{p}]$;
 - (b) we have $\varpi \cdot \lambda = \alpha^\vee \circ \lambda^* \circ \alpha$; and
 - (c) the K^p -orbit of maps $v \mapsto \alpha_* \circ \eta^p(v)$ for $v \in V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$ coincides with η^{p^*} .

Two decuples $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^*, \lambda^*, \eta^{p^*}; \alpha)$ and $(A'_0, \lambda'_0, \eta_0^{p'}; A', \lambda', \eta^{p'}; A'^*, \lambda'^*, \eta^{p'^*}; \alpha')$ are equivalent if there are prime-to- p O_F -linear quasi-isogenies $\varphi_0: A_0 \rightarrow A'_0$, $\varphi: A \rightarrow A'$, and $\varphi^*: A^* \rightarrow A'^*$ such that

- φ_0 carries η_0^p to $\eta_0^{p'}$;
- there exists $c \in \mathbb{Z}_{(p)}^\times$ such that $\varphi_0^\vee \circ \lambda'_0 \circ \varphi_0 = c\lambda_0$, $\varphi^\vee \circ \lambda' \circ \varphi = c\lambda$, and $\varphi^{*\vee} \circ \lambda'^* \circ \varphi^* = c\lambda^*$;
- the K^p -orbit of maps $v \mapsto \varphi_* \circ \eta^p(v) \circ (\varphi_{0*})^{-1}$ for $v \in V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$ coincides with $\eta^{p'}$;
- the K^p -orbit of maps $v \mapsto \varphi_*^* \circ \eta^{p^*}(v) \circ (\varphi_{0*}^*)^{-1}$ for $v \in V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$ coincides with $\eta^{p'^*}$;
- $\varphi^* \circ \alpha = \alpha' \circ \varphi$ holds.

On the level of morphisms,

- a morphism $g \in K^p \setminus U(V)(\mathbb{A}_F^{\infty, p})/K^{p'}$ of $\mathfrak{K}(V)^p$ maps $B_p(V, K^p)(S)$ to $B_p(V, K^{p'})(S)$ by changing η^p, η^{p^*} to $\eta^p \circ g, \eta^{p^*} \circ g$, respectively; and
- a morphism a of \mathfrak{T} acts on $M_p(V, K^p)(S)$ by changing η_0^p to $\eta_0^p \circ a$.

We obtain in the obvious way a correspondence

$$(4.3) \quad S_p(V, -) \xleftarrow{\pi} B_p(V, -) \xrightarrow{\iota} M_p(V, -)$$

in $\text{Fun}(\mathfrak{K}(V)^p \times \mathfrak{T}, \text{PSch}'_{/\mathbb{F}_p^\Phi})_{/T_p}$.

Definition 4.2.4 (Basic correspondence). We refer to (4.3) as the *basic correspondence*⁴ on $M_p(V, -)$, with $S_p(V, -)$ being the *source* of the basic correspondence.

Theorem 4.2.5. *In the diagram (4.3), take a point*

$$s^* = (A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p^*}) \in S_p(V, K^p)(\kappa)$$

where κ is a field containing \mathbb{F}_p^Φ . Put $B_{s^*} := \pi^{-1}(s^*)$, and denote by $(\mathcal{A}, \lambda, \eta^p; \alpha)$ the universal object over the fiber B_{s^*} .

⁴We adopt this terminology since the image of ι is in fact the basic locus of $M_p(V, -)$.

- (1) The fiber B_{s^*} is a smooth scheme over κ , with a canonical isomorphism for its tangent bundle

$$\mathcal{T}_{B_{s^*}/\kappa} \simeq \mathcal{H}om(\omega_{A^\vee, \tau_\infty}, \ker \alpha_{*, \tau_\infty} / \omega_{A^\vee, \tau_\infty}).$$

- (2) The restriction of ι to B_{s^*} is locally on B_{s^*} a closed immersion, with a canonical isomorphism for its normal bundle

$$\mathcal{N}_{\iota|_{B_{s^*}}} \simeq \mathcal{H}om(\omega_{A^\vee, \tau_\infty}, \text{im } \alpha_{*, \tau_\infty}).$$

- (3) The assignment sending a point $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^*, \lambda^*, \eta^{p^*}; \alpha) \in B_{s^*}(S)$ for every $S \in \text{Sch}'_\kappa$ to the subbundle

$$H := (\check{\alpha}_{*, \tau_\infty})^{-1} \omega_{A^\vee/S, \tau_\infty} \subseteq H_1^{\text{dR}}(A^*/S)_{\tau_\infty} = H_1^{\text{dR}}(A^*/\kappa)_{\tau_\infty} \otimes_\kappa \mathcal{O}_S = (\mathcal{V}_{s^*})_S,$$

where $\check{\alpha}: A^* \rightarrow A$ is the (unique) O_F -linear quasi- p -isogeny such that $\check{\alpha} \circ \alpha = \varpi \cdot \text{id}_A$, induces an isomorphism

$$\zeta_{s^*}: B_{s^*} \xrightarrow{\sim} \text{DL}_{s^*} = \text{DL}(\mathcal{V}_{s^*}, \{ , \}_{s^*}, \lceil \frac{N+1}{2} \rceil).$$

In particular, B_{s^*} is a geometrically irreducible projective smooth scheme in $\text{Sch}/_\kappa$ of dimension $\lfloor \frac{N-1}{2} \rfloor$ by Lemma 4.2.2. In particular, ι is of pure codimension $\lfloor \frac{N}{2} \rfloor$.

Proof. For an object $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^*, \lambda^*, \eta^{p^*}; \alpha) \in B_p(\mathbb{V}, \mathbb{K}^p)(S)$, Definition 4.2.3(a) implies that there is a (unique) O_F -linear quasi- p -isogeny $\check{\alpha}: A^* \rightarrow A$ such that $\check{\alpha} \circ \alpha = \varpi \cdot \text{id}_A$ hence $\alpha \circ \check{\alpha} = \varpi \cdot \text{id}_{A^*}$. Moreover, we have the following properties from Definition 4.2.3:

- (a') $\ker \check{\alpha}[p^\infty]$ is contained in $A^*[p]$;
- (b') we have $\varpi \cdot \lambda^* = \check{\alpha}^\vee \circ \lambda \circ \check{\alpha}$; and
- (c') the \mathbb{K}^p -orbit of maps $v \mapsto \varpi^{-1} \check{\alpha}_* \circ \eta^{*p}(v)$ for $v \in \mathbb{V} \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$ coincides with η^p .

First, we show (1). It is clear that B_{s^*} is a scheme of finite type over κ . Consider a closed immersion $S \hookrightarrow \hat{S}$ in Sch'_κ defined by an ideal sheaf \mathcal{I} satisfying $\mathcal{I}^2 = 0$. Take a point $x = (A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^*, \lambda^*, \eta^{p^*}; \alpha) \in B_{s^*}(S)$. To compute lifting of x to \hat{S} , we use the Serre–Tate and Grothendieck–Messing theories. Note that lifting α is equivalent to lifting both α and $\check{\alpha}$, satisfying (b,c) in Definition 4.2.3 and (b',c') above, respectively. Thus, by Proposition 3.4.8, to lift x to an \hat{S} -point is equivalent to lifting

- $\omega_{A^\vee/S, \tau_\infty}$ to a subbundle $\hat{\omega}_{A^\vee, \tau_\infty}$ of $H_1^{\text{cris}}(A/\hat{S})_{\tau_\infty}$ (of rank 1),
- $\omega_{A^\vee/S, \tau_\infty^c}$ to a subbundle $\hat{\omega}_{A^\vee, \tau_\infty^c}$ of $H_1^{\text{cris}}(A/\hat{S})_{\tau_\infty^c}$ (of rank $N-1$),

subject to the following requirements

- (a'') $\hat{\omega}_{A^\vee, \tau_\infty}$ and $\hat{\omega}_{A^\vee, \tau_\infty^c}$ are orthogonal under $\langle , \rangle_{\lambda, \tau_\infty}^{\text{cris}}$ (3.3); and
- (b'') $\check{\alpha}_{*, \tau_\infty^c} H_1^{\text{cris}}(A^*/\hat{S})_{\tau_\infty^c}$ is contained in $\hat{\omega}_{A^\vee, \tau_\infty^c}$.

Since $\langle , \rangle_{\lambda, \tau_\infty}^{\text{cris}}$ is a perfect pairing, $\hat{\omega}_{A^\vee, \tau_\infty}$ uniquely determines $\hat{\omega}_{A^\vee, \tau_\infty^c}$ by (a''). Moreover, by Property (b') above, we know that $\ker \alpha_{*, \tau_\infty}$ and $\text{im } \check{\alpha}_{*, \tau_\infty^c}$ are orthogonal complement to each other under $\langle , \rangle_{\lambda, \tau_\infty}^{\text{cris}}$. Thus, (b'') is equivalent to

- (c'') $\hat{\omega}_{A^\vee, \tau_\infty}$ is contained in the kernel of $\alpha_{*, \tau_\infty}: H_1^{\text{cris}}(A/\hat{S})_{\tau_\infty} \rightarrow H_1^{\text{cris}}(A^*/\hat{S})_{\tau_\infty}$.

To summarize, lifting x to an \hat{S} -point is equivalent to lifting $\omega_{A^\vee/S, \tau_\infty}$ to a subbundle $\hat{\omega}_{A^\vee, \tau_\infty}$ of $\ker \alpha_{*, \tau_\infty}$. In other words, the subset of $B_{s^*}(\hat{S})$ above x is canonically a torsor over $\text{Hom}_{\mathcal{O}_S}(\omega_{A^\vee, \tau_\infty}, (\ker \alpha_{*, \tau_\infty} / \omega_{A^\vee, \tau_\infty}) \otimes_{\mathcal{O}_S} \mathcal{I})$. Thus, (1) follows.

Next, we show (2). By Theorem 4.1.3, we have a canonical isomorphism

$$\iota_\kappa^* \mathcal{T}_{M_p(\mathbb{V}, \mathbb{K}^p)/\kappa}|_{B_{s^*}} \simeq \mathcal{H}om(\omega_{A^\vee, \tau_\infty}, H_1^{\text{dR}}(\mathcal{A})_{\tau_\infty} / \omega_{A^\vee, \tau_\infty}),$$

and the induced map $\mathcal{T}_{\mathbb{B}_{s^*}/\kappa} \rightarrow \iota_{\kappa}^* \mathcal{T}_{\mathbb{M}_p(\mathbb{V}, K^p)}|_{\mathbb{B}_{s^*}}$ is identified with the canonical map

$$\mathcal{H}om(\omega_{A^\vee, \tau_\infty}, \ker \alpha_{*, \tau_\infty} / \omega_{A^\vee, \tau_\infty}) \rightarrow \mathcal{H}om(\omega_{A^\vee, \tau_\infty}, H_1^{\text{dR}}(\mathcal{A})_{\tau_\infty} / \omega_{A^\vee, \tau_\infty}).$$

It is clearly injective, with cokernel canonically isomorphic to

$$\mathcal{H}om(\omega_{A^\vee, \tau_\infty}, \text{im } \alpha_{*, \tau_\infty}).$$

Thus, (2) follows.

Finally, we show (3). We first show that ζ_{s^*} has the correct image, namely, H is a locally free \mathcal{O}_S -module of rank $\lceil \frac{N+1}{2} \rceil$, and satisfies $(\mathbf{F}H^{(p)})^\perp \subseteq H$. Lemma 3.4.13(1,2,3) implies that H is locally free, and

$$\begin{aligned} \text{rank}_{\mathcal{O}_S}(\ker \alpha_{*, \tau_\infty}) - \text{rank}_{\mathcal{O}_S}(\ker \alpha_{*, \tau_\infty^c}) &= 1, \\ \text{rank}_{\mathcal{O}_S}(\ker \alpha_{*, \tau_\infty}) + \text{rank}_{\mathcal{O}_S}(\ker \alpha_{*, \tau_\infty^c}) &= 2\lceil \frac{N}{2} \rceil - 1. \end{aligned}$$

Thus, we have $\text{rank}_{\mathcal{O}_S}(\ker \alpha_{*, \tau_\infty}) = \lceil \frac{N}{2} \rceil$ and

$$\text{rank}_{\mathcal{O}_S}(\ker \check{\alpha}_{*, \tau_\infty}) = N - \text{rank}_{\mathcal{O}_S}(\ker \alpha_{*, \tau_\infty}) = \lceil \frac{N-1}{2} \rceil.$$

On the other hand, as $\omega_{A^\vee/S, \tau_\infty}$ has rank 1 and $\omega_{A^{\vee\vee}/S, \tau_\infty}$ has rank 0, $\omega_{A^\vee/S, \tau_\infty}$ is contained in the kernel of α_{*, τ_∞} hence in the image of $\check{\alpha}_{*, \tau_\infty}$. Together, we obtain $\text{rank}_{\mathcal{O}_S} H = \lceil \frac{N+1}{2} \rceil$. From the equalities

$$\begin{aligned} \check{\alpha}_{*, \tau_\infty^c}(\mathbf{F}H^{(p)}) &= \check{\alpha}_{*, \tau_\infty^c} \mathbf{F}A^* \left((\check{\alpha}_{*, \tau_\infty})^{-1} \omega_{A^\vee/S, \tau_\infty} \right)^{(p)} = \check{\alpha}_{*, \tau_\infty^c} \mathbf{F}A^* (\check{\alpha}_{*, \tau_\infty^c}^{(p)})^{-1} \omega_{A^{(p)\vee}/S, \tau_\infty^c} \\ &= \mathbf{F}A \check{\alpha}_{*, \tau_\infty^c}^{(p)} (\check{\alpha}_{*, \tau_\infty^c}^{(p)})^{-1} \omega_{A^{(p)\vee}/S, \tau_\infty^c} = \mathbf{F}A \omega_{A^{(p)\vee}/S, \tau_\infty^c} = 0 \end{aligned}$$

and the fact that $\mathbf{F}H^{(p)}$ and $\ker \check{\alpha}_{*, \tau_\infty^c}$ are both subbundles of $H_1^{\text{dR}}(A^*/S)_{\tau_\infty^c}$ of rank $\lceil \frac{N+1}{2} \rceil$, we know $\mathbf{F}H^{(p)} = \ker \check{\alpha}_{*, \tau_\infty^c}$. By Definition 4.2.3(b) and the definition of $\check{\alpha}$, we have

$$\langle \ker \check{\alpha}_{*, \tau_\infty^c}, \text{im } \alpha_{*, \tau_\infty} \rangle_{\lambda^*, \tau_\infty^c} = \langle \check{\alpha}_{*, \tau_\infty^c} \ker \check{\alpha}_{*, \tau_\infty^c}, H_1^{\text{dR}}(A/S)_{\tau_\infty} \rangle_{\lambda, \tau_\infty^c} = 0,$$

which implies

$$\ker \check{\alpha}_{*, \tau_\infty} = \text{im } \alpha_{*, \tau_\infty} \subseteq (\ker \check{\alpha}_{*, \tau_\infty^c})^\perp = (\mathbf{F}H^{(p)})^\perp.$$

As both sides are subbundles of $H_1^{\text{dR}}(A^*/S)_{\tau_\infty}$ of rank $\lceil \frac{N-1}{2} \rceil$, we must have $\ker \check{\alpha}_{*, \tau_\infty} = (\mathbf{F}H^{(p)})^\perp$. In particular, we have $(\mathbf{F}H^{(p)})^\perp \subseteq H$. Thus, ζ_{s^*} is defined as we claim.

Since the target of ζ_{s^*} is smooth over κ by Lemma 4.2.2, to see that ζ_{s^*} is an isomorphism, it suffices to check that for every algebraically closed field κ' containing κ

(3.1) ζ_{s^*} induces a bijection on κ' -points; and

(3.2) ζ_{s^*} induces an isomorphism on the tangent spaces at every κ' -point.

To ease notation, we may assume that $\kappa' = \kappa$ hence is perfect in particular.

For (3.1), we construct an inverse to the map $\zeta_{s^*}(\kappa)$. Take a point $y \in \text{DL}_{s^*}(\kappa)$ represented by a κ -linear subspace $H \subseteq \mathcal{Y}_{s^*} = H_1^{\text{dR}}(A^*/\kappa)_{\tau_\infty}$. We regard \mathbf{F} and \mathbf{V} as those sesquilinear maps in Remark 3.4.10. In particular, we have $(\mathbf{F}H)^\perp \subseteq H$. For every $\tau \in \Sigma_\infty$, we define a $W(\kappa)$ -submodule $\mathcal{D}_{A, \tau} \subseteq \mathcal{D}(A^*)_\tau$ as follows.

- If $\tau \notin \{\tau_\infty, \tau_\infty^c\}$, then $\mathcal{D}_{A, \tau} = \mathcal{D}(A^*)_\tau$.
- We set $\mathcal{D}_{A, \tau_\infty} := \mathbf{V}^{-1} \check{H}^c$, where \check{H}^c is the preimage of H^\perp under the reduction map $\mathcal{D}(A^*)_{\tau_\infty^c} \rightarrow \mathcal{D}(A^*)_{\tau_\infty^c} / p\mathcal{D}(A^*)_{\tau_\infty^c} = H_1^{\text{dR}}(A^*)_{\tau_\infty^c}$.
- We set $\mathcal{D}_{A, \tau_\infty^c} := \mathbf{F}H$, where \check{H} is the preimage of H under the reduction map $\mathcal{D}(A^*)_{\tau_\infty} \rightarrow \mathcal{D}(A^*)_{\tau_\infty} / p\mathcal{D}(A^*)_{\tau_\infty} = H_1^{\text{dR}}(A^*)_{\tau_\infty}$.

Finally, put $\mathcal{D}_A := \bigoplus_{\tau \in \Sigma_\infty} \mathcal{D}_{A, \tau}$ as a $W(\kappa)$ -submodule of $\mathcal{D}(A^*)$. We show that it is stable under \mathbf{F} and \mathbf{V} . It suffices to show that both \mathbf{F} and \mathbf{V} stabilize $\mathcal{D}_{A, \tau_\infty} \oplus \mathcal{D}_{A, \tau_\infty^c}$, which breaks into checking that

- $\mathbf{F}\mathcal{D}_{A,\tau_\infty} \subseteq \mathcal{D}_{A,\tau_\infty^c}$, that is, $\mathbf{F}\mathbf{V}^{-1}\tilde{H}^c \subseteq \mathbf{F}\tilde{H}$. It suffices to show that $\mathbf{V}^{-1}(H^\perp)$ (as a subspace of $\mathbf{H}_1^{\text{dR}}(A^*)_{\tau_\infty}$) is contained in H . However, $\mathbf{V}^{-1}(H^\perp) = (\mathbf{F}H)^\perp$, which is contained in H .
- $\mathbf{F}\mathcal{D}_{A,\tau_\infty^c} \subseteq \mathcal{D}_{A,\tau_\infty}$, that is, $\mathbf{F}\tilde{H} \subseteq \mathbf{V}^{-1}\tilde{H}^c$. It suffices to show $p\mathbf{F}\tilde{H} \subseteq \tilde{H}^c$, which obviously holds.
- $\mathbf{V}\mathcal{D}_{A,\tau_\infty} \subseteq \mathcal{D}_{A,\tau_\infty^c}$, that is, $\mathbf{V}\mathbf{V}^{-1}\tilde{H}^c \subseteq \mathbf{F}\tilde{H}$. It suffices to show $H^\perp \subseteq \mathbf{F}H$ as subspaces of $\mathbf{H}_1^{\text{dR}}(A^*)_{\tau_\infty^c}$, which follows from $(\mathbf{F}H)^\perp \subseteq H$.
- $\mathbf{V}\mathcal{D}_{A,\tau_\infty^c} \subseteq \mathcal{D}_{A,\tau_\infty}$, that is, $\mathbf{V}\tilde{H} \subseteq \mathbf{V}^{-1}\tilde{H}^c$. It is obvious as $\mathbf{V}^{-1}\tilde{H}^c$ contains $p\mathcal{D}(A^*)_{\tau_\infty}$.

Thus, $(\mathcal{D}_A, \mathbf{F}, \mathbf{V})$ is a Dieudonné module over $W(\kappa)$. By the Dieudonné theory, there is an O_F -abelian scheme A over κ with $\mathcal{D}(A)_\tau = \mathcal{D}_{A,\tau}$ for every $\tau \in \Sigma_\infty$, and an O_F -linear p -isogeny $\alpha: A \rightarrow A^*$ inducing the inclusion of Dieudonné modules $\mathcal{D}(A) = \mathcal{D}_A \subseteq \mathcal{D}(A^*)$. Moreover, since $p\mathcal{D}(A^*) \subseteq \mathcal{D}(A)$, we have $\ker \alpha[p^\infty] \subseteq A[\mathfrak{p}]$.

Let $\lambda: A \rightarrow A^\vee$ be the unique quasi-polarization such that $\varpi\lambda = \alpha^\vee \circ \lambda^* \circ \alpha$. We claim that λ is p -principal. It is enough to show the induced pairing

$$p^{-1} \cdot \langle \cdot, \cdot \rangle_{\lambda, \tau_\infty}: \mathcal{D}(A)_{\tau_\infty} \times \mathcal{D}(A)_{\tau_\infty^c} \rightarrow W(\kappa)$$

(Notation 3.4.12) is non-degenerate. Since \tilde{H} is $W(\kappa)$ -dual to $p^{-1}\tilde{H}^c$, hence $\mathcal{D}(A)_{\tau_\infty^c} = \mathbf{F}\tilde{H}$ is dual to $\mathbf{V}^{-1}(p^{-1}\tilde{H}^c) = p^{-1}\mathbf{V}^{-1}\tilde{H}^c = p^{-1}\mathcal{D}(A)_{\tau_\infty}$, the above pairing is non-degenerate.

It is an easy consequence of Lemma 3.4.13(2,3) that the O_F -abelian scheme A has signature type $N\Phi - \tau_\infty + \tau_\infty^c$. Finally, let η^p be the unique K^p -level structure such that Definition 4.2.3(c) is satisfied. Putting together, we obtain a point $x = (A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^*, \lambda^*, \eta^{p*}; \alpha) \in \mathbf{B}_{s^*}(\kappa)$ such that $\zeta_{s^*}(x) = y$. It is easy to see that such assignment gives rise to an inverse of $\zeta_{s^*}(\kappa)$; hence (3.1) follows immediately.

For (3.2), let \mathcal{T}_x and \mathcal{T}_y be the tangent spaces at x and y as in (3.1), respectively. By (1) and Lemma 4.2.2, we have canonical isomorphisms

$$\mathcal{T}_x \simeq \text{Hom}_\kappa(\omega_{A^\vee, \tau_\infty}, \ker \alpha_{*, \tau_\infty} / \omega_{A^\vee, \tau_\infty}), \quad \mathcal{T}_y \simeq \text{Hom}_\kappa(H / (\mathbf{F}H)^\perp, \mathbf{H}_1^{\text{dR}}(A^*)_{\tau_\infty} / H).$$

Moreover, by the definition of ζ_{s^*} , the map $(\zeta_{s^*})_*: \mathcal{T}_x \rightarrow \mathcal{T}_y$ is induced by the following two maps

$$\begin{aligned} H / (\mathbf{F}H)^\perp &= (\check{\alpha}_{*, \tau_\infty})^{-1} \omega_{A^\vee, \tau_\infty} / \ker \check{\alpha}_{*, \tau_\infty} \xrightarrow{\check{\alpha}_{*, \tau_\infty}} \omega_{A^\vee, \tau_\infty}, \\ \mathbf{H}_1^{\text{dR}}(A^*)_{\tau_\infty} / H &= \mathbf{H}_1^{\text{dR}}(A^*)_{\tau_\infty} / (\check{\alpha}_{*, \tau_\infty})^{-1} \omega_{A^\vee, \tau_\infty} \xrightarrow{\check{\alpha}_{*, \tau_\infty}} \ker \alpha_{*, \tau_\infty} / \omega_{A^\vee, \tau_\infty}, \end{aligned}$$

both being isomorphisms. Thus, (3.2) hence (3) follow. \square

Remark 4.2.6. In Theorem 4.2.5, when K^p is sufficiently small, the restriction of ι to \mathbf{B}_{s^*} is a closed immersion for every point $s^* \in \mathbf{S}_p(\mathbf{V}, K^p)(\kappa)$ and every field κ containing \mathbb{F}_p^Φ .

4.3. Source of basic correspondence and Tate cycles. In this subsection, we study the source $\mathbf{S}_p(\mathbf{V}, \mathbf{-})$ of the basic correspondence. We will describe the set $\mathbf{S}_p(\mathbf{V}, \mathbf{-})(\overline{\mathbb{F}}_p)$ in terms of certain Shimura set and study its Galois action. Such a description is not canonical, which depends on the choice of a definite uniformization datum defined as follows.

Definition 4.3.1. We define a *definite uniformization datum for \mathbf{V} (at \mathfrak{p})* to be a collection of $(\mathbf{V}^*, \mathbf{i}, \{\Lambda_{\mathfrak{q}}^*\}_{\mathfrak{q}|p})$ where

- \mathbf{V}^* is a standard definite hermitian space over F of rank N ;
- $\mathbf{i}: \mathbf{V} \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p} \rightarrow \mathbf{V}^* \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$ is an isometry;
- for every prime \mathfrak{q} of F^+ above p other than \mathfrak{p} , $\Lambda_{\mathfrak{q}}^*$ is a self-dual $O_{F_{\mathfrak{q}}}$ -lattice in $\mathbf{V}^* \otimes_F F_{\mathfrak{q}}$; and
- $\Lambda_{\mathfrak{p}}^*$ is an $O_{F_{\mathfrak{p}}}$ -lattice in $\mathbf{V}^* \otimes_F F_{\mathfrak{p}}$ satisfying $p\Lambda_{\mathfrak{p}}^* \subseteq (\Lambda_{\mathfrak{p}}^*)^\vee$ such that $(\Lambda_{\mathfrak{p}}^*)^\vee / p\Lambda_{\mathfrak{p}}^*$ has length 0 (resp. 1) if N is odd (resp. even).

By the Hasse principle for hermitian spaces, there exists a definite uniformization datum for which we fix one. Let K_q^* be the stabilizer of Λ_q^* for every q over p ; and put $K_p^* := \prod_{q|p} K_q^*$. The isometry \mathbf{i} induces an equivalence of categories $\mathbf{i}: \mathfrak{K}(V)^p \xrightarrow{\sim} \mathfrak{K}(V^*)^p$.

Construction 4.3.2. We now construct a *uniformization map*, denoted by the Greek letter *upsilon*

$$(4.4) \quad v: S_p(V, \mathbf{-})(\overline{\mathbb{F}}_p) \rightarrow \mathrm{Sh}(V^*, (\mathbf{i}\mathbf{-})K_p^*) \times T_p(\overline{\mathbb{F}}_p)$$

in $\mathrm{Fun}(\mathfrak{K}(V)^p \times \mathfrak{T}, \mathrm{Set})_{/T_p(\overline{\mathbb{F}}_p)}$, which turns out to be an isomorphism.

Take a point $s^* = (A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p^*}) \in S_p(V, K^p)(\overline{\mathbb{F}}_p)$. Let

$$V_{s^*} := \mathrm{Hom}_{O_F}(A_0, A^*) \otimes \mathbb{Q}$$

be the space of O_F -linear quasi-homomorphisms. We equip with V_{s^*} a pairing

$$(x, y) = \varpi^{-1} \cdot \lambda_0^{-1} \circ y^\vee \circ \lambda^* \circ x \in \mathrm{End}_{\overline{\mathbb{F}}_p}(A_0) \otimes \mathbb{Q} = F,$$

which becomes a hermitian space over F . Note that we have an extra factor ϖ^{-1} in the above pairing. Moreover, for every prime q of F^+ above p , put

$$\Lambda_{s^*, q} := \mathrm{Hom}_{O_F}(A_0[\mathfrak{q}^\infty], A^*[\mathfrak{q}^\infty])$$

which is an O_{F_q} -lattice in $(V_{s^*})_q$ since A^* is isogenous to A_0^N .

Now we construct v , whose process is very similar to Remark 4.1.5. Note that we have an isometry

$$\rho: V_{s^*} \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p} \xrightarrow{\sim} \mathrm{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}}^{\varpi \lambda_0, \lambda^*}(\mathrm{H}_1^{\mathrm{ét}}(A_0, \mathbb{A}^{\infty, p}), \mathrm{H}_1^{\mathrm{ét}}(A^*, \mathbb{A}^{\infty, p})).$$

By Lemma 4.3.3 below, we can choose an isometry $\eta_{\mathrm{rat}}: V_{s^*} \rightarrow V^*$. Thus, we obtain an isometry

$$g^p := \eta_{\mathrm{rat}} \circ \rho^{-1} \circ \eta^{p^*} \circ \mathbf{i}^{-1}: V^* \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p} \rightarrow V^* \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$$

as an element in $U(V^*)(\mathbb{A}_{F^+}^{\infty, p})$. By Lemma 4.3.3(1,2), for every q above p , there exists an element $g_q \in U(V^*)(F_q^+)$ such that $g_q \Lambda_q^* = \eta_{\mathrm{rat}} \Lambda_{s^*, q}$. Together, we obtain an element $g_{s^*} := (g^p, (g_q)_{q|p}) \in U(V^*)(\mathbb{A}_{F^+}^{\infty, p})$ such that the double coset $U(V^*)(F)g(\mathbf{i}K^p)K_p^*$ depends only on the point s^* . Thus, it allows us to define

$$v(s^*) := \left(U(V^*)(F)g_{s^*}(\mathbf{i}K^p)K_p^*, (A_0, \lambda_0, \eta_0^p) \right) \in \mathrm{Sh}(V^*, (\mathbf{i}K^p)K_p^*) \times T_p(\overline{\mathbb{F}}_p).$$

Lemma 4.3.3. *The hermitian spaces V_{s^*} and V^* are isomorphic. Moreover,*

- (1) *for every prime q of F^+ above p other than \mathfrak{p} , the lattice $\Lambda_{s^*, q}$ is self-dual;*
- (2) *the lattice $\Lambda_{s^*, \mathfrak{p}}$ satisfies $p\Lambda_{s^*, \mathfrak{p}} \subseteq (\Lambda_{s^*, \mathfrak{p}})^\vee$ such that $(\Lambda_{s^*, \mathfrak{p}})^\vee / p\Lambda_{s^*, \mathfrak{p}}$ has length 0 (resp. 1) if N is odd (resp. even).*

Proof. We first prove (1) and (2).

For (1), note that $A^*[\mathfrak{q}^\infty]$ is isomorphic to $(A_0[\mathfrak{q}^\infty])^N$, equipped with the polarization $\lambda^*[\mathfrak{q}^\infty]$ that is principal. Thus, $\Lambda_{s^*, q}$ is self-dual as $\lambda_0[\mathfrak{q}^\infty]$ is principal and $\mathrm{val}_q(\varpi) = 0$.

For (2), note that $A^*[\mathfrak{p}^\infty]$ is isomorphic to $(A_0[\mathfrak{p}^\infty])^N$, equipped with the polarization $\lambda^*[\mathfrak{p}^\infty]$ satisfying such that $\ker \lambda^*[\mathfrak{p}^\infty]$ is trivial (resp. contained in $A^*[\mathfrak{p}]$ of rank p^2) if N is odd (resp. even). Thus, the statement follows as $\lambda_0[\mathfrak{p}^\infty]$ is principal and $\mathrm{val}_p(\varpi) = 1$.

Now to prove the main statement, it suffices to show that

- (i) V_{s^*} is totally positive definite; and
- (ii) the hermitian spaces $V_{s^*} \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$ and $V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$ are isomorphic.

For (i), it follows from the same argument in [KR14, Lemma 2.7].

For (ii), we have a map

$$V_{s^*} \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \rightarrow \mathrm{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}}^{\varpi \lambda_0, \lambda^*} (H_1^{\acute{e}t}(A_0, \mathbb{A}^{\infty,p}), H_1^{\acute{e}t}(A^*, \mathbb{A}^{\infty,p}))$$

of hermitian spaces, which is injective. As both sides have rank N and the right-hand side is isomorphic to $V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$, (ii) follows. \square

Proposition 4.3.4. *The uniformization map v (4.4) is an isomorphism. Moreover, the induced action of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^{\Phi})$ on the target of v factors through the projection map*

$$\mathrm{Sh}(V^*, (\mathbf{i}-)K_p^*) \times T_p(\overline{\mathbb{F}}_p) \rightarrow T_p(\overline{\mathbb{F}}_p).$$

Proof. We first show that v is an isomorphism. Take a point $t = (A_0, \lambda_0, \eta_0^p) \in T_p(\overline{\mathbb{F}}_p)$. It suffices to show that, for every $K^p \in \mathfrak{K}(V)^p$, the restriction

$$v: S_p(V, K^p)(\overline{\mathbb{F}}_p)/t \rightarrow \mathrm{Sh}(V^*, (\mathbf{i}K^p)K_p^*)$$

to the fiber over t is an isomorphism. The injectivity follows directly from the definition. For the surjectivity, it suffices to show that for every $g \in U(V^*)(\mathbb{A}_{F^+}^{\infty,p})$, there is an object $s^* = (A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p^*}) \in S_p(V, K^p)(\overline{\mathbb{F}}_p)/t$ whose image under v is the image of g in $\mathrm{Sh}(V^*, (\mathbf{i}K^p)K_p^*)$. To construct s^* , we take an O_F -lattice Λ^* in V^* satisfying $O_F \otimes_F F_p = \Lambda_p^*$. Put $A^* := A_0 \otimes_{O_F} \Lambda^*$, which is equipped with a unique quasi-polarization λ^* such that the canonical isomorphism

$$V^* \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \simeq \mathrm{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}} (H_1^{\acute{e}t}(A_0, \mathbb{A}^{\infty,p}), H_1^{\acute{e}t}(A^*, \mathbb{A}^{\infty,p}))$$

of $F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ -modules is an isometry of hermitian spaces. We let η^{p^*} be the map

$$V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \xrightarrow{g \circ \mathbf{i}} V^* \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} = \mathrm{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}}^{\varpi \lambda_0, \lambda^*} (H_1^{\acute{e}t}(A_0, \mathbb{A}^{\infty,p}), H_1^{\acute{e}t}(A^*, \mathbb{A}^{\infty,p})).$$

Then $v(s^*) = g$ in $\mathrm{Sh}(V^*, (\mathbf{i}K^p)K_p^*)$. Thus, v is an isomorphism.

Since v is an isomorphism, the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^{\Phi})$ acts on the target of v . We show that it acts trivially on the first factor of the target of v . Take an element $\varsigma \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^{\Phi})$ and a point $s^* = (A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p^*}) \in S_p(V, K^p)(\overline{\mathbb{F}}_p)$. Then ςs^* is simply represented by $(A_0^{\varsigma}, \lambda_0^{\varsigma}, \eta_0^{p^{\varsigma}}; A^{*\varsigma}, \lambda^{*\varsigma}, \eta^{p^{*\varsigma}})$, the ς -twist of the previous object. We then have a canonical isomorphism

$$V_{\varsigma s^*} = \mathrm{Hom}_{O_F}(A_0^{\varsigma}, A^{*\varsigma}) \otimes \mathbb{Q} \simeq \mathrm{Hom}_{O_F}(A_0, A^*) \otimes \mathbb{Q} = V_{s^*}$$

of hermitian spaces. Unraveling the definition, we see that $g_{s^*} = g_{\varsigma s^*}$. Thus, we have

$$v(\varsigma s^*) := \left(U(V^*)(F)g_{s^*}(\mathbf{i}K^p)K_p^*, (A_0^{\varsigma}, \lambda_0^{\varsigma}, \eta_0^{p^{\varsigma}}) \right).$$

The proposition follows. \square

Next, we define an action of the Hecke algebra $\mathbb{Z}[K_p^* \backslash U(V^*)(F_p^+)/K_p^*]$ on $S_p(V, \mathbf{-})$ via finite étale correspondences, that is compatible with the uniformization map (4.4).

Construction 4.3.5. For every element $g \in K_p^* \backslash U(V^*)(F_p^+)/K_p^*$, we define a functor

$$\begin{aligned} S_p(V, \mathbf{-})_g: \mathfrak{K}(V)^p \times \mathfrak{T} &\rightarrow \mathrm{PSch}'_{/\mathbb{F}_p^{\Phi}} \\ K^p &\mapsto S_p(V, K^p)_g \end{aligned}$$

such that for every $S \in \mathrm{Sch}'_{/\mathbb{F}_p^{\Phi}}$, $S_p(V, K^p)_g(S)$ is the set of equivalence classes of decuples $(A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p^*}; A_g^*, \lambda_g^*, \eta_g^{p^*}; \phi^*)$ where

- $(A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p^*})$ and $(A_0, \lambda_0, \eta_0^p; A_g^*, \lambda_g^*, \eta_g^{p^*})$ are both elements in $S_p(V, K^p)(S)$; and
- $\phi^*: A^* \rightarrow A_g^*$ is an O_F -linear quasi-isogeny such that
 - (a) $\phi^{*\vee} \circ \lambda_g^* \circ \phi^* = \lambda^*$;

- (b) $\phi^*[\mathfrak{p}^\infty]: A^*[\mathfrak{p}^\infty] \rightarrow A_g^*[\mathfrak{p}^\infty]$ is a quasi-isogeny of height zero under which the two lattices $\text{Hom}_{O_F}(A_{0s}[\mathfrak{p}^\infty], A_s^*[\mathfrak{p}^\infty])$ and $\text{Hom}_{O_F}(A_{0s}[\mathfrak{p}^\infty], A_{gs}^*[\mathfrak{p}^\infty])$ are at the relative position determined by g for every geometric point s of S ;
- (c) $\phi^*[\mathfrak{q}^\infty]$ is an isomorphism for every prime \mathfrak{q} of F^+ above p that is not \mathfrak{p} ; and
- (d) the K^p -orbit of maps $v \mapsto \phi_*^* \circ \eta^{p*}(v)$ for $v \in V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$ coincides with η_g^{p*} .

The equivalence relation and the action of morphisms in $\mathfrak{K}(V)^p \times \mathfrak{T}$ are defined similarly as in Definition 4.2.3. Then we construct the *Hecke correspondence* (of g) to be the morphism

$$(4.5) \quad \text{Hk}_g: \text{S}_p(V, -)_g \rightarrow \text{S}_p(V, -) \times \text{S}_p(V, -)$$

in $\text{Fun}(\mathfrak{K}(V)^p \times \mathfrak{T}, \text{PSch}'_{/\mathbb{F}_p})_{/\mathbb{T}_p}$ induced by the assignment

$$(A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p*}; A_g^*, \lambda_g^*, \eta_g^{p*}; \phi^*) \mapsto ((A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p*}), (A_0, \lambda_0, \eta_0^p; A_g^*, \lambda_g^*, \eta_g^{p*})).$$

Here, the product in (4.5) is also taken in the category $\text{Fun}(\mathfrak{K}(V)^p \times \mathfrak{T}, \text{PSch}'_{/\mathbb{F}_p})_{/\mathbb{T}_p}$, that is, $\text{S}_p(V, -) \times \text{S}_p(V, -)$ is a functor sending K^p to $\text{S}_p(V, K^p) \times_{\mathbb{T}_p} \text{S}_p(V, K^p)$ on which \mathfrak{T} acts diagonally.

Proposition 4.3.6. *For every $g \in K_p^* \backslash \text{U}(V^*)(F_p^+)/K_p^*$, we have*

- (1) *The morphism Hk_g (4.5) is finite étale; in particular, it is a morphism in $\text{Fun}(\mathfrak{K}(V)^p \times \mathfrak{T}, \text{Sch}_{/\mathbb{F}_p})_{/\mathbb{T}_p}$.*
- (2) *The uniformization map v (4.4) lifts uniquely to an isomorphism making the diagram*

$$\begin{array}{ccc} \text{S}_p(V, -)_g(\overline{\mathbb{F}}_p) & \xrightarrow{v} & \text{Sh}(V^*, (\mathbf{i}-)(gK_p^*g^{-1} \cap K_p^*)) \times \mathbb{T}_p(\overline{\mathbb{F}}_p) \\ \text{Hk}_g(\overline{\mathbb{F}}_p) \downarrow & & \downarrow \\ \text{S}_p(V, -)(\overline{\mathbb{F}}_p) \times_{\mathbb{T}_p(\overline{\mathbb{F}}_p)} \text{S}_p(V, -)(\overline{\mathbb{F}}_p) & \xrightarrow{v \times v} & (\text{Sh}(V^*, (\mathbf{i}-)K_p^*) \times \text{Sh}(V^*, (\mathbf{i}-)K_p^*)) \times \mathbb{T}_p(\overline{\mathbb{F}}_p) \end{array}$$

in $\text{Fun}(\mathfrak{K}(V)^p \times \mathfrak{T}, \text{Set})_{/\mathbb{T}_p(\overline{\mathbb{F}}_p)}$ commutative, where the right vertical map is induced by the set-theoretical Hecke correspondence of g .

Proof. For (1), it suffices to consider those $K^p \in \mathfrak{K}(V)^p$ that are sufficiently small. Then the morphism $\text{Hk}_g: \text{S}_p(V, K^p)_g \rightarrow \text{S}_p(V, K^p) \times_{\mathbb{T}_p} \text{S}_p(V, K^p)$ is closed, hence represented by a finite étale scheme. Part (2) follows directly from the definition. \square

Remark 4.3.7. In fact, the proof of Proposition 4.3.6(1) together with Proposition 4.3.4 imply that Hk_g is a local isomorphism.

Remark 4.3.8. Note that since K_p^* is a special maximal open compact subgroup of $\text{U}(V^*)(F_p^+)$, the algebra $\mathbb{Z}[K_p^* \backslash \text{U}(V^*)(F_p^+)/K_p^*]$ is commutative. Moreover, when N is odd, $\Lambda_{s^*, p}$ is a self-dual lattice under the pairing $\varpi \cdot (\ ,)_{V^*}$; hence $\mathbb{Z}[K_p^* \backslash \text{U}(V^*)(F_p^+)/K_p^*]$ is canonically isomorphic to $\mathbb{T}_{N, p}$.

Let L be a p -coprime coefficient ring. The uniformization map (4.4) induces an isomorphism

$$L[\text{Sh}(V^*, (\mathbf{i}-)K_p^*)] \simeq H_{\mathbb{Z}}^0(\overline{\text{S}}_p(V, -), L) = H_{\mathbb{Z}}^0(\text{S}_p(V, -), L)$$

in $\text{Fun}(\mathfrak{K}(V)^p, \text{Mod}(L[K_p^* \backslash \text{U}(V^* \otimes_F F_p)/K_p^*]))$ by Proposition 4.3.6. Recall from Theorem 4.2.5(3) that the morphism ι in (4.3) is of pure codimension $\lfloor \frac{N}{2} \rfloor$.

Construction 4.3.9. Put $r := \lfloor \frac{N}{2} \rfloor \geq 0$. We construct a pair of maps

$$\left\{ \begin{array}{l} \text{inc}_!^*: L[\text{Sh}(V^*, (\mathbf{i}-)K_p^*)] \xrightarrow{\sim} H_{\mathbb{Z}}^0(\text{S}_p(V, -), L) \\ \quad \xrightarrow{\pi^*} H_{\mathbb{Z}}^0(\text{B}_p(V, -), L) \xrightarrow{\iota} H_{\mathbb{Z}}^{2r}(\text{M}_p(V, -), L(r)), \\ \text{inc}_*^*: H_{\mathbb{Z}}^{2(N-r-1)}(\text{M}_p(V, -), L(N-r-1)) \xrightarrow{\iota^*} H_{\mathbb{Z}}^{2(N-r-1)}(\text{B}_p(V, -), L(N-r-1)) \\ \quad \xrightarrow{\pi_!} H_{\mathbb{Z}}^0(\text{S}_p(V, -), L) \xrightarrow{\sim} L[\text{Sh}(V^*, (\mathbf{i}-)K_p^*)], \end{array} \right.$$

in $\text{Fun}(\mathfrak{K}(V)^p, \text{Mod}(L))$.

Theorem 4.3.10. *Suppose $N = 2r + 1$ odd with $r \geq 0$. Then the composite map $\text{inc}_\star^* \circ \text{inc}_\dagger^*$ is equal to the Hecke operator*

$$\mathbb{T}_{N,p}^\star := \sum_{\delta=0}^r \mathbf{d}_{r-\delta,p} \cdot \mathbb{T}_{N,p;\delta} \in \mathbb{T}_{N,p}$$

in which the numbers $\mathbf{d}_{r-\delta,p}$ are introduced in Notation 1.3.2, and the Hecke operators $\mathbb{T}_{N,p;\delta}$ are introduced in Notation B.2.1 (as $\mathbb{T}_{N;\delta}^\circ$).

Note that by Remark 4.3.8, $L[\text{Sh}(V^\star, (\mathbf{i}-)K_p^\star)]$ is a $\mathbb{T}_{N,p}$ -module when N is odd.

Proof. This is [XZ, Theorem 9.3.5]. □

4.4. Functoriality under special morphisms. In this subsection, we study the behavior of various moduli schemes under the special morphisms, which is closely related to the Rankin–Selberg motives for $\text{GL}_n \times \text{GL}_{n+1}$.

We start from the datum $(V_n, \{\Lambda_{n,q}\}_{q|p})$ as in the beginning of Subsection 4.1, but with V_n of rank $n \geq 1$. We then have the induced datum

$$(V_{n+1}, \{\Lambda_{n+1,q}\}_{q|p}) := ((V_n)_\# , \{(\Lambda_{n,q})_\#\}_{q|p})$$

of rank $n + 1$ by Definition 3.1.7. For $N \in \{n, n + 1\}$, we let $K_{N,q}$ be the stabilizer of $\Lambda_{N,q}$, and put $K_{N,p} := \prod_{q|p} K_{N,q}$. Recall the category $\mathfrak{K}(V_n)_{\text{sp}}^p$ and functors $\mathbf{-}_b, \mathbf{-}_\#$ from Definition 3.1.11. To unify notation, we put $\mathbf{-}_n := \mathbf{-}_b$ and $\mathbf{-}_{n+1} := \mathbf{-}_\#$. There are five stages of functoriality we will consider.

The first stage concerns Shimura varieties. The canonical inclusions

$$V_n \hookrightarrow V_{n+1}, \quad \{\Lambda_{n,q} \hookrightarrow \Lambda_{n+1,q}\}_{q|p}$$

induce a morphism

$$(4.6) \quad \text{sh}_\dagger: \text{Sh}(V_n, \mathbf{-}_n K_{n,p}) \rightarrow \text{Sh}(V_{n+1}, \mathbf{-}_{n+1} K_{n+1,p})$$

in $\text{Fun}(\mathfrak{K}(V_n)_{\text{sp}}^p, \text{Sch}/_F)$, known as the *special morphism*.

For the second stage of functoriality, we have a morphism

$$(4.7) \quad \mathbf{m}_\dagger: \mathbf{M}_p(V_n, \mathbf{-}_n) \rightarrow \mathbf{M}_p(V_{n+1}, \mathbf{-}_{n+1})$$

in $\text{Fun}(\mathfrak{K}(V_n)_{\text{sp}}^p \times \mathfrak{T}, \text{Sch}/_{\mathbb{Z}_p^\Phi})/\mathbb{T}_p$ sending an object $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p) \in \mathbf{M}_p(V_n, K_n^p)(S)$ to the object $(A_0, \lambda_0, \eta_0^p; A \times A_0, \lambda \times \lambda_0, \eta^p \oplus (\text{id}_{A_0})_*) \in \mathbf{M}_p(V_{n+1}, K_{n+1}^p)(S)$. We then have the following commutative diagram

$$(4.8) \quad \begin{array}{ccc} \mathbf{M}_p^\eta(V_{n+1}, \mathbf{-}_{n+1}) & \xrightarrow{(4.2)} & \text{Sh}(V_{n+1}, \mathbf{-}_{n+1} K_{n+1,p}) \times_{\text{Spec } F} \mathbf{T}_p^\eta \\ \mathbf{m}_\dagger^\eta \uparrow & & \uparrow \text{sh}_\dagger \times \text{id} \\ \mathbf{M}_p^\eta(V_n, \mathbf{-}_n) & \xrightarrow{(4.2)} & \text{Sh}(V_n, \mathbf{-}_n K_{n,p}) \times_{\text{Spec } F} \mathbf{T}_p^\eta \end{array}$$

in $\text{Fun}(\mathfrak{K}(V_n)_{\text{sp}}^p \times \mathfrak{T}, \text{Sch}/_{\mathbb{Q}_p^\Phi})/\mathbb{T}_p^\eta$.

At the third stage of functoriality, we study the basic correspondence (4.3) under the special morphisms. We will complete a commutative diagram in $\text{Fun}(\mathfrak{K}(V_n)_{\text{sp}}^p \times \mathfrak{T}, \text{Sch}_{/\mathbb{F}_p^\Phi})_{/T_p}$ as follows

$$(4.9) \quad \begin{array}{ccccc} \text{S}_p(V_{n+1}, \mathbf{-}_{n+1}) & \xleftarrow{\pi_{n+1}} & \text{B}_p(V_{n+1}, \mathbf{-}_{n+1}) & \xrightarrow{\iota_{n+1}} & \text{M}_p(V_{n+1}, \mathbf{-}_{n+1}) \\ \uparrow s_\uparrow & & \uparrow b_\uparrow & & \uparrow m_\uparrow \\ \text{S}_p(V_n, \mathbf{-})_{\text{sp}} & \xleftarrow{\pi_{\text{sp}}} & \text{B}_p(V_n, \mathbf{-})_{\text{sp}} & & \\ \downarrow s_\downarrow & \square & \downarrow b_\downarrow & & \\ \text{S}_p(V_n, \mathbf{-}_n) & \xleftarrow{\pi_n} & \text{B}_p(V_n, \mathbf{-}_n) & \xrightarrow{\iota_n} & \text{M}_p(V_n, \mathbf{-}_n) \end{array}$$

in which the lower-left square is Cartesian; and the lower (resp. upper) line is the basic correspondences on $\text{M}_p(V_n, \mathbf{-}_n)$ (resp. $\text{M}_p(V_{n+1}, \mathbf{-}_{n+1})$) as introduced in Definition 4.2.4.

Definition 4.4.1. We define a functor

$$\begin{aligned} \text{S}_p(V_n, \mathbf{-})_{\text{sp}} : \mathfrak{K}(V_n)_{\text{sp}}^p \times \mathfrak{T} &\rightarrow \text{PSch}'_{/\mathbb{F}_p^\Phi} \\ K^p &\mapsto \text{S}_p(V_n, K^p)_{\text{sp}} \end{aligned}$$

such that for every $S \in \text{Sch}'_{/\mathbb{F}_p^\Phi}$, $\text{S}_p(V_n, K^p)_{\text{sp}}(S)$ is the set of equivalence classes of decuples $(A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p*}; A_\natural^*, \lambda_\natural^*, \eta_\natural^{p*}; \delta^*)$ where

- $(A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p*})$ is an element in $\text{S}_p(V_n, K_n^p)(S)$;
- $(A_0, \lambda_0, \eta_0^p; A_\natural^*, \lambda_\natural^*, \eta_\natural^{p*})$ is an element in $\text{S}_p(V_{n+1}, K_{n+1}^p)(S)$;
- $\delta^* : A^* \times A_0 \rightarrow A_\natural^*$ is an O_F -linear quasi- p -isogeny (Definition 3.4.5) such that
 - (a) $\ker \delta^*[p^\infty]$ is contained in $(A^* \times A_0)[\mathfrak{p}]$;
 - (b) we have $\lambda^* \times \varpi \lambda_0 = \delta^{*\vee} \circ \lambda_\natural^* \circ \delta^*$; and
 - (c) the K_{n+1}^p -orbit of maps $v \mapsto \delta_*^* \circ (\eta^{p*} \oplus (\text{id}_{A_0})_*)(v)$ for $v \in V_{n+1} \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$ coincides with η_\natural^{p*} .

The equivalence relation and the action of morphisms in $\mathfrak{K}(V_n)_{\text{sp}}^p \times \mathfrak{T}$ are defined similarly as in Definition 4.2.3.

We have apparently the forgetful morphism

$$\text{S}_p(V_n, \mathbf{-})_{\text{sp}} \rightarrow T_p$$

in $\text{Fun}(\mathfrak{K}(V_n)_{\text{sp}}^p \times \mathfrak{T}, \text{PSch}'_{/\mathbb{F}_p^\Phi})$ which is represented by finite and étale schemes. By definition, we have the two forgetful morphisms

$$\begin{aligned} s_\downarrow : \text{S}_p(V_n, \mathbf{-})_{\text{sp}} &\rightarrow \text{S}_p(V_n, \mathbf{-}_n), \\ s_\uparrow : \text{S}_p(V_n, \mathbf{-})_{\text{sp}} &\rightarrow \text{S}_p(V_{n+1}, \mathbf{-}_{n+1}) \end{aligned}$$

in $\text{Fun}(\mathfrak{K}(V_n)_{\text{sp}}^p \times \mathfrak{T}, \text{Sch}_{/\mathbb{F}_p^\Phi})_{/T_p}$.

Lemma 4.4.2. *We have the following properties concerning s_\downarrow .*

- (1) *When n is odd, s_\downarrow is an isomorphism, and the morphism*

$$s_\uparrow \circ s_\downarrow^{-1} : \text{S}_p(V_n, \mathbf{-}_n) \rightarrow \text{S}_p(V_{n+1}, \mathbf{-}_{n+1})$$

is given by the assignment

$$(A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p*}) \mapsto (A_0, \lambda_0, \eta_0^p; A^* \times A_0, \lambda^* \times \varpi \lambda_0, \eta^{p*} \times (\text{id}_{A_0})_*).$$

- (2) *When n is even, s_\downarrow is finite étale of degree $p + 1$.*

Proof. Take an object K^p of $\mathfrak{K}(V_n)_{\text{sp}}^p$, and a point $x = (A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p*}) \in S_{\mathfrak{p}}(V_n, K_n^p)(\kappa)$ for some perfect field κ containing \mathbb{F}_p^Φ .

For (1), it suffices to show that the fibre $s_{\downarrow}^{-1}(x)$ consists of the single point with the extra datum $(A_{\mathfrak{h}}^*, \lambda_{\mathfrak{h}}^*, \eta_{\mathfrak{h}}^{p*}; \delta^*) = (A^* \times A_0, \lambda^* \times \varpi \lambda_0, \eta^{p*} \times \eta_0^p; \text{id})$. This follows from the fact that δ^* as in Definition 4.4.1 induces an equivalence between $(A_{\mathfrak{h}}^*, \lambda_{\mathfrak{h}}^*, \eta_{\mathfrak{h}}^{p*})$ and $(A^* \times A_0, \lambda^* \times \varpi \lambda_0, \eta^{p*} \times \eta_0^p)$.

For (2), we note first that a point in the fibre $s_{\downarrow}^{-1}(x)$ is determined by the quasi- p -isogeny δ^* , which is in turn determined, up to equivalence, by a totally isotropic (O_F/\mathfrak{p}) -subgroup of $\ker(\lambda^* \times \varpi \lambda_0)$ of order p^2 . We classify such subgroups by using Dieudonné theory. Let $\mathcal{D}(A^* \times A_0)_{\tau_{\infty}^c}^{\vee}$ be the dual lattice of $\mathcal{D}(A^* \times A_0)_{\tau_{\infty}^c}$ (Notation 3.4.12) but with respect to the quasi-polarization $\lambda^* \times \varpi \lambda_0$. The quotient $\mathscr{W}_x := \mathcal{D}(A^* \times A_0)_{\tau_{\infty}^c}^{\vee} / \mathcal{D}(A^* \times A_0)_{\tau_{\infty}^c}$ is κ -vector space of dimension 2 equipped with an induced *nondegenerate* hermitian pairing. Then the hermitian space \mathscr{W}_x is admissible in the sense of Definition A.1.1 with underlying hermitian space over \mathbb{F}_{p^2} given by $\mathscr{W}_{x,0} := \mathscr{W}_x^{\mathbb{V}^{-1}\mathbb{F}=1}$. Then $\mathscr{W}_{x,0}$ is an \mathbb{F}_{p^2} -vector space of dimension 2. By the classical Dieudonné theory for finite group schemes over κ , the set of totally isotropic (O_F/\mathfrak{p}) -subgroups of $\ker(\lambda^* \times \varpi \lambda_0)$ of order p^2 is in natural bijection with the set of isotropic \mathbb{F}_{p^2} -lines in $\mathscr{W}_{x,0}$, which has cardinality $p + 1$. \square

Definition 4.4.3. We define $B_{\mathfrak{p}}(V_n, \mathbf{-})_{\text{sp}}$ to be the fiber product indicated in the following Cartesian diagram

$$\begin{array}{ccc} B_{\mathfrak{p}}(V_n, \mathbf{-})_{\text{sp}} & \xrightarrow{\pi_{\text{sp}}} & S_{\mathfrak{p}}(V_n, \mathbf{-})_{\text{sp}} \\ \text{b}_{\downarrow} \downarrow & & \downarrow \text{s}_{\downarrow} \\ B_{\mathfrak{p}}(V_n, \mathbf{-}_n) & \xrightarrow{\pi_n} & S_{\mathfrak{p}}(V_n, \mathbf{-}_n) \end{array}$$

in $\text{Fun}(\mathfrak{K}(V_n)_{\text{sp}}^p \times \mathfrak{T}, \text{Sch}_{/\mathbb{F}_p^\Phi})_{/\mathbb{T}_p}$.

Lemma 4.4.4. *The assignment sending an object*

$$((A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^*, \lambda^*, \eta^{p*}; \alpha), (A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p*}; A_{\mathfrak{h}}^*, \lambda_{\mathfrak{h}}^*, \eta_{\mathfrak{h}}^{p*}; \delta^*))$$

of $B_{\mathfrak{p}}(V_n, K^p)_{\text{sp}}(S)$ to

$$(4.10) \quad (A_0, \lambda_0, \eta_0^p; A \times A_0, \lambda \times \lambda_0, \eta^p \oplus (\text{id}_{A_0})_*; A_{\mathfrak{h}}^*, \lambda_{\mathfrak{h}}^*, \eta_{\mathfrak{h}}^{p*}; \delta^* \circ (\alpha \times \text{id}_{A_0}))$$

defines a morphism

$$\text{b}_{\uparrow}: B_{\mathfrak{p}}(V_n, \mathbf{-})_{\text{sp}} \rightarrow B_{\mathfrak{p}}(V_{n+1}, \mathbf{-}_{n+1})$$

in $\text{Fun}(\mathfrak{K}(V_n)_{\text{sp}}^p \times \mathfrak{T}, \text{Sch}_{/\mathbb{F}_p^\Phi})_{/\mathbb{T}_p}$.

Proof. The lemma amounts to showing that (4.10) is an object of $B_{\mathfrak{p}}(V_{n+1}, K_{n+1}^p)(S)$. Put $\alpha_{\mathfrak{h}} := \delta^* \circ (\alpha \times \text{id}_{A_0}): A \times A_0 \rightarrow A_{\mathfrak{h}}^*$. The only nontrivial condition in Definition 4.2.3 to check is that $\ker \alpha_{\mathfrak{h}}[p^\infty]$ is contained in $(A \times A_0)[\mathfrak{p}]$. For this, we may assume $S = \text{Spec } \kappa$ for a perfect field κ containing \mathbb{F}_p^Φ .

Consider the following injective maps of Dieudonné modules

$$\mathcal{D}(A)_{\tau} \oplus \mathcal{D}(A_0)_{\tau} \xrightarrow{\alpha_{*,\tau} \oplus \text{id}} \mathcal{D}(A^*)_{\tau} \oplus \mathcal{D}(A_0)_{\tau} \xrightarrow{\delta_{*,\tau}^*} \mathcal{D}(A_{\mathfrak{h}}^*)_{\tau}$$

for every $\tau \in \Sigma_{\infty}$. We have the inclusion $\mathcal{D}(A_{\mathfrak{h}}^*)_{\tau} \subseteq \mathcal{D}(A^*)_{\tau}^{\vee} \oplus \varpi^{-1} \mathcal{D}(A_0)_{\tau}$ (Notation 3.4.12). Thus, it suffices to show $p \mathcal{D}(A^*)_{\tau}^{\vee} \subseteq \mathcal{D}(A)_{\tau}$ for every $\tau \in \Sigma_{\infty}$. For $\tau \notin \{\tau_{\infty}, \tau_{\infty}^c\}$, we have $\mathcal{D}(A^*)_{\tau}^{\vee} = \mathcal{D}(A)_{\tau}$. It remains to show $p \mathcal{D}(A^*)_{\tau}^{\vee} \subseteq \mathcal{D}(A)_{\tau}$ for $\tau \in \{\tau_{\infty}, \tau_{\infty}^c\}$. Recall the subspace

$H := (\check{\alpha}_{*,\tau_\infty})^{-1}\omega_{A^\vee/\kappa,\tau_\infty} \subseteq H_1^{\text{dR}}(A^*/\kappa)_{\tau_\infty}$ from Theorem 4.2.5. Under the notation in proof of Theorem 4.2.5, since $(\mathbf{F}H)^\perp \subseteq H$, we have $p\mathcal{D}(A^*)_{\tau_\infty}^\vee \subseteq \tilde{H}$ hence $p\tilde{\mathcal{D}}(A^*)_{\tau_\infty}^\vee \subseteq \tilde{H}^c$. Thus, we have

$$\begin{aligned} p\mathcal{D}(A^*)_{\tau_\infty}^\vee &= p\mathbf{V}^{-1}(\mathcal{D}(A^*)_{\tau_\infty}^\vee) \subseteq \mathbf{V}^{-1}\tilde{H}^c = \mathcal{D}(A)_{\tau_\infty}, \\ p\mathcal{D}(A^*)_{\tau_\infty}^\vee &= p\mathbf{F}(\mathcal{D}(A^*)_{\tau_\infty}^\vee) \subseteq \mathbf{F}\tilde{H} = \mathcal{D}(A)_{\tau_\infty}^c. \end{aligned}$$

The lemma follows. \square

By the above lemma, we obtain our desired diagram (4.9). Moreover, we have the following result.

Proposition 4.4.5. *When n is even, the square*

$$\begin{array}{ccc} \mathbf{B}_p(\mathbf{V}_{n+1}, \mathbf{-}_{n+1}) & \xrightarrow{\iota_{n+1}} & \mathbf{M}_p(\mathbf{V}_{n+1}, \mathbf{-}_{n+1}) \\ \mathbf{b}_\uparrow \uparrow & & \uparrow \mathbf{m}_\uparrow \\ \mathbf{B}_p(\mathbf{V}_n, \mathbf{-})_{\text{sp}} & \xrightarrow{\iota_n \circ \mathbf{b}_\downarrow} & \mathbf{M}_p(\mathbf{V}_n, \mathbf{-}) \end{array}$$

extracted from the diagram (4.9) is Cartesian.

We remark that the above proposition is not correct on the nose when n is odd and at least 3.

Proof. The square in the proposition induces a morphism

$$\iota_{\text{sp}}: \mathbf{B}_p(\mathbf{V}_n, \mathbf{-})_{\text{sp}} \rightarrow \mathbf{B}_p(\mathbf{V}_{n+1}, \mathbf{-}_{n+1}) \times_{\mathbf{M}_p(\mathbf{V}_{n+1}, \mathbf{-}_{n+1})} \mathbf{M}_p(\mathbf{V}_n, \mathbf{-}).$$

We need to prove that ι_{sp} is an isomorphism. By Theorem 4.2.5, we know that ι_{sp} is locally for the Zariski topology on the source a closed immersion, such that both the source and the target are smooth. Thus, it suffices to show that for a given algebraically closed field κ containing \mathbb{F}_p^Φ , we have that

- (1) $\iota_{\text{sp}}(\kappa)$ is an isomorphism in $\mathbf{Fun}(\mathfrak{K}(\mathbf{V}_n)_{\text{sp}}^p \times \mathfrak{T}, \mathbf{Set})$; and
- (2) for every $K^p \in \mathfrak{K}(\mathbf{V}_n)_{\text{sp}}^p$ and every $x \in \mathbf{B}_p(\mathbf{V}_n, K^p)_{\text{sp}}(\kappa)$, the induced diagram

$$(4.11) \quad \begin{array}{ccc} \mathcal{T}_{\mathbf{b}_\uparrow(x)} & \xrightarrow{\iota_{n+1*}} & \mathcal{T}_{\iota_{n+1}(\mathbf{b}_\uparrow(x))} \\ \mathbf{b}_\uparrow^* \uparrow & & \uparrow \mathbf{m}_\uparrow^* \\ \mathcal{T}_x & \xrightarrow{\iota_n^* \circ \mathbf{b}_\downarrow^*} & \mathcal{T}_{\iota(\mathbf{b}_\downarrow(x))} \end{array}$$

of tangent spaces is a Cartesian square of κ -modules.

For (1), we take an object $K^p \in \mathfrak{K}(\mathbf{V}_n)_{\text{sp}}^p$ and construct an inverse of $\iota_{\text{sp}}(\kappa)$. Take a point

$$(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A_{\mathfrak{q}}^*, \lambda_{\mathfrak{q}}^*, \eta_{\mathfrak{q}}^{p*}; \alpha_{\mathfrak{q}})$$

in the target of $\iota_{\text{sp}}(\kappa)$. Then $\alpha_{\mathfrak{q}}$ induces an inclusion

$$\mathcal{D}(A)_\tau \oplus \mathcal{D}(A_0)_\tau \subseteq \mathcal{D}(A_{\mathfrak{q}}^*)_\tau$$

of Dieudonné modules, which is an equality if $\tau \notin \{\tau_\infty, \tau_\infty^c\}$. We put

$$\mathcal{D}_{A^*} := \bigoplus_{\tau \in \Sigma_\infty} \mathcal{D}_{A^*, \tau}$$

where $\mathcal{D}_{A^*, \tau} = \mathcal{D}(A)_\tau$ for $\tau \notin \{\tau_\infty, \tau_\infty^c\}$ and $\mathcal{D}_{A^*, \tau} = \mathcal{D}(A_{\mathfrak{q}}^*)_\tau \cap p^{-1}\mathcal{D}(A)_\tau$ for $\tau \in \{\tau_\infty, \tau_\infty^c\}$. Then \mathcal{D}_{A^*} is a Dieudonné module containing $\mathcal{D}(A)$. By the Dieudonné theory, there is an O_F -abelian

scheme A^* over κ with $\mathcal{D}(A^*)_\tau = \mathcal{D}_{A^*,\tau}$ for every $\tau \in \Sigma_\infty$, and an O_F -linear isogeny $\alpha: A \rightarrow A^*$ inducing the inclusion of Dieudonné modules $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$. We factor $\alpha_{\mathfrak{q}}$ as

$$A \times A_0 \xrightarrow{\alpha \times \text{id}_{A_0}} A^* \times A_0 \xrightarrow{\delta^*} A_{\mathfrak{q}}^*.$$

It is clear that there is a unique quasi-polarization λ^* of A^* such that $\lambda^* \times \varpi \lambda_0 = \delta^{*\vee} \circ \lambda_{\mathfrak{q}}^* \circ \delta^*$. Let η^{p^*} be the K_n^p -level structure induced from η^p under α . We claim that the datum

$$((A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^*, \lambda^*, \eta^{p^*}; \alpha), (A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p^*}; A_{\mathfrak{q}}^*, \lambda_{\mathfrak{q}}^*, \eta_{\mathfrak{q}}^{p^*}; \delta^*))$$

gives rise to an element in $B_p(V_n, K^p)_{\text{sp}}(\kappa)$. It suffices to show that $(A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p^*})$ is an element in $S_p(V_n, K_n^p)(\kappa)$. Moreover precisely, we need to show that

- (1.1) the O_F -abelian scheme A^* has signature type $n\Phi$; and
- (1.2) $\ker \lambda^*[p^\infty]$ is contained in $A^*[p]$ of degree p^2 .

To prove these, we add two auxiliary properties

- (1.3) the composite map $\mathcal{D}(A_{\mathfrak{q}}^*)_\tau \subseteq p^{-1}\mathcal{D}(A)_\tau \oplus p^{-1}\mathcal{D}(A_0)_\tau \rightarrow p^{-1}\mathcal{D}(A_0)_\tau$ is surjective for $\tau \in \{\tau_\infty, \tau_\infty^c\}$; and
- (1.4) the cokernel of the inclusion $\mathcal{D}(A^*)_\tau \oplus \mathcal{D}(A_0)_\tau \subseteq \mathcal{D}(A_{\mathfrak{q}}^*)_\tau$ is isomorphic to κ for $\tau \in \{\tau_\infty, \tau_\infty^c\}$.

For (1.3), if not surjective, then we have $\mathcal{D}(A_{\mathfrak{q}}^*)_\tau \subseteq p^{-1}\mathcal{D}(A)_\tau \oplus \mathcal{D}(A_0)_\tau$ for both $\tau \in \{\tau_\infty, \tau_\infty^c\}$. As $\varpi \lambda \times \varpi \lambda_0 = \alpha_{\mathfrak{q}}^\vee \circ \lambda_{\mathfrak{q}}^* \circ \alpha_{\mathfrak{q}}$, this contradicts with the fact that $\lambda_{\mathfrak{q}}^*$ is p -principal.

For (1.4), it follows (1.3) and the fact that the kernel of $\mathcal{D}(A_{\mathfrak{q}}^*)_\tau \rightarrow p^{-1}\mathcal{D}(A_0)_\tau$ is $\mathcal{D}(A^*)_\tau$ for $\tau \in \{\tau_\infty, \tau_\infty^c\}$.

For (1.1), it amounts to showing that $F: \mathcal{D}(A^*)_\tau \rightarrow \mathcal{D}(A^*)_{\tau^c}$ is an isomorphism for every $\tau \in \Phi$. This is obvious for $\tau \neq \tau_\infty$. When $\tau = \tau_\infty$, this follows from (iv) and the fact that both $F: \mathcal{D}(A_{\mathfrak{q}}^*)_\tau \rightarrow \mathcal{D}(A_{\mathfrak{q}}^*)_{\tau^c}$ and $F: \mathcal{D}(A_0)_\tau \rightarrow \mathcal{D}(A_0)_{\tau^c}$ are isomorphisms.

For (1.2), it follows from (1.4) and the fact that $\lambda_{\mathfrak{q}}^*$ is p -principal.

Thus, (1) is proved.

For (2), the diagram (4.11) is identified with

$$\begin{array}{ccc} \text{Hom}_\kappa(\omega_{A^\vee, \tau_\infty}, \ker \alpha_{\mathfrak{q}^*, \tau_\infty} / \omega_{A^\vee, \tau_\infty}) & \longrightarrow & \text{Hom}_\kappa(\omega_{A^\vee, \tau_\infty}, H_1^{\text{dR}}(A \times A_0)_{\tau_\infty} / \omega_{A^\vee, \tau_\infty}) \\ \uparrow & & \uparrow \\ \text{Hom}_\kappa(\omega_{A^\vee, \tau_\infty}, \ker \alpha_{*, \tau_\infty} / \omega_{A^\vee, \tau_\infty}) & \longrightarrow & \text{Hom}_\kappa(\omega_{A^\vee, \tau_\infty}, H_1^{\text{dR}}(A)_{\tau_\infty} / \omega_{A^\vee, \tau_\infty}) \end{array}$$

by Theorem 4.1.3 and Theorem 4.2.5. However, it is an easy consequence of (1.3) that $\ker \alpha_{\mathfrak{q}^*, \tau_\infty} \cap H_1^{\text{dR}}(A)_{\tau_\infty} = \ker \alpha_{*, \tau_\infty}$. Thus, the above diagram is Cartesian; and (2) follows. \square

At the fourth stage of functoriality, we compare the special morphisms for basic correspondences and for Deligne–Lusztig varieties. Take a point

$$s^* = (A_0, \lambda_0, \eta_0^p; A^*, \lambda^*, \eta^{p^*}; A_{\mathfrak{q}}^*, \lambda_{\mathfrak{q}}^*, \eta_{\mathfrak{q}}^{p^*}; \delta^*) \in S_p(V_n, K^p)_{\text{sp}}(\kappa)$$

for a field κ containing \mathbb{F}_p^Φ . Put

$$s_n^* := s_\downarrow(s^*), \quad s_{n+1}^* := s_\uparrow(s^*);$$

and denote by B_{s^*} , $B_{s_n^*}$, and $B_{s_{n+1}^*}$ their preimages under π_{sp} , π_n , and π_{n+1} , respectively. By Lemma 4.2.2, we have admissible pairs $(\mathcal{V}_{s_n^*}, \{ , \}_{s_n^*})$ and $(\mathcal{V}_{s_{n+1}^*}, \{ , \}_{s_{n+1}^*})$. As in Construction A.1.5, we extend the pair $(\mathcal{V}_{s_n^*}, \{ , \}_{s_n^*})$ to $(\mathcal{V}_{s_n^*, \#}, \{ , \}_{s_n^*, \#})$. Then the homomorphism $\delta^*: A^* \times A_0 \rightarrow A_{\mathfrak{q}}^*$ induces a κ -linear map

$$\delta_{s^*}: \mathcal{V}_{s_n^*, \#} \rightarrow \mathcal{V}_{s_{n+1}^*}$$

satisfying $\{\delta_{s^*}(x), \delta_{s^*}(y)\}_{s_{n+1}^*} = \{x, y\}_{s_n^*, \#}$ for every $x, y \in \mathcal{V}_{s_n^*, \#}$. By Construction A.1.5, we obtain a morphism

$$\delta_{s^* \uparrow}: \mathrm{DL}_{s_n^*} = \mathrm{DL}(\mathcal{V}_{s_n^*}, \{ , \}_{s_n^*}, [\frac{n+1}{2}]) \rightarrow \mathrm{DL}_{s_{n+1}^*} = \mathrm{DL}(\mathcal{V}_{s_{n+1}^*}, \{ , \}_{s_{n+1}^*}, [\frac{n+2}{2}])$$

of corresponding Deligne–Lusztig varieties.

Proposition 4.4.6. *Let the notation be as above. The following diagram*

$$\begin{array}{ccc} \mathrm{B}_{s_{n+1}^*} & \xrightarrow[\simeq]{\zeta_{s_{n+1}^*}} & \mathrm{DL}_{s_{n+1}^*} \\ \mathrm{b}_\uparrow \uparrow & & \uparrow \delta_{s^* \uparrow} \\ \mathrm{B}_{s^*} & \xrightarrow[\simeq]{\zeta_{s^*} \circ \mathrm{b}_\downarrow} & \mathrm{DL}_{s^*} \end{array}$$

in Sch/κ commutes, where $\zeta_{s_n^*}$ and $\zeta_{s_{n+1}^*}$ are the isomorphisms in Theorem 4.2.5(3). In particular, $\mathrm{b}_\uparrow: \mathrm{B}_{s^*} \rightarrow \mathrm{B}_{s_{n+1}^*}$ is an isomorphism if n is odd, and is a regular embedding of codimension one if n is even.

Proof. Note that by Lemma 4.4.2, the restricted morphism $\mathrm{b}_\downarrow: \mathrm{B}_{s^*} \rightarrow \mathrm{B}_{s_n^*}$ is an isomorphism. Thus, the last claim follows from the commutativity and Proposition A.1.6.

When n is odd, the commutativity is obvious. When n is even, it suffices to show that for every point

$$(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^*, \lambda^*, \eta^{p^*}; \alpha) \in \mathrm{B}_{s^*}(S),$$

we have

$$(4.12) \quad \delta_{*, \tau_\infty}^* \left((\check{\alpha}_{*, \tau_\infty})^{-1} \omega_{A^\vee/S, \tau_\infty} \oplus \mathrm{H}_1^{\mathrm{dR}}(A_0/S)_{\tau_\infty} \right) = (\check{\alpha}_{\mathfrak{q}^*, \tau_\infty})^{-1} \omega_{A^\vee \times A_0^\vee/S, \tau_\infty}$$

in view of the diagram

$$\begin{array}{ccc} A \times A_0 & \xlongequal{\quad} & A \times A_0 \\ \alpha \times \mathrm{id}_{A_0} \downarrow & & \downarrow \alpha_{\mathfrak{q}} := \delta^* \circ (\alpha \times \mathrm{id}_{A_0}) \\ A^* \times A_0 & \xrightarrow{\delta^*} & A_{\mathfrak{q}}^* \\ \check{\alpha} \times \varpi \mathrm{id}_{A_0} \downarrow & & \downarrow \check{\alpha}_{\mathfrak{q}} \\ A \times A_0 & \xlongequal{\quad} & A \times A_0 \end{array}$$

in which $\check{\alpha} \circ \alpha = \varpi \cdot \mathrm{id}_A$ and $\check{\alpha}_{\mathfrak{q}} \circ \alpha_{\mathfrak{q}} = \varpi \cdot \mathrm{id}_{A \times A_0}$. Since both sides of (4.12) have the same rank, it suffices to show that

$$\check{\alpha}_{\mathfrak{q}^*, \tau_\infty} \left(\delta_{*, \tau_\infty}^* \left((\check{\alpha}_{*, \tau_\infty})^{-1} \omega_{A^\vee/S, \tau_\infty} \oplus \mathrm{H}_1^{\mathrm{dR}}(A_0/S)_{\tau_\infty} \right) \right) \subseteq \omega_{A^\vee \times A_0^\vee/S, \tau_\infty},$$

which is obvious as ϖ annihilates $\mathrm{H}_1^{\mathrm{dR}}(A_0/S)_{\tau_\infty}$. The proposition is proved. \square

At the final stage of functoriality, we relate the special morphisms for sources of basic correspondences to Shimura sets under the uniformization map v (4.4).

Notation 4.4.7. As in Definition 4.3.1, we choose a definite uniformization datum $(V_n^*, \mathfrak{i}_n, \{\Lambda_{n, \mathfrak{q}}^*\}_{\mathfrak{q}|p})$ for V . We also fix a definite uniformization datum $(V_{n+1}^*, \mathfrak{i}_{n+1}, \{\Lambda_{n+1, \mathfrak{q}}^*\}_{\mathfrak{q}|p})$ for V_{n+1} satisfying

- $\circ V_{n+1}^* = (V_n^*)_{\#}$ and $\mathfrak{i}_{n+1} = (\mathfrak{i}_n)_{\#}$;
- $\circ \Lambda_{n+1, \mathfrak{q}}^* = (\Lambda_{n, \mathfrak{q}}^*)_{\#}$ for $\mathfrak{q} \neq \mathfrak{p}$; and
- $\circ (\Lambda_{n, \mathfrak{p}}^*)_{\#} \subseteq \Lambda_{n+1, \mathfrak{p}}^* \subseteq p^{-1}(\Lambda_{n, \mathfrak{p}}^*)_{\#}^{\vee}$.

Let $K_{n+1, \mathfrak{q}}^*$ be the stabilizer of $\Lambda_{n+1, \mathfrak{q}}^*$ for every \mathfrak{q} over p ; and put $K_{n+1, p}^* := \prod_{\mathfrak{q}|p} K_{n+1, \mathfrak{q}}^*$. Moreover, we put $K_{\mathrm{sp}, p}^* := K_{n, \mathfrak{p}}^* \cap K_{n+1, \mathfrak{p}}^*$ (as a subgroup of $K_{n, \mathfrak{p}}^*$) and $K_{\mathrm{sp}, p}^* := K_{\mathrm{sp}, p}^* \times \prod_{\mathfrak{q} \neq \mathfrak{p}} K_{n, \mathfrak{q}}^*$.

Remark 4.4.8. When n is odd, since $(\Lambda_{n,p}^*)^\vee = p\Lambda_{n,p}^*$, we must have $\Lambda_{n+1,p}^* = (\Lambda_{n,p}^*)_\#$ as well hence $K_{\text{sp},p}^* = K_{n,p}^*$. When n is even, the number of choices of $\Lambda_{n+1,p}^*$ is $p+1$.

Similar to Construction 4.3.2, we may construct a uniformization map

$$(4.13) \quad v_{\text{sp}} : \mathbb{S}_p(V_n, \mathbf{-})_{\text{sp}}(\overline{\mathbb{F}}_p) \rightarrow \text{Sh}(V_n^*, (\mathbf{i}_n \mathbf{-}_n) K_{\text{sp},p}^*) \times \mathbb{T}_p(\overline{\mathbb{F}}_p)$$

in $\text{Fun}(\mathfrak{K}(V_n)_{\text{sp}}^p \times \mathfrak{T}, \text{Set})_{/\mathbb{T}_p(\overline{\mathbb{F}}_p)}$ which is an isomorphism, whose details we leave to readers.

Proposition 4.4.9. *The following diagram*

$$\begin{array}{ccc} \mathbb{S}_p(V_{n+1}, \mathbf{-}_{n+1})(\overline{\mathbb{F}}_p) & \xrightarrow[(4.4)]{v_{n+1}} & \text{Sh}(V_{n+1}^*, (\mathbf{i}_{n+1} \mathbf{-}_{n+1}) K_{n+1,p}^*) \times \mathbb{T}_p(\overline{\mathbb{F}}_p) \\ \uparrow s_\uparrow(\overline{\mathbb{F}}_p) & & \uparrow \text{sh}_\uparrow^* \times \text{id} \\ \mathbb{S}_p(V_n, \mathbf{-})_{\text{sp}}(\overline{\mathbb{F}}_p) & \xrightarrow[(4.13)]{v_{\text{sp}}} & \text{Sh}(V_n^*, (\mathbf{i}_n \mathbf{-}_n) K_{\text{sp},p}^*) \times \mathbb{T}_p(\overline{\mathbb{F}}_p) \\ \downarrow s_\downarrow(\overline{\mathbb{F}}_p) & & \downarrow \text{sh}_\downarrow^* \times \text{id} \\ \mathbb{S}_p(V_n, \mathbf{-}_n)(\overline{\mathbb{F}}_p) & \xrightarrow[(4.4)]{v_n} & \text{Sh}(V_n^*, (\mathbf{i}_n \mathbf{-}_n) K_{n,p}^*) \times \mathbb{T}_p(\overline{\mathbb{F}}_p) \end{array}$$

in $\text{Fun}(\mathfrak{K}(V_n)_{\text{sp}}^p \times \mathfrak{T}, \text{Set})_{/\mathbb{T}_p(\overline{\mathbb{F}}_p)}$ commutes, where sh_\downarrow^* and sh_\uparrow^* are obvious maps on Shimura sets. Moreover, the induced actions of $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^\Phi)$ on all terms on the right-hand side factor through the projection to the factor $\mathbb{T}_p(\overline{\mathbb{F}}_p)$.

Proof. The commutativity follows directly from definition. The proof of the last claim is same to Proposition 4.3.4. \square

4.5. Second geometric reciprocity law. In this subsection, we state and prove a theorem we call *second geometric reciprocity law*, which can be regarded a geometric template for the second explicit reciprocity law studied in Subsection 7.3 once throw the automorphic input.

We keep the setup in Subsection 4.4. However, we allow $\mathbf{-} = (\mathbf{-}_n, \mathbf{-}_{n+1})$ to be an object of $\mathfrak{K}(V_n)^p \times \mathfrak{K}(V_{n+1})^p$, rather than $\mathfrak{K}(V_n)_{\text{sp}}^p$. Denote by n_0 and n_1 the unique even and odd numbers in $\{n, n+1\}$, respectively. Write $n_0 = 2r_0$ and $n_1 = 2r_1 + 1$ for unique integers $r_0, r_1 \geq 1$. In particular, we have $n = r_0 + r_1$. Let L be a p -coprime coefficient ring.

To ease notation, we put $X_{n_\alpha}^? := X_p^?(V_{n_\alpha}, \mathbf{-}_{n_\alpha})$ for meaningful triples $(X, ?, \alpha) \in \{\mathbf{M}, \mathbf{M}, \mathbf{B}, \mathbf{S}\} \times \{, \eta\} \times \{0, 1\}$.

Construction 4.5.1. We construct two maps and two graphs.

(1) For every integers i, j , we define

$$\text{loc}'_p : H_{\text{ét}}^i(\text{Sh}(V_{n_0}, \mathbf{-}_{n_0} K_{n_0,p}) \times_{\text{Spec } F} \text{Sh}(V_{n_1}, \mathbf{-}_{n_1} K_{n_1,p}), L(j)) \rightarrow H_{\mathfrak{X}}^i(M_{n_0} \times_{\mathbb{T}_p} M_{n_1}, L(j))$$

to be the composition of the localization map

$$\begin{aligned} \text{loc}_p : H_{\text{ét}}^i(\text{Sh}(V_{n_0}, \mathbf{-}_{n_0} K_{n_0,p}) \times_{\text{Spec } F} \text{Sh}(V_{n_1}, \mathbf{-}_{n_1} K_{n_1,p}), L(j)) \\ \rightarrow H_{\text{ét}}^i((\text{Sh}(V_{n_0}, \mathbf{-}_{n_0} K_{n_0,p}) \times_{\text{Spec } F} \text{Sh}(V_{n_1}, \mathbf{-}_{n_1} K_{n_1,p})) \otimes_F \mathbb{Q}_p^\Phi, L(j)), \end{aligned}$$

the pullback map

$$H_{\text{ét}}^i((\text{Sh}(V_{n_0}, \mathbf{-}_{n_0} K_{n_0,p}) \times_{\text{Spec } F} \text{Sh}(V_{n_1}, \mathbf{-}_{n_1} K_{n_1,p})) \otimes_F \mathbb{Q}_p^\Phi, L(j)) \rightarrow H_{\mathfrak{X}}^i(\mathbf{M}_{n_0}^\eta \times_{\mathbb{T}_p^\eta} \mathbf{M}_{n_1}^\eta, L(j))$$

induced from (4.2), and the isomorphism

$$H_{\mathfrak{X}}^i(M_{n_0} \times_{\mathbb{T}_p} M_{n_1}, R\Psi L(j)) \xrightarrow{\sim} H_{\mathfrak{X}}^i(M_{n_0} \times_{\mathbb{T}_p} M_{n_1}, L(j))$$

due to the fact $L \simeq R\Psi L$ by Theorem 4.1.3.

(2) Analogous to Construction 4.3.9, we define the map

$$\begin{aligned} \text{inc}_!^{*,*} &: L[\text{Sh}(V_{n_0}^*, (\mathbf{i}_{n_0} - \mathbf{n}_0)K_{n_0,p}^*)] \otimes_L L[\text{Sh}(V_{n_1}^*, (\mathbf{i}_{n_1} - \mathbf{n}_1)K_{n_1,p}^*)] \\ &\xrightarrow{\sim} H_{\mathfrak{X}}^0(S_{n_0}, L) \otimes_L H_{\mathfrak{X}}^0(S_{n_1}, L) = H_{\mathfrak{X}}^0(S_{n_0} \times_{\mathbb{T}_p} S_{n_1}, L) \\ &\xrightarrow{(\pi_{n_0} \times \pi_{n_1})^*} H_{\mathfrak{X}}^0(B_{n_0} \times_{\mathbb{T}_p} B_{n_1}, L) \xrightarrow{(\iota_{n_0} \times \iota_{n_1})_!} H_{\mathfrak{X}}^{2n}(M_{n_0} \times_{\mathbb{T}_p} M_{n_1}, L(n)) \end{aligned}$$

in $\text{Fun}(\mathfrak{K}(V_n)^p \times \mathfrak{K}(V_{n+1})^p, \text{Mod}(L))$.

Suppose that $\mathbf{-}$ is taken in the subcategory $\mathfrak{K}(V_n)_{\text{sp}}^p$.

(3) We define $\Delta \text{Sh}(V_n, \mathbf{-}_n K_{n,p})$ to be the graph of the morphism sh_{\uparrow} (4.6), as a closed subscheme of $\text{Sh}(V_{n_0}, \mathbf{-}_{n_0} K_{n_0,p}) \times_{\text{Spec } F} \text{Sh}(V_{n_1}, \mathbf{-}_{n_1} K_{n_1,p})$, which gives rise to a class

$$[\Delta \text{Sh}(V_n, \mathbf{-}_n K_{n,p})] \in H_{\text{ét}}^{2n}(\text{Sh}(V_{n_0}, \mathbf{-}_{n_0} K_{n_0,p}) \times_{\text{Spec } F} \text{Sh}(V_{n_1}, \mathbf{-}_{n_1} K_{n_1,p}), L(n))$$

by the absolute cycle class map.

(4) We define $\Delta \text{Sh}(V_n^*, (\mathbf{i}_n - \mathbf{n})K_{\text{sp},p}^*)$ to be the graph of the correspondence $(\text{sh}_{\downarrow}^*, \text{sh}_{\uparrow}^*)$, which is a subset of $\text{Sh}(V_{n_0}^*, (\mathbf{i}_{n_0} - \mathbf{n}_0)K_{n_0,p}^*) \times \text{Sh}(V_{n_1}^*, (\mathbf{i}_{n_1} - \mathbf{n}_1)K_{n_1,p}^*)$.

The following theorem, which we call the *second geometric reciprocity law*, relates the class $[\Delta \text{Sh}(V_n, \mathbf{-}_n K_{n,p})]$ with an explicit class coming from the Shimura set.

Theorem 4.5.2 (Second geometric reciprocity law). *Suppose that $\mathbf{-}$ is taken in the subcategory $\mathfrak{K}(V_n)_{\text{sp}}^p$. We have*

$$\mathbb{T}_{n_1,p}^* \cdot (\text{id} \times \pi_{n_1})_! (\text{id} \times \iota_{n_1})^* \text{loc}'_p([\Delta \text{Sh}(V_n, \mathbf{-}_n K_{n,p})]) = (\text{id} \times \pi_{n_1})_! (\text{id} \times \iota_{n_1})^* \text{inc}_!^{*,*}(\mathbb{1}_{\Delta \text{Sh}(V_n^*, (\mathbf{i}_n - \mathbf{n})K_{\text{sp},p}^*)})$$

in $H_{\mathfrak{X}}^{2r_0}(M_{n_0} \times_{\mathbb{T}_p} S_{n_1}, L(r_0))$, where $\mathbb{T}_{n_1,p}^* \in \mathbb{T}_{n_1,p}$ is the Hecke operator appeared in Theorem 4.3.10.

Note that by Proposition 4.3.6 and Remark 4.3.8, $H_{\mathfrak{X}}^{2r_0}(M_{n_0} \times_{\mathbb{T}_p} S_{n_1}, L(r_0))$ is a $\mathbb{T}_{n_1,p}$ -module. For readers' convenience, we illustrate the identity in the above theorem through the following diagram

$$\begin{array}{ccccc} H_{\text{ét}}^{2n}(\text{Sh}(V_{n_0}, \mathbf{-}_{n_0} K_{n_0,p}) \times_{\text{Spec } F} \text{Sh}(V_{n_1}, \mathbf{-}_{n_1} K_{n_1,p}), L(n)) & \xrightarrow{\text{loc}'_p} & H_{\mathfrak{X}}^{2n}(M_{n_0} \times_{\mathbb{T}_p} M_{n_1}, L(n)) & \xleftarrow{\text{inc}_!^{*,*}} & L[\text{Sh}(V_{n_0}^*, (\mathbf{i}_{n_0} - \mathbf{n}_0)K_{n_0,p}^*)] \otimes_L L[\text{Sh}(V_{n_1}^*, (\mathbf{i}_{n_1} - \mathbf{n}_1)K_{n_1,p}^*)] \\ \downarrow \Psi & & \downarrow (\text{id} \times \iota_{n_1})^* & & \downarrow \Psi \\ [\Delta \text{Sh}(V_n, \mathbf{-}_n K_{n,p})] & & H_{\mathfrak{X}}^{2n}(M_{n_0} \times_{\mathbb{T}_p} B_{n_1}, L(n)) & & \mathbb{1}_{\Delta \text{Sh}(V_n^*, (\mathbf{i}_n - \mathbf{n})K_{\text{sp},p}^*)} \\ & & \downarrow (\text{id} \times \pi_{n_1})_! & & \\ & & H_{\mathfrak{X}}^{2r_0}(M_{n_0} \times_{\mathbb{T}_p} S_{n_1}, L(r_0)) & & \\ & & \downarrow \Psi & & \\ & & \mathbb{T}_{n_1,p}^* \cdot \sim = \sim & & \end{array}$$

Proof. We denote

$$\mathbf{m}_{\Delta} : \mathbf{M}_n \rightarrow \mathbf{M}_n \times_{\mathbb{T}_p} \mathbf{M}_{n+1} = \mathbf{M}_{n_0} \times_{\mathbb{T}_p} \mathbf{M}_{n_1}$$

the diagonal morphism of the correspondence $(\text{id}, \mathbf{m}_{\uparrow})$ (4.7) in $\text{Fun}(\mathfrak{K}(V_n)_{\text{sp}}^p \times \mathfrak{I}, \text{Sch}/_{\mathbb{Z}_p^{\Phi}})_{/\mathbb{T}_p}$. Then we have the identity

$$\text{loc}'_p([\Delta \text{Sh}(V_n, \mathbf{-}_n K_{n,p})]) = \mathbf{m}_{\Delta}![\mathbf{M}_n] \in H_{\mathfrak{X}}^{2n}(M_n \times_{\mathbb{T}_p} M_{n+1}, L(n))$$

by the commutative diagram (4.8).

Put $B_{\text{sp}} := B_p(V_n, \mathbf{-})_{\text{sp}}$ for short, and denote

$$\mathbf{b}_{\Delta} := (\mathbf{b}_{\downarrow}, \mathbf{b}_{\uparrow}) : B_{\text{sp}} \rightarrow B_n \times_{\mathbb{T}_p} B_{n+1} = B_{n_0} \times_{\mathbb{T}_p} B_{n_1}$$

the diagonal morphism of the correspondence $(b_\downarrow, b_\uparrow)$. By Proposition 4.4.5 (resp. Lemma 4.4.2) when $n = n_0$ (resp. $n = n_1$), the following commutative diagram

$$\begin{array}{ccc} B_{\text{sp}} & \xrightarrow{(\iota_{n_0} \times \text{id}) \circ b_\Delta} & M_{n_0} \times_{T_p} B_{n_1} \\ \downarrow \iota_n \circ b_\downarrow & & \downarrow \text{id} \times \iota_{n_1} \\ M_n & \xrightarrow{m_\Delta} & M_{n_0} \times_{T_p} M_{n_1} \end{array}$$

is Cartesian. Then by Proper Base Change, we have

$$\begin{aligned} \mathbf{T}_{n_1, \mathfrak{p}}^*(\text{id} \times \pi_{n_1})_!(\text{id} \times \iota_{n_1})^* m_{\Delta!}[M_n] &= \mathbf{T}_{n_1, \mathfrak{p}}^*(\text{id} \times \pi_{n_1})_!((\iota_{n_0} \times \text{id}) \circ b_\Delta)_!(\iota_n \circ b_\downarrow)^*[M_n] \\ &= \mathbf{T}_{n_1, \mathfrak{p}}^*(\text{id} \times \pi_{n_1})_!((\iota_{n_0} \times \text{id}) \circ b_\Delta)_![B_{\text{sp}}]. \end{aligned}$$

The commutative diagram

$$\begin{array}{ccc} B_{\text{sp}} & \xrightarrow{(\iota_{n_0} \times \text{id}) \circ b_\Delta} & M_{n_0} \times_{T_p} B_{n_1} \\ \downarrow (\text{id} \times \pi_{n_1}) \circ b_\Delta & & \downarrow \text{id} \times \pi_1 \\ B_{n_0} \times_{T_p} S_{n_1} & \xrightarrow{\iota_{n_0} \times \text{id}} & M_{n_0} \times_{T_p} S_{n_1} \end{array}$$

implies the identity

$$\mathbf{T}_{n_1, \mathfrak{p}}^*(\text{id} \times \pi_{n_1})_!((\iota_{n_0} \times \text{id}) \circ b_\Delta)_![B_{\text{sp}}] = \mathbf{T}_{n_1, \mathfrak{p}}^*(\iota_{n_0} \times \text{id})_!((\text{id} \times \pi_{n_1}) \circ b_\Delta)_![B_{\text{sp}}].$$

Now by the definition of B_{sp} (Definition 4.4.3), we have

$$((\text{id} \times \pi_{n_1}) \circ b_\Delta)_![B_{\text{sp}}] = (\pi_{n_0} \times \text{id})^*(\mathbb{1}_{\Delta \text{Sh}(V_n^*, (i_n - n)K_{\text{sp}, p}^*)}).$$

In all, we have

$$\mathbf{T}_{n_1, \mathfrak{p}}^*(\text{id} \times \pi_{n_1})_!(\text{id} \times \iota_{n_1})^* m_{\Delta!}[M_n] = (\iota_{n_0} \times \text{id})_!(\pi_{n_0} \times \text{id})^*(\mathbf{T}_{n_1, \mathfrak{p}}^* \mathbb{1}_{\Delta \text{Sh}(V_n^*, (i_n - n)K_{\text{sp}, p}^*)}),$$

which, by Theorem 4.3.10, equals

$$\begin{aligned} &(\iota_{n_0} \times \text{id})_!(\pi_{n_0} \times \text{id})^*(\text{id} \times \pi_{n_1})_!(\text{id} \times \iota_{n_1})^*(\text{id} \times \iota_{n_1})_!(\text{id} \times \pi_{n_1})^*(\mathbb{1}_{\Delta \text{Sh}(V_n^*, (i_n - n)K_{\text{sp}, p}^*)}) \\ &= (\text{id} \times \pi_{n_1})_!(\text{id} \times \iota_{n_1})^* \text{inc}_!^{*,*}(\mathbb{1}_{\Delta \text{Sh}(V_n^*, (i_n - n)K_{\text{sp}, p}^*)}). \end{aligned}$$

The theorem follows. \square

5. UNITARY MODULI SCHEMES: SEMISTABLE CASE

In this section, we define and study certain semistable integral moduli scheme whose generic fiber is the product of a unitary Shimura variety and an auxiliary CM moduli. Since the materials in this section are strictly in the linear order, we will leave the summary of contents to each subsection.

We fix a special inert prime (Definition 3.3.4) \mathfrak{p} of F^+ (with the underlying rational prime p). We take the prescribed subring \mathbb{P} in Definition 3.4.2 to be $\mathbb{Z}_{(p)}$. We choose following data

- a CM type Φ containing τ_∞ ;
- a rational skew-hermitian space W_0 over $O_F \otimes \mathbb{Z}_{(p)}$ of rank 1 and type Φ (Definition 3.5.3);
- a neat open compact subgroup $K_0^p \subseteq T_0(\mathbb{A}^{\infty, p})$;
- an isomorphism $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$ that induces the place \mathfrak{p} of F^+ ;
- an element $\varpi \in O_{F^+}$ that is totally positive and satisfies $\text{val}_{\mathfrak{p}}(\varpi) = 1$, and $\text{val}_{\mathfrak{q}}(\varpi) = 0$ for every prime $\mathfrak{q} \neq \mathfrak{p}$ of F^+ above p .

We adopt Notation 3.3.6. In particular, \mathbb{F}_p^Φ contains \mathbb{F}_{p^2} . Since W_0 and K_0^p are insensitive and will never be changed in the remaining part of this section, we will not include them in all notations. However, we will keep the prime \mathfrak{p} in notations as later in application, we need to choose different primes in a crucial step. Put $\mathbf{T}_p := \mathbf{T}_p(W_0, K_0^p) \otimes_{O_{F_\Phi} \otimes \mathbb{Z}_{(p)}} \mathbb{Z}_p^\Phi$.

5.1. Construction of moduli schemes. In this subsection, we construct our initial moduli schemes. We start from the datum $(V^\circ, \{\Lambda_{\mathfrak{q}}^\circ\}_{\mathfrak{q}|p})$ where

- V° is a standard *definite* hermitian space (Definition 3.1.7) over F of rank $N \geq 1$, and
- for every prime \mathfrak{q} of F^+ above p , a self-dual $O_{F_\mathfrak{q}}$ -lattice $\Lambda_{\mathfrak{q}}^\circ$ in $V^\circ \otimes_F F_\mathfrak{q}$.

Definition 5.1.1. We define a functor

$$\begin{aligned} \mathbf{M}_p(V^\circ, -) : \mathfrak{K}(V^\circ)^p \times \mathfrak{T} &\rightarrow \text{PSch}'_{/\mathbb{Z}_p^\Phi} \\ K^{p^\circ} &\mapsto \mathbf{M}_p(V^\circ, K^{p^\circ}) \end{aligned}$$

such that for every $S \in \text{Sch}'_{/\mathbb{Z}_p^\Phi}$, $\mathbf{M}_p(V^\circ, K^{p^\circ})(S)$ is the set of equivalence classes of sextuples $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p)$ where

- $(A_0, \lambda_0, \eta_0^p)$ is an element in $\mathbf{T}_p(S)$;
- (A, λ) is a unitary O_F -abelian scheme of signature type $N\Phi - \tau_\infty + \tau_\infty^c$ over S (Definitions 3.4.2 and 3.4.3) such that $\ker \lambda[p^\infty]$ is contained in $A[\mathfrak{p}]$ of rank p^2 ;
- η^p is a K^{p° -level structure, that is, for a chosen geometric point s on every connected component of S , a $\pi_1(S, s)$ -invariant K^{p° -orbit of isomorphisms

$$\eta^p : V^\circ \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p} \rightarrow \text{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}}^{\lambda_0, \lambda} (H_1^{\text{ét}}(A_{0s}, \mathbb{A}^{\infty, p}), H_1^{\text{ét}}(A_s, \mathbb{A}^{\infty, p}))$$

of hermitian spaces over $F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p} = F \otimes_{F^+} \mathbb{A}_{F^+}^{\infty, p}$. See Construction 3.4.4 (with $\square = \{\infty, p\}$) for the right-hand side.

The equivalence relation and the action of morphisms in $\mathfrak{K}(V^\circ)^p \times \mathfrak{T}$ are defined similarly as in Definition 4.1.2.

Remark 5.1.2. In the definition of the moduli functor $\mathbf{M}_p(V^\circ, -)$, we use the *definite* hermitian space V° to define the tame level structure – this is different from the usual treatment. The reason for doing this is to make the uniformization map (5.4) for certain stratum in the special fiber of $\mathbf{M}_p(V^\circ, -)$ *canonical*, since our main interest is the Shimura set $\text{Sh}(V^\circ, -K_p^\circ)$, while the trade-off is that the relation between the generic fiber of $\mathbf{M}_p(V^\circ, -)$ and unitary Shimura varieties cannot be made canonical (see Definition 5.1.6).

We have apparently the forgetful morphism

$$(5.1) \quad \mathbf{M}_p(V^\circ, -) \rightarrow \mathbf{T}_p$$

in $\text{Fun}(\mathfrak{K}(V^\circ)^p \times \mathfrak{T}, \text{PSch}'_{/\mathbb{Z}_p^\Phi})$, which is representable by quasi-projective schemes. According to Notation 3.3.6, we shall denote by the base change of (5.1) to \mathbb{F}_p^Φ by $\mathbf{M}_p(V^\circ, -) \rightarrow \mathbf{T}_p$, which is a morphism in $\text{Fun}(\mathfrak{K}(V^\circ)^p \times \mathfrak{T}, \text{Sch}_{/\mathbb{F}_p^\Phi})$.

Definition 5.1.3. For every $K^{p^\circ} \in \mathfrak{K}(V^\circ)^p$, let $(\mathcal{A}_0, \lambda_0, \eta_0^p; \mathcal{A}, \lambda, \eta^p)$ be the universal object over $\mathbf{M}_p(V^\circ, K^{p^\circ})$. We define

- (1) $\mathbf{M}_p^\circ(V^\circ, K^{p^\circ})$ to be the locus of $\mathbf{M}_p(V^\circ, K^{p^\circ})$ on which $\omega_{\mathcal{A}^\vee, \tau_\infty}$ coincides with $H_1^{\text{dR}}(\mathcal{A})_{\tau_\infty}^\perp$, which we call the *balloon stratum*;⁵

⁵This terminology is borrowed from an unpublished note by Kudla and Rapoport, where they study the corresponding Rapoport–Zink space. The intuition becomes clear after Theorem 5.2.4 where we show that this stratum is a projective space fibration over a zero-dimensional scheme.

- (2) $M_p^\bullet(V^\circ, K^{p^\circ})$ to be the locus of $M_p(V^\circ, K^{p^\circ})$ on which $H_1^{\text{dR}}(\mathcal{A})_{\tau_\infty}^\perp$ is a line subbundle of $\omega_{\mathcal{A}^\vee, \tau_\infty^\circ}$, which we call the *ground stratum*;
- (3) $M_p^\dagger(V^\circ, K^{p^\circ})$ to be $M_p^\circ(V^\circ, K^{p^\circ}) \cap M_p^\bullet(V^\circ, K^{p^\circ})$, which we call the *link stratum*.⁶

We denote

$$\begin{aligned} m^{\circ\dagger}: M_p^\dagger(V^\circ, -) &\rightarrow M_p^\circ(V^\circ, -), \\ m^{\dagger\bullet}: M_p^\dagger(V^\circ, -) &\rightarrow M_p^\bullet(V^\circ, -), \end{aligned}$$

the obvious inclusion morphisms.

Remark 5.1.4. When $N = 1$, the ground stratum and link stratum are both empty.

Theorem 5.1.5. *For every $K^{p^\circ} \in \mathfrak{K}(V^\circ)^p$, we have*

- (1) *The scheme $M_p(V^\circ, K^{p^\circ})$ is quasi-projective and strictly semi-stable over T_p of relative dimension $N - 1$; and we have*

$$M_p(V^\circ, K^{p^\circ}) = M_p^\circ(V^\circ, K^{p^\circ}) \cup M_p^\bullet(V^\circ, K^{p^\circ}).$$

Moreover, (5.1) is projective if and only if its base change to \mathbb{Q}_p^Φ is.

- (2) *The loci $M_p^\circ(V^\circ, K^{p^\circ})$ and $M_p^\bullet(V^\circ, K^{p^\circ})$ are both closed subsets of $M_p(V^\circ, K^{p^\circ})$, (whose induced reduced schemes are) smooth over T_p .*
- (3) *We have a canonical isomorphism for the relative tangent sheaf*

$$\mathcal{T}_{M_p^\circ(V^\circ, K^{p^\circ})/T_p} \simeq \mathcal{H}om(\omega_{\mathcal{A}^\vee, \tau_\infty^\circ}, \text{Lie}_{\mathcal{A}, \tau_\infty^\circ}).$$

- (4) *When $N \geq 2$, the relative tangent sheaf $\mathcal{T}_{M_p^\dagger(V^\circ, K^{p^\circ})/T_p}$ fits canonically into a sequence*

$$0 \longrightarrow \mathcal{H}om(\omega_{\mathcal{A}^\vee, \tau_\infty}, \omega_{\mathcal{A}^\vee, \tau_\infty}^\perp / \omega_{\mathcal{A}^\vee, \tau_\infty}) \longrightarrow \mathcal{T}_{M_p^\dagger(V^\circ, K^{p^\circ})/T_p} \longrightarrow \mathcal{H}om(\omega_{\mathcal{A}^\vee, \tau_\infty} / H_1^{\text{dR}}(\mathcal{A})_{\tau_\infty}^\perp, \text{Lie}_{\mathcal{A}, \tau_\infty^\circ}) \longrightarrow 0.$$

- (5) *When $N \geq 2$, we have a canonical isomorphism for the relative tangent sheaf*

$$\mathcal{T}_{M_p^\dagger(V^\circ, K^{p^\circ})/T_p} \simeq \mathcal{H}om(\omega_{\mathcal{A}^\vee, \tau_\infty} / H_1^{\text{dR}}(\mathcal{A})_{\tau_\infty}^\perp, \text{Lie}_{\mathcal{A}, \tau_\infty^\circ}).$$

Proof. For (1), the (quasi-)projectiveness part is well-known. We consider the remaining assertions. Take a point $x = (A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p) \in M_p(V^\circ, K^{p^\circ})(\kappa)$ for a perfect field κ containing \mathbb{F}_p^Φ , and denote by the completed local ring of $M_p(V^\circ, K^{p^\circ})$ at x by \mathcal{O}_x . We have a $W(\kappa)$ -bilinear pairing $\langle \cdot, \cdot \rangle_{\lambda, \tau_\infty}: \mathcal{D}(A)_{\tau_\infty} \times \mathcal{D}(A)_{\tau_\infty^\circ} \rightarrow W(\kappa)$ as in Notation 3.4.12. By repeatedly applying Proposition 3.4.8, we have for every commutative Artinian $W(\kappa)$ -algebra R that $\text{Hom}_{W(\kappa)}(\mathcal{O}_x, R)$ is the set of R -subbundle

$$M_{\tau_\infty} \subseteq \mathcal{D}(A)_{\tau_\infty} \otimes_{W(\kappa)} R, \quad M_{\tau_\infty^\circ} \subseteq \mathcal{D}(A)_{\tau_\infty^\circ} \otimes_{W(\kappa)} R$$

of ranks 1 and $N - 1$ lifting $\omega_{\mathcal{A}^\vee/\kappa, \tau_\infty}$ and $\omega_{\mathcal{A}^\vee/\kappa, \tau_\infty^\circ}$, respectively, such that $\langle M_{\tau_\infty}, M_{\tau_\infty^\circ} \rangle_{\lambda, \tau_\infty} = 0$. We choose isomorphisms $\mathcal{D}(A)_{\tau_\infty} \simeq W(\kappa)^{\oplus N}$ and $\mathcal{D}(A)_{\tau_\infty^\circ} \simeq W(\kappa)^{\oplus N}$ under which the pairing $\langle \cdot, \cdot \rangle_{\lambda, \tau_\infty}$ is given by

$$\langle (x_1, \dots, x_N), (y_1, \dots, y_N) \rangle_{\lambda, \tau_\infty} = px_1y_1 + x_2y_2 + \dots + x_Ny_N.$$

There are four possible cases.

- (i) If $\omega_{\mathcal{A}^\vee/\kappa, \tau_\infty}$ is generated by $(1, 0, \dots, 0)$ and $\omega_{\mathcal{A}^\vee/\kappa, \tau_\infty^\circ}$ contains $(1, 0, \dots, 0)$, then possibly after changing coordinates, we may assume that $\omega_{\mathcal{A}^\vee/\kappa, \tau_\infty^\circ} = \{(y_1, \dots, y_{N-1}, 0)\}$. Then we have $\mathcal{O}_x \simeq W(\kappa)[[x_1, \dots, x_{N-1}, x_N]]/(x_1x_N - p)$.
- (ii) If $\omega_{\mathcal{A}^\vee/\kappa, \tau_\infty}$ is generated by $(1, 0, \dots, 0)$ and $\omega_{\mathcal{A}^\vee/\kappa, \tau_\infty^\circ}$ does not contain $(1, 0, \dots, 0)$, then possibly after changing coordinates, we may assume that $\omega_{\mathcal{A}^\vee/\kappa, \tau_\infty^\circ} = \{(0, y_2, \dots, y_N)\}$. It is clear that M_{τ_∞} is determined by $M_{\tau_\infty^\circ}$; and $\mathcal{O}_x \simeq W(\kappa)[[x_2, \dots, x_N]]$.

⁶This is the stratum linking balloons to the ground.

- (iii) If $\omega_{A^\vee/\kappa, \tau_\infty}$ is not generated by $(1, 0, \dots, 0)$ and $\omega_{A^\vee/\kappa, \tau_\infty^c}$ contains $(1, 0, \dots, 0)$, then possibly after changing coordinates, we may assume that $\omega_{A^\vee/\kappa, \tau_\infty}$ is generated by $(0, \dots, 0, 1)$. It is clear that $M_{\tau_\infty^c}$ is determined by M_{τ_∞} ; and $\mathcal{O}_x \simeq W(\kappa)[[x_1, \dots, x_{N-1}]]$.
- (iv) If $\omega_{A^\vee/\kappa, \tau_\infty}$ is not generated by $(1, 0, \dots, 0)$ and $\omega_{A^\vee/\kappa, \tau_\infty^c}$ does not contain $(1, 0, \dots, 0)$, then this would not happen.

Together with the fact that $\mathbf{M}_p(V^\circ, K^{p^\circ}) \otimes \mathbb{Q}$ is smooth of dimension $N - 1$, $\mathbf{M}_p(V^\circ, K^{p^\circ})$ is strictly semi-stable over \mathbf{T}_p of relative dimension $N - 1$. Moreover, $\mathbf{M}_p^\circ(V^\circ, K^{p^\circ})$ is the locus where (i) or (ii) happens; and $\mathbf{M}_p^\bullet(V^\circ, K^{p^\circ})$ is the locus where (i) or (iii) happens. Thus, both (1) and (2) follow.

For (3–5), we will use deformation theory. For common use, we consider a closed immersion $S \hookrightarrow \hat{S}$ in $\mathbf{Sch}/\mathbf{T}_p$ defined by an ideal sheaf \mathcal{I} with $\mathcal{I}^2 = 0$. Take an S -point $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p)$ in various schemes we will consider. By Proposition 3.4.8, we need to lift $\omega_{A^\vee, \tau_\infty}$ and $\omega_{A^\vee, \tau_\infty^c}$ to subbundles $\hat{\omega}_{A^\vee, \tau_\infty} \subseteq H_1^{\text{cris}}(A/\hat{S})_{\tau_\infty}$ and $\hat{\omega}_{A^\vee, \tau_\infty^c} \subseteq H_1^{\text{cris}}(A/\hat{S})_{\tau_\infty^c}$, respectively, that are orthogonal to each other under the pairing (3.3).

For (3), since we require $\langle \hat{\omega}_{A^\vee, \tau_\infty}, H_1^{\text{cris}}(A/\hat{S})_{\tau_\infty^c} \rangle_{\lambda, \tau_\infty}^{\text{cris}} = 0$, it remains to lift $\hat{\omega}_{A^\vee, \tau_\infty}$ without restriction. Thus, (3) follows by Remark 3.4.6.

For (4), we need to first find lifting $\hat{\omega}_{A^\vee, \tau_\infty^c}$ that contains $H_1^{\text{cris}}(A/\hat{S})_{\tau_\infty^c}^\perp$; and then find lifting $\hat{\omega}_{A^\vee, \tau_\infty}$ satisfying $\langle \hat{\omega}_{A^\vee, \tau_\infty}, \hat{\omega}_{A^\vee, \tau_\infty^c} \rangle_{\lambda, \tau_\infty}^{\text{cris}} = 0$. Thus, (4) follows by Remark 3.4.6.

For (5), we only need to find lifting $\hat{\omega}_{A^\vee, \tau_\infty}$ that contains $H_1^{\text{cris}}(A/\hat{S})_{\tau_\infty}^\perp$, which implies (5). \square

In the remaining part of this subsection, we discuss the relation between $\mathbf{M}_p(V^\circ, -)$ and unitary Shimura varieties. Since we use a standard definite hermitian space to parameterize the level structures, such relation is not canonical, which depends on the choice of an indefinite uniformization datum defined as follows.

Definition 5.1.6. We define an *indefinite uniformization datum* for V° (at \mathfrak{p}) to be a collection of $(V', j, \{\Lambda'_q\}_{q|p})$ where

- \circ V' is a standard indefinite hermitian space over F of rank N ;
- \circ $j: V^\circ \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p} \rightarrow V' \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$ is an isometry;
- \circ for every prime \mathfrak{q} of F^+ above p other than \mathfrak{p} , Λ'_q is a self-dual O_{F_q} -lattice in $V' \otimes_F F_q$; and
- \circ Λ'_p is an O_{F_p} -lattice in $V' \otimes_F F_p$ satisfying $\Lambda'_p \subseteq (\Lambda'_p)^\vee$ and $(\Lambda'_p)^\vee/\Lambda'_p$ has length 1.

By the Hasse principle for hermitian spaces, there exists an indefinite uniformization datum for which we fix one. Let K'_q be the stabilizer of Λ'_q for every \mathfrak{q} over p ; and put $K'_p := \prod_{q|p} K'_q$. The isometry j induces an equivalence of categories $j: \mathfrak{K}(V^\circ)^p \xrightarrow{\sim} \mathfrak{K}(V')^p$.

Then similar to Remark 4.1.5, we obtain a “moduli interpretation” isomorphism

$$(5.2) \quad \mathbf{M}_p^\eta(V^\circ, -) \xrightarrow{\sim} \text{Sh}(V', j-K'_p) \times_{\text{Spec } F} \mathbf{T}_p^\eta$$

in $\text{Fun}(\mathfrak{K}(V^\circ)^p \times \mathfrak{T}, \text{Sch}/\mathbb{Q}_p^\Phi)/\mathbf{T}_p^\eta$, where \mathfrak{T} acts on $\text{Sh}(V', j-K'_p) \times_{\text{Spec } F} \mathbf{T}_p^\eta$ via the second factor.

Lemma 5.1.7. *Let L be a p -coprime coefficient ring. The two specialization maps*

$$\begin{aligned} H_{\mathfrak{X}, c}^i(\mathbf{M}_p(V^\circ, -) \otimes_{\mathbb{Z}_p^\Phi} \overline{\mathbb{Q}}_p, L) &\rightarrow H_{\mathfrak{X}, c}^i(\overline{\mathbf{M}}_p(V^\circ, -), R\Psi L), \\ H_{\mathfrak{X}}^i(\mathbf{M}_p(V^\circ, -) \otimes_{\mathbb{Z}_p^\Phi} \overline{\mathbb{Q}}_p, L) &\rightarrow H_{\mathfrak{X}}^i(\overline{\mathbf{M}}_p(V^\circ, -), R\Psi L), \end{aligned}$$

are both isomorphisms. In particular, (5.2) induces isomorphisms

$$\begin{aligned} H_{\text{ét}, c}^i(\text{Sh}(V', j-K'_p)_{\overline{F}}, L) &\simeq H_{\mathfrak{X}, c}^i(\overline{\mathbf{M}}_p(V^\circ, -), R\Psi L), \\ H_{\text{ét}}^i(\text{Sh}(V', j-K'_p)_{\overline{F}}, L) &\simeq H_{\mathfrak{X}}^i(\overline{\mathbf{M}}_p(V^\circ, -), R\Psi L), \end{aligned}$$

in $\text{Fun}(\mathfrak{K}(V^\circ)^p, \text{Mod}(L[\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^\Phi)])$) for every $i \in \mathbb{Z}$. Here, $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^\Phi)$ is regarded as a subgroup of $\text{Gal}(\overline{F}/F)$ under our fixed isomorphism $\iota_p: \mathbb{C} \simeq \overline{\mathbb{Q}}_p$.

Proof. When $\mathbf{M}_p(V, \mathbf{-})$ is proper, this is simply the proper base change. When $\mathbf{M}_p(V, \mathbf{-})$ is not proper, this follows from [LS18, Corollary 5.20]. \square

Remark 5.1.8. When $[F^+ : \mathbb{Q}] > 1$, the Shimura variety $\text{Sh}(V', K^{p'}K'_p)$ is proper over F for $K^{p'} \in \mathfrak{K}(V')^p$. We explain that $\text{Sh}(V', K^{p'}K'_p)$ has proper smooth reduction at every place w of F above $\Sigma_p^+ \setminus \{\mathfrak{p}\}$.

Take a place w of F above $\Sigma_p^+ \setminus \{\mathfrak{p}\}$. Choose a CM type Φ containing τ_∞ and an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$ that induces w (not the unique place above $\mathfrak{p}!$). Put $\mathbf{T}_w := \mathbf{T}_p(W_0, K_0^p) \otimes_{O_{F_\Phi} \otimes \mathbb{Z}_{(p)}} \mathbb{Z}_p^\Phi$. We define a functor $\mathbf{M}_w(V', K^{p'})$ on $\text{Sch}'_{/\mathbb{Z}_p^\Phi}$ such that for every $S \in \text{Sch}'_{/\mathbb{Z}_p^\Phi}$, $\mathbf{M}_w(V', K^{p'})(S)$ is the set of equivalence classes of sextuples $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p)$ where

- $(A_0, \lambda_0, \eta_0^p)$ is an element in $\mathbf{T}_w(S)$;
- (A, λ) is a unitary O_F -abelian scheme of signature type $N\Phi - \tau_\infty + \tau_\infty^\varepsilon$ over S (Definitions 3.4.2 and 3.4.3) such that $\ker \lambda[p^\infty]$ is contained in $A[\mathfrak{p}]$ of rank p^2 ;
- η^p is a $K^{p'}$ -level structure, similarly defined as in Definition 5.1.1.

Then $\mathbf{M}_w(V', K^{p'})$ is represented by a projective scheme over \mathbb{Z}_p^Φ . An easy computation of the tangent sheaf as in Theorem 4.1.3 shows that $\mathbf{M}_w(V', K^{p'})$ is smooth of relative dimension $N - 1$. Moreover, we have a canonical isomorphism

$$\mathbf{M}_w^\eta(V', K^{p'}) \simeq \text{Sh}(V', K^{p'}K'_p) \times_{\text{Spec } F} \mathbf{T}_w^\eta$$

over \mathbf{T}_w^η . Thus, $\text{Sh}(V', K^{p'}K'_p)$ has proper smooth reduction at w as \mathbf{T}_w is finite étale over O_{F_w} .

5.2. Basic correspondence on balloon stratum. In this subsection, we construct and study the basic correspondence on the balloon stratum $\mathbf{M}_p^\circ(V^\circ, \mathbf{-})$.

Definition 5.2.1. We define a functor

$$\begin{aligned} \mathbf{S}_p^\circ(V^\circ, \mathbf{-}): \mathfrak{K}(V^\circ)^p \times \mathfrak{T} &\rightarrow \text{PSch}'_{/\mathbb{F}_p^\Phi} \\ K^{p^\circ} &\mapsto \mathbf{S}_p^\circ(V^\circ, K^{p^\circ}) \end{aligned}$$

such that for every $S \in \text{Sch}'_{/\mathbb{F}_p^\Phi}$, $\mathbf{S}_p^\circ(V^\circ, K^{p^\circ})(S)$ is the set of equivalence classes of sextuples $(A_0, \lambda_0, \eta_0^p; A^\circ, \lambda^\circ, \eta^{p^\circ})$ where

- $(A_0, \lambda_0, \eta_0^p)$ is an element in $\mathbf{T}_p(S)$;
- (A°, λ°) is a unitary O_F -abelian scheme of signature type $N\Phi$ over S such that λ° is p -principal;
- η^{p° is, for a chosen geometric point s on every connected component of S , a $\pi_1(S, s)$ -invariant K^{p° -orbit of isomorphisms

$$\eta^{p^\circ}: V^\circ \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p} \rightarrow \text{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}}^{\lambda_0, \lambda^\circ}(\mathbf{H}_1^{\text{ét}}(A_{0s}, \mathbb{A}^{\infty, p}), \mathbf{H}_1^{\text{ét}}(A_s^\circ, \mathbb{A}^{\infty, p}))$$

of hermitian spaces over $F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p} = F \otimes_{F^+} \mathbb{A}_{F^+}^{\infty, p}$.

The equivalence relation and the action of morphisms in $\mathfrak{K}(V^\circ)^p \times \mathfrak{T}$ are defined similarly as in Definition 4.1.2.

We have apparently the forgetful morphism

$$\mathbf{S}_p^\circ(V^\circ, \mathbf{-}) \rightarrow \mathbf{T}_p$$

in $\text{Fun}(\mathfrak{K}(V^\circ)^p \times \mathfrak{T}, \text{PSch}'_{/\mathbb{F}_p^\Phi})$ which is represented by finite and étale schemes.

Now we take a point $s^\circ = (A_0, \lambda_0, \eta_0^p; A^\circ, \lambda^\circ, \eta^{p^\circ}) \in \mathbb{S}_p^\circ(\mathbb{V}^\circ, \mathbb{K}^{p^\circ})(\kappa)$ where κ is a perfect field containing \mathbb{F}_p^Φ . By Remark 3.4.10, the (κ, σ^{-1}) -linear Verschiebung map

$$\mathbb{V}: \mathbb{H}_1^{\text{dR}}(A^\circ/\kappa)_{\tau_\infty} \rightarrow \mathbb{H}_1^{\text{dR}}(A^\circ/\kappa)_{\sigma^{-1}\tau_\infty} = \mathbb{H}_1^{\text{dR}}(A^\circ/\kappa)_{\tau_\infty^c}$$

is an isomorphism. Thus, we obtain a (κ, σ) -linear isomorphism

$$\mathbb{V}^{-1}: \mathbb{H}_1^{\text{dR}}(A^\circ/\kappa)_{\tau_\infty^c} \rightarrow \mathbb{H}_1^{\text{dR}}(A^\circ/\kappa)_{\tau_\infty}.$$

We define a non-degenerate pairing

$$\{ , \}_s: \mathbb{H}_1^{\text{dR}}(A^\circ/\kappa)_{\tau_\infty^c} \times \mathbb{H}_1^{\text{dR}}(A^\circ/\kappa)_{\tau_\infty} \rightarrow \kappa$$

by the formula $\{x, y\}_{s^\circ} := \langle \mathbb{V}^{-1}x, y \rangle_{\lambda^\circ, \tau_\infty}$ (Notation 3.4.7). To ease notation, we put

$$\mathcal{V}_{s^\circ} := \mathbb{H}_1^{\text{dR}}(A^\circ/\kappa)_{\tau_\infty^c}.$$

By the same proof of Lemma 4.2.2, we know that $(\mathcal{V}_{s^\circ}, \{ , \}_s)$ is admissible. Thus, we have the Deligne–Lusztig variety $\text{DL}_{s^\circ} := \text{DL}(\mathcal{V}_{s^\circ}, \{ , \}_s, N-1)$ (Definition A.1.2).

Definition 5.2.2. We define a functor

$$\begin{aligned} \mathbb{B}_p^\circ(\mathbb{V}^\circ, -): \mathfrak{K}(\mathbb{V}^\circ)^p \times \mathfrak{T} &\rightarrow \text{PSch}'_{/\mathbb{F}_p^\Phi} \\ \mathbb{K}^{p^\circ} &\mapsto \mathbb{B}_p^\circ(\mathbb{V}^\circ, \mathbb{K}^{p^\circ}) \end{aligned}$$

such that for every $S \in \text{Sch}'_{/\mathbb{F}_p^\Phi}$, $\mathbb{B}_p^\circ(\mathbb{V}^\circ, \mathbb{K}^{p^\circ})(S)$ is the set of equivalence classes of decuples $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^\circ, \lambda^\circ, \eta^{p^\circ}; \beta)$ where

- $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p)$ is an element of $\mathbb{M}_p^\circ(\mathbb{V}^\circ, \mathbb{K}^{p^\circ})(S)$;
- $(A_0, \lambda_0, \eta_0^p; A^\circ, \lambda^\circ, \eta^{p^\circ})$ is an element of $\mathbb{S}_p^\circ(\mathbb{V}^\circ, \mathbb{K}^{p^\circ})(S)$;
- $\beta: A \rightarrow A^\circ$ is an O_F -linear quasi- p -isogeny (Definition 3.4.5) such that
 - (a) $\ker \beta[p^\infty]$ is contained in $A[p]$;
 - (b) we have $\lambda = \beta^\vee \circ \lambda^\circ \circ \beta$; and
 - (c) the \mathbb{K}^{p° -orbit of maps $v \mapsto \beta_* \circ \eta^p(v)$ for $v \in \mathbb{V}^\circ \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$ coincides with η^{p° .

The equivalence relation and the action of morphisms in $\mathfrak{K}(\mathbb{V}^\circ)^p \times \mathfrak{T}$ are defined similarly as in Definition 4.2.3.

We obtain in the obvious way a correspondence

$$(5.3) \quad \mathbb{S}_p^\circ(\mathbb{V}^\circ, -) \xleftarrow{\pi^\circ} \mathbb{B}_p^\circ(\mathbb{V}^\circ, -) \xrightarrow{\iota^\circ} \mathbb{M}_p^\circ(\mathbb{V}^\circ, -)$$

in $\text{Fun}(\mathfrak{K}(\mathbb{V}^\circ)^p \times \mathfrak{T}, \text{PSch}'_{/\mathbb{F}_p^\Phi})_{/\mathbb{T}_p}$.

Definition 5.2.3 (Basic correspondence). We refer to (5.3) as the *basic correspondence* on the balloon stratum $\mathbb{M}_p^\circ(\mathbb{V}^\circ, -)$, with $\mathbb{S}_p^\circ(\mathbb{V}^\circ, -)$ being the *source* of the basic correspondence.

Theorem 5.2.4. *In the diagram (5.3), ι° is an isomorphism. Moreover, for every point $s^\circ = (A_0, \lambda_0, \eta_0^p; A^\circ, \lambda^\circ, \eta^{p^\circ}) \in \mathbb{S}_p^\circ(\mathbb{V}^\circ, \mathbb{K}^{p^\circ})(\kappa)$ where κ is a perfect field containing \mathbb{F}_p^Φ , if we put $\mathbb{B}_{s^\circ}^\circ := \pi^{\circ-1}(s^\circ)$, then the assignment sending $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^\circ, \lambda^\circ, \eta^{p^\circ}; \beta) \in \mathbb{B}_{s^\circ}^\circ(S)$ to the subbundle*

$$H := \beta_{*, \tau_\infty^c} \omega_{A^\vee/S, \tau_\infty^c} \subseteq \mathbb{H}_1^{\text{dR}}(A^\circ/S)_{\tau_\infty^c} = \mathbb{H}_1^{\text{dR}}(A^\circ/\kappa)_{\tau_\infty^c} \otimes_{\kappa} \mathcal{O}_S = (\mathcal{V}_{s^\circ})_S$$

induces an isomorphism $\zeta_{s^\circ}^\circ: \mathbb{B}_{s^\circ}^\circ \xrightarrow{\sim} \mathbb{P}(\mathcal{V}_{s^\circ})$ satisfying that

(1) $\zeta_{s^\circ}^\circ$ restricts to an isomorphism

$$\zeta_{s^\circ}^\circ: \mathbb{B}_{s^\circ}^\circ \cap \iota^{\circ-1} \mathbb{M}_p^\dagger(\mathbb{V}^\circ, \mathbb{K}^{p^\circ}) \xrightarrow{\sim} \text{DL}_{s^\circ} = \text{DL}(\mathcal{V}_{s^\circ}, \{ , \}_s, N-1);$$

(2) we have an isomorphism

$$\mathrm{Hom}\left(\omega_{A^\vee, \tau_\infty}, \omega_{A^\vee, \tau_\infty}^\perp / \omega_{A^\vee, \tau_\infty}\right) \simeq (\zeta_{s^\circ}^\circ)^* \mathcal{O}_{\mathbb{P}(\mathcal{V}_{s^\circ})}(-(p+1)).$$

In particular, $B_{s^\circ}^\circ \cap \iota^{\circ-1} M_p^\dagger(V^\circ, K^{p^\circ})$ is a Fermat hypersurface in $B_{s^\circ}^\circ \simeq \mathbb{P}(\mathcal{V}_{s^\circ})$.

Proof. Take an object $K^{p^\circ} \in \mathfrak{K}(V^\circ)^p$. It is clear that $B_p^\circ(V^\circ, \mathbf{-})$ is a scheme. We denote by $(\mathcal{A}_0, \lambda_0, \eta_0^p; \mathcal{A}, \lambda, \eta^p; \mathcal{A}^\circ, \lambda^\circ, \eta^{p^\circ}; \beta)$ the universal object over $B_p^\circ(V^\circ, K^{p^\circ})$.

First, we show that ι° is an isomorphism. It is an easy exercise from Grothendieck–Messing theory that the canonical map $\mathcal{T}_{B_p^\circ(V^\circ, K^{p^\circ})/\mathbb{T}_p} \rightarrow \iota^{\circ*} \mathcal{T}_{M_p^\circ(V^\circ, K^{p^\circ})/\mathbb{T}_p}$ is an isomorphism. Thus, it suffices to show that $\iota^\circ(\kappa')$ is a bijection for every algebraically closed field κ' containing κ . To ease notation, we may assume $\kappa' = \kappa$. We construct an inverse of $\iota^\circ(\kappa)$. Take a point $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p) \in M_p^\circ(V^\circ, K^{p^\circ})(\kappa)$. Write $\tilde{\omega}_{A^\vee, \tau_\infty}$ the preimage of $\omega_{A^\vee, \tau_\infty}$ under the reduction map $\mathcal{D}(A)_{\tau_\infty} \rightarrow H_1^{\mathrm{dR}}(A/\kappa)_{\tau_\infty}$. As $\langle \omega_{A^\vee, \tau_\infty}, H_1^{\mathrm{dR}}(A/\kappa)_{\tau_\infty} \rangle_{\lambda, \tau_\infty} = 0$, we have $\mathcal{D}(A)_{\tau_\infty}^\vee = p^{-1} \tilde{\omega}_{A^\vee, \tau_\infty}$. Now we put $\mathcal{D}_{A^\circ, \tau} := \mathcal{D}(A)_\tau$ for $\tau \neq \tau_\infty$, and $\mathcal{D}_{A^\circ, \tau_\infty} := p^{-1} \tilde{\omega}_{A^\vee, \tau_\infty}$. We claim that $\mathcal{D}_{A^\circ} := \bigoplus_{\tau \in \Sigma_\infty} \mathcal{D}_{A^\circ, \tau}$ is a Dieudonné module, which amounts to the inclusions $F\mathcal{D}_{A^\circ, \tau_\infty} \subseteq \mathcal{D}_{A^\circ, \tau_\infty}$ and $V\mathcal{D}_{A^\circ, \tau_\infty} \subseteq \mathcal{D}_{A^\circ, \tau_\infty}$. The first one is obvious; and the second one is equivalent to the first one as $\mathcal{D}_{A^\circ, \tau_\infty}$ and $\mathcal{D}_{A^\circ, \tau_\infty}^\circ$ are integrally dual under $\langle \cdot, \cdot \rangle_{\lambda, \tau_\infty}^{\mathrm{cris}}$. Then by the Dieudonné theory, there is an O_F -abelian scheme A° over κ with $\mathcal{D}(A^\circ)_\tau = \mathcal{D}_{A^\circ, \tau}$ for every $\tau \in \Sigma_\infty$, and an O_F -linear isogeny $\beta: A \rightarrow A^\circ$ inducing the inclusion of Dieudonné modules $\mathcal{D}(A) \subseteq \mathcal{D}(A^\circ)$. By Lemma 3.4.13(2,4), the O_F -abelian scheme A° has signature type $N\Phi$. Let λ° be the unique quasi-polarization of A° satisfying $\lambda = \beta^\vee \circ \lambda^\circ \circ \beta$, which is p -principal as $\mathcal{D}_{A^\circ, \tau_\infty} = \mathcal{D}_{A^\circ, \tau_\infty}^\vee$. Finally, we let η^{p° be the map sending $v \in V^\circ \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$ to $\beta_* \circ \eta^p(v)$. Thus, we obtain an object $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^\circ, \lambda^\circ, \eta^{p^\circ}; \beta) \in S_p^\circ(V^\circ, K^{p^\circ})(\kappa)$. It is straightforward to check that such assignment gives rise to an inverse of $\iota^\circ(\kappa)$.

Second, we show that $\zeta_{s^\circ}^\circ$ is well-defined, namely, H is a subbundle of rank $N-1$. By Lemma 3.4.13(2,4) and Definition 5.2.2(b), we have $\mathrm{rank}_{\mathcal{O}_S}(\ker \beta_{*, \tau_\infty}) - \mathrm{rank}_{\mathcal{O}_S}(\ker \beta_{*, \tau_\infty}^\circ) = 1$ and $\mathrm{rank}_{\mathcal{O}_S}(\ker \beta_{*, \tau_\infty}) + \mathrm{rank}_{\mathcal{O}_S}(\ker \beta_{*, \tau_\infty}^\circ) = 1$. Thus, $\beta_{*, \tau_\infty}^\circ$ is an isomorphism, hence H is a subbundle of rank $N-1$.

Third, we show that $\zeta_{s^\circ}^\circ$ is an isomorphism. Denote by $\mathcal{H} \subseteq (\mathcal{V}_{s^\circ})_{\mathbb{P}(\mathcal{V}_{s^\circ})}$ the universal subbundle (of rank $N-1$). Then we have a canonical isomorphism

$$\mathcal{T}_{\mathbb{P}(\mathcal{V}_{s^\circ})/\kappa} \simeq \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}(\mathcal{V}_{s^\circ})}}(\mathcal{H}, H_1^{\mathrm{dR}}(A^\circ/\kappa)_{\tau_\infty}^\circ / \mathcal{H}).$$

By Theorem 5.1.5(1) and the fact that $\beta_{*, \tau_\infty}^\circ$ is an isomorphism, we obtain an isomorphism

$$(\iota^{\circ*} \mathcal{T}_{M_p^\circ(V^\circ, K^{p^\circ})/\mathbb{T}_p})|_{B_{s^\circ}^\circ} \xrightarrow{\sim} \zeta_{s^\circ}^{\circ*} \mathcal{T}_{\mathbb{P}(\mathcal{V}_{s^\circ})/\kappa}.$$

Thus, to show that $\zeta_{s^\circ}^\circ: B_{s^\circ}^\circ \rightarrow \mathbb{P}(\mathcal{V}_{s^\circ})$ is an isomorphism, it suffices to construct an inverse of $\zeta_{s^\circ}^\circ(\kappa')$ for every algebraically closed field κ' containing κ . To ease notation, we may assume $\kappa' = \kappa$. Take a κ -linear subspace $H \subseteq \mathcal{V}_{s^\circ} = H_1^{\mathrm{dR}}(A^\circ)_{\tau_\infty}^\circ$ of rank $N-1$. Let \tilde{H} denote by its preimage under the reduction map $\mathcal{D}(A^\circ)_{\tau_\infty} \rightarrow H_1^{\mathrm{dR}}(A^\circ)_{\tau_\infty}^\circ$. We put $\mathcal{D}_{A, \tau} := \mathcal{D}(A^\circ)_\tau$ for $\tau \neq \tau_\infty$, and $\mathcal{D}_{A, \tau_\infty} := V^{-1} \tilde{H} \subseteq \mathcal{D}(A^\circ)_{\tau_\infty}$. It is clear that $\mathcal{D}_A := \bigoplus_{\tau \in \Sigma_\infty} \mathcal{D}_{A, \tau}$ is a Dieudonné module. By the Dieudonné theory, there is an O_F -abelian scheme A over κ with $\mathcal{D}(A)_\tau = \mathcal{D}_{A, \tau}$ for every $\tau \in \Sigma_\infty$, and an O_F -linear isogeny $\beta: A \rightarrow A^\circ$ inducing the inclusion of Dieudonné modules $\mathcal{D}(A) \subseteq \mathcal{D}(A^\circ)$. By a similar argument as for ι° , we obtain a point $(A, \lambda, \eta^p; \beta) \in B_{s^\circ}^\circ(\kappa)$; and it follows that such assignment is an inverse of $\zeta_{s^\circ}^\circ(\kappa)$.

Finally, we check the two properties of $\zeta_{s^\circ}^\circ$.

For (1), we check that the closed subscheme $\zeta_{s^\circ}^\circ(B_{s^\circ}^\circ \cap \iota^{\circ-1} M_p^\dagger(V^\circ, K^{p^\circ}))$ coincides with $\mathrm{DL}(\mathcal{V}_{s^\circ}, \{ \cdot, \cdot \}_{s^\circ}, N-1)$. Recall that $M_p^\dagger(V^\circ, K^{p^\circ})$ is defined by the condition

$$H_{\mathrm{dR}}^1(A/S)_{\tau_\infty}^\perp \subseteq \omega_{A^\vee/S, \tau_\infty}^\circ.$$

Note that we have $H = \beta_{*,\tau_\infty^c} \omega_{A^\vee/S,\tau_\infty^c}$ and $\mathbf{V}^{-1}H^{(p)} = \beta_{*,\tau_\infty} \mathbf{H}_1^{\text{dR}}(A/S)_{\tau_\infty}$, which implies $(\mathbf{V}^{-1}H^{(p)})^\perp = (\beta_{*,\tau_\infty} \mathbf{H}_1^{\text{dR}}(A/S)_{\tau_\infty})^\perp = \beta_{*,\tau_\infty^c} (\mathbf{H}_1^{\text{dR}}(A/S)_{\tau_\infty}^\perp)$. Applying the isomorphism β_{*,τ_∞^c} , the above condition is equivalent to

$$(\mathbf{V}^{-1}H^{(p)})^\perp \subseteq H,$$

which is the condition defining $\text{DL}(\mathcal{V}_{s^\circ}, \{ , \}_{s^\circ}, N-1)$.

For (2), we have

$$\omega_{A^\vee,\tau_\infty} = \ker \beta_{*,\tau_\infty} \simeq \mathbf{H}_1^{\text{dR}}(A^\circ/S)_{\tau_\infty} / \beta_{*,\tau_\infty} \mathbf{H}_1^{\text{dR}}(A/S)_{\tau_\infty} = \mathbf{H}_1^{\text{dR}}(A^\circ/S)_{\tau_\infty} / \mathbf{V}^{-1}H^{(p)}$$

and

$$\omega_{A^\vee,\tau_\infty}^\perp / \omega_{A^\vee,\tau_\infty} \simeq \beta_{*,\tau_\infty} \omega_{A^\vee,\tau_\infty}^\perp = (\beta_{*,\tau_\infty^c} \omega_{A^\vee/S,\tau_\infty^c})^\perp = H^\perp.$$

Thus, we have

$$\omega_{A^\vee,\tau_\infty} \simeq \zeta_{s^\circ}^{\circ*} \mathcal{O}_{\mathbb{P}(\mathcal{V}_{s^\circ})}(p), \quad \omega_{A^\vee,\tau_\infty}^\perp / \omega_{A^\vee,\tau_\infty} \simeq \zeta_{s^\circ}^{\circ*} \mathcal{O}_{\mathbb{P}(\mathcal{V}_{s^\circ})}(-1)$$

from which (2) follows.

The theorem is all proved. \square

Corollary 5.2.5. *When $N \geq 2$, the normal bundle of the closed immersion*

$$\mathbf{m}^{\dagger\bullet} : \mathbf{M}_p^\dagger(\mathbf{V}^\circ, \mathbf{K}^{p^\circ}) \rightarrow \mathbf{M}_p^\bullet(\mathbf{V}^\circ, \mathbf{K}^{p^\circ})$$

is isomorphic to $(\mathbf{m}^{\dagger\circ})^* \mathcal{O}_{\mathbf{M}_p^\circ(\mathbf{V}^\circ, \mathbf{K}^{p^\circ})}(-(p+1))$.

Proof. By Theorem 5.1.5(4,5), we have that the normal bundle is isomorphic to

$$\mathcal{H}om(\omega_{A^\vee,\tau_\infty}, \omega_{A^\vee,\tau_\infty}^\perp / \omega_{A^\vee,\tau_\infty}).$$

Thus, the claim follows from Theorem 5.2.4. We can also argue that the normal bundle of $\mathbf{m}^{\dagger\bullet}$ is dual to the normal bundle of $\mathbf{m}^{\dagger\circ}$ which is isomorphic to $(\mathbf{m}^{\dagger\circ})^* \mathcal{O}_{\mathbf{M}_p^\circ(\mathbf{V}^\circ, \mathbf{K}^{p^\circ})}(p+1)$ by Theorem 5.2.4. \square

Construction 5.2.6. Let \mathbf{K}_q° be the stabilizer of Λ_q° for every $q \mid p$; and put $\mathbf{K}_p^\circ := \prod_{q \mid p} \mathbf{K}_q^\circ$. Similar to Construction 4.3.2, we may construct a *uniformization map, canonical* this time,

$$(5.4) \quad v^\circ : \mathbf{S}_p^\circ(\mathbf{V}^\circ, -)(\overline{\mathbb{F}}_p) \xrightarrow{\sim} \text{Sh}(\mathbf{V}^\circ, -\mathbf{K}_p^\circ) \times \mathbf{T}_p(\overline{\mathbb{F}}_p)$$

in $\text{Fun}(\mathfrak{K}(\mathbf{V}^\circ)^p \times \mathfrak{T}, \text{Set})_{/\mathbf{T}_p(\overline{\mathbb{F}}_p)}$ which is an isomorphism, under which the induced action of $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^\Phi)$ on the target is trivial on $\text{Sh}(\mathbf{V}^\circ, -\mathbf{K}_p^\circ)$.

Moreover, similar to Construction 4.3.5 and Proposition 4.3.6, for every $g \in \mathbf{K}_p^\circ \setminus \mathbf{U}(\mathbf{V}^\circ)(F_p^+)/\mathbf{K}_p^\circ$, we may construct the Hecke correspondence

$$\text{Hk}_g : \mathbf{S}_p^\circ(\mathbf{V}^\circ, -)_g \rightarrow \mathbf{S}_p^\circ(\mathbf{V}^\circ, -) \times \mathbf{S}_p^\circ(\mathbf{V}^\circ, -)$$

as a morphism in $\text{Fun}(\mathfrak{K}(\mathbf{V}^\circ)^p \times \mathfrak{T}, \text{Sch}_{/\mathbb{F}_p^\Phi})_{/\mathbf{T}_p}$ that is finite étale and compatible with the uniformization map.

5.3. Basic correspondence on ground stratum. In this subsection, we construct and study the basic correspondence on the ground stratum $\mathbf{M}_p^\bullet(\mathbf{V}^\circ, -)$. We assume $N \geq 2$.

Definition 5.3.1. We define a functor

$$\begin{aligned} \mathbf{S}_p^\bullet(\mathbf{V}^\circ, -) : \mathfrak{K}(\mathbf{V}^\circ)^p \times \mathfrak{T} &\rightarrow \text{PSch}'_{/\mathbb{F}_p^\Phi} \\ \mathbf{K}^{p^\circ} &\mapsto \mathbf{S}_p^\bullet(\mathbf{V}^\circ, \mathbf{K}^{p^\circ}) \end{aligned}$$

such that for every $S \in \text{Sch}'_{/\mathbb{F}_p^\Phi}$, $\mathbf{S}_p^\bullet(\mathbf{V}^\circ, \mathbf{K}^{p^\circ})(S)$ is the set of equivalence classes of sextuples $(A_0, \lambda_0, \eta_0^p; A^\bullet, \lambda^\bullet, \eta^{p^\bullet})$ where

$\circ (A_0, \lambda_0, \eta_0^p)$ is an element in $\mathbf{T}_p(S)$;

- $(A^\bullet, \lambda^\bullet)$ is a unitary O_F -abelian scheme of signature type $N\Phi$ over S such that $\ker \lambda^\bullet[p^\infty]$ is trivial (resp. contained in $A^\bullet[p]$ of rank p^2) if N is even (resp. odd);
- η^{p^\bullet} is, for a chosen geometric point s on every connected component of S , a $\pi_1(S, s)$ -invariant K^{p^\bullet} -orbit of isomorphisms

$$\eta^{p^\bullet}: V^\circ \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p} \rightarrow \text{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}}^{\varpi \lambda_0, \lambda^\bullet} (H_1^{\text{ét}}(A_{0s}, \mathbb{A}^{\infty, p}), H_1^{\text{ét}}(A_s^\bullet, \mathbb{A}^{\infty, p}))$$

of hermitian spaces over $F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p} = F \otimes_{F^+} \mathbb{A}_{F^+}^{\infty, p}$.

The equivalence relation and the action of morphisms in $\mathfrak{K}(V^\circ)^p \times \mathfrak{T}$ are defined similarly as in Definition 4.1.2.

We have apparently the forgetful morphism

$$S_{\mathfrak{p}}^\bullet(V^\circ, -) \rightarrow T_{\mathfrak{p}}$$

in $\text{Fun}(\mathfrak{K}(V^\circ)^p \times \mathfrak{T}, \text{PSch}'_{/\mathbb{F}_p^\Phi})$ which is represented by finite and étale schemes.

Now we take a point $s^\bullet = (A_0, \lambda_0, \eta_0^p; A^\bullet, \lambda^\bullet, \eta^{p^\bullet}) \in S_{\mathfrak{p}}^\bullet(V^\circ, K^{p^\circ})(\kappa)$ where κ is a perfect field containing \mathbb{F}_p^Φ . By Remark 3.4.10, the (κ, σ^{-1}) -linear Verschiebung map

$$V: H_1^{\text{dR}}(A^\bullet/\kappa)_{\tau_\infty} \rightarrow H_1^{\text{dR}}(A^\bullet/\kappa)_{\sigma^{-1}\tau_\infty} = H_1^{\text{dR}}(A^\bullet/\kappa)_{\tau_\infty^\varepsilon},$$

is an isomorphism. Thus, we obtain a (κ, σ) -linear isomorphism

$$V^{-1}: H_1^{\text{dR}}(A^\bullet/\kappa)_{\tau_\infty^\varepsilon} \rightarrow H_1^{\text{dR}}(A^\bullet/\kappa)_{\tau_\infty}.$$

We define a pairing

$$\{ , \}_{s^\bullet}: H_1^{\text{dR}}(A^\bullet/\kappa)_{\tau_\infty^\varepsilon} \times H_1^{\text{dR}}(A^\bullet/\kappa)_{\tau_\infty} \rightarrow \kappa$$

by the formula $\{x, y\}_{s^\bullet} := \langle V^{-1}x, y \rangle_{\lambda^\bullet, \tau_\infty}$ (Notation 3.4.7). To ease notation, we put

$$\mathcal{V}_{s^\bullet} := H_1^{\text{dR}}(A^\bullet/\kappa)_{\tau_\infty^\varepsilon}.$$

By the same proof of Lemma 4.2.2, we know that $(\mathcal{V}_{s^\bullet}, \{ , \}_{s^\bullet})$ is admissible. Thus, we have the Deligne–Lusztig variety $\text{DL}_{s^\bullet}^\bullet := \text{DL}^\bullet(\mathcal{V}_{s^\bullet}, \{ , \}_{s^\bullet})$ (Definition A.2.1). Moreover, $\dim_\kappa \mathcal{V}_{s^\bullet}^\perp$ is equal to 0 (resp. 1) when N is even (resp. odd).

Definition 5.3.2. We define a functor

$$\begin{aligned} B_{\mathfrak{p}}^\bullet(V^\circ, -): \mathfrak{K}(V^\circ)^p \times \mathfrak{T} &\rightarrow \text{PSch}'_{/\mathbb{F}_p^\Phi} \\ K^{p^\circ} &\mapsto B_{\mathfrak{p}}^\bullet(V^\circ, K^{p^\circ}) \end{aligned}$$

such that for every $S \in \text{Sch}'_{/\mathbb{F}_p^\Phi}$, $B_{\mathfrak{p}}^\bullet(V^\circ, K^{p^\circ})(S)$ is the set of equivalence classes of decuples $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^\bullet, \lambda^\bullet, \eta^{p^\bullet}; \gamma)$ where

- $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p)$ is an element of $M_{\mathfrak{p}}^\bullet(V^\circ, K^{p^\circ})(S)$;
- $(A_0, \lambda_0, \eta_0^p; A^\bullet, \lambda^\bullet, \eta^{p^\bullet})$ is an element of $S_{\mathfrak{p}}^\bullet(V^\circ, K^{p^\circ})(S)$;
- $\gamma: A \rightarrow A^\bullet$ is an O_F -linear quasi- p -isogeny (Definition 3.4.5) such that
 - (a) $\ker \gamma[p^\infty]$ is contained in $A[p]$;
 - (b) $(\ker \gamma_{*, \tau_\infty})^\perp$ is contained in $\omega_{A^\vee/S, \tau_\infty^\varepsilon}$;
 - (c) we have $\varpi \cdot \lambda = \gamma^\vee \circ \lambda^\bullet \circ \gamma$; and
 - (d) the K^{p° -orbit of maps $v \mapsto \gamma_* \circ \eta^p(v)$ for $v \in V^\circ \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$ coincides with η^{p^\bullet} .

The equivalence relation and the action of morphisms in $\mathfrak{K}(V^\circ)^p \times \mathfrak{T}$ are defined similarly as in Definition 4.2.3.

We obtain in the obvious way a correspondence

$$(5.5) \quad \mathbf{S}_p^\bullet(V^\circ, -) \xleftarrow{\pi^\bullet} \mathbf{B}_p^\bullet(V^\circ, -) \xrightarrow{\iota^\bullet} \mathbf{M}_p^\bullet(V^\circ, -)$$

in $\text{Fun}(\mathfrak{K}(V^\circ)^p \times \mathfrak{T}, \text{PSch}'_{/\mathbb{F}_p^\Phi})_{/\mathbb{T}_p}$.

Definition 5.3.3 (Basic correspondence). We refer to (5.5) as the *basic correspondence* on the ground stratum $\mathbf{M}_p^\bullet(V^\circ, -)$, with $\mathbf{S}_p^\bullet(V^\circ, -)$ being the *source* of the basic correspondence.

Theorem 5.3.4. *In the diagram (5.5), take a point*

$$s^\bullet = (A_0, \lambda_0, \eta_0^p; A^\bullet, \lambda^\bullet, \eta^{p^\bullet}) \in \mathbf{S}_p^\bullet(V^\circ, K^{p^\circ})(\kappa)$$

where κ is a perfect field containing \mathbb{F}_p^Φ . Put $\mathbf{B}_{s^\bullet}^\bullet := \pi^{\bullet-1}(s^\bullet)$, and denote by $(\mathcal{A}, \lambda, \eta^p; \gamma)$ the universal object over the fiber $\mathbf{B}_{s^\bullet}^\bullet$.

(1) The fiber $\mathbf{B}_{s^\bullet}^\bullet$ is a smooth scheme over κ , whose tangent sheaf $\mathcal{T}_{\mathbf{B}_{s^\bullet}^\bullet/\kappa}$ fits canonically into an exact sequence

$$0 \rightarrow \mathcal{H}om\left(\omega_{\mathcal{A}^\vee, \tau_\infty}, \omega_{\mathcal{A}^\vee, \tau_\infty}^\perp / \omega_{\mathcal{A}^\vee, \tau_\infty}\right) \rightarrow \mathcal{T}_{\mathbf{B}_{s^\bullet}^\bullet/\kappa} \rightarrow \mathcal{H}om\left(\omega_{\mathcal{A}^\vee, \tau_\infty} / (\ker \gamma_{*, \tau_\infty})^\perp, \text{Lie}_{\mathcal{A}^\vee, \tau_\infty}\right) \rightarrow 0.$$

(2) The restriction of ι_κ^\bullet to $\mathbf{B}_{s^\bullet}^\bullet$ is locally on $\mathbf{B}_{s^\bullet}^\bullet$ a closed immersion, with a canonical isomorphism for its normal sheaf

$$\mathcal{N}_{\iota_\kappa^\bullet|_{\mathbf{B}_{s^\bullet}^\bullet}} \simeq \mathcal{H}om\left((\ker \gamma_{*, \tau_\infty})^\perp / H_1^{\text{dR}}(\mathcal{A})_{\tau_\infty}^\perp, \text{Lie}_{\mathcal{A}^\vee, \tau_\infty}\right) \simeq (\text{im } \gamma_{*, \tau_\infty}) \otimes_{\mathcal{O}_{\mathbf{B}_{s^\bullet}^\bullet}} \text{Lie}_{\mathcal{A}^\vee, \tau_\infty}.$$

(3) The assignment sending $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^\bullet, \lambda^\bullet, \eta^{p^\bullet}; \gamma) \in \mathbf{B}_{s^\bullet}^\bullet(S)$ to the subbundles

$$H_1 := ((\check{\gamma}_{*, \tau_\infty})^{-1} \omega_{A^\vee/S, \tau_\infty})^\perp \subseteq H_1^{\text{dR}}(A^\bullet/S)_{\tau_\infty} = H_1^{\text{dR}}(A^\bullet/\kappa)_{\tau_\infty} \otimes_\kappa \mathcal{O}_S = (\mathcal{V}_{s^\bullet})_S,$$

$$H_2 := \gamma_{*, \tau_\infty} \omega_{A^\vee/S, \tau_\infty} \subseteq H_1^{\text{dR}}(A^\bullet/S)_{\tau_\infty} = H_1^{\text{dR}}(A^\bullet/\kappa)_{\tau_\infty} \otimes_\kappa \mathcal{O}_S = (\mathcal{V}_{s^\bullet})_S,$$

where $\check{\gamma}: A^\bullet \rightarrow A$ is the (unique) O_F -linear quasi- p -isogeny such that $\check{\gamma} \circ \gamma = \varpi \cdot \text{id}_A$, induces an isomorphism

$$\zeta_{s^\bullet}^\bullet: \mathbf{B}_{s^\bullet}^\bullet \xrightarrow{\sim} \text{DL}_{s^\bullet}^\bullet = \text{DL}^\bullet(\mathcal{V}_{s^\bullet}, \{ \cdot, \cdot \}_{s^\bullet}).$$

In particular, $\mathbf{B}_{s^\bullet}^\bullet$ is a geometrically irreducible projective smooth scheme in $\text{Sch}_{/\kappa}$ of dimension $\lfloor \frac{N}{2} \rfloor$.

(4) If we denote by $(\mathcal{H}_{s^\bullet \cdot 1}, \mathcal{H}_{s^\bullet \cdot 2})$ the universal object over $\text{DL}_{s^\bullet}^\bullet$, then there is a canonical isomorphism

$$\zeta_{s^\bullet}^{\bullet*} \left(\mathcal{H}_{s^\bullet \cdot 1}^\perp / \mathcal{H}_{s^\bullet \cdot 2} \right) \simeq \iota^{\bullet*} \text{Lie}_{\mathcal{A}, \tau_\infty}$$

of line bundles on $\mathbf{B}_{s^\bullet}^\bullet$.

Proof. For an object $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^\bullet, \lambda^\bullet, \eta^{p^\bullet}; \gamma) \in \mathbf{B}_p^\bullet(V^\circ, K^{p^\circ})(S)$, Definition 5.3.2(a) implies that there is a (unique) O_F -linear quasi- p -isogeny $\check{\gamma}: A^\bullet \rightarrow A$ such that $\check{\gamma} \circ \gamma = \varpi \cdot \text{id}_A$ hence $\gamma \circ \check{\gamma} = \varpi \cdot \text{id}_{A^\bullet}$. Moreover, we have the following properties from Definition 5.3.2:

- (a') $\ker \check{\gamma}[p^\infty]$ is contained in $A^\bullet[\mathfrak{p}]$;
- (b') $(\text{im } \check{\gamma}_{*, \tau_\infty})^\perp$ is contained in $\omega_{\mathcal{A}^\vee, \tau_\infty}$;
- (c') we have $\varpi \cdot \lambda^\bullet = \check{\gamma}^\vee \circ \lambda \circ \check{\gamma}$; and
- (d') the K^p -orbit of maps $v \mapsto \varpi^{-1} \check{\gamma}_* \circ \eta^{p^\bullet}(v)$ for $v \in V^\circ \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$ coincides with η^p .

First, we show (1). It is clear that $\mathbf{B}_{s^\bullet}^\bullet$ is a scheme of finite type over κ . Consider a closed immersion $S \hookrightarrow \hat{S}$ in $\text{Sch}'_{/\kappa}$ defined by an ideal sheaf \mathcal{I} satisfying $\mathcal{I}^2 = 0$. Take a point $x = (A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^\bullet, \lambda^\bullet, \eta^{p^\bullet}; \gamma) \in \mathbf{B}_{s^\bullet}^\bullet(S)$. To compute lifting of x to \hat{S} , we use the Serre–Tate and Grothendieck–Messing theories. Note that lifting γ is equivalent to lifting both γ and $\check{\gamma}$, satisfying (b,c,d) in Definition 5.3.2 and (b',c',d') above, respectively. Thus, by Proposition 3.4.8, to lift x to an \hat{S} -point is equivalent to lifting

- $\omega_{A^\vee/S, \tau_\infty}$ to a subbundle $\hat{\omega}_{A^\vee, \tau_\infty}$ of $H_1^{\text{cris}}(A/\hat{S})_{\tau_\infty}$ (of rank 1),
- $\omega_{A^\vee/S, \tau_\infty^\varepsilon}$ to a subbundle $\hat{\omega}_{A^\vee, \tau_\infty^\varepsilon}$ of $H_1^{\text{cris}}(A/\hat{S})_{\tau_\infty^\varepsilon}$ (of rank $N - 1$),

subject to the following requirements

- (a'') $\hat{\omega}_{A^\vee, \tau_\infty}$ and $\hat{\omega}_{A^\vee, \tau_\infty^\varepsilon}$ are orthogonal under $\langle \cdot, \cdot \rangle_{\lambda, \tau_\infty}^{\text{cris}}$ (3.3);
- (b'') $(\check{\gamma}_{*, \tau_\infty} H_1^{\text{cris}}(A^\bullet/\hat{S})_{\tau_\infty})^\perp$ is contained in $\hat{\omega}_{A^\vee, \tau_\infty^\varepsilon}$.

As $\check{\gamma}_{*, \tau_\infty} H_1^{\text{cris}}(A^\bullet/\hat{S})_{\tau_\infty} = \ker \gamma_{*, \tau_\infty} \subseteq H_1^{\text{cris}}(A/\hat{S})_{\tau_\infty}$, (b'') is equivalent to

- (c'') $(\ker \gamma_{*, \tau_\infty})^\perp$ is contained in $\hat{\omega}_{A^\vee, \tau_\infty^\varepsilon}$.

To summarize, lifting x to an \hat{S} -point is equivalent to lifting $\omega_{A^\vee/S, \tau_\infty^\varepsilon}$ to a subbundle $\hat{\omega}_{A^\vee, \tau_\infty^\varepsilon}$ of $H_1^{\text{cris}}(A/\hat{S})_{\tau_\infty^\varepsilon}$ containing $(\ker \gamma_{*, \tau_\infty})^\perp$, and then lifting $\omega_{A^\vee/S, \tau_\infty}$ to a subbundle $\hat{\omega}_{A^\vee, \tau_\infty}$ of $\hat{\omega}_{A^\vee, \tau_\infty^\varepsilon}^\perp$. Thus, (1) follows.

Next, we show (2). By Theorem 5.1.5(4), the map $\mathcal{T}_{\mathbb{B}_s^\bullet/\kappa} \rightarrow \iota^{\bullet*} \mathcal{T}_{\mathbb{M}_p^\bullet(V^\circ, K^{p^\circ})/\kappa}|_{\mathbb{B}_s^\bullet}$ is induced by the canonical map

$$\mathcal{H}om\left(\omega_{A^\vee, \tau_\infty^\varepsilon}/(\ker \gamma_{*, \tau_\infty})^\perp, \text{Lie}_{A^\vee, \tau_\infty^\varepsilon}\right) \rightarrow \mathcal{H}om\left(\omega_{A^\vee, \tau_\infty^\varepsilon}/H_1^{\text{dR}}(\mathcal{A})_{\tau_\infty}^\perp, \text{Lie}_{A^\vee, \tau_\infty^\varepsilon}\right).$$

It is clearly injective, whose cokernel is canonically isomorphic to

$$\begin{aligned} & \mathcal{H}om\left((\ker \gamma_{*, \tau_\infty})^\perp/H_1^{\text{dR}}(\mathcal{A})_{\tau_\infty}^\perp, \text{Lie}_{A^\vee, \tau_\infty^\varepsilon}\right) \\ & \simeq \mathcal{H}om\left((\text{im } \gamma_{*, \tau_\infty})^\vee, \text{Lie}_{A^\vee, \tau_\infty^\varepsilon}\right) \simeq (\text{im } \gamma_{*, \tau_\infty}) \otimes_{\mathcal{O}_{\mathbb{B}_s^\bullet}} \text{Lie}_{A^\vee, \tau_\infty^\varepsilon}. \end{aligned}$$

We obtain (2).

Third, we show (3). We first show that ζ_s^\bullet has the correct image, namely, we check

- $\text{rank}_{\mathcal{O}_S} H_1 = \lceil \frac{N}{2} \rceil$ and $\text{rank}_{\mathcal{O}_S} H_2 = \lceil \frac{N}{2} \rceil - 1$: By Lemma 3.4.13(2,3) and Definition 5.3.2, we have

$$\begin{aligned} \text{rank}_{\mathcal{O}_S}(\ker \gamma_{*, \tau_\infty}) + \text{rank}_{\mathcal{O}_S}(\ker \gamma_{*, \tau_\infty^\varepsilon}) &= 2\lceil \frac{N}{2} \rceil + 1, \\ \text{rank}_{\mathcal{O}_S}(\ker \gamma_{*, \tau_\infty}) - \text{rank}_{\mathcal{O}_S}(\ker \gamma_{*, \tau_\infty^\varepsilon}) &= 1, \end{aligned}$$

which imply

$$(5.6) \quad \text{rank}_{\mathcal{O}_S}(\ker \gamma_{*, \tau_\infty}) = \lceil \frac{N+1}{2} \rceil, \quad \text{rank}_{\mathcal{O}_S}(\ker \gamma_{*, \tau_\infty^\varepsilon}) = \lceil \frac{N-1}{2} \rceil.$$

Thus, we obtain $\text{rank}_{\mathcal{O}_S} H_1 = \lceil \frac{N}{2} \rceil$. Since $\ker \gamma_{*, \tau_\infty^\varepsilon} \subseteq (\ker \gamma_{*, \tau_\infty})^\perp \subseteq \omega_{A^\vee/S, \tau_\infty^\varepsilon}$, we have $H_2 = \gamma_{*, \tau_\infty^\varepsilon} \omega_{A^\vee/S, \tau_\infty^\varepsilon} \simeq \omega_{A^\vee/S, \tau_\infty^\varepsilon} / \ker \gamma_{*, \tau_\infty^\varepsilon}$. Thus, we obtain $\text{rank}_{\mathcal{O}_S} H_2 = \lceil \frac{N}{2} \rceil - 1$.

- $H_1^{\text{dR}}(A^\bullet/S)_{\tau_\infty}^\perp \subseteq H_2$: By Definition 5.3.2(c) and the definition of $\check{\gamma}$, we have $\lambda \circ \check{\gamma} = \gamma^\vee \circ \lambda^\bullet$, which implies

$$(5.7) \quad (\ker \gamma_{*, \tau_\infty})^\perp = \gamma_{*, \tau_\infty^\varepsilon}^{-1} (H_1^{\text{dR}}(A^\bullet/S)_{\tau_\infty}^\perp).$$

By Definition 5.3.2(b), H_2 contains $\gamma_{*, \tau_\infty^\varepsilon} (\ker \gamma_{*, \tau_\infty})^\perp$ in which the latter coincides with $H_1^{\text{dR}}(A^\bullet/S)_{\tau_\infty}^\perp$ by (5.7).

- $H_2 \subseteq H_1$: As $\lambda \circ \check{\gamma} = \gamma^\vee \circ \lambda^\bullet$, we have

$$\langle (\check{\gamma}_{*, \tau_\infty})^{-1} \omega_{A^\vee/S, \tau_\infty}, \gamma_{*, \tau_\infty^\varepsilon} \omega_{A^\vee/S, \tau_\infty^\varepsilon} \rangle_{\lambda^\bullet, \tau_\infty} = \langle \check{\gamma}_{*, \tau_\infty} (\check{\gamma}_{*, \tau_\infty})^{-1} \omega_{A^\vee/S, \tau_\infty}, \omega_{A^\vee/S, \tau_\infty^\varepsilon} \rangle_{\lambda, \tau_\infty} = 0.$$

Thus, we have $H_2 \subseteq H_1$.

- $H_2 \subseteq H_1^\perp$: Note that we have

$$\text{im } \gamma_{*, \tau_\infty^\varepsilon} = \ker \check{\gamma}_{*, \tau_\infty^\varepsilon} = (\check{\gamma}_{*, \tau_\infty^\varepsilon})^{-1} (\mathbf{F}\omega_{A^\vee/S, \tau_\infty}^{(p)}) \subseteq \mathbf{F}((\check{\gamma}_{*, \tau_\infty})^{-1} \omega_{A^\vee/S, \tau_\infty}) = \mathbf{F}((H_1^{(p)})^\perp).$$

Thus, $(\mathbf{F}((H_1^{(p)})^\perp))^\perp \subseteq (\mathrm{im} \gamma_{*,\tau_\infty^c})^\perp$, which in turn implies $H_1^{(p)} \subseteq \mathbf{V}((\mathrm{im} \gamma_{*,\tau_\infty^c})^\perp)$, which further implies $\mathbf{V}^{-1}H_1^{(p)} \subseteq (\mathrm{im} \gamma_{*,\tau_\infty^c})^\perp$, which implies $\mathrm{im} \gamma_{*,\tau_\infty^c} \subseteq H_1^\perp$. By comparing ranks via (5.6), we obtain

$$(5.8) \quad \mathrm{im} \gamma_{*,\tau_\infty^c} = H_1^\perp.$$

In particular, H_1^\perp contains H_2 as $\mathrm{im} \gamma_{*,\tau_\infty^c}$ does.

- $H_1 \subseteq H_2^\perp$: Note that $H_2^{(p)} = \gamma_{*,\tau_\infty}(\mathbf{V}H_1^{\mathrm{dR}}(A/S)_{\tau_\infty^c}) = \mathbf{V}(\mathrm{im} \gamma_{*,\tau_\infty}) = \mathbf{V}(\ker \check{\gamma}_{*,\tau_\infty}) \subseteq \mathbf{V}(H_1^\perp)$. Thus, $\mathbf{V}^{-1}H_2^{(p)} \subseteq H_1^\perp$, which implies $H_1 \subseteq (\mathbf{V}^{-1}H_2^{(p)})^\perp = H_2^\perp$.
- $H_1^\perp \subseteq H_2^\perp$: It follows from $H_2 \subseteq H_1$.

Since the target of ζ_s^\bullet is smooth over κ by Proposition A.2.2, to see that ζ_s^\bullet is an isomorphism, it suffices to check that for every algebraically closed field κ' containing κ

(3.1) ζ_s^\bullet induces a bijection on κ' -points; and

(3.2) ζ_s^\bullet induces an isomorphism on the tangent spaces at every κ -point.

To ease notation, we may assume $\kappa' = \kappa$.

For (3.1), we construct an inverse to the map $\zeta_s^\bullet(\kappa)$. Take a point $y \in \mathrm{DL}_s^\bullet(\kappa)$ represented by κ -linear subspaces

$$H_1^{\mathrm{dR}}(A^\bullet)_{\tau_\infty}^\perp \subseteq H_2 \subseteq H_1 \subseteq \mathcal{V}_s^\bullet = H_1^{\mathrm{dR}}(A^\bullet)_{\tau_\infty^c}.$$

We regard \mathbf{F} and \mathbf{V} as those sesquilinear maps in Remark 3.4.10. For every $\tau \in \Sigma_\infty$, we define a $W(\kappa)$ -submodule $\mathcal{D}_{A,\tau} \subseteq \mathcal{D}(A^\bullet)_\tau$ as follows.

- If $\tau \notin \{\tau_\infty, \tau_\infty^c\}$, then $\mathcal{D}_{A,\tau} = \mathcal{D}(A^\bullet)_\tau$.
- We set $\mathcal{D}_{A,\tau_\infty} := \mathbf{V}^{-1}\tilde{H}_2$, where \tilde{H}_2 is the preimage of H_2 under the reduction map $\mathcal{D}(A^\bullet)_{\tau_\infty^c} \rightarrow \mathcal{D}(A^\bullet)_{\tau_\infty^c}/p\mathcal{D}(A^\bullet)_{\tau_\infty^c} = H_1^{\mathrm{dR}}(A^\bullet)_{\tau_\infty^c}$.
- We set $\mathcal{D}_{A,\tau_\infty^c} := \mathbf{F}\tilde{H}_1^c$, where \tilde{H}_1^c is the preimage of H_1^\perp under the reduction map $\mathcal{D}(A^\bullet)_{\tau_\infty} \rightarrow \mathcal{D}(A^\bullet)_{\tau_\infty}/p\mathcal{D}(A^\bullet)_{\tau_\infty} = H_1^{\mathrm{dR}}(A^\bullet)_{\tau_\infty}$.

Finally, put $\mathcal{D}_A := \bigoplus_{\tau \in \Sigma_\infty} \mathcal{D}_{A,\tau}$ as a $W(\kappa)$ -submodule of $\mathcal{D}(A^\bullet)$. We show that it is stable under \mathbf{F} and \mathbf{V} . It suffices to show that both \mathbf{F} and \mathbf{V} stabilize $\mathcal{D}_{A,\tau_\infty} \oplus \mathcal{D}_{A,\tau_\infty^c}$, which breaks into checking that

- $\mathbf{F}\mathcal{D}_{A,\tau_\infty} \subseteq \mathcal{D}_{A,\tau_\infty^c}$, that is, $\mathbf{F}\mathbf{V}^{-1}\tilde{H}_2 \subseteq \mathbf{F}\tilde{H}_1^c$. It suffices to show that $\mathbf{V}^{-1}H_2$ (as a subspace of $H_1^{\mathrm{dR}}(A^\bullet)_{\tau_\infty}$) is contained in H_1^\perp , which follow from the relation $H_1 \subseteq H_2^\perp$.
- $\mathbf{F}\mathcal{D}_{A,\tau_\infty^c} \subseteq \mathcal{D}_{A,\tau_\infty}$, that is, $\mathbf{F}\tilde{H}_1^c \subseteq \mathbf{V}^{-1}\tilde{H}_2$. It suffices to show $p\tilde{H}_1^c \subseteq \tilde{H}_2$, which obviously holds.
- $\mathbf{V}\mathcal{D}_{A,\tau_\infty} \subseteq \mathcal{D}_{A,\tau_\infty^c}$, that is, $\mathbf{V}\mathbf{V}^{-1}\tilde{H}_2 \subseteq \mathbf{F}\tilde{H}_1^c$. It suffices to show $H_2 \subseteq \mathbf{F}H_1^\perp$, which follows from the identity $\mathbf{F}H_1^\perp = (\mathbf{V}^{-1}H_1)^\perp$ and the relation $H_2 \subseteq H_1^\perp$.
- $\mathbf{V}\mathcal{D}_{A,\tau_\infty^c} \subseteq \mathcal{D}_{A,\tau_\infty}$, that is, $\mathbf{V}\tilde{H}_1^c \subseteq \mathbf{V}^{-1}\tilde{H}_2$. It is obvious as $\mathbf{V}^{-1}\tilde{H}_2$ contains $p\mathcal{D}(A^\bullet)_{\tau_\infty}$.

Thus, $(\mathcal{D}_A, \mathbf{F}, \mathbf{V})$ is a Dieudonné module over $W(\kappa)$. By the Dieudonné theory, there is an O_F -abelian scheme A over κ with $\mathcal{D}(A)_\tau = \mathcal{D}_{A,\tau}$ for every $\tau \in \Sigma_\infty$, and an O_F -linear isogeny $\gamma: A \rightarrow A^\bullet$ inducing the inclusion of Dieudonné modules $\mathcal{D}(A) = \mathcal{D}_A \subseteq \mathcal{D}(A^\bullet)$. Moreover, since $p\mathcal{D}(A^\bullet) \subseteq \mathcal{D}(A)$, we have $\ker \gamma[p^\infty] \subseteq A[p]$. Now we check that $(\ker \gamma_{*,\tau_\infty})^\perp$ is contained in $\omega_{A^\vee/S, \tau_\infty^c}$, which is equivalent to that $p\mathcal{D}(A^\bullet)_{\tau_\infty}^\vee \cap \mathcal{D}(A)_{\tau_\infty^c} \subseteq \mathbf{V}\mathcal{D}(A)_{\tau_\infty}$. However, as H_2 contains $H_1^{\mathrm{dR}}(A^\bullet)_{\tau_\infty}^\perp$, we have $p\mathcal{D}(A^\bullet)_{\tau_\infty}^\vee \subseteq \tilde{H}_2 = \mathbf{V}\mathcal{D}(A)_{\tau_\infty}$.

Let $\lambda: A \rightarrow A^\vee$ be the unique quasi-polarization such that $\varpi\lambda = \gamma^\vee \circ \lambda^\bullet \circ \gamma$. We claim that $\lambda[p^\infty]$ is a polarization whose kernel is contained in $A[p]$ of rank p^2 . Since $H_2 \subseteq H_1$, we have $\langle \tilde{H}_1^c, \tilde{H}_2 \rangle_{\lambda^\bullet, \tau_\infty} \subseteq pW(\kappa)$, which implies $\langle \mathcal{D}(A)_{\tau_\infty}, \mathcal{D}(A)_{\tau_\infty^c} \rangle_{\lambda^\bullet, \tau_\infty} \subseteq pW(\kappa)$. It is enough to show that the inclusion $\mathcal{D}(A)_{\tau_\infty^c} \rightarrow \mathcal{D}(A)_{\tau_\infty}^\vee$ induced from $\langle \ , \ \rangle_{\lambda^\bullet, \tau_\infty}$ has cokernel of length $N + 1$. This

follows from the facts that the cokernel of $\mathcal{D}(A^\bullet)_{\tau_\infty^c} \hookrightarrow \mathcal{D}(A^\bullet)_{\tau_\infty^c}^\vee$ has length $N - 2\lfloor \frac{N}{2} \rfloor$, and the cokernel of $\mathcal{D}(A)_{\tau_\infty} \oplus \mathcal{D}(A)_{\tau_\infty^c} \hookrightarrow \mathcal{D}(A^\bullet)_{\tau_\infty} \oplus \mathcal{D}(A^\bullet)_{\tau_\infty^c}$ has length $2\lfloor \frac{N}{2} \rfloor + 1$.

It is an easy consequence of Lemma 3.4.13(2) that the O_F -abelian scheme A has signature type $N\Phi - \tau_\infty + \tau_\infty^c$. Finally, let η^p be the unique K^p -level structure such that Definition 4.2.3(d) is satisfied. Putting together, we obtain a point $x = (A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^\bullet, \lambda^\bullet, \eta^{p^\bullet}; \gamma) \in \mathbf{B}_{s^\bullet}(\kappa)$ such that $\zeta_{s^\bullet}^\bullet(x) = y$. It is easy to see that such assignment gives rise to an inverse of $\zeta_{s^\bullet}^\bullet(\kappa)$; hence (3.1) follows immediately.

For (3.2), let \mathcal{T}_x and \mathcal{T}_y be the tangent spaces at x and y as in (3.1), respectively. By Proposition A.2.2 and the construction, the induced map $(\zeta_{s^\bullet}^\bullet)_*: \mathcal{T}_x \rightarrow \mathcal{T}_y$ fits into a commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_\kappa \left(\omega_{A^\vee, \tau_\infty}, \omega_{A^\vee, \tau_\infty}^\perp / \omega_{A^\vee, \tau_\infty} \right) & \longrightarrow & \mathcal{T}_x & \longrightarrow & \mathrm{Hom}_\kappa \left(\omega_{A^\vee, \tau_\infty} / (\ker \gamma_{*, \tau_\infty})^\perp, \mathrm{Lie}_{A^\vee, \tau_\infty} \right) \\ \downarrow & & \downarrow (\zeta_{s^\bullet}^\bullet)_* & & \downarrow \\ \mathrm{Hom}_\kappa \left(H_1/H_2, H_2^\perp/H_1 \right) & \longrightarrow & \mathcal{T}_y & \longrightarrow & \mathrm{Hom}_\kappa (H_2/\mathcal{V}_{s^\bullet}^\perp, H_1^\perp/H_2) \end{array}$$

in $\mathrm{Mod}(\kappa)$. The right vertical arrow is induced by maps

$$\omega_{A^\vee, \tau_\infty} / (\ker \gamma_{*, \tau_\infty})^\perp \xrightarrow{\gamma_{*, \tau_\infty}} H_2/\mathcal{V}_{s^\bullet}^\perp, \quad \mathrm{Lie}_{A^\vee, \tau_\infty} \simeq H_1^{\mathrm{dR}}(A)_{\tau_\infty} / \omega_{A^\vee, \tau_\infty} \xrightarrow{\gamma_{*, \tau_\infty}} H_1^\perp/H_2$$

which are both isomorphisms by (5.7) and (5.8), respectively. The left vertical arrow is the composition

$$\mathrm{Hom}_\kappa \left(\omega_{A^\vee, \tau_\infty}, \omega_{A^\vee, \tau_\infty}^\perp / \omega_{A^\vee, \tau_\infty} \right) \rightarrow \mathrm{Hom}_\kappa \left(H_1^\perp/V^{-1}H_2, H_2^\perp/H_1^\perp \right) \xrightarrow{\sim} \mathrm{Hom}_\kappa \left(H_1/H_2, H_2^\perp/H_1 \right)$$

in which the first arrow is induced by maps

$$H_1^\perp/V^{-1}H_2 \xrightarrow{\check{\gamma}_{*, \tau_\infty}} \omega_{A^\vee, \tau_\infty}, \quad H_2^\perp/H_1^\perp \xrightarrow{\check{\gamma}_{*, \tau_\infty}} \omega_{A^\vee, \tau_\infty}^\perp / \omega_{A^\vee, \tau_\infty}$$

which are both isomorphisms as $\check{\gamma}_{*, \tau_\infty}(H_1^\perp) = \omega_{A^\vee, \tau_\infty}$, $\check{\gamma}_{*, \tau_\infty}(V^{-1}H_2) = 0$, and $\check{\gamma}_{*, \tau_\infty}(H_2^\perp) = \omega_{A^\vee, \tau_\infty}^\perp$. Thus, $(\zeta_{s^\bullet}^\bullet)_*: \mathcal{T}_x \rightarrow \mathcal{T}_y$ is an isomorphism by the Five Lemma; hence (3.2) and (3) follow.

Finally, (4) is a consequence of (5.8). \square

Remark 5.3.5. We have the following remarks concerning Theorem 5.3.4.

- (1) When K^{p° is sufficiently small, the restriction of ι_κ^\bullet to $\mathbf{B}_{s^\bullet}^\bullet$ is a closed immersion for every point $s^\bullet \in \mathbf{S}_p^\bullet(V^\circ, K^{p^\circ})(\kappa)$ and every perfect field κ containing \mathbb{F}_p^Φ .
- (2) In fact, one can show that the union of $\mathbf{M}_p^\dagger(V^\circ, K^{p^\circ})$ and the image of $\iota^\bullet: \mathbf{B}_p^\bullet(V^\circ, K^{p^\circ}) \rightarrow \mathbf{M}_p^\bullet(V^\circ, K^{p^\circ})$ is exactly the basic locus of $\mathbf{M}_p^\bullet(V^\circ, K^{p^\circ})$. In particular, as long as $N \geq 5$, the basic locus of $\mathbf{M}_p^\bullet(V^\circ, K^{p^\circ})$ is *not* equidimensional.

Construction 5.3.6. To construct a uniformization map for $\mathbf{S}_p^\bullet(V^\circ, -)$, we need to choose an O_{F_p} -lattice Λ_p^\bullet in $V^\circ \otimes_F F_p$ satisfying

- $\circ \Lambda_p^\circ \subseteq \Lambda_p^\bullet \subseteq p^{-1}\Lambda_p^\circ$, and
- $\circ p\Lambda_p^\bullet \subseteq (\Lambda_p^\bullet)^\vee$ such that $(\Lambda_p^\bullet)^\vee/p\Lambda_p^\bullet$ has length 0 (resp. 1) if N is even (resp. odd).

Let K_p^\bullet be the stabilizer of Λ_p^\bullet ; and put $K_p^\bullet := K_p^\bullet \times \prod_{q|p, q \neq p} K_q^\circ$. Similar to Construction 4.3.2, we may construct a *uniformization map*

$$(5.9) \quad v^\bullet: \mathbf{S}_p^\bullet(V^\circ, -)(\overline{\mathbb{F}}_p) \xrightarrow{\sim} \mathrm{Sh}(V^\circ, -K_p^\bullet) \times T_p(\overline{\mathbb{F}}_p)$$

in $\mathrm{Fun}(\mathfrak{K}(V^\circ)^p \times \mathfrak{T}, \mathrm{Set})_{/T_p(\overline{\mathbb{F}}_p)}$ which is an isomorphism, under which the induced action of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^\Phi)$ on the target is trivial on $\mathrm{Sh}(V^\circ, -K_p^\bullet)$.

Moreover, similar to Construction 4.3.5 and Proposition 4.3.6, for every $g \in K_p^\bullet \backslash U(V^\circ)(F_p^+)/K_p^\bullet$, we may construct the Hecke correspondence

$$\mathrm{Hk}_g: S_p^\bullet(V^\circ, -)_g \rightarrow S_p^\bullet(V^\circ, -) \times S_p^\bullet(V^\circ, -)$$

as a morphism in $\mathrm{Fun}(\mathfrak{K}(V^\circ)^p \times \mathfrak{T}, \mathrm{Sch}/_{\mathbb{F}_p^\Phi})_{/\mathbb{T}_p}$ that is finite étale and compatible with the uniformization map.

5.4. Basic correspondence on link stratum. In this subsection, we construct and study the basic correspondence on the link stratum $M_p^\dagger(V^\circ, -)$. We also discuss its relation with the two previously constructed basic correspondences. We assume $N \geq 2$.

Definition 5.4.1. We define a functor

$$\begin{aligned} S_p^\dagger(V^\circ, -): \mathfrak{K}(V^\circ)^p \times \mathfrak{T} &\rightarrow \mathrm{PSch}'_{/\mathbb{F}_p^\Phi} \\ K^{p^\circ} &\mapsto S_p^\dagger(V^\circ, K^{p^\circ}) \end{aligned}$$

such that for every $S \in \mathrm{Sch}'_{/\mathbb{F}_p^\Phi}$, $S_p^\dagger(V^\circ, K^{p^\circ})(S)$ is the set of equivalence classes of decuples $(A_0, \lambda_0, \eta_0^p; A^\circ, \lambda^\circ, \eta^{p^\circ}; A^\bullet, \lambda^\bullet, \eta^{p^\bullet}; \psi)$ where

- $(A_0, \lambda_0, \eta_0^p; A^\circ, \lambda^\circ, \eta^{p^\circ})$ is an element in $S_p^\circ(V^\circ, K^{p^\circ})(S)$;
- $(A_0, \lambda_0, \eta_0^p; A^\bullet, \lambda^\bullet, \eta^{p^\bullet})$ is an element in $S_p^\bullet(V^\circ, K^{p^\circ})(S)$;
- $\psi: A^\circ \rightarrow A^\bullet$ is an O_F -linear quasi- p -isogeny (Definition 3.4.5) such that
 - (a) $\ker \psi[p^\infty]$ is contained in $A^\circ[p]$;
 - (b) we have $\varpi \cdot \lambda^\circ = \psi^\vee \circ \lambda^\bullet \circ \psi$; and
 - (c) the K^{p° -orbit of maps $v \mapsto \psi_* \circ \eta^{p^\circ}(v)$ for $v \in V^\circ \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$ coincides with η^{p^\bullet} .

The equivalence relation and the action of morphisms in $\mathfrak{K}(V^\circ)^p \times \mathfrak{T}$ are defined similarly as in Definition 4.2.3.

We have apparently the forgetful morphism

$$S_p^\dagger(V^\circ, -) \rightarrow \mathbb{T}_p$$

in $\mathrm{Fun}(\mathfrak{K}(V^\circ)^p \times \mathfrak{T}, \mathrm{PSch}'_{/\mathbb{F}_p^\Phi})$ which is represented by finite and étale schemes.

By definition, we have the two forgetful morphisms

$$s^{\dagger^\circ}: S_p^\dagger(V^\circ, -) \rightarrow S_p^\circ(V^\circ, -), \quad s^{\dagger^\bullet}: S_p^\dagger(V^\circ, -) \rightarrow S_p^\bullet(V^\circ, -)$$

in $\mathrm{Fun}(\mathfrak{K}(V^\circ)^p \times \mathfrak{T}, \mathrm{Sch}/_{\mathbb{F}_p^\Phi})_{/\mathbb{T}_p}$.

Definition 5.4.2. We define $B_p^\dagger(V^\circ, -)$ to be the limit of the following diagram

$$\begin{array}{ccccc} S_p^\circ(V^\circ, -) & \xleftarrow{\pi^\circ} & B_p^\circ(V^\circ, -) & \xrightarrow{\iota^\circ} & M_p^\circ(V^\circ, -) \\ \uparrow s^{\dagger^\circ} & & & & \uparrow m^{\dagger^\circ} \\ S_p^\dagger(V^\circ, -) & & & & M_p^\dagger(V^\circ, -) \\ \downarrow s^{\dagger^\bullet} & & & & \downarrow m^{\dagger^\bullet} \\ S_p^\bullet(V^\circ, -) & \xleftarrow{\pi^\bullet} & B_p^\bullet(V^\circ, -) & \xrightarrow{\iota^\bullet} & M_p^\bullet(V^\circ, -) \end{array}$$

in the category $\mathrm{Fun}(\mathfrak{K}(V^\circ)^p \times \mathfrak{T}, \mathrm{Sch}/_{\mathbb{F}_p^\Phi})_{/\mathbb{T}_p}$.

From the definition above, we have the following commutative diagram

$$(5.10) \quad \begin{array}{ccccc} S_p^\circ(V^\circ, -) & \xleftarrow{\pi^\circ} & B_p^\circ(V^\circ, -) & \xrightarrow[\sim]{\iota^\circ} & M_p^\circ(V^\circ, -) \\ & \swarrow s^{\dagger\circ} & \swarrow b^{\dagger\circ} & \swarrow m^{\dagger\circ} & \\ & & S_p^\dagger(V^\circ, -) & \xleftarrow{\pi^\dagger} & B_p^\dagger(V^\circ, -) & \xrightarrow{\iota^\dagger} & M_p^\dagger(V^\circ, -) \\ & & \searrow s^{\dagger\bullet} & \searrow b^{\dagger\bullet} & \searrow m^{\dagger\bullet} & & \\ & & & & S_p^\bullet(V^\circ, -) & \xleftarrow{\pi^\bullet} & B_p^\bullet(V^\circ, -) & \xrightarrow{\iota^\bullet} & M_p^\bullet(V^\circ, -) \end{array}$$

in $\text{Fun}(\mathfrak{K}(V^\circ)^p \times \mathfrak{T}, \text{Sch}/\mathbb{F}_p^\Phi)_{/T_p}$, together with the four new morphisms from $B_p^\dagger(V^\circ, -)$ as indicated. It will be clear in Subsection 5.10 why we draw the diagram oblique.

Theorem 5.4.3. *In the diagram (5.10), we have*

(1) *The square*

$$\begin{array}{ccc} B_p^\dagger(V^\circ, -) & \xrightarrow{\iota^\dagger} & M_p^\dagger(V^\circ, -) \\ \downarrow b^{\dagger\bullet} & & \downarrow m^{\dagger\bullet} \\ B_p^\bullet(V^\circ, -) & \xrightarrow{\iota^\bullet} & M_p^\bullet(V^\circ, -) \end{array}$$

is a Cartesian diagram.

(2) *Take a point $s^\dagger = (A_0, \lambda_0, \eta_0^p; A^\circ, \lambda^\circ, \eta^{p^\circ}; A^\bullet, \lambda^\bullet, \eta^{p^\bullet}; \psi) \in S_p^\dagger(V^\circ, K^{p^\circ})(\kappa)$ where κ is a perfect field containing \mathbb{F}_p^Φ . Put $B_{s^\dagger}^\dagger := \pi^{\dagger-1}(s^\dagger)$ and $\mathcal{V}_{s^\dagger} := (\text{im } \psi_{*, \tau_\infty^\circ}) / H_1^{\text{dR}}(A^\bullet / \kappa)_{\tau_\infty}^\perp$ which has dimension $\lfloor \frac{N}{2} \rfloor$. Then the assignment sending*

$$\begin{aligned} & ((A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^\circ, \lambda^\circ, \eta^{p^\circ}; \beta), (A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^\bullet, \lambda^\bullet, \eta^{p^\bullet}; \gamma)) \in B_{s^\dagger}^\dagger(S) \\ & \text{(with } \gamma = \psi \circ \beta) \text{ to } (\gamma_{*, \tau_\infty^\circ} \omega_{A^\vee/S, \tau_\infty^\circ}) / H_1^{\text{dR}}(A^\bullet/S)_{\tau_\infty}^\perp \end{aligned}$$

$$\zeta_{s^\dagger}^\dagger : B_{s^\dagger}^\dagger \xrightarrow{\sim} \mathbb{P}(\mathcal{V}_{s^\dagger}).$$

Proof. For (1), unravelling all the definitions, it suffices to show that for every object

$$((A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^\circ, \lambda^\circ, \eta^{p^\circ}; \beta), (A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^\bullet, \lambda^\bullet, \eta^{p^\bullet}; \gamma))$$

of $M_p^\dagger(V^\circ, K^{p^\circ})(S) \times_{M_p^\bullet(V^\circ, K^{p^\circ})(S)} B_p^\bullet(V^\circ, K^{p^\circ})(S)$, the quasi-isogeny $\psi := \gamma \circ \beta^{-1} : A^\circ \rightarrow A^\bullet$ is a quasi- p -isogeny. However, we know that $\beta_{*, \tau_\infty^\circ} : H_1^{\text{dR}}(A)_{\tau_\infty^\circ} \rightarrow H_1^{\text{dR}}(A^\circ)_{\tau_\infty^\circ}$ is an isomorphism; and $\ker \beta_{*, \tau_\infty^\circ} = \omega_{A^\vee, \tau_\infty^\circ}$. Thus, it suffices to show that $\omega_{A^\vee, \tau_\infty^\circ}$ is contained in $\ker \gamma$, which is clear since $\omega_{A^\bullet, \tau_\infty^\circ} = 0$.

For (2), we first show that for a point

$$x^\bullet = (A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^\bullet, \lambda^\bullet, \eta^{p^\bullet}; \gamma) \in B_p^\bullet(V^\circ, K^{p^\circ})(S),$$

$\iota^\bullet(x^\bullet)$ belongs to $M_p^\dagger(V^\circ, K^{p^\circ})(S)$ if and only if $H_1 = H_1^{-1}$, where we recall from Theorem 5.3.4 that $H_1 := ((\check{\gamma}_{*, \tau_\infty^\circ})^{-1} \omega_{A^\vee, \tau_\infty^\circ})^\perp$. In fact, by Definition 5.1.3, $\iota^\bullet(x^\bullet) \in M_p^\dagger(V^\circ, K^{p^\circ})(S)$ if and only if $\omega_{A^\vee, \tau_\infty^\circ} = H_{\text{dR}}^1(A)_{\tau_\infty^\circ}^\perp$. In the proof of Theorem 5.3.4, we see $\text{im } \gamma_{*, \tau_\infty^\circ} = H_1^{-1}$ (5.8). As $\lambda \circ \check{\gamma} = \gamma^\vee \circ \lambda^\bullet$, we have $(\text{im } \gamma_{*, \tau_\infty^\circ})^\perp = (\check{\gamma}_{*, \tau_\infty^\circ})^{-1} H_{\text{dR}}^1(A)_{\tau_\infty^\circ}^\perp$. Thus, if $\omega_{A^\vee, \tau_\infty^\circ} = H_{\text{dR}}^1(A)_{\tau_\infty^\circ}^\perp$, then $H_1 = ((\text{im } \gamma_{*, \tau_\infty^\circ})^\perp)^\perp$ which equals $\text{im } \gamma_{*, \tau_\infty^\circ} = H_1^{-1}$, as $\text{im } \gamma_{*, \tau_\infty^\circ}$ contains $H_1^{\text{dR}}(A^\bullet)_{\tau_\infty^\circ}^\perp$. On the other hand, if $H_1 = H_1^{-1}$, then $(\check{\gamma}_{*, \tau_\infty^\circ})^{-1} \omega_{A^\vee, \tau_\infty^\circ} = (\text{im } \gamma_{*, \tau_\infty^\circ})^\perp = (\check{\gamma}_{*, \tau_\infty^\circ})^{-1} H_{\text{dR}}^1(A)_{\tau_\infty^\circ}^\perp$, which implies easily that $\omega_{A^\vee, \tau_\infty^\circ} = H_{\text{dR}}^1(A)_{\tau_\infty^\circ}^\perp$.

Second, we show $H_1 = \text{im } \psi_{*, \tau_\infty^\circ}$ if $x^\bullet \in B_{s^\dagger}^\dagger(S)$. Since $\gamma = \psi \circ \beta$, we have $\text{im } \gamma_{*, \tau_\infty^\circ} \subseteq \text{im } \psi_{*, \tau_\infty^\circ}$. As $\text{im } \gamma_{*, \tau_\infty^\circ} = H_1^{-1} = H_1$, we have $H_1 \subseteq \text{im } \psi_{*, \tau_\infty^\circ}$. On the other hand, it follows easily from Lemma 3.4.13(2,3) that $\text{im } \psi_{*, \tau_\infty^\circ}$ has rank $\lfloor \frac{N}{2} \rfloor$. Thus, we must have $H_1 = \text{im } \psi_{*, \tau_\infty^\circ}$.

The above two claims together with Theorem 5.3.4(3) imply (2). \square

Remark 5.4.4. It follows from the proof of Theorem 5.4.3 that for every $s^\dagger \in S_p^\dagger(V^\circ, K^{p^\circ})(\kappa)$, if we put $s^\circ := s^{\dagger\circ}(s^\dagger)$ and $s^\bullet := s^{\dagger\bullet}(s^\dagger)$, then

- (1) the morphism $\zeta_{s^\circ}^\circ \circ b^{\dagger\circ} \circ (\zeta_{s^\dagger}^\dagger)^{-1}$ identifies $\mathbb{P}(\mathcal{V}_{s^\dagger})$ as a closed subscheme of $\mathbb{P}(\mathcal{V}_{s^\circ})$ induced by the obvious κ -linear (surjective) map $\mathcal{V}_{s^\circ} \rightarrow \mathcal{V}_{s^\dagger}$; and
- (2) the morphism $\zeta_{s^\bullet}^\bullet \circ b^{\dagger\bullet} \circ (\zeta_{s^\dagger}^\dagger)^{-1}$ identifies $\mathbb{P}(\mathcal{V}_{s^\dagger})$ as a closed subscheme (of codimension one) of $\text{DL}^\bullet(\mathcal{V}_{s^\bullet}, \{ , \}_s)$ defined by the condition $H_1 = H_1^{-1}$.

Construction 5.4.5. Put $K_p^\dagger := K_p^\circ \cap K_p^\bullet$. Similar to Construction 4.3.2, we construct a *uniformization map*

$$(5.11) \quad v^\dagger: S_p^\dagger(V^\circ, -)(\overline{\mathbb{F}}_p) \xrightarrow{\sim} \text{Sh}(V^\circ, -K_p^\dagger) \times T_p(\overline{\mathbb{F}}_p)$$

in $\text{Fun}(\mathfrak{K}(V^\circ)^p \times \mathfrak{T}, \text{Set})_{/T_p(\overline{\mathbb{F}}_p)}$ which is an isomorphism, under which the induced action of $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^\Phi)$ on the target is trivial on $\text{Sh}(V^\circ, -K_p^\dagger)$.

5.5. Cohomology of link stratum. In this subsection, we study the cohomology of the link stratum. We assume $N \geq 2$.

We first construct certain Hecke correspondences for $B_p^\circ(V^\circ, -)$ extending Construction 5.2.6. Unlike the functor $S_p^\circ(V^\circ, -)$, the natural action of $K_p^\circ = U(\Lambda_p^\circ)(O_{F_p^+})$ on the functor $B_p^\circ(V^\circ, -)$ is nontrivial. However, as we will see, such action factors through the quotient $U(\Lambda_p^\circ)(O_{F_p^+}) \rightarrow U(\Lambda_p^\circ)(\mathbb{F}_p)$. Let K_{p1}° be the kernel of the reduction map $K_p^\circ = U(\Lambda_p^\circ)(O_{F_p^+}) \rightarrow U(\Lambda_p^\circ)(\mathbb{F}_p)$.

Construction 5.5.1. We first define a functor

$$\begin{aligned} S_{p1}^\circ(V^\circ, -): \mathfrak{K}(V^\circ)^p \times \mathfrak{T} &\rightarrow \text{PSch}'_{/\mathbb{F}_p^\Phi} \\ K^{p^\circ} &\mapsto S_p^\circ(V^\circ, K^{p^\circ}) \end{aligned}$$

such that for every $S \in \text{Sch}'_{/\mathbb{F}_p^\Phi}$, $S_{p1}^\circ(V^\circ, K^{p^\circ})(S)$ is the set of equivalence classes of septuples $(A_0, \lambda_0, \eta_0^p; A^\circ, \lambda^\circ, \eta^{p^\circ}; \eta_p^\circ)$ where

- $\circ (A_0, \lambda_0, \eta_0^p; A^\circ, \lambda^\circ, \eta^{p^\circ})$ is an element in $S_p^\circ(V^\circ, K^{p^\circ})(S)$;
- $\circ \eta_p^\circ$ is, for a chosen geometric point s on every connected component of S , an isomorphism

$$\eta_p^\circ: \Lambda_p^\circ \otimes \mathbb{F}_p \rightarrow \text{Hom}_{O_F}(A_{0s}[\mathfrak{p}], A_s^\circ[\mathfrak{p}])$$

of hermitian spaces over $O_{F_p} \otimes \mathbb{F}_p$, where $\text{Hom}_{O_F}(A_{0s}[\mathfrak{p}], A_s^\circ[\mathfrak{p}])$ is equipped with the hermitian form constructed similarly as in Construction 3.4.4 with respect to $(\lambda_0, \lambda^\circ)$.

The equivalence relation and the action of morphisms in $\mathfrak{K}(V^\circ)^p \times \mathfrak{T}$ are defined similarly as in Definition 4.1.2. In fact, we have a further action of $U(\Lambda_p^\circ)(\mathbb{F}_p)$ on $S_{p1}^\circ(V^\circ, -)$. Moreover, similar to Construction 4.3.5 and Proposition 4.3.6, for every $g \in K_{p1}^\circ \backslash U(V^\circ)(F_p^+)/K_{p1}^\circ$, we may construct the Hecke correspondence

$$(5.12) \quad \text{Hk}_g: S_{p1}^\circ(V^\circ, -)_g \rightarrow S_{p1}^\circ(V^\circ, -) \times S_{p1}^\circ(V^\circ, -)$$

as a morphism in $\text{Fun}(\mathfrak{K}(V^\circ)^p \times \mathfrak{T}, \text{Sch}_{/\mathbb{F}_p^\Phi})_{/T_p}$ that is finite étale.

On the other hand, Theorem 5.2.4 implies that we have a canonical isomorphism

$$B_p^\circ(V^\circ, -) \simeq S_{p1}^\circ(V^\circ, -) \times^{U(\Lambda_p^\circ)(\mathbb{F}_p)} \mathbb{P}(\Lambda_p^\circ \otimes \mathbb{F}_p)$$

in the category $\text{Fun}(\mathfrak{K}(V^\circ)^p \times \mathfrak{T}, \text{Sch}_{/\mathbb{F}_p^\Phi})_{/T_p}$. Thus, for every $g \in K_{p1}^\circ \backslash U(V^\circ)(F_p^+)/K_{p1}^\circ$, we obtain from (5.12) the Hecke correspondence

$$\text{Hk}_g: B_p^\circ(V^\circ, -)_g \rightarrow B_p^\circ(V^\circ, -) \times B_p^\circ(V^\circ, -)$$

as a morphism in $\text{Fun}(\mathfrak{K}(V^\circ)^p \times \mathfrak{Z}, \text{Sch}/\mathbb{F}_p^\Phi)_{/\mathbb{T}_p}$ that is finite étale.

Now we study cohomology.

Lemma 5.5.2. *Consider a p -coprime coefficient ring L .*

(1) *If $p + 1$ is invertible in L , then the restriction map*

$$(m^{\dagger\circ})^*: H_{\mathfrak{X}}^i(\overline{M}_p^\circ(V^\circ, -), L) \rightarrow H_{\mathfrak{X}}^i(\overline{M}_p^\dagger(V^\circ, -), L)$$

is an isomorphism for every integer $i \notin \{N - 2, 2N - 2\}$. In particular, $H_{\mathfrak{X}}^i(\overline{M}_p^\circ(V^\circ, -), L)$ and $H_{\mathfrak{X}}^i(\overline{M}_p^\dagger(V^\circ, -), L)$ vanish if i is odd and different from $N - 2$.

(2) *For every $i \in \mathbb{Z}$, both $H_{\mathfrak{X}}^i(\overline{M}_p^\circ(V^\circ, -), L)$ and $H_{\mathfrak{X}}^i(\overline{M}_p^\dagger(V^\circ, -), L)$ are free L -modules.*

(3) *When N is even, the action of $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^\Phi)$ on $H_{\mathfrak{X}}^{N-2}(\overline{M}_p^\dagger(V^\circ, -), L(\frac{N-2}{2}))$ is trivial.*

Proof. By Theorem 5.2.4, for every $K^{p^\circ} \in \mathfrak{K}(V^\circ)^p$ and every $s^\circ \in S_p^\circ(V^\circ, K^{p^\circ})(\overline{\mathbb{F}}_p)$, the restriction of $(m^{\dagger\circ})^*$ to the fibers over s° is a morphism appeared in Lemma A.1.4.

Part (1) then follows from Lemma A.1.4(2). Part (2) follows from Lemma A.1.4(3). Part (3) follows from Lemma A.1.4(4) and Construction 5.2.6. \square

Definition 5.5.3. Let $\xi \in H_{\mathfrak{X}}^2(\overline{B}_p^\circ(V^\circ, -), L(1))$ be the first Chern class of the tautological quotient line bundle on $\overline{B}_p^\circ(V^\circ, -)$ (that is, in the situation of Theorem 5.2.4, the restriction of ξ to $B_{s^\circ}^\circ$ is isomorphic to $\zeta_{s^\circ}^{\circ*} \mathcal{O}_{\mathbb{P}(\mathcal{V}_{s^\circ})}(1)$ for every $K^{p^\circ} \in \mathfrak{K}(V^\circ)^p$ and every $s^\circ \in S_p^\circ(V^\circ, K^{p^\circ})(\overline{\mathbb{F}}_p)$). We define the *primitive cohomology* $H^{\text{prim}}(\overline{M}_p^\dagger(V^\circ, -), L(i))$ to be the kernel of the map

$$\cup(m^{\dagger\circ*} \iota_1^\circ \xi): H_{\mathfrak{X}}^{N-2}(\overline{M}_p^\dagger(V^\circ, -), L(i)) \rightarrow H_{\mathfrak{X}}^N(\overline{M}_p^\dagger(V^\circ, -), L(i+1)).$$

Proposition 5.5.4. *Take an object $K^{p^\circ} \in \mathfrak{K}(V^\circ)^p$ and a rational prime $\ell \neq p$. Let $\pi^{\infty,p}$ be an irreducible admissible representation of $U(V^\circ)(\mathbb{A}_{F^+}^{\infty,p})$ with coefficients in $\overline{\mathbb{Q}}_\ell$ such that $(\pi^{\infty,p})^{K^{p^\circ}}$ is a constituent of $H^{\text{prim}}(\overline{M}_p^\dagger(V^\circ, K^{p^\circ}), \overline{\mathbb{Q}}_\ell)$. Then one can complete $\pi^{\infty,p}$ to an automorphic representation $\pi = \pi^{\infty,p} \otimes \pi_\infty \otimes \prod_{\mathfrak{q}|p} \pi_{\mathfrak{q}}$ of $U(V^\circ)(\mathbb{A}_{F^+})$ such that π_∞ is trivial; $\pi_{\mathfrak{q}}$ is unramified for $\mathfrak{q} \neq \mathfrak{p}$; and*

- (1) *when N is even, $\pi_{\mathfrak{p}}$ is a constituent of an unramified principal series;*
- (2) *when N is odd, $\text{BC}(\pi_{\mathfrak{p}})$ is a constituent of an unramified principal series of $\text{GL}_N(F_{\mathfrak{p}})$ whose Satake parameter contains $\{-p, -p^{-1}\}$.*

Proof. Put $K_{p1}^\circ := K_{p1}^\circ \times \prod_{\mathfrak{q}|p, \mathfrak{q} \neq \mathfrak{p}} K_{\mathfrak{q}}^\circ$. By Construction 5.5.1, the cohomology $H_{\mathfrak{X}}^{N-2}(\overline{M}_p^\dagger(V^\circ, K^{p^\circ}), \overline{\mathbb{Q}}_\ell)$ is an $\overline{\mathbb{Q}}_\ell[K^{p^\circ} K_{p1}^\circ \backslash U(V^\circ)(\mathbb{A}_{F^+}^\infty)/K^{p^\circ} K_{p1}^\circ]$ -module for which $H^{\text{prim}}(\overline{M}_p^\dagger(V^\circ, K^{p^\circ}), \overline{\mathbb{Q}}_\ell)$ is a submodule.

In the uniformization map (5.4), we let $s_0 \in S_p^\circ(V^\circ, K^{p^\circ})(\overline{\mathbb{F}}_p)$ be the point corresponding to the unit element on the right-hand side. Put

$$H_{s_0}^{\text{prim}}(\overline{M}_p^\dagger(V^\circ, K^{p^\circ}), \overline{\mathbb{Q}}_\ell) := H^{\text{prim}}(\overline{M}_p^\dagger(V^\circ, K^{p^\circ}), \overline{\mathbb{Q}}_\ell) \cap H^{N-2}(\overline{M}_p^\dagger(V^\circ, K^{p^\circ}) \cap \pi^\circ(s_0), \overline{\mathbb{Q}}_\ell).$$

Then $H_{s_0}^{\text{prim}}(\overline{M}_p^\dagger(V^\circ, K^{p^\circ}), \overline{\mathbb{Q}}_\ell)$ is a representation of $U(\Lambda_p^\circ)(\mathbb{F}_p) = K_{\mathfrak{p}}^\circ/K_{p1}^\circ$, which is (isomorphic to) the representation Ω_N studied in Subsection C.2. Then we may identify $H^{\text{prim}}(\overline{M}_p^\dagger(V^\circ, K^{p^\circ}), \overline{\mathbb{Q}}_\ell)$ with

$$(5.13) \quad \text{Map}_{K_{p1}^\circ} \left(U(V^\circ)(F^+) \backslash U(V^\circ)(\mathbb{A}_{F^+}^\infty)/K^{p^\circ} \prod_{\mathfrak{q}|p, \mathfrak{q} \neq \mathfrak{p}} K_{\mathfrak{q}}^\circ, \Omega_N \right)$$

as $\overline{\mathbb{Q}}_\ell[K^{p^\circ} K_{p1}^\circ \backslash U(V^\circ)(\mathbb{A}_{F^+}^\infty)/K^{p^\circ} K_{p1}^\circ]$ -modules. It is well-known (see, for example, [HM78]) that the representation Ω_N is irreducible and determined by N up to isomorphism. Therefore, (5.13) is a submodule of $\text{Map}(U(V^\circ)(F^+) \backslash U(V^\circ)(\mathbb{A}_{F^+}^\infty)/K^{p^\circ} K_{p1}^\circ, \overline{\mathbb{Q}}_\ell)$. In particular, we can complete $\pi^{\infty,p}$ to

an automorphic representation $\pi = \pi^{\infty, p} \otimes \pi_\infty \otimes \prod_{\mathfrak{q}|p} \pi_{\mathfrak{q}}$ of $U(V^\circ)(\mathbb{A}_{F^+})$ such that π_∞ is trivial; $\pi_{\mathfrak{q}}$ is unramified for $\mathfrak{q} \neq \mathfrak{p}$; and $\pi_{\mathfrak{p}}|_{K_{\mathfrak{p}}^\circ}$ contains Ω_N .

In case (1), by Proposition C.2.1(2), we know that Ω_N has nonzero Borel fixed vectors. Thus, $\pi_{\mathfrak{p}}$ is a constituent of an unramified principal series.

In case (2), we first consider the case where $N = 3$. As $\pi_{\mathfrak{p}}|_{K_{\mathfrak{p}}^\circ}$ contains Ω_3 , it has to be $c\text{-Ind}_{K_3^{\text{U}_3}} \Omega_3$ by Proposition C.2.1(3) and [MP96, Theorem 6.11(2)]. Thus, by [MP96, Proposition 6.6], $\pi_{\mathfrak{p}}|_{K_{\mathfrak{p}}^\circ}$ is irreducible supercuspidal, which is actually the unique supercuspidal unipotent representation of $U(V^\circ)(F_{\mathfrak{p}}^+)$. In fact, $c\text{-Ind}_{K_3^{\text{U}_3}} \Omega_3$ is the representation $\pi^s(\mathbf{1})$ appeared in [Rog90, Proposition 13.1.3(d)], after identifying $\overline{\mathbb{Q}}_\ell$ with \mathbb{C} . By [Rog90, Proposition 13.2.2(c)], $\text{BC}(\pi^s(\mathbf{1}))$ is the tempered constituent of the unramified principal series of $\text{GL}_3(F_{\mathfrak{p}})$ with the Satake parameter $\{-p, 1, -p^{-1}\}$. Now for general $N = 2r + 1$, as $\pi_{\mathfrak{p}}|_{K_{\mathfrak{p}}^\circ}$ contains Ω_N , by Proposition C.2.1(4) and [MP96, Theorem 6.11(2)], $\pi_{\mathfrak{p}}$ is a constituent the normalized parabolic induction of $\pi^s(\mathbf{1}) \boxtimes \chi_1 \boxtimes \cdots \boxtimes \chi_{r-1}$ for some unramified characters $\chi_1, \dots, \chi_{r-1}$ of F^\times . Therefore, by the compatibility of local base change and induction, $\text{BC}(\pi_{\mathfrak{p}})$ is a constituent of an unramified principal series of $\text{GL}_N(F_{\mathfrak{p}})$ whose Satake parameter contains $\{-p, -p^{-1}\}$.

The proposition is proved. \square

5.6. Intersection on ground stratum. In this subsection, we describe certain scheme-theoretical intersection on the ground stratum, which will be used in the next subsection. We assume $N \geq 2$.

Take an object $K^{p^\circ} \in \mathfrak{K}(V^\circ)^p$. Given two (possibly same) points $s_1^\bullet, s_2^\bullet \in S_{\mathfrak{p}}^\bullet(V^\circ, K^{p^\circ})(\kappa)$ for a perfect field κ containing \mathbb{F}_p^Φ , we put

$$B_{s_1^\bullet, s_2^\bullet}^\bullet := B_{s_1^\bullet}^\bullet \times_{M_{\mathfrak{p}}^\bullet(V^\circ, K^{p^\circ})_\kappa} B_{s_2^\bullet}^\bullet$$

as the (possibly empty) fiber product of $\iota_\kappa^\bullet | B_{s_1^\bullet}^\bullet$ and $\iota_\kappa^\bullet | B_{s_2^\bullet}^\bullet$. To describe $B_{s_1^\bullet, s_2^\bullet}^\bullet$, we need to use some particular cases of the Hecke correspondences introduced in Construction 5.3.6. We now give more details.

Definition 5.6.1. For every integer $0 \leq j \leq N$, we define a functor

$$\begin{aligned} S_{\mathfrak{p}}^\bullet(V^\circ, -)_j: \mathfrak{K}(V^\circ)^p \times \mathfrak{T} &\rightarrow \text{PSch}'_{/\mathbb{F}_p^\Phi} \\ K^{p^\circ} &\mapsto S_{\mathfrak{p}}^\bullet(V^\circ, K^{p^\circ})_j \end{aligned}$$

such that for every $S \in \text{Sch}'_{/\mathbb{F}_p^\Phi}$, $S_{\mathfrak{p}}^\bullet(V^\circ, K^{p^\circ})_j(S)$ is the set of equivalence classes of decuples $(A_0, \lambda_0, \eta_0^p; A_1^\bullet, \lambda_1^\bullet, \eta_1^{p^\bullet}; A_2^\bullet, \lambda_2^\bullet, \eta_2^{p^\bullet}; \phi^\bullet)$ where

- $(A_0, \lambda_0, \eta_0^p; A_i^\bullet, \lambda_i^\bullet, \eta_i^{p^\bullet})$ for $i = 1, 2$ are two elements in $S_{\mathfrak{p}}^\bullet(V^\circ, K^{p^\circ})(S)$; and
- $\phi^\bullet: A_1^\bullet \rightarrow A_2^\bullet$ is an O_F -linear quasi-isogeny such that
 - (a) $p\phi^\bullet \circ \lambda_1^{\bullet-1}$ is a quasi- p -isogeny; and $\ker(p\phi^\bullet)[\mathfrak{p}]$ has rank $p^{2(N-j)}$;
 - (b) $\phi^\bullet[\mathfrak{q}^\infty]$ is an isomorphism for every prime \mathfrak{q} of F^+ above p that is not \mathfrak{p} ;
 - (c) we have $\phi^{\bullet\vee} \circ \lambda_2^\bullet \circ \phi^\bullet = \lambda_1^\bullet$; and
 - (d) the K^{p° -orbit of maps $v \mapsto \phi_*^\bullet \circ \eta_1^{p^\bullet}(v)$ for $v \in V^\circ \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$ coincides with $\eta_2^{p^\bullet}$.

The equivalence relation and the action of morphisms in $\mathfrak{K}(V^\circ)^p \times \mathfrak{T}$ are defined similarly as in Definition 4.2.3. Finally, we denote

$$\text{Hk}_j: S_{\mathfrak{p}}^\bullet(V^\circ, -)_j \rightarrow S_{\mathfrak{p}}^\bullet(V^\circ, -) \times S_{\mathfrak{p}}^\bullet(V^\circ, -)$$

the morphism in $\text{Fun}(\mathfrak{K}(V^\circ)^p \times \mathfrak{T}, \text{Sch}'_{/\mathbb{F}_p^\Phi})_{/T_{\mathfrak{p}}}$ induced by the assignment

$$(A_0, \lambda_0, \eta_0^p; A_1^\bullet, \lambda_1^\bullet, \eta_1^{p^\bullet}; A_2^\bullet, \lambda_2^\bullet, \eta_2^{p^\bullet}; \phi^\bullet) \mapsto ((A_0, \lambda_0, \eta_0^p; A_1^\bullet, \lambda_1^\bullet, \eta_1^{p^\bullet}), (A_0, \lambda_0, \eta_0^p; A_2^\bullet, \lambda_2^\bullet, \eta_2^{p^\bullet})).$$

Remark 5.6.2. When K^{p° is sufficiently small, the morphism

$$\mathrm{Hk}_j : \mathbf{S}_p^\bullet(V^\circ, K^{p^\circ})_j \rightarrow \mathbf{S}_p^\bullet(V^\circ, K^{p^\circ}) \times \mathbf{S}_p^\bullet(V^\circ, K^{p^\circ})$$

is a closed immersion for every j ; and the images of Hk_j for all j are mutually disjoint.

Now we take a point $s^\bullet = (A_0, \lambda_0, \eta_0^p; A_1^\bullet, \lambda_1^\bullet, \eta_1^{p^\bullet}; A_2^\bullet, \lambda_2^\bullet, \eta_2^{p^\bullet}; \phi^\bullet) \in \mathbf{S}_p^\bullet(V^\circ, K^{p^\circ})_j(\kappa)$ where κ is a perfect field containing \mathbb{F}_p^Φ . By Definition 5.6.1(c), we have $(p\phi^\bullet \circ \lambda_1^{\bullet-1})^\vee = p\phi^{\bullet-1} \circ \lambda_2^{\bullet-1}$. Thus, $p\phi^{\bullet-1} \circ \lambda_2^{\bullet-1}$ hence $p\phi^{\bullet-1}$ are quasi- p -isogenies as well. In particular, for every $\tau \in \Sigma_\infty$, we may consider

$$\begin{aligned} \ker(p\phi^\bullet)_{*,\tau} &:= \ker\left((p\phi^\bullet)_{*,\tau} : H_1^{\mathrm{dR}}(A_1^\bullet/\kappa)_\tau \rightarrow H_1^{\mathrm{dR}}(A_2^\bullet/\kappa)_\tau\right), \\ \mathrm{im}(p\phi^{\bullet-1})_{*,\tau} &:= \mathrm{im}\left((p\phi^{\bullet-1})_{*,\tau} : H_1^{\mathrm{dR}}(A_2^\bullet/\kappa)_\tau \rightarrow H_1^{\mathrm{dR}}(A_1^\bullet/\kappa)_\tau\right). \end{aligned}$$

Lemma 5.6.3. *We have*

- (1) $\mathrm{im}(p\phi^{\bullet-1})_{*,\tau} \subseteq \ker(p\phi^\bullet)_{*,\tau}$ for every $\tau \in \Sigma_\infty$;
- (2) $\dim_\kappa \ker(p\phi^\bullet)_{*,\tau} = N - j$ for $\tau \in \{\tau_\infty, \tau_\infty^c\}$;
- (3) $\mathrm{im}(p\phi^{\bullet-1})_{*,\tau} \cap H_1^{\mathrm{dR}}(A_1^\bullet/\kappa)_{\tau^c}^\perp = 0$ for $\tau \in \{\tau_\infty, \tau_\infty^c\}$;
- (4) $(\mathrm{im}(p\phi^{\bullet-1})_{*,\tau})^\perp = \ker(p\phi^\bullet)_{*,\tau^c}$ for $\tau \in \{\tau_\infty, \tau_\infty^c\}$; and
- (5) $\dim_\kappa \mathrm{im}(p\phi^{\bullet-1})_{*,\tau} = j$ for $\tau \in \{\tau_\infty, \tau_\infty^c\}$.

In particular, $\mathbf{S}_p^\bullet(V^\circ, K^{p^\circ})_j$ is empty if $j > \lfloor \frac{N}{2} \rfloor$.

Proof. For (1), it is obvious since $(p\phi^\bullet) \circ (p\phi^{\bullet-1}) = p^2$.

For (2), by Definition 5.6.1(a), we have $\dim_\kappa \ker(p\phi^\bullet)_{*,\tau_\infty} + \dim_\kappa \ker(p\phi^\bullet)_{*,\tau_\infty^c} = 2(N - j)$. Using the isomorphisms $\mathbf{V} : H_1^{\mathrm{dR}}(A_1^\bullet/\kappa)_{\tau_\infty} \rightarrow H_1^{\mathrm{dR}}(A_1^\bullet/\kappa)_{\tau_\infty^c}$ and $\mathbf{V} : H_1^{\mathrm{dR}}(A_2^\bullet/\kappa)_{\tau_\infty} \rightarrow H_1^{\mathrm{dR}}(A_2^\bullet/\kappa)_{\tau_\infty^c}$, we have $\dim_\kappa \ker(p\phi^\bullet)_{*,\tau_\infty} = \dim_\kappa \ker(p\phi^\bullet)_{*,\tau_\infty^c}$ hence both equal to $N - j$.

For (3), it suffices to consider $\tau = \tau_\infty$ due to the isomorphism \mathbf{V} . Via ϕ^\bullet , we regard $\mathcal{D}(A_2^\bullet)$ as a lattice in $\mathcal{D}(A_1^\bullet)_\mathbb{Q}$. By Definition 5.6.1(a), we have $p\mathcal{D}(A_2^\bullet)_{\tau_\infty} \subseteq \mathcal{D}(A_1^\bullet)_{\tau_\infty} \subseteq \mathcal{D}(A_2^\bullet)_{\tau_\infty}^\vee$ (Notation 3.4.12). Suppose $H_1^{\mathrm{dR}}(A_1^\bullet/\kappa)_{\tau_\infty}^\perp \subseteq \mathrm{im}(p\phi^{\bullet-1})_{*,\tau_\infty}$. Then one can find $x_2 \in \mathcal{D}(A_2^\bullet)_{\tau_\infty}$ and $x_1 \in \mathcal{D}(A_1^\bullet)_{\tau_\infty}^\vee \setminus \mathcal{D}(A_1^\bullet)_{\tau_\infty}$ such that $px_1 = px_2$. It follows that $\langle x_2, \mathbf{V}x_2 \rangle_{\lambda_2^\bullet, \tau_\infty} = \langle x_1, \mathbf{V}x_1 \rangle_{\lambda_1^\bullet, \tau_\infty}$ does not belong to $W(\kappa)$, which is a contradiction. Here, we regard \mathbf{V} as Verschiebung maps on for Dieudonné modules of A_1^\bullet and A_2^\bullet , which are isomorphisms.

For (4), as $\lambda_1^\bullet \circ \phi^{\bullet-1} = \phi^{\bullet\vee} \circ \lambda_2^\bullet$, we have for $\tau \in \{\tau_\infty, \tau_\infty^c\}$ that

$$(\mathrm{im}(p\phi^{\bullet-1})_{*,\tau})^\perp = ((p\phi^\bullet)_{*,\tau^c})^{-1} H_1^{\mathrm{dR}}(A_1^\bullet/\kappa)_\tau^\perp,$$

which equals $\ker(p\phi^\bullet)_{*,\tau^c}$ by (3).

For (5), by (2,3,4), we have $\dim_\kappa \mathrm{im}(p\phi^{\bullet-1})_{*,\tau} = j$ for $\tau \in \{\tau_\infty, \tau_\infty^c\}$.

The last claim follows from (1,2,5). □

By Lemma 5.6.3(1,4), for $\tau \in \{\tau_\infty, \tau_\infty^c\}$, we may put

$$H_1^{\mathrm{dR}}(\phi^\bullet)_\tau := \frac{\ker(p\phi^\bullet)_{*,\tau}}{\mathrm{im}(p\phi^{\bullet-1})_{*,\tau}};$$

and we have the induced κ -bilinear pairing

$$\langle \cdot, \cdot \rangle_{\lambda_1^\bullet, \tau_\infty} : H_1^{\mathrm{dR}}(\phi^\bullet)_{\tau_\infty} \times H_1^{\mathrm{dR}}(\phi^\bullet)_{\tau_\infty^c} \rightarrow \kappa.$$

On the other hand, the (κ, σ^{-1}) -linear Verschiebung map $\mathbf{V} : H_1^{\mathrm{dR}}(A_1^\bullet/\kappa)_{\tau_\infty} \rightarrow H_1^{\mathrm{dR}}(A_1^\bullet/\kappa)_{\tau_\infty^c}$ induces a (κ, σ^{-1}) -linear isomorphism $\mathbf{V} : H_1^{\mathrm{dR}}(\phi^\bullet)_{\tau_\infty} \rightarrow H_1^{\mathrm{dR}}(\phi^\bullet)_{\tau_\infty^c}$. We define a pairing

$$\{ \cdot, \cdot \}_{s^\bullet} : H_1^{\mathrm{dR}}(\phi^\bullet)_{\tau_\infty^c} \times H_1^{\mathrm{dR}}(\phi^\bullet)_{\tau_\infty} \rightarrow \kappa$$

by the formula $\{x, y\}_{s^\bullet} := \langle V^{-1}x, y \rangle_{\lambda_1^\bullet, \tau_\infty}$. To ease notation, we put

$$\mathcal{V}_{s^\bullet} := H_1^{\text{dR}}(\phi^\bullet)_{\tau_\infty^\varepsilon}.$$

Lemma 5.6.4. *Suppose $j \leq \lfloor \frac{N}{2} \rfloor - 1$. The pair $(\mathcal{V}_{s^\bullet}, \{ \cdot, \cdot \}_{s^\bullet})$ is admissible of rank $N - 2j$ (Definition A.1.1) satisfying $\dim_\kappa \mathcal{V}_{s^\bullet}^\perp = N - 2\lfloor \frac{N}{2} \rfloor$. In particular, we have the geometrically irreducible smooth projective scheme $\text{DL}^\bullet(\mathcal{V}_{s^\bullet}, \{ \cdot, \cdot \}_{s^\bullet}) \in \text{Sch}/_\kappa$ of dimension $\lfloor \frac{N}{2} \rfloor - j$ as introduced in Definition A.2.1.*

Proof. By Lemma 5.6.3(2,5), we have $\dim_\kappa \mathcal{V}_{s^\bullet} = N - 2j$. By Lemma 5.6.3(3,4), we have $\dim_\kappa \mathcal{V}_{s^\bullet}^\perp = N - 2\lfloor \frac{N}{2} \rfloor$. The lemma follows by Proposition A.2.2. \square

Now consider a connected scheme $S \in \text{Sch}'/\kappa$ and a point $x \in B_{s_1^\bullet, s_2^\bullet}^\bullet(S)$ represented by a quatordecuple $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A_1^\bullet, \lambda_1^\bullet, \eta_1^{p^\bullet}; \gamma_1; A_2^\bullet, \lambda_2^\bullet, \eta_2^{p^\bullet}; \gamma_2)$.

Lemma 5.6.5. *There exists a unique integer j satisfying $0 \leq j \leq \lfloor \frac{N}{2} \rfloor - 1$ such that $s^\bullet := (A_0, \lambda_0, \eta_0^p; A_1^\bullet, \lambda_1^\bullet, \eta_1^{p^\bullet}; A_2^\bullet, \lambda_2^\bullet, \eta_2^{p^\bullet}; \phi^\bullet)$ is an element in $S_p^\bullet(\mathbb{V}^\circ, \mathbb{K}^{p^\circ})_j(S)$, where $\phi^\bullet := \gamma_2 \circ \gamma_1^{-1}$. Moreover, we have*

$$(5.14) \quad \text{im}(p\phi^{\bullet-1})_{*, \tau_\infty^\varepsilon} \subseteq H_2 \subseteq H_1 \subseteq \ker(p\phi^\bullet)_{*, \tau_\infty^\varepsilon},$$

where $H_2 \subseteq H_1 \subseteq H_1^{\text{dR}}(A_1^\bullet/S)_{\tau_\infty^\varepsilon}$ are subbundles in Theorem 5.3.4 for the image of x in $B_{s_1^\bullet}^\bullet(S)$.

Proof. First, by definition, we have $\ker(p\phi^\bullet)[\mathfrak{p}] = \ker(\gamma_2 \circ \check{\gamma}_1)[\mathfrak{p}]$, which is an O_F -stable finite flat subgroup of $A_1^\bullet[\mathfrak{p}]$. Thus, as S is connected, there is a unique integer j satisfying $0 \leq j \leq N$ such that $\ker(p\phi^\bullet)[\mathfrak{p}]$ has rank $p^{2(N-j)}$.

Second, we show that $p\phi^\bullet \circ \lambda_1^{\bullet-1}$ is a quasi- p -isogeny, that is, $\gamma_2 \circ \check{\gamma}_1 \circ \lambda_1^{\bullet-1}$ is a quasi- p -isogeny. By Theorem 5.3.4(3), $\gamma_{1*, \tau_\infty^\varepsilon} \omega_{A^\vee/S, \tau_\infty^\varepsilon}$ contains $H_1^{\text{dR}}(A_1^\bullet)_{\tau_\infty^\varepsilon}^\perp$, which implies $\check{\gamma}_{1*, \tau_\infty^\varepsilon} H_1^{\text{dR}}(A_1^\bullet)_{\tau_\infty^\varepsilon}^\perp = 0$. In other words, $\ker \lambda_1^\bullet[\mathfrak{p}^\infty]$ is contained in $\ker \check{\gamma}_1[\mathfrak{p}^\infty]$. Thus, $\check{\gamma}_1 \circ \lambda_1^{\bullet-1}$ is already a quasi- p -isogeny; so is $p\phi^\bullet \circ \lambda_1^{\bullet-1}$.

Third, we show that j is at most $\lfloor \frac{N}{2} \rfloor - 1$. (Note that Lemma 5.6.3 already implies that $j \leq \lfloor \frac{N}{2} \rfloor$.) Theorem 5.3.4 implies $\text{rank}_{\mathcal{O}_S} H_2 + 1 = \text{rank}_{\mathcal{O}_S} H_1$ and $H_1^{\text{dR}}(A_1^\bullet/S)_{\tau_\infty^\varepsilon}^\perp \subseteq H_2$. Lemma 5.6.3(3) implies $\text{rank}_{\mathcal{O}_S} H_2 \geq \text{rank}_{\mathcal{O}_S} \text{im}(p\phi^{\bullet-1})_{*, \tau_\infty^\varepsilon} + 1$. Thus, by Lemma 5.6.3(2,5) and (5.14), we have $(N - j) - j \geq 2$, that is, $j \leq \lfloor \frac{N}{2} \rfloor - 1$.

Definition 5.6.1(b,c,d) are obvious. Thus, it remains to check (5.14). On one hand, we have

$$\begin{aligned} \text{im}(p\phi^{\bullet-1})_{*, \tau_\infty^\varepsilon} &= \text{im}(\gamma_1 \circ \check{\gamma}_2)_{*, \tau_\infty^\varepsilon} = \gamma_{1*, \tau_\infty^\varepsilon} \check{\gamma}_{2*, \tau_\infty^\varepsilon} H_1^{\text{dR}}(A_2^\bullet/S)_{\tau_\infty^\varepsilon} \\ &= \gamma_{1*, \tau_\infty^\varepsilon} \check{\gamma}_{2*, \tau_\infty^\varepsilon} \omega_{A_2^\vee/S, \tau_\infty^\varepsilon} \subseteq \gamma_{1*, \tau_\infty^\varepsilon} \omega_{A_1^\vee/S, \tau_\infty^\varepsilon} = H_2. \end{aligned}$$

On the other hand, since $\check{\gamma}_{1*, \tau_\infty^\varepsilon} \text{im}(p\phi^{\bullet-1})_{*, \tau_\infty^\varepsilon} = \check{\gamma}_{1*, \tau_\infty^\varepsilon} \text{im}(\gamma_1 \circ \check{\gamma}_2)_{*, \tau_\infty^\varepsilon} = 0$, we have the inclusion $\text{im}(p\phi^{\bullet-1})_{*, \tau_\infty^\varepsilon} \subseteq (\check{\gamma}_{1*, \tau_\infty^\varepsilon})^{-1} \omega_{A^\vee, \tau_\infty^\varepsilon}$. Thus, $H_1 = ((\check{\gamma}_{1*, \tau_\infty^\varepsilon})^{-1} \omega_{A^\vee, \tau_\infty^\varepsilon})^\perp$ is contained in $(\text{im}(p\phi^{\bullet-1})_{*, \tau_\infty^\varepsilon})^\perp$, which is $\ker(p\phi^\bullet)_{*, \tau_\infty^\varepsilon}$ by Lemma 5.6.3(4). The lemma is proved. \square

Definition 5.6.6. By Lemma 5.6.5, we have a morphism

$$B_{s_1^\bullet, s_2^\bullet}^\bullet \rightarrow \prod_{j=0}^{\lfloor \frac{N}{2} \rfloor - 1} \text{Hk}_j^{-1}(s_1^\bullet, s_2^\bullet).$$

For a point $s^\bullet \in \text{Hk}_j^{-1}(s_1^\bullet, s_2^\bullet)(\kappa)$ for some $0 \leq j \leq \lfloor \frac{N}{2} \rfloor - 1$, we denote by $B_{s^\bullet}^\bullet$ the inverse image under the above morphism, which is an open and closed subscheme of $B_{s_1^\bullet, s_2^\bullet}^\bullet$.

Theorem 5.6.7. *Let $s_1^\bullet, s_2^\bullet \in S_p^\bullet(V^\circ, K^{p^\circ})(\kappa)$ be two points for a perfect field κ containing \mathbb{F}_p^Φ . We have*

$$B_{s_1^\bullet, s_2^\bullet}^\bullet = \prod_{j=0}^{\lfloor \frac{N}{2} \rfloor - 1} \prod_{s^\bullet \in \text{Hk}_j^{-1}(s_1^\bullet, s_2^\bullet)(\kappa)} B_{s^\bullet}^\bullet.$$

Take $s^\bullet = (A_0, \lambda_0, \eta_0^p; A_1^\bullet, \lambda_1^\bullet, \eta_1^{p^\bullet}; A_2^\bullet, \lambda_2^\bullet, \eta_2^{p^\bullet}; \phi^\bullet) \in \text{Hk}_j^{-1}(s_1^\bullet, s_2^\bullet)(\kappa)$ for some $0 \leq j \leq \lfloor \frac{N}{2} \rfloor - 1$.

(1) Denote by \bar{H}_i the image of H_i in $H_1^{\text{dR}}(\phi^\bullet)_{\tau_\infty} \otimes_\kappa \mathcal{O}_S = (\mathcal{V}_{s^\bullet})_S$ for $i = 1, 2$. Then The assignment sending $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A_1^\bullet, \lambda_1^\bullet, \eta_1^{p^\bullet}; \gamma_1; A_2^\bullet, \lambda_2^\bullet, \eta_2^{p^\bullet}; \gamma_2) \in B_{s^\bullet}^\bullet(S)$ to (\bar{H}_1, \bar{H}_2) induces an isomorphism

$$\zeta_{s^\bullet}^\bullet : B_{s^\bullet}^\bullet \rightarrow \text{DL}^\bullet(\mathcal{V}_{s^\bullet}, \{ , \}_{s^\bullet})$$

(Definition A.2.1) in Sch/κ .

(2) The cokernel of the map

$$\mathcal{T}_{B_{s_1^\bullet/\kappa}^\bullet | B_{s_2^\bullet/\kappa}^\bullet} \oplus \mathcal{T}_{B_{s_2^\bullet/\kappa}^\bullet | B_{s_1^\bullet/\kappa}^\bullet} \rightarrow \iota^{**} \mathcal{T}_{M_p^\bullet(V^\circ, K^{p^\circ})/\kappa} | B_{s^\bullet}^\bullet$$

is canonically isomorphic to

$$\zeta_{s^\bullet}^{\bullet*} \left((\sigma^* \bar{\mathcal{H}}_{s^\bullet 2}) \otimes_{\mathcal{O}_{\text{DL}^\bullet(\mathcal{V}_{s^\bullet}, \{ , \}_{s^\bullet})}} (\bar{\mathcal{H}}_{s^\bullet 1}^{-1} / \bar{\mathcal{H}}_{s^\bullet 2}) \right)$$

where $(\bar{\mathcal{H}}_{s^\bullet 1}, \bar{\mathcal{H}}_{s^\bullet 2})$ is the universal object over $\text{DL}^\bullet(\mathcal{V}_{s^\bullet}, \{ , \}_{s^\bullet})$.

Proof. The decomposition of $B_{s_1^\bullet, s_2^\bullet}^\bullet$ follows directly from the definition and the fact that $\text{Hk}_j^{-1}(s_1^\bullet, s_2^\bullet)$ is isomorphic to a finite disjoint union of $\text{Spec } \kappa$.

Now we show (1). We first notice that Lemma 5.6.3 implies that (\bar{H}_1, \bar{H}_2) is an element in $\text{DL}^\bullet(\mathcal{V}_{s^\bullet}, \{ , \}_{s^\bullet})(S)$.

Since the target of $\zeta_{s^\bullet}^\bullet$ is smooth over κ by Lemma 5.6.4, to see that $\zeta_{s^\bullet}^\bullet$ is an isomorphism, it suffices to check that for every algebraically closed field κ' containing κ

(1.1) $\zeta_{s^\bullet}^\bullet$ induces a bijection on κ' -points; and

(1.2) $\zeta_{s^\bullet}^\bullet$ induces an isomorphism on the tangent spaces at every κ' -point.

To ease notation, we may assume $\kappa' = \kappa$.

For (1.1), we construct an inverse to the map $\zeta_{s^\bullet}^\bullet(\kappa)$. Take a point $y \in \text{DL}^\bullet(\mathcal{V}_{s^\bullet}, \{ , \}_{s^\bullet})(\kappa)$ represented by κ -linear subspaces $\mathcal{V}_{s^\bullet}^{-1} \subseteq \bar{H}_2 \subseteq \bar{H}_1 \subseteq \mathcal{V}_{s^\bullet}$, or equivalently, subspaces

$$\text{im}(p\phi^{\bullet-1})_{*, \tau_\infty} \oplus H_1^{\text{dR}}(A_1^\bullet/\kappa)_{\tau_\infty}^\perp \subseteq H_2 \subseteq H_1 \subseteq \ker(p\phi^\bullet)_{*, \tau_\infty} \subseteq H_1^{\text{dR}}(A_1^\bullet/\kappa)_{\tau_\infty}.$$

These give rise to a point $y_1 \in \text{DL}^\bullet(\mathcal{V}_{s_1^\bullet}, \{ , \}_{s_1^\bullet})(\kappa)$. By Theorem 5.3.4(3), we obtain a unique point $x_1 = (A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A_1^\bullet, \lambda_1^\bullet, \eta_1^{p^\bullet}; \gamma_1) \in B_{s_1^\bullet}^\bullet(\kappa)$ such that $\zeta_{s_1^\bullet}^\bullet(x_1) = y_1$. Put $\gamma_2 := \phi^\bullet \circ \gamma_1 : A \rightarrow A_2^\bullet$. We claim that γ_2 is a quasi- p -isogeny. In fact, as $\lambda \circ \check{\gamma}_1 = \gamma_1^\vee \circ \lambda_1^\bullet$, $\langle \text{im } \gamma_{1*, \tau_\infty}, \text{im } \gamma_{1*, \tau_\infty}^c \rangle_{\lambda_1^\bullet, \tau_\infty} = 0$. Thus, we have

$$\text{im } \gamma_{1*, \tau_\infty} \subseteq (\text{im } \gamma_{1*, \tau_\infty})^\perp = (V^{-1} \gamma_{1*, \tau_\infty} \omega_{A^\vee, \tau_\infty})^\perp = H_2^\perp \subseteq \ker(p\phi^\bullet)_{*, \tau_\infty}.$$

By the isomorphisms $V : H_1^{\text{dR}}(A_1^\bullet/\kappa)_{\tau_\infty} \rightarrow H_1^{\text{dR}}(A_1^\bullet/\kappa)_{\tau_\infty}^c$ and $V : H_1^{\text{dR}}(A_2^\bullet/\kappa)_{\tau_\infty} \rightarrow H_1^{\text{dR}}(A_2^\bullet/\kappa)_{\tau_\infty}^c$, we obtain $\text{im } \gamma_{1*, \tau_\infty} \subseteq \ker(p\phi^\bullet)_{*, \tau_\infty}$. In particular, $\text{im}(p\phi^\bullet \circ \gamma_1)_{*, \tau} = 0$ for every $\tau \in \Sigma_\infty$; in other words, γ_2 is a quasi- p -isogeny. Now we show that $x_2 := (A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A_2^\bullet, \lambda_2^\bullet, \eta_2^{p^\bullet}; \gamma_2)$ satisfies Definition 5.3.2(a–d).

For (a), it suffices to show that $p\gamma_2^{-1}$ is a quasi- p -isogeny, equivalently, $\gamma_1^{-1} \circ (p\phi^{\bullet-1})$ is a quasi- p -isogeny. However, we have $\text{im}(p\phi^{\bullet-1})_{*, \tau_\infty} = V^{-1} \text{im}(p\phi^{\bullet-1})_{*, \tau_\infty}^c \subseteq V^{-1} H_2 = \text{im } \gamma_{1*, \tau_\infty}$, hence $\text{im}(p\phi^{\bullet-1})_{*, \tau_\infty} \subseteq \text{im } \gamma_{1*, \tau_\infty}$ using the action of V , which together imply that $\gamma_1^{-1} \circ (p\phi^{\bullet-1})$ is a quasi- p -isogeny.

For (b), we identify $\mathcal{D}(A)$ as submodules of both $\mathcal{D}(A_1^\bullet)$ and $\mathcal{D}(A_2^\bullet)$ via γ_1 and γ_2 , respectively. Then we need to show that $p\mathcal{D}(A_2^\bullet)_{\tau_\infty}^\vee \cap \mathcal{D}(A)_{\tau_\infty} \subseteq \mathbf{VD}(A)_{\tau_\infty}$. As $p\phi^{\bullet-1} \circ \lambda_2^{\bullet-1}$ is a quasi- p -isogeny, we have $p\mathcal{D}(A_2^\bullet)_{\tau_\infty}^\vee \subseteq \mathcal{D}(A_1^\bullet)_{\tau_\infty}$. Moreover, the image of $p\mathcal{D}(A_2^\bullet)_{\tau_\infty}^\vee$ in $\mathcal{D}(A_1^\bullet)_{\tau_\infty}/p\mathcal{D}(A_1^\bullet)_{\tau_\infty} = H_1^{\text{dR}}(A_1^\bullet)_{\tau_\infty}$ is contained in $\text{im}(p\phi^{\bullet-1})_{*,\tau_\infty} \oplus H_1^{\text{dR}}(A_1^\bullet/\kappa)_{\tau_\infty}^\perp$, which is further contained in H_2 . Thus, $p\mathcal{D}(A_2^\bullet)_{\tau_\infty}^\vee \cap \mathcal{D}(A)_{\tau_\infty} \subseteq \mathbf{VD}(A)_{\tau_\infty}$ as $\mathbf{VD}(A)_{\tau_\infty}$ is the inverse image of H_2 in $\mathcal{D}(A_1^\bullet)_{\tau_\infty}$.

For (c) and (d), they follow obviously.

To summarize, x_2 belongs to $B_{s_2}^\bullet(\kappa)$; and $x := (x_1, x_2)$ is an element in $B_s^\bullet(\kappa)$ such that $\zeta_s^\bullet(x) = y$. It is easy to see that such assignment gives rise to an inverse of $\zeta_s^\bullet(\kappa)$; hence (1.1) follows immediately.

For (1.2), let \mathcal{T}_x and \mathcal{T}_y be the tangent spaces at x and y as in (1.1), respectively. By Theorem 5.3.4(1), we have a canonical short exact sequence

$$0 \rightarrow \text{Hom}_\kappa \left(\omega_{A^\vee, \tau_\infty}, \frac{\omega_{A^\vee, \tau_\infty}^\perp}{\omega_{A^\vee, \tau_\infty}} \right) \rightarrow \mathcal{T}_x \rightarrow \text{Hom}_\kappa \left(\frac{\omega_{A^\vee, \tau_\infty}}{(\ker \gamma_{1*, \tau_\infty})^\perp + (\ker \gamma_{2*, \tau_\infty})^\perp}, \text{Lie}_{A^\vee, \tau_\infty} \right) \rightarrow 0.$$

Then by Proposition A.2.2 and the construction, the induced map $(\zeta_s^\bullet)_*: \mathcal{T}_x \rightarrow \mathcal{T}_y$ fits into a commutative diagram

$$\begin{array}{ccccc} \text{Hom}_\kappa \left(\omega_{A^\vee, \tau_\infty}, \frac{\omega_{A^\vee, \tau_\infty}^\perp}{\omega_{A^\vee, \tau_\infty}} \right) & \longrightarrow & \mathcal{T}_x & \longrightarrow & \text{Hom}_\kappa \left(\frac{\omega_{A^\vee, \tau_\infty}}{(\ker \gamma_{1*, \tau_\infty})^\perp + (\ker \gamma_{2*, \tau_\infty})^\perp}, \text{Lie}_{A^\vee, \tau_\infty} \right) \\ \downarrow & & \downarrow (\zeta_s^\bullet)_* & & \downarrow \\ \text{Hom}_\kappa \left(\bar{H}_1/\bar{H}_2, \bar{H}_2^\perp/\bar{H}_1 \right) & \longrightarrow & \mathcal{T}_y & \longrightarrow & \text{Hom}_\kappa \left(\bar{H}_2/\mathcal{V}_{s^\bullet}^{-1}, \bar{H}_1^\perp/\bar{H}_2 \right) \end{array}$$

in $\text{Mod}(\kappa)$. The left vertical arrow is the composition

$$\begin{aligned} \text{Hom}_\kappa \left(\omega_{A^\vee, \tau_\infty}, \omega_{A^\vee, \tau_\infty}^\perp/\omega_{A^\vee, \tau_\infty} \right) &\rightarrow \text{Hom}_\kappa \left(H_1^\perp/\mathbf{V}^{-1}H_2, H_2^\perp/H_1^\perp \right) \\ &\xrightarrow{\sim} \text{Hom}_\kappa \left(H_1/H_2, H_2^\perp/H_1 \right) \simeq \text{Hom}_\kappa \left(\bar{H}_1/\bar{H}_2, \bar{H}_2^\perp/\bar{H}_1 \right), \end{aligned}$$

which is an isomorphism. The right vertical arrow is induced by maps

$$\begin{aligned} \frac{\omega_{A^\vee, \tau_\infty}}{(\ker \gamma_{1*, \tau_\infty})^\perp + (\ker \gamma_{2*, \tau_\infty})^\perp} &\xrightarrow{\gamma_{1*, \tau_\infty}} \frac{H_2}{\text{im}(p\phi^{\bullet-1})_{*, \tau_\infty} \oplus H_1^{\text{dR}}(A_1^\bullet/\kappa)_{\tau_\infty}^\perp} \simeq \bar{H}_2/\mathcal{V}_{s^\bullet}^{-1}, \\ \text{Lie}_{A^\vee, \tau_\infty} \simeq H_1^{\text{dR}}(A)_{\tau_\infty}/\omega_{A^\vee, \tau_\infty} &\xrightarrow{\gamma_{1*, \tau_\infty}} H_1^\perp/H_2 \simeq \bar{H}_1^\perp/\bar{H}_2. \end{aligned}$$

As $\ker \gamma_{2*, \tau_\infty} = \text{im } \check{\gamma}_{2*, \tau_\infty}$, we have

$$\frac{\omega_{A^\vee, \tau_\infty}}{(\ker \gamma_{1*, \tau_\infty})^\perp + (\ker \gamma_{2*, \tau_\infty})^\perp} \simeq \frac{\gamma_{1*, \tau_\infty} \omega_{A^\vee, \tau_\infty}}{\text{im}(\gamma_1 \circ \check{\gamma}_2)_{*, \tau_\infty} + H_1^{\text{dR}}(A_1^\bullet/\kappa)_{\tau_\infty}^\perp} = \frac{H_2}{\text{im}(p\phi^{\bullet-1})_{*, \tau_\infty} \oplus H_1^{\text{dR}}(A_1^\bullet/\kappa)_{\tau_\infty}^\perp},$$

which implies that the first map is an isomorphism. By Theorem 5.3.4(4), the second map is an isomorphism as well. Thus, $(\zeta_s^\bullet)_*: \mathcal{T}_x \rightarrow \mathcal{T}_y$ is an isomorphism by the Five Lemma; hence (1.2) and (1) follow.

Then we show (2). Theorem 5.3.4(2) implies that the cokernel of the map

$$\mathcal{T}_{B_{s_1}^\bullet/\kappa} |_{B_s^\bullet} \oplus \mathcal{T}_{B_{s_2}^\bullet/\kappa} |_{B_s^\bullet} \rightarrow \iota^{**} \mathcal{T}_{M_p^\bullet(\mathbf{V}^\circ, K^{p^\circ})/\kappa} |_{B_s^\bullet}$$

is canonically isomorphic to

$$(5.15) \quad \text{Hom} \left((\ker \gamma_{1*, \tau_\infty} + \ker \gamma_{2*, \tau_\infty})^\perp / H_1^{\text{dR}}(\mathcal{A})_{\tau_\infty}^\perp, \text{Lie}_{A^\vee, \tau_\infty} \right).$$

As $\ker \gamma_{2^*, \tau_\infty} = \text{im } \check{\gamma}_{2^*, \tau_\infty}$, we have

$$(5.16) \quad \frac{H_1^{\text{dR}}(\mathcal{A})_{\tau_\infty}}{\ker \gamma_{1^*, \tau_\infty} + \ker \gamma_{2^*, \tau_\infty}} \simeq \frac{\text{im } \gamma_{1^*, \tau_\infty}}{\text{im}(\gamma_1 \circ \check{\gamma}_2)_{*, \tau_\infty}} = \frac{\text{im } \gamma_{1^*, \tau_\infty}}{\text{im}(p\phi^{\bullet-1})_{*, \tau_\infty}} \simeq \frac{\text{V im } \gamma_{1^*, \tau_\infty}}{\text{V im}(p\phi^{\bullet-1})_{*, \tau_\infty}}.$$

However, we have $\text{V im } \gamma_{1^*, \tau_\infty} = (\gamma_{1^*, \tau_\infty} \omega_{\mathcal{A}, \tau_\infty}^c)^{(p)}$ and $\text{V im}(p\phi^{\bullet-1})_{*, \tau_\infty} = (\text{im}(p\phi^{\bullet-1})_{*, \tau_\infty}^c)^{(p)}$. Thus, (5.16) is isomorphic to $\sigma^* \bar{\mathcal{H}}_{s^\bullet, 2}$, hence

$$(5.15) \simeq \mathcal{H}om\left((\sigma^* \bar{\mathcal{H}}_{s^\bullet, 2})^\vee, \text{Lie}_{\mathcal{A}^\vee, \tau_\infty^c}\right) \simeq (\sigma^* \bar{\mathcal{H}}_{s^\bullet, 2}) \otimes_{\mathcal{O}_{\text{DL}^\bullet(\mathcal{V}_s^\bullet, \{ \cdot \}, \{ \cdot \}, \{ \cdot \})}} (\bar{\mathcal{H}}_{s^\bullet, 1}^\vee / \bar{\mathcal{H}}_{s^\bullet, 2}),$$

where we use Theorem 5.3.4(4) for the last isomorphism. We have proved (2) and the theorem. \square

We also need a description for

$$B_{s^\bullet}^\dagger := B_{s^\bullet} \times_{M_{\mathfrak{p}}^\bullet(\mathbb{V}^\circ, K^{p^\circ})} M_{\mathfrak{p}}^\dagger(\mathbb{V}^\circ, K^{p^\circ})$$

for $s^\bullet \in \text{Hk}_j^{-1}(s_1^\bullet, s_2^\bullet)(\kappa)$. It is clear that if we put

$$B_{s_i^\bullet}^\dagger := B_{s_i^\bullet} \times_{M_{\mathfrak{p}}^\bullet(\mathbb{V}^\circ, K^{p^\circ})} M_{\mathfrak{p}}^\dagger(\mathbb{V}^\circ, K^{p^\circ})$$

for $i = 1, 2$, then

$$B_{s^\bullet}^\dagger = B_{s_1^\bullet}^\dagger \times_{M_{\mathfrak{p}}^\dagger(\mathbb{V}^\circ, K^{p^\circ})} B_{s_2^\bullet}^\dagger.$$

By definition, for every $S \in \text{Sch}/\kappa$, $B_{s^\bullet}^\dagger(S)$ is the set of equivalence classes of unvigtintuples

$$(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^\circ, \lambda^\circ, \eta^{p^\circ}; A_1^\bullet, \lambda_1^\bullet, \eta_1^{p^\bullet}; A_2^\bullet, \lambda_2^\bullet, \eta_2^{p^\bullet}; \beta, \gamma_1, \gamma_2, \psi_1, \psi_2, \phi^\bullet)$$

rendering the diagram

$$\begin{array}{ccc} A_1^\bullet & \xrightarrow{\phi^\bullet} & A_2^\bullet \\ \swarrow \gamma_1 & & \searrow \gamma_2 \\ & A & \\ \swarrow \psi_1 & & \searrow \psi_2 \\ & A^\circ & \end{array}$$

commute. Here, the letters remain the same meaning as in our previous moduli problems. Put

$$S_{s^\bullet}^\dagger := \{s^\bullet\} \times_{S_{\mathfrak{p}}^\bullet(\mathbb{V}^\circ, K^{p^\circ}) \times S_{\mathfrak{p}}^\bullet(\mathbb{V}^\circ, K^{p^\circ})} \left(S_{\mathfrak{p}}^\dagger(\mathbb{V}^\circ, K^{p^\circ}) \times S_{\mathfrak{p}}^\dagger(\mathbb{V}^\circ, K^{p^\circ}) \right) \times_{S_{\mathfrak{p}}^\circ(\mathbb{V}^\circ, K^{p^\circ}) \times S_{\mathfrak{p}}^\circ(\mathbb{V}^\circ, K^{p^\circ})} S_{\mathfrak{p}}^\circ(\mathbb{V}^\circ, K^{p^\circ})$$

where $S_{\mathfrak{p}}^\circ(\mathbb{V}^\circ, K^{p^\circ}) \rightarrow S_{\mathfrak{p}}^\circ(\mathbb{V}^\circ, K^{p^\circ}) \times S_{\mathfrak{p}}^\circ(\mathbb{V}^\circ, K^{p^\circ})$ is the diagonal morphism. Then we have a canonical map

$$\pi_{s^\bullet}^\dagger : B_{s^\bullet}^\dagger \rightarrow S_{s^\bullet}^\dagger$$

of κ -schemes by forgetting (A, λ, η^p) and related morphisms.

Theorem 5.6.8. *Let $s_1^\bullet, s_2^\bullet \in S_{\mathfrak{p}}^\bullet(\mathbb{V}^\circ, K^{p^\circ})(\kappa)$ be two points for a perfect field κ containing \mathbb{F}_p^Φ . Take $s^\bullet \in \text{Hk}_j^{-1}(s_1^\bullet, s_2^\bullet)(\kappa)$ for some $0 \leq j \leq \lfloor \frac{N}{2} \rfloor - 1$. Then the scheme $S_{s^\bullet}^\dagger$ is a disjoint of $(p+1)(p^3+1) \cdots (p^{2\lfloor \frac{N}{2} \rfloor - 2j - 1} + 1)$ copies of $\text{Spec } \kappa$.*

Take a point $t^\dagger = (A_0, \lambda_0, \eta_0^p; A^\circ, \lambda^\circ, \eta^{p^\circ}; A_1^\bullet, \lambda_1^\bullet, \eta_1^{p^\bullet}; A_2^\bullet, \lambda_2^\bullet, \eta_2^{p^\bullet}; \psi_1, \psi_2, \phi^\bullet) \in S_{s^\bullet}^\dagger(\kappa)$.

(1) *The assignment sending*

$$(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^\circ, \lambda^\circ, \eta^{p^\circ}; A_1^\bullet, \lambda_1^\bullet, \eta_1^{p^\bullet}; A_2^\bullet, \lambda_2^\bullet, \eta_2^{p^\bullet}; \beta, \gamma_1, \gamma_2, \psi_1, \psi_2, \phi^\bullet) \in B_{s^\bullet}^\dagger(S)$$

to $H_2/(\text{im}(p\phi^{\bullet-1})_{, \tau_\infty} + H_1^{\text{dR}}(A^\bullet/S)_{\tau_\infty}^\perp)$ induces an isomorphism*

$$\zeta_{t^\dagger}^\dagger : (\pi_{s^\bullet}^\dagger)^{-1}(t^\dagger) \xrightarrow{\sim} \mathbb{P}(\mathcal{V}_{t^\dagger})$$

where we put

$$\mathcal{V}_{t^\dagger} := \frac{\mathrm{im}(\psi_1)_{*,\tau_\infty^\circ}}{\mathrm{im}(p\phi^{\bullet-1})_{*,\tau_\infty^\circ} + H_1^{\mathrm{dR}}(A^\bullet/S)_{\tau_\infty^\circ}^\perp}$$

which has dimension $\lfloor \frac{N}{2} \rfloor - j$.

(2) The cokernel of the map

$$\mathcal{T}_{B_{s_1^\dagger}^\dagger/\kappa} |_{(\pi_{s_1^\dagger}^\dagger)^{-1}(t^\dagger)} \bigoplus \mathcal{T}_{B_{s_2^\dagger}^\dagger/\kappa} |_{(\pi_{s_2^\dagger}^\dagger)^{-1}(t^\dagger)} \rightarrow \iota^{\bullet*} \mathcal{T}_{M_p^\dagger(V^\circ, K^{p^\circ})/\kappa} |_{(\pi_{s^\dagger}^\dagger)^{-1}(t^\dagger)}$$

is canonically isomorphic to

$$\zeta_{t^\dagger}^{\dagger*} \left((\sigma^* \mathcal{H}_{t^\dagger}) \otimes_{\mathcal{O}_{\mathbb{P}(\mathcal{V}_{t^\dagger})}} \mathcal{O}_{\mathbb{P}(\mathcal{V}_{t^\dagger})}(1) \right)$$

where \mathcal{H}_{t^\dagger} is the universal object, namely, the tautological bundle on $\mathbb{P}(\mathcal{V}_{t^\dagger})$.

Proof. In fact, the assignment sending $(A_0, \lambda_0, \eta_0^p; A^\circ, \lambda^\circ, \eta^{p^\circ}; A_1^\bullet, \lambda_1^\bullet, \eta_1^{p^\bullet}; A_2^\bullet, \lambda_2^\bullet, \eta_2^{p^\bullet}; \psi_1, \psi_2, \phi^\bullet) \in S_{s^\dagger}^\bullet(S)$ to $\mathrm{im}(\psi_1)_{*,\tau_\infty^\circ}$ induces a bijection from $S_{s^\dagger}^\bullet(S)$ to the subbundles $H \subseteq H_1^{\mathrm{dR}}(A^\bullet/S)_{\tau_\infty^\circ}$ of rank $\lfloor \frac{N}{2} \rfloor$ satisfying $\mathrm{im}(p\phi^{\bullet-1})_{*,\tau_\infty^\circ} \otimes_\kappa \mathcal{O}_S \subseteq H \subseteq \ker(p\phi^\bullet)_{*,\tau_\infty^\circ} \otimes_\kappa \mathcal{O}_S$ and $\langle V^{-1}H, H \rangle_{\tau_\infty^\circ} = 0$. Thus, we know that $S_{s^\dagger}^\bullet$ is a disjoint of $(p+1)(p^3+1) \dots (p^{2\lfloor \frac{N}{2} \rfloor - 2j - 1} + 1)$ copies of $\mathrm{Spec} \kappa$.

For (1), we denote by s_1^\dagger the image of t^\dagger in $S_p^\dagger(V^\circ, K^{p^\circ})(\kappa)$ in the first factor. Then a point $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^\circ, \lambda^\circ, \eta^{p^\circ}; A_1^\bullet, \lambda_1^\bullet, \eta_1^{p^\bullet}; \beta, \gamma_1) \in B_{s_1^\dagger}^\dagger(S)$ belongs to $B_{s^\dagger}^\dagger(S)$ if and only if H_2 contains $\mathrm{im}(p\phi^{\bullet-1})_{*,\tau_\infty^\circ} \otimes_\kappa \mathcal{O}_S$. Thus, (1) follows from Theorem 5.4.3(2).

For (2), it follows from Theorem 5.6.7(2) and the isomorphism

$$\left(\bar{\mathcal{H}}_{s^\dagger}^{-1} / \bar{\mathcal{H}}_{s^\dagger} \right) |_{\mathbb{P}(\mathcal{V}_{t^\dagger})} = \left(\bar{\mathcal{H}}_{s^\dagger} / \bar{\mathcal{H}}_{s^\dagger} \right) |_{\mathbb{P}(\mathcal{V}_{t^\dagger})} \simeq \mathcal{O}_{\mathbb{P}(\mathcal{V}_{t^\dagger})}(1).$$

□

5.7. Incidence maps on ground stratum. In this subsection, we define and study the incidence maps on ground stratum. We assume $N \geq 2$. In order to have a uniformization map for $S_p^\bullet(V^\circ, -)$, we also choose data as in Construction 5.3.6.

Definition 5.7.1. We denote

- $\mathbb{T}_{N,p}^\circ$ the Hecke algebra $\mathbb{Z}[K_p^\circ \backslash U(V^\circ)(F_p^+) / K_p^\circ]$;
- $\mathbb{T}_{N,p}^\bullet$ the Hecke algebra $\mathbb{Z}[K_p^\bullet \backslash U(V^\circ)(F_p^+) / K_p^\bullet]$;
- $\mathbb{T}_{N,p}^{\circ\bullet} \in \mathbb{Z}[K_p^\bullet \backslash U(V^\circ)(F_p^+) / K_p^\circ]$ the characteristic function of $K_p^\bullet K_p^\circ$; and
- $\mathbb{T}_{N,p}^{\bullet\circ} \in \mathbb{Z}[K_p^\circ \backslash U(V^\circ)(F_p^+) / K_p^\bullet]$ the characteristic function of $K_p^\circ K_p^\bullet$.

Moreover, we define the *intertwining Hecke operator* to be

$$\mathbb{I}_{N,p}^\circ := \mathbb{T}_{N,p}^{\bullet\circ} \circ \mathbb{T}_{N,p}^\circ \in \mathbb{T}_{N,p}^\circ$$

where the composition is taken as composition of cosets.

Remark 5.7.2. We remind the readers that according to our convention, the unit elements of $\mathbb{Z}[K_p^\circ \backslash U(V^\circ)(F_p^+) / K_p^\circ]$ and $\mathbb{Z}[K_p^\bullet \backslash U(V^\circ)(F_p^+) / K_p^\bullet]$ are $\mathbb{1}_{K_p^\circ}$ and $\mathbb{1}_{K_p^\bullet}$, respectively. However, when N is odd, K_p° and K_p^\bullet have different volumes under a common Haar measure on $U(V^\circ)(F_p^+)$; in other words, the convolution products on the two Hecke algebras are not induced by the same Haar measure on $U(V^\circ)(F_p^+)$.

Let L be a p -coprime coefficient ring. By Construction 5.2.6 and Construction 5.3.6, we have canonical isomorphisms

$$\begin{aligned} L[\mathrm{Sh}(V^\circ, -K_p^\circ)] &\simeq H_x^0(\bar{S}_p^\circ(V^\circ, -), L), \\ L[\mathrm{Sh}(V^\circ, -K_p^\bullet)] &\simeq H_x^0(\bar{S}_p^\bullet(V^\circ, -), L), \end{aligned}$$

in $\text{Fun}(\mathfrak{K}(\mathbb{V}^\circ)^p, \text{Mod}(L[\mathbb{K}_p^\circ \setminus \mathbb{U}(\mathbb{V}^\circ)(F_p^+)/\mathbb{K}_p^\circ]))$ and in $\text{Fun}(\mathfrak{K}(\mathbb{V}^\circ)^p, \text{Mod}(L[\mathbb{K}_p^\bullet \setminus \mathbb{U}(\mathbb{V}^\circ)(F_p^+)/\mathbb{K}_p^\bullet]))$, induced by v° (5.4) and v^\bullet (5.9), respectively.

Construction 5.7.3. Recall from Definition 5.5.3 the class $\xi \in H_{\mathfrak{X}}^2(\overline{\mathbb{B}}_p^\circ(\mathbb{V}^\circ, -), L(1))$, which is the first Chern class of the tautological quotient line bundle on $\overline{\mathbb{B}}_p^\circ(\mathbb{V}^\circ, -)$. Put $r := \lfloor \frac{N}{2} \rfloor \geq 1$. We construct three pairs of maps in $\text{Fun}(\mathfrak{K}(\mathbb{V}_n^\circ)_{\text{sp}}^p, \text{Mod}(L))$ as follows:

$$\left\{ \begin{array}{l} \text{inc}_!^\circ : L[\text{Sh}(\mathbb{V}^\circ, -\mathbb{K}_p^\circ)] \xrightarrow{\sim} H_{\mathfrak{X}}^0(\overline{\mathbb{S}}_p^\circ(\mathbb{V}^\circ, -), L) \xrightarrow{\pi^{\circ*}} H_{\mathfrak{X}}^0(\overline{\mathbb{B}}_p^\circ(\mathbb{V}^\circ, -), L) \\ \quad \xrightarrow{\cup \xi^{N-r-1}} H_{\mathfrak{X}}^{2(N-r-1)}(\overline{\mathbb{B}}_p^\circ(\mathbb{V}^\circ, -), L(N-r-1)) \\ \quad \xrightarrow{\iota_!^\circ} H_{\mathfrak{X}}^{2(N-r-1)}(\overline{\mathbb{M}}_p^\circ(\mathbb{V}^\circ, -), L(N-r-1)), \\ \text{inc}_\circ^* : H_{\mathfrak{X}}^{2r}(\overline{\mathbb{M}}_p^\circ(\mathbb{V}^\circ, -), L(r)) \xrightarrow{\iota^{\circ*}} H_{\mathfrak{X}}^{2r}(\overline{\mathbb{B}}_p^\circ(\mathbb{V}^\circ, -), L(r)) \\ \quad \xrightarrow{\cup \xi^{N-r-1}} H_{\mathfrak{X}}^{2(N-1)}(\overline{\mathbb{B}}_p^\circ(\mathbb{V}^\circ, -), L(N-1)) \\ \quad \xrightarrow{\pi_!^\circ} H_{\mathfrak{X}}^0(\overline{\mathbb{S}}_p^\circ(\mathbb{V}^\circ, -), L) \xrightarrow{\sim} L[\text{Sh}(\mathbb{V}^\circ, -\mathbb{K}_p^\circ)]; \end{array} \right.$$

$$\left\{ \begin{array}{l} \text{inc}_!^\dagger : L[\text{Sh}(\mathbb{V}^\circ, -\mathbb{K}_p^\circ)] \xrightarrow{\sim} H_{\mathfrak{X}}^0(\overline{\mathbb{S}}_p^\circ(\mathbb{V}^\circ, -), L) \xrightarrow{\pi^{\circ*}} H_{\mathfrak{X}}^0(\overline{\mathbb{B}}_p^\circ(\mathbb{V}^\circ, -), L) \\ \quad \xrightarrow{\cup \xi^{N-r-2}} H_{\mathfrak{X}}^{2(N-r-2)}(\overline{\mathbb{B}}_p^\circ(\mathbb{V}^\circ, -), L(N-r-2)) \\ \quad \xrightarrow{\iota_!^\circ} H_{\mathfrak{X}}^{2(N-r-2)}(\overline{\mathbb{M}}_p^\circ(\mathbb{V}^\circ, -), L(N-r-2)) \\ \quad \xrightarrow{m^{\dagger\circ*}} H_{\mathfrak{X}}^{2(N-r-2)}(\overline{\mathbb{M}}_p^\dagger(\mathbb{V}^\circ, -), L(N-r-2)) \\ \quad \xrightarrow{m_!^{\dagger\bullet}} H_{\mathfrak{X}}^{2(N-r-1)}(\overline{\mathbb{M}}_p^\bullet(\mathbb{V}^\circ, -), L(N-r-1)), \\ \text{inc}_\dagger^* : H_{\mathfrak{X}}^{2r}(\overline{\mathbb{M}}_p^\bullet(\mathbb{V}^\circ, -), L(r)) \xrightarrow{m^{\dagger\bullet*}} H_{\mathfrak{X}}^{2r}(\overline{\mathbb{M}}_p^\dagger(\mathbb{V}^\circ, -), L(r)) \\ \quad \xrightarrow{m_!^{\dagger\circ}} H_{\mathfrak{X}}^{2(r+1)}(\overline{\mathbb{M}}_p^\circ(\mathbb{V}^\circ, -), L(r+1)) \\ \quad \xrightarrow{\iota^{\circ*}} H_{\mathfrak{X}}^{2(r+1)}(\overline{\mathbb{B}}_p^\circ(\mathbb{V}^\circ, -), L(r+1)) \\ \quad \xrightarrow{\cup \xi^{N-r-2}} H_{\mathfrak{X}}^{2(N-1)}(\overline{\mathbb{B}}_p^\circ(\mathbb{V}^\circ, -), L(N-1)) \\ \quad \xrightarrow{\pi_!^\circ} H_{\mathfrak{X}}^0(\overline{\mathbb{S}}_p^\circ(\mathbb{V}^\circ, -), L) \xrightarrow{\sim} L[\text{Sh}(\mathbb{V}^\circ, -\mathbb{K}_p^\circ)]; \end{array} \right.$$

$$\left\{ \begin{array}{l} \text{inc}_!^\bullet : L[\text{Sh}(\mathbb{V}^\circ, -\mathbb{K}_p^\bullet)] \xrightarrow{\sim} H_{\mathfrak{X}}^0(\overline{\mathbb{S}}_p^\bullet(\mathbb{V}^\circ, -), L) \xrightarrow{\pi^{\bullet*}} H_{\mathfrak{X}}^0(\overline{\mathbb{B}}_p^\bullet(\mathbb{V}^\circ, -), L) \\ \quad \xrightarrow{\iota_!^\bullet} H_{\mathfrak{X}}^{2(N-r-1)}(\overline{\mathbb{M}}_p^\bullet(\mathbb{V}^\circ, -), L(N-r-1)), \\ \text{inc}_\bullet^* : H_{\mathfrak{X}}^{2r}(\overline{\mathbb{M}}_p^\bullet(\mathbb{V}^\circ, -), L(r)) \xrightarrow{\iota^{\bullet*}} H_{\mathfrak{X}}^{2r}(\overline{\mathbb{B}}_p^\bullet(\mathbb{V}^\circ, -), L(r)) \\ \quad \xrightarrow{\pi_!^\bullet} H_{\mathfrak{X}}^0(\overline{\mathbb{S}}_p^\bullet(\mathbb{V}^\circ, -), L) \xrightarrow{\sim} L[\text{Sh}(\mathbb{V}^\circ, -\mathbb{K}_p^\bullet)]. \end{array} \right.$$

Note that the construction of the second pair only makes sense when $N \geq 3$; and when $N = 2$, we regard $\text{inc}_!^\dagger$ and inc_\dagger^* as zero maps. In fact, the two maps in each pair are essentially Poincaré dual to each other.

Definition 5.7.4. Suppose $N = 2r + 1$ odd with $r \geq 1$. We define the *incidence map (on the ground stratum)* to be the map

$$\text{inc} : L[\text{Sh}(\mathbb{V}^\circ, -\mathbb{K}_p^\circ)] \oplus L[\text{Sh}(\mathbb{V}^\circ, -\mathbb{K}_p^\bullet)] \rightarrow L[\text{Sh}(\mathbb{V}^\circ, -\mathbb{K}_p^\circ)] \oplus L[\text{Sh}(\mathbb{V}^\circ, -\mathbb{K}_p^\bullet)]$$

in $\text{Fun}(\mathfrak{K}(V^\circ)^p, \text{Mod}(L))$ given by the matrix

$$\begin{pmatrix} \text{inc}_\dagger^* \circ \text{inc}_\dagger^\dagger & \text{inc}_\dagger^* \circ \text{inc}_\dagger^\bullet \\ \text{inc}_\bullet^* \circ \text{inc}_\dagger^\dagger & \text{inc}_\bullet^* \circ \text{inc}_\dagger^\bullet \end{pmatrix}$$

if we write elements in the column form.

Remark 5.7.5. The construction of the incidence map can be encoded in the following diagram

$$\begin{array}{ccc} L[\text{Sh}(V^\circ, -K_p^\circ)] & & L[\text{Sh}(V^\circ, -K_p^\bullet)] \\ \downarrow & & \downarrow \\ H_{\mathbb{Z}}^{2r-2}(\overline{M}_p^\dagger(V^\circ, -), L(r-1)) & & H_{\mathbb{Z}}^0(\overline{B}_p^\bullet(V^\circ, -), L) \\ & \swarrow \text{m}_\dagger^\bullet & \nwarrow \iota_\dagger^\bullet \\ & H_{\mathbb{Z}}^{2r}(\overline{M}_p^\bullet(V^\circ, -), L(r)) & \\ & \swarrow \text{m}^{\dagger\bullet\bullet} & \searrow \iota^{\bullet\bullet} \\ H_{\mathbb{Z}}^{2r}(\overline{M}_p^\dagger(V^\circ, -), L(r)) & & H_{\mathbb{Z}}^{2r}(\overline{B}_p^\bullet(V^\circ, -), L(r)) \\ \downarrow & & \downarrow \\ L[\text{Sh}(V^\circ, -K_p^\circ)] & & L[\text{Sh}(V^\circ, -K_p^\bullet)] \end{array}$$

in $\text{Fun}(\mathfrak{K}(V^\circ)^p, \text{Mod}(L))$.

Proposition 5.7.6. *Suppose $N = 2r + 1$ odd with $r \geq 1$. Then the incidence map inc is given by the matrix*

$$\begin{pmatrix} -(p+1)^2 & \mathbf{T}_{N,p}^{\bullet\bullet} \\ \mathbf{T}_{N,p}^{\bullet\circ} & \mathbf{T}_{N,p}^{\bullet\bullet} \end{pmatrix}$$

where

$$\mathbf{T}_{N,p}^{\bullet\bullet} := \sum_{\delta=0}^{r-1} \mathbf{d}_{r-\delta,p}^\bullet \cdot \mathbf{T}_{N,p;\delta}^{\bullet\bullet}$$

in which the numbers $\mathbf{d}_{r-\delta,p}^\bullet$ are introduced in Notation 1.3.2, and the Hecke operators $\mathbf{T}_{N,p;\delta}^{\bullet\bullet}$ are introduced in Notation B.2.1 (as $\mathbf{T}_{N;\delta}^{\bullet\bullet}$).

Proof. Take an object $K^{p^\circ} \in \mathfrak{K}(V^\circ)^p$.

First, we show $\text{inc}_\dagger^* \circ \text{inc}_\dagger^\dagger = -(p+1)^2$. Since $\text{m}^{\dagger\bullet\bullet} \mathcal{O}_{\overline{M}_p^\circ(V^\circ, K^{p^\circ})}(1)$ has degree $p+1$, it follows from Corollary 5.2.5.

Second, we show $\text{inc}_\dagger^* \circ \text{inc}_\dagger^\bullet = \mathbf{T}_{N,p}^{\bullet\circ}$ and $\text{inc}_\bullet^* \circ \text{inc}_\dagger^\dagger = \mathbf{T}_{N,p}^{\bullet\circ}$. However, these are consequences of Theorem 5.4.3 and Construction 5.4.5.

Finally, we show $\text{inc}_\bullet^* \circ \text{inc}_\dagger^\bullet = \mathbf{T}_{N,p}^{\bullet\bullet}$. By Theorem 5.6.7(1), it suffices to show that for every $s_1^\bullet, s_2^\bullet \in \mathbf{S}_p^\bullet(V^\circ, K^{p^\circ})(\overline{\mathbb{F}}_p)$ and every $s^\bullet \in \text{Hk}_j^{-1}(s_1^\bullet, s_2^\bullet)$, the intersection multiplicity of $\mathbf{B}_{s_1^\bullet}^\bullet$ and $\mathbf{B}_{s_2^\bullet}^\bullet$ at the component $\mathbf{B}_{s^\bullet}^\bullet$ equals $\mathbf{d}_{r-j,p}^\bullet$. However, this is true by Theorem 5.6.7(2), Proposition A.2.4(1), and the excessive intersection formula.

The proposition is proved. \square

Now we assume that $N = 2r$ is even with $r \geq 2$. The reader may notice that the situation is different from Definition 5.7.4 since now $\overline{M}_p^\bullet(V^\circ, -)$ has dimension $2r-1$ while $\overline{B}_p^\bullet(V^\circ, -)$ still has dimension r . Thus to obtain a similar diagram as in Remark 5.7.5, we have to insert a map

$$\Theta: H_{\mathbb{Z}}^{2r-2}(\overline{M}_p^\bullet(V^\circ, -), L(r-1)) \rightarrow H_{\mathbb{Z}}^{2r}(\overline{M}_p^\bullet(V^\circ, -), L(r))$$

to obtain a diagram like

$$\begin{array}{ccccc}
L[\mathrm{Sh}(V^\circ, -K_p^\circ)] & & & & L[\mathrm{Sh}(V^\circ, -K_p^\bullet)] \\
\downarrow & & & & \downarrow \\
H_{\mathbb{Z}}^{2r-4}(\overline{M}_p^\dagger(V^\circ, -), L(r-2)) & & & & H_{\mathbb{Z}}^0(\overline{B}_p^\bullet(V^\circ, -), L) \\
\searrow^{m_!^\bullet} & & & & \swarrow^{i_!^\bullet} \\
& H_{\mathbb{Z}}^{2r-2}(\overline{M}_p^\bullet(V^\circ, -), L(r-1)) & & & \\
& \downarrow \Theta & & & \\
& H_{\mathbb{Z}}^{2r}(\overline{M}_p^\bullet(V^\circ, -), L(r)) & & & \\
& \swarrow^{m_!^{\bullet\bullet}} & & & \searrow^{i_!^{\bullet\bullet}} \\
H_{\mathbb{Z}}^{2r}(\overline{M}_p^\dagger(V^\circ, -), L(r)) & & & & H_{\mathbb{Z}}^{2r}(\overline{B}_p^\bullet(V^\circ, -), L(r)) \\
\downarrow & & & & \downarrow \\
L[\mathrm{Sh}(V^\circ, -K_p^\circ)] & & & & L[\mathrm{Sh}(V^\circ, -K_p^\bullet)].
\end{array}$$

Definition 5.7.7. For every line bundle \mathcal{L} on $M_p^\bullet(V^\circ, -)$,⁷ we denote

$$\Theta_{\mathcal{L}}: H_{\mathbb{Z}}^{2r-2}(\overline{M}_p^\bullet(V^\circ, -), L(r-1)) \rightarrow H_{\mathbb{Z}}^{2r}(\overline{M}_p^\bullet(V^\circ, -), L(r))$$

the map by taking cup product with $c_1(\mathcal{L})$, and define the \mathcal{L} -incidence map (on the ground stratum) to be the map

$$\mathrm{inc}_{\mathcal{L}}: L[\mathrm{Sh}(V^\circ, -K_p^\circ)] \oplus L[\mathrm{Sh}(V^\circ, -K_p^\bullet)] \rightarrow L[\mathrm{Sh}(V^\circ, -K_p^\circ)] \oplus L[\mathrm{Sh}(V^\circ, -K_p^\bullet)]$$

in $\mathrm{Fun}(\mathfrak{K}(V^\circ)^p, \mathrm{Mod}(L))$ given by the matrix

$$\begin{pmatrix} \mathrm{inc}_!^* \circ \Theta_{\mathcal{L}} \circ \mathrm{inc}_!^\dagger & \mathrm{inc}_!^* \circ \Theta_{\mathcal{L}} \circ \mathrm{inc}_!^\bullet \\ \mathrm{inc}_\bullet^* \circ \Theta_{\mathcal{L}} \circ \mathrm{inc}_!^\dagger & \mathrm{inc}_\bullet^* \circ \Theta_{\mathcal{L}} \circ \mathrm{inc}_!^\bullet \end{pmatrix}$$

if we write elements in the column form.

There are three natural choices of \mathcal{L} , which are $\mathcal{O}(M_p^\dagger(V^\circ, -))$, $\mathrm{Lie}_{\mathcal{A}, \tau_\infty^\circ}$, and $\omega_{\mathcal{A}^\vee, \tau_\infty}$. We now compute $\Theta_{\mathcal{L}}$ for the first two.⁸

Proposition 5.7.8. *Suppose $N = 2r$ with $r \geq 2$. Let L be a p -coprime coefficient ring. For $\mathcal{L} = \mathcal{O}(M_p^\dagger(V^\circ, -))$, the incidence map $\mathrm{inc}_{\mathcal{L}}$ is given by*

$$\begin{pmatrix} (p+1)^3 & -(p+1)\mathbf{T}_{N,p}^{\circ\bullet} \\ -(p+1)\mathbf{T}_{N,p}^{\bullet\circ} & \mathbf{R}_{N,p}^{\bullet\bullet} \end{pmatrix}$$

where

$$\mathbf{R}_{N,p}^{\bullet\bullet} := \sum_{\delta=0}^{r-1} \frac{1 - (-p)^{r-\delta}}{p+1} (p+1)(p+3) \cdots (p^{2(r-\delta)-1} + 1) \cdot \mathbf{T}_{N,p;\delta}^{\bullet\bullet}$$

in which the Hecke operators $\mathbf{T}_{N,p;\delta}^{\bullet\bullet}$ are introduced in Notation B.2.1 (as $\mathbf{T}_{N;\delta}^{\bullet\bullet}$).

⁷A line bundle \mathcal{L} on $M_p^\bullet(V^\circ, -)$ is a collection of a line bundle $\mathcal{L}(K^{p^\circ})$ on every $M_p^\bullet(V^\circ, K^{p^\circ})$, compatible with respect to pullbacks.

⁸In fact, the third one is a linear combination of the first two.

Proof. Take an object $K^{p^\circ} \in \mathfrak{K}(V^\circ)^p$.

First, we show $\text{inc}_\dagger^* \circ \Theta_{\mathcal{L}} \circ \text{inc}_\dagger^\dagger = (p+1)^3$. Since $m^{\dagger \circ *}\mathcal{O}_{\overline{M}_p^\circ(V^\circ, K^{p^\circ})}(1)$ has degree $p+1$, it follows from Corollary 5.2.5.

Second, we show $\text{inc}_\dagger^* \circ \Theta_{\mathcal{L}} \circ \text{inc}_\dagger^\bullet = -(p+1)\mathbf{T}_{N,p}^\circ$ and $\text{inc}_\bullet^* \circ \Theta_{\mathcal{L}} \circ \text{inc}_\dagger^\dagger = -(p+1)\mathbf{T}_{N,p}^{\circ\bullet}$. However, these are consequences of Corollary 5.2.5, Theorem 5.4.3, and Construction 5.4.5.

It remains to compute $\text{inc}_\bullet^* \circ \Theta_{\mathcal{L}} \circ \text{inc}_\dagger^\bullet$. By Theorem 5.6.7(1), it suffices to show that for every $s_1^\bullet, s_2^\bullet \in \mathbf{S}_p^\bullet(V^\circ, K^{p^\circ})(\overline{\mathbb{F}}_p)$ and every $s^\bullet \in \text{Hk}_j^{-1}(s_1^\bullet, s_2^\bullet)$, the intersection multiplicity of $B_{s_1^\bullet}^\dagger$ and $B_{s_2^\bullet}^\dagger$ at the component $B_{s^\bullet}^\dagger$ equals

$$\frac{1 - (-p)^{r-j}}{p+1} (p+1)(p+3) \cdots (p^{2(r-j)-1} + 1).$$

By Theorem 5.6.8 and the excessive intersection formula, such intersection multiplicity equals

$$\sum_{t^\dagger \in \mathbf{S}_s^\dagger(\overline{\mathbb{F}}_p)} \int_{\mathbb{P}(\mathcal{Y}_{t^\dagger})} c_{r-j-1} \left((\sigma^* \mathcal{H}_{t^\dagger}) \otimes_{\mathcal{O}_{\mathbb{P}(\mathcal{Y}_{t^\dagger})}} \mathcal{O}_{\mathbb{P}(\mathcal{Y}_{t^\dagger})}(1) \right).$$

A simple exercise shows that

$$\int_{\mathbb{P}(\mathcal{Y}_{t^\dagger})} c_{r-j-1} \left((\sigma^* \mathcal{H}_{t^\dagger}) \otimes_{\mathcal{O}_{\mathbb{P}(\mathcal{Y}_{t^\dagger})}} \mathcal{O}_{\mathbb{P}(\mathcal{Y}_{t^\dagger})}(1) \right) = \frac{1 - (-p)^{r-j}}{p+1}$$

for every $t^\dagger \in \mathbf{S}_s^\dagger(\overline{\mathbb{F}}_p)$. Thus, the claim follows from Theorem 5.6.8. \square

Proposition 5.7.9. *Suppose $N = 2r$ with $r \geq 2$. Let L be a p -coprime coefficient ring. For $\mathcal{L} = \text{Lie}_{\mathcal{A}, \tau_\infty^\circ}$, the incidence map $\text{inc}_{\mathcal{L}}$ is given by*

$$\begin{pmatrix} -(p+1)^2 & \mathbf{T}_{N,p}^\circ \\ \mathbf{T}_{N,p}^\circ & \mathbf{T}_{N,p}^{\circ\bullet} \end{pmatrix}$$

where

$$\mathbf{T}_{N,p}^\bullet := \sum_{\delta=0}^{r-1} \mathbf{d}_{r-\delta,p}^\bullet \cdot \mathbf{T}_{N,p;\delta}^\bullet$$

in which the numbers $\mathbf{d}_{r-\delta,p}^\bullet$ are introduced in Notation 1.3.2, and the Hecke operators $\mathbf{T}_{N,p;\delta}^\bullet$ are introduced in Notation B.2.1 (as $\mathbf{T}_{N;\delta}^\bullet$).

Proof. Take an object $K^{p^\circ} \in \mathfrak{K}(V^\circ)^p$. By Theorem 5.2.4, we have an isomorphism

$$(5.17) \quad \iota^{**} \text{Lie}_{\mathcal{A}, \tau_\infty^\circ} \simeq \mathcal{O}_{\overline{M}_p^\circ(V^\circ, K^{p^\circ})}(1)$$

of line bundles on $\overline{M}_p^\circ(V^\circ, K^{p^\circ})$.

First, we show $\text{inc}_\dagger^* \circ \Theta_{\mathcal{L}} \circ \text{inc}_\dagger^\dagger = -(p+1)^2$. This is a consequence of (5.17), Corollary 5.2.5 and the fact that $m^{\dagger \circ *}\mathcal{O}_{\overline{M}_p^\circ(V^\circ, K^{p^\circ})}(1)$ has degree $p+1$.

Second, we show $\text{inc}_\dagger^* \circ \Theta_{\mathcal{L}} \circ \text{inc}_\dagger^\bullet = \mathbf{T}_{N,p}^\circ$ and $\text{inc}_\bullet^* \circ \Theta_{\mathcal{L}} \circ \text{inc}_\dagger^\dagger = \mathbf{T}_{N,p}^{\circ\bullet}$. However, these are consequences of (5.17) and Corollary 5.2.5, Theorem 5.4.3, and Construction 5.4.5.

It remains to compute $\text{inc}_\bullet^* \circ \Theta_{\mathcal{L}} \circ \text{inc}_\dagger^\bullet$. By Theorem 5.6.7 and the excessive intersection formula, it suffices to show that for every $s_1^\bullet, s_2^\bullet \in \mathbf{S}_p^\bullet(V^\circ, K^{p^\circ})(\overline{\mathbb{F}}_p)$ and every $s^\bullet \in \text{Hk}_j^{-1}(s_1^\bullet, s_2^\bullet)$, we have

$$(5.18) \quad \int_{\text{DL}^\bullet(\mathcal{Y}_{s^\bullet}, \{ , \}_{s^\bullet})} c_{r-1} \left((\sigma^* \overline{\mathcal{H}}_{s^\bullet, 2}) \otimes_{\mathcal{O}_{\text{DL}^\bullet(\mathcal{Y}_{s^\bullet}, \{ , \}_{s^\bullet})}} (\overline{\mathcal{H}}_{s^\bullet, 1}^\dagger / \overline{\mathcal{H}}_{s^\bullet, 2}) \right) \cdot c_1 \left((\zeta_{s^\bullet}^\bullet)_* \text{Lie}_{\mathcal{A}, \tau_\infty^\circ} \right) = \mathbf{d}_{r,p}^\bullet,$$

where $(\overline{\mathcal{H}}_{s^\bullet, 1}, \overline{\mathcal{H}}_{s^\bullet, 2})$ is the universal object over $\text{DL}^\bullet(\mathcal{Y}_{s^\bullet}, \{ , \}_{s^\bullet})$. However, by Theorem 5.3.4(4), we have $(\zeta_{s^\bullet}^\bullet)_* \text{Lie}_{\mathcal{A}, \tau_\infty^\circ} \simeq \overline{\mathcal{H}}_{s^\bullet, 1}^\dagger / \overline{\mathcal{H}}_{s^\bullet, 2}$. Thus, (5.18) follows from Proposition A.2.4(2). The proposition is proved. \square

5.8. Weight spectral sequence. In this subsection, we study the weight spectral sequence associated to $\mathbf{M}_p(V^\circ, \mathbf{-})$. We keep the setup in Subsection 5.7. In particular, N is an integer at least 2 with $r := \lfloor \frac{N}{2} \rfloor \geq 1$, and L is a p -coprime coefficient ring. To ease notation, we put $X_N^? := X_p^?(V^\circ, \mathbf{-})$ for meaningful pairs $(X, ?) \in \{\mathbf{M}, \mathbf{M}, \mathbf{B}, \mathbf{S}\} \times \{, \circ, \bullet, \dagger\}$.

Construction 5.8.1. By Theorem 5.1.5(1), we have the weight spectral sequence $(E_s^{p,q}, d_s^{p,q})$, with terms in the category $L[\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)]$, abutting to the cohomology $H_{\mathbb{F}}^{p+q}(\overline{\mathbf{M}}_N, R\Psi L(r))$. In particular, we have

$$E_1^{0,2d} = H_{\mathbb{F}}^{2d}(\overline{\mathbf{M}}_N^\circ, L(r)) \bigoplus H_{\mathbb{F}}^{2d}(\overline{\mathbf{M}}_N^\bullet, L(r)).$$

Thus, the six maps in Construction 5.7.3 give rise to another six maps

$$\begin{cases} \text{Inc}_!^\circ : L[\text{Sh}(V^\circ, \mathbf{-K}_p^\circ)] \rightarrow E_1^{0,2(N-r-1)}(N-2r-1), \\ \text{Inc}_!^\dagger : L[\text{Sh}(V^\circ, \mathbf{-K}_p^\circ)] \rightarrow E_1^{0,2(N-r-1)}(N-2r-1), \\ \text{Inc}_!^\bullet : L[\text{Sh}(V^\circ, \mathbf{-K}_p^\bullet)] \rightarrow E_1^{0,2(N-r-1)}(N-2r-1), \\ \text{Inc}_\circ^* : E_1^{0,2r} \rightarrow L[\text{Sh}(V^\circ, \mathbf{-K}_p^\circ)], \\ \text{Inc}_\dagger^* : E_1^{0,2r} \rightarrow L[\text{Sh}(V^\circ, \mathbf{-K}_p^\circ)], \\ \text{Inc}_\bullet^* : E_1^{0,2r} \rightarrow L[\text{Sh}(V^\circ, \mathbf{-K}_p^\bullet)], \end{cases}$$

in $\text{Fun}(\mathcal{K}(V^\circ)^p, \text{Mod}(L))$.

In the future, we will have to study the composite maps

$$\begin{pmatrix} \text{Inc}_\circ^* \\ \text{Inc}_\dagger^* \\ \text{Inc}_\bullet^* \end{pmatrix} (\text{Inc}_!^\circ \quad \text{Inc}_!^\dagger \quad \text{Inc}_!^\bullet), \quad \begin{pmatrix} \text{Inc}_\circ^* \\ \text{Inc}_\dagger^* \\ \text{Inc}_\bullet^* \end{pmatrix} \circ d_1^{-1,2r} \circ d_1^{0,2r-2}(-1) \circ (\text{Inc}_!^\circ \quad \text{Inc}_!^\dagger \quad \text{Inc}_!^\bullet)$$

when N is odd and even, respectively. In the next two lemmas, we will study the spectral sequence and prove two formulae related to the above maps, according to the parity of N .

Lemma 5.8.2. *Suppose $N = 2r + 1$ odd with $r \geq 1$.*

(1) *The first page of $E_s^{p,q}$ is as follows:*

$q \geq 2r + 2$	$\dots \longrightarrow \dots \longrightarrow \dots$
$q = 2r + 1$	$H_{\mathbb{F}}^{2r-1}(\overline{\mathbf{M}}_N^\dagger, L(r-1)) \xrightarrow{d_1^{-1,2r+1}} H_{\mathbb{F}}^{2r+1}(\overline{\mathbf{M}}_N^\circ, L(r)) \oplus H_{\mathbb{F}}^{2r+1}(\overline{\mathbf{M}}_N^\bullet, L(r)) \xrightarrow{d_1^{0,2r+1}} H_{\mathbb{F}}^{2r+1}(\overline{\mathbf{M}}_N^\dagger, L(r))$
$q = 2r$	$H_{\mathbb{F}}^{2r-2}(\overline{\mathbf{M}}_N^\dagger, L(r-1)) \xrightarrow{d_1^{-1,2r}} H_{\mathbb{F}}^{2r}(\overline{\mathbf{M}}_N^\circ, L(r)) \oplus H_{\mathbb{F}}^{2r}(\overline{\mathbf{M}}_N^\bullet, L(r)) \xrightarrow{d_1^{0,2r}} H_{\mathbb{F}}^{2r}(\overline{\mathbf{M}}_N^\dagger, L(r))$
$q = 2r - 1$	$H_{\mathbb{F}}^{2r-3}(\overline{\mathbf{M}}_N^\dagger, L(r-1)) \xrightarrow{d_1^{-1,2r-1}} H_{\mathbb{F}}^{2r-1}(\overline{\mathbf{M}}_N^\circ, L(r)) \oplus H_{\mathbb{F}}^{2r-1}(\overline{\mathbf{M}}_N^\bullet, L(r)) \xrightarrow{d_1^{0,2r-1}} H_{\mathbb{F}}^{2r-1}(\overline{\mathbf{M}}_N^\dagger, L(r))$
$q \leq 2r - 2$	$\dots \longrightarrow \dots \longrightarrow \dots$
$E_1^{p,q}$	$p = -1 \qquad \qquad \qquad p = 0 \qquad \qquad \qquad p = 1$

with $d_1^{-1,i} = (m_!^\dagger, -m_!^\bullet)$, $d_1^{0,i} = (m^\dagger)^\circ - (m^\bullet)^\circ$ for every $i \in \mathbb{Z}$; and $E_1^{p,q} = 0$ if $|p| > 1$.

(2) We have

$$\begin{pmatrix} \text{Inc}_\circ^* \\ \text{Inc}_\dagger^* \\ \text{Inc}_\bullet^* \end{pmatrix} \begin{pmatrix} \text{Inc}_\circ^\circ & \text{Inc}_\dagger^\dagger & \text{Inc}_\bullet^\bullet \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -(p+1)^2 & \mathbf{T}_{N,p}^{\circ\bullet} \\ 0 & \mathbf{T}_{N,p}^{\bullet\circ} & \mathbf{T}_{N,p}^{\bullet\bullet} \end{pmatrix}.$$

(3) We have $(\mathbf{T}_{N,p}^{\bullet\circ} \circ \text{Inc}_\dagger^* + (p+1)^2 \text{Inc}_\bullet^*) \circ d_1^{-1,2r} = 0$.

Proof. Part (1) is immediate. Part (2) is a consequence of Proposition 5.7.6.

For (3), note that under the composite isomorphism

$$\begin{aligned} i: L[\text{Sh}(V^\circ, -\mathbf{K}_p^\circ)] &\xrightarrow{\sim} H_{\mathbb{X}}^0(\overline{S}_N^\circ, L) \xrightarrow{\pi^{\circ*}} H_{\mathbb{X}}^0(\overline{B}_N^\circ, L) \xrightarrow{\cup \xi^{r-1}} H_{\mathbb{X}}^{2r-2}(\overline{B}_N^\circ, L(r-1)) \\ &\xrightarrow{i_\circ^\circ} H_{\mathbb{X}}^{2r-2}(\overline{M}_N^\circ, L(r-1)) \xrightarrow{m^{\dagger\circ*}} H_{\mathbb{X}}^{2r-2}(\overline{M}_N^\dagger, L(r-1)) = E_1^{-1,2r}, \end{aligned}$$

the map $d_1^{-1,2r} \circ i: L[\text{Sh}(V^\circ, -\mathbf{K}_p^\circ)] \rightarrow E_1^{0,2r}$ coincides with $(p+1)\text{Inc}_\circ^\circ - \text{Inc}_\dagger^\dagger$. Thus, (3) follows by (2) as we have

$$\begin{pmatrix} 0 & \mathbf{T}_{N,p}^{\bullet\circ} & (p+1)^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -(p+1)^2 & \mathbf{T}_{N,p}^{\circ\bullet} \\ 0 & \mathbf{T}_{N,p}^{\bullet\circ} & \mathbf{T}_{N,p}^{\bullet\bullet} \end{pmatrix} \begin{pmatrix} p+1 \\ -1 \\ 0 \end{pmatrix} = 0.$$

The lemma is proved. \square

For N even, we first recall that there is an (increasing) monodromy filtration $F_\bullet \mathbf{R}\Psi L(r)$ of $\mathbf{R}\Psi L(r)$. Such filtration induces a filtration $F_\bullet H_{\mathbb{X}}^i(\overline{M}_N, \mathbf{R}\Psi L(r))$ of the cohomology $H_{\mathbb{X}}^i(\overline{M}_N, \mathbf{R}\Psi L(r))$, and a corresponding filtration $F_\bullet H^1(I_{\mathbb{Q}_p^\Phi}, H_{\mathbb{X}}^i(\overline{M}_N, \mathbf{R}\Psi L(r)))$ of the quotient module $H^1(I_{\mathbb{Q}_p^\Phi}, H_{\mathbb{X}}^i(\overline{M}_N, \mathbf{R}\Psi L(r)))$.

Lemma 5.8.3. *Suppose $N = 2r$ even with $r \geq 1$.*

(1) *The first page of $E_s^{p,q}$ is as follows:*

$q \geq 2r + 1$	$\dots \longrightarrow \dots \longrightarrow \dots$
$q = 2r$	$H_{\mathbb{X}}^{2r-2}(\overline{M}_N^\dagger, L(r-1)) \xrightarrow{d_1^{-1,2r}} H_{\mathbb{X}}^{2r}(\overline{M}_N^\circ, L(r)) \oplus H_{\mathbb{X}}^{2r}(\overline{M}_N^\bullet, L(r)) \xrightarrow{d_1^{0,2r}} H_{\mathbb{X}}^{2r}(\overline{M}_N^\dagger, L(r))$
$q = 2r - 1$	$0 \longrightarrow H_{\mathbb{X}}^{2r-1}(\overline{M}_N^\bullet, L(r)) \longrightarrow 0$
$q = 2r - 2$	$H_{\mathbb{X}}^{2r-4}(\overline{M}_N^\dagger, L(r-1)) \xrightarrow{d_1^{-1,2r-2}} H_{\mathbb{X}}^{2r-2}(\overline{M}_N^\circ, L(r)) \oplus H_{\mathbb{X}}^{2r-2}(\overline{M}_N^\bullet, L(r)) \xrightarrow{d_1^{0,2r-2}} H_{\mathbb{X}}^{2r-2}(\overline{M}_N^\dagger, L(r))$
$q \leq 2r - 3$	$\dots \longrightarrow \dots \longrightarrow \dots$
$E_1^{p,q}$	$p = -1 \qquad \qquad \qquad p = 0 \qquad \qquad \qquad p = 1$

with $d_1^{-1,i} = (m_\dagger^{\circ\bullet}, -m_\dagger^{\bullet\circ})$, $d_1^{0,i} = (m^{\dagger\circ})^* - (m^{\dagger\bullet})^*$ for every $i \in \mathbb{Z}$; and $E_1^{p,q} = 0$ if $|p| > 1$.

(2) *The spectral sequence $E_s^{p,q}$ degenerates at the second page.*

(3) In the (three-step) filtration $F_{\bullet} H_{\mathbb{F}_p}^{2r-1}(\overline{M}_N, R\Psi L(r))$, we have canonical isomorphisms

$$\begin{cases} F_{-1} H_{\mathbb{F}_p}^{2r-1}(\overline{M}_N, R\Psi L(r)) \simeq E_2^{1,2r-2} = \text{coker } d_1^{0,2r-2}, \\ \frac{F_0 H_{\mathbb{F}_p}^{2r-1}(\overline{M}_N, R\Psi L(r))}{F_{-1} H_{\mathbb{F}_p}^{2r-1}(\overline{M}_N, R\Psi L(r))} \simeq E_2^{0,2r-1} = H_{\mathbb{F}_p}^{2r-1}(\overline{M}_N^{\bullet}, L(r)), \\ \frac{H_{\mathbb{F}_p}^{2r-1}(\overline{M}_N, R\Psi L(r))}{F_0 H_{\mathbb{F}_p}^{2r-1}(\overline{M}_N, R\Psi L(r))} \simeq E_2^{-1,2r} = \ker d_1^{-1,2r-2}, \end{cases}$$

in $\text{Fun}(\mathfrak{K}(V^{\circ})^p, \text{Mod}(L[\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^{\Phi})]))$.

(4) The monodromy map on $H_{\mathbb{F}_p}^{2r-1}(\overline{M}_N, R\Psi L(r))$ factors through $F_0 H_{\mathbb{F}_p}^{2r-1}(\overline{M}_N, R\Psi L(r))$ and is given by the composite map

$$E_2^{-1,2r} \xrightarrow{\mu} E_2^{1,2r-2} \hookrightarrow H_{\mathbb{F}_p}^{2r-1}(\overline{M}_N, R\Psi L(r)),$$

where μ is the map induced from the identity map on $H_{\mathbb{F}_p}^{2r-2}(\overline{M}_N^{\dagger}, L(r-1))$.

(5) We have a canonical isomorphism

$$F_{-1} H^1(I_{\mathbb{Q}_p^{\Phi}}, H_{\mathbb{F}_p}^{2r-1}(\overline{M}_N, R\Psi L(r))) \simeq \left(\frac{E_2^{1,2r-2}}{\mu E_2^{-1,2r}} \right) (-1);$$

in $\text{Fun}(\mathfrak{K}(V^{\circ})^p, \text{Mod}(L[\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^{\Phi})]))$; and the map $d_1^{-1,2r}$ induces an isomorphism

$$\left(\frac{E_2^{1,2r-2}}{\mu E_2^{-1,2r}} \right) (-1) \simeq \frac{\text{im } d_1^{-1,2r}}{\text{im}(d_1^{-1,2r} \circ d_1^{0,2r-2}(-1))}$$

in $\text{Fun}(\mathfrak{K}(V^{\circ})^p, \text{Mod}(L[\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^{\Phi})]))$.

(6) If $p^2 - 1$ is invertible in L , then we have a canonical short exact sequence

$$0 \longrightarrow F_{-1} H^1(I_{\mathbb{Q}_p^{\Phi}}, H_{\mathbb{F}_p}^{2r-1}(\overline{M}_N, R\Psi L(r))) \longrightarrow H_{\text{sing}}^1(\mathbb{Q}_p^{\Phi}, H_{\mathbb{F}_p}^{2r-1}(\overline{M}_N, R\Psi L(r))) \longrightarrow H_{\mathbb{F}_p}^{2r-1}(\overline{M}_N^{\bullet}, L(r-1))^{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^{\Phi})} \rightarrow 0$$

in $\text{Fun}(\mathfrak{K}(V^{\circ})^p, \text{Mod}(L))$.

(7) The composite map

$$\begin{pmatrix} \text{Inc}_{\circ}^* \\ \text{Inc}_{\dagger}^* \\ \text{Inc}_{\bullet}^* \end{pmatrix} \circ d_1^{-1,2r} \circ d_1^{0,2r-2}(-1) \circ \begin{pmatrix} \text{Inc}_{\circ}^{\circ} & \text{Inc}_{\dagger}^{\dagger} & \text{Inc}_{\bullet}^{\bullet} \end{pmatrix}$$

coincides with

$$\begin{pmatrix} p+1 & (p+1)^2 & -T_{N,p}^{\circ\bullet} \\ (p+1)^2 & (p+1)^3 & -(p+1)T_{N,p}^{\circ\bullet} \\ -T_{N,p}^{\circ\bullet} & -(p+1)T_{N,p}^{\circ\bullet} & R_{N,p}^{\bullet} \end{pmatrix}, \quad \begin{pmatrix} p+1 & 0 & -T_{N,p}^{\circ\bullet} \\ 0 & 0 & 0 \\ -T_{N,p}^{\circ\bullet} & 0 & R_{N,p}^{\bullet} \end{pmatrix}$$

when $N \geq 4$ and when $N = 2$, respectively.

(8) The image of the map

$$\begin{aligned} & (T_{N,p}^{\circ\bullet} \circ \text{Inc}_{\circ}^* + (p+1)\text{Inc}_{\bullet}^*) \circ d_1^{-1,2r} \circ d_1^{0,2r-2}(-1) \circ (\text{Inc}_{\circ}^{\circ} + \text{Inc}_{\dagger}^{\dagger} + \text{Inc}_{\bullet}^{\bullet}): \\ & L[\text{Sh}(V^{\circ}, -K_p^{\circ})]^{\oplus 2} \bigoplus L[\text{Sh}(V^{\circ}, -K_p^{\bullet})] \rightarrow L[\text{Sh}(V^{\circ}, -K_p^{\bullet})] \end{aligned}$$

is exactly $((p+1)R_{N,p}^{\bullet} - T_{N,p}^{\circ\bullet} \circ T_{N,p}^{\circ\bullet})L[\text{Sh}(V^{\circ}, -K_p^{\bullet})]$, where $R_{N,p}^{\bullet}$ is introduced in Proposition 5.7.8.

Proof. For (1), note that by Lemma 5.5.2(1), both $H_{\mathbb{F}_p}^i(\overline{M}_N^{\dagger}, L)$ and $H_{\mathbb{F}_p}^i(\overline{M}_N^{\circ}, L)$ vanish for i odd. Thus, (1) follows.

Parts (2–4) follow directly from the description of $E_1^{p,q}$ and [Sai03, Corollary 2.8](2) for the description of the monodromy map⁹. Part (5) follows from (1–4).

For (6), by Lemma 5.5.2(3), we know that the action of $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^\Phi)$ on $E_2^{1,2r-2}(-1)$ is trivial. As $p^2 - 1$ is invertible in L , we further have $E_2^{-1,2r}(-1)^{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^\Phi)} = 0$ and

$$H^1(\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^\Phi), F_{-1}H^1(I_{\mathbb{Q}_p^\Phi}, H_{\mathbb{Z}}^{2r-1}(\overline{M}_N, R\Psi L(r)))) = 0.$$

In particular, we have the isomorphism

$$\begin{aligned} H_{\text{sing}}^1(\mathbb{Q}_p^\Phi, H_{\mathbb{Z}}^{2r-1}(\overline{M}_N, R\Psi L(r))) &\simeq H^1(I_{\mathbb{Q}_p^\Phi}, H_{\mathbb{Z}}^{2r-1}(\overline{M}_N, R\Psi L(r)))^{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^\Phi)} \\ &\simeq F_0H^1(I_{\mathbb{Q}_p^\Phi}, H_{\mathbb{Z}}^{2r-1}(\overline{M}_N, R\Psi L(r)))^{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^\Phi)}, \end{aligned}$$

and that (6) follows from the induced long exact sequence.

For (7), when $N \geq 4$ (that is, $r \geq 2$), it follows from Theorem 5.2.4(2) and Proposition 5.7.8. When $N = 2$, it follows from a direct computation.

For (8), we have the identity

$$\begin{aligned} & \left(\mathbf{T}_{N,p}^{\bullet\circ} \quad 0 \quad p+1 \right) \begin{pmatrix} \text{Inc}_\circ^* \\ \text{Inc}_\dagger^* \\ \text{Inc}_\bullet^* \end{pmatrix} \circ d_1^{-1,2r} \circ d_1^{0,2r-2}(-1) \circ \left(\text{Inc}_\dagger^\circ \quad \text{Inc}_\dagger^\dagger \quad \text{Inc}_\bullet^\circ \right) \\ &= \left(0 \quad 0 \quad (p+1)\mathbf{R}_{N,p}^\bullet - \mathbf{T}_{N,p}^{\bullet\circ} \circ \mathbf{T}_{N,p}^{\circ\bullet} \right) \end{aligned}$$

by (7), which implies (8).

The lemma is all proved. \square

Construction 5.8.4. We construct

- (1) when $N = 2r + 1$ is odd, the map

$$\nabla^1: E_2^{0,2r} \rightarrow L[\text{Sh}(V_N^\circ, K_N^\circ)]$$

to be restriction of the map

$$\mathbf{T}_{N,p}^{\bullet\circ} \circ \text{Inc}_\dagger^* + (p+1)\text{Inc}_\bullet^*: E_1^{0,2r-1} \rightarrow L[\text{Sh}(V_N^\circ, K_N^\bullet)],$$

to $\ker d_1^{0,2r-1}$, which factors through $E_2^{0,2r}$ by Lemma 5.8.2(3), composed with the map $\mathbf{T}_{N,p}^{\bullet\circ}: L[\text{Sh}(V_N^\circ, K_N^\bullet)] \rightarrow L[\text{Sh}(V_N^\circ, K_N^\circ)]$;

- (2) when $N = 2r$ is even, the map

$$\nabla^0: \ker d_1^{0,2r} \rightarrow L[\text{Sh}(V_N^\circ, K_N^\circ)]$$

to be restriction of the map

$$\mathbf{T}_{N,p}^{\bullet\circ} \circ \text{Inc}_\circ^* + (p+1)\text{Inc}_\bullet^*: E_1^{0,2r} \rightarrow L[\text{Sh}(V_N^\circ, K_N^\bullet)]$$

in Lemma 5.8.3(8) to $\ker d_1^{0,2r}$, composed with the map $\mathbf{T}_{N,p}^{\bullet\circ}: L[\text{Sh}(V_N^\circ, K_N^\bullet)] \rightarrow L[\text{Sh}(V_N^\circ, K_N^\circ)]$.

Remark 5.8.5. By the descriptions of the Galois actions in Construction 5.2.6 and Construction 5.3.6, the map ∇^1 factors through the quotient map $E_2^{0,2r} \rightarrow (E_2^{0,2r})_{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^\Phi)}$.

To temporarily end the discussion on weight spectral sequences, we record the following easy lemma, which will be used later.

⁹The description of the monodromy map does not require that the scheme is proper over the base.

Lemma 5.8.6. *Suppose $N \geq 3$. The following diagram*

$$\begin{array}{ccc} E_1^{0,2r} & \xrightarrow{(\text{Inc}_\circ^*, \text{Inc}_\dagger^*, \text{Inc}_\bullet^*)} & L[\text{Sh}(V^\circ, -K_p^\circ)]^{\oplus 2} \oplus L[\text{Sh}(V^\circ, -K_p^\bullet)] \\ d_1^{0,2r} \downarrow & & \downarrow (p+1, -1, 0) \\ E_1^{1,2r} & \xrightarrow{\quad\quad\quad} & L[\text{Sh}(V^\circ, -K_p^\circ)] \end{array}$$

is commutative, where the lower arrow is the composite map

$$\begin{aligned} H_{\mathfrak{X}}^{2r}(\overline{M}_p^\dagger(V^\circ, -), L(r)) &\xrightarrow{m_1^{\dagger\circ}} H_{\mathfrak{X}}^{2(r+1)}(\overline{M}_p^\circ(V^\circ, -), L(r+1)) \xrightarrow{\iota^{\circ*}} H_{\mathfrak{X}}^{2(r+1)}(\overline{B}_p^\circ(V^\circ, -), L(r+1)) \\ &\xrightarrow{\cup \xi^{N-r-2}} H_{\mathfrak{X}}^{2(N-1)}(\overline{B}_p^\circ(V^\circ, -), L(N-1)) \xrightarrow{\pi_1^\circ} H_{\mathfrak{X}}^0(\overline{S}_p^\circ(V^\circ, -), L) \xrightarrow{\sim} L[\text{Sh}(V^\circ, -K_p^\circ)], \end{aligned}$$

which is an isomorphism.

Proof. The commutativity of the diagram follows from the formula $d_1^{0,2r} = (m^{\dagger\circ})^* - (m^{\dagger\bullet})^*$, and the fact that $M_p^\dagger(V^\circ, -)$ is a hypersurface in $M_p^\circ(V^\circ, -)$ of degree $p+1$ by Theorem 5.2.4 and Lemma A.1.4(1). By Lemma 5.5.2 and Poincaré duality, the lower arrow is an isomorphism. \square

5.9. Special results in the rank 3 case. In this subsection, we study some special properties of the ground stratum $M_p^\bullet(V^\circ, -)$ when $N = 3$. The results here will only be used in the situation (b) of Lemma 8.1.4 and are only necessary for the main theorems in Subsection 1.1 in the case where $n = 2$ and $F^+ = \mathbb{Q}$, so readers may skip this subsection at this moment.

To begin with, we recall the following definition.

Definition 5.9.1 ([FK88, Chapter I. Definition 3.7 & Note 3.10]). A proper morphism $f: X \rightarrow Y$ of schemes of characteristic p is *purely inseparable* if the following two equivalent conditions hold:

- (1) For every (scheme-theoretical) point y of Y , there lies exactly one point x of X , and the residue field extension is purely inseparable.
- (2) For every algebraically field κ of characteristic p , the induced map $f(\kappa): X(\kappa) \rightarrow Y(\kappa)$ is a bijection.

We now assume $\dim_F V^\circ = N = 3$.

Definition 5.9.2. We define a functor

$$\begin{aligned} \mathbf{M}'_p(V^\circ, -): \mathfrak{K}(V^\circ)^p \times \mathfrak{T} &\rightarrow \text{PSch}'_{/\mathbb{Z}_p^\Phi} \\ K^{p^\circ} &\mapsto \mathbf{M}'_p(V^\circ, K^{p^\circ}) \end{aligned}$$

such that for every $S \in \text{Sch}'_{/\mathbb{Z}_p^\Phi}$, $\mathbf{M}'_p(V^\circ, K^{p^\circ})(S)$ is the set of equivalence classes of sextuples $(A_0, \lambda_0, \eta_0^p; A', \lambda', \eta^{p'})$ where

- $(A_0, \lambda_0, \eta_0^p)$ is an element in $\mathbf{T}_p(S)$;
- (A', λ') is a unitary O_F -abelian scheme of signature type $3\Phi - 2\tau_\infty + 2\tau_\infty^c$ over S (Definitions 3.4.2 and 3.4.3) such that λ' is p -principal;
- $\eta^{p'}$ is a K^{p° -level structure, that is, for a chosen geometric point s on every connected component of S , a $\pi_1(S, s)$ -invariant K^{p° -orbit of isomorphisms

$$\eta^{p'}: V^\circ \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p} \rightarrow \text{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}}^{\lambda_0, \lambda'}(H_1^{\text{ét}}(A'_{0s}, \mathbb{A}^{\infty, p}), H_1^{\text{ét}}(A'_s, \mathbb{A}^{\infty, p}))$$

of hermitian spaces over $F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p} = F \otimes_{F^+} \mathbb{A}_{F^+}^{\infty, p}$. See Construction 3.4.4 (with $\square = \{\infty, p\}$) for the right-hand side.

The equivalence relation and the action of morphisms in $\mathfrak{K}(V^\circ)^p \times \mathfrak{T}$ are defined similarly as in Definition 4.1.2.

We have apparently the forgetful morphism

$$(5.19) \quad \mathbf{M}'_{\mathfrak{p}}(\mathbf{V}^\circ, -) \rightarrow \mathbf{T}_{\mathfrak{p}}$$

in $\text{Fun}(\mathfrak{K}(\mathbf{V}^\circ)^p \times \mathfrak{T}, \text{PSch}'_{/\mathbb{F}_p^\Phi})$. By a similar proof of Theorem 4.1.3, the morphism (5.19) is represented by quasi-projective smooth schemes of relative dimension 2. We denote by the base change of (5.19) to \mathbb{F}_p^Φ by $\mathbf{M}'_{\mathfrak{p}}(\mathbf{V}^\circ, -) \rightarrow \mathbf{T}_{\mathfrak{p}}$, which is a morphism in $\text{Fun}(\mathfrak{K}(\mathbf{V}^\circ)^p \times \mathfrak{T}, \text{Sch}_{/\mathbb{F}_p^\Phi})$.

Definition 5.9.3. We define a functor

$$\begin{aligned} \mathbf{N}_{\mathfrak{p}}(\mathbf{V}^\circ, -) : \mathfrak{K}(\mathbf{V}^\circ)^p \times \mathfrak{T} &\rightarrow \text{PSch}'_{/\mathbb{F}_p^\Phi} \\ K^{p^\circ} &\mapsto \mathbf{N}_{\mathfrak{p}}(\mathbf{V}^\circ, K^{p^\circ}) \end{aligned}$$

such that for every $S \in \text{Sch}'_{/\mathbb{F}_p^\Phi}$, $\mathbf{N}_{\mathfrak{p}}(\mathbf{V}^\circ, K^{p^\circ})(S)$ is the set of equivalence classes of decuples $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A', \lambda', \eta^{p'}; \delta)$ where

- $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p)$ is an element of $\mathbf{M}_{\mathfrak{p}}(\mathbf{V}^\circ, K^{p^\circ})(S)$;
- $(A_0, \lambda_0, \eta_0^p; A', \lambda', \eta^{p'})$ is an element of $\mathbf{M}'_{\mathfrak{p}}(\mathbf{V}^\circ, K^{p^\circ})(S)$;
- $\delta: A \rightarrow A'$ is an \mathcal{O}_F -linear quasi- p -isogeny (Definition 3.4.5) such that
 - (a) $\ker \delta[p^\infty]$ is contained in $A[\mathfrak{p}]$;
 - (b) we have $\lambda = \delta^\vee \circ \lambda' \circ \delta$; and
 - (c) the K^{p° -orbit of maps $v \mapsto \delta_* \circ \eta^p(v)$ for $v \in \mathbf{V}^\circ \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$ coincides with $\eta^{p'}$.

The equivalence relation and the action of morphisms in $\mathfrak{K}(\mathbf{V}^\circ)^p \times \mathfrak{T}$ are defined similarly as in Definition 4.2.3.

By definition, we have the following two obvious forgetful morphisms.

$$\begin{array}{ccc} & \mathbf{N}_{\mathfrak{p}}(\mathbf{V}^\circ, -) & \\ \mu \swarrow & & \searrow \mu' \\ \mathbf{M}_{\mathfrak{p}}(\mathbf{V}^\circ, -) & & \mathbf{M}'_{\mathfrak{p}}(\mathbf{V}^\circ, -) \end{array}$$

in $\text{Fun}(\mathfrak{K}(\mathbf{V}^\circ)^p \times \mathfrak{T}, \text{Sch}_{/\mathbb{F}_p^\Phi})$. By the extension property of isogeny, it is clear that both μ and μ' are proper. We apply the Stein factorization to the morphism μ' and obtain the following diagram

$$(5.20) \quad \begin{array}{ccc} & \mathbf{N}_{\mathfrak{p}}(\mathbf{V}^\circ, -) \xrightarrow{\nu} \mathbf{N}'_{\mathfrak{p}}(\mathbf{V}^\circ, -) & \\ \mu \swarrow & & \searrow \nu' \\ \mathbf{M}_{\mathfrak{p}}(\mathbf{V}^\circ, -) & & \mathbf{M}'_{\mathfrak{p}}(\mathbf{V}^\circ, -) \end{array}$$

in $\text{Fun}(\mathfrak{K}(\mathbf{V}^\circ)^p \times \mathfrak{T}, \text{Sch}_{/\mathbb{F}_p^\Phi})$. For every $K^{p^\circ} \in \mathfrak{K}(\mathbf{V}^\circ)^p$ a perfect field κ containing \mathbb{F}_p^Φ , we say that a point $(A_0, \lambda_0, \eta_0^p; A', \lambda', \eta^{p'}) \in \mathbf{M}'_{\mathfrak{p}}(\mathbf{V}^\circ, K^{p^\circ})(\kappa)$ is *special* if we have $\text{FH}_1^{\text{dR}}(A'/\kappa)_{\tau_\infty} = \text{VH}_1^{\text{dR}}(A'/\kappa)_{\tau_\infty}$. We denote by $\mathbf{M}'_{\mathfrak{p}}(\mathbf{V}^\circ, K^{p^\circ})_{\text{sp}}$ the locus of special points in $\mathbf{M}'_{\mathfrak{p}}(\mathbf{V}^\circ, K^{p^\circ})$, regarded as a Zariski closed subset, and by $\mathbf{N}'_{\mathfrak{p}}(\mathbf{V}^\circ, K^{p^\circ})_{\text{sp}}$ the (set-theoretical) inverse image of $\mathbf{M}'_{\mathfrak{p}}(\mathbf{V}^\circ, K^{p^\circ})_{\text{sp}}$ under ν' . An easy deformation argument shows that $\mathbf{M}'_{\mathfrak{p}}(\mathbf{V}^\circ, K^{p^\circ})_{\text{sp}}$ is of dimension zero.

Proposition 5.9.4. *In (5.20), for every $K^{p^\circ} \in \mathfrak{K}(\mathbf{V}^\circ)^p$, we have*

- (1) *The morphism $\mu: \mathbf{N}_{\mathfrak{p}}(\mathbf{V}^\circ, K^{p^\circ}) \rightarrow \mathbf{M}_{\mathfrak{p}}(\mathbf{V}^\circ, K^{p^\circ})$ induces a purely inseparable morphism onto its image which is $\mathbf{M}_{\mathfrak{p}}^\bullet(\mathbf{V}^\circ, K^{p^\circ})$.*
- (2) *The morphism $\nu': \mathbf{N}'_{\mathfrak{p}}(\mathbf{V}^\circ, K^{p^\circ}) \rightarrow \mathbf{M}'_{\mathfrak{p}}(\mathbf{V}^\circ, K^{p^\circ})$ is purely inseparable.*
- (3) *The morphism $\nu: \mathbf{N}_{\mathfrak{p}}(\mathbf{V}^\circ, K^{p^\circ}) \rightarrow \mathbf{N}'_{\mathfrak{p}}(\mathbf{V}^\circ, K^{p^\circ})$ is the blow-up along $\mathbf{N}'_{\mathfrak{p}}(\mathbf{V}^\circ, K^{p^\circ})_{\text{sp}}$.¹⁰*

¹⁰Note that blow-up along a zero-dimensional closed subscheme Z of a regular scheme depends only on the underlying closed subset of Z .

Proof. For (1), it suffices to show that for every algebraically closed field κ containing \mathbb{F}_p^Φ , $\mu(\kappa)$ is an isomorphism from $N_p(V^\circ, K^{p^\circ})(\kappa)$ to $M_p^\bullet(V^\circ, K^{p^\circ})(\kappa)$.

We first show that the image of $\mu(\kappa)$ is contained in $M_p^\bullet(V^\circ, K^{p^\circ})(\kappa)$. Take a point $y = (A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A', \lambda', \eta^{p'}; \delta) \in N_p(V^\circ, K^{p^\circ})(\kappa)$. By Lemma 3.4.13(2,4) and the relation $\lambda = \delta^\vee \circ \lambda' \circ \delta$, we know that $\delta_{*,\tau}: H_1^{\text{dR}}(A/\kappa)_\tau \rightarrow H_1^{\text{dR}}(A'/\kappa)_\tau$ is an isomorphism if $\tau \neq \tau_\infty^c$; and $\ker \delta_{*,\tau_\infty^c}$ has dimension 1. Moreover, since λ' is p -principal, we have $\ker \delta_{*,\tau_\infty^c} = H_1^{\text{dR}}(A/\kappa)_{\tau_\infty^c}^\perp$. By the signature condition again, we also have $\ker \delta_{*,\tau_\infty^c} \subseteq \omega_{A',\tau_\infty^c}$. Thus, $\mu(y)$ belongs to $M_p^\bullet(V^\circ, K^{p^\circ})(\kappa)$ by Definition 5.1.3.

It remains to construct an inverse to $\mu(\kappa): N_p(V^\circ, K^{p^\circ})(\kappa) \rightarrow M_p^\bullet(V^\circ, K^{p^\circ})(\kappa)$. Take a point $x = (A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p) \in M_p^\bullet(V^\circ, K^{p^\circ})(\kappa)$. Write $\tilde{H}_1^{\text{dR}}(A/\kappa)_{\tau_\infty}^\perp$ the preimage of $H_1^{\text{dR}}(A/\kappa)_{\tau_\infty}^\perp$ under the reduction map $\mathcal{D}(A)_{\tau_\infty^c} \rightarrow H_1^{\text{dR}}(A/\kappa)_{\tau_\infty^c}$. As $\langle H_1^{\text{dR}}(A/\kappa)_{\tau_\infty}^\perp, H_1^{\text{dR}}(A/\kappa)_{\tau_\infty} \rangle_{\lambda, \tau_\infty^c} = 0$, we have $\mathcal{D}(A)_{\tau_\infty^c}^\vee = p^{-1} \tilde{H}_1^{\text{dR}}(A/\kappa)_{\tau_\infty}^\perp$. Now we put $\mathcal{D}_{A',\tau} := \mathcal{D}(A)_\tau$ for $\tau \neq \tau_\infty^c$, and $\mathcal{D}_{A',\tau_\infty^c} := p^{-1} \tilde{H}_1^{\text{dR}}(A/\kappa)_{\tau_\infty}^\perp$. We claim that $\mathcal{D}_{A'} := \bigoplus_{\tau \in \Sigma_\infty} \mathcal{D}_{A',\tau}$ is a Dieudonné module, which amounts to the inclusions $F\mathcal{D}_{A',\tau_\infty^c} \subseteq \mathcal{D}_{A',\tau_\infty^c}$ and $V\mathcal{D}_{A',\tau_\infty^c} \subseteq \mathcal{D}_{A',\tau_\infty^c}$. The first one follows from the relation $F(H_1^{\text{dR}}(A/\kappa)_{\tau_\infty}^\perp) \subseteq F\omega_{A',\tau_\infty^c} = 0$ in which the first inclusion is due to Definition 5.1.3; and the second one is equivalent to the first one as $\mathcal{D}_{A',\tau_\infty^c}$ and $\mathcal{D}_{A',\tau_\infty}$ are integrally dual under $\langle \cdot, \cdot \rangle_{\lambda, \tau_\infty^c}^{\text{cris}}$. Then by the Dieudonné theory, there is an O_F -abelian scheme A' over κ with $\mathcal{D}(A')_\tau = \mathcal{D}_{A',\tau}$ for every $\tau \in \Sigma_\infty$, and an O_F -linear isogeny $\delta: A \rightarrow A'$ inducing the inclusion of Dieudonné modules $\mathcal{D}(A) \subseteq \mathcal{D}(A')$. By Lemma 3.4.13(2,4), the O_F -abelian scheme A' has signature type $3\Phi - 2\tau_\infty + 2\tau_\infty^c$. Let λ' be the unique quasi-polarization of A' satisfying $\lambda = \delta^\vee \circ \lambda' \circ \delta$, which is p -principal as $\mathcal{D}_{A',\tau_\infty} = \mathcal{D}_{A',\tau_\infty}^\vee$. Finally, we let $\eta^{p'}$ be the map sending $v \in V^\circ \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ to $\delta_* \circ \eta^p(v)$. Thus, we obtain an object $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A', \lambda', \eta^{p'}; \delta) \in N_p(V^\circ, K^{p^\circ})(\kappa)$. It is straightforward to check that such assignment gives rise to an inverse of $\mu(\kappa)$.

We now consider (2) and (3) simultaneously. Let $N_p(V^\circ, K^{p^\circ})_{\text{sp}}$ be the inverse image of $M_p^\bullet(V^\circ, K^{p^\circ})_{\text{sp}}$ under μ' . By Lemma 5.9.5(1) below, the induced morphism

$$\mu': N_p(V^\circ, K^{p^\circ}) \setminus N_p(V^\circ, K^{p^\circ})_{\text{sp}} \rightarrow M_p^\bullet(V^\circ, K^{p^\circ}) \setminus M_p^\bullet(V^\circ, K^{p^\circ})_{\text{sp}}$$

is purely inseparable. Thus, the induced morphism

$$\nu: N_p(V^\circ, K^{p^\circ}) \setminus N_p(V^\circ, K^{p^\circ})_{\text{sp}} \rightarrow N_p'(V^\circ, K^{p^\circ}) \setminus N_p'(V^\circ, K^{p^\circ})_{\text{sp}}$$

is an isomorphism, and the induced morphism

$$\nu': N_p'(V^\circ, K^{p^\circ}) \setminus N_p'(V^\circ, K^{p^\circ})_{\text{sp}} \rightarrow M_p'(V^\circ, K^{p^\circ}) \setminus M_p'(V^\circ, K^{p^\circ})_{\text{sp}}$$

is purely inseparable. Since $N_p(V^\circ, K^{p^\circ})$ is quasi-projective, ν is projective. Thus, ν is a projective birational morphism, which has to be the blow-up along a subset Z of $N_p'(V^\circ, K^{p^\circ})_{\text{sp}}$ (see, for example, [Liu02, Theorem 8.1.24]). Now we take a point x' of $M_p'(V^\circ, K^{p^\circ})_{\text{sp}}$ with the residue field κ , which is a finite extension of \mathbb{F}_p^Φ . Since ν' is a finite morphism, the inverse image of x' consists of finitely many points y'_1, \dots, y'_n of $N_p'(V^\circ, K^{p^\circ})_{\text{sp}}$ with residue fields $\kappa_1, \dots, \kappa_n$, respectively. By Lemma 5.9.5(2) below, the residue field extension κ_i/κ is trivial for every $1 \leq i \leq n$; and moreover, Z has nonempty intersection with $\{y'_1, \dots, y'_n\}$. Thus, $\mu'^{-1}(x')$ has cardinality at least $|\kappa| + n$. But we know that $\mu'^{-1}(x')$ has cardinality exactly $|\kappa| + 1$. Therefore, we must have $n = 1$. We immediately have both (2) and (3). \square

Lemma 5.9.5. *Consider an element $x' \in M_p'(V^\circ, K^{p^\circ})(\kappa)$ for some $K^{p^\circ} \in \mathfrak{K}(V^\circ)^p$ and a perfect field κ containing \mathbb{F}_p^Φ . We have*

- (1) *If the image of x' is not special, then $\mu'^{-1}(x')$ is a singleton.*
- (2) *If the image of x' is special, then $\mu'^{-1}(x')$ is isomorphic to $\mathbb{P}^1(\kappa)$.*

Proof. Write $x' = (A_0, \lambda_0, \eta_0^p; A', \lambda', \eta^{p'})$. By the Dieudonné theory and Lemma 3.4.13(2,4), we see that $\mu'^{-1}(x')$ is bijective to Dieudonné submodules $\mathcal{D}_A \subseteq \mathcal{D}(A')$ satisfying $\mathcal{D}_{A,\tau} = \mathcal{D}(A')_\tau$ for $\tau \neq \tau_\infty^c$, and that $\mathcal{D}(A')_{\tau_\infty^c}/\mathcal{D}_{A,\tau_\infty^c}$ is a vector space over κ of dimension 1. This amounts to the subspaces of $H_1^{\text{dR}}(A'/\kappa)_{\tau_\infty^c}$ of dimension 2 containing $\text{FH}_1^{\text{dR}}(A'/\kappa)_{\tau_\infty^c} + \text{VH}_1^{\text{dR}}(A'/\kappa)_{\tau_\infty^c}$. Since both $\text{FH}_1^{\text{dR}}(A'/\kappa)_{\tau_\infty^c}$ and $\text{VH}_1^{\text{dR}}(A'/\kappa)_{\tau_\infty^c}$ have dimension 1, the lemma follows by the definition of special points. \square

Remark 5.9.6. In fact, one can show that μ induces an isomorphism from $N_p(V^\circ, \mathbf{-})$ to $M_p^\bullet(V^\circ, \mathbf{-})$; and ν' is purely inseparable of degree p . But we do not need these facts.

5.10. Functoriality under special morphisms. In this subsection, we study the behavior of various moduli schemes under the special morphisms, which is closely related to the Rankin–Selberg motives for $\text{GL}_n \times \text{GL}_{n+1}$.

We start from the datum $(V_n^\circ, \{\Lambda_{n,q}^\circ\}_{q|p})$ as in the beginning of Subsection 5.1, but with V_n° of rank $n \geq 2$. (See Remark 5.10.15 below for the case $n = 1$.) We then have the induced datum

$$(V_{n+1}^\circ, \{\Lambda_{n+1,q}^\circ\}_{q|p}) := ((V_n^\circ)_\sharp, \{(\Lambda_{n,q}^\circ)_\sharp\}_{q|p})$$

of rank $n + 1$ by Definition 3.1.7. For $N \in \{n, n + 1\}$, we let $K_{N,q}^\circ$ be the stabilizer of $\Lambda_{N,q}^\circ$, and put $K_{N,p}^\circ := \prod_{q|p} K_{N,q}^\circ$. Recall the category $\mathfrak{K}(V_n^\circ)_{\text{sp}}^p$ and functors $\mathbf{-}_p, \mathbf{-}_\sharp$ from Definition 3.1.11. To unify notation, we put $\mathbf{-}_n := \mathbf{-}_p$ and $\mathbf{-}_{n+1} := \mathbf{-}_\sharp$. Similar to the case of smooth moduli schemes considered in Subsection 4.4, there are five stages of functoriality we will consider.

The first stage concerns Shimura varieties.

Notation 5.10.1. We choose an indefinite uniformization datum $(V'_n, j_n, \{\Lambda'_{n,q}\}_{q|p})$ for V_n° as in Definition 5.1.6. Put $V'_{n+1} := (V'_n)_\sharp$, $j_{n+1} := (j_n)_\sharp$, and $\Lambda'_{n+1,q} := (\Lambda'_{n,q})_\sharp$. Then $(V'_{n+1}, j_{n+1}, \{\Lambda'_{n+1,q}\}_{q|p})$ is an indefinite uniformization datum for V_{n+1}° . For $N \in \{n, n + 1\}$, we let $K'_{N,q}$ be the stabilizer of $\Lambda'_{N,q}$, and put $K'_{N,p} := \prod_{q|p} K'_{N,q}$.

We obtain a morphism

$$\text{sh}'_\dagger: \text{Sh}(V'_n, j_n \mathbf{-}_n K'_{n,p}) \rightarrow \text{Sh}(V'_{n+1}, j_{n+1} \mathbf{-}_{n+1} K'_{n+1,p})$$

in $\text{Fun}(\mathfrak{K}(V_n^\circ)_{\text{sp}}^p, \text{Sch}_F)$.

For the second stage of functoriality, we have a morphism

$$(5.21) \quad \mathbf{m}_\dagger: \mathbf{M}_p(V_n^\circ, \mathbf{-}_n) \rightarrow \mathbf{M}_p(V_{n+1}^\circ, \mathbf{-}_{n+1})$$

in $\text{Fun}(\mathfrak{K}(V_n^\circ)_{\text{sp}}^p \times \mathfrak{T}, \text{Sch}_{/\mathbb{Z}_p^\times})_{/\mathbb{T}_p}$ sending an object $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p) \in \mathbf{M}_p(V_n^\circ, K_n^{p^\circ})(S)$ to the object $(A_0, \lambda_0, \eta_0^p; A \times A_0, \lambda \times \lambda_0, \eta^p \oplus (\text{id}_{A_0})_*) \in \mathbf{M}_p(V_{n+1}^\circ, K_{n+1}^{p^\circ})(S)$. It is clear that \mathbf{m}_\dagger restricts to three morphisms

$$(5.22) \quad \begin{cases} \mathbf{m}_\dagger^\circ: \mathbf{M}_p^\circ(V_n^\circ, \mathbf{-}_n) \rightarrow \mathbf{M}_p^\circ(V_{n+1}^\circ, \mathbf{-}_{n+1}), \\ \mathbf{m}_\dagger^\dagger: \mathbf{M}_p^\dagger(V_n^\circ, \mathbf{-}_n) \rightarrow \mathbf{M}_p^\dagger(V_{n+1}^\circ, \mathbf{-}_{n+1}), \\ \mathbf{m}_\dagger^\bullet: \mathbf{M}_p^\bullet(V_n^\circ, \mathbf{-}_n) \rightarrow \mathbf{M}_p^\bullet(V_{n+1}^\circ, \mathbf{-}_{n+1}). \end{cases}$$

Moreover, we have the following commutative diagram

$$(5.23) \quad \begin{array}{ccc} \mathbf{M}_p^\eta(V_{n+1}^\circ, \mathbf{-}_{n+1}) & \xrightarrow{(5.2)} & \text{Sh}(V'_{n+1}, j_{n+1} \mathbf{-}_{n+1} K'_{n+1,p}) \times_{\text{Spec } F} \mathbf{T}_p^\eta \\ \mathbf{m}_\dagger^\eta \uparrow & & \uparrow \text{sh}'_\dagger \times \text{id} \\ \mathbf{M}_p^\eta(V_n^\circ, \mathbf{-}_n) & \xrightarrow{(5.2)} & \text{Sh}(V'_n, j_n \mathbf{-}_n K'_{n,p}) \times_{\text{Spec } F} \mathbf{T}_p^\eta \end{array}$$

in $\text{Fun}(\mathfrak{K}(V_n^\circ)_{\text{sp}}^p \times \mathfrak{T}, \text{Sch}_{/\mathbb{Q}_p^\times})_{/\mathbb{T}_p^\eta}$.

First, we consider the basic correspondences on the balloon strata, that is, the back layer of the diagram (5.24).

We define $s_{\uparrow}^{\circ}: S_{\mathfrak{p}}^{\circ}(V_n^{\circ}, \mathbf{-}_n) \rightarrow S_{\mathfrak{p}}^{\circ}(V_{n+1}^{\circ}, \mathbf{-}_{n+1})$ to be the morphism sending an object

$$(A_0, \lambda_0, \eta_0^p; A^{\circ}, \lambda^{\circ}, \eta^{p^{\circ}}) \in S_{\mathfrak{p}}^{\circ}(V_n^{\circ}, K_n^{p^{\circ}})(S)$$

to the object

$$(A_0, \lambda_0, \eta_0^p; A^{\circ} \times A_0, \lambda^{\circ} \times \lambda_0, \eta^{p^{\circ}} \oplus (\text{id}_{A_0})_*) \in S_{\mathfrak{p}}^{\circ}(V_{n+1}^{\circ}, K_{n+1}^{p^{\circ}})(S).$$

Remark 5.10.2. The canonical inclusions

$$V_n^{\circ} \hookrightarrow V_{n+1}^{\circ}, \quad \{\Lambda_{n,q}^{\circ} \hookrightarrow \Lambda_{n+1,q}^{\circ}\}_{q|p}$$

induce a morphism

$$\text{sh}_{\uparrow}^{\circ}: \text{Sh}(V_n^{\circ}, \mathbf{-}_n K_{n,p}^{\circ}) \rightarrow \text{Sh}(V_{n+1}^{\circ}, \mathbf{-}_{n+1} K_{n+1,p}^{\circ})$$

in $\text{Fun}(\mathfrak{K}(V_n^{\circ})_{\text{sp}}^p, \text{Set})$. It is clear that the following diagram

$$\begin{array}{ccc} S_{\mathfrak{p}}^{\circ}(V_{n+1}^{\circ}, \mathbf{-}_{n+1})(\overline{\mathbb{F}}_p) & \xrightarrow{v_{n+1}^{\circ}} & \text{Sh}(V_{n+1}^{\circ}, \mathbf{-}_{n+1} K_{n+1,p}^{\circ}) \times T_{\mathfrak{p}}(\overline{\mathbb{F}}_p) \\ \uparrow s_{\uparrow}^{\circ}(\overline{\mathbb{F}}_p) & & \uparrow \text{sh}_{\uparrow}^{\circ} \times \text{id}_{T_{\mathfrak{p}}(\overline{\mathbb{F}}_p)} \\ S_{\mathfrak{p}}^{\circ}(V_n^{\circ}, \mathbf{-}_n)(\overline{\mathbb{F}}_p) & \xrightarrow{v_n^{\circ}} & \text{Sh}(V_n^{\circ}, \mathbf{-}_n K_{n,p}^{\circ}) \times T_{\mathfrak{p}}(\overline{\mathbb{F}}_p) \end{array}$$

in $\text{Fun}(\mathfrak{K}(V_n^{\circ})_{\text{sp}}^p, \text{Set})_{/T_{\mathfrak{p}}(\overline{\mathbb{F}}_p)}$ commutes, where v_{n+1}° and v_n° are uniformization maps in Construction 5.2.6.

We define $b_{\uparrow}^{\circ}: B_{\mathfrak{p}}^{\circ}(V_n^{\circ}, \mathbf{-}_n) \rightarrow B_{\mathfrak{p}}^{\circ}(V_{n+1}^{\circ}, \mathbf{-}_{n+1})$ to be the morphism sending an object

$$(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^{\circ}, \lambda^{\circ}, \eta^{p^{\circ}}; \beta) \in B_{\mathfrak{p}}^{\circ}(V_n^{\circ}, K_n^{p^{\circ}})(S)$$

to the object

$$(A_0, \lambda_0, \eta_0^p; A \times A_0, \lambda \times \lambda_0, \eta^p \oplus (\text{id}_{A_0})_*; A^{\circ} \times A_0, \lambda^{\circ} \times \lambda_0, \eta^{p^{\circ}} \oplus (\text{id}_{A_0})_*; \beta \times \text{id}_{A_0}) \in B_{\mathfrak{p}}^{\circ}(V_{n+1}^{\circ}, K_{n+1}^{p^{\circ}})(S).$$

Second, we consider the basic correspondences on the ground strata, that is, the front layer of the diagram (5.24).

Definition 5.10.3. We define a functor

$$\begin{aligned} S_{\mathfrak{p}}^{\bullet}(V_n^{\circ}, \mathbf{-})_{\text{sp}} &: \mathfrak{K}(V_n^{\circ})_{\text{sp}}^p \times \mathfrak{T} \rightarrow \text{PSch}'_{/\mathbb{F}_p^{\Phi}} \\ K^{p^{\circ}} &\mapsto S_{\mathfrak{p}}^{\bullet}(V_n^{\circ}, K^{p^{\circ}})_{\text{sp}} \end{aligned}$$

such that for every $S \in \text{Sch}'_{/\mathbb{F}_p^{\Phi}}$, $S_{\mathfrak{p}}^{\bullet}(V_n^{\circ}, K^{p^{\circ}})_{\text{sp}}(S)$ is the set of equivalence classes of decuples $(A_0, \lambda_0, \eta_0^p; A^{\bullet}, \lambda^{\bullet}, \eta^{p^{\bullet}}; A_{\mathfrak{q}}^{\bullet}, \lambda_{\mathfrak{q}}^{\bullet}, \eta_{\mathfrak{q}}^{p^{\bullet}}; \delta^{\bullet})$ where

- $(A_0, \lambda_0, \eta_0^p; A^{\bullet}, \lambda^{\bullet}, \eta^{p^{\bullet}})$ is an element in $S_{\mathfrak{p}}^{\bullet}(V_n^{\circ}, K_n^{p^{\circ}})(S)$;
- $(A_0, \lambda_0, \eta_0^p; A_{\mathfrak{q}}^{\bullet}, \lambda_{\mathfrak{q}}^{\bullet}, \eta_{\mathfrak{q}}^{p^{\bullet}})$ is an element in $S_{\mathfrak{p}}^{\bullet}(V_{n+1}^{\circ}, K_{n+1}^{p^{\circ}})(S)$;
- $\delta^{\bullet}: A^{\bullet} \times A_0 \rightarrow A_{\mathfrak{q}}^{\bullet}$ is an O_F -linear quasi- p -isogeny (Definition 3.4.5) such that
 - (a) $\ker \delta^{\bullet}[p^{\infty}]$ is contained in $(A^{\bullet} \times A_0)[\mathfrak{p}]$;
 - (b) we have $\lambda^{\bullet} \times \varpi \lambda_0 = \delta^{\bullet \vee} \circ \lambda_{\mathfrak{q}}^{\bullet} \circ \delta^{\bullet}$; and
 - (c) the K_{n+1}^p -orbit of maps $v \mapsto \delta_{*}^{\bullet} \circ (\eta^{p^{\bullet}} \oplus (\text{id}_{A_0})_*)(v)$ for $v \in V_{\mathfrak{q}}^{\circ} \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$ coincides with $\eta_{\mathfrak{q}}^{p^{\bullet}}$.

The equivalence relation and the action of morphisms in $\mathfrak{K}(V_n^{\circ})_{\text{sp}}^p \times \mathfrak{T}$ are defined similarly as in Definition 4.2.3.

We have apparently the forgetful morphism

$$S_p^\bullet(V_n^\circ, -)_{\text{sp}} \rightarrow T_p$$

in $\text{Fun}(\mathcal{K}(V_n^\circ)_{\text{sp}}^p \times \mathfrak{T}, \text{PSch}'_{/\mathbb{F}_p})$ which is represented by finite and étale schemes. By definition, we have the two forgetful morphisms

$$s_\downarrow^\bullet: S_p^\bullet(V_n^\circ, -)_{\text{sp}} \rightarrow S_p^\bullet(V_n^\circ, -_n), \quad s_\uparrow^\bullet: S_p^\bullet(V_n^\circ, -)_{\text{sp}} \rightarrow S_p^\bullet(V_{n+1}^\circ, -_{n+1})$$

in $\text{Fun}(\mathcal{K}(V_n^\circ)_{\text{sp}}^p \times \mathfrak{T}, \text{Sch}_{/\mathbb{F}_p})/T_p$.

Lemma 5.10.4. *We have the following properties concerning s_\downarrow^\bullet .*

(1) *When n is even, s_\downarrow^\bullet is an isomorphism, and the morphism*

$$s_\uparrow^\bullet \circ s_\downarrow^{\bullet-1}: S_p^\bullet(V_n^\circ, -_n) \rightarrow S_p^\bullet(V_{n+1}^\circ, -_{n+1})$$

is given by the assignment

$$(A_0, \lambda_0, \eta_0^p; A^\bullet, \lambda^\bullet, \eta^{p^\bullet}) \mapsto (A_0, \lambda_0, \eta_0^p; A^\bullet \times A_0, \lambda^\bullet \times \varpi \lambda_0, \eta^{p^\bullet} \times (\text{id}_{A_0})_*).$$

(2) *When n is odd, s_\downarrow^\bullet is finite étale of degree $p + 1$.*

Proof. The proof is very similar to Lemma 4.4.2, which we leave to readers. \square

Definition 5.10.5. We define $B_p^\bullet(V_n^\circ, -)_{\text{sp}}$ to be the fiber product indicated in the following Cartesian diagram

$$\begin{array}{ccc} B_p^\bullet(V_n^\circ, -)_{\text{sp}} & \xrightarrow{\pi_{\text{sp}}^\bullet} & S_p^\bullet(V_n^\circ, -)_{\text{sp}} \\ b_\downarrow^\bullet \downarrow & & \downarrow s_\downarrow^\bullet \\ B_p^\bullet(V_n^\circ, -_n) & \xrightarrow{\pi_n^\bullet} & S_p^\bullet(V_n^\circ, -_n) \end{array}$$

in $\text{Fun}(\mathcal{K}(V_n^\circ)_{\text{sp}}^p \times \mathfrak{T}, \text{Sch}_{/\mathbb{F}_p})/T_p$. We define $b_\uparrow^\bullet: B_p^\bullet(V_n^\circ, -)_{\text{sp}} \rightarrow B_p^\bullet(V_{n+1}^\circ, -_{n+1})$ to be the morphism sending an object

$$((A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^\bullet, \lambda^\bullet, \eta^{p^\bullet}; \gamma), (A_0, \lambda_0, \eta_0^p; A^\bullet, \lambda^\bullet, \eta^{p^\bullet}; A_\natural^\bullet, \lambda_\natural^\bullet, \eta_\natural^{p^\bullet}; \delta^\bullet)) \in B_p^\bullet(V_n^\circ, K^{p^\circ})_{\text{sp}}(S)$$

to $(A_0, \lambda_0, \eta_0^p; A \times A_0, \lambda \times \lambda_0, \eta^p \oplus (\text{id}_{A_0})_*; A_\natural^\bullet, \lambda_\natural^\bullet, \eta_\natural^{p^\bullet}; \delta^\bullet \circ (\gamma \times \text{id}_{A_0}))$, which is an object of $B_p^\bullet(V_{n+1}^\circ, K_{n+1}^{p^\circ})(S)$ by a similar argument of Lemma 4.4.4.

We have the following result.

Proposition 5.10.6. *When n is odd, the square*

$$\begin{array}{ccc} B_p^\bullet(V_{n+1}^\circ, -_{n+1}) & \xrightarrow{\iota_{n+1}^\bullet} & M_p^\bullet(V_{n+1}^\circ, -_{n+1}) \\ b_\uparrow^\bullet \uparrow & & \uparrow m_\uparrow^\bullet \\ B_p^\bullet(V_n^\circ, -)_{\text{sp}} & \xrightarrow{\iota_n^\bullet \circ b_\downarrow^\bullet} & M_p^\bullet(V_n^\circ, -_n) \end{array}$$

extracted from the diagram (5.24) is Cartesian.

Proof. The proof is very similar to Proposition 4.4.5, which we leave to readers. \square

Third, we consider the basic correspondences on the link strata, that is, the middle (vertical) layer of the diagram (5.24).

Definition 5.10.7. We define $S_p^\dagger(V_n^\circ, -)_{\text{sp}}$ to be the fiber product indicated in the following Cartesian diagram

$$\begin{array}{ccc} S_p^\dagger(V_n^\circ, -)_{\text{sp}} & \xrightarrow{s_{\text{sp}}^\dagger} & S_p^\bullet(V_n^\circ, -)_{\text{sp}} \\ s_\downarrow^\dagger \downarrow & & \downarrow s_\downarrow^\bullet \\ S_p^\dagger(V_n^\circ, -_n) & \xrightarrow{s_n^\dagger} & S_p^\bullet(V_n^\circ, -_n) \end{array}$$

in $\text{Fun}(\mathfrak{K}(V_n^\circ)_{\text{sp}}^p \times \mathfrak{T}, \text{Sch}/\mathbb{F}_p^\Phi)/T_p$. By Lemma 5.10.4, we know that s_\downarrow^\dagger is an isomorphism (resp. finite étale of degree $p+1$) when n is even (resp. odd). We define $s_\uparrow^\dagger: S_p^\dagger(V_n^\circ, -)_{\text{sp}} \rightarrow S_p^\dagger(V_{n+1}^\circ, -_{n+1})$ to be the morphism sending an object

$$((A_0, \lambda_0, \eta_0^p; A^\circ, \lambda^\circ, \eta^{p^\circ}; A^\bullet, \lambda^\bullet, \eta^{p^\bullet}; \psi), (A_0, \lambda_0, \eta_0^p; A^\bullet, \lambda^\bullet, \eta^{p^\bullet}; A_\natural^\bullet, \lambda_\natural^\bullet, \eta_\natural^{p^\bullet}; \delta^\bullet)) \in S_p^\dagger(V_n^\circ, K^{p^\circ})_{\text{sp}}(S)$$

to the object

$$(A_0, \lambda_0, \eta_0^p; A^\circ \times A_0, \lambda^\circ \times \lambda_0, \eta^{p^\circ} \oplus (\text{id}_{A_0})_*; A_\natural^\bullet, \lambda_\natural^\bullet, \eta_\natural^{p^\bullet}; \delta^\bullet \circ (\psi \times \text{id}_{A_0})) \in S_p^\dagger(V_{n+1}^\circ, K_{n+1}^{p^\circ})(S).$$

Lemma 5.10.8. *We have*

(1) *When n is even, the square*

$$\begin{array}{ccc} S_p^\dagger(V_{n+1}^\circ, -_{n+1}) & \xrightarrow{s_{n+1}^\dagger} & S_p^\bullet(V_{n+1}^\circ, -_{n+1}) \\ \uparrow s_\uparrow^\dagger & & \uparrow s_\uparrow^\bullet \\ S_p^\dagger(V_n^\circ, -)_{\text{sp}} & \xrightarrow{s_{\text{sp}}^\dagger} & S_p^\bullet(V_n^\circ, -)_{\text{sp}} \end{array}$$

extracted from (5.24) is a Cartesian diagram.

(2) *When n is odd, the square*

$$\begin{array}{ccc} S_p^\circ(V_{n+1}^\circ, -_{n+1}) & \xleftarrow{s_{n+1}^\circ} & S_p^\dagger(V_{n+1}^\circ, -_{n+1}) \\ \uparrow s_\uparrow^\circ & & \uparrow s_\uparrow^\dagger \\ S_p^\circ(V_n^\circ, -_n) & \xleftarrow{s_n^\circ \circ s_\downarrow^\dagger} & S_p^\dagger(V_n^\circ, -)_{\text{sp}} \end{array}$$

extracted from (5.24) is a Cartesian diagram.

Proof. Let $S_p^\dagger(V_n^\circ, -)_{\text{sp}}$ be the actual fiber product in both cases. Take an object $K^{p^\circ} \in \mathfrak{K}(V_n^\circ)_{\text{sp}}^p$. We have to show that the natural morphism $s^\dagger: S_p^\dagger(V_n^\circ, K^{p^\circ})_{\text{sp}} \rightarrow S_p^\dagger(V_n^\circ, K^{p^\circ})_{\text{sp}}$ is an isomorphism. Since s^\dagger is a morphism of étale schemes over \mathbb{F}_p^Φ , it suffices to show that $s^\dagger(\kappa)$ is an isomorphism for every perfect field κ containing \mathbb{F}_p^Φ .

For (1), by Lemma 5.10.4(1), an object in $S_p^\dagger(V_n^\circ, K^{p^\circ})_{\text{sp}}(S)$ is given by a pair of objects:

$$\begin{aligned} (A_0, \lambda_0, \eta_0^p; A^\bullet, \lambda^\bullet, \eta^{p^\bullet}; A^\bullet \times A_0, \lambda^\bullet \times \varpi \lambda_0, \eta^{p^\bullet} \times (\text{id}_{A_0})_*) &\in S_p^\bullet(V_n^\circ, K^{p^\circ})_{\text{sp}}(\kappa), \\ (A_0, \lambda_0, \eta_0^p; A_\natural^\circ, \lambda_\natural^\circ, \eta_\natural^{p^\circ}; A^\bullet \times A_0, \lambda^\bullet \times \varpi \lambda_0, \eta^{p^\bullet} \times (\text{id}_{A_0})_*; \psi_\natural) &\in S_p^\dagger(V_{n+1}^\circ, K_{n+1}^{p^\circ})(\kappa). \end{aligned}$$

Let A° be the cokernel of the kernel of the composite map $A_\natural^\circ \xrightarrow{\psi_\natural} A^\bullet \times A_0 \rightarrow A^\bullet$, and $\psi: A^\circ \rightarrow A^\bullet$ the induced map. Let λ° be the unique quasi-polarization of A° satisfying $\varpi \cdot \lambda^\circ = \psi^\vee \circ \lambda^\bullet \circ \psi$. Since λ_\natural° is p -principal and we have $\varpi \cdot \lambda_\natural^\circ = \psi_\natural^\vee \circ (\lambda^\bullet \times \varpi \cdot \lambda_0) \circ \psi_\natural$, the composite map $A_\natural^\circ \xrightarrow{\psi_\natural} A^\bullet \times A_0 \rightarrow A_0$

splits. Thus, the natural map $A_{\natural}^{\circ} \rightarrow A^{\circ} \times A_0$ is an isomorphism. Then λ° is p -principal, and we obtain an object

$$(A_0, \lambda_0, \eta_0^p; A^{\circ}, \lambda^{\circ}, \eta^{p^{\circ}}; A^{\bullet}, \lambda^{\bullet}, \eta^{p^{\bullet}}; \psi) \in S_{\mathfrak{p}}^{\dagger}(V_n^{\circ}, K_n^{p^{\circ}})(\kappa) = S_{\mathfrak{p}}^{\dagger}(V_n^{\circ}, K^{p^{\circ}})_{\text{sp}}(\kappa),$$

where $\eta^{p^{\circ}}$ is chosen so that Definition 5.4.1(c) is satisfied. In other words, we obtain a morphism from $S_{\mathfrak{p}}^{\dagger}(V_n^{\circ}, K^{p^{\circ}})_{\text{sp}}(\kappa)$ to $S_{\mathfrak{p}}^{\dagger}(V_n^{\circ}, K^{p^{\circ}})_{\text{sp}}(\kappa)$. It is straightforward to check that it is an inverse to the morphism $s^{\dagger}(\kappa)$.

For (2), an object in $S_{\mathfrak{p}}^{\dagger}(V_n^{\circ}, K^{p^{\circ}})_{\text{sp}}(\kappa)$ is given by a pair of objects:

$$(A_0, \lambda_0, \eta_0^p; A^{\circ}, \lambda^{\circ}, \eta^{p^{\circ}}) \in S_{\mathfrak{p}}^{\circ}(V_n^{\circ}, K_n^{p^{\circ}})(\kappa), \\ (A_0, \lambda_0, \eta_0^p; A^{\circ} \times A_0, \lambda^{\circ} \times \lambda_0, \eta^{p^{\circ}} \times (\text{id}_{A_0})_*; A_{\natural}^{\bullet}, \lambda_{\natural}^{\bullet}, \eta_{\natural}^{p^{\bullet}}; \psi_{\natural}) \in S_{\mathfrak{p}}^{\dagger}(V_{n+1}^{\circ}, K_{n+1}^{p^{\circ}})(\kappa).$$

Let $A^{\bullet\vee}$ be the cokernel of the kernel of the composite map $A_{\natural}^{\bullet\vee} \xrightarrow{\psi_{\natural}^{\vee}} A^{\circ\vee} \times A_0^{\vee} \rightarrow A^{\circ\vee}$, and $\psi^{\vee}: A^{\circ\vee} \rightarrow A^{\bullet\vee}$ the induced map. Taking dual, we obtain a map $\psi: A^{\circ} \rightarrow A^{\bullet}$ and an induced map $\delta^{\bullet}: A^{\bullet} \times A_0 \rightarrow A_{\natural}^{\bullet}$. Let λ^{\bullet} be the unique quasi-polarization of A^{\bullet} satisfying $\varpi \cdot \lambda^{\circ} = \psi^{\vee} \circ \lambda^{\bullet\vee} \circ \psi$. Since $\lambda_{\natural}^{\bullet}$ is p -principal and we have $\lambda^{\bullet} \times \varpi \cdot \lambda_0 = \delta^{\bullet\vee} \circ \lambda_{\natural}^{\bullet\vee} \circ \delta^{\bullet}$, we know that $\ker \lambda^{\bullet}[p^{\infty}]$ is contained in $A^{\bullet}[p]$ of rank p^2 , and we obtain an object

$$\left((A_0, \lambda_0, \eta_0^p; A^{\circ}, \lambda^{\circ}, \eta^{p^{\circ}}; A^{\bullet}, \lambda^{\bullet}, \eta^{p^{\bullet}}; \psi), (A_0, \lambda_0, \eta_0^p; A^{\circ}, \lambda^{\circ}, \eta^{p^{\circ}}; A_{\natural}^{\bullet}, \lambda_{\natural}^{\bullet}, \eta_{\natural}^{p^{\bullet}}; \delta^{\bullet}) \right) \in S_{\mathfrak{p}}^{\dagger}(V_n^{\circ}, K^{p^{\circ}})_{\text{sp}}(\kappa),$$

where $\eta^{p^{\bullet}}$ is chosen so that Definition 5.4.1(c) is satisfied. In other words, we obtain a morphism from $S_{\mathfrak{p}}^{\dagger}(V_n^{\circ}, K^{p^{\circ}})_{\text{sp}}(\kappa)$ to $S_{\mathfrak{p}}^{\dagger}(V_n^{\circ}, K^{p^{\circ}})_{\text{sp}}(\kappa)$. It is straightforward to check that it is an inverse to the morphism $s^{\dagger}(\kappa)$. \square

Definition 5.10.9. We define $B_{\mathfrak{p}}^{\dagger}(V_n^{\circ}, -)_{\text{sp}}$ to be the fiber product indicated in the following Cartesian diagram

$$\begin{array}{ccc} B_{\mathfrak{p}}^{\dagger}(V_n^{\circ}, -)_{\text{sp}} & \xrightarrow{\pi_{\text{sp}}^{\dagger}} & S_{\mathfrak{p}}^{\dagger}(V_n^{\circ}, -)_{\text{sp}} \\ b_{\downarrow}^{\dagger} \downarrow & & \downarrow s_{\downarrow}^{\dagger} \\ B_{\mathfrak{p}}^{\dagger}(V_n^{\circ}, -_n) & \xrightarrow{\pi_n^{\dagger}} & S_{\mathfrak{p}}^{\dagger}(V_n^{\circ}, -_n) \end{array}$$

in $\text{Fun}(\mathfrak{K}(V_n^{\circ})_{\text{sp}}^p \times \mathfrak{T}, \text{Sch}_{/\mathbb{F}_p})_{/T_p}$.

By the universal property of Cartesian diagrams, we obtain a unique morphism

$$b_{\text{sp}}^{\dagger\bullet}: B_{\mathfrak{p}}^{\dagger}(V_n^{\circ}, -)_{\text{sp}} \rightarrow B_{\mathfrak{p}}^{\bullet}(V_n^{\circ}, -)_{\text{sp}}$$

rendering the front lower-left cube of (5.24) commute. Finally, an easy diagram chasing indicates that we have a unique morphism

$$b_{\uparrow}^{\dagger}: B_{\mathfrak{p}}^{\dagger}(V_n^{\circ}, -)_{\text{sp}} \rightarrow B_{\mathfrak{p}}^{\dagger}(V_{n+1}^{\circ}, -_{n+1})$$

rendering the entire diagram (5.24) commute. Thus, we obtain our desired diagram (5.24).

Remark 5.10.10. By Proposition 5.10.6 and Theorem 5.4.3(1), one can show that when n is odd, the square

$$\begin{array}{ccc} B_{\mathfrak{p}}^{\dagger}(V_{n+1}^{\circ}, -_{n+1}) & \xrightarrow{\iota_{n+1}^{\dagger}} & M_{\mathfrak{p}}^{\dagger}(V_{n+1}^{\circ}, -_{n+1}) \\ b_{\uparrow}^{\dagger} \uparrow & & \uparrow m_{\uparrow}^{\dagger} \\ B_{\mathfrak{p}}^{\dagger}(V_n^{\circ}, -)_{\text{sp}} & \xrightarrow{\iota_n^{\dagger} \circ b_{\downarrow}^{\dagger}} & M_{\mathfrak{p}}^{\dagger}(V_n^{\circ}, -_n) \end{array}$$

extracted from the diagram (5.24) is Cartesian.

In particular, $b_{\uparrow}^{\bullet}: B_{s^{\bullet}}^{\bullet} \rightarrow B_{s_{n+1}^{\bullet}}^{\bullet}$ is an isomorphism when n is even.

Proof. The proof is very similar to Proposition 4.4.6, which we leave to readers. The last assertion follows as $b_{\downarrow}^{\bullet}: B_{s^{\bullet}}^{\bullet} \rightarrow B_{s_n^{\bullet}}^{\bullet}$ is always an isomorphism, and $\delta_{s^{\bullet}, \uparrow}$ is an isomorphism when n is even. \square

At the final stage of functoriality, we relate the special morphisms for sources of basic correspondences to Shimura sets under the uniformization maps v° (5.4), v^{\bullet} (5.9), and v^{\dagger} (5.11). Recall that we have data $(V_n^{\circ}, \{\Lambda_{n,q}^{\circ}\}_{q|p})$ and $(V_{n+1}^{\circ}, \{\Lambda_{n+1,q}^{\circ}\}_{q|p})$.

Notation 5.10.13. As in Construction 5.3.6, we choose a lattice chain $\Lambda_{n,p}^{\circ} \subseteq \Lambda_{n,p}^{\bullet} \subseteq p^{-1}\Lambda_{n,p}^{\circ}$ of $V_n^{\circ} \otimes_F F_p$, and a lattice chain $\Lambda_{n+1,p}^{\circ} \subseteq \Lambda_{n+1,p}^{\bullet} \subseteq p^{-1}\Lambda_{n+1,p}^{\circ}$ of $V_{n+1}^{\circ} \otimes_F F_p$, for which we assume that $(\Lambda_{n,p}^{\bullet})_{\#} \subseteq \Lambda_{n+1,p}^{\bullet} \subseteq p^{-1}(\Lambda_{n,p}^{\bullet})_{\#}^{\vee}$ holds. We now introduce various open compact subgroups at p .

- For $N \in \{n, n+1\}$, we have $K_{N,p}^{\circ}$ from Construction 5.2.6, $K_{N,p}^{\bullet}$ from Construction 5.3.6, and $K_{N,p}^{\dagger} = K_{N,p}^{\circ} \cap K_{N,p}^{\bullet}$ from Construction 5.4.5.
- Put $K_{\text{sp},p}^{\bullet} := K_{n,p}^{\bullet} \cap K_{n+1,p}^{\bullet}$ (as a subgroup of $K_{n,p}^{\bullet}$) and $K_{\text{sp},p}^{\circ} := K_{\text{sp},p}^{\bullet} \times \prod_{q|p, q \neq p} K_{n,q}^{\circ}$.
- Put $K_{\text{sp},p}^{\dagger} := K_{\text{sp},p}^{\circ} \cap K_{n,p}^{\circ}$.

For later use, we also introduce natural maps

$$\left\{ \begin{array}{l} \text{sh}_{\uparrow}^{\circ}: \text{Sh}(V_n^{\circ}, -_n K_{n,p}^{\circ}) \rightarrow \text{Sh}(V_{n+1}^{\circ}, -_{n+1} K_{n+1,p}^{\circ}), \\ \text{sh}_{\uparrow}^{\bullet}: \text{Sh}(V_n^{\circ}, -_n K_{n,p}^{\bullet}) \rightarrow \text{Sh}(V_n^{\circ}, -_n K_{\text{sp},p}^{\bullet}), \\ \text{sh}_{\downarrow}^{\bullet}: \text{Sh}(V_n^{\circ}, -_n K_{\text{sp},p}^{\bullet}) \rightarrow \text{Sh}(V_{n+1}^{\circ}, -_{n+1} K_{n+1,p}^{\bullet}), \\ \text{sh}_{\uparrow}^{\dagger}: \text{Sh}(V_n^{\circ}, -_n K_{n,p}^{\dagger}) \rightarrow \text{Sh}(V_n^{\circ}, -_n K_{\text{sp},p}^{\dagger}), \\ \text{sh}_{\downarrow}^{\dagger}: \text{Sh}(V_n^{\circ}, -_n K_{\text{sp},p}^{\dagger}) \rightarrow \text{Sh}(V_{n+1}^{\circ}, -_{n+1} K_{n+1,p}^{\dagger}), \\ \text{sh}_n^{\circ}: \text{Sh}(V_n^{\circ}, -_n K_{n,p}^{\circ}) \rightarrow \text{Sh}(V_n^{\circ}, -_n K_{n,p}^{\circ}), \\ \text{sh}_n^{\bullet}: \text{Sh}(V_n^{\circ}, -_n K_{n,p}^{\dagger}) \rightarrow \text{Sh}(V_n^{\circ}, -_n K_{n,p}^{\bullet}), \\ \text{sh}_{n+1}^{\circ}: \text{Sh}(V_{n+1}^{\circ}, -_{n+1} K_{n+1,p}^{\dagger}) \rightarrow \text{Sh}(V_{n+1}^{\circ}, -_{n+1} K_{n+1,p}^{\circ}), \\ \text{sh}_{n+1}^{\bullet}: \text{Sh}(V_{n+1}^{\circ}, -_{n+1} K_{n+1,p}^{\dagger}) \rightarrow \text{Sh}(V_{n+1}^{\circ}, -_{n+1} K_{n+1,p}^{\bullet}), \\ \text{sh}_{\text{sp}}^{\dagger}: \text{Sh}(V_n^{\circ}, -_n K_{\text{sp},p}^{\dagger}) \rightarrow \text{Sh}(V_n^{\circ}, -_n K_{\text{sp},p}^{\bullet}), \end{array} \right.$$

in $\text{Fun}(\mathfrak{K}(V^{\circ})_{\text{sp}}^p, \text{Set})$. Note that $\text{sh}_{\uparrow}^{\circ}$ has already appeared in Remark 5.10.2.

Similar to Construction 4.3.2, we may construct two uniformization maps

$$(5.25) \quad v_{\text{sp}}^{\bullet}: S_{\mathfrak{p}}^{\bullet}(V_n, -)_{\text{sp}}(\overline{\mathbb{F}}_p) \rightarrow \text{Sh}(V_n^{\circ}, -_n K_{\text{sp},p}^{\bullet}) \times T_{\mathfrak{p}}(\overline{\mathbb{F}}_p)$$

$$(5.26) \quad v_{\text{sp}}^{\dagger}: S_{\mathfrak{p}}^{\dagger}(V_n, -)_{\text{sp}}(\overline{\mathbb{F}}_p) \rightarrow \text{Sh}(V_n^{\circ}, -_n K_{\text{sp},p}^{\dagger}) \times T_{\mathfrak{p}}(\overline{\mathbb{F}}_p)$$

in $\text{Fun}(\mathfrak{K}(V_n^{\circ})_{\text{sp}}^p \times \mathfrak{T}, \text{Set})_{/T_{\mathfrak{p}}(\overline{\mathbb{F}}_p)}$, which are isomorphisms. We leave the details to readers.

Remark 5.10.15. When $n = 1$, we have the diagram (5.24) in which all terms not in the top or back layers are empty. Propositions 5.10.12 and 5.10.14 can be modified in the obvious way.

5.11. First geometric reciprocity law. In this subsection, we state and prove a theorem we call *first geometric reciprocity law*, which can be regarded a geometric template for the first explicit reciprocity law studied in Subsection 7.2 once throw the automorphic input.

We maintain the setup in Subsection 5.10. However, we allow $\mathbf{-} = (\mathbf{-}_n, \mathbf{-}_{n+1})$ to be an object of $\mathfrak{K}(V_n^\circ)^p \times \mathfrak{K}(V_{n+1}^\circ)^p$, rather than $\mathfrak{K}(V_n^\circ)_{\text{sp}}^p$. Denote by n_0 and n_1 the unique even and odd numbers in $\{n, n+1\}$, respectively. Write $n_0 = 2r_0$ and $n_1 = 2r_1 + 1$ for unique integers $r_0, r_1 \geq 1$. In particular, we have $n = r_0 + r_1$. Let L be a p -coprime coefficient ring.

To ease notation, we put $X_{n_\alpha}^? := X_p^?(V_{n_\alpha}^\circ, \mathbf{-}_{n_\alpha})$ for meaningful triples $(X, ?, \alpha) \in \{\mathbf{M}, \mathbf{M}, \mathbf{B}, \mathbf{S}\} \times \{, \eta, \circ, \bullet, \dagger\} \times \{0, 1\}$.

Notation 5.11.1. We introduce following objects.

- (1) Put $\mathbf{P} := \mathbf{M}_{n_0} \times_{\mathbf{T}_p} \mathbf{M}_{n_1}$.
- (2) For $(?, ?_1) \in \{\circ, \bullet, \dagger\}^2$, put $\mathbf{P}^{?, ?_1} := \mathbf{M}_{n_0}^{?_0} \times_{\mathbf{T}_p} \mathbf{M}_{n_1}^{?_1}$, which is a closed subscheme of \mathbf{P} .¹¹
- (3) Let $\sigma: \mathbf{Q} \rightarrow \mathbf{P}$ be the blow-up along the subscheme $\mathbf{P}^{\circ, \circ}$, which is a morphism in $\text{Fun}(\mathfrak{K}(V_n^\circ)^p \times \mathfrak{K}(V_{n+1}^\circ)^p \times \mathfrak{T}, \text{Sch}/\mathbb{Z}_p^\Phi)/\mathbf{T}_p$.
- (4) For $(?, ?_1) \in \{\circ, \bullet, \dagger\}^2$, let $\mathbf{Q}^{?, ?_1}$ be the strict transform of $\mathbf{P}^{?, ?_1}$ under σ , which is a closed subscheme of \mathbf{Q} .
- (5) Let $\gamma_{?, ?_1}^{?, ?_1}: \mathbf{P}^{?, ?_1} \rightarrow \mathbf{P}^{?', ?'_1}$ be the closed embedding if $\mathbf{P}^{?, ?_1}$ is contained in $\mathbf{P}^{?', ?'_1}$, and $\delta_{?, ?_1}^{?', ?'_1}: \mathbf{Q}^{?, ?_1} \rightarrow \mathbf{Q}^{?', ?'_1}$ the closed embedding if $\mathbf{Q}^{?, ?_1}$ is contained in $\mathbf{Q}^{?', ?'_1}$.

Suppose that $\mathbf{-}$ is taken in the subcategory $\mathfrak{K}(V_n^\circ)_{\text{sp}}^p$.

- (6) Let \mathbf{P}_Δ be the graph of $\mathbf{m}_\dagger: \mathbf{M}_n \rightarrow \mathbf{M}_{n+1}$ (5.21) over \mathbf{T}_p in $\text{Fun}(\mathfrak{K}(V_n^\circ)_{\text{sp}}^p \times \mathfrak{T}, \text{Sch}/\mathbb{Z}_p^\Phi)/\mathbf{T}_p$, as a closed subscheme of \mathbf{P} .
- (7) For $? = \bullet, \circ$, let $\mathbf{P}_\Delta^?$ be the graph of $\mathbf{m}_\dagger^?: \mathbf{M}_n^? \rightarrow \mathbf{M}_{n+1}^?$ (5.22) over \mathbf{T}_p in $\text{Fun}(\mathfrak{K}(V_n^\circ)_{\text{sp}}^p \times \mathfrak{T}, \text{Sch}/\mathbb{Z}_p^\Phi)/\mathbf{T}_p$, as a closed subscheme of $\mathbf{P}^{?, ?}$.
- (8) Let \mathbf{Q}_Δ be the strict transform of \mathbf{P}_Δ under σ , which is a closed subscheme of \mathbf{Q} .

Lemma 5.11.2. *The two specialization maps*

$$\begin{aligned} H_{\mathfrak{T}, c}^i(\mathbf{Q} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p, L) &\rightarrow H_{\mathfrak{T}, c}^i(\overline{\mathbb{Q}}, R\Psi L), \\ H_{\mathfrak{T}}^i(\mathbf{Q} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p, L) &\rightarrow H_{\mathfrak{T}}^i(\overline{\mathbb{Q}}, R\Psi L), \end{aligned}$$

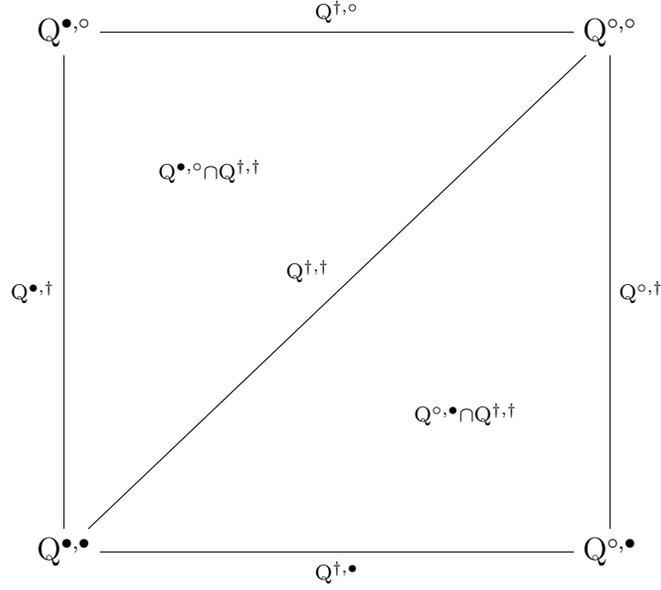
are both isomorphisms.

Proof. When \mathbf{Q} is proper, this is simply the proper base change. When \mathbf{Q} is not proper, this again follows from [LS18, Corollary 5.20]. \square

Lemma 5.11.3. *The scheme \mathbf{Q} (valued at any object of $\mathfrak{K}(V_n^\circ)_{\text{sp}}^p$) is strictly semistable over \mathbb{Z}_p^Φ of relative dimension $2n - 1$. Moreover, we have*

¹¹Recall from Notation 3.3.6(5) that \mathbf{P} is $\mathbf{P} \otimes_{\mathbb{Z}_p^\Phi} \mathbb{F}_p^\Phi$.

(1) The reduction graph of \mathbf{Q} is as follows



so that

$$\begin{cases} Q^{(0)} = Q^{\circ,\circ} \amalg Q^{\circ,\bullet} \amalg Q^{\bullet,\bullet} \amalg Q^{\bullet,\circ}, \\ Q^{(1)} = Q^{\circ,\dagger} \amalg Q^{\dagger,\bullet} \amalg Q^{\bullet,\dagger} \amalg Q^{\dagger,\circ} \amalg Q^{\dagger,\dagger}, \\ Q^{(2)} = (Q^{\bullet,\circ} \cap Q^{\dagger,\dagger}) \amalg (Q^{\circ,\bullet} \cap Q^{\dagger,\dagger}), \\ Q^{(c)} = \emptyset, \text{ for } c \geq 3. \end{cases}$$

Here, $Q^{(c)}$ denotes the union of the strata of \mathbf{Q} of codimension c .

(2) For the morphism σ , we have that

- the induced morphism $\sigma: Q^{?_0,?_1} \rightarrow P^{?_0,?_1}$ is an isomorphism if $?_0 \neq ?_1$;
- the induced morphism $\sigma: Q^{?_0,?_1} \rightarrow P^{?_0,?_1}$ is the blow-up along $P^{\dagger,\dagger}$ if $(?_0, ?_1) \in \{(\circ, \circ), (\bullet, \bullet)\}$;
- the induced morphism $\sigma: Q^{\dagger,\dagger} \rightarrow P^{\dagger,\dagger}$ is a trivial \mathbb{P}^1 -bundle;
- the induced morphisms $\sigma: Q^{\bullet,\circ} \cap Q^{\dagger,\dagger} \rightarrow P^{\dagger,\dagger}$ and $\sigma: Q^{\circ,\bullet} \cap Q^{\dagger,\dagger} \rightarrow P^{\dagger,\dagger}$ are both isomorphisms.

(3) The natural map

$$\sigma^*: H_{\mathfrak{X}}^i(\overline{P}^{?_0,?_1}, O_\lambda) \rightarrow H_{\mathfrak{X}}^i(\overline{Q}^{?_0,?_1}, O_\lambda)$$

is injective, and moreover an isomorphism if $?_0 \neq ?_1$.

(4) For $(?_0, ?_1) \in \{(\circ, \circ), (\bullet, \bullet)\}$, the map

$$(\delta_{?_0,?_1}^{\dagger,\dagger})! \circ \sigma^*: H_{\mathfrak{X}}^{i-2}(\overline{P}^{\dagger,\dagger}, O_\lambda(-1)) \rightarrow H_{\mathfrak{X}}^i(\overline{Q}^{?_0,?_1}, O_\lambda)$$

is injective; and we have

$$H_{\mathfrak{X}}^i(\overline{Q}^{?_0,?_1}, O_\lambda) = \sigma^* H_{\mathfrak{X}}^i(\overline{P}^{?_0,?_1}, O_\lambda) \bigoplus (\delta_{?_0,?_1}^{\dagger,\dagger})! \sigma^* H_{\mathfrak{X}}^{i-2}(\overline{P}^{\dagger,\dagger}, O_\lambda(-1)).$$

(5) If we denote by $\mathfrak{f} \in H_{\mathfrak{X}}^2(\overline{Q}^{\dagger,\dagger}, O_\lambda(1))$ the cycle class of an arbitrary \mathfrak{X} -orbit of fibers of the trivial \mathbb{P}^1 -fibration $\overline{\sigma}: \overline{Q}^{\dagger,\dagger} \rightarrow \overline{P}^{\dagger,\dagger}$, then the map

$$(\mathfrak{f} \cup) \circ \sigma^*: H_{\mathfrak{X}}^{i-2}(\overline{P}^{\dagger,\dagger}, O_\lambda(-1)) \rightarrow H_{\mathfrak{X}}^i(\overline{Q}^{\dagger,\dagger}, O_\lambda)$$

is injective; and we have

$$H_{\mathfrak{X}}^i(\overline{Q}^{\dagger,\dagger}, O_\lambda) = \sigma^* H_{\mathfrak{X}}^i(\overline{P}^{\dagger,\dagger}, O_\lambda) \bigoplus \mathfrak{f} \cup \sigma^* H_{\mathfrak{X}}^{i-2}(\overline{P}^{\dagger,\dagger}, O_\lambda(-1)).$$

$\mathfrak{K}(\mathbf{V}_{n+1}^\circ)^p, \text{Mod}(L)$) as follows:

$$\left\{ \begin{array}{l} \text{inc}_!^{\circ, \dagger} : L[\text{Sh}(\mathbf{V}_{n_0}^\circ, \mathbf{-}_{n_0} \mathbf{K}_{n_0,p}^\circ)] \otimes_L L[\text{Sh}(\mathbf{V}_{n_1}^\circ, \mathbf{-}_{n_1} \mathbf{K}_{n_1,p}^\circ)] \\ \quad \xrightarrow{\sim} \mathbf{H}_{\mathfrak{X}}^0(\overline{\mathbf{S}}_{n_0}^\circ, L) \otimes_L \mathbf{H}_{\mathfrak{X}}^0(\overline{\mathbf{S}}_{n_1}^\circ, L) = \mathbf{H}_{\mathfrak{X}}^0(\overline{\mathbf{S}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{S}}_{n_1}^\circ, L) \\ \quad \xrightarrow{(\pi_{n_0}^\circ \times \pi_{n_1}^\circ)^*} \mathbf{H}_{\mathfrak{X}}^0(\overline{\mathbf{B}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{B}}_{n_1}^\circ, L) \\ \quad \xrightarrow{\cup \xi_0^{r_0-1} \cup \xi_1^{r_1-1}} \mathbf{H}_{\mathfrak{X}}^{2(n-2)}(\overline{\mathbf{B}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{B}}_{n_1}^\circ, L(n-2)) \\ \quad \xrightarrow{(\iota_{n_0}^\circ \times \iota_{n_1}^\circ)!} \mathbf{H}_{\mathfrak{X}}^{2(n-2)}(\overline{\mathbf{M}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{M}}_{n_1}^\circ, L(n-2)) \\ \quad \xrightarrow{(\text{id} \times \mathbf{m}_{n_1}^{\dagger \circ})^*} \mathbf{H}_{\mathfrak{X}}^{2(n-2)}(\overline{\mathbf{M}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{M}}_{n_1}^\dagger, L(n-2)) \\ \quad \xrightarrow{(\text{id} \times \mathbf{m}_{n_1}^{\dagger \bullet})!} \mathbf{H}_{\mathfrak{X},c}^{2(n-1)}(\overline{\mathbf{M}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{M}}_{n_1}^\bullet, L(n-1)) = \mathbf{H}_{\mathfrak{X},c}^{2(n-1)}(\overline{\mathbf{P}}^{\circ, \bullet}, L(n-1)), \\ \text{inc}_{\circ, \dagger}^* : \mathbf{H}_{\mathfrak{X}}^{2n}(\overline{\mathbf{P}}^{\circ, \bullet}, L(n)) = \mathbf{H}_{\mathfrak{X}}^{2n}(\overline{\mathbf{M}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{M}}_{n_1}^\bullet, L(n)) \\ \quad \xrightarrow{(\text{id} \times \mathbf{m}_{n_1}^{\dagger \bullet})^*} \mathbf{H}_{\mathfrak{X}}^{2n}(\overline{\mathbf{M}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{M}}_{n_1}^\dagger, L(n)) \\ \quad \xrightarrow{(\text{id} \times \mathbf{m}_{n_1}^{\dagger \circ})!} \mathbf{H}_{\mathfrak{X}}^{2n+2}(\overline{\mathbf{M}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{M}}_{n_1}^\circ, L(n+1)) \\ \quad \xrightarrow{(\iota_{n_0}^\circ \times \iota_{n_1}^\circ)^*} \mathbf{H}_{\mathfrak{X}}^{2n+2}(\overline{\mathbf{B}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{B}}_{n_1}^\circ, L(n+1)) \\ \quad \xrightarrow{\cup \xi_0^{r_0-1} \cup \xi_1^{r_1-1}} \mathbf{H}_{\mathfrak{X}}^{4n-2}(\overline{\mathbf{B}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{B}}_{n_1}^\circ, L(2n-1)) \\ \quad \xrightarrow{(\pi_{n_0}^\circ \times \pi_{n_1}^\circ)!} \mathbf{H}_{\mathfrak{X}}^0(\overline{\mathbf{S}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{S}}_{n_1}^\circ, L) = \mathbf{H}_{\mathfrak{X}}^0(\overline{\mathbf{S}}_{n_0}^\circ, L) \otimes_L \mathbf{H}_{\mathfrak{X}}^0(\overline{\mathbf{S}}_{n_1}^\circ, L) \\ \quad \xrightarrow{\sim} L[\text{Sh}(\mathbf{V}_{n_0}^\circ, \mathbf{-}_{n_0} \mathbf{K}_{n_0,p}^\circ)] \otimes_L L[\text{Sh}(\mathbf{V}_{n_1}^\circ, \mathbf{-}_{n_1} \mathbf{K}_{n_1,p}^\circ)]; \end{array} \right.$$

$$\left\{ \begin{array}{l} \text{inc}_!^{\circ, \bullet} : L[\text{Sh}(\mathbf{V}_{n_0}^\circ, \mathbf{-}_{n_0} \mathbf{K}_{n_0,p}^\circ)] \otimes_L L[\text{Sh}(\mathbf{V}_{n_1}^\circ, \mathbf{-}_{n_1} \mathbf{K}_{n_1,p}^\bullet)] \\ \quad \xrightarrow{\sim} \mathbf{H}_{\mathfrak{X}}^0(\overline{\mathbf{S}}_{n_0}^\circ, L) \otimes_L \mathbf{H}_{\mathfrak{X}}^0(\overline{\mathbf{S}}_{n_1}^\bullet, L) = \mathbf{H}_{\mathfrak{X}}^0(\overline{\mathbf{S}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{S}}_{n_1}^\bullet, L) \\ \quad \xrightarrow{(\pi_{n_0}^\circ \times \pi_{n_1}^\bullet)^*} \mathbf{H}_{\mathfrak{X}}^0(\overline{\mathbf{B}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{B}}_{n_1}^\bullet, L) \\ \quad \xrightarrow{\cup \xi_0^{r_0-1}} \mathbf{H}_{\mathfrak{X}}^{2(r_0-1)}(\overline{\mathbf{B}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{B}}_{n_1}^\bullet, L(r_0-1)) \\ \quad \xrightarrow{(\iota_{n_0}^\circ \times \iota_{n_1}^\bullet)!} \mathbf{H}_{\mathfrak{X},c}^{2(n-1)}(\overline{\mathbf{M}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{M}}_{n_1}^\bullet, L(n-1)) = \mathbf{H}_{\mathfrak{X},c}^{2(n-1)}(\overline{\mathbf{P}}^{\circ, \bullet}, L(n-1)), \\ \text{inc}_{\circ, \bullet}^* : \mathbf{H}_{\mathfrak{X}}^{2n}(\overline{\mathbf{P}}^{\circ, \bullet}, L(n)) = \mathbf{H}_{\mathfrak{X}}^{2n}(\overline{\mathbf{M}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{M}}_{n_1}^\bullet, L(n)) \\ \quad \xrightarrow{(\iota_{n_0}^\circ \times \iota_{n_1}^\bullet)^*} \mathbf{H}_{\mathfrak{X}}^{2n}(\overline{\mathbf{B}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{B}}_{n_1}^\bullet, L(n)) \\ \quad \xrightarrow{\cup \xi_0^{r_0-1}} \mathbf{H}_{\mathfrak{X}}^{2(n_0-1+r_1)}(\overline{\mathbf{B}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{B}}_{n_1}^\bullet, L(n_0-1+r_1)) \\ \quad \xrightarrow{(\pi_{n_0}^\circ \times \pi_{n_1}^\bullet)!} \mathbf{H}_{\mathfrak{X}}^0(\overline{\mathbf{S}}_{n_0}^\circ \times_{\overline{\mathbf{T}}_p} \overline{\mathbf{S}}_{n_1}^\bullet, L) = \mathbf{H}_{\mathfrak{X}}^0(\overline{\mathbf{S}}_{n_0}^\circ, L) \otimes_L \mathbf{H}_{\mathfrak{X}}^0(\overline{\mathbf{S}}_{n_1}^\bullet, L) \\ \quad \xrightarrow{\sim} L[\text{Sh}(\mathbf{V}_{n_0}^\circ, \mathbf{-}_{n_0} \mathbf{K}_{n_0,p}^\circ)] \otimes_L L[\text{Sh}(\mathbf{V}_{n_1}^\circ, \mathbf{-}_{n_1} \mathbf{K}_{n_1,p}^\bullet)]; \end{array} \right.$$

$$\left\{ \begin{array}{l}
\text{inc}_!^{\bullet, \dagger}: L[\text{Sh}(V_{n_0}^\circ, -_{n_0} K_{n_0, p}^\bullet)] \otimes_L L[\text{Sh}(V_{n_1}^\circ, -_{n_1} K_{n_1, p}^\circ)] \\
\quad \xrightarrow{\sim} H_{\mathfrak{X}}^0(\overline{S}_{n_0}^\bullet, L) \otimes_L H_{\mathfrak{X}}^0(\overline{S}_{n_1}^\circ, L) = H_{\mathfrak{X}}^0(\overline{S}_{n_0}^\bullet \times_{\overline{T}_p} \overline{S}_{n_1}^\circ, L) \\
\quad \xrightarrow{(\pi_{n_0}^\circ \times \pi_{n_1}^\circ)^*} H_{\mathfrak{X}}^0(\overline{B}_{n_0}^\bullet \times_{\overline{T}_p} \overline{B}_{n_1}^\circ, L) \\
\quad \xrightarrow{\cup \xi_1^{r_1-1}} H_{\mathfrak{X}}^{2r_1-2}(\overline{B}_{n_0}^\bullet \times_{\overline{T}_p} \overline{B}_{n_1}^\circ, L(r_1-1)) \\
\quad \xrightarrow{(\text{id} \times \iota_{n_1}^\circ)!} H_{\mathfrak{X}}^{2r_1-2}(\overline{B}_{n_0}^\bullet \times_{\overline{T}_p} \overline{M}_{n_1}^\circ, L(r_1-1)) \\
\quad \xrightarrow{(\text{id} \times m_{n_1}^\dagger)^*} H_{\mathfrak{X}}^{2r_1-2}(\overline{B}_{n_0}^\bullet \times_{\overline{T}_p} \overline{M}_{n_1}^\dagger, L(r_1-1)) \\
\quad \xrightarrow{(\iota_{n_0}^\bullet \times m_{n_1}^\dagger)!} H_{\mathfrak{X}, c}^{2(n-1)}(\overline{M}_{n_0}^\bullet \times_{\overline{T}_p} \overline{M}_{n_1}^\bullet, L(n-1)) = H_{\mathfrak{X}, c}^{2(n-1)}(\overline{P}^{\bullet, \bullet}, L(n-1)), \\
\text{inc}_{\bullet, \dagger}^*: H_{\mathfrak{X}}^{2n}(\overline{P}^{\bullet, \bullet}, L(n)) = H_{\mathfrak{X}}^{2n}(\overline{M}_{n_0}^\bullet \times_{\overline{T}_p} \overline{M}_{n_1}^\bullet, L(n)) \\
\quad \xrightarrow{(\iota_{n_0}^\bullet \times m_{n_1}^\dagger)^*} H_{\mathfrak{X}}^{2n}(\overline{B}_{n_0}^\bullet \times_{\overline{T}_p} \overline{M}_{n_1}^\dagger, L(n)) \\
\quad \xrightarrow{(\text{id} \times m_{n_1}^\dagger)!} H_{\mathfrak{X}}^{2n+2}(\overline{B}_{n_0}^\bullet \times_{\overline{T}_p} \overline{M}_{n_1}^\circ, L(n+1)) \\
\quad \xrightarrow{(\text{id} \times \iota_{n_1}^\circ)^*} H_{\mathfrak{X}}^{2n+2}(\overline{B}_{n_0}^\bullet \times_{\overline{T}_p} \overline{B}_{n_1}^\circ, L(n+1)) \\
\quad \xrightarrow{\cup \xi_1^{r_1-1}} H_{\mathfrak{X}}^{2(r_0+n_1-1)}(\overline{B}_{n_0}^\bullet \times_{\overline{T}_p} \overline{B}_{n_1}^\circ, L(r_0+n_1-1)) \\
\quad \xrightarrow{(\pi_{n_0}^\bullet \times \pi_{n_1}^\circ)!} H_{\mathfrak{X}}^0(\overline{S}_{n_0}^\bullet \times_{\overline{T}_p} \overline{S}_{n_1}^\circ, L) = H_{\mathfrak{X}}^0(\overline{S}_{n_0}^\bullet, L) \otimes_L H_{\mathfrak{X}}^0(\overline{S}_{n_1}^\circ, L) \\
\quad \xrightarrow{\sim} L[\text{Sh}(V_{n_0}^\circ, -_{n_0} K_{n_0, p}^\bullet)] \otimes_L L[\text{Sh}(V_{n_1}^\circ, -_{n_1} K_{n_1, p}^\circ)]; \\
\text{inc}_!^{\bullet, \bullet}: L[\text{Sh}(V_{n_0}^\circ, -_{n_0} K_{n_0, p}^\bullet)] \otimes_L L[\text{Sh}(V_{n_1}^\circ, -_{n_1} K_{n_1, p}^\bullet)] \\
\quad \xrightarrow{\sim} H_{\mathfrak{X}}^0(\overline{S}_{n_0}^\bullet, L) \otimes_L H_{\mathfrak{X}}^0(\overline{S}_{n_1}^\bullet, L) = H_{\mathfrak{X}}^0(\overline{S}_{n_0}^\bullet \times_{\overline{T}_p} \overline{S}_{n_1}^\bullet, L) \\
\quad \xrightarrow{(\pi_{n_0}^\bullet \times \pi_{n_1}^\bullet)^*} H_{\mathfrak{X}}^0(\overline{B}_{n_0}^\bullet \times_{\overline{T}_p} \overline{B}_{n_1}^\bullet, L) \\
\quad \xrightarrow{(\iota_{n_0}^\bullet \times \iota_{n_1}^\bullet)!} H_{\mathfrak{X}, c}^{2(n-1)}(\overline{M}_{n_0}^\bullet \times_{\overline{T}_p} \overline{M}_{n_1}^\bullet, L(n-1)) = H_{\mathfrak{X}, c}^{2(n-1)}(\overline{P}^{\bullet, \bullet}, L(n-1)), \\
\text{inc}_{\bullet, \bullet}^*: H_{\mathfrak{X}}^{2n}(\overline{P}^{\bullet, \bullet}, L(n)) = H_{\mathfrak{X}}^{2n}(\overline{M}_{n_0}^\bullet \times_{\overline{T}_p} \overline{M}_{n_1}^\bullet, L(n)) \\
\quad \xrightarrow{(\iota_{n_0}^\bullet \times \iota_{n_1}^\bullet)^*} H_{\mathfrak{X}}^{2n}(\overline{B}_{n_0}^\bullet \times_{\overline{T}_p} \overline{B}_{n_1}^\bullet, L(n)) \\
\quad \xrightarrow{(\pi_{n_0}^\bullet \times \pi_{n_1}^\bullet)!} H_{\mathfrak{X}}^0(\overline{S}_{n_0}^\bullet \times_{\overline{T}_p} \overline{S}_{n_1}^\bullet, L) = H_{\mathfrak{X}}^0(\overline{S}_{n_0}^\bullet, L) \otimes_L H_{\mathfrak{X}}^0(\overline{S}_{n_1}^\bullet, L) \\
\quad \xrightarrow{\sim} L[\text{Sh}(V_{n_0}^\circ, -_{n_0} K_{n_0, p}^\bullet)] \otimes_L L[\text{Sh}(V_{n_1}^\circ, -_{n_1} K_{n_1, p}^\bullet)].
\end{array} \right.$$

In fact, the two maps in each pair are Poincaré dual to each other.

Theorem 5.11.5 (First geometric reciprocity law). *Take an object $K^{p^\circ} \in \mathfrak{K}(V_n^\circ)_{\text{sp}}^p$. For the class $\text{cl}(P_\Delta^\bullet) \in H_{\mathfrak{X}}^{2n}(\overline{P}^{\bullet, \bullet}, L(n))$, we have*

(1) *For $f \in L[\text{Sh}(V_{n_0}^\circ, K_{n_0}^{p^\circ} K_{n_0, p}^\bullet)] \otimes_L L[\text{Sh}(V_{n_1}^\circ, K_{n_1}^{p^\circ} K_{n_1, p}^\bullet)]$, the identity*

$$\int_{\overline{P}^{\bullet, \bullet}}^{\mathfrak{X}} \text{cl}(P_\Delta^\bullet) \cup \text{inc}_!^{\bullet, \dagger}(f) = \sum_{s \in \text{Sh}(V_n^\circ, K_n^{p^\circ} K_{\text{sp}, p}^\bullet)} (\mathbb{T}_{n_1, p}^{\bullet, \bullet} f)(\text{sh}_\downarrow^\bullet(s), \text{sh}_\uparrow^\bullet(s))$$

holds.

(2) For $f \in L[\text{Sh}(V_{n_0}^\circ, K_{n_0}^{p^\circ} K_{n_0,p}^\bullet)] \otimes_L L[\text{Sh}(V_{n_1}^\circ, K_{n_1}^{p^\circ} K_{n_1,p}^\bullet)]$, the identity

$$\int_{\overline{\mathbb{P}}^{\bullet,\bullet}} \text{cl}(\mathbb{P}_\Delta^\bullet) \cup \text{inc}_!^{\bullet,\bullet}(f) = \sum_{s \in \text{Sh}(V_n^\circ, K_n^{p^\circ} K_{sp,p}^\bullet)} (\mathbb{T}_{n_1,p}^\bullet f)(\text{sh}_\downarrow^\bullet(s), \text{sh}_\uparrow^\bullet(s))$$

holds.

(3) For $f \in L[\text{Sh}(V_{n_0}^\circ, K_{n_0}^{p^\circ} K_{n_0,p}^\circ)] \otimes_L L[\text{Sh}(V_{n_1}^\circ, K_{n_1}^{p^\circ} K_{n_1,p}^\circ)]$, the identity

$$\begin{aligned} & \int_{\overline{\mathbb{P}}^{\bullet,\bullet}} \text{cl}(\mathbb{P}_\Delta^\bullet) \cup \left(\text{inc}_!^{\bullet,\dagger}(\mathbb{T}_{n_0,p}^\circ \otimes \mathbb{I}_{n_1,p}^\circ f) + (p+1)^2 \text{inc}_!^{\bullet,\bullet}(\mathbb{T}_{n_0,p}^\circ \otimes \mathbb{T}_{n_1,p}^\circ f) \right) \\ &= \sum_{s \in \text{Sh}(V_n^\circ, K_n^{p^\circ} K_{n,p}^\circ)} (\mathbb{I}_{n_0,p}^\circ \otimes \mathbb{T}_{n_1,p}^\circ f)(s, \text{sh}_\uparrow^\circ(s)) \end{aligned}$$

holds.

Here, $\int_{\overline{\mathbb{P}}^{\bullet,\bullet}}$ denotes the \mathfrak{T} -trace map in Definition 3.5.8; and sh_\uparrow° , $\text{sh}_\uparrow^\bullet$, and $\text{sh}_\downarrow^\bullet$ are maps in Notation 5.10.13.

The intersection number in (3) is the actual one that is responsible for the first explicit reciprocity law which will be discussed in Subsection 7.2.

Proof. We first show (3) assuming (1) and (2). By (1), (2), and Lemma B.4.4, we have for $f \in L[\text{Sh}(V_{n_0}^\circ, K_{n_0}^{p^\circ} K_{n_0,p}^\circ)] \otimes_L L[\text{Sh}(V_{n_1}^\circ, K_{n_1}^{p^\circ} K_{n_1,p}^\circ)]$,

$$\begin{aligned} & \int_{\overline{\mathbb{P}}^{\bullet,\bullet}} \text{cl}(\mathbb{P}_\Delta^\bullet) \cup \left(\text{inc}_!^{\bullet,\dagger}(\mathbb{T}_{n_0,p}^\circ \otimes \mathbb{I}_{n_1,p}^\circ f) + (p+1)^2 \text{inc}_!^{\bullet,\bullet}(\mathbb{T}_{n_0,p}^\circ \otimes \mathbb{T}_{n_1,p}^\circ f) \right) \\ &= \sum_{s \in \text{Sh}(V_n^\circ, K_n^{p^\circ} K_{sp,p}^\circ)} (\mathbb{T}_{n_0,p}^\circ \otimes (\mathbb{T}_{n_1,p}^\circ \circ \mathbb{I}_{n_1,p}^\circ) f)(\text{sh}_\downarrow^\bullet(s), \text{sh}_\uparrow^\bullet(s)) \\ & \quad + \sum_{s \in \text{Sh}(V_n^\circ, K_n^{p^\circ} K_{sp,p}^\circ)} (\mathbb{T}_{n_0,p}^\circ \otimes ((p+1)^2 \mathbb{T}_{n_1,p}^\circ \circ \mathbb{T}_{n_1,p}^\circ) f)(\text{sh}_\downarrow^\bullet(s), \text{sh}_\uparrow^\bullet(s)) \\ &= \sum_{s \in \text{Sh}(V_n^\circ, K_n^{p^\circ} K_{sp,p}^\circ)} (\mathbb{T}_{n_0,p}^\circ \otimes (\mathbb{T}_{n_1,p}^\circ \circ \mathbb{I}_{n_1,p}^\circ) f)(\text{sh}_\downarrow^\bullet(s), \text{sh}_\uparrow^\bullet(s)) \\ & \quad + \sum_{s \in \text{Sh}(V_n^\circ, K_n^{p^\circ} K_{sp,p}^\circ)} (\mathbb{T}_{n_0,p}^\circ \otimes (\mathbb{T}_{n_1,p}^\circ \circ \mathbb{T}_{n_1,p}^\circ - \mathbb{T}_{n_1,p}^\circ \circ \mathbb{I}_{n_1,p}^\circ) f)(\text{sh}_\downarrow^\bullet(s), \text{sh}_\uparrow^\bullet(s)) \\ &= \sum_{s \in \text{Sh}(V_n^\circ, K_n^{p^\circ} K_{sp,p}^\circ)} (\mathbb{T}_{n_0,p}^\circ \otimes (\mathbb{T}_{n_1,p}^\circ \circ \mathbb{T}_{n_1,p}^\circ) f)(\text{sh}_\downarrow^\bullet(s), \text{sh}_\uparrow^\bullet(s)) \end{aligned}$$

which, by Lemma 5.11.6 below, equals

$$\sum_{s \in \text{Sh}(V_n^\circ, K_n^{p^\circ} K_{n,p}^\circ)} (\mathbb{I}_{n_0,p}^\circ \otimes \mathbb{T}_{n_1,p}^\circ f)(s, \text{sh}_\uparrow^\circ(s)).$$

Thus, (3) is proved.

Now we consider (1) and (2) simultaneously. Similar to the maps $\text{inc}_!^\bullet$ and $\text{inc}_!^\dagger$ in Construction 5.7.3, we have maps

$$\begin{aligned} \text{inc}_\alpha^\bullet &: L[\text{Sh}(V_{n_\alpha}^\circ, K_{n_\alpha}^{p^\circ} K_{n_\alpha,p}^\bullet)] \rightarrow H_{\mathfrak{T},c}^{2(r_\alpha + \alpha - 1)}(\overline{\mathbb{M}}_{n_\alpha}^\bullet, L(r_\alpha + \alpha - 1)), \\ \text{inc}_\alpha^\dagger &: L[\text{Sh}(V_{n_\alpha}^\circ, K_{n_\alpha}^{p^\circ} K_{n_\alpha,p}^\circ)] \rightarrow H_{\mathfrak{T},c}^{2(r_\alpha + \alpha - 1)}(\overline{\mathbb{M}}_{n_\alpha}^\bullet, L(r_\alpha + \alpha - 1)), \end{aligned}$$

for $\alpha = 0, 1$. Note that we now take $H_{\mathfrak{T},c}$ for the target of the maps rather than $H_{\mathfrak{T}}$. Moreover, in the calculation below, we will frequently use the following formula for intersection number pairings:

for a finite morphism $i: X \rightarrow Y$ of smooth schemes over an algebraically closed field, and proper smooth subschemes X' of X and Y' of Y , we have

$$\langle X_\Delta, X' \times Y' \rangle_{X \times Y} = \langle X'_\Delta, X' \times Y' \rangle_{X' \times Y} = \langle i_* X', Y' \rangle_Y$$

where X_Δ and X'_Δ denote by the graphs of i and $i|_{X'}$, respectively. The proof for (1) and (2) differs by the parity of n .

We first consider the case where $n = n_0$ is even. By Lemma 5.10.4(1) and Proposition 5.10.14, $\mathrm{sh}_\dagger^\bullet$ is an isomorphism. Take a point $s_n^\bullet \in \mathrm{Sh}(V_n^\circ, K_n^{p^\circ} K_{n,p}^\bullet)$. Let s^\bullet be the unique element in $\mathrm{Sh}(V_n^\circ, K_n^{p^\circ} K_{\mathrm{sp},p}^\bullet)$ such that $s_n^\bullet = \mathrm{sh}_\dagger^\bullet(s^\bullet)$, and put $s_{n+1}^\bullet := \mathrm{sh}_\dagger^\bullet(s^\bullet)$. By (the last assertion in) Proposition 5.10.12, we have

$$m_{\dagger!}^\bullet \mathrm{inc}_0^\bullet(1_{s_n^\bullet}) = \mathrm{inc}_1^\bullet(1_{s_{n+1}^\bullet}).$$

For (1), we have for every $s'_{n+1} \in \mathrm{Sh}(V_{n+1}^\circ, K_{n+1}^{p^\circ} K_{n+1,p}^\bullet)$ the identity

$$\begin{aligned} \int_{\overline{\mathbb{P}}^{\bullet,\bullet}} \mathrm{cl}(\mathbb{P}_\Delta^\bullet) \cup \mathrm{inc}_!^{\bullet,\dagger}(1_{(s_n^\bullet, s'_{n+1})}) &= \int_{\overline{\mathbb{M}}_{n+1}^\bullet} \left(m_{\dagger!}^\bullet \mathrm{inc}_0^\bullet(1_{s_n^\bullet}) \right) \cup \mathrm{inc}_1^\dagger(1_{s'_{n+1}}) \\ &= \int_{\overline{\mathbb{M}}_{n+1}^\bullet} \mathrm{inc}_0^\bullet(1_{s_{n+1}^\bullet}) \cup \mathrm{inc}_1^\dagger(1_{s'_{n+1}}). \end{aligned}$$

Thus, (1) follows from Proposition 5.7.6. For (2), we have for every $s'_{n+1} \in \mathrm{Sh}(V_{n+1}^\circ, K_{n+1}^{p^\circ} K_{n+1,p}^\bullet)$ the identity

$$\begin{aligned} \int_{\overline{\mathbb{P}}^{\bullet,\bullet}} \mathrm{cl}(\mathbb{P}_\Delta^\bullet) \cup \mathrm{inc}_!^{\bullet,\bullet}(1_{(s_n^\bullet, s'_{n+1})}) &= \int_{\overline{\mathbb{M}}_{n+1}^\bullet} \left(m_{\dagger!}^\bullet \mathrm{inc}_0^\bullet(1_{s_n^\bullet}) \right) \cup \mathrm{inc}_1^\bullet(1_{s'_{n+1}}) \\ &= \int_{\overline{\mathbb{M}}_{n+1}^\bullet} \mathrm{inc}_0^\bullet(1_{s_{n+1}^\bullet}) \cup \mathrm{inc}_1^\bullet(1_{s'_{n+1}}). \end{aligned}$$

Thus, (2) follows from Proposition 5.7.6.

We then consider the case where $n = n_1$ is odd. Take a point $s_{n+1}^\bullet \in \mathrm{Sh}(V_{n+1}^\circ, K_{n+1}^{p^\circ} K_{n+1,p}^\bullet)$. By Proposition 5.10.6, Proposition 5.10.12, and Proposition 5.10.14, we have

$$m_{\dagger!}^{\bullet*} \mathrm{inc}_0^\bullet(1_{s_{n+1}^\bullet}) = \mathrm{inc}_1^\bullet(\mathrm{sh}_{\dagger!}^\bullet \mathrm{sh}_\dagger^{\bullet*} 1_{s_{n+1}^\bullet}).$$

For (1), we have for every $s'_n \in \mathrm{Sh}(V_n^\circ, K_n^{p^\circ} K_{n,p}^\bullet)$ the identity

$$\begin{aligned} \int_{\overline{\mathbb{P}}^{\bullet,\bullet}} \mathrm{cl}(\mathbb{P}_\Delta^\bullet) \cup \mathrm{inc}_!^{\bullet,\dagger}(1_{(s'_{n+1}, s'_n)}) &= \int_{\overline{\mathbb{M}}_n^\bullet} \left(m_{\dagger!}^{\bullet*} \mathrm{inc}_0^\bullet(1_{s_{n+1}^\bullet}) \right) \cup \mathrm{inc}_1^\dagger(1_{s'_n}) \\ &= \int_{\overline{\mathbb{M}}_n^\bullet} \mathrm{inc}_1^\bullet(\mathrm{sh}_{\dagger!}^\bullet \mathrm{sh}_\dagger^{\bullet*} 1_{s_{n+1}^\bullet}) \cup \mathrm{inc}_1^\dagger(1_{s'_n}). \end{aligned}$$

Thus, (1) follows from Proposition 5.7.6. For (2), we have for every $s'_n \in \mathrm{Sh}(V_n^\circ, K_n^{p^\circ} K_{n,p}^\bullet)$ the identity

$$\begin{aligned} \int_{\overline{\mathbb{P}}^{\bullet,\bullet}} \mathrm{cl}(\mathbb{P}_\Delta^\bullet) \cup \mathrm{inc}_!^{\bullet,\bullet}(1_{(s'_{n+1}, s'_n)}) &= \int_{\overline{\mathbb{M}}_n^\bullet} \left(m_{\dagger!}^{\bullet*} \mathrm{inc}_0^\bullet(1_{s_{n+1}^\bullet}) \right) \cup \mathrm{inc}_1^\bullet(1_{s'_n}) \\ &= \int_{\overline{\mathbb{M}}_n^\bullet} \mathrm{inc}_1^\bullet(\mathrm{sh}_{\dagger!}^\bullet \mathrm{sh}_\dagger^{\bullet*} 1_{s_{n+1}^\bullet}) \cup \mathrm{inc}_1^\bullet(1_{s'_n}). \end{aligned}$$

Thus, (1) follows from Proposition 5.7.6.

The theorem is proved. \square

Lemma 5.11.6. *For every $f \in L[\mathrm{Sh}(V_{n_0}^\circ, K_{n_0}^{p^\circ} K_{n_0,p}^\bullet)] \otimes_L L[\mathrm{Sh}(V_{n_1}^\circ, K_{n_1}^{p^\circ} K_{n_1,p}^\bullet)]$, we have*

$$\sum_{s \in \mathrm{Sh}(V_n^\circ, K_n^{p^\circ} K_{\mathrm{sp},p}^\bullet)} (\mathbb{T}_{n_1,p}^{\bullet\circ} f)(\mathrm{sh}_\dagger^\bullet(s), \mathrm{sh}_\dagger^\bullet(s)) = \sum_{s \in \mathrm{Sh}(V_n^\circ, K_n^{p^\circ} K_{n,p}^\bullet)} (\mathbb{T}_{n_0,p}^{\bullet\circ} f)(s, \mathrm{sh}_\dagger^\bullet(s)).$$

Proof. There are two cases.

When n is even, by Lemma 5.10.8(1) and Proposition 5.10.14, we have

$$\begin{aligned} \sum_{s \in \text{Sh}(V_n^\circ, K_n^{p^\circ} K_{\text{sp},p}^\bullet)} (\mathbb{T}_{n_1,p}^{\circ\bullet} f)(\text{sh}_\downarrow^\bullet(s), \text{sh}_\uparrow^\bullet(s)) &= \sum_{s \in \text{Sh}(V_n^\circ, K_n^{p^\circ} K_{\text{sp},p}^\dagger)} f(\text{sh}_\downarrow^\dagger(\text{sh}_n^{\dagger\bullet}(s)), \text{sh}_{n+1}^{\dagger\circ}(\text{sh}_\uparrow^\dagger(s))) \\ &= \sum_{s \in \text{Sh}(V_n^\circ, K_n^{p^\circ} K_{\text{sp},p}^\dagger)} f(\text{sh}_\downarrow^\dagger(\text{sh}_n^{\dagger\bullet}(s)), \text{sh}_\uparrow^\circ(\text{sh}_n^{\dagger\circ}(\text{sh}_\downarrow^\dagger(s)))), \end{aligned}$$

which, by Lemma 5.10.4(1), Definition 5.10.7, and Proposition 5.10.14, equals

$$\sum_{s \in \text{Sh}(V_n^\circ, K_n^{p^\circ} K_{\text{sp},p}^\dagger)} f(\text{sh}_n^{\dagger\bullet}(s), \text{sh}_\uparrow^\circ(\text{sh}_n^{\dagger\circ}(s))) = \sum_{s \in \text{Sh}(V_n^\circ, K_n^{p^\circ} K_{n,p}^\circ)} (\mathbb{T}_{n_0,p}^{\circ\bullet} f)(s, \text{sh}_\uparrow^\circ(s)).$$

When n is odd, by Definition 5.10.7 and Proposition 5.10.14, we have

$$\begin{aligned} \sum_{s \in \text{Sh}(V_n^\circ, K_n^{p^\circ} K_{\text{sp},p}^\bullet)} (\mathbb{T}_{n_1,p}^{\circ\bullet} f)(\text{sh}_\downarrow^\bullet(s), \text{sh}_\uparrow^\bullet(s)) &= \sum_{s \in \text{Sh}(V_n^\circ, K_n^{p^\circ} K_{\text{sp},p}^\dagger)} f(\text{sh}_n^{\dagger\circ}(\text{sh}_\downarrow^\dagger(s)), \text{sh}_\uparrow^\bullet(\text{sh}_{\text{sp}}^{\dagger\bullet}(s))) \\ &= \sum_{s \in \text{Sh}(V_n^\circ, K_n^{p^\circ} K_{\text{sp},p}^\dagger)} f(\text{sh}_n^{\dagger\circ}(\text{sh}_\downarrow^\dagger(s)), \text{sh}_{n+1}^{\dagger\bullet}(\text{sh}_\uparrow^\dagger(s))), \end{aligned}$$

which, by Lemma 5.10.8(2) and Proposition 5.10.14, equals

$$\sum_{s \in \text{Sh}(V_n^\circ, K_n^{p^\circ} K_{n,p}^\circ)} (\mathbb{T}_{n_0,p}^{\circ\bullet} f)(s, \text{sh}_\uparrow^\circ(s)).$$

The lemma is proved. \square

Construction 5.11.7. We constructs maps

$$\left\{ \begin{array}{l} \text{Inc}_{\circ,\dagger}^* : H_{\mathbb{X}}^{2n}(\overline{\mathbb{Q}}^{(0)}, L(n)) \rightarrow H_{\mathbb{X}}^{2n}(\overline{\mathbb{Q}}^{\circ,\bullet}, L(n)) \xrightarrow{\sigma_1} H_{\mathbb{X}}^{2n}(\overline{\mathbb{P}}^{\circ,\bullet}, L(n)) \\ \quad \xrightarrow{\text{inc}_{\circ,\dagger}^*} L[\text{Sh}(V_{n_0}^\circ, -_{n_0} K_{n_0,p}^\circ)] \otimes_L L[\text{Sh}(V_{n_1}^\circ, -_{n_1} K_{n_1,p}^\circ)], \\ \text{Inc}_{\circ,\bullet}^* : H_{\mathbb{X}}^{2n}(\overline{\mathbb{Q}}^{(0)}, L(n)) \rightarrow H_{\mathbb{X}}^{2n}(\overline{\mathbb{Q}}^{\circ,\bullet}, L(n)) \xrightarrow{\sigma_1} H_{\mathbb{X}}^{2n}(\overline{\mathbb{P}}^{\circ,\bullet}, L(n)) \\ \quad \xrightarrow{\text{inc}_{\circ,\bullet}^*} L[\text{Sh}(V_{n_0}^\circ, -_{n_0} K_{n_0,p}^\circ)] \otimes_L L[\text{Sh}(V_{n_1}^\circ, -_{n_1} K_{n_1,p}^\bullet)], \\ \text{Inc}_{\bullet,\dagger}^* : H_{\mathbb{X}}^{2n}(\overline{\mathbb{Q}}^{(0)}, L(n)) \rightarrow H_{\mathbb{X}}^{2n}(\overline{\mathbb{Q}}^{\bullet,\bullet}, L(n)) \xrightarrow{\sigma_1} H_{\mathbb{X}}^{2n}(\overline{\mathbb{P}}^{\bullet,\bullet}, L(n)) \\ \quad \xrightarrow{\text{inc}_{\bullet,\dagger}^*} L[\text{Sh}(V_{n_0}^\circ, -_{n_0} K_{n_0,p}^\bullet)] \otimes_L L[\text{Sh}(V_{n_1}^\circ, -_{n_1} K_{n_1,p}^\circ)], \\ \text{Inc}_{\bullet,\bullet}^* : H_{\mathbb{X}}^{2n}(\overline{\mathbb{Q}}^{(0)}, L(n)) \rightarrow H_{\mathbb{X}}^{2n}(\overline{\mathbb{Q}}^{\bullet,\bullet}, L(n)) \xrightarrow{\sigma_1} H_{\mathbb{X}}^{2n}(\overline{\mathbb{P}}^{\bullet,\bullet}, L(n)) \\ \quad \xrightarrow{\text{inc}_{\bullet,\bullet}^*} L[\text{Sh}(V_{n_0}^\circ, -_{n_0} K_{n_0,p}^\bullet)] \otimes_L L[\text{Sh}(V_{n_1}^\circ, -_{n_1} K_{n_1,p}^\bullet)]. \end{array} \right.$$

Define the map

$$\nabla : H_{\mathbb{X}}^{2n}(\overline{\mathbb{Q}}^{(0)}, L(n)) \rightarrow L[\text{Sh}(V_{n_0}^\circ, -_{n_0} K_{n_0,p}^\circ)] \otimes_L L[\text{Sh}(V_{n_1}^\circ, -_{n_1} K_{n_1,p}^\circ)]$$

to be the sum of the following four maps

$$\begin{aligned} &(\mathbb{I}_{n_0,p}^\circ \otimes \mathbb{I}_{n_1,p}^\circ) \circ \text{Inc}_{\circ,\dagger}^*, & (p+1)^2 (\mathbb{I}_{n_0,p}^\circ \otimes \mathbb{T}_{n_1,p}^{\circ\bullet}) \circ \text{Inc}_{\circ,\bullet}^*, \\ &(p+1) (\mathbb{T}_{n_0,p}^{\circ\bullet} \otimes \mathbb{I}_{n_1,p}^\circ) \circ \text{Inc}_{\bullet,\dagger}^*, & (p+1)^3 (\mathbb{T}_{n_0,p}^{\circ\bullet} \otimes \mathbb{T}_{n_1,p}^{\circ\bullet}) \circ \text{Inc}_{\bullet,\bullet}^*. \end{aligned}$$

At last, we recall the construction of potential map from [Liu19, Section 2.2]. For $r \in \mathbb{Z}$, put

$$B^r(\mathbb{Q}, L) := \ker \left(\delta_0^* : H_{\mathbb{X}}^{2r}(\overline{\mathbb{Q}}^{(0)}, L(r)) \rightarrow H_{\mathbb{X}}^{2r}(\overline{\mathbb{Q}}^{(1)}, L(r)) \right)$$

and

$$B_r(\mathbb{Q}, L) := \text{coker} \left(\delta_{1!} : H_{\mathbb{Z}}^{2(2n-r-2)}(\overline{\mathbb{Q}}^{(1)}, L(2n-r-2)) \rightarrow H_{\mathbb{Z}}^{2(2n-r-1)}(\overline{\mathbb{Q}}^{(0)}, L(2n-r-1)) \right).$$

We define $B^r(\mathbb{Q}, L)^0$ and $B_{2n-r-1}(\mathbb{Q}, L)_0$ to be the kernel and the cokernel of the tautological map

$$B^r(\mathbb{Q}, L) \rightarrow B_{2n-r-1}(\mathbb{Q}, L),$$

respectively. By [Liu19, Lemma 2.4], the composite map

$$H_{\mathbb{Z}}^{2(r-1)}(\overline{\mathbb{Q}}^{(0)}, L(r-1)) \xrightarrow{\delta_0^*} H_{\mathbb{Z}}^{2(r-1)}(\overline{\mathbb{Q}}^{(1)}, L(r-1)) \xrightarrow{\delta_{1!}} H_{\mathbb{Z}}^{2r}(\overline{\mathbb{Q}}^{(0)}, L(r))$$

factors through a unique map

$$B_{2n-r}(\mathbb{Q}, L)_0 \rightarrow B^r(\mathbb{Q}, L)^0$$

in $\text{Fun}(\mathfrak{K}(V_n^\circ)^p \times \mathfrak{K}(V_{n+1}^\circ)^p, \text{Mod}(L[\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^\Phi)]))$. Put

$$C^r(\mathbb{Q}, L) := B^r(\mathbb{Q}, L)_{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^\Phi)}^0, \quad C_r(\mathbb{Q}, L) := B_r(\mathbb{Q}, L)_0^{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^\Phi)}.$$

Then we obtain the *potential map*

$$(5.28) \quad \Delta^r : C_{2n-r}(\mathbb{Q}, L) \rightarrow C^r(\mathbb{Q}, L)$$

in $\text{Fun}(\mathfrak{K}(V_n^\circ)^p \times \mathfrak{K}(V_{n+1}^\circ)^p, \text{Mod}(L))$.¹³ We will be most interested in the case where $r = n$.

Remark 5.11.8. By the descriptions of the Galois actions in Construction 5.2.6 and Construction 5.3.6, the map ∇ in Construction 5.11.7 factors through the quotient map

$$H_{\mathbb{Z}}^{2n}(\overline{\mathbb{Q}}^{(0)}, L(n)) \rightarrow H_{\mathbb{Z}}^{2n}(\overline{\mathbb{Q}}^{(0)}, L(n))_{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^\Phi)},$$

hence restricts to a map

$$\nabla : C^m(\mathbb{Q}, L) \rightarrow L[\text{Sh}(V_{n_0}^\circ, \mathbf{-}_{n_0}K_{n_0,p}^\circ)] \otimes_L L[\text{Sh}(V_{n_1}^\circ, \mathbf{-}_{n_1}K_{n_1,p}^\circ)]$$

in $\text{Fun}(\mathfrak{K}(V_n^\circ)^p \times \mathfrak{K}(V_{n+1}^\circ)^p, \text{Mod}(L))$, via the canonical map $C^n(\mathbb{Q}, L) \rightarrow H_{\mathbb{Z}}^{2n}(\overline{\mathbb{Q}}^{(0)}, L(n))_{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^\Phi)}$.

6. TATE CLASSES AND ARITHMETIC LEVEL RAISING

In this section, we study two important arithmetic properties of semistable moduli schemes introduced in Section 5. The first is the existence of Tate cycles when the rank is odd, studied in Subsection 6.2. The second is the arithmetic level raising when the rank is even, studied in Subsections 6.3 and 6.4. In Subsection 6.1, we collect some preliminaries on automorphic representations and their motives.

Let $N \geq 2$ be an integer with $r := \lfloor \frac{N}{2} \rfloor$.

6.1. Preliminaries on automorphic representations. In this subsection, we consider

- a relevant representation Π of $\text{GL}_N(\mathbb{A}_F)$ (Definition 1.1.3),
- a strong coefficient field $E \subseteq \mathbb{C}$ of Π (Definition 3.2.5),
- a finite set Σ_{\min}^+ of nonarchimedean places of F^+ containing Σ_{Π}^+ (Notation 3.1.4),
- a (possibly empty) finite set Σ_{lr}^+ ¹⁴ of nonarchimedean places of F^+ that are inert in F , strongly disjoint from Σ_{\min}^+ (Definition 1.3.3),
- a finite set Σ^+ of nonarchimedean places of F^+ containing $\Sigma_{\min}^+ \cup \Sigma_{\text{lr}}^+$.

¹³In [Liu19], $C^r(\mathbb{Q}, L)$ and $C_r(\mathbb{Q}, L)$ are denoted by $A^r(\mathbb{Q}, L)^0$ and $A_r(\mathbb{Q}, L)_0$, respectively.

¹⁴Here, the subscript “lr” stands for “level-raising”.

We then have, by Construction 3.1.10, the homomorphism

$$\phi_{\Pi}: \mathbb{T}_N^{\Sigma^+} \rightarrow O_E.$$

For every prime λ of E , we have a continuous homomorphism

$$\rho_{\Pi, \lambda}: \Gamma_F \rightarrow \mathrm{GL}_N(E_{\lambda})$$

from Proposition 3.2.4(2) and Definition 3.2.5, such that $\rho_{\Pi, \lambda}^{\circ}$ and $\rho_{\Pi, \lambda}^{\vee}(1 - N)$ are conjugate.

We choose

- a finite place λ of E (with the underlying rational prime ℓ) satisfying $\ell \nmid \|v\|(\|v\|^2 - 1)$ for every $v \in \Sigma_{\mathrm{lr}}^+$,
- a positive integer m ,
- a standard definite hermitian space V_N° of rank N over F , together with a self-dual $\prod_{v \notin \Sigma_{\infty}^+ \cup \Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}}^+} O_{F_v}$ -lattice Λ_N° in $V_N^{\circ} \otimes_F \mathbb{A}_F^{\Sigma_{\infty}^+ \cup \Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}}^+}$, satisfying that $(V_N^{\circ})_v$ is not split for $v \in \Sigma_{\mathrm{lr}}^+$ when N is even,
- an object $K_N^{\circ} \in \mathfrak{K}(V_N^{\circ})$ of the form

$$K_N^{\circ} = \prod_{v \in \Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}}^+} (K_N^{\circ})_v \times \prod_{v \notin \Sigma_{\infty}^+ \cup \Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}}^+} \mathrm{U}(\Lambda_N^{\circ})(O_{F_v^+}),$$

satisfying that when N is even, $(K_N^{\circ})_v$ is a transferable open compact subgroup of $\mathrm{U}(V_N^{\circ})(F_v^+)$ (Definition D.2.1)¹⁵ for $v \in \Sigma_{\mathrm{min}}^+$ and is a special maximal subgroup of $\mathrm{U}(V_N^{\circ})(F_v^+)$ for $v \in \Sigma_{\mathrm{lr}}^+$,

- a special inert prime (Definition 3.3.4) \mathfrak{p} of F^+ (with the underlying rational prime p) satisfying
 - (P1): Σ^+ does not contain p -adic places;
 - (P2): ℓ does not divide $p(p^2 - 1)$;
 - (P3): there exists a CM type Φ containing τ_{∞} as in the initial setup of Section 5 satisfying $\mathbb{Q}_p^{\Phi} = \mathbb{Q}_{p^2}$;
 - (P4): if N is even, then $P_{\alpha(\Pi_{\mathfrak{p}})} \bmod \lambda^m$ is level-raising special at \mathfrak{p} (Definition 3.1.5); if N is odd, then $P_{\alpha(\Pi_{\mathfrak{p}})} \bmod \lambda$ is Tate generic at \mathfrak{p} (Definition 3.1.5);
 - (P5): $P_{\alpha(\Pi_{\mathfrak{p}})} \bmod \lambda$ is intertwining generic at \mathfrak{p} (Definition 3.1.5);
 - (P6): if N is even, the natural map

$$\frac{O_E/\lambda^m[\mathrm{Sh}(V_N^{\circ}, K_N^{\circ})]}{\mathbb{T}_N^{\Sigma^+ \cup \Sigma_{\mathfrak{p}}^+} \cap \ker \phi_{\Pi}} \rightarrow \frac{O_E/\lambda^m[\mathrm{Sh}(V_N^{\circ}, K_N^{\circ})]}{\ker \phi_{\Pi}}$$

is an isomorphism;

(So we can and will adopt the setup in Section 5 to the datum $(V_N^{\circ}, \{\Lambda_{N, q}^{\circ}\}_{q|p})$.)

- remaining data in the initial setup of Section 5 with $\mathbb{Q}_p^{\Phi} = \mathbb{Q}_{p^2}$;
- a definite uniformization datum as in Construction 5.3.6; and
- an indefinite uniformization datum $(V'_N, j_N, \{\Lambda'_{q, N}\}_{q|p})$ for V_N° as in Definition 5.1.6.

Put $K_N^{p\circ} := (K_N^{\circ})^p$ and $K_N^{\bullet} := K_N^{p\circ} \times K_N^{\circ}$. Like in Subsection 5.8, we put $X_N^? := X_{\mathfrak{p}}^?(V_N^{\circ}, K_N^{p\circ})$ for meaningful pairs $(X, ?) \in \{\mathbf{M}, \mathbf{M}, \mathbf{B}, \mathbf{S}\} \times \{, \eta, \circ, \bullet, \dagger\}$. Let $(E_s^{p, q}, d_s^{p, q})$ be the weight spectral sequence abutting to the cohomology $H_{\mathfrak{X}}^{p+q}(\overline{M}_N, R\Psi O_{\lambda}(r))$ from Subsection 5.8.

Remark 6.1.1. By Construction 3.1.10 and (P2) (namely, $\ell \neq p$), we know that $P_{\alpha(\Pi_{\mathfrak{p}})}$ is a polynomial with coefficients in O_{λ} .

Remark 6.1.2. Note that when $N = 2$, (P2) and (P4) together imply (P5).

¹⁵By Lemma D.2.2(3), every sufficiently small $(K_N^{\circ})_v$ is transferable. So the readers may ignore this technical requirement.

Notation 6.1.3. We introduce the following ideas of $\mathbb{T}_N^{\Sigma^+ \cup \Sigma_p^+}$

$$\begin{cases} \mathfrak{m} := \mathbb{T}_N^{\Sigma^+ \cup \Sigma_p^+} \cap \ker \left(\mathbb{T}_N^{\Sigma^+} \xrightarrow{\phi_\Pi} O_E \rightarrow O_E/\lambda \right), \\ \mathfrak{n} := \mathbb{T}_N^{\Sigma^+ \cup \Sigma_p^+} \cap \ker \left(\mathbb{T}_N^{\Sigma^+} \xrightarrow{\phi_\Pi} O_E \rightarrow O_E/\lambda^m \right). \end{cases}$$

We then introduce following assumptions.

Assumption 6.1.4. We have $H_{\mathfrak{z}}^i(\overline{M}_N, R\Psi O_\lambda)_{\mathfrak{m}} = 0$ for $i \neq N - 1$, and that $H_{\mathfrak{z}}^{N-1}(\overline{M}_N, R\Psi O_\lambda)_{\mathfrak{m}}$ is a finite free O_λ -module.

Assumption 6.1.5. The Galois representation $\rho_{\Pi, \lambda}$ is residually absolutely irreducible.

Remark 6.1.6. Under Assumption 6.1.5, we obtain a homomorphism

$$\bar{\rho}_{\Pi, \lambda}: \Gamma_F \rightarrow \mathrm{GL}_N(O_\lambda/\lambda)$$

from the residual homomorphism of $\rho_{\Pi, \lambda}$, which is unique to to conjugation, absolutely irreducible, and $(1 - N)$ -polarizable (Definition 2.4.7). From Construction 2.4.8 or Lemma E.1.3(3), we then have an extension

$$\bar{\rho}_{\Pi, \lambda, +}: \Gamma_{F^+} \rightarrow \mathcal{G}_N(O_\lambda/\lambda)$$

of $\bar{\rho}_{\Pi, \lambda}$. For a different extension $\bar{\rho}_{\Pi, \lambda, +'}$, there exist elements $g \in \mathrm{GL}_N(O_\lambda/\lambda)$ and $a \in (O_\lambda/\lambda)^\times$ such that $\bar{\rho}_{\Pi, \lambda, +'}(x) = g\bar{\rho}_{\Pi, \lambda, +}(x)g^{-1}$ for $x \in \Gamma_F$, and $\bar{\rho}_{\Pi, \lambda, +'}(c) = (aB, \mu, \mathfrak{c})$ if $\bar{\rho}_{\Pi, \lambda, +}(c) = (B, \mu, \mathfrak{c})$. The discussion below on the extension $\bar{\rho}_{\Pi, \lambda, +}$ is independent of such ambiguity.

We now fix an isomorphism $\iota_\ell: \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$ that induces the place λ of E , till the end of this section.

Definition 6.1.7. Let π be an automorphic representation of $U(V_N^\circ)(\mathbb{A}_{F^+})$. We say that π is Π -congruent (outside $\Sigma^+ \cup \Sigma_p^+$) if π_∞ is trivial, and for every nonarchimedean place v of F^+ not in $\Sigma^+ \cup \Sigma_p^+ \cup \Sigma_\ell^+$, the two homomorphisms $\iota_\ell \phi_{\alpha(\mathrm{BC}(\pi_v))}$ and $\iota_\ell \phi_{\alpha(\Pi_v)}$ from $\mathbb{T}_{N, v}$ to $\overline{\mathbb{Q}}_\ell$, which in fact take values in $\overline{\mathbb{Z}}_\ell$, coincide in $\overline{\mathbb{F}}_\ell$.

Lemma 6.1.8. *The two maps*

$$\begin{aligned} \mathbb{T}_{N, \mathfrak{p}}^\circ: O_E[\mathrm{Sh}(V_N^\circ, K_N^\circ)]_{\mathfrak{m}} &\rightarrow O_E[\mathrm{Sh}(V_N^\circ, K_N^\bullet)]_{\mathfrak{m}} \\ \mathbb{T}_{N, \mathfrak{p}}^\bullet: O_E[\mathrm{Sh}(V_N^\circ, K_N^\bullet)]_{\mathfrak{m}} &\rightarrow O_E[\mathrm{Sh}(V_N^\circ, K_N^\circ)]_{\mathfrak{m}} \end{aligned}$$

are both isomorphisms, where $\mathbb{T}_{N, \mathfrak{p}}^\circ$ and $\mathbb{T}_{N, \mathfrak{p}}^\bullet$ are introduced in Definition 5.7.1.

Proof. By Proposition B.4.3(1) (resp. Proposition B.3.5(1)) when N is odd (resp. even) and (P5), we know that the endomorphism $\mathbb{I}_{N, \mathfrak{p}}^\circ = \mathbb{T}_{N, \mathfrak{p}}^\bullet \circ \mathbb{T}_{N, \mathfrak{p}}^\circ$ of $O_E[\mathrm{Sh}(V_N^\circ, K_N^\circ)]_{\mathfrak{m}}$ is an isomorphism. Thus, it suffices to show that the free O_λ -modules $O_E[\mathrm{Sh}(V_N^\circ, K_N^\circ)]_{\mathfrak{m}}$ and $O_E[\mathrm{Sh}(V_N^\circ, K_N^\bullet)]_{\mathfrak{m}}$ have the same rank. We show that $O_E[\mathrm{Sh}(V_N^\circ, K_N^\circ)]_{\mathfrak{m}} \otimes_{O_\lambda} \overline{\mathbb{Q}}_\ell$ and $O_E[\mathrm{Sh}(V_N^\circ, K_N^\bullet)]_{\mathfrak{m}} \otimes_{O_\lambda} \overline{\mathbb{Q}}_\ell$ have the same dimension. We have

$$\begin{aligned} O_E[\mathrm{Sh}(V_N^\circ, K_N^\circ)]_{\mathfrak{m}} \otimes_{O_\lambda} \overline{\mathbb{Q}}_\ell &\simeq \bigoplus_{\pi} m(\pi) \cdot \pi^{K_N^\circ}, \\ O_E[\mathrm{Sh}(V_N^\circ, K_N^\bullet)]_{\mathfrak{m}} \otimes_{O_\lambda} \overline{\mathbb{Q}}_\ell &\simeq \bigoplus_{\pi} m(\pi) \cdot \pi^{K_N^\bullet}, \end{aligned}$$

where π runs over all automorphic representations of $U(V_N^\circ)(\mathbb{A}_{F^+})$ (with coefficients in $\overline{\mathbb{Q}}_\ell$) that are Π -congruent; and $m(\pi)$ denotes the automorphic multiplicity of π .¹⁶ It suffices to show that if in the second direct sum $\pi_{\mathfrak{p}}^{K_N^\bullet} \neq \{0\}$, which has to be of dimension one since K_N^\bullet is special

¹⁶Although we know that $m(\pi) = 1$ by Proposition C.3.1(2), we do not need this fact here.

maximal, then $\pi_p^{K_N^\circ} \neq \{0\}$ as well. Since π_p is semistable, then its Satake parameter α does not contain the pair $\{-1, -1\}$ (resp. $\{-p, -p^{-1}\}$) when N is even (resp. odd) by (P5). Let π'_p be the unique constituent of the principal series of α such that $(\pi'_p)^{K_N^\circ} \neq \{0\}$, then by Proposition B.4.3(1) (resp. Proposition B.3.5(1)) when N is odd (resp. even) again, we see that $(\pi'_p)^{K_N^\circ} \neq \{0\}$. Thus, we must have $\pi_p = \pi'_p$ as K_N^\bullet is special maximal. The lemma follows. \square

Lemma 6.1.9. *Let π be an automorphic representation of $U(V_N^\circ)(\mathbb{A}_{F^+})$ that is Π -congruent. If Assumption 6.1.5 holds, then π is stable.*

Proof. By Proposition 3.2.8, we know that $\text{BC}(\pi)$ exists as (V_N°, π) is a relevant pair. Let $\rho_{\text{BC}(\pi), \iota_\ell} : \Gamma_F \rightarrow \text{GL}_N(\overline{\mathbb{Q}}_\ell)$ be the associated Galois representation from Proposition 3.2.3(2), which is the direct sum of the associated Galois representation of each isobaric factors. If $\text{BC}(\pi)$ is not cuspidal, then $\rho_{\text{BC}(\pi), \iota_\ell}$ is decomposable. Since π is Π -congruent, by the Chebotarev density theorem, $\rho_{\text{BC}(\pi), \iota_\ell}$ admits a lattice whose residual representation is isomorphic to $\bar{\rho}_{\Pi, \lambda} \otimes_{O_\lambda/\lambda} \overline{\mathbb{F}}_\ell$, which is irreducible. This is a contradiction. Thus, the lemma follows. \square

Lemma 6.1.10. *Assume Assumption 6.1.5. Then the natural maps*

$$\begin{aligned} H_{\text{ét}, c}^i(\text{Sh}(V'_N, j_N K_N^{p\circ} K'_{p, N})_{\overline{F}}, O_\lambda)_\mathfrak{m} &\rightarrow H_{\text{ét}}^i(\text{Sh}(V'_N, j_N K_N^{p\circ} K'_{p, N})_{\overline{F}}, O_\lambda)_\mathfrak{m}, \\ H_{\mathfrak{X}, c}^i(\overline{M}_N^\bullet, O_\lambda)_\mathfrak{m} &\rightarrow H_{\mathfrak{X}}^i(\overline{M}_N^\bullet, O_\lambda)_\mathfrak{m}, \end{aligned}$$

are both isomorphisms for every $i \in \mathbb{Z}$.

Proof. By Lemma 5.1.7, and the description of the weight spectral sequence $(E_s^{p, q}, d_s^{p, q})$ in Lemma 5.8.2 (for N odd) and Lemma 5.8.3 (for N even), it suffices to show that the natural map

$$(6.1) \quad H_{\text{ét}, c}^i(\text{Sh}(V'_N, j_N K_N^{p\circ} K'_{p, N})_{\overline{F}}, O_\lambda)_\mathfrak{m} \rightarrow H_{\text{ét}}^i(\text{Sh}(V'_N, j_N K_N^{p\circ} K'_{p, N})_{\overline{F}}, O_\lambda)_\mathfrak{m}$$

is an isomorphism for every $i \in \mathbb{Z}$. This is trivial when $\text{Sh}(V'_N, j_N K_N^{p\circ} K'_{p, N})$ is proper.

If $\text{Sh}(V'_N, j_N K_N^{p\circ} K'_{p, N})$ is not proper, then the Witt index of V'_N is 1; and $\text{Sh}(V'_N, j_N K_N^{p\circ} K'_{p, N})$ has a *canonical* toroidal compactification $\widetilde{\text{Sh}}(V'_N, j_N K_N^{p\circ} K'_{p, N})$, which is smooth over F . As $j_N K_N^{p\circ} K'_{p, N}$ is neat, the boundary $Z := \widetilde{\text{Sh}}(V'_N, j_N K_N^{p\circ} K'_{p, N}) \setminus \text{Sh}(V'_N, j_N K_N^{p\circ} K'_{p, N})$ is geometrically isomorphic to a disjoint union of abelian varieties (of dimension $N - 2$). In particular, $H_{\text{ét}}^i(Z_{\overline{F}}, O_\lambda)$ is a free O_λ -module (of finite rank). Let π'^∞ be an irreducible admissible representation of $U(V'_N)(\mathbb{A}_{F^+})$ that appears in $H_{\text{ét}}^i(Z_{\overline{F}}, O_\lambda) \otimes_{O_\lambda, \iota} \mathbb{C}$. Then π'^∞ extends to an automorphic representation π' of $U(V'_N)(\mathbb{A}_{F^+})$ that is a subquotient of the parabolic induction of a cuspidal automorphic representation of $L(\mathbb{A}_{F^+})$ where L is the unique proper Levi subgroup of $U(V'_N)$ up to conjugation. In particular, π' is not stable. Thus, by (the same argument of) Lemma 6.1.9, we have $H_{\text{ét}}^i(Z_{\overline{F}}, O_\lambda)_\mathfrak{m} = 0$ for every $i \in \mathbb{Z}$. This implies that (6.1) is an isomorphism. \square

6.2. Tate classes in the odd rank case. In this section, we assume that $N = 2r + 1$ is odd with $r \geq 1$. We study the properties of the localized spectral sequence $E_{s, \mathfrak{m}}^{p, q}$, after Lemma 5.8.2.

Lemma 6.2.1. *We have*

$$H_{\mathfrak{X}}^i(\overline{M}_N^\dagger, O_\lambda)_\mathfrak{m} = 0$$

for every odd integer i .

Proof. For $i \neq 2r - 1$, it follows from Lemma 5.5.2(1). Now we assume $i = 2r - 1$.

Let $\pi^{\infty, p}$ be an irreducible admissible representation of $U(V_N^\circ)(\mathbb{A}_{F^+}^\infty)$ that appears in the cohomology $H_{\mathfrak{X}}^{2r-1}(\overline{M}_N^\dagger, O_\lambda)_\mathfrak{m} \otimes_{O_\lambda, \iota} \mathbb{C}$. By Proposition 5.5.4, we may complete $\pi^{\infty, p}$ to an automorphic representation π of $U(V_N^\circ)(\mathbb{A}_{F^+})$ as in that proposition, such that π is Π -congruent and such that $\text{BC}(\pi_p)$ is a constituent of an unramified principal series of $\text{GL}_N(F_p)$, whose Satake parameter contains $\{-p, -p^{-1}\}$ which is different from $\alpha(\Pi_p)$ in $\overline{\mathbb{F}}_\ell$ by (P5).

On the other hand, by the Chebotarev density theorem, both $\rho_{\text{BC}(\pi), \ell_\ell}$ and $\rho_{\Pi, \lambda} \otimes_{E_\lambda} \overline{\mathbb{Q}}_\ell$ each admits a lattice such that their reductions are isomorphic. However, this is not possible by Proposition C.3.1(2) and Proposition 3.2.4(2). Therefore, we must have $H_{\overline{\mathbb{Z}}}^{2r-1}(\overline{\mathbb{M}}_N^\dagger, O_\lambda)_\mathfrak{m} = 0$. The lemma is proved. \square

Lemma 6.2.2. *Assume Assumption 6.1.4. We have*

- (1) $E_{1, \mathfrak{m}}^{p, q} = 0$ if q is odd;
- (2) $E_{1, \mathfrak{m}}^{p, q}$ is a free O_λ -module for every $(p, q) \in \mathbb{Z}^2$;
- (3) $E_{2, \mathfrak{m}}^{p, q} = 0$ unless $(p, q) = (0, 2r)$;
- (4) $E_{2, \mathfrak{m}}^{0, 2r}$ is canonically isomorphic to $H_{\overline{\mathbb{Z}}}^{2r}(\overline{\mathbb{M}}_N, \text{R}\Psi O_\lambda(r))_\mathfrak{m}$, which is a free O_λ -module;
- (5) $E_{s, \mathfrak{m}}^{0, 2r}$ degenerates at the second page.

Proof. Part (1) follows from Lemma 6.2.1 and Assumption 6.1.4. Part (3) follows since $d_1^{-1, 2r}$ is injective and $d_1^{0, 2r}$ is surjective. The remaining parts are immediate consequences of (1) and Assumption 6.1.4. \square

Theorem 6.2.3. *The map*

$$\nabla_\mathfrak{m}^1 : E_{2, \mathfrak{m}}^{0, 2r} \rightarrow O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_\mathfrak{m}$$

(Construction 5.8.4) is surjective. Moreover, if we assume Assumptions 6.1.4, 6.1.5, and Hypothesis 3.2.9 for N , then we have

- (1) The generalized Frobenius eigenvalues of the $O_\lambda/\lambda[\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})]$ -module $E_{2, \mathfrak{m}}^{0, 2r} \otimes_{O_\lambda} O_\lambda/\lambda$ is contained in the set of roots of $P_{\alpha(\Pi_p)} \bmod \lambda$ in a finite extension of O_λ/λ .
- (2) The $O_\lambda[\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})]$ -module $E_{2, \mathfrak{m}}^{0, 2r}$ is weakly semisimple (Definition 2.1.2).
- (3) The map $\nabla_\mathfrak{m}^1$ induces an isomorphism

$$\nabla_\mathfrak{m}^1 : (E_{2, \mathfrak{m}}^{0, 2r})_{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})} \xrightarrow{\sim} O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_\mathfrak{m}.$$

By Remark 5.8.5, the map $\nabla_\mathfrak{m}^1$ always factors through the quotient map $E_{2, \mathfrak{m}}^{0, 2r} \rightarrow (E_{2, \mathfrak{m}}^{0, 2r})_{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})}$.

Proof. We first show that $\nabla_\mathfrak{m}^1$ is surjective. From Construction 5.8.1, we have a map

$$(\text{Inc}_i^\circ, \text{Inc}_i^\dagger, \text{Inc}_i^\bullet \circ \mathbf{T}_p^{\bullet\circ}) := O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]^{\oplus 3} \rightarrow E_1^{0, 2r}$$

which induces a map

$$\ker(d_1^{0, 2r} \circ (\text{Inc}_i^\circ, \text{Inc}_i^\dagger, \text{Inc}_i^\bullet \circ \mathbf{T}_p^{\bullet\circ})) \rightarrow \ker d_1^{0, 2r}.$$

However, by Lemma 5.8.6, the former kernel is simply the kernel of the map

$$(p+1 \quad -1 \quad 0) \begin{pmatrix} \text{Inc}_i^\circ \\ \text{Inc}_i^\dagger \\ \text{Inc}_i^\bullet \end{pmatrix} (\text{Inc}_i^\circ \quad \text{Inc}_i^\dagger \quad \text{Inc}_i^\bullet \circ \mathbf{T}_p^{\bullet\circ}).$$

Now since $(p+1, -1, 0)$ and $(0, \mathbf{T}_p^{\bullet\circ} \circ \mathbf{T}_p^{\bullet\circ}, (p+1)^2 \mathbf{T}_p^{\bullet\circ}) \otimes O_\lambda$ are linearly independent, by Nakayama's lemma, $\nabla_\mathfrak{m}^1$ is surjective if the following matrix

$$\begin{pmatrix} \text{Inc}_i^\circ \\ \text{Inc}_i^\dagger \\ \mathbf{T}_p^{\bullet\circ} \circ \text{Inc}_i^\bullet \end{pmatrix} (\text{Inc}_i^\circ \quad \text{Inc}_i^\dagger \quad \text{Inc}_i^\bullet \circ \mathbf{T}_p^{\bullet\circ})$$

in $\mathbb{T}_{N, p}^\circ$ is nondegenerate modulo \mathfrak{m} . However, by Lemma 5.8.2(2), the above matrix equals

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -(p+1)^2 & \mathbf{I}_{N, p}^\circ \\ 0 & \mathbf{I}_{N, p}^\circ & \mathbf{T}_{N, p}^{\bullet\circ} \circ \mathbf{T}_{N, p}^{\bullet\circ} \circ \mathbf{T}_{N, p}^{\bullet\circ} \end{pmatrix},$$

whose non-degeneracy modulo \mathfrak{m} follows from Lemma B.4.2, Proposition B.4.3, and (P4,P5).

Now we consider the three remaining assertions. By Lemma 5.1.7 and Lemma 6.2.2, we have an isomorphism

$$E_{2,\mathfrak{m}}^{0,2r} \simeq H_{\text{ét}}^{2r}(\text{Sh}(V', j_N K_N^{\text{po}} K'_{p,N})_{\overline{F}}, O_\lambda(r))_{\mathfrak{m}}$$

of $O_\lambda[\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^2})]$ -modules. By Lemma 6.1.10, Proposition C.3.1(2), and Hypothesis 3.2.9, we have

$$H_{\text{ét}}^{2r}(\text{Sh}(V', j_N K_N^{\text{po}} K'_{p,N})_{\overline{F}}, O_\lambda(r))_{\mathfrak{m}} \otimes_{O_\lambda} \overline{\mathbb{Q}}_\ell \simeq \bigoplus_{\pi'} \rho_{\text{BC}(\pi'), \iota_\ell}^{\mathfrak{c}}(r)^{\oplus d(\pi')}$$

of representations of Γ_F with coefficients in $\overline{\mathbb{Q}}_\ell$, where $d(\pi') := \dim(\pi'^{\infty, p})^{j_N K_N^{\text{po}}}$, and the direct sum is taken over all stable automorphic representations π' of $U(V')(\mathbb{A}_{F^+})$ that is Π -congruent and such that π'_{τ_∞} is a holomorphic discrete series representation of $U(V')(F_{\tau_\infty}^+)$ with the Harish-Chandra parameter $\{r, r-1, \dots, 1-r, -r\}$; and π'_{τ} is trivial for every archimedean place $\tau \neq \tau_\infty$.

For the proof of (1–3), we may replace E_λ by a finite extension inside $\overline{\mathbb{Q}}_\ell$ such that $\rho_{\text{BC}(\pi'), \iota_\ell}^{\mathfrak{c}}$ is defined over E_λ for every π' appeared in the previous direct sum. Now we regard $\rho_{\text{BC}(\pi'), \iota_\ell}^{\mathfrak{c}}$ as a representation over E_λ . Then $\rho_{\text{BC}(\pi'), \iota_\ell}^{\mathfrak{c}}(r)$ admits a Γ_F -stable O_λ -lattice $R_{\text{BC}(\pi')}$, unique up to homothety, whose reduction $\overline{R}_{\text{BC}(\pi')}$ is isomorphic to $\overline{\rho}_{\Pi, \lambda}(r)$. Moreover, we have an inclusion

$$E_{2,\mathfrak{m}}^{0,2r} \simeq H_{\text{ét}}^{2r}(\text{Sh}(V', j_N K_N^{\text{po}} K'_{p,N})_{\overline{F}}, O_\lambda(r))_{\mathfrak{m}} \subseteq \bigoplus_{\pi'} (R_{\text{BC}(\pi')}^{\mathfrak{c}})^{\oplus d(\pi')}$$

of $O_\lambda[\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})]$ -modules. This already implies (1).

By (P4), we know that $\overline{\rho}_{\Pi, \lambda}^{\mathfrak{c}}(r)$ is weakly semisimple and

$$\dim_{O_\lambda/\lambda} \overline{\rho}_{\Pi, \lambda}^{\mathfrak{c}}(r)^{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})} = 1.$$

On the other hand, we have

$$\dim_{E_\lambda} \rho_{\text{BC}(\pi'), \iota_\ell}^{\mathfrak{c}}(r)^{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})} \geq 1.$$

Thus by Lemma 2.1.5, for every π' in the previous direct sum, $R_{\text{BC}(\pi')}^{\mathfrak{c}}$ is weakly semisimple, and

$$\dim_{E_\lambda} \rho_{\text{BC}(\pi'), \iota_\ell}^{\mathfrak{c}}(r)^{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})} = 1.$$

This implies (2) by Lemma 2.1.4(1).

The above discussion also implies that, for (3), it suffices to show

$$\sum_{\pi'} d(\pi') \leq \dim_{E_\lambda} O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_{\mathfrak{m}} \otimes_{O_\lambda} E_\lambda$$

where π' is taken over the same set as in the previous direct sum. However, this follows from Corollary C.3.3 and Lemma 6.1.8. The theorem is proved. \square

6.3. Arithmetic level raising in the even rank case. In this subsection, we assume that $N = 2r$ is even with $r \geq 1$. We study the properties of the localized spectral sequence $E_{s,\mathfrak{m}}^{p,q}$, after Lemma 5.8.3.

Proposition 6.3.1. *Suppose $N \geq 4$. Assume Assumptions 6.1.4, 6.1.5, and Hypothesis 3.2.9 for N . Then we have*

(1) *The map*

$$(\text{Inc}_!^\circ + \text{Inc}_!^\dagger + \text{Inc}_!^\bullet)_{\mathfrak{m}}: O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_{\mathfrak{m}}^{\oplus 2} \bigoplus O_\lambda[\text{Sh}(V_N^\circ, K_N^\bullet)]_{\mathfrak{m}} \rightarrow E_{1,\mathfrak{m}}^{0,2r-2}(-1)$$

from Construction 5.8.1 is an isomorphism.

(2) The map

$$(\text{Inc}_\circ^*, \text{Inc}_\dagger^*, \text{Inc}_\bullet^*)_{\mathfrak{m}} : \mathbb{E}_{1,\mathfrak{m}}^{0,2r} \rightarrow O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_{\mathfrak{m}}^{\oplus 2} \bigoplus O_\lambda[\text{Sh}(V_N^\circ, K_N^\bullet)]_{\mathfrak{m}}$$

from Construction 5.8.1 is surjective, whose kernel is the O_λ -torsion of $\mathbb{H}_{\mathfrak{F}}^{2r}(\overline{M}_N^\bullet, O_\lambda(r))_{\mathfrak{m}}$.

(3) The map $\nabla_{\mathfrak{m}}^0 : d_{1,\mathfrak{m}}^{0,2r} \rightarrow O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_{\mathfrak{m}}$ (Construction 5.8.4) is surjective.

(4) The map $\nabla_{\mathfrak{m}}^0 \circ d_{1,\mathfrak{m}}^{-1,2r} \circ d_{1,\mathfrak{m}}^{0,2r-2}(-1)$ induces a map

$$F_{-1}H^1(\mathbb{I}_{\mathbb{Q}_{p^2}}, \mathbb{H}_{\mathfrak{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r))_{\mathfrak{m}}) \rightarrow O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_{\mathfrak{m}} / ((p+1)\mathbb{R}_{N,p}^\circ - \mathbb{I}_{N,p}^\circ)$$

which is surjective, whose kernel is canonically the O_λ -torsion of $\mathbb{H}_{\mathfrak{F}}^{2r}(\overline{M}_N^\bullet, O_\lambda(r))_{\mathfrak{m}}$.

Proof. We first claim that the map

$$(\text{inc}_\dagger^\dagger + \text{inc}_\dagger^\bullet \circ \mathbb{T}_{N,p}^\circ)_{\mathfrak{m}} : O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_{\mathfrak{m}}^{\oplus 2} \rightarrow \mathbb{H}_{\mathfrak{F}}^{2r-2}(\overline{M}_N^\bullet, O_\lambda(r-1))_{\mathfrak{m}}$$

is an isomorphism. In fact, by Lemma 6.3.2 below, it suffices to find a line bundle \mathcal{L} as in Definition 5.7.7 such that $(\text{inc}_\mathcal{L})_{\mathfrak{m}}$ is surjective, where

$$\text{inc}_\mathcal{L} := (\text{inc}_\dagger^* \circ \mathbb{T}_{N,p}^\circ \circ \text{inc}_\bullet^*) \circ \Theta_\mathcal{L} \circ (\text{inc}_\dagger^\dagger + \text{inc}_\dagger^\bullet \circ \mathbb{T}_{N,p}^\circ).$$

We take \mathcal{L} to be $\mathcal{O}(M_N^\dagger)^{\otimes 2} \otimes (\text{Lie}_{\mathcal{A},\tau_\infty^c})^{\otimes p+1}$. Then by Proposition 5.7.8 and Proposition 5.7.9, the endomorphism $\text{inc}_\mathcal{L}$ is given the matrix

$$\begin{pmatrix} (p+1)^3 & (p+1)\mathbb{I}_{N,p}^\circ \\ (p+1)\mathbb{I}_{N,p}^\circ & \mathbb{T}_{N,p}^\circ \circ (\mathbb{R}_{N,p}^\bullet + (\mathbb{R}_{N,p}^\bullet + (p+1)\mathbb{T}_{N,p}^\bullet)) \circ \mathbb{T}_{N,p}^\circ \end{pmatrix}$$

in $\mathbb{T}_{N,p}^\circ$. Now, by Lemma B.3.6 and Proposition B.3.5, the determinant of the above matrix mod \mathfrak{m} is equal to

$$-p^{r^2} \prod_{i=1}^r \left(\alpha_i + \frac{1}{\alpha_i} + 2 \right) \cdot \left((p+1)^2 p^{r^2} \prod_{i=1}^r \left(\alpha_i + \frac{1}{\alpha_i} - p - \frac{1}{p} \right) + (p+1)^3 (p^{r^2+1} - p^{r^2-1}) \sum_{j=1}^r \prod_{\substack{i=1 \\ i \neq j}}^r \left(\alpha_i + \frac{1}{\alpha_i} - p - \frac{1}{p} \right) \right)$$

where $\{\alpha_r, \dots, \alpha_1, \alpha_1^{-1}, \dots, \alpha_r^{-1}\}$ are the roots of $P_{\alpha(\Pi_{N,p})} \bmod \lambda$ in a finite extension of O_λ/λ . By (P2), we have

$$p^{r^2} (p+1)^3 (p^{r^2+1} - p^{r^2-1}) \not\equiv 0 \pmod{\lambda};$$

by (P4), we have

$$\prod_{i=1}^r \left(\alpha_i + \frac{1}{\alpha_i} - p - \frac{1}{p} \right) \equiv 0 \pmod{\lambda}, \quad \sum_{j=1}^r \prod_{\substack{i=1 \\ i \neq j}}^r \left(\alpha_i + \frac{1}{\alpha_i} - p - \frac{1}{p} \right) \not\equiv 0 \pmod{\lambda};$$

and by (P5), we have

$$\prod_{i=1}^r \left(\alpha_i + \frac{1}{\alpha_i} + 2 \right) \not\equiv 0 \pmod{\lambda}.$$

In particular, the matrix representing $\text{inc}_\mathcal{L}$ is nondegenerate modulo \mathfrak{m} ; hence the claim follows from Nakayama's lemma.

Part (1) follows immediately from the above claim and Lemma 6.1.8. Part (2) follows from (1) by Poincaré duality, together with Lemma 6.1.10.

For (3), by definition, $\Delta_{\mathfrak{m}}^0$ is the composition of

$$(\mathbb{T}_{N,p}^\circ \text{Inc}_\circ^*, \mathbb{T}_{N,p}^\circ \circ \text{Inc}_\dagger^*, \text{Inc}_\bullet^*)_{\mathfrak{m}} : \mathbb{E}_{1,\mathfrak{m}}^{0,2r} \rightarrow O_\lambda[\text{Sh}(V_N^\circ, K_N^\bullet)]_{\mathfrak{m}}^{\oplus 3},$$

which is surjective by (2) and Lemma 6.1.8, and the obviously surjective map

$$(1, 0, p+1) : O_\lambda[\text{Sh}(V_N^\circ, K_N^\bullet)]_{\mathfrak{m}}^{\oplus 3} \rightarrow O_\lambda[\text{Sh}(V_N^\circ, K_N^\bullet)]_{\mathfrak{m}}.$$

Thus, (3) follows.

Now we consider (4). Let $(E_{1,m}^{0,2r})_{\text{fr}}$ be the free O_λ -quotient of $E_{1,m}^{0,2r}$, which is simply the quotient by the O_λ -torsion $(H_{\mathfrak{X}}^{2r}(\overline{M}_N^\bullet, O_\lambda(r))_{\text{m}})_{\text{tor}}$ of $H_{\mathfrak{X}}^{2r}(\overline{M}_N^\bullet, O_\lambda(r))_{\text{m}}$. Thus by (2), we obtain an isomorphism

$$(\text{Inc}_\circ^*, \text{Inc}_\dagger^*, \text{Inc}_\bullet^*)_{\text{m}}: (E_{1,m}^{0,2r})_{\text{fr}} \xrightarrow{\sim} O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_{\text{m}}^{\oplus 2} \bigoplus O_\lambda[\text{Sh}(V_N^\circ, K_N^\bullet)]_{\text{m}}$$

through which we identify the two sides. If we let $(\ker d_{1,m}^{0,2r})_{\text{fr}}$ be the free O_λ -quotient of $\ker d_{1,m}^{0,2r}$, then by Lemma 5.8.6, the above isomorphism maps the submodule $(\ker d_{1,m}^{0,2r})_{\text{fr}}$ to the kernel of the map

$$(p+1, -1, 0): O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_{\text{m}}^{\oplus 2} \bigoplus O_\lambda[\text{Sh}(V_N^\circ, K_N^\bullet)]_{\text{m}} \rightarrow O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_{\text{m}}.$$

By Assumption 6.1.4, we have $\text{im } d_{1,m}^{-1,2r} = \ker d_{1,m}^{0,2r}$. Combining Lemma 5.8.3(5), we see that the map $d_{1,m}^{-1,2r}$ induces a canonical isomorphism

$$F_{-1}H^1(I_{\mathbb{Q}_{p^2}}, H_{\mathfrak{X}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r))_{\text{m}}) \simeq \frac{\text{im } d_{1,m}^{-1,2r}}{\text{im}(d_{1,m}^{-1,2r} \circ d_{1,m}^{0,2r-2}(-1))} = \frac{\ker d_{1,m}^{0,2r}}{\text{im}(d_{1,m}^{-1,2r} \circ d_{1,m}^{0,2r-2}(-1))}$$

induced by $d_{1,m}^{-1,2r}$. Thus, we have a canonical surjective map

$$F_{-1}H^1(I_{\mathbb{Q}_{p^2}}, H_{\mathfrak{X}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r))_{\text{m}}) \rightarrow \frac{(\ker d_{1,m}^{0,2r})_{\text{fr}}}{\text{im}(d_{1,m}^{-1,2r} \circ d_{1,m}^{0,2r-2}(-1))}$$

whose kernel is

$$\frac{(H_{\mathfrak{X}}^{2r}(\overline{M}_N^\bullet, O_\lambda(r))_{\text{m}})_{\text{tor}}}{(H_{\mathfrak{X}}^{2r}(\overline{M}_N^\bullet, O_\lambda(r))_{\text{m}})_{\text{tor}} \cap \text{im}(d_{1,m}^{-1,2r} \circ d_{1,m}^{0,2r-2}(-1))}.$$

By Lemma 6.1.8 and Lemma 5.8.3(7), we see that $(\ker d_{1,m}^{0,2r})_{\text{fr}} \cap \ker \nabla_{\text{m}}^0$ is contained in the image $d_{1,m}^{-1,2r} \circ d_{1,m}^{0,2r-2}(-1)$, as modules of $(E_{1,m}^{0,2r})_{\text{fr}}$. Thus, by (3), the map ∇_{m}^0 induces an isomorphism

$$\frac{(\ker d_{1,m}^{0,2r})_{\text{fr}}}{\text{im}(d_{1,m}^{-1,2r} \circ d_{1,m}^{0,2r-2}(-1))} \xrightarrow{\sim} \frac{O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_{\text{m}}}{\text{im}(\nabla_{\text{m}}^0 \circ d_{1,m}^{-1,2r} \circ d_{1,m}^{0,2r-2}(-1))}.$$

By Lemma 5.8.3(8), $\text{im}(\nabla_{\text{m}}^0 \circ d_{1,m}^{-1,2r} \circ d_{1,m}^{0,2r-2}(-1))$ coincides with the submodule

$$\left(\mathbf{T}_{N,p}^\bullet \circ ((p+1)\mathbf{R}_{N,p}^\bullet - \mathbf{T}_{N,p}^\circ \circ \mathbf{T}_{N,p}^\bullet) \circ \mathbf{T}_{N,p}^\circ \right) \cdot O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_{\text{m}}.$$

Note that, by Lemma B.3.6, we have

$$\mathbf{T}_{N,p}^\circ \circ ((p+1)\mathbf{R}_{N,p}^\bullet - \mathbf{T}_{N,p}^\circ \circ \mathbf{T}_{N,p}^\bullet) \circ \mathbf{T}_{N,p}^\circ = \mathbf{I}_{N,p}^\circ \left((p+1)\mathbf{R}_{N,p}^\circ - \mathbf{I}_{N,p}^\circ \right).$$

Thus, to conclude (4), it remains to show that

$$(6.2) \quad (H_{\mathfrak{X}}^{2r}(\overline{M}_N^\bullet, O_\lambda(r))_{\text{m}})_{\text{tor}} \cap \text{im}(d_{1,m}^{-1,2r} \circ d_{1,m}^{0,2r-2}(-1)) = 0.$$

By Lemma 5.1.7, Hypothesis 3.2.9, and Proposition C.3.1(2), we know that the $\overline{\mathbb{Q}}_\ell[\Gamma_F]$ -module $H_{\mathfrak{X}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r))_{\text{m}} \otimes_{O_\lambda} \overline{\mathbb{Q}}_\lambda$ is isomorphic to a direct sum of $\rho_{\Pi', \iota_\ell}(r)$ for some relevant representations Π' of $\text{GL}_N(\mathbb{A}_F)$. By Proposition 3.2.4(1) and [TY07, Lemma 1.4(3)], we know that $\rho_{\Pi', \iota_\ell}(r)$ is pure of weight -1 at \mathfrak{p} (Definition 2.4.4). In particular, we have $H^1(\mathbb{Q}_{p^2}, \rho_{\Pi', \iota_\ell}(r)) = 0$ by [Nek07, Proposition 4.2.2(1)], hence that both sides of the inclusion

$$F_{-1}H^1(I_{\mathbb{Q}_{p^2}}, H_{\mathfrak{X}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r))_{\text{m}}) \subseteq H_{\text{sing}}^1(\mathbb{Q}_{p^2}, H_{\mathfrak{X}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r))_{\text{m}})$$

are torsion O_λ -modules. Thus, the O_λ -rank of $\text{im}(d_{1,m}^{-1,2r} \circ d_{1,m}^{0,2r-2}(-1))$ is equal to the O_λ -rank of $\ker d_{1,m}^{0,2r}$, which in turn is equal to the sum of O_λ -ranks of $O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_{\text{m}}$ and $O_\lambda[\text{Sh}(V_N^\circ, K_N^\bullet)]_{\text{m}}$. However, the source of the map $d_{1,m}^{-1,2r} \circ d_{1,m}^{0,2r-2}(-1)$, which is $E_{1,m}^{0,2r-2} / \text{im } d_{1,m}^{-1,2r}$, is also a free O_λ -module of the same rank. Therefore, we must have (6.2). Part (4) is proved. \square

Lemma 6.3.2. *Suppose $N \geq 4$. Assume Assumptions 6.1.4, 6.1.5, and Hypothesis 3.2.9 for N . Then $H_{\mathfrak{S}}^{2r-2}(\overline{M}_N^\bullet, O_\lambda)_\mathfrak{m}$ is a free O_λ -module; and its rank over O_λ is at most twice the rank of the (free) O_λ -module $O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_\mathfrak{m}$.*

Proof. By Assumption 6.1.4, Lemma 5.8.3(2), and Lemma 5.5.2(2), we have an injective map

$$H_{\mathfrak{S}}^{2r-2}(\overline{M}_N^\bullet, O_\lambda)_\mathfrak{m} \hookrightarrow H_{\mathfrak{S}}^{2r-2}(\overline{M}_N^\dagger, O_\lambda)_\mathfrak{m}$$

induced by $d_1^{0,2r-2}$. For the target, we have an isomorphism

$$H_{\mathfrak{S}}^{2r-2}(\overline{M}_N^\dagger, O_\lambda)_\mathfrak{m} \simeq O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_\mathfrak{m} \oplus H^{\text{prim}}(\overline{M}_N^\dagger, O_\lambda)_\mathfrak{m}.$$

In particular, $H_{\mathfrak{S}}^{2r-2}(\overline{M}_N^\dagger, O_\lambda)_\mathfrak{m}$ hence $H_{\mathfrak{S}}^{2r-2}(\overline{M}_N^\bullet, O_\lambda)_\mathfrak{m}$ are free O_λ -modules.

Let $\pi^{\infty,p}$ be an irreducible admissible representation of $U(V_N^\circ)(\mathbb{A}_{F^+}^{\infty,p})$ that appears in $H_{\mathfrak{S}}^{2r-2}(\overline{M}_N^\bullet, O_\lambda)_\mathfrak{m} \otimes_{O_\lambda, \iota} \mathbb{C}$. Then, by Proposition 5.5.4, one can complete $\pi^{\infty,p}$ to an automorphic representation $\pi = \pi^{\infty,p} \otimes \pi_\infty \otimes \prod_{\mathfrak{q}|p} \pi_{\mathfrak{q}}$ such that π_∞ is trivial; $\pi_{\mathfrak{q}}$ is unramified for $\mathfrak{q} \neq \mathfrak{p}$; and $\pi_{\mathfrak{p}}$ is a constituent of an unramified principal series. Moreover, π is Π -congruent. By Assumption 6.1.5 and Lemma 6.1.9, we know that π is stable.

To prove the lemma, it suffices to show that for such π as above, we have

$$(6.3) \quad \dim_{\overline{\mathbb{Q}}_\ell} H_{\mathfrak{S}}^{2r-2}(\overline{M}_N^\bullet, \overline{\mathbb{Q}}_\ell)[\iota_\ell \pi^\infty] \leq 2 \dim_{\overline{\mathbb{Q}}_\ell} \overline{\mathbb{Q}}_\ell[\text{Sh}(V_N^\circ, K_N^\circ)][\iota_\ell \pi^\infty].$$

Note that as in the proof of Proposition 5.5.4, we have an isomorphism

$$(6.4) \quad \iota_\ell^{-1} H^{\text{prim}}(\overline{M}_N^\dagger, \overline{\mathbb{Q}}_\ell) \simeq \text{Map}_{K_{N,\mathfrak{p}}^\circ} \left(U(V_N^\circ)(F^+) \backslash U(V_N^\circ)(\mathbb{A}_{F^+}^\infty) / K_N^{p^\circ} \prod_{\mathfrak{q}|p, \mathfrak{q} \neq \mathfrak{p}} K_{N,\mathfrak{q}}^\circ, \Omega_N \right).$$

By Proposition C.3.1(2), we have $\text{BC}(\pi_{\mathfrak{p}}) \simeq \text{BC}(\pi)_{\mathfrak{p}}$. Let $\rho_{\text{BC}(\pi), \iota_\ell} : \Gamma_F \rightarrow \text{GL}_N(\overline{\mathbb{Q}}_\ell)$ be the associated Galois representation. Since π is Π -congruent, by the Chebotarev density theorem, $\rho_{\text{BC}(\pi), \iota_\ell}$ admits a lattice whose residual representation is isomorphic to $\bar{\rho}_{\Pi, \lambda} \otimes_{O_\lambda/\lambda} \overline{\mathbb{F}}_\ell$, which is irreducible by Assumption 6.1.5. Thus, by Proposition 3.2.4(2), $\alpha(\text{BC}(\pi_{\mathfrak{p}}))$ does not contain $\{-1, -1\}$ due to (P5) and contains $\{p, p^{-1}\}$ with multiplicity at most one by (P4). We now have three cases.

Case 1: $\pi_{\mathfrak{p}}$ is unramified. Then (6.3) follows by (6.4) and the fact that the multiplicity of Ω_N in $\pi_{\mathfrak{p}}|_{K_{N,\mathfrak{p}}^\circ}$ is at most 1 by Proposition C.2.1(2).

Case 2: $\pi_{\mathfrak{p}}$ is not unramified and $\pi_{\mathfrak{p}} \notin \mathcal{S}$. Then by Lemma C.2.3(1), $\pi_{\mathfrak{p}}|_{K_{N,\mathfrak{p}}^\circ}$ does not contain Ω_N . Thus, both sides of (6.3) are zero by (6.4).

Case 3: $\pi_{\mathfrak{p}}$ belongs to \mathcal{S} . Then we have $\overline{\mathbb{Q}}_\ell[\text{Sh}(V_N^\circ, K_N^\circ)][\iota_\ell \pi^\infty] = 0$, hence an inclusion

$$(6.5) \quad \iota_\ell^{-1} H_{\mathfrak{S}}^{2r-2}(\overline{M}_N^\bullet, \overline{\mathbb{Q}}_\ell)[\pi^\infty] \hookrightarrow \text{Map}_{K_{N,\mathfrak{p}}^\circ} \left(U(V_N^\circ)(F^+) \backslash U(V_N^\circ)(\mathbb{A}_{F^+}^\infty) / K_N^{p^\circ} \prod_{\mathfrak{q}|p, \mathfrak{q} \neq \mathfrak{p}} K_{N,\mathfrak{q}}^\circ, \Omega_N \right) [\pi^\infty]$$

by (6.4). Note that, by Proposition C.2.1(2), the multiplicity of Ω_N in $\pi_{\mathfrak{p}}|_{K_{N,\mathfrak{p}}^\circ}$ is one, hence we have

$$\text{Map}_{K_{N,\mathfrak{p}}^\circ} \left(U(V_N^\circ)(F^+) \backslash U(V_N^\circ)(\mathbb{A}_{F^+}^\infty) / K_N^{p^\circ} \prod_{\mathfrak{q}|p, \mathfrak{q} \neq \mathfrak{p}} K_{N,\mathfrak{q}}^\circ, \Omega_N \right) [\pi^\infty] \simeq (\pi^{\infty,p})^{K_N^{p^\circ}}$$

by Proposition C.3.1(2).

On the other hand, by Lemma 6.1.10, Proposition C.3.1(2), Corollary C.3.2, and Hypothesis 3.2.9, we know that the $\overline{\mathbb{Q}}_\ell[\Gamma_F]$ -module

$$H_{\text{ét}}^{2r-1}(\text{Sh}(V_N', j_N K_N^{p^\circ} K'_{p,N})_{\overline{F}}, \overline{\mathbb{Q}}_\ell)[\iota_\ell \pi^{\infty,p}]$$

is isomorphic to $\dim(\pi^{\infty,p})^{K_N^{p^\circ}}$ copies of $\rho_{\text{BC}(\pi), \iota_\ell}^c$. By Proposition 3.2.4(2), we know that $\rho_{\text{BC}(\pi), \iota_\ell}^c|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^2})}$ has nontrivial monodromy action. Thus, by Lemma 5.1.7 and the spectral

sequence $E_s^{p,q}$, the cokernel of (6.5) has dimension $\dim(\pi^{\infty,p})^{K_N^{p\circ}}$, which forces the target of (6.5) to vanish. In particular, (6.3) holds.

The lemma is proved. \square

Remark 6.3.3. Following the well-known computation of level raising of Shimura curves (see, for example, Step 5 of the proof of [Liu19, Proposition 3.32]), we know that Proposition 6.3.1(4) also holds when $N = 2$. Moreover, as \overline{M}_2^\bullet is a disjoint union of projective lines, the kernel of the map is trivial hence the map is an isomorphism.

Recall that we have fixed a positive integer m at the beginning of Subsection 6.1, and introduced the ideal \mathfrak{n} in Notation 6.1.3.

Theorem 6.3.4. *Assume Assumptions 6.1.4, 6.1.5, and Hypothesis 3.2.9 for N . Moreover, if $N \geq 4$, we further assume that*

- (a) $\ell \geq 2(N + 1)$ and ℓ is unramified in F ;
- (b) $\bar{\rho}_{\Pi,\lambda,+}$ (Remark 6.1.5) is rigid for $(\Sigma_{\min}^+, \Sigma_{\text{lr}}^+)$ (Definition E.7.1), and $\bar{\rho}_{\Pi,\lambda}|_{\text{Gal}(\overline{F}/F(\zeta_\ell))}$ is absolutely irreducible;
- (c) the composite homomorphism $\mathbb{T}_N^{\Sigma^+} \xrightarrow{\phi_\Pi} O_E \rightarrow O_E/\lambda$ is cohomologically generic (Definition D.1.1); and
- (d) $O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_{\mathfrak{m}}$ is nontrivial.

Then we have

- (1) $H_{\mathfrak{I}}^i(\overline{M}_N^\bullet, O_\lambda)_{\mathfrak{m}}$ is a free O_λ -module for every $i \in \mathbb{Z}$.
- (2) $E_{2,\mathfrak{m}}^{p,q}$ is a free O_λ -module, and vanishes if $(p, q) \notin \{(-1, 2r), (0, 2r - 1), (1, 2r - 2)\}$.
- (3) If we denote by $\{\alpha_1^{\pm 1}, \dots, \alpha_r^{\pm 1}\}$ the roots of $P_{\alpha(\Pi_p)} \bmod \lambda$ in a finite extension of O_λ/λ , then the generalized Frobenius eigenvalues of the $O_\lambda/\lambda[\text{Gal}(\overline{F}_p/\mathbb{F}_{p^2})]$ -module $H_{\mathfrak{I}}^{2r-1}(\overline{M}_N^\bullet, O_\lambda(r))_{\mathfrak{m}} \otimes_{O_\lambda} O_\lambda/\lambda$ is contained in $\{p\alpha_1^{\pm 1}, \dots, p\alpha_r^{\pm 1}\} \setminus \{1, p^2\}$.
- (4) The map in Proposition 6.3.1(4) (see Remark 6.3.3 for $N = 2$) factors through a map

$$\nabla_{/\mathfrak{n}}^0: F_{-1}H^1(I_{\mathbb{Q}_{p^2}}, H_{\mathfrak{I}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r)))/\mathfrak{n} \rightarrow O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]/\mathfrak{n}$$

which is an isomorphism. The map from Lemma 5.8.3(6) induces a canonical isomorphism

$$F_{-1}H^1(I_{\mathbb{Q}_{p^2}}, H_{\mathfrak{I}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r)))/\mathfrak{n} \rightarrow H_{\text{sing}}^1(\mathbb{Q}_{p^2}, H_{\mathfrak{I}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r)))/\mathfrak{n}.$$

- (5) There exist finitely many positive integers m_1, \dots, m_μ at most m such that we have an isomorphism

$$H_{\text{ét}}^{2r-1}(\text{Sh}(V_N', j_N K_N^{p\circ} K'_{p,N})_{\overline{F}}, O_\lambda(r))/\mathfrak{n} \simeq \bigoplus_{i=1}^{\mu} \overline{R}^{(m_i)c}$$

of $O_\lambda[\Gamma_F]$ -modules, where R is the Γ_F -stable O_λ -lattice in $\rho_{\Pi,\lambda}(r)$, unique up to homothety.

Remark 6.3.5. In fact, from the proof, one sees that when $N \geq 4$, we can take $m_1 = \dots = m_\mu = m$ due to our strong extra assumptions.

6.4. Proof of Theorem 6.3.4. The proof differs according to $N = 2$ or $N \geq 4$ as we can see from the extra assumptions. We start from the much more difficult case where $N \geq 4$.

We apply the discussion of Section E to the pair (\bar{r}, χ) as in Subsection E.7, where

$$\bar{r} := \bar{\rho}_{\Pi,\lambda,+}: \Gamma_{F^+} \rightarrow \mathcal{G}_N(O_\lambda/\lambda),$$

and the similitude character $\chi = \eta_{F/F^+}^\mu \epsilon_\ell^{1-N}: \Gamma_{F^+} \rightarrow O_\lambda^\times$ for some $\mu \in \mathbb{Z}/2\mathbb{Z}$.¹⁷ Then \bar{r} is rigid for $(\Sigma_{\min}^+, \Sigma_{\text{lr}}^+)$, and also for $(\Sigma_{\min}^+, \Sigma_{\text{lr}}^+ \cup \{\mathfrak{p}\})$ by (P4).

¹⁷In fact, it will follow from Theorem E.7.3(3) that $\mu = 0$; but we do not need this fact a priori.

For $? = \text{mix}, \text{unr}, \text{ram}$, consider a global deformation problem (Definition E.2.6)

$$\mathcal{S}^? := (\bar{r}, \eta_{F/F^+}^\mu \epsilon_\ell^{1-N}, \Sigma_{\min}^+ \cup \Sigma_{\text{lr}}^+ \cup \{\mathfrak{p}\} \cup \Sigma_\ell^+, \{\mathcal{D}_v\}_{v \in \Sigma_{\min}^+ \cup \Sigma_{\text{lr}}^+ \cup \{\mathfrak{p}\} \cup \Sigma_\ell^+})$$

where

- for $v \in \Sigma_{\min}^+$, \mathcal{D}_v is the local deformation problem classifying all liftings of \bar{r}_v ;
- for $v \in \Sigma_{\text{lr}}^+$, \mathcal{D}_v is the local deformation problem $\mathcal{D}_v^{\text{ram}}$ of \bar{r}_v from Definition E.6.1;
- for $v = \mathfrak{p}$, \mathcal{D}_v is the local deformation problem $\mathcal{D}_v^?$ of \bar{r}_v from Definition E.6.1;
- for $v \in \Sigma_\ell^+$, \mathcal{D}_v is the local deformation problem $\mathcal{D}_v^{\text{FL}}$ of \bar{r}_v from Definition E.3.6.

Then we have the *global universal deformation ring* $R_{\mathcal{S}^?}^{\text{univ}}$ from Proposition E.2.7. Put $R^? := R_{\mathcal{S}^?}^{\text{univ}}$ for short. Then we have canonical surjective homomorphisms $R^{\text{mix}} \rightarrow R^{\text{unr}}$ and $R^{\text{mix}} \rightarrow R^{\text{ram}}$ of O_λ -algebras. Finally, put

$$R^{\text{cong}} := R^{\text{unr}} \otimes_{R^{\text{mix}}} R^{\text{ram}}.$$

We fix a universal lifting

$$r_{\text{mix}}: \Gamma_{F^+} \rightarrow \mathcal{G}_N(R^{\text{mix}})$$

of \bar{r} , which induce a continuous homomorphism

$$r_{\text{mix}}^\natural: \Gamma_F \rightarrow \text{GL}_N(R^{\text{mix}})$$

by restriction (Notation E.1.2). By pushforward, R^{cong} also induces homomorphisms

$$r_{\text{unr}}: \Gamma_{F^+} \rightarrow \mathcal{G}_N(R^{\text{unr}}), \quad r_{\text{ram}}: \Gamma_{F^+} \rightarrow \mathcal{G}_N(R^{\text{ram}}).$$

Denote by $P_{F_p^+}$ the maximal closed subgroup of the inertia subgroup $I_{F_p^+} \subseteq \Gamma_{F_p^+}$ of pro-order coprime to ℓ , as in Subsection E.5. Then $\Gamma_{F_p^+}/P_{F_p^+} \simeq t^{\mathbb{Z}_\ell} \times \phi_p^{\widehat{\mathbb{Z}}}$ is a p -tame group (Definition E.4.1). By definition, the homomorphism r_{mix}^\natural is trivial on $P_{F_p^+}$. Let \bar{v} and \bar{v}' be eigenvectors in $(O_\lambda/\lambda)^{\oplus N}$ for $r_{\text{mix}}^\natural(\phi_p^2)$ with eigenvalues p^{-2r} and p^{-2r+2} , respectively. By Hensel's lemma, \bar{v} and \bar{v}' lift to eigenvectors v and v' in $(R^{\text{mix}})^{\oplus N}$ for $r_{\text{mix}}^\natural(\phi_p^2)$, with eigenvalues s and s' in R^{mix} lifting p^{-2r} and p^{-2r+2} , respectively. Let $x \in R^{\text{mix}}$ be the unique element such that $r_{\text{mix}}^\natural(t)v' = xv + v'$. Then we must have $x(s - p^{-2r}) = 0$. By Definition E.6.1, we have

$$R^{\text{unr}} = R^{\text{mix}}/(x), \quad R^{\text{ram}} = R^{\text{mix}}/(s - p^{-2r}), \quad R^{\text{cong}} = R^{\text{mix}}/(s - p^{-2r}, x).$$

Let T^{unr} be the image of $T_N^{\Sigma^+}$ in $\text{End}_{O_\lambda}(O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)])$. By (d) in Theorem 6.3.4, we know that $T_m^{\text{unr}} \neq 0$. Thus by Proposition E.7.3, we have a canonical isomorphism $R^{\text{unr}} \xrightarrow{\sim} T_m^{\text{unr}}$ so that $O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_m$ is canonically a free R^{unr} -module of rank $d_{\text{unr}} > 0$. We may write the eigenvalues of $r_{\text{unr}}^\natural(\phi_p^2)$ by $\{p^{-2r+1}\alpha_1^{\pm 1}, \dots, p^{-2r+1}\alpha_{r-1}^{\pm 1}, s, s' = p^{-4r+2}s^{-1}\}$ with $\alpha_1, \dots, \alpha_{r-1}$ in certain finite flat extension of R^{unr} that are not congruent to p or p^{-1} in O_λ/λ . By Proposition B.3.5(2), we have

$$((p+1)R_{N,p}^\circ - I_{N,p}^\circ).O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_m = (s - p^{-2r}).O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_m.$$

In particular, we have

$$O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_m / ((p+1)R_{N,p}^\circ - I_{N,p}^\circ) = O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_m \otimes_{R^{\text{unr}}} R^{\text{cong}},$$

which is a free R^{cong} -module of rank d_{unr} .

On the other hand, let T^{ram} be the image of $T_N^{\Sigma^+ \cap \Sigma_p^+}$ in $\text{End}_{O_\lambda}(H_{\mathbb{F}_\ell}^{2r-1}(\overline{M}_N, R\Psi O_\lambda))$. By Proposition 6.3.1(4) and Lemma 5.8.3(6), we know that $T_m^{\text{ram}} \neq 0$. Thus by Lemma 5.1.7 and Theorem E.7.3 (with $(\Sigma_{\min}^+, \Sigma_{\text{lr}}^+, \Sigma^+)$ replaced by $(\Sigma_{\min}^+, \Sigma_{\text{lr}}^+ \cup \{\mathfrak{p}\}, \Sigma^+ \cup \Sigma_p^+)$), we have a canonical isomorphism $R^{\text{ram}} \xrightarrow{\sim} T_m^{\text{ram}}$ so that $H_{\mathbb{F}_\ell}^{2r-1}(\overline{M}_N, R\Psi O_\lambda)_m$ is canonically a free R^{ram} -module. Define the R^{ram} -module

$$H := \text{Hom}_{\Gamma_F} \left((R^{\text{ram}})^{\oplus N}, H_{\mathbb{F}_\ell}^{2r-1}(\overline{M}_N, R\Psi O_\lambda)_m \right)$$

where Γ_F acts on $(\mathbf{R}^{\text{ram}})^{\oplus N}$ via the homomorphism $r_{\text{ram}}^{\natural, \mathbf{c}}$. By the same argument for [Sch18, Theorem 5.6] (using Proposition C.3.1 and Hypothesis 3.2.9, here), we have a canonical isomorphism

$$H_{\mathbb{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda)_m \simeq H \otimes_{\mathbf{R}^{\text{ram}}} (\mathbf{R}^{\text{ram}})^{\oplus N}$$

of $\mathbf{R}^{\text{ram}}[\Gamma_F]$ -modules. Since \mathbf{R}^{ram} is a local ring, H is a free \mathbf{R}^{ram} -module, say of rank d_{ram} . If we still denote by \mathbf{v} and \mathbf{v}' for their projection in $(\mathbf{R}^{\text{ram}})^{\oplus N}$, then it is easy to see that

$$H_{\text{sing}}^1(\mathbb{Q}_{p^2}, (\mathbf{R}^{\text{ram}})^{\oplus N}(r)) = \mathbf{R}^{\text{ram}}_{\mathbf{v}/\mathbf{v}'} \simeq \mathbf{R}^{\text{ram}}/(\mathbf{x}) = \mathbf{R}^{\text{cong}}.$$

Thus, we obtain

$$H_{\text{sing}}^1(\mathbb{Q}_{p^2}, H_{\mathbb{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r))_m) \simeq H \otimes_{\mathbf{R}^{\text{ram}}} H_{\text{sing}}^1(\mathbb{Q}_{p^2}, (\mathbf{R}^{\text{ram}})^{\oplus N}(r)) \simeq H \otimes_{\mathbf{R}^{\text{ram}}} \mathbf{R}^{\text{cong}},$$

which is a free \mathbf{R}^{cong} -module of rank $d_{\text{ram}} > 0$.

Proposition 6.4.1. *Under the assumptions of Theorem 6.3.4, we have $d_{\text{unr}} = d_{\text{ram}}$. In particular, the two canonical maps*

$$\begin{aligned} F_{-1}H^1(I_{\mathbb{Q}_{p^2}}, H_{\mathbb{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r))_m) &\rightarrow O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_m / ((p+1)\mathbf{R}_{N,p}^\circ - \mathbf{I}_{N,p}^\circ), \\ F_{-1}H^1(I_{\mathbb{Q}_{p^2}}, H_{\mathbb{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r))_m) &\rightarrow H_{\text{sing}}^1(\mathbb{Q}_{p^2}, H_{\mathbb{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r))_m), \end{aligned}$$

from Proposition 6.3.1(4) and Lemma 5.8.3(6), respectively, are both isomorphisms.

Proof. By Proposition 6.3.1(4), the first map is surjective. By Lemma 5.8.3(6), the second map is injective. Thus, we must have $d_{\text{ram}} \geq d_{\text{unr}} > 0$ by the previous discussion.

Take a geometric point $\eta_1 \in (\text{Spec } \mathbf{R}^{\text{unr}})(\overline{\mathbb{Q}}_\ell)$ in the support of $O_\lambda[\text{Sh}(V_N^\circ, K_N^\circ)]_m$, which corresponds to a relevant representation Π_1 of $\text{GL}_N(\mathbb{A}_F)$ such that $\rho_{\Pi_1, \ell}$ is residually isomorphic to $\bar{\rho}_{\Pi, \lambda} \otimes_{O_{\lambda/\lambda}} \overline{\mathbb{F}}_\ell$. Then we have

$$d_{\text{unr}} = \dim \overline{\mathbb{Q}}_\ell[\text{Sh}(V_N^\circ, K_N^\circ)][\iota_\ell \phi_{\Pi_1}].$$

Take a geometric point $\eta_2 \in (\text{Spec } \mathbf{R}^{\text{ram}})(\overline{\mathbb{Q}}_\ell)$ in the support of $H_{\mathbb{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda)_m$, which corresponds to a relevant representation Π_2 of $\text{GL}_N(\mathbb{A}_F)$ such that $\rho_{\Pi_2, \ell}$ is residually isomorphic to $\bar{\rho}_{\Pi, \lambda} \otimes_{O_{\lambda/\lambda}} \overline{\mathbb{F}}_\ell$. Then we have

$$d_{\text{ram}} = \dim H_{\mathbb{F}}^{2r-1}(\overline{M}_N, R\Psi \overline{\mathbb{Q}}_\ell)[\iota_\ell \phi_{\Pi_2}] = \dim H_{\text{ét}}^{2r-1}(\text{Sh}(V'_N, j_N K_N^{p\circ} K'_{p,N})_{\overline{F}}, \overline{\mathbb{Q}}_\ell)[\iota_\ell \phi_{\Pi_2}]$$

by Lemma 5.1.7. By Proposition D.2.3 and Lemma 6.4.2 below, we have $d_{\text{unr}} = d_{\text{ram}}$. The proposition follows. \square

Lemma 6.4.2. *Let Π_1 and Π_2 be two relevant representations of $\text{GL}_N(\mathbb{A}_F)$ such that the associated Galois representations $\rho_{\Pi_1, \ell}$ and $\rho_{\Pi_2, \ell}$ are both residually isomorphic to $\bar{\rho}_{\Pi, \lambda} \otimes_{O_{\lambda/\lambda}} \overline{\mathbb{F}}_\ell$. For every $v \in \Sigma_{\text{min}}^+$ (so that every lifting of $\bar{\rho}_{\Pi, \lambda, +, v}$ is minimally ramified), if we realize $\Pi_{1, v}$ and $\Pi_{2, v}$ on vector spaces V_1 and V_2 , respectively, then there exist normalized intertwining operators $A_{\Pi_{1, v}}$ and $A_{\Pi_{2, v}}$ for $\Pi_{1, v}$ and $\Pi_{2, v}$ [Shi11, Section 4.1], respectively, such that we have an $\text{GL}_N(O_{F_v})$ -equivariant isomorphism $i: V_1 \xrightarrow{\sim} V_2$ satisfying $i \circ A_{\Pi_{1, v}} = A_{\Pi_{2, v}} \circ i$.*

Proof. We will give the proof when v does not split in F , and leave the other similar case to the readers. Let w be the unique place of F above v .

By Proposition 3.2.4(1), both $\Pi_{1, w}$ and $\Pi_{2, w}$ are tempered. Thus by the Bernstein–Zelevinsky classification, for $\alpha = 1, 2$, we can write

$$\Pi_{\alpha, w} = I_{P_\alpha}^{\text{GL}_N(F_w)} (\sigma_{\alpha, -t_\alpha} \boxtimes \cdots \boxtimes \sigma_{\alpha, -1} \boxtimes \sigma_{\alpha, 0} \boxtimes \sigma_{\alpha, 1} \boxtimes \cdots \boxtimes \sigma_{\alpha, t_\alpha})$$

for some integer $t_\alpha \geq 0$, some standard parabolic subgroup $P_\alpha \subseteq \text{GL}_N(F_w)$, and some (unitary) discrete series representations $\{\sigma_{\alpha, -t_\alpha}, \dots, \sigma_{\alpha, t_\alpha}\}$ satisfying $\sigma_{\alpha, -i} \simeq \sigma_{\alpha, i}^{\vee, \mathbf{c}}$. See Subsection C.1 for the notation on parabolic induction.

By Proposition E.5.11(3) and [BLGGT14, Lemma 1.3.4(2)], we know that $\rho_{\Pi_1, \ell}|_{\mathbb{I}_{F_w}}$ and $\rho_{\Pi_2, \ell}|_{\mathbb{I}_{F_w}}$ are conjugate. Thus, by [Yao, Lemma 3.6], we have $P_1 = P_2$ (say P) and $t_1 = t_2$ (say t), and we assume that there are unramified (unitary) characters $\{\chi_{-t}, \dots, \chi_t\}$ of F_w^\times satisfying $\chi_{-i} \simeq \chi_i^{-1}$ such that $\sigma_{2,i} = \sigma_{1,i} \otimes \chi_i$. For every i , we choose a vector space W_i on which $\sigma_{1,i}$ realizes (and also realize $\sigma_{1,i}^{\vee c}$ on W_i via $g \mapsto {}^t g^{-1, c}$), and fix a linear map $A_i: W_i \rightarrow W_{-i}$ intertwining σ_i and $\sigma_{-i}^{\vee c}$ satisfying $A_{-i} \circ A_i = \text{id}_{W_i}$. Put $\sigma := \boxtimes_{i=-t}^t \sigma_{1,i}$ regarded as a representation of P by inflation, which realizes on the space $W := \bigotimes_{i=-t}^t W_i$; and put $A_\sigma := \bigotimes_{i=-t}^t A_i \in \text{End}(W)$. Choose an element $w \in \text{GL}_N(F_w)$ satisfying $w = {}^t w^c$, that $wPw^{-1} \cap P$ is the standard Levi subgroup of P , and that for $(a_{-t}, \dots, a_t) \in wPw^{-1} \cap P$, we have $w(a_{-t}, \dots, a_t)w^{-1} = (a_t, \dots, a_{-t})$.

We realize $\Pi_{1,w}$ on the space

$$V_1 := \{f: \text{GL}_n(F_w) \rightarrow W \mid f(pg) = \delta_P^{1/2}(p)\sigma(p)f(g), p \in P, g \in \text{GL}_n(F_w)\}.$$

Define a linear map $A_{\Pi_{1,w}}: V_1 \rightarrow V_1$ by the formula

$$\left(A_{\Pi_{1,w}}(f)\right)(g) = A_\sigma\left(f(w {}^t g^{-1, c})\right).$$

Then it is clear that $A_{\Pi_{1,w}}$ is an intertwining operator for $\Pi_{1,w}$ satisfying $A_{\Pi_{1,w}}^2 = 1$. Similarly, we realize $\Pi_{2,w}$ on the space

$$V_2 := \{f: \text{GL}_n(F_w) \rightarrow W \mid f(pg) = \delta_P^{1/2}(p)\chi(p)\sigma(p)f(g), p \in P, g \in \text{GL}_n(F_w)\},$$

where we put $\chi := \boxtimes_{i=-t}^t \chi_i$ regarded as a character of P . We define $A_{\Pi_{2,w}}: V_2 \rightarrow V_2$ by the same formula, which is a normalized intertwining operator for $\Pi_{2,w}$. The desired isomorphism i is the map sending $f \in V_1$ to the unique function $i(f)$ such that $i(f)(g) = f(g)$ for $g \in \text{GL}_N(O_{F_w})$. The lemma is proved. \square

Now we can prove Theorem 6.3.4 when $N \geq 4$.

Proof of Theorem 6.3.4 when $N \geq 4$. For (1), Assumption 6.1.4, Lemma 5.5.2, and the spectral sequence in Lemma 5.8.3 imply that $H_{\mathbb{F}}^i(\overline{M}_N^\bullet, O_\lambda)_\mathfrak{m}$ is O_λ -torsion free for $i \neq 2r - 1, 2r$. By Proposition 6.3.1(4) and Proposition 6.4.1, we know that $H_{\mathbb{F}}^{2r}(\overline{M}_N^\bullet, O_\lambda)_\mathfrak{m}$ is O_λ -torsion free. It remains to show that $H_{\mathbb{F}}^{2r-1}(\overline{M}_N^\bullet, O_\lambda)_\mathfrak{m}$ is O_λ -torsion free as well.

By definition, the universal homomorphism $r_{\text{ram}}^\natural: \Gamma_F \rightarrow \text{GL}_N(\mathbb{R}^{\text{ram}})$ has a direct sum decomposition $(\mathbb{R}^{\text{ram}})^{\oplus N} = \mathbb{R}_1 \oplus \mathbb{R}_2$ in which \mathbb{R}_1 is a free \mathbb{R}^{ram} -submodule of rank $N - 2$, and \mathbb{R}_2 is the free \mathbb{R}^{ram} -submodule of rank 2 generated by (the image of) \mathbf{v} and \mathbf{v}' . We have

$$F_{-1}H_{\mathbb{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda)_\mathfrak{m} \subseteq H \otimes_{\mathbb{R}^{\text{ram}}} \mathbb{R}^{\text{ram}}\mathbf{v}.$$

On the other hand, we have

$$H_{\mathbb{F}}^{2r-1}(\overline{M}_N^\bullet, O_\lambda)_\mathfrak{m} = \frac{F_0 H_{\mathbb{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda)_\mathfrak{m}}{F_{-1} H_{\mathbb{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda)_\mathfrak{m}},$$

and that the quotient $\frac{F_1 H_{\mathbb{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda)_\mathfrak{m}}{F_0 H_{\mathbb{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda)_\mathfrak{m}}$ is torsion free by Lemma 5.5.2. Thus, the O_λ -torsion of $H_{\mathbb{F}}^{2r-1}(\overline{M}_N^\bullet, O_\lambda)_\mathfrak{m}$ coincides with

$$H \otimes_{\mathbb{R}^{\text{ram}}} \mathbb{R}^{\text{ram}}\mathbf{v} / F_{-1} H_{\mathbb{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda)_\mathfrak{m},$$

which is fixed by $\text{Gal}(\overline{\mathbb{F}}_p / \mathbb{F}_{p^2})$. However, by Lemma 5.8.3(6) and Proposition 6.4.1, the O_λ -torsion of $H_{\mathbb{F}}^{2r-1}(\overline{M}_N^\bullet, O_\lambda)_\mathfrak{m}$ has to vanish. Part (1) is proved.

Part (2) is an immediate consequence of (1), Assumption 6.1.4, Lemma 5.5.2, and the spectral sequence in Lemma 5.8.3.

Part (3) is a consequence of (1) and (P4) that $P_{\alpha(\Pi_p)} \bmod \lambda^m$ is level-raising special at \mathfrak{p} . In fact, we have an isomorphism

$$H_{\mathfrak{F}}^{2r-1}(\overline{M}_N^\bullet, O_\lambda(r)) \simeq H \otimes_{\mathbf{R}^{\text{ram}}} \mathbf{R}_1(r)$$

of $O_\lambda[\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})]$ -modules.

For (4), by Proposition 6.4.1, it suffices to show that the two natural maps

$$\begin{aligned} F_{-1}H^1(I_{\mathbb{Q}_{p^2}}, H_{\mathfrak{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r))_{\mathfrak{m}})/\mathfrak{n} &\rightarrow F_{-1}H^1(I_{\mathbb{Q}_{p^2}}, H_{\mathfrak{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r))/\mathfrak{n}), \\ H_{\text{sing}}^1(\mathbb{Q}_{p^2}, H_{\mathfrak{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r))_{\mathfrak{m}})/\mathfrak{n} &\rightarrow H_{\text{sing}}^1(\mathbb{Q}_{p^2}, H_{\mathfrak{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r))/\mathfrak{n}), \end{aligned}$$

are both isomorphisms. Note that we have a short exact sequence

$$0 \rightarrow F_{-1}H^1(I_{\mathbb{Q}_{p^2}}, H_{\mathfrak{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r))_{\mathfrak{m}}) \rightarrow H^1(I_{\mathbb{Q}_{p^2}}, H_{\mathfrak{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r))_{\mathfrak{m}}) \rightarrow \frac{H_{\mathfrak{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r))_{\mathfrak{m}}}{F_{-1}H_{\mathfrak{F}}^{2r-1}(\overline{M}_N, R\Psi O_\lambda(r))_{\mathfrak{m}}} \rightarrow 0$$

of $\mathbb{T}_N^{\Sigma^+ \cup \Sigma_p^+}$ -modules, which is split by considering $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$ actions and (3). Thus, the first isomorphism is confirmed. The second one is also confirmed as, by (3), one can replace $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$ -invariants by $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$ -coinvariants. Part (4) is proved.

For (5), we have

$$H_{\text{ét}}^{2r-1}(\text{Sh}(V'_N, j_N K_N^{p\circ} K'_{p,N})_{\overline{F}}, O_\lambda(r))/\mathfrak{n} \simeq H \otimes_{\mathbf{R}^{\text{ram}}/\mathfrak{n}} (\mathbf{R}^{\text{ram}}/\mathfrak{n})^{\oplus N}(r)$$

by Lemma 5.1.7. Here, we regard \mathfrak{n} as its image in $\mathbb{T}_m^{\text{ram}}$, where the latter is canonically isomorphic to \mathbf{R}^{ram} . We claim that $O_\lambda/\lambda^m = \mathbf{R}^{\text{ram}}/\mathfrak{n}$ and $(\mathbf{R}^{\text{ram}}/\mathfrak{n})^{\oplus N}(r) \simeq \overline{R}^{(m)c}$ as $O_\lambda/\lambda^m[\Gamma_F]$ -modules, where we recall that Γ_F acts on $(\mathbf{R}^{\text{ram}}/\mathfrak{n})^{\oplus N}$ via $r_{\text{ram}}^{\mathfrak{h},c}$. By definition, \mathfrak{n} is the kernel of homomorphism

$$\mathbb{T}_N^{\Sigma^+ \cup \Sigma_p^+} \xrightarrow{\phi_\Pi} O_E \rightarrow O_E/\lambda^m,$$

which satisfies $\mathfrak{n} \cap O_\lambda = \lambda^m O_\lambda$. Thus, the structure homomorphism $O_\lambda \rightarrow \mathbf{R}^{\text{ram}}$ induces an equality $O_\lambda/\lambda^m = \mathbf{R}^{\text{ram}}/\mathfrak{n}$. Now by the Chebotarev density theorem, and a result of Mazur and Carayol (see [Kis09, Theorem 1.4.1]), we know that the two liftings $(\mathbf{R}^{\text{ram}}/\mathfrak{n})^{\oplus N}(r)$ and $\overline{R}^{(m)c}$ of $\overline{\rho}_{\Pi,\lambda}^c(r)$ to O_λ/λ^m have to be isomorphic.

Theorem 6.3.4 is all proved when $N \geq 4$. \square

Proof of Theorem 6.3.4 when $N = 2$. Part (1) is trivial since \overline{M}_2^\bullet is a disjoint union of projective lines.

Part (2) follows from (1) by the same reason as for $N \geq 4$.

Part (3) is trivial as $H_{\mathfrak{F}}^1(\overline{M}_2^\bullet, O_\lambda(1)) = 0$.

For (4), from Remark 6.3.3, we know that the natural map

$$F_{-1}H^1(I_{\mathbb{Q}_{p^2}}, H_{\mathfrak{F}}^1(\overline{M}_2, R\Psi O_\lambda(1))_{\mathfrak{m}}) \rightarrow O_\lambda[\text{Sh}(V_2^\circ, K_2^\circ)]_{\mathfrak{m}}/((p+1)\mathbf{R}_{2,p}^\circ - I_{2,p}^\circ)$$

is an isomorphism. By (3) and Lemma 5.8.3(6), the natural map

$$F_{-1}H^1(I_{\mathbb{Q}_{p^2}}, H_{\mathfrak{F}}^1(\overline{M}_2, R\Psi O_\lambda(1))_{\mathfrak{m}}) \rightarrow H_{\text{sing}}^1(\mathbb{Q}_{p^2}, H_{\mathfrak{F}}^1(\overline{M}_2, R\Psi O_\lambda(1))_{\mathfrak{m}})$$

is an isomorphism as well. Passing to the quotient by \mathfrak{n} follows from the same argument as for $N \geq 4$.

For (5), let \mathbb{T}^{ram} be the image of $\mathbb{T}_2^{\Sigma^+ \cap \Sigma_p^+}$ in $\text{End}_{O_\lambda}(H_{\mathfrak{F}}^1(\overline{M}_2, R\Psi O_\lambda(1)))$. Then by the same argument for [Sch18, Theorem 5.6], one have an isomorphism

$$H_{\mathfrak{F}}^1(\overline{M}_2, R\Psi O_\lambda(1))_{\mathfrak{m}} \simeq H \otimes_{\mathbb{T}_m^{\text{ram}}} (\mathbb{T}_m^{\text{ram}})^{\oplus 2}$$

of $\mathbb{T}_m^{\text{ram}}[\Gamma_F]$ -modules, where H is a (finitely generated) $\mathbb{T}_m^{\text{ram}}$ -module, and Γ_F acts on the factor $(\mathbb{T}_m^{\text{ram}})^{\oplus 2}$ by some continuous homomorphism which lifts $\overline{\rho}_{\Pi,\lambda}^c(r)$ (from $O_\lambda/\lambda = \mathbb{T}_m^{\text{ram}}/\mathfrak{m}$ to $\mathbb{T}_m^{\text{ram}}$).

Clearly, the natural homomorphism $O_\lambda/\lambda^m \rightarrow \mathbb{T}_m^{\text{ram}}/\mathfrak{n}$ is an isomorphism. Then as an O_λ/λ^m -module, $\mathbb{H} \otimes_{\mathbb{T}_m^{\text{ram}}} \mathbb{T}_m^{\text{ram}}/\mathfrak{n}$ is isomorphic to $\bigoplus_{i=1}^\mu O_\lambda/\lambda^{m_i}$ for some positive integers m_1, \dots, m_μ at most m . Thus, it remains to show that $(\mathbb{T}_m^{\text{ram}}/\mathfrak{n})^{\oplus 2}$ and $\bar{R}^{(m)c}$ are isomorphic as deformations of $\bar{\rho}_{\Pi, \lambda}^c(r)$. But this is a consequence of the Eichler–Shimura relation for the unitary Shimura curve $\text{Sh}(V'_2, j_2 K_2^{p^\circ} K'_{p,2})$ [Liu, Corollary D.9], the Chebotarev density theorem, and a result of Mazur and Carayol (see [Kis09, Theorem 1.4.1]).

Theorem 6.3.4 is all proved when $N = 2$. □

7. EXPLICIT RECIPROCITY LAWS FOR RANKIN–SELBERG MOTIVES

In this section, we state and prove the two explicit reciprocity laws for automorphic Rankin–Selberg motives. In Subsection 7.1, we setup the stage for automorphic Rankin–Selberg motives, which will be used until the end of the next section. In Subsections 7.2 and 7.3, we state and prove our first and second explicit reciprocity law, respectively.

7.1. Setup for automorphic Rankin–Selberg motives. Let $n \geq 2$ be an integer. We denote by n_0 and n_1 the unique even and odd numbers in $\{n, n + 1\}$, respectively. Write $n_0 = 2r_0$ and $n_1 = 2r_1 + 1$ for unique integers $r_0, r_1 \geq 1$. In particular, we have $n = r_0 + r_1$.

In this and the next sections, we consider

- for $\alpha = 0, 1$, a relevant representation Π_α of $\text{GL}_{n_\alpha}(\mathbb{A}_F)$ (Definition 1.1.3),
- a strong coefficient field $E \subseteq \mathbb{C}$ of both Π_0 and Π_1 (Definition 3.2.5).

Put $\Sigma_{\min}^+ := \Sigma_{\Pi_0}^+ \cup \Sigma_{\Pi_1}^+$ (Notation 3.1.4). We then have the homomorphism

$$\phi_{\Pi_\alpha} : \mathbb{T}_{n_\alpha}^{\Sigma_{\min}^+} \rightarrow O_E$$

for $\alpha = 0, 1$. For $\alpha = 0, 1$ and every prime λ of E , we have a continuous homomorphism

$$\rho_{\Pi_\alpha, \lambda} : \Gamma_F \rightarrow \text{GL}_{n_\alpha}(E_\lambda)$$

from Proposition 3.2.4(2) and Definition 3.2.5, such that $\rho_{\Pi_\alpha, \lambda}^c$ and $\rho_{\Pi_\alpha, \lambda}^\vee(1 - n_\alpha)$ are conjugate.

Assumption 7.1.1. For $\alpha = 0, 1$, the Galois representation $\rho_{\Pi_\alpha, \lambda}$ is residually absolutely irreducible.

7.2. First explicit reciprocity law. We start by choosing

- a finite place λ of E (with the underlying rational prime ℓ),
- a positive integer m ,
- a (possibly empty) finite set $\Sigma_{\text{r}, \text{I}}^+$ ¹⁸ of nonarchimedean places of F^+ that are inert in F , strongly disjoint from Σ_{\min}^+ (Definition 1.3.3), satisfying $\ell \nmid \|v\|(\|v\|^2 - 1)$ for $v \in \Sigma_{\text{r}, \text{I}}^+$,
- a finite set Σ_{I}^+ of nonarchimedean places of F^+ containing $\Sigma_{\min}^+ \cup \Sigma_{\text{r}, \text{I}}^+$,
- a standard definite hermitian space V_n° of rank n over F , together with a self-dual $\prod_{v \notin \Sigma_\infty^+ \cup \Sigma_{\min}^+ \cup \Sigma_{\text{r}, \text{I}}^+} O_{F_v}$ -lattice Λ_n° in $V_n^\circ \otimes_F \mathbb{A}_F^{\Sigma_\infty^+ \cup \Sigma_{\min}^+ \cup \Sigma_{\text{r}, \text{I}}^+}$ (and put $V_{n+1}^\circ := (V_n^\circ)_\#$ and $\Lambda_{n+1}^\circ := (\Lambda_n^\circ)_\#$), satisfying that $(V_{n_0}^\circ)_v$ is not split for $v \in \Sigma_{\text{r}, \text{I}}^+$,

¹⁸Here, the subscript “I” (Roman number one) stands for the “first”. In the next subsection, we will have $\Sigma_{\text{r}, \text{II}}^+$ for the second reciprocity law.

○ an object $K_n^\circ \in \mathfrak{K}(V_n^\circ)$ and an object $(K_{\text{sp}}^\circ, K_{n+1}^\circ) \in \mathfrak{K}(V_n^\circ)_{\text{sp}}$ of the forms

$$\begin{aligned} K_n^\circ &= \prod_{v \in \Sigma_{\min}^+ \cup \Sigma_{\text{lr}, \mathbb{I}}^+} (K_n^\circ)_v \times \prod_{v \notin \Sigma_\infty^+ \cup \Sigma_{\min}^+ \cup \Sigma_{\text{lr}, \mathbb{I}}^+} \text{U}(\Lambda_n^\circ)(O_{F_v^+}), \\ K_{\text{sp}}^\circ &= \prod_{v \in \Sigma_{\min}^+ \cup \Sigma_{\text{lr}, \mathbb{I}}^+} (K_{\text{sp}}^\circ)_v \times \prod_{v \notin \Sigma_\infty^+ \cup \Sigma_{\min}^+ \cup \Sigma_{\text{lr}, \mathbb{I}}^+} \text{U}(\Lambda_n^\circ)(O_{F_v^+}), \\ K_{n+1}^\circ &= \prod_{v \in \Sigma_{\min}^+ \cup \Sigma_{\text{lr}, \mathbb{I}}^+} (K_{n+1}^\circ)_v \times \prod_{v \notin \Sigma_\infty^+ \cup \Sigma_{\min}^+ \cup \Sigma_{\text{lr}, \mathbb{I}}^+} \text{U}(\Lambda_{n+1}^\circ)(O_{F_v^+}), \end{aligned}$$

satisfying $(K_{\text{sp}}^\circ)_v = (K_n^\circ)_v$ for $v \in \Sigma_{\min}^+$, $(K_{\text{sp}}^\circ)_v \subseteq (K_n^\circ)_v$ for $v \in \Sigma_{\text{lr}, \mathbb{I}}^+$, and that $(K_{n_0}^\circ)_v$ is a transferable open compact subgroup (Definition D.2.1) of $\text{U}(V_{n_0}^\circ)(F_v^+)$ for $v \in \Sigma_{\min}^+$ and is a special maximal subgroup of $\text{U}(V_{n_0}^\circ)(F_v^+)$ for $v \in \Sigma_{\text{lr}, \mathbb{I}}^+$,

○ a special inert prime (Definition 3.3.4) \mathfrak{p} of F^+ (with the underlying rational prime p) satisfying

(PI1): $\Sigma_{\mathbb{I}}^+$ does not contain p -adic places;

(PI2): ℓ does not divide $p(p^2 - 1)$;

(PI3): there exists a CM type Φ containing τ_∞ as in the initial setup of Section 5 satisfying

$$\mathbb{Q}_p^\Phi = \mathbb{Q}_{p^2};$$

(PI4): $P_{\alpha(\Pi_{0, \mathfrak{p}})} \bmod \lambda^m$ is level-raising special at \mathfrak{p} (Definition 3.1.5);

$P_{\alpha(\Pi_{1, \mathfrak{p}})} \bmod \lambda$ is Tate generic at \mathfrak{p} (Definition 3.1.5);

(PI5): $P_{\alpha(\Pi_{\alpha, \mathfrak{p}})} \bmod \lambda$ is intertwining generic at \mathfrak{p} (Definition 3.1.5) for $\alpha = 0, 1$;

(PI6): the natural map

$$\frac{O_E/\lambda^m[\text{Sh}(V_{n_\alpha}^\circ, K_{n_\alpha}^\circ)]}{\mathbb{T}_{n_\alpha}^{\Sigma_{\mathbb{I}}^+ \cup \Sigma_p^+} \cap \ker \phi_{\Pi_\alpha}} \rightarrow \frac{O_E/\lambda^m[\text{Sh}(V_{n_\alpha}^\circ, K_{n_\alpha}^\circ)]}{\mathbb{T}_{n_\alpha}^{\Sigma_{\mathbb{I}}^+} \cap \ker \phi_{\Pi_\alpha}}$$

is an isomorphism of *nontrivial* O_E/λ^m -modules for $\alpha = 0, 1$;

(PI7): $P_{\alpha(\Pi_{0, \mathfrak{p}}) \otimes \alpha(\Pi_{1, \mathfrak{p}})} \bmod \lambda^m$ is level-raising special at \mathfrak{p} (Definition 3.1.5);

(So we can and will adopt the setup in Subsection 5.10 to the datum $(V_n^\circ, \{\Lambda_{n, q}^\circ\}_{q|p})$.)

○ remaining data in the initial setup of Section 5 with $\mathbb{Q}_p^\Phi = \mathbb{Q}_{p^2}$; and

○ a definite uniformization datum as in Notation 5.10.13.

Put $K_{\text{sp}}^{p^\circ} := (K_{\text{sp}}^\circ)^p$ and $K_{\text{sp}}^\bullet := K_{\text{sp}}^{p^\circ} \times K_{n_0, p}^\bullet$; put $K_{n_\alpha}^{p^\circ} := (K_{n_\alpha}^\circ)^p$ and $K_{n_\alpha}^\bullet := K_{n_\alpha}^{p^\circ} \times K_{n_\alpha, p}^\bullet$ for $\alpha = 0, 1$. Like in Subsection 5.11, we put $X_{n_\alpha}^\alpha := X_{\mathfrak{p}}^\alpha(V_{n_\alpha}^\circ, K_{n_\alpha}^{p^\circ})$ for meaningful triples $(X, \alpha, \mathfrak{p}) \in \{\mathbf{M}, \mathbf{M}, \mathbf{B}, \mathbf{S}\} \times \{, \eta, \circ, \bullet, \dagger\} \times \{0, 1\}$. For $\alpha = 0, 1$, let $({}^\alpha E_s^{p, q}, {}^\alpha d_s^{p, q})$ be the weight spectral sequence abutting to the cohomology $H_{\mathfrak{I}}^{p+q}(\overline{\mathbf{M}}_{n_\alpha}, R\Psi O_\lambda(r_\alpha))$ from Subsection 5.8.

Notation 7.2.1. We introduce the following ideas of $\mathbb{T}_{n_\alpha}^{\Sigma_{\mathbb{I}}^+ \cup \Sigma_p^+}$, for $\alpha = 0, 1$

$$\begin{cases} \mathfrak{m}_\alpha := \mathbb{T}_{n_\alpha}^{\Sigma_{\mathbb{I}}^+ \cup \Sigma_p^+} \cap \ker \left(\mathbb{T}_{n_\alpha}^{\Sigma^+} \xrightarrow{\phi_{\Pi_\alpha}} O_E \rightarrow O_E/\lambda \right), \\ \mathfrak{n}_\alpha := \mathbb{T}_{n_\alpha}^{\Sigma_{\mathbb{I}}^+ \cup \Sigma_p^+} \cap \ker \left(\mathbb{T}_{n_\alpha}^{\Sigma^+} \xrightarrow{\phi_{\Pi_\alpha}} O_E \rightarrow O_E/\lambda^m \right). \end{cases}$$

We then introduce following assumptions.

Assumption 7.2.2. For $\alpha = 0, 1$, we have $H_{\mathfrak{I}}^i(\overline{\mathbf{M}}_{n_\alpha}, R\Psi O_\lambda)_{\mathfrak{m}_\alpha} = 0$ for $i \neq n_\alpha - 1$, and that $H_{\mathfrak{I}}^{n_\alpha-1}(\overline{\mathbf{M}}_{n_\alpha}, R\Psi O_\lambda)_{\mathfrak{m}_\alpha}$ is a finite free O_λ -module.

Assumption 7.2.3. Under Assumption 7.1.1, if $n_0 \geq 4$, then

(a) $\ell \geq 2(n_0 + 1)$ and ℓ is unramified in F ;

- (b) $\bar{\rho}_{\Pi_0, \lambda, +}$ (Remark 6.1.5) is rigid for $(\Sigma_{\min}^+, \Sigma_{\text{fr}, I}^+)$ (Definition E.7.1), and $\bar{\rho}_{\Pi_0, \lambda}|_{\text{Gal}(\bar{F}/F(\zeta_\ell))}$ is absolutely irreducible; and
- (c) the composite homomorphism $\mathbb{T}_{n_0}^{\Sigma_{\min}^+} \xrightarrow{\phi_{\Pi_0}} O_E \rightarrow O_E/\lambda$ is cohomologically generic (Definition D.1.1).

Now we apply constructions in Subsection 5.11, evaluating on the object $(K_n^{p^\circ}, K_{n+1}^{p^\circ})$ of $\mathfrak{K}(V_n^\circ)^p \times \mathfrak{K}(V_{n+1}^\circ)^p$. In particular, we have the blow-up morphism $\sigma: \mathbf{Q} \rightarrow \mathbf{P}$ from Notation 5.11.1, and the localized spectral sequence $(\mathbb{E}_{s, (m_0, m_1)}^{p, q}, d_{s, (m_0, m_1)}^{p, q})$ from (5.27).

Lemma 7.2.4. *Assume Assumptions 7.1.1, 7.2.3, 7.2.2, and Hypothesis 3.2.9 for both n and $n+1$. Then*

- (1) For $(?_0, ?_1) \in \{\circ, \bullet, \dagger\}^2$ and $i \in \mathbb{Z}$, we have a canonical isomorphism

$$H_{\mathfrak{X}}^i(\bar{P}^{?_0, ?_1}, O_\lambda)_{(m_0, m_1)} \simeq \bigoplus_{i_0+i_1=i} H_{\mathfrak{X}}^{i_0}(\bar{M}_{n_0}^{?_0}, O_\lambda)_{m_0} \otimes_{O_\lambda} H_{\mathfrak{X}}^{i_1}(\bar{M}_{n_1}^{?_1}, O_\lambda)_{m_1}$$

in $\text{Mod}(\text{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_{p^2}), O_\lambda)_{\text{fr}}$.

- (2) We have $\mathbb{E}_{2, (m_0, m_1)}^{p, q} = 0$ if $(p, q) \notin \{(-1, 2n), (0, 2n-1), (1, 2n-2)\}$, and canonical isomorphisms

$$\begin{cases} \mathbb{E}_{2, (m_0, m_1)}^{-1, 2n} \simeq {}^0E_{2, m_0}^{-1, 2r_0} \otimes_{O_\lambda} {}^1E_{2, m_1}^{0, 2r_1}, \\ \mathbb{E}_{2, (m_0, m_1)}^{0, 2n-1} \simeq {}^0E_{2, m_0}^{0, 2r_0-1} \otimes_{O_\lambda} {}^1E_{2, m_1}^{0, 2r_1}, \\ \mathbb{E}_{2, (m_0, m_1)}^{1, 2n-2} \simeq {}^0E_{2, m_0}^{1, 2r_0-2} \otimes_{O_\lambda} {}^1E_{2, m_1}^{0, 2r_1}, \end{cases}$$

in $\text{Mod}(\text{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_{p^2}), O_\lambda)_{\text{fr}}$.

- (3) If $\mathbb{E}_{2, (m_0, m_1)}^{i, 2n-1-i}(-1)$ has a nontrivial subquotient on which $\text{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_{p^2})$ acts trivially, then $i = 1$.
- (4) For $(?_0, ?_1) \in \{\circ, \bullet, \dagger\}^2$ and $i \in \mathbb{Z}$, both $H_{\mathfrak{X}}^{2i}(\bar{P}^{?_0, ?_1}, O_\lambda(i))_{(m_0, m_1)}$ and $H_{\mathfrak{X}}^{2i}(\bar{Q}^{?_0, ?_1}, O_\lambda(i))_{(m_0, m_1)}$ are weakly semisimple.
- (5) We have $H_{\mathfrak{X}}^i(\bar{Q}, R\Psi O_\lambda)_{(m_0, m_1)} = 0$ for $i \neq 2n-1$.
- (6) The canonical map $H_{\mathfrak{X}, c}^i(\bar{Q}^{(c)}, O_\lambda)_{(m_0, m_1)} \rightarrow H_{\mathfrak{X}}^i(\bar{Q}^{(c)}, O_\lambda)_{(m_0, m_1)}$ is an isomorphism for every integers c and i .

Proof. For (1), by Lemma 5.5.2, Lemma 6.2.2(2), Theorem 6.3.4(1), we know that $H_{\mathfrak{X}}^{i_\alpha}(\bar{M}_{n_\alpha}^{?_\alpha}, O_\lambda)_{m_\alpha}$ is a finitely generated free O_λ -module for $\alpha = 0, 1$ and every $i_\alpha \in \mathbb{Z}$. Thus, (1) follows from Lemma 6.1.10 and the Künneth formula.

For (2), we first show that $\mathbb{E}_{s, (m_0, m_1)}^{p, q}$ degenerates at the second page. By (1), Lemma 5.11.3(2), Lemma 5.5.2, and Lemma 6.2.1, the composition of $d_{1, (m_0, m_1)}^{-2, q}$ and the natural projection

$$\mathbb{E}_{1, (m_0, m_1)}^{-1, q} \rightarrow H_{\mathfrak{X}}^{q-2}(\bar{Q}^{\dagger, \dagger}, O_\lambda(n-1)) \bigoplus H_{\mathfrak{X}}^{q-2}(\bar{Q}^{\dagger, \circ}, O_\lambda(n-1))$$

is injective for every $q \in \mathbb{Z}$. Thus, $d_{1, (m_0, m_1)}^{-2, q}$ is injective, which implies $\mathbb{E}_{2, (m_0, m_1)}^{-2, q} = 0$ for every $q \in \mathbb{Z}$. By a dual argument, we have $\mathbb{E}_{2, (m_0, m_1)}^{2, q} = 0$ for every $q \in \mathbb{Z}$ as well. For the degeneration, it suffices to show that $d_{1, (m_0, m_1)}^{-1, q}$ is injective and $d_{1, (m_0, m_1)}^{0, q}$ is surjective for q odd. By Lemma 5.11.3(2), Lemma 5.5.2, and Lemma 6.2.2(1), we have $H_{\mathfrak{X}}^{q-2}(\bar{Q}^{(1)}, O_\lambda(n-1)) = H_{\mathfrak{X}}^{q-2}(\bar{Q}^{\bullet, \dagger}, O_\lambda(n-1))$ for q odd, which easily implies the injectivity of $d_{1, (m_0, m_1)}^{-1, q}$. By a dual argument, $d_{1, (m_0, m_1)}^{0, q}$ is surjective for q odd.

Now for every $q \in \mathbb{Z}$, the morphism σ induces a map

$$\sigma_1^*: \bigoplus_{q_0+q_1=q} {}^0E_{1, m_0}^{*, q_0} \otimes_{O_\lambda} {}^1E_{1, m_1}^{*, q_1} \rightarrow \mathbb{E}_{1, (m_0, m_1)}^{*, q}$$

of complexes of $O_\lambda[\text{Gal}(\overline{\mathbb{F}}/\mathbb{F}_{p^2})]$ -modules, hence a map

$$\sigma_2^*: \bigoplus_{p_0+p_1=p} \bigoplus_{q_0+q_1=q} {}^0\mathbb{E}_{2,m_0}^{p_0,q_0} \otimes_{O_\lambda} {}^1\mathbb{E}_{2,m_1}^{p_1,q_1} \rightarrow \mathbb{E}_{2,(m_0,m_1)}^{p,q}$$

of $O_\lambda[\text{Gal}(\overline{\mathbb{F}}/\mathbb{F}_{p^2})]$ -modules for $(p, q) \in \mathbb{Z}^2$. By Lemma 6.2.2 and Theorem 6.3.4(2), to show (2), it suffices to show that σ_2^* is an isomorphism, or the natural map

$$\bigoplus_{i_0+i_1=i} \mathbb{H}_{\mathbb{F}}^{i_0}(\overline{M}_{n_0}, R\Psi O_\lambda(r_0))_{m_0} \otimes_{O_\lambda} \mathbb{H}_{\mathbb{F}}^{i_1}(\overline{M}_{n_1}, R\Psi O_\lambda(r_1))_{m_1} \rightarrow \mathbb{H}_{\mathbb{F}}^i(\overline{Q}, R\Psi O_\lambda(n))_{(m_0,m_1)}$$

induced by σ is an isomorphism for every $i \in \mathbb{Z}$. By Lemma 5.1.7 and Lemma 5.11.2, the above map is identified with

$$\bigoplus_{i_0+i_1=i} \mathbb{H}_{\mathbb{F}}^{i_0}(\mathbf{M}_{n_0}^\eta \otimes_{\mathbb{Q}_{p^2}} \overline{\mathbb{Q}}_p, O_\lambda(r_0))_{m_0} \otimes_{O_\lambda} \mathbb{H}_{\mathbb{F}}^{i_1}(\mathbf{M}_{n_1}^\eta \otimes_{\mathbb{Q}_{p^2}} \overline{\mathbb{Q}}_p, O_\lambda(r_1))_{m_1} \rightarrow \mathbb{H}_{\mathbb{F}}^i(\mathbf{Q}^\eta \otimes_{\mathbb{Q}_{p^2}} \overline{\mathbb{Q}}_p, O_\lambda(n))_{(m_0,m_1)},$$

which is an isomorphism by Lemma 6.1.10, and the Künneth formula. Thus, (2) follows.

For (3), let $\{\alpha_{0,1}^{\pm 1}, \dots, \alpha_{0,r_0}^{\pm 1}\}$ and $\{\alpha_{1,1}^{\pm 1}, \dots, \alpha_{1,r_1}^{\pm 1}, 1\}$ be the roots of $P_{\alpha(\Pi_{0,p})} \bmod \lambda$ and $P_{\alpha(\Pi_{1,p})} \bmod \lambda$ in a finite extension of O_λ/λ , respectively. By (PI4), we may assume $\alpha_{0,r_0} = p$. By (2), Theorem 6.2.3(1), and Theorem 6.3.4(3), the generalized Frobenius eigenvalues of the $O_\lambda/\lambda[\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})]$ -modules $\mathbb{E}_{2,(m_0,m_1)}^{-1,2n}(-1) \otimes_{O_\lambda} O_\lambda/\lambda$ and $\mathbb{E}_{2,(m_0,m_1)}^{0,2n-1}(-1) \otimes_{O_\lambda} O_\lambda/\lambda$ are contained in $\{p^{-2}\alpha_{1,1}^{\pm 1}, \dots, p^{-2}\alpha_{1,r_1}^{\pm 1}, p^{-2}\}$ and $\{p^{-1}\alpha_{0,1}^{\pm 1}\alpha_{1,1}^{\pm 1}, \dots, p^{-1}\alpha_{0,r_0-1}^{\pm 1}\alpha_{1,r_1}^{\pm 1}\} \cup \{p^{-1}\alpha_{0,1}^{\pm 1}, \dots, p^{-1}\alpha_{0,r_0-1}^{\pm 1}\}$, respectively. By (PI2), we have $p^2 \neq 1$ in O_λ/λ . By (PI7), we have $\alpha_{1,i_1} \notin \{p^2, p^{-2}\}$ for $1 \leq i_1 \leq r_1$, which implies $1 \notin \{p^{-2}\alpha_{1,1}^{\pm 1}, \dots, p^{-2}\alpha_{1,r_1}^{\pm 1}, p^{-2}\}$. Again by (PI7), we have $\alpha_{0,i_0}\alpha_{1,i_1} \notin \{p, p^{-1}\}$ for $1 \leq i_0 < r_0$ and $1 \leq i_1 \leq r_1$, which implies $1 \notin \{p^{-1}\alpha_{0,1}^{\pm 1}\alpha_{1,1}^{\pm 1}, \dots, p^{-1}\alpha_{0,r_0-1}^{\pm 1}\alpha_{1,r_1}^{\pm 1}\}$. By (PI4), we have $\alpha_{0,i_0} \notin \{p, p^{-1}\}$ for $1 \leq i_0 < r_0$, which implies $1 \notin \{p^{-1}\alpha_{0,1}^{\pm 1}, \dots, p^{-1}\alpha_{0,r_0-1}^{\pm 1}\}$. Thus, (3) follows.

For (4), by Lemma 5.11.3 (3–5) and Lemma 2.1.4(1), it suffices to show that $\mathbb{H}_{\mathbb{F}}^{2i}(\overline{P}^{?0,?1}, O_\lambda(i))_{(m_0,m_1)}$ is weakly semisimple. By (1) and Lemma 6.2.2(1), it suffices to show that $\mathbb{H}_{\mathbb{F}}^{2i_0}(\overline{M}_{n_0}^{?0}, O_\lambda(i_0))_{m_0} \otimes_{O_\lambda} \mathbb{H}_{\mathbb{F}}^{2i_1}(\overline{M}_{n_1}^{?1}, O_\lambda(i_1))_{m_1}$ is weakly semisimple for $i_0, i_1 \in \mathbb{Z}$. By Lemma 5.5.2, the action of $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$ on $\mathbb{H}_{\mathbb{F}}^{2i_\alpha}(\overline{M}_{n_\alpha}^{?}, O_\lambda(i_\alpha))_{m_\alpha}$ is trivial for $\alpha = 0, 1, ? = \circ, \dagger$, and every $i_\alpha \in \mathbb{Z}$. On the other hand, it is a consequence of Theorem 6.3.4(2) (for i_0) and Lemma 6.2.2(3) (for i_1) that the action of $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$ on $\mathbb{H}_{\mathbb{F}}^{2i_\alpha}(\overline{M}_{n_\alpha}^\bullet, O_\lambda(i_\alpha))_{m_\alpha}$ is trivial if $i_0 \notin \{r_0 - 1, r_0\}$ or $i_1 \neq r_1$. By Proposition 6.3.1(1,2) and Theorem 6.3.4(1), the actions of $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$ on both $\mathbb{H}_{\mathbb{F}}^{2r_0-2}(\overline{M}_{n_0}^\bullet, O_\lambda(r_0 - 1))_{m_0}$ and $\mathbb{H}_{\mathbb{F}}^{2r_0}(\overline{M}_{n_0}^\bullet, O_\lambda(r_0))_{m_0}$ are also trivial. Thus, by Lemma 2.1.4(1), it remains to show that $\mathbb{H}_{\mathbb{F}}^{2r_1}(\overline{M}_{n_1}^\bullet, O_\lambda(r_1))_{m_1}$ is weakly semisimple, which follows from Theorem 6.2.3(2) as it is isomorphic to the direct sum of ${}^1\mathbb{E}_{2,m_1}^{0,2r_1}$ and $\mathbb{H}_{\mathbb{F}}^{2r_1}(\overline{M}_{n_1}^\dagger, O_\lambda(r_1))_{m_1}$.

Part (5) is a direct consequence of (2).

Part (6) follows from (1), Lemma 6.1.10, and Lemma 5.11.3(3–5). \square

Remark 7.2.5. In fact, Lemma 7.2.4(5) holds under only Assumption 7.2.2; and Lemma 7.2.4(6) holds under only Assumption 7.1.1.

Lemma 7.2.4(5) induces a coboundary map

$$\text{AJ}_{\mathbf{Q}}: \mathbb{Z}_{\mathbb{F}}^n(\mathbf{Q}^\eta) \rightarrow \mathbb{H}^1(\mathbb{Q}_{p^2}, \mathbb{H}_{\mathbb{F}}^{2n-1}(\overline{Q}, R\Psi O_\lambda(n))_{(m_0,m_1)}).$$

We also recall the singular quotient map

$$\partial: \mathbb{H}^1(\mathbb{Q}_{p^2}, \mathbb{H}_{\mathbb{F}}^{2n-1}(\overline{Q}, R\Psi O_\lambda(n))_{(m_0,m_1)}) \rightarrow \mathbb{H}_{\text{sing}}^1(\mathbb{Q}_{p^2}, \mathbb{H}_{\mathbb{F}}^{2n-1}(\overline{Q}, R\Psi O_\lambda(n))_{(m_0,m_1)}).$$

By our choice of K_n° and $(K_{\text{sp}}, K_{n+1}^\circ)$, we obtain a morphism

$$\mathbf{M}_p(V_n^\circ, K_{\text{sp}}^\circ) \rightarrow \mathbf{P}$$

which is finite. Denote by \mathbf{P}_{sp} the corresponding cycle; and let \mathbf{Q}_{sp} be the strict transform of \mathbf{P}_{sp} under σ , which is a \mathbf{T}_p -invariant cycle of \mathbf{Q} . Our main goal is to compute $\partial \text{AJ}_{\mathbf{Q}}(\mathbf{Q}_{\text{sp}}^\eta)$ in $H_{\text{sing}}^1(\mathbb{Q}_{p^2}, H_{\mathbb{F}}^{2n-1}(\overline{\mathbf{Q}}, R\Psi O_\lambda(n)))/(\mathbf{n}_0, \mathbf{n}_1)$. Recall the map Δ^n (5.28); the cycle \mathbf{Q}_{sp} gives rise to a class $\text{cl}(\mathbf{Q}_{\text{sp}}) \in C^n(\mathbf{Q}, L)$ (see Subsection 5.11 for the target).

Proposition 7.2.6. *Assume Assumptions 7.1.1, 7.2.3, 7.2.2, and Hypothesis 3.2.9 for both n and $n + 1$. There is a canonical isomorphism*

$$H_{\text{sing}}^1(\mathbb{Q}_{p^2}, H_{\mathbb{F}}^{2n-1}(\overline{\mathbf{Q}}, R\Psi O_\lambda(n))_{(\mathbf{m}_0, \mathbf{m}_1)}) \simeq \text{coker } \Delta_{(\mathbf{m}_0, \mathbf{m}_1)}^n$$

under which $\partial \text{AJ}_{\mathbf{Q}}(\mathbf{Q}_{\text{sp}}^\eta)$ coincides with the image of $\text{cl}(\mathbf{Q}_{\text{sp}})$ in $\text{coker } \Delta_{(\mathbf{m}_0, \mathbf{m}_1)}^n$.

Proof. By [Liu19, Theorem 2.16 and Theorem 2.18],¹⁹ it suffices to show that O_λ is a very nice coefficient for $\mathbb{E}_{s, (\mathbf{m}_0, \mathbf{m}_1)}^{p, q}$ in the sense of [Liu19, Definition 2.15]. In fact, in [Liu19, Definition 2.15], (N1) is satisfied due to Lemma 7.2.4(2); (N2) is satisfied due to Lemma 7.2.4(3); and (N3) is satisfied due to Lemma 7.2.4(4) and Lemma 2.1.4(2).

The proposition is proved. \square

By Construction 5.11.7 and Remark 5.11.8, we have a map

$$\nabla: C^n(\mathbf{Q}, O_\lambda) \rightarrow O_\lambda[\text{Sh}(V_{n_0}^\circ, K_{n_0}^\circ)] \otimes_{O_\lambda} O_\lambda[\text{Sh}(V_{n_1}^\circ, K_{n_1}^\circ)].$$

Theorem 7.2.7 (First explicit reciprocity law). *Assume Assumptions 7.1.1, 7.2.3, 7.2.2, and Hypothesis 3.2.9 for both n and $n + 1$.*

- (1) *The image of the composite map $\nabla_{(\mathbf{m}_0, \mathbf{m}_1)} \circ \Delta_{(\mathbf{m}_0, \mathbf{m}_1)}^n$ is contained in $\mathbf{n}_0 \cdot O_\lambda[\text{Sh}(V_{n_0}^\circ, K_{n_0}^\circ)]_{\mathbf{m}_0} \otimes_{O_\lambda} O_\lambda[\text{Sh}(V_{n_1}^\circ, K_{n_1}^\circ)]_{\mathbf{m}_1}$.*
- (2) *In view of (1), the induced map*

$$\nabla_{\mathbf{m}_1/\mathbf{n}_0}: \text{coker } \Delta_{\mathbf{m}_1}^n / \mathbf{n}_0 \rightarrow O_\lambda[\text{Sh}(V_{n_0}^\circ, K_{n_0}^\circ)] / \mathbf{n}_0 \otimes_{O_\lambda} O_\lambda[\text{Sh}(V_{n_1}^\circ, K_{n_1}^\circ)]_{\mathbf{m}_1}$$

is an isomorphism.

- (3) *Under the natural pairing*

$$O_\lambda[\text{Sh}(V_{n_0}^\circ, K_{n_0}^\circ)] / \mathbf{n}_0 \otimes_{O_\lambda} O_\lambda[\text{Sh}(V_{n_1}^\circ, K_{n_1}^\circ)]_{\mathbf{m}_1} \times O_\lambda / \lambda^m [\text{Sh}(V_{n_0}^\circ, K_{n_0}^\circ)][\mathbf{n}_0] \otimes_{O_\lambda} O_\lambda[\text{Sh}(V_{n_1}^\circ, K_{n_1}^\circ)]_{\mathbf{m}_1} \rightarrow O_\lambda / \lambda^m$$

obtained by taking inner product, the pairing of $\nabla_{/(\mathbf{n}_0, \mathbf{n}_1)}(\partial \text{AJ}_{\mathbf{Q}}(\mathbf{Q}_{\Delta}^\eta))$ and every function $f \in O_\lambda / \lambda^m [\text{Sh}(V_{n_0}^\circ, K_{n_0}^\circ)][\mathbf{n}_0] \otimes_{O_\lambda} O_\lambda / \lambda^m [\text{Sh}(V_{n_1}^\circ, K_{n_1}^\circ)][\mathbf{n}_1]$ is equal to

$$(p+1) \cdot \phi_{\Pi_0}(\mathbf{I}_{n_0, p}^\circ) \cdot \phi_{\Pi_1}(\mathbf{T}_{n_1, p}^\circ) \cdot \sum_{s \in \text{Sh}(V_n^\circ, K_{\text{sp}}^\circ)} f(s, \text{sh}_\dagger^\circ(s)).$$

Here, we regard $\partial \text{AJ}_{\mathbf{Q}}(\mathbf{Q}_{\text{sp}}^\eta)$ as an element in $\text{coker } \Delta_{(\mathbf{m}_0, \mathbf{m}_1)}^n$ (hence in $\text{coker } \Delta_{\mathbf{m}_1}^n / \mathbf{n}_0$) via the canonical isomorphism in Proposition 7.2.6.

Proof. We first consider (1). By Lemma 5.11.3(3,4), we have

$$H_{\mathbb{F}}^{2(n-1)}(\overline{\mathbf{Q}}^{(0)}, O_\lambda(n-1))_{(\mathbf{m}_0, \mathbf{m}_1)} = \bigoplus_{(?_0, ?_1) \in \{\circ, \bullet\}^2} \sigma^* H_{\mathbb{F}}^{2(n-1)}(\overline{\mathbf{P}}^{?_0, ?_1}, O_\lambda(n-1))_{(\mathbf{m}_0, \mathbf{m}_1)}$$

$$\bigoplus (\delta_{\circ, \circ}^{\dagger, \dagger})! \sigma^* H_{\mathbb{F}}^{2(n-2)}(\overline{\mathbf{P}}^{\dagger, \dagger}, O_\lambda(n-2))_{(\mathbf{m}_0, \mathbf{m}_1)} \bigoplus (\delta_{\bullet, \bullet}^{\dagger, \dagger})! \sigma^* H_{\mathbb{F}}^{2(n-2)}(\overline{\mathbf{P}}^{\dagger, \dagger}, O_\lambda(n-2))_{(\mathbf{m}_0, \mathbf{m}_1)}.$$

Thus, it suffices to show that

- (1a) The image of $\sigma^* H_{\mathbb{F}}^{2(n-1)}(\overline{\mathbf{P}}^{\circ, \bullet}, O_\lambda(n-1))_{(\mathbf{m}_0, \mathbf{m}_1)} \bigoplus \sigma^* H_{\mathbb{F}}^{2(n-1)}(\overline{\mathbf{P}}^{\bullet, \bullet}, O_\lambda(n-1))_{(\mathbf{m}_0, \mathbf{m}_1)}$ under the map $(\nabla \circ \delta_{1!} \circ \delta_0^*)_{(\mathbf{m}_0, \mathbf{m}_1)}$ is contained in $\mathbf{n}_0 \cdot O_\lambda[\text{Sh}(V_{n_0}^\circ, K_{n_0}^\circ)]_{\mathbf{m}_0} \otimes_{O_\lambda} O_\lambda[\text{Sh}(V_{n_1}^\circ, K_{n_1}^\circ)]_{\mathbf{m}_1}$.

¹⁹Although it is assumed that the underlying strictly semistable scheme X is proper over the base in [Liu19], the proof of relevant results works without change in our case even when \mathbf{Q} is not proper in view of Lemma 7.2.4(6).

- (1b) The image of $\sigma^* H_{\mathbb{X}}^{2(n-1)}(\overline{P}^{\circ, \circ}, O_{\lambda}(n-1))_{(m_0, m_1)} \oplus \sigma^* H_{\mathbb{X}}^{2(n-1)}(\overline{P}^{\bullet, \circ}, O_{\lambda}(n-1))_{(m_0, m_1)}$ under the map $(\nabla \circ \delta_{1!} \circ \delta_0^*)_{(m_0, m_1)}$ is zero.
- (1c) The image of $(\delta_{\circ, \circ}^{\dagger, \dagger})! \sigma^* H_{\mathbb{X}}^{2(n-2)}(\overline{P}^{\dagger, \dagger}, O_{\lambda}(n-2))_{(m_0, m_1)}$ under the map $(\nabla \circ \delta_{1!} \circ \delta_0^*)_{(m_0, m_1)}$ is zero.
- (1d) The image of $(\delta_{\bullet, \bullet}^{\dagger, \dagger})! \sigma^* H_{\mathbb{X}}^{2(n-2)}(\overline{P}^{\dagger, \dagger}, O_{\lambda}(n-2))_{(m_0, m_1)}$ under the map $(\nabla \circ \delta_{1!} \circ \delta_0^*)_{(m_0, m_1)}$ is zero.

For (1a), we have a commutative diagram

$$\begin{array}{ccc} H_{\mathbb{X}}^{2(n-1)}(\overline{P}^{\circ, \bullet}, O_{\lambda}(n-1))_{(m_0, m_1)} \oplus H_{\mathbb{X}}^{2(n-1)}(\overline{P}^{\bullet, \bullet}, O_{\lambda}(n-1))_{(m_0, m_1)} & \longrightarrow & {}^0 E_{1, m_0}^{0, 2r_0-2} \otimes_{O_{\lambda}} H_{\mathbb{X}}^{2r_1}(\overline{M}_{n_1}^{\bullet}, O_{\lambda}(r_1))_{m_1} \\ \sigma^* \downarrow & & \downarrow \\ H_{\mathbb{X}}^{2(n-1)}(\overline{Q}^{\circ, \bullet}, O_{\lambda}(n-1))_{(m_0, m_1)} \oplus H_{\mathbb{X}}^{2(n-1)}(\overline{Q}^{\bullet, \bullet}, O_{\lambda}(n-1))_{(m_0, m_1)} & \longrightarrow & O_{\lambda}[\text{Sh}(V_{n_0}^{\circ}, K_{n_0}^{\circ})]_{m_0} \otimes_{O_{\lambda}} O_{\lambda}[\text{Sh}(V_{n_1}^{\circ}, K_{n_1}^{\circ})]_{m_1} \end{array}$$

in which

○ the upper horizontal arrow is the map

$$\begin{aligned} & H_{\mathbb{X}}^{2(n-1)}(\overline{P}^{\circ, \bullet}, O_{\lambda}(n-1))_{(m_0, m_1)} \oplus H_{\mathbb{X}}^{2(n-1)}(\overline{P}^{\bullet, \bullet}, O_{\lambda}(n-1))_{(m_0, m_1)} \\ & \rightarrow H_{\mathbb{X}}^{2(r_0-1)}(\overline{M}_{n_0}^{\circ}, O_{\lambda}(r_0-1))_{m_0} \otimes_{O_{\lambda}} H_{\mathbb{X}}^{2r_1}(\overline{M}_{n_1}^{\bullet}, O_{\lambda}(r_1))_{m_1} \\ & \quad \oplus H_{\mathbb{X}}^{2(r_0-1)}(\overline{M}_{n_0}^{\bullet}, O_{\lambda}(r_0-1))_{m_0} \otimes_{O_{\lambda}} H_{\mathbb{X}}^{2r_1}(\overline{M}_{n_1}^{\bullet}, O_{\lambda}(r_1))_{m_1} \\ & = {}^0 E_{1, m_0}^{0, 2r_0-2} \otimes_{O_{\lambda}} H_{\mathbb{X}}^{2r_1}(\overline{M}_{n_1}^{\bullet}, O_{\lambda}(r_1))_{m_1} \end{aligned}$$

given by Lemma 7.2.4(1) and the Künneth formula;

○ the right vertical arrow is

$$(\nabla^0 \circ {}^0 d_1^{-1, 2r_0} \circ {}^0 d_1^{0, 2r_0-2}(-1))_{m_0} \otimes (I_{n_1, p}^{\circ} \circ \text{inc}_{\dagger}^* + (p+1)^2 T_{n_1, p}^{\circ \bullet} \circ \text{inc}_{\bullet}^*)_{m_1};$$

and

○ the lower horizontal arrow is $(\nabla \circ \delta_{1!} \circ \delta_0^*)_{(m_0, m_1)}$.

For (1a), by Proposition B.3.5(2) and (PI4), we have

$$((p+1)R_{N, p}^{\circ} - I_{N, p}^{\circ}) \cdot O_{\lambda}[\text{Sh}(V_{n_0}^{\circ}, K_{n_0}^{\circ})]_{m_0} \subseteq \mathfrak{n}_0 \cdot O_{\lambda}[\text{Sh}(V_{n_0}^{\circ}, K_{n_0}^{\circ})]_{m_0}.$$

Thus, (1a) follows from Proposition 6.3.1(4) and Lemma 5.11.3(3).

For (1b) and (1c), both images are actually contained in the sum of

$$(I_{n_1, p}^{\circ} \circ \text{inc}_{\circ, \dagger}^* + (p+1)^2 T_{n_1, p}^{\circ \bullet} \circ \text{inc}_{\circ, \bullet}^*)(\gamma_{\circ, \bullet}^{\circ, \dagger})! H_{\mathbb{X}}^{2(n-1)}(\overline{P}^{\circ, \dagger}, O_{\lambda}(n-1))_{(m_0, m_1)}$$

and

$$(I_{n_1, p}^{\circ} \circ \text{inc}_{\circ, \dagger}^* + (p+1)^2 T_{n_1, p}^{\circ \bullet} \circ \text{inc}_{\bullet, \bullet}^*)(\gamma_{\bullet, \bullet}^{\circ, \dagger})! H_{\mathbb{X}}^{2(n-1)}(\overline{P}^{\bullet, \dagger}, O_{\lambda}(n-1))_{(m_0, m_1)},$$

which by Lemma 7.2.4(1) coincide with

$$H_{\mathbb{X}}^{2r_0}(\overline{M}_{n_0}^{\circ}, O_{\lambda}(r_0))_{m_0} \otimes_{O_{\lambda}} \left((I_{n_1, p}^{\circ} \circ \text{Inc}_{\dagger}^* + (p+1)^2 T_{n_1, p}^{\circ \bullet} \circ \text{Inc}_{\bullet}^*) {}^1 d_1^{-1, 2r_1} H_{\mathbb{X}}^{2(r_1-1)}(\overline{M}_{n_1}^{\dagger}, O_{\lambda}(r_1-1))_{m_1} \right)$$

and

$$H_{\mathbb{X}}^{2r_0}(\overline{M}_{n_0}^{\bullet}, O_{\lambda}(r_0))_{m_0} \otimes_{O_{\lambda}} \left((I_{n_1, p}^{\circ} \circ \text{Inc}_{\dagger}^* + (p+1)^2 T_{n_1, p}^{\circ \bullet} \circ \text{Inc}_{\bullet}^*) {}^1 d_1^{-1, 2r_1} H_{\mathbb{X}}^{2(r_1-1)}(\overline{M}_{n_1}^{\dagger}, O_{\lambda}(r_1-1))_{m_1} \right),$$

respectively. However, they vanish by Lemma 5.8.2(3). Thus, (1b) and (1c) follow.

For (1d), by [Liu19, Lemma 2.4], it follows from (1c). Thus, (1) is proved.

Now we consider (2). We claim that the map $\nabla_{(m_0, m_1)}$ (with domain $C^n(Q, O_{\lambda})_{(m_0, m_1)}$) is surjective. In fact, consider the submodule

$$\ker {}^0 d_{1, m_0}^{0, 2r_0} \otimes_{O_{\lambda}} \ker {}^1 d_{1, m_1}^{0, 2r_1} \subseteq \bigoplus_{(?_0, ?_1) \in \{\circ, \bullet\}^2} H_{\mathbb{X}}^{2(n-1)}(\overline{P}^{?_0, ?_1}, O_{\lambda}(n-1))_{(m_0, m_1)}$$

in view of Lemma 7.2.4(1). Then $\sigma^* \left(\ker^0 d_{1,m_0}^{0,2r_0} \otimes_{O_\lambda} \ker^1 d_{1,m_1}^{0,2r_1} \right)$ is contained in $C^m(\mathbb{Q}, O_\lambda)_{(m_0, m_1)}$. On the other hand, the map $\nabla_{(m_0, m_1)} \circ \sigma^*$ (with domain $\ker^0 d_{1,m_0}^{0,2r_0} \otimes_{O_\lambda} \ker^1 d_{1,m_1}^{0,2r_1}$) coincides with $\nabla_{m_0}^0 \otimes \nabla_{m_1}^1$, which is surjective by Proposition 6.3.1(3) and Theorem 6.2.3. The claim follows.

Thus, it remains to show that the domain and the target of ∇_{m_1/n_0} have the same cardinality. By Proposition 7.2.6, we have an isomorphism

$$(7.1) \quad \text{coker } \Delta_{m_1/n_0}^n / \mathfrak{n}_0 = \text{coker } \Delta_{(m_0, m_1)}^n / \mathfrak{n}_0 \simeq H_{\text{sing}}^1(\mathbb{Q}_{p^2}, H_{\bar{\mathbb{Q}}}^{2n-1}(\bar{\mathbb{Q}}, R\Psi O_\lambda(n))_{(m_0, m_1)}) / \mathfrak{n}_0$$

of O_λ/λ^m -modules. By Lemma 7.2.4(2,3) and Theorem 6.2.3(2), we have

$$H_{\text{sing}}^1(\mathbb{Q}_{p^2}, H_{\bar{\mathbb{Q}}}^{2n-1}(\bar{\mathbb{Q}}, R\Psi O_\lambda(n))_{(m_0, m_1)}) \simeq H_{\text{sing}}^1(\mathbb{Q}_{p^2}, H_{\bar{\mathbb{Q}}}^{2r_0-1}(\bar{M}_{n_0}, R\Psi O_\lambda(r_0))_{m_0}) \otimes_{O_\lambda} ({}^1E_{2, m_1}^{0, 2r_1})^{\text{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_{p^2})}.$$

Then by Theorem 6.2.3(3) and Theorem 6.3.4(4), we have

$$(7.1) \simeq O_\lambda[\text{Sh}(V_{n_0}^\circ, K_{n_0}^\circ)] / \mathfrak{n}_0 \otimes_{O_\lambda} O_\lambda[\text{Sh}(V_{n_1}^\circ, K_{n_1}^\circ)] / \mathfrak{n}_1.$$

Thus, (2) is proved.

Finally we consider (3). As Q_{sp} does not intersect with Q^\bullet , we have

$$\text{cl}(Q_{\text{sp}}) = \text{cl}(Q_\Delta^\bullet) \in H_{\bar{\mathbb{Q}}}^{2n}(\bar{\mathbb{Q}}^{\bullet, \bullet}, O_\lambda(n)),$$

and by Construction 5.11.7,

$$\nabla(\text{cl}(Q_{\text{sp}})) = \left((p+1)(T_{n_0, p}^{\circ, \bullet} \otimes I_{n_1, p}^\circ) \circ \text{inc}_{\bullet, \dagger}^* + (p+1)^3(T_{n_0, p}^{\circ, \bullet} \otimes T_{n_1, p}^{\circ, \bullet}) \circ \text{inc}_{\bullet, \bullet}^* \right) (\text{cl}(P_{\text{sp}}^\bullet)).$$

Applying Theorem 5.11.5(3) to the object $(K_{\text{sp}}^\circ, K_{n+1}^\circ) \in \mathfrak{K}(V_n^\circ)_{\text{sp}}$ followed by pushforward, we know that the pairing between $\nabla_{m_1/n_0}(\text{cl}(Q_{\text{sp}}))$ and any function

$$f \in O_\lambda/\lambda^m[\text{Sh}(V_{n_0}^\circ, K_{n_0}^\circ)] / \mathfrak{n}_0 \otimes_{O_\lambda} O_\lambda/\lambda^m[\text{Sh}(V_{n_1}^\circ, K_{n_1}^\circ)] / \mathfrak{n}_1$$

is given by the formula

$$(p+1) \cdot \phi_{\Pi_0}(I_{n_0, p}^\circ) \cdot \phi_{\Pi_1}(T_{n_1, p}^\circ) \cdot \sum_{s \in \text{Sh}(V_n^\circ, K_{\text{sp}}^\circ)} f(s, \text{sh}_\dagger^\circ(s))$$

in view of (PI6). We then obtain (3) by Proposition 7.2.6.

The theorem is proved. \square

We state a corollary for later application. We choose an indefinite uniformization datum as in Notation 5.10.1, and put $\text{Sh}'_{n_\alpha} := \text{Sh}(V'_{n_\alpha}, j_{n_\alpha} K_{n_\alpha}^{p_\circ} K'_{n_\alpha, p})$ for $\alpha = 0, 1$.

Assume Assumptions 7.1.1, 7.2.2. By Lemma 6.1.10, Lemma 5.1.7, and the Künneth formula, we have $H_{\text{ét}}^i((\text{Sh}'_{n_0} \times_{\text{Spec } F} \text{Sh}'_{n_1})_{\bar{F}}, O_\lambda)_{(m_0, m_1)} = 0$ if $i \neq 2n-1$. In particular, we obtain the Abel–Jacobi map

$$\text{AJ}: Z^n(\text{Sh}'_{n_0} \times_{\text{Spec } F} \text{Sh}'_{n_1}) \rightarrow H^1(F, H^{2n-1}((\text{Sh}'_{n_0} \times_{\text{Spec } F} \text{Sh}'_{n_1})_{\bar{F}}, O_\lambda(n)) / (\mathfrak{n}_0, \mathfrak{n}_1)).$$

Let Sh'_{sp} be the cycle given by the finite morphism $\text{Sh}(V'_n, j_n K_{\text{sp}}^{p_\circ} K'_{n, p}) \rightarrow \text{Sh}'_n \times_{\text{Spec } F} \text{Sh}'_{n+1}$, which is an element in $Z^n(\text{Sh}'_{n_0} \times_{\text{Spec } F} \text{Sh}'_{n_1})$.

Corollary 7.2.8. *Assume Assumptions 7.1.1, 7.2.3, 7.2.2, and Hypothesis 3.2.9 for both n and $n+1$. Then we have*

$$\begin{aligned} & \exp_\lambda \left(\partial_p \text{loc}_p \text{AJ}(\text{Sh}'_{\text{sp}}), H_{\text{sing}}^1(F_p, H^{2n-1}((\text{Sh}'_{n_0} \times_{\text{Spec } F} \text{Sh}'_{n_1})_{\bar{F}}, O_\lambda(n)) / (\mathfrak{n}_0, \mathfrak{n}_1)) \right) \\ &= \exp_\lambda \left(\mathbb{1}_{\text{Sh}(V_n^\circ, K_{\text{sp}}^\circ)}, O_\lambda[\text{Sh}(V_{n_0}^\circ, K_{n_0}^\circ) \times \text{Sh}(V_{n_1}^\circ, K_{n_1}^\circ)] / (\mathfrak{n}_0, \mathfrak{n}_1) \right) \end{aligned}$$

where \exp_λ is introduced in Definition 2.1.6. Here, we regard $\mathbb{1}_{\text{Sh}(V_n^\circ, K_{\text{sp}}^\circ)}$ as the pushforward of the characteristic function along the map $\text{Sh}(V_n^\circ, K_{\text{sp}}^\circ) \rightarrow \text{Sh}(V_n^\circ, K_n^\circ) \times \text{Sh}(V_{n+1}^\circ, K_{n+1}^\circ)$.

Proof. Note that the isomorphism (5.2) induces a map

$$H^{2n-1}((\mathrm{Sh}'_{n_0} \times_{\mathrm{Spec} F} \mathrm{Sh}'_{n_1})_{\overline{F}}, O_\lambda(n))_{(m_0, m_1)} \rightarrow H_{\overline{\mathfrak{F}}}^{2n-1}(\overline{\mathbb{Q}}, R\Psi O_\lambda(n))_{(m_0, m_1)}$$

of $O_\lambda[\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_{p^2})]$ -modules, which is an isomorphism by Lemma 5.11.2. Combining with the diagram (5.23), we have

$$\begin{aligned} & \exp_\lambda \left(\partial_p \mathrm{loc}_p \mathrm{AJ}(\mathrm{Sh}'_{\mathrm{sp}}), H_{\mathrm{sing}}^1(F_{\mathfrak{p}}, H^{2n-1}((\mathrm{Sh}'_{n_0} \times_{\mathrm{Spec} F} \mathrm{Sh}'_{n_1})_{\overline{F}}, O_\lambda(n)) / (\mathbf{n}_0, \mathbf{n}_1)) \right) \\ &= \exp_\lambda \left(\partial \mathrm{AJ}_{\mathbf{Q}}(\mathbf{Q}_{\mathrm{sp}}^\eta), H_{\mathrm{sing}}^1(\mathbb{Q}_{p^2}, H_{\overline{\mathfrak{F}}}^{2n-1}(\overline{\mathbb{Q}}, R\Psi O_\lambda(n)) / (\mathbf{n}_0, \mathbf{n}_1)) \right). \end{aligned}$$

Now Theorem 7.2.7 implies

$$\begin{aligned} & \exp_\lambda \left(\partial \mathrm{AJ}_{\mathbf{Q}}(\mathbf{Q}_{\mathrm{sp}}^\eta), H_{\mathrm{sing}}^1(\mathbb{Q}_{p^2}, H_{\overline{\mathfrak{F}}}^{2n-1}(\overline{\mathbb{Q}}, R\Psi O_\lambda(n)) / (\mathbf{n}_0, \mathbf{n}_1)) \right) \\ &= \exp_\lambda \left((p+1)\phi_{\Pi_0}(\mathbb{I}_{n_0, \mathfrak{p}}^\circ)\phi_{\Pi_1}(\mathbb{T}_{n_1, \mathfrak{p}}^\circ)\mathbb{1}_{\mathrm{Sh}(V_{n_0}^\circ, K_{n_0}^\circ)}, O_\lambda[\mathrm{Sh}(V_{n_0}^\circ, K_{n_0}^\circ)]/\mathbf{n}_0 \otimes_{O_\lambda} O_\lambda[\mathrm{Sh}(V_{n_1}^\circ, K_{n_1}^\circ)]/\mathbf{n}_1 \right). \end{aligned}$$

Note that $(p+1)$ is invertible in O_λ by (PI2); $\phi_{\Pi_0}(\mathbb{I}_{n_0, \mathfrak{p}}^\circ)$ is invertible in O_λ by (PI5) and Proposition B.3.5(1); and $\phi_{\Pi_1}(\mathbb{T}_{n_1, \mathfrak{p}}^\circ)$ is invertible in O_λ by (PI4) and Proposition B.4.3(2). Thus, the corollary follows. \square

7.3. Second explicit reciprocity law. We start by choosing

- a finite place λ of E (with the underlying rational prime ℓ),
- a positive integer m ,
- a (possibly empty) finite set $\Sigma_{\mathrm{lr}, \mathrm{II}}^+$ of nonarchimedean places of F^+ that are inert in F , strongly disjoint from Σ_{min}^+ (Definition 1.3.3), satisfying $\ell \nmid \|v\|(\|v\|^2 - 1)$ for $v \in \Sigma_{\mathrm{lr}, \mathrm{II}}^+$,
- a finite set Σ_{II}^+ of nonarchimedean places of F^+ containing $\Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}, \mathrm{II}}^+$,
- a standard indefinite hermitian space V_n of rank n over F , together with a self-dual $\prod_{v \notin \Sigma_\infty^+ \cup \Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}, \mathrm{II}}^+} O_{F_v}$ -lattice Λ_n in $V_n \otimes_F \mathbb{A}_F^{\Sigma_\infty^+ \cup \Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}, \mathrm{II}}^+}$ (and put $V_{n+1} := (V_n)_\sharp$ and $\Lambda_{n+1} := (\Lambda_n)_\sharp$), satisfying that $(V_{n_0})_v$ is not split for $v \in \Sigma_{\mathrm{lr}, \mathrm{II}}^+$,
- an object $K_n \in \mathfrak{K}(V_n)$ and an object $(K_{\mathrm{sp}}, K_{n+1}) \in \mathfrak{K}(V_n)_{\mathrm{sp}}$ of the forms

$$\begin{aligned} K_n &= \prod_{v \in \Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}, \mathrm{II}}^+} (K_n)_v \times \prod_{v \notin \Sigma_\infty^+ \cup \Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}, \mathrm{II}}^+} \mathrm{U}(\Lambda_n)(O_{F_v^+}), \\ K_{\mathrm{sp}} &= \prod_{v \in \Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}, \mathrm{II}}^+} (K_{\mathrm{sp}})_v \times \prod_{v \notin \Sigma_\infty^+ \cup \Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}, \mathrm{II}}^+} \mathrm{U}(\Lambda_n)(O_{F_v^+}), \\ K_{n+1} &= \prod_{v \in \Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}, \mathrm{II}}^+} (K_{n+1})_v \times \prod_{v \notin \Sigma_\infty^+ \cup \Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}, \mathrm{II}}^+} \mathrm{U}(\Lambda_{n+1})(O_{F_v^+}), \end{aligned}$$

satisfying $(K_{\mathrm{sp}})_v = (K_n)_v$ for $v \in \Sigma_{\mathrm{min}}^+$, $(K_{\mathrm{sp}})_v \subseteq (K_n)_v$ for $v \in \Sigma_{\mathrm{lr}, \mathrm{II}}^+$, and that $(K_{n_0})_v$ is a transferable open compact subgroup (Definition D.2.1) of $\mathrm{U}(V_{n_0}^\circ)(F_v^+)$ for $v \in \Sigma_{\mathrm{min}}^+$ and is a special maximal subgroup of $\mathrm{U}(V_{n_0}^\circ)(F_v^+)$ for $v \in \Sigma_{\mathrm{lr}, \mathrm{II}}^+$,

- a special inert prime (Definition 3.3.4) \mathfrak{p} of F^+ (with the underlying rational prime p) satisfying
 - (PII1): Σ_{II}^+ does not contain p -adic places;
 - (PII2): ℓ does not divide $p(p^2 - 1)$;
 - (PII3): there exists a CM type Φ containing τ_∞ as in the initial setup of Section 5 satisfying $\mathbb{Q}_p^\Phi = \mathbb{Q}_{p^2}$;
 - (PII4): $P_{\alpha(\Pi_0, \mathfrak{p})} \bmod \lambda^m$ is level-raising special at \mathfrak{p} (Definition 3.1.5);
 $P_{\alpha(\Pi_1, \mathfrak{p})} \bmod \lambda$ is Tate generic at \mathfrak{p} (Definition 3.1.5);
 - (PII7): $P_{\alpha(\Pi_0, \mathfrak{p}) \otimes \alpha(\Pi_1, \mathfrak{p})} \bmod \lambda^m$ is level-raising special at \mathfrak{p} (Definition 3.1.5);
- (So we can and will adopt the setup in Subsection 4.4 to the datum $(V_n, \{\Lambda_{n, q}\}_{q|p})$.)

- remaining data in the initial setup of Section 4 with $\mathbb{Q}_p^\Phi = \mathbb{Q}_{p^2}$; and
- a definite uniformization datum as in Notation 4.4.7.

Put $K_{\text{sp}}^* := (i_n K_{\text{sp}}^p) \times K_{n,p}^*$, and $K_{n_\alpha}^* := (i_{n_\alpha} K_{n_\alpha}^p) \times K_{n_\alpha,p}^*$ for $\alpha = 0, 1$. Put $K_{\text{sp,sp}}^* := (i_n K_{\text{sp}}^p) \times K_{\text{sp,p}}^*$ and $K_{n,\text{sp}}^* := (i_n K_n^p) \times K_{\text{sp,p}}^*$. Like in Subsection 4.5, we put $X_{n_\alpha}^? := X_{\mathfrak{p}}^?(V_{n_\alpha}, K_{n_\alpha}^p)$ for meaningful triples $(X, ?, \alpha) \in \{\mathbf{M}, \mathbf{M}, \mathbf{B}, \mathbf{S}\} \times \{, \eta\} \times \{0, 1\}$.

Notation 7.3.1. We introduce the following ideas of $\mathbb{T}_{n_\alpha}^{\Sigma_{\Pi}^+ \cup \Sigma_p^+}$, for $\alpha = 0, 1$

$$\begin{cases} \mathfrak{m}_\alpha := \mathbb{T}_{n_\alpha}^{\Sigma_{\Pi}^+ \cup \Sigma_p^+} \cap \ker \left(\mathbb{T}_{n_\alpha}^{\Sigma^+} \xrightarrow{\phi_{\Pi_\alpha}} O_E \rightarrow O_E/\lambda \right), \\ \mathfrak{n}_\alpha := \mathbb{T}_{n_\alpha}^{\Sigma_{\Pi}^+ \cup \Sigma_p^+} \cap \ker \left(\mathbb{T}_{n_\alpha}^{\Sigma^+} \xrightarrow{\phi_{\Pi_\alpha}} O_E \rightarrow O_E/\lambda^m \right). \end{cases}$$

We then introduce following assumption.

Assumption 7.3.2. For $\alpha = 0, 1$, we have $H_{\mathfrak{T}}^i(\overline{\mathbf{M}}_{n_\alpha}, O_\lambda)_{\mathfrak{m}_\alpha} = 0$ for $i \neq n_\alpha - 1$, and that $H_{\mathfrak{T}}^{n_\alpha-1}(\overline{\mathbf{M}}_{n_\alpha}, O_\lambda)_{\mathfrak{m}_\alpha}$ is a finite free O_λ -module.

Lemma 7.3.3. *Assume Assumptions 7.1.1, 7.3.2, and Hypothesis 3.2.9 for n_1 .*

- (1) *The $O_\lambda[\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})]$ -module $H_{\mathfrak{T}}^{2r_1}(\overline{\mathbf{M}}_{n_1}, O_\lambda(r_1))_{\mathfrak{m}_1}$ is weakly semisimple (Definition 2.1.2).*
- (2) *The map*

$$\pi_{n_1!} \circ \iota_{n_1}^* : (H_{\mathfrak{T}}^{2r_1}(\overline{\mathbf{M}}_{n_1}, O_\lambda(r_1))_{\mathfrak{m}_1})_{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})} \rightarrow H_{\mathfrak{T}}^0(\overline{\mathbf{S}}_{n_1}, O_\lambda)_{\mathfrak{m}_1}$$

is an isomorphism.

Proof. The proof of the lemma is similar to Theorem 6.2.3. For readers' convenience, we reproduce the details under the current setup.

For (1), by Lemma 5.1.7, we have an isomorphism

$$H_{\mathfrak{T}}^{2r_1}(\overline{\mathbf{M}}_{n_1}, O_\lambda(r_1))_{\mathfrak{m}_1} \simeq H_{\text{ét}}^{2r_1}(\text{Sh}(V_{n_1}, K_{n_1})_{\overline{F}}, O_\lambda(r_1))_{\mathfrak{m}_1}$$

of $O_\lambda[\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^2})]$ -modules. By Lemma 6.1.10, Proposition C.3.1(2), and Hypothesis 3.2.9, we have an isomorphism

$$H_{\text{ét}}^{2r_1}(\text{Sh}(V_{n_1}, K_{n_1})_{\overline{F}}, O_\lambda(r_1))_{\mathfrak{m}_1} \otimes_{O_\lambda} \overline{\mathbb{Q}}_\ell \simeq \bigoplus_{\pi_1} \rho_{\text{BC}(\pi_1), \iota_\ell}^{\mathfrak{c}}(r_1)^{\oplus d(\pi_1)}$$

of representations of Γ_F with coefficients in $\overline{\mathbb{Q}}_\ell$, where $d(\pi_1) := \dim(\pi_1^{\infty, p})^{K_{n_1}^p}$. Here, the direct sum is taken over all stable automorphic representations π_1 of $\text{U}(V_{n_1})(\mathbb{A}_{F^+})$ that is Π_1 -congruent and such that $\pi_{1_{\mathcal{T}_\infty}}$ is a holomorphic discrete series representation of $\text{U}(V_{n_1})(F_{\mathcal{T}_\infty}^+)$ with the Harish-Chandra parameter $\{r_1, r_1 - 1, \dots, 1 - r_1, -r_1\}$; and $\pi_{1_{\mathcal{T}}}$ is trivial for every archimedean place $\mathcal{T} \neq \mathcal{T}_\infty$. We may replace E_λ by a finite extension inside $\overline{\mathbb{Q}}_\ell$ such that $\rho_{\text{BC}(\pi_1), \iota_\ell}$ is defined over E_λ for every π_1 appeared in the previous direct sum. Now we regard $\rho_{\text{BC}(\pi_1), \iota_\ell}$ as a representation over E_λ . Then $\rho_{\text{BC}(\pi_1), \iota_\ell}(r_1)$ admits a Γ_F -stable O_λ -lattice $\mathbf{R}_{\text{BC}(\pi_1)}$, unique up to homothety, whose reduction $\overline{\mathbf{R}}_{\text{BC}(\pi_1)}$ is isomorphic to $\overline{\rho}_{\Pi_1, \lambda}(r_1)$. Moreover, we have an inclusion

$$H_{\text{ét}}^{2r_1}(\text{Sh}(V_{n_1}, K_{n_1})_{\overline{F}}, O_\lambda(r_1))_{\mathfrak{m}_1} \subseteq \bigoplus_{\pi_1} (\mathbf{R}_{\text{BC}(\pi_1)}^{\mathfrak{c}})^{\oplus d(\pi_1)}$$

of $O_\lambda[\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})]$ -modules. By (PII4), we know that $\overline{\rho}_{\Pi_1, \lambda}^{\mathfrak{c}}(r_1)$ is weakly semisimple and

$$\dim_{O_\lambda/\lambda} \overline{\rho}_{\Pi_1, \lambda}^{\mathfrak{c}}(r_1)^{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})} = 1.$$

On the other hand, we have

$$\dim_{E_\lambda} \rho_{\text{BC}(\pi_1), \iota_\ell}^{\mathfrak{c}}(r_1)^{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})} \geq 1.$$

Thus by Lemma 2.1.5, for every π_1 in the previous direct sum, $R_{\text{BC}(\pi_1)}^\varepsilon$ is weakly semisimple. Thus, $H_{\overline{\mathbb{F}}}^{2r_1}(\overline{M}_{n_1}, O_\lambda(r_1))_{\mathfrak{m}_1}$ is weakly semisimple by Lemma 2.1.4(1). Thus, (1) follows.

For (2), we note that in (1) we have also proved that $(H_{\overline{\mathbb{F}}}^{2r_1}(\overline{M}_{n_1}, O_\lambda(r_1))_{\mathfrak{m}_1})_{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})}$ is a free O_λ -module of rank $\sum_{\pi_1} d(\pi_1)$. By Theorem 4.3.10, Proposition B.4.3(2), and (PII4), we know that $\pi_{n_1!} \circ \iota_{n_1}^*$ is surjective. Thus, it remains to show that

$$\sum_{\pi_1} d(\pi_1) \leq \dim_{E_\lambda} H_{\overline{\mathbb{F}}}^0(\overline{S}_{n_1}, O_\lambda)_{\mathfrak{m}_1} \otimes_{O_\lambda} E_\lambda.$$

However, the above inequality is a consequence of Proposition 4.3.4 and Corollary C.3.3.

The lemma is proved. \square

We have a finite morphism $\text{Sh}(V_n, K_{\text{sp}}) \rightarrow \text{Sh}(V_n, K_n) \times_{\text{Spec } F} \text{Sh}(V_{n+1}, K_{n+1})$, which gives rise to a class

$$[\text{Sh}(V_n, K_{\text{sp}})] \in H_{\text{ét}}^{2n}(\text{Sh}(V_{n_0}, K_{n_0}) \times_{\text{Spec } F} \text{Sh}(V_{n_1}, K_{n_1}), O_\lambda(n))$$

by the absolute cycle class map.

Theorem 7.3.4 (Second explicit reciprocity law). *Assume Assumptions 7.1.1, 7.3.2, and Hypothesis 3.2.9 for both n and $n+1$. Then we have*

$$\begin{aligned} & \exp_\lambda \left(\text{loc}_p([\text{Sh}(V_n, K_{\text{sp}})]), H_{\text{ét}}^{2n}((\text{Sh}(V_{n_0}, K_{n_0}) \times_{\text{Spec } F} \text{Sh}(V_{n_1}, K_{n_1}))_{F_p}, O_\lambda(n)) / (\mathfrak{n}_0, \mathfrak{n}_1) \right) \\ & \leq \exp_\lambda \left(\mathbb{1}_{\text{Sh}(V_n^*, K_{\text{sp}}^*)}, O_\lambda[\text{Sh}(V_{n_0}^*, K_{n_0}^*) \times \text{Sh}(V_{n_1}^*, K_{n_1}^*)] / (\mathfrak{n}_0, \mathfrak{n}_1) \right) \end{aligned}$$

where loc_p is introduced in Construction 4.5.1; \exp_λ is introduced in Definition 2.1.6; and the element $\mathbb{1}_{\text{Sh}(V_n^*, K_{\text{sp}}^*)}$ is regarded as the pushforward of the characteristic function along the map $\text{Sh}(V_n^*, K_{\text{sp}}^*) \rightarrow \text{Sh}(V_n^*, K_n^*) \times \text{Sh}(V_{n+1}^*, K_{n+1}^*)$.

Proof. We claim that

- (1) the action of $\mathbf{T}_{n_1, p}^*$ on $H_{\overline{\mathbb{F}}}^{2r_0}(M_{n_0} \times_{\mathbb{T}_p} S_{n_1}, O_\lambda(r_0))_{(\mathfrak{m}_0, \mathfrak{m}_1)}$ is invertible; and
- (2) the composite map

$$\begin{aligned} & (\text{id} \times \pi_{n_1})! \circ (\text{id} \times \iota_{n_1})^* : H_{\overline{\mathbb{F}}}^{2n}(M_{n_0} \times_{\mathbb{T}_p} M_{n_1}, O_\lambda(n))_{(\mathfrak{m}_0, \mathfrak{m}_1)} \rightarrow H_{\overline{\mathbb{F}}}^{2r_0}(M_{n_0} \times_{\mathbb{T}_p} S_{n_1}, O_\lambda(r_0))_{(\mathfrak{m}_0, \mathfrak{m}_1)} \\ & \text{is an isomorphism.} \end{aligned}$$

We prove the theorem assuming these two claims. Take a uniformizer λ_0 of E_λ . Suppose $\lambda_0^e \mathbb{1}_{\text{Sh}(V_n^*, K_{\text{sp}}^*)} = 0$ in $O_\lambda[\text{Sh}(V_{n_0}^*, K_{n_0}^*) \times \text{Sh}(V_{n_1}^*, K_{n_1}^*)] / (\mathfrak{n}_0, \mathfrak{n}_1)$ for some integer $e \geq 0$. Applying Theorem 4.5.2 to the object $(K_{\text{sp}}, K_{n+1}) \in \mathfrak{K}(V_n)_{\text{sp}}$ followed by pushforward, we have

$$\lambda_0^e \mathbf{T}_{n_1, p}^* \cdot (\text{id} \times \pi_{n_1})! (\text{id} \times \iota_{n_1})^* \text{loc}'_p([\text{Sh}(V_n, K_{\text{sp}})]) = 0$$

in $H_{\overline{\mathbb{F}}}^{2n}(M_{n_0} \times_{\mathbb{T}_p} S_{n_1}, O_\lambda(n)) / (\mathfrak{n}_0, \mathfrak{n}_1)$. By the above two claims, we must have

$$\lambda_0^e \text{loc}'_p([\text{Sh}(V_n, K_{\text{sp}})]) = 0$$

in $H_{\overline{\mathbb{F}}}^{2n}(M_{n_0} \times_{\mathbb{T}_p} M_{n_1}, O_\lambda(n)) / (\mathfrak{n}_0, \mathfrak{n}_1)$. Thus, we have

$$\lambda_0^e \text{loc}_p([\text{Sh}(V_n, K_{\text{sp}})]) = 0$$

as the map $H_{\text{ét}}^{2n}((\text{Sh}(V_{n_0}, K_{n_0}) \times_{\text{Spec } F} \text{Sh}(V_{n_1}, K_{n_1}))_{F_p}, O_\lambda(n)) \rightarrow H_{\overline{\mathbb{F}}}^{2n}(M_{n_0} \times_{\mathbb{T}_p} M_{n_1}, O_\lambda(n))$ is an isomorphism. The theorem follows.

Now we consider the two claims. By the Hochschild–Serre spectral sequence, we have a short exact sequence

$$0 \longrightarrow H^1(\mathbb{F}_{p^2}, H_{\text{ét}}^{2n-1}(\overline{M}_{n_0} \times_{\overline{\mathbb{T}}_p} \overline{M}_{n_1}, O_\lambda(n))_{(\mathfrak{m}_0, \mathfrak{m}_1)}) \longrightarrow H_{\overline{\mathbb{F}}}^{2n}(M_{n_0} \times_{\mathbb{T}_p} M_{n_1}, O_\lambda(n))_{(\mathfrak{m}_0, \mathfrak{m}_1)} \longrightarrow H_{\overline{\mathbb{F}}}^{2n}(\overline{M}_{n_0} \times_{\overline{\mathbb{T}}_p} \overline{M}_{n_1}, O_\lambda(n))_{(\mathfrak{m}_0, \mathfrak{m}_1)}^{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})} \longrightarrow 0$$

of O_λ -modules. By the Künneth formula and (an analog of) Lemma 6.1.10, we have

$$H_{\mathbb{F}}^i(\overline{M}_{n_0} \times_{\overline{\mathbb{T}}_p} \overline{M}_{n_1}, O_\lambda)_{(m_0, m_1)} \simeq \bigoplus_{i_0+i_1=i} H_{\mathbb{F}}^{i_0}(\overline{M}_{n_0}, O_\lambda) \otimes_{O_\lambda} H_{\mathbb{F}}^{i_1}(\overline{M}_{n_1}, O_\lambda)$$

for every $i \in \mathbb{Z}$. This implies $H_{\mathbb{F}}^{2n}(\overline{M}_{n_0} \times_{\overline{\mathbb{T}}_p} \overline{M}_{n_1}, O_\lambda(n))_{(m_0, m_1)} = 0$ and

$$H_{\mathbb{F}}^{2n-1}(\overline{M}_{n_0} \times_{\overline{\mathbb{T}}_p} \overline{M}_{n_1}, O_\lambda(n))_{(m_0, m_1)} \simeq H_{\mathbb{F}}^{2r_0-1}(\overline{M}_{n_0}, O_\lambda(r_0))_{m_0} \otimes_{O_\lambda} H_{\mathbb{F}}^{2r_1}(\overline{M}_{n_1}, O_\lambda(r_1))_{m_1}.$$

In particular, we have a canonical isomorphism

$$(7.2) \quad H_{\mathbb{F}}^{2n}(\overline{M}_{n_0} \times_{\overline{\mathbb{T}}_p} \overline{M}_{n_1}, O_\lambda(n))_{(m_0, m_1)} \simeq H^1(\mathbb{F}_{p^2}, H_{\mathbb{F}}^{2r_0-1}(\overline{M}_{n_0}, O_\lambda(r_0))_{m_0} \otimes_{O_\lambda} H_{\mathbb{F}}^{2r_1}(\overline{M}_{n_1}, O_\lambda(r_1))_{m_1}).$$

Similarly, we have

$$(7.3) \quad \begin{aligned} H_{\mathbb{F}}^{2r_0}(\overline{M}_{n_0} \times_{\overline{\mathbb{T}}_p} S_{n_1}, O_\lambda(r_0))_{(m_0, m_1)} &\simeq H^1(\mathbb{F}_{p^2}, H_{\mathbb{F}}^{2r_0-1}(\overline{M}_{n_0}, O_\lambda(r_0))_{m_0} \otimes_{O_\lambda} H_{\mathbb{F}}^0(\overline{S}_{n_1}, O_\lambda)_{m_1}) \\ &= H^1(\mathbb{F}_{p^2}, H_{\mathbb{F}}^{2r_0-1}(\overline{M}_{n_0}, O_\lambda(r_0))_{m_0} \otimes_{O_\lambda} H_{\mathbb{F}}^0(\overline{S}_{n_1}, O_\lambda)_{m_1}). \end{aligned}$$

For claim (1), note that the action of $\mathbb{T}_{n_1, p}$ on $H_{\mathbb{F}}^{2r_0}(\overline{M}_{n_0} \times_{\overline{\mathbb{T}}_p} S_{n_1}, O_\lambda(r_0))_{(m_0, m_1)}$ factors through the second factor under the isomorphism (7.3). By Proposition B.4.3(2) and (PII4), we know that the action of $\mathbb{T}_{n_1, p}^*$ on $H_{\mathbb{F}}^0(\overline{S}_{n_1}, O_\lambda)_{m_1}$ is invertible. Thus, (1) follows.

For claim (2), by (PII7) and a similar argument for the proof of Lemma 7.2.4(3), we know that the $O_\lambda[\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})]$ -module

$$H_{\mathbb{F}}^{2r_0-1}(\overline{M}_{n_0}, O_\lambda(r_0))_{m_0} \otimes_{O_\lambda} \ker \left((H_{\mathbb{F}}^{2r_1}(\overline{M}_{n_1}, O_\lambda(r_1))_{m_1}) \rightarrow (H_{\mathbb{F}}^{2r_1}(\overline{M}_{n_1}, O_\lambda(r_1))_{m_1})_{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})} \right)$$

has zero $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$ -coinvariants. Combining with Lemma 7.3.3, we obtain an isomorphism

$$H_{\mathbb{F}}^{2n}(\overline{M}_{n_0} \times_{\overline{\mathbb{T}}_p} \overline{M}_{n_1}, O_\lambda(n))_{(m_0, m_1)} \simeq H^1(\mathbb{F}_{p^2}, H_{\mathbb{F}}^{2r_0-1}(\overline{M}_{n_0}, O_\lambda(r_0))_{m_0} \otimes_{O_\lambda} (H_{\mathbb{F}}^{2r_1}(\overline{M}_{n_1}, O_\lambda(r_1))_{m_1})_{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})})$$

from (7.2), under which the map $(\text{id} \times \pi_{n_1})_! \circ (\text{id} \times \iota_{n_1})^*$ coincides with $\text{id} \otimes (\pi_{n_1} \circ \iota_{n_1}^*)$. Thus, (2) follows.

The theorem is proved. \square

8. PROOF OF MAIN THEOREMS

In the section, we prove our main theorems on bounding Selmer groups. In Subsection 8.1, we introduce the notation of (weakly) admissible primes for the coefficient field, and make some additional preparation for the main theorem. In Subsections 8.2 and 8.3, we prove our main theorems in the (Selmer) rank 0 and 1 cases, respectively.

8.1. Admissible primes for coefficient field. We keep the setup in Subsection 7.1.

Definition 8.1.1. We introduce following assumptions on a prime λ of E with the underlying rational prime ℓ (and the ring of integers O_λ of E_λ):

- (L1): $\ell > 4n$ and is unramified in F ;
- (L2): Σ_{\min}^+ does not contain ℓ -adic places;
- (L3): the Galois representation $\rho_{\Pi_0, \lambda} \otimes_{E_\lambda} \rho_{\Pi_1, \lambda}$ is absolutely irreducible;
- (L4): Assumption 7.1.1 is satisfied, that is, $\rho_{\Pi_0, \lambda}$ and $\rho_{\Pi_1, \lambda}$ are both residually absolutely irreducible;

(L5): under (L4), for $\alpha = 0, 1$, we have a Γ_F -stable O_λ -lattice R_α in $\rho_{\Pi_\alpha, \lambda}(r_\alpha)$, unique up to homothety, that is $(1-\alpha)$ -polarizable, for which we choose a $(1-\alpha)$ -polarization $\Xi_\alpha: R_\alpha^c \xrightarrow{\sim} R_\alpha^\vee(1-\alpha)$ and an isomorphism $R_\alpha \simeq O_\lambda^{\oplus n_\alpha}$ of O_λ -modules.²⁰ After adopting the notation in Subsection 2.6, we have

(L5-1): either one of the two assumptions in Lemma 2.3.4 is satisfied;

(L5-2): Lemma 2.6.1(3) holds with $F' = F_{\text{rfx}}^+$ (Definition 3.3.2) and $\mathcal{P}(T) = T^2 - 1$ (see Remark 8.1.2 below for a more explicit description);

(L6): under (L4), the homomorphism $\bar{\rho}_{\Pi_0, \lambda, +}$ (Remark 6.1.5) is rigid for $(\Sigma_{\min}^+, \emptyset)$ (Definition E.7.1), and $\bar{\rho}_{\Pi_0, \lambda}|_{\text{Gal}(\bar{F}/F(\zeta_\ell))}$ is absolutely irreducible;

(L7): for $\alpha = 0, 1$, the composite homomorphism $\mathbb{T}_{n_\alpha}^{\Sigma_{\min}^+} \xrightarrow{\phi_{\Pi_\alpha}} O_E \rightarrow O_E/\lambda$ is cohomologically generic (Definition D.1.1).

Finally, we say that

- (1) λ is *weakly admissible* (with respect to (Π_0, Π_1)) if (L1–L5) are satisfied;
- (2) λ is *admissible* (with respect to (Π_0, Π_1)) if (L1–L7) are satisfied.

Remark 8.1.2. In Definition 8.1.1, (L5-2) is equivalent to the following assertion: the image of the restriction of the homomorphism

$$(\bar{\rho}_{0+}, \bar{\rho}_{1+}, \bar{\epsilon}_\ell): \Gamma_{F^+} \rightarrow \mathcal{G}_{n_0}(O_\lambda/\lambda) \times \mathcal{G}_{n_1}(O_\lambda/\lambda) \times (O_\lambda/\lambda)^\times$$

(see Notation 2.5.1 for the notation) to $\text{Gal}(\bar{F}/F_{\text{rfx}}^+)$ contains an element $(\gamma_0, \gamma_1, \xi)$ satisfying

- (a) $\xi^2 - 1 \neq 0$;
- (b) for $\alpha = 0, 1$, γ_α belongs to $(\text{GL}_{n_\alpha}(O_\lambda/\lambda) \times (O_\lambda/\lambda)^\times)\mathfrak{c}$ with order coprime to ℓ ;
- (c) 1 appears in the eigenvalues of each of h_{γ_0} , h_{γ_1} , and $h_{\gamma_0} \otimes h_{\gamma_1}$ (Notation 2.5.2) with multiplicity one;
- (d) h_{γ_0} does not have an eigenvalue that is equal to -1 in O_λ/λ ;
- (e) h_{γ_1} does not have an eigenvalue that is equal to $-\xi$ in O_λ/λ .

Lemma 8.1.3. *The representation $\rho_{\Pi_0, \lambda} \otimes_{E_\lambda} \rho_{\Pi_1, \lambda}(n)$ is pure of weight -1 at every nonarchimedean place w of F not above ℓ (Definition 2.4.4).*

Proof. This is a consequence of Proposition 3.2.4(1) and [TY07, Lemma 1.4(3)]. \square

Lemma 8.1.4. *Assume Hypothesis 3.2.9 for n_1 . Let V_{n_1} be a standard indefinite hermitian space of rank n_1 over F , Λ_{n_1} a self-dual $\prod_{v \notin \Sigma_\infty^+ \cup \Sigma_{\min}^+} O_{F_v}$ -lattice in $V_{n_1} \otimes_F \mathbb{A}_F^{\Sigma_\infty^+ \cup \Sigma_{\min}^+}$, and λ a prime of E . Consider a finite set \mathfrak{P} of special inert primes of F^+ whose underlying rational primes are distinct and coprime to Σ_{\min}^+ , and an object $K_{n_1} \in \mathfrak{K}(V_{n_1})$ of the form $(K_{n_1})_{\Sigma_{\min}^+} \times \prod_{v \notin \Sigma_\infty^+ \cup \Sigma_{\min}^+} \text{U}(\Lambda_{n_1})(O_{F_v}^+)$. Put*

$$\mathfrak{m}_1 := \mathbb{T}_{n_1}^{\Sigma_{\min}^+ \cup \Sigma_{\mathfrak{P}}^+} \cap \ker \left(\mathbb{T}_{n_1}^{\Sigma_{\min}^+} \xrightarrow{\phi_{\Pi_1}} O_E \rightarrow O_E/\lambda \right)$$

where $\Sigma_{\mathfrak{P}}^+$ is the union of Σ_p^+ for all underlying rational primes p of \mathfrak{P} . Suppose that $P_{\alpha(\Pi_1, \mathfrak{p})} \bmod \lambda$ is intertwining generic (Definition 3.1.5) for every $\mathfrak{p} \in \mathfrak{P}$; and that

- (a) either the composite homomorphism $\mathbb{T}_{n_1}^{\Sigma_{\min}^+} \xrightarrow{\phi_{\Pi_1}} O_E \rightarrow O_E/\lambda$ is cohomologically generic;
- (b) or $n_1 = 3$ and $H_{\text{ét}}^i(\text{Sh}(V_3, K_3), O_E/\lambda)_{\mathfrak{m}_1} = 0$ if $i \neq 2$.

Then for every special maximal subgroup $K'_{n_1, \mathfrak{P}}$ of $\prod_{\mathfrak{p} \in \mathfrak{P}} \text{U}(V_{n_1})(F_{\mathfrak{p}}^+)$ and every $i \in \mathbb{Z}$, we have an isomorphism

$$H_{\text{ét}}^i(\text{Sh}(V_{n_1}, K_{n_1}), O_\lambda)_{\mathfrak{m}_1} \simeq H_{\text{ét}}^i(\text{Sh}(V_{n_1}, K'_{n_1, \mathfrak{P}}), O_\lambda)_{\mathfrak{m}_1}$$

²⁰In fact, (L5) does not depend on the choice of Ξ_α and the basis, since Ξ_α is unique up to units in O_λ and the basis is unique up to conjugation in $\text{GL}_{n_\alpha}(O_\lambda)$.

of $O_\lambda[\Gamma_F]$ -modules.

Note that (a) is stronger than (b) when $n_1 = 3$.

Proof. We first note that for every $\mathfrak{p} \in \mathfrak{P}$, $U(V_{n_1})(F_{\mathfrak{p}}^+)$ has two special maximal subgroups up to conjugation, exact one of which is hyperspecial maximal.

Take an element $\mathfrak{p} \in \mathfrak{P}$, a special maximal subgroup $K_{n_1, \mathfrak{P}}^{\mathfrak{p}}$ of $\prod_{\mathfrak{p}' \in \mathfrak{P} \setminus \{\mathfrak{p}\}} U(V_{n_1})(F_{\mathfrak{p}'}^+)$, a hyperspecial maximal subgroup $K_{n_1, \mathfrak{p}}^\circ$ of $U(V_{n_1})(F_{\mathfrak{p}}^+)$, and a non-hyperspecial special maximal subgroup $K_{n_1, \mathfrak{p}}^\bullet$ of $U(V_{n_1})(F_{\mathfrak{p}}^+)$. We claim that if $H_{\text{ét}}^i(\text{Sh}(V_{n_1}, K_{n_1, \mathfrak{P}}^{\mathfrak{p}} K_{n_1, \mathfrak{p}}^\circ), O_E/\lambda)_{\mathfrak{m}_1} = 0$ for $i \neq 2r_1$, then there is an isomorphism

$$H_{\text{ét}}^i(\text{Sh}(V_{n_1}, K_{n_1, \mathfrak{P}}^{\mathfrak{p}} K_{n_1, \mathfrak{p}}^\circ), O_\lambda)_{\mathfrak{m}_1} \simeq H_{\text{ét}}^i(\text{Sh}(V_{n_1}, K_{n_1, \mathfrak{P}}^{\mathfrak{p}} K_{n_1, \mathfrak{p}}^\bullet), O_\lambda)_{\mathfrak{m}_1}$$

of $O_\lambda[\Gamma_F]$ -modules for every $i \in \mathbb{Z}$.

Fix an embedding $E_\lambda \hookrightarrow \overline{\mathbb{Q}}_\ell$. We first show that there is an isomorphism

$$(8.1) \quad H_{\text{ét}}^i(\text{Sh}(V_{n_1}, K_{n_1, \mathfrak{P}}^{\mathfrak{p}} K_{n_1, \mathfrak{p}}^\circ), O_\lambda)_{\mathfrak{m}_1} \otimes_{O_\lambda} \overline{\mathbb{Q}}_\ell \simeq H_{\text{ét}}^i(\text{Sh}(V_{n_1}, K_{n_1, \mathfrak{P}}^{\mathfrak{p}} K_{n_1, \mathfrak{p}}^\bullet), O_\lambda)_{\mathfrak{m}_1} \otimes_{O_\lambda} \overline{\mathbb{Q}}_\ell$$

of $\overline{\mathbb{Q}}_\ell[\Gamma_F]$ -modules for every $i \in \mathbb{Z}$. Let $\Lambda_{n_1, \mathfrak{p}}^\circ$ be the self-dual $O_{F_{\mathfrak{p}}}$ -lattice in $V_{n_1} \otimes_F F_{\mathfrak{p}}$ whose stabilizer is $K_{n_1, \mathfrak{p}}^\circ$. Without loss of generality, we may assume that $K_{n_1, \mathfrak{p}}^\bullet$ is the stabilizer of a lattice $\Lambda_{n_1, \mathfrak{p}}^\bullet$ satisfying $\Lambda_{n_1, \mathfrak{p}}^\circ \subseteq \Lambda_{n_1, \mathfrak{p}}^\bullet$ and $(\Lambda_{n_1, \mathfrak{p}}^\bullet)^\vee / \mathfrak{p} \Lambda_{n_1, \mathfrak{p}}^\bullet \simeq \mathbb{F}_{p^2}$. To show (8.1), it suffices to show that for every (necessarily cuspidal) automorphic representation π_1 of $U(V_{n_1})(\mathbb{A}_{F^+})$ that appears in either side of (8.1), the maps

$$\mathbf{T}_{n_1, \mathfrak{p}}^{\bullet \circ} : \pi_{1, \mathfrak{p}}^{K_{n_1, \mathfrak{p}}^\circ} \rightarrow \pi_{1, \mathfrak{p}}^{K_{n_1, \mathfrak{p}}^\bullet}, \quad \mathbf{T}_{n_1, \mathfrak{p}}^{\circ \bullet} : \pi_{1, \mathfrak{p}}^{K_{n_1, \mathfrak{p}}^\bullet} \rightarrow \pi_{1, \mathfrak{p}}^{K_{n_1, \mathfrak{p}}^\circ}$$

are both isomorphisms. Hypothesis 3.2.9 and the Chebotarev density theorem imply that $\rho_{\text{BC}(\pi_1), \iota_\ell}$ and $\rho_{\Pi_1, \lambda} \otimes_{E_\lambda} \overline{\mathbb{Q}}_\ell$ have the isomorphic (irreducible) residual representations. In particular, the Satake parameter of $\text{BC}(\pi_1)_{\mathfrak{p}}$ does not contain $\{-p, -p^{-1}\}$ by Proposition 3.2.4(2) and the assumption that $P_{\alpha(\Pi_1, \mathfrak{p})} \bmod \lambda$ is intertwining generic. Thus, we obtain the isomorphism (8.1) by Proposition B.4.3(2).

To prove the claim it suffices to show that $H_{\text{ét}}^i(\text{Sh}(V_{n_1}, K_{n_1, \mathfrak{P}}^{\mathfrak{p}} K_{n_1, \mathfrak{p}}^\bullet), O_\lambda)_{\mathfrak{m}_1}$ is a free O_λ -module for every $i \in \mathbb{Z}$. If we assume (a), then this follows immediately. If we assume (b), then this follows from Proposition 5.9.4, Lemma 6.2.1, and a straightforward computation on the spectral sequence in Lemma 5.8.2.

The lemma follows immediately from the above claim by induction on the number of primes $\mathfrak{p} \in \mathfrak{P}$ for which $K'_{n_1, \mathfrak{P}}$ is not hyperspecial maximal at \mathfrak{p} . Note that the initial induction hypothesis is satisfied by either (a) or (b). \square

Proposition 8.1.5. *Suppose $E = \mathbb{Q}$ and that there are two elliptic curves A_0 and A_1 over F^+ such that for every rational prime ℓ of E and $\alpha = 0, 1$, we have $\rho_{\Pi_\alpha, \ell} \simeq \text{Sym}^{n_\alpha - 1} H_{\text{ét}}^1(A_{\alpha \overline{F}}, \mathbb{Q}_\ell)|_{\Gamma_F}$. If $A_{0 \overline{F}}$ and $A_{1 \overline{F}}$ are not isogenous to each other and $\text{End}(A_{0 \overline{F}}) = \text{End}(A_{1 \overline{F}}) = \mathbb{Z}$, then all but finitely many rational primes ℓ are weakly admissible; and when $[F^+ : \mathbb{Q}] > 1$, all but finitely many rational primes ℓ are admissible.*

Proof. We need to show that every condition in Definition 8.1.1 excludes only finitely many ℓ (for (L7) we assume $[F^+ : \mathbb{Q}] > 1$). By [Ser72, Théorème 6], for sufficiently large ℓ , the homomorphisms

$$\Gamma_{F^+} \rightarrow \text{GL}(H_{\text{ét}}^1(A_{\alpha \overline{F}}, \mathbb{F}_\ell)) \simeq \text{GL}_2(\mathbb{F}_\ell)$$

are both surjective for $\alpha = 0, 1$. Thus, we may assume that this is the case.

For (L1) and (L2), this is trivial.

For (L3), (L4), and (L5), this has been proved in Proposition 2.6.3.

For (L6), by Proposition E.5.12, the condition that $\bar{\rho}_{\Pi_0, \lambda, +}$ is rigid for $(\Sigma_{\min}^+, \emptyset)$ excludes only finitely many ℓ . It is clear that the remaining two conditions also exclude only finitely many ℓ .

For (L7), this follows from Corollary D.1.4. \square

8.2. Main theorem in the Selmer rank 0 case.

Theorem 8.2.1. *Keep the setup in Subsection 7.1. Assume Hypothesis 3.2.9 for both n and $n+1$. If $L(\frac{1}{2}, \Pi_0 \times \Pi_1) \neq 0$, then for all admissible primes λ of E , and for all but finitely many weakly admissible primes λ of E when $n = 2$, we have*

$$H_f^1(F, \rho_{\Pi_0, \lambda} \otimes_{E_\lambda} \rho_{\Pi_1, \lambda}(n)) = 0.$$

Proof. By Lemma 8.2.2 below, we may fix the choices of $V_n^\circ, \Lambda_n^\circ, (K_n^\circ, K_{n+1}^\circ)$ in that lemma such that

$$\sum_{s \in \text{Sh}(V_n^\circ, K_n^\circ)} f(s, \text{sh}_\uparrow(s)) \neq 0$$

for some $f \in O_E[\text{Sh}(V_{n_0}^\circ, K_{n_0}^\circ)][\ker \phi_{\Pi_0}] \otimes_{O_E} O_E[\text{Sh}(V_{n_1}^\circ, K_{n_1}^\circ)][\ker \phi_{\Pi_1}]$. Moreover, by Lemma D.2.2(3), we may assume that $(K_{n_0}^\circ)_v$ is transferable (Definition D.2.1) for $v \in \Sigma_{\min}^+$.

We take a prime λ of E with the underlying rational prime ℓ . We adopt notation in Subsection 2.6 with the initial data in Definition 8.1.1. Define two nonnegative integers m_{per} and m_{lat} as follows.

- (1) Let m_{per} be the largest (nonnegative) integer such that

$$\sum_{s \in \text{Sh}(V_n^\circ, K_n^\circ)} f(s, \text{sh}_\uparrow(s)) \in \lambda^{m_{\text{per}}} O_E$$

for every $f \in O_E[\text{Sh}(V_{n_0}^\circ, K_{n_0}^\circ)][\ker \phi_{\Pi_0}] \otimes_{O_E} O_E[\text{Sh}(V_{n_1}^\circ, K_{n_1}^\circ)][\ker \phi_{\Pi_1}]$.

- (2) We choose a standard *indefinite* hermitian space V_{n_1} over F of rank n_1 , together with an identification $U((V_{n_1}^\circ)^\infty) \simeq U(V_{n_1}^\infty)$ of reductive groups over $\mathbb{A}_{F^+}^\infty$.²¹ In particular, we have the Shimura variety $\text{Sh}(V_{n_1}, K_{n_1}^\circ)$. By Hypothesis 3.2.9, we have an isomorphism

$$H_{\text{ét}}^{2r_1}(\text{Sh}(V_{n_1}, K_{n_1}^\circ)_{\overline{F}}, E_\lambda(r_1)) / \ker \phi_{\Pi_1} \simeq (R_1^\circ \otimes_{O_\lambda} E_\lambda)^{\oplus \mu_1}$$

of $E_\lambda[\Gamma_F]$ -modules for some integer $\mu_1 > 0$. We fix a map

$$H_{\text{ét}}^{2r_1}(\text{Sh}(V_{n_1}, K_{n_1}^\circ)_{\overline{F}}, O_\lambda(r_1)) / \ker \phi_{\Pi_1} \rightarrow (R_1^\circ)^{\oplus \mu_1}$$

of $O_\lambda[\Gamma_F]$ -modules whose kernel and cokernel are both O_λ -torsion. Then we let m_{lat} be the smallest nonnegative integer such that both the kernel and the cokernel are annihilated by $\lambda^{m_{\text{lat}}}$.

Now we assume that either λ is admissible, or $n = 2$ and λ is weakly admissible and satisfies $H_{\text{ét}}^i(\text{Sh}(V_3, K_3^\circ)_{\overline{F}}, O_E/\lambda) / \ker \phi_{\Pi_1} = 0$ for $i \neq 2$ (which only excludes finitely many primes). Note that in both cases we have $H_{\text{ét}}^i(\text{Sh}(V_{n_1}, K_{n_1}^\circ)_{\overline{F}}, O_E/\lambda) / \ker \phi_{\Pi_1} = 0$ if $i \neq 2r_1$.

We start to prove the theorem by contradiction, hence assume

$$\dim_{E_\lambda} H_f^1(F, \rho_{\Pi_0, \lambda} \otimes_{E_\lambda} \rho_{\Pi_1, \lambda}(n)) \geq 1.$$

Take a sufficiently large positive integer m which will be determined later. By Lemma 8.1.3, we may apply Proposition 2.4.6 by taking Σ to be the set of places of F above $\Sigma_{\min}^+ \cup \Sigma_\ell^+$. Then we obtain a submodule S of $H_{f, \text{R}}^1(F, \overline{\text{R}}^{(m)})$ that is free of rank 1 over $O_\lambda/\lambda^{m-m_\Sigma}$ such that $\text{loc}_w|_S = 0$ for every nonarchimedean place $w \in \Sigma$ not above ℓ . Now we apply the discussion in Subsection 2.3 to the submodule $S \subseteq H^1(F, \overline{\text{R}}^{(m)})$. By (L5-1) and Lemma 2.3.6, we obtain an injective map

$$\theta_S: \text{Gal}(F_S/F_{\overline{\rho}^{(m)}}) \rightarrow \text{Hom}_{O_\lambda}(S, \overline{\text{R}}^{(m)})$$

²¹There are many choices of such V_{n_1} and the isomorphism. We choose one only to get some control on the discrepancy of the integral cohomology of Shimura varieties and the lattice coming from Galois representations.

whose image generates an O_λ -submodule containing $\lambda^{\mathfrak{R}(m)} \text{Hom}_{O_\lambda}(S, \bar{\mathfrak{R}}^{(m)})$, which further contains $\lambda^{\mathfrak{R}} \text{Hom}_{O_\lambda}(S, \bar{\mathfrak{R}}^{(m)})$ by Lemma 2.3.3 and (L3). By (L5-2) and Lemma 2.6.1, we may choose an element $(\gamma_1, \gamma_2, \xi)$ in the image of $(\bar{\rho}_{1+}^{(m)}, \bar{\rho}_{2+}^{(m)}, \bar{\epsilon}_\ell^{(m)})|_{\text{Gal}(\bar{F}/F_{\text{rffx}}^+)}$ satisfying Lemma 2.6.1(2). It gives rise to an element $\gamma \in (\text{GL}_{n_0 n_1}(O_\lambda/\lambda^m) \times (O_\lambda/\lambda^m)^\times) \mathfrak{c}$ as in Notation 2.5.2 such that $(\bar{\mathfrak{R}}^{(m)})^{h_\gamma}$ is a free O_λ/λ^m -module of rank 1 by Lemma 2.6.2 and (2.4). Now we apply the discussion in Subsection 2.5. By Proposition 2.5.5 (with $m_0 = m_\Sigma$, $r_\gamma = 1$, $r_S = 1$), we may fix an (S, γ) -abundant element $\Psi \in G_{S, \gamma}$ (Definition 2.5.6).

We apply the discussion and notation in Subsection 7.2 to our situation with λ , m , $\Sigma_{\text{r}, \text{I}}^+ = \emptyset$, $\Sigma_{\text{I}}^+ = \Sigma_{\text{min}}^+$, $(V_n^\circ, \Lambda_n^\circ)$, K_n° and $(K_n^\circ, K_{n+1}^\circ)$. By the Chebotarev density theorem, we can choose a γ -abundant place $w_+^{(m)}$ of $F_+^{(m)}$ satisfying $\Psi_{w^{(m)}} = \Psi$ and whose underlying prime \mathfrak{p} of F^+ (and the underlying rational prime p) is a special inert prime satisfying (PI1)–(PI7) and

(PI8): the natural map

$$\text{H}_{\text{ét}}^{2r_1}(\text{Sh}(V_{n_1}, K_{n_1}^\circ)_{\bar{F}}, O_\lambda(r_1))/(\mathbb{T}_{n_1}^{\Sigma_{\text{I}}^+ \cup \Sigma_p^+} \cap \ker \phi_{\Pi_1}) \rightarrow \text{H}_{\text{ét}}^{2r_1}(\text{Sh}(V_{n_1}, K_{n_1}^\circ)_{\bar{F}}, O_\lambda(r_1))/\ker \phi_{\Pi_1}$$

is an isomorphism.

We also choose remaining data in the initial setup of Section 5 with $\mathbb{Q}_p^\Phi = \mathbb{Q}_{p^2}$, a definite uniformization datum as in Notation 5.10.13, and an indefinite uniformization datum as in Notation 5.10.1. By the definition of m_{per} , we have

$$(8.2) \quad \exp_\lambda \left(\mathbb{1}_{\text{Sh}(V_n^\circ, K_{\text{sp}}^\circ)}, O_E[\text{Sh}(V_{n_0}^\circ, K_{n_0}^\circ) \times \text{Sh}(V_{n_1}^\circ, K_{n_1}^\circ)]/(\mathfrak{n}_0, \mathfrak{n}_1) \right) \geq m - m_{\text{per}},$$

where we recall that

$$\mathfrak{n}_\alpha = \mathbb{T}_{n_\alpha}^{\Sigma_{\text{I}}^+ \cup \Sigma_p^+} \cap \ker \left(\mathbb{T}_{n_\alpha}^{\Sigma_{\text{min}}^+} \xrightarrow{\phi_{\Pi_\alpha}} O_E \rightarrow O_E/\lambda^m \right)$$

for $\alpha = 0, 1$. Here, $\mathbb{1}_{\text{Sh}(V_n^\circ, K_{\text{sp}}^\circ)}$ is nothing but the characteristic function of the graph $\Delta \text{Sh}(V_n^\circ, K_n^\circ)$ of the map $\text{Sh}(V_n^\circ, K_n^\circ) \rightarrow \text{Sh}(V_{n+1}^\circ, K_{n+1}^\circ)$.

We claim that there exists an element $c_1 \in \text{H}^1(F, \bar{\mathfrak{R}}^{(m_1)\mathfrak{c}})$ for some positive integer $m_1 \leq m$ satisfying

$$(8.3) \quad \exp_\lambda \left(\partial_{\mathfrak{p}} \text{loc}_{\mathfrak{p}}(c_1), \text{H}_{\text{sing}}^1(F_{\mathfrak{p}}, \bar{\mathfrak{R}}^{(m_1)\mathfrak{c}}) \right) \geq m - m_{\text{per}} - m_{\text{lat}};$$

and such that for every nonarchimedean place w of F not above $\Sigma^+ \cup \{\mathfrak{p}\}$,

$$(8.4) \quad \text{loc}_w(c_1) \in \text{H}_f^1(F_w, \bar{\mathfrak{R}}^{(m_1)\mathfrak{c}})$$

holds.

We first prove the theorem assuming the existence of such c_1 . Fix a generator of the submodule $S \subseteq \text{H}_{f, \mathfrak{R}}^1(F, \bar{\mathfrak{R}}^{(m)})$ and denote by its image in $\text{H}^1(F, \bar{\mathfrak{R}}^{(m_1)})$ by s_1 . We also identify $\bar{\mathfrak{R}}^{(m_1)\mathfrak{c}}$ with $(\bar{\mathfrak{R}}^{(m_1)})^*$ via the polarization Ξ . Now we compute the local Tate pairing $\langle s_1, c_1 \rangle_w$ (2.1) for every nonarchimedean place w of F .

- Suppose that w is above Σ_{min}^+ . Then we have $\text{loc}_w(s_1) = 0$ by our choice of S . Thus, $\langle s_1, c_1 \rangle_w = 0$.
- Suppose that w is above Σ_ℓ^+ . Then by (L2), $\mathfrak{R}_{\mathbb{Q}}$ is crystalline with Hodge–Tate weights in $[-n, n-1]$. Thus, we have $\text{loc}_w(s_1) \in \text{H}_f^1(F_w, \bar{\mathfrak{R}}^{(m_1)})$ by Lemma 2.4.3(2) and (L1). By (8.4), Lemma 2.2.6 and (L1), we have $\lambda^{m_{\text{dif}}} \langle s_1, c_1 \rangle_w = 0$ where $\mathfrak{d}_\lambda = \lambda^{m_{\text{dif}}} \subseteq O_\lambda$ is the different ideal of $E_\lambda/\mathbb{Q}_\ell$.
- Suppose that w is not above $\Sigma_{\text{min}}^+ \cup \Sigma_\ell^+ \cup \{\mathfrak{p}\}$. Then by (L2), \mathfrak{R} is unramified. Thus, we have $\text{loc}_w(s_1) \in \text{H}_f^1(F_w, \bar{\mathfrak{R}}^{(m_1)})$ by Lemma 2.4.3(1). By (8.4) and Lemma 2.2.3, we have $\langle s_1, c_1 \rangle_w = 0$.

○ Suppose that w is the unique place above \mathfrak{p} . Then since $w_+^{(m)}$ is abundant, we have

$$\exp_\lambda \left(\text{loc}_w(s_1), H_f^1(F_w, \bar{R}^{(m_1)}) \right) \geq m_1 - m_\Sigma - \mathfrak{r}_R.$$

By (8.3) and Lemma 2.2.3 again, we have

$$\exp_\lambda \left(\langle s_1, c_1 \rangle_w, O_\lambda / \lambda^{m_1} \right) \geq m - m_{\text{per}} - m_{\text{lat}} - m_\Sigma - \mathfrak{r}_R.$$

Therefore, as long as we take m such that $m > m_{\text{per}} + m_{\text{lat}} + m_\Sigma + \mathfrak{r}_R + m_{\text{dif}}$, we will have a contradiction to the relation

$$\sum_w \langle s_1, c_1 \rangle_w = 0,$$

where the sum is taken over all nonarchimedean places w of F . The theorem is proved.

Now we consider the claim. By (L4), (L6), and Theorem 6.3.4(5), we have an isomorphism

$$(8.5) \quad H_{\text{ét}}^{2r_0-1} \left((\text{Sh}(V'_{n_0}, j_{n_0} K_{n_0}^{p_0} K'_{n_0,p})_{\bar{F}}, O_\lambda(r_0)) / \mathfrak{n}_0 \right) \xrightarrow{\sim} \bigoplus_{i=1}^{\mu_0} \bar{R}_0^{(m_i)c}$$

of $O_\lambda[\Gamma_F]$ -modules, for finitely many positive integers m_1, \dots, m_{μ_0} at most m . Assumption 7.2.2 for $\alpha = 0$ is satisfied by (L7) if $n \geq 3$ and by (L4) if $n = 2$.

By Lemma 8.1.4, we have an isomorphism

$$H_{\text{ét}}^i \left(\text{Sh}(V_{n_1}, K_{n_1}^c)_{\bar{F}}, O_\lambda \right)_{\mathfrak{m}_1} \simeq H_{\text{ét}}^i \left(\text{Sh}(V'_{n_1}, j_{n_1} K_{n_1}^{p_0} K'_{n_1,p})_{\bar{F}}, O_\lambda \right)_{\mathfrak{m}_1}$$

of $O_\lambda[\Gamma_F]$ -modules. Assumption 7.2.2 for $\alpha = 1$ is satisfied by (L7) if $n \geq 3$ and by (PI8) if $n = 2$. Moreover, by (PI8), we may fix a map

$$H_{\text{ét}}^{2r_1} \left(\text{Sh}(V'_{n_1}, j_{n_1} K_{n_1}^{p_0} K'_{n_1,p})_{\bar{F}}, O_\lambda(r_1) \right) / (\mathbb{T}_{n_1}^{\Sigma_1^+ \cup \Sigma_p^+} \cap \ker \phi_{\Pi_1}) \rightarrow (R_1^c)^{\oplus \mu_1}$$

of $O_\lambda[\Gamma_F]$ -modules whose kernel and cokernel are both annihilated by $\lambda^{m_{\text{lat}}}$. Taking quotient by λ^m , we obtain a map

$$(8.6) \quad H_{\text{ét}}^{2r_1} \left(\text{Sh}(V'_{n_1}, j_{n_1} K_{n_1}^{p_0} K'_{n_1,p})_{\bar{F}}, O_\lambda(r_1) \right) / \mathfrak{n}_1 \rightarrow (\bar{R}_1^{(m)c})^{\oplus \mu_1}$$

of $O_\lambda[\Gamma_F]$ -modules whose kernel and cokernel are both annihilated by $\lambda^{m_{\text{lat}}}$.

To continue, we adopt the notational abbreviation prior to Corollary 7.2.8. By Lemma 6.1.10 and the Künneth formula, we obtain a map

$$(8.7) \quad \Upsilon: H_{\text{ét}}^{2n-1} \left((\text{Sh}'_{n_0} \times_{\text{Spec } F} \text{Sh}'_{n_1})_{\bar{F}}, O_\lambda(n) \right) / (\mathfrak{n}_0, \mathfrak{n}_1) \rightarrow \bigoplus_{i=1}^{\mu} \bar{R}^{(m_i)c}$$

of $O_\lambda[\Gamma_F]$ -modules whose kernel and cokernel are both annihilated by $\lambda^{m_{\text{lat}}}$, from (8.5) and (8.6). Here, we have re-indexed μ_1 copies of $\{m_1, \dots, m_{\mu_0}\}$ into $\mu := \mu_0 \mu_1$ positive integers at most m . Recall that we have a class

$$\text{AJ}(\text{Sh}'_{\text{sp}}) \in H^1(F, H_{\text{ét}}^{2n-1} \left((\text{Sh}'_{n_0} \times_{\text{Spec } F} \text{Sh}'_{n_1})_{\bar{F}}, O_\lambda(n) \right) / (\mathfrak{n}_0, \mathfrak{n}_1)),$$

where Sh'_{sp} is nothing but the graph of the morphism $\text{Sh}'_n \rightarrow \text{Sh}'_{n+1}$. By Corollary 7.2.8 and (8.2), we have

$$(8.8) \quad \exp_\lambda \left(\partial_p \text{loc}_p \text{AJ}(\text{Sh}'_{\text{sp}}), H_{\text{sing}}^1(F_p, H_{\text{ét}}^{2n-1} \left((\text{Sh}'_{n_0} \times_{\text{Spec } F} \text{Sh}'_{n_1})_{\bar{F}}, O_\lambda(n) \right) / (\mathfrak{n}_0, \mathfrak{n}_1)) \right) \geq m - m_{\text{per}}.$$

For every $1 \leq i \leq \mu$, let

$$\Upsilon_i: H_{\text{ét}}^{2n-1} \left((\text{Sh}'_{n_0} \times_{\text{Spec } F} \text{Sh}'_{n_1})_{\bar{F}}, O_\lambda(n) \right) / (\mathfrak{n}_0, \mathfrak{n}_1) \rightarrow \bar{R}^{(m_i)c}$$

be the composition of Υ (8.7) with the projection to the i -th factor; and put

$$c_i := H^1(F, \Upsilon_i(\text{AJ}(\text{Sh}'_{\text{sp}}))) \in H^1(F, \bar{R}^{(m_i)c}).$$

Then (8.8) implies

$$\max_{1 \leq i \leq \mu} \exp_{\lambda} \left(\partial_{\mathfrak{p}} \text{loc}_{\mathfrak{p}}(c_i), H_{\text{sing}}^1(F_{\mathfrak{p}}, \bar{R}^{(m_i)c}) \right) \geq m - m_{\text{per}} - m_{\text{lat}}.$$

Without loss of generality, we obtain (8.3). On the other hand, as both Sh'_n and Sh'_{n+1} have smooth models over O_{F_w} for which (an analogue of) Lemma 4.1.4 holds, we obtain (8.4). \square

Lemma 8.2.2. *Let Π_0 and Π_1 be as in Theorem 8.2.1 such that for $\alpha = 0, 1$, $\rho_{\Pi_{\alpha}, \lambda}$ is absolutely irreducible for some prime λ of E . Suppose $L(\frac{1}{2}, \Pi_0 \times \Pi_1) \neq 0$. Then there exist*

- a standard definite hermitian space V_n° of rank n over F , together with a self-dual $\prod_{v \notin \Sigma_{\infty}^+ \cup \Sigma_{\min}^+} O_{F_v}$ -lattice Λ_n° in $V_n^{\circ} \otimes_F \mathbb{A}_F^{\Sigma_{\infty}^+ \cup \Sigma_{\min}^+}$ (and put $V_{n+1}^{\circ} := (V_n^{\circ})_{\sharp}$ and $\Lambda_{n+1}^{\circ} := (\Lambda_n^{\circ})_{\sharp}$),
- an object $(K_n^{\circ}, K_{n+1}^{\circ}) \in \mathfrak{K}(V_n^{\circ})_{\text{sp}}$ in which $K_{n\alpha}^{\circ}$ is of the form

$$K_{n\alpha}^{\circ} = \prod_{v \in \Sigma_{\min}^+} (K_{n\alpha}^{\circ})_v \times \prod_{v \notin \Sigma_{\infty}^+ \cup \Sigma_{\min}^+} \text{U}(\Lambda_{n\alpha}^{\circ})(O_{F_v^+})$$

for $\alpha = 0, 1$,

such that

$$\sum_{s \in \text{Sh}(V_n^{\circ}, K_n^{\circ})} f(s, \text{sh}_{\uparrow}(s)) \neq 0$$

for some element $f \in O_E[\text{Sh}(V_{n_0}^{\circ}, K_{n_0}^{\circ})][\ker \phi_{\Pi_0}] \otimes_{O_E} O_E[\text{Sh}(V_{n_1}^{\circ}, K_{n_1}^{\circ})][\ker \phi_{\Pi_1}]$.

Proof. This follows from the direction (1) \Rightarrow (2) of [BPLZZ, Theorem 1.7], together with [BPLZZ, Remark 4.15]. Note that since our Π_0 and Π_1 are relevant representations of $\text{GL}_{n_0}(\mathbb{A}_F)$ and $\text{GL}_{n_1}(\mathbb{A}_F)$, respectively, both members in the pair of hermitian spaces in (2) of [BPLZZ, Theorem 1.7] have to be standard definite. \square

Corollary 8.2.3. *Keep the setup in Subsection 7.1. Suppose that*

- (a) *there exists a very special inert prime \mathfrak{p} of F^+ (Definition 3.3.4) such that $\Pi_{0, \mathfrak{p}}$ is Steinberg, and $\Pi_{1, \mathfrak{p}}$ is unramified whose Satake parameter contains 1 exactly once;*
- (b) *for $\alpha = 0, 1$, there exists a nonarchimedean place w_{α} of F such that $\Pi_{\alpha, w_{\alpha}}$ is supercuspidal;*
- (c) *$[F^+ : \mathbb{Q}] > 1$ if $n \geq 3$.*

If $L(\frac{1}{2}, \Pi_0 \times \Pi_1) \neq 0$, then for all but finitely many primes λ of E , we have

$$H_f^1(F, \rho_{\Pi_0, \lambda} \otimes_{E_{\lambda}} \rho_{\Pi_1, \lambda}(n)) = 0.$$

Proof. By Theorem 8.2.1, it suffices to show that all but finitely many primes λ of E are admissible (or weakly admissible if $n = 2$). It suffices to show that each of conditions (L1–L6) in Definition 8.1.1 excludes only finitely many λ , and also for (L7) if $n \neq 3$.

For (L1) and (L2), this is trivial.

For (L4), this follows from Lemma E.8.3 by (b).

For (L3), this follows from Lemma 8.2.4 below by (L4) and (a).

For (L6), this follows from Theorem E.8.4 by (a) and (b).

For (L7), this follows from Corollary D.1.4 by (c).

For (L5-1), let λ be a prime of E satisfying (L4) and (L6), whose underlying rational prime is at least $2n(n+1) - 1$. Then by (a), $\bar{\rho}_{\Pi_0, \lambda}$ and $\bar{\rho}_{\Pi_1, \lambda}$ satisfy the assumptions in Lemma 8.2.4 below, with $k = O_{\lambda}/\lambda$ and $\Gamma = \Gamma_F$. Thus, by Lemma 8.2.4(2), assumption (b) of Lemma 2.3.4 hence (L5-1) hold.

For (L5-2), take an arithmetic Frobenius element $\phi_{\mathfrak{p}} \in \Gamma_{F_{\mathfrak{p}}^+}$. By Definition 3.3.4, $\phi_{\mathfrak{p}}$ belongs to $\text{Gal}(\bar{F}/F_{\text{rlx}}^+)$. For $\alpha = 0, 1$, put $r_{\alpha} := \lfloor \frac{n_{\alpha}}{2} \rfloor$ as always. By (a), the Satake parameter of $\Pi_{0, \mathfrak{p}}$ is

$\{p^{\pm 1}, \dots, p^{\pm(2r_0-1)}\}$; and we may write the Satake parameter of $\Pi_{1,p}$ as $\{1, \alpha_1^{\pm 1}, \dots, \alpha_{r_1}^{\pm 1}\}$ in which α_i is an algebraic number other than 1 for $1 \leq i \leq r_1$. For our purpose, we may replace E by a finite extension in \mathbb{C} so that $\alpha_i \in E$ for $1 \leq i \leq r_1$. By Proposition 3.2.4(1), we have $|\alpha_i| = 1$ for $1 \leq i \leq r_1$. Therefore, for all but finitely many prime λ of E , we have

- $\{p, \alpha_1, \dots, \alpha_{r_1}\}$ is contained in O_λ^\times ;
- $\{p^{\pm 1} \bmod \lambda, \dots, p^{\pm(2r_0-1)} \bmod \lambda\}$ consists of distinct elements and does not contain -1 ;
- $\{\alpha_i \bmod \lambda \mid 1 \leq i \leq r_1\}$ is disjoint from $\{1, -p, -p^{-1}\}$;
- $\{p^{\pm 1} \alpha_i \bmod \lambda, \dots, p^{\pm(2r_0-1)} \alpha_i \bmod \lambda \mid 1 \leq i \leq r_1\}$ is disjoint from $\{p, p^{-1}\}$.

Then for every prime λ satisfying (L4) and the above properties, (L5-2) is satisfied by taking the element $(\bar{\rho}_{0+}, \bar{\rho}_{1+}, \bar{\epsilon}_\ell)(\phi_p)$ in Lemma 2.6.1(3).

The corollary follows. \square

Lemma 8.2.4. *Let Γ be a group, and k a field of characteristic either zero or at least $2n(n+1)-1$. Let $\rho_0: \Gamma \rightarrow \mathrm{GL}_{n_0}(k)$ and $\rho_1: \Gamma \rightarrow \mathrm{GL}_{n_1}(k)$ be two homomorphisms that are absolutely irreducible. Suppose that there exists an element $t \in \Gamma$ such that $\rho_0(t) = 1 + J_{n_0}$ and $\rho_1(t) = 1$. Then we have*

- (1) $\rho_0 \otimes \rho_1$ is absolutely irreducible;
- (2) $\rho_0 \otimes \rho_1$ is not a subquotient of $\mathrm{ad}(\rho_0 \otimes \rho_1)$.

Proof. We may assume that k is algebraically closed. For $\alpha = 0, 1$, let $V_i = k^{\oplus n_i}$ be the space which Γ acts on through ρ_α . By [Ser94, Corollaire 1], we know that both $\rho_0 \otimes \rho_1$ and $\mathrm{ad}(\rho_0 \otimes \rho_1)$ are semisimple.

For (1), thanks to the element t , every subrepresentation of $V_0 \otimes_k V_1$ is of the form $V_0 \otimes_k V'_1$ where V'_1 is a subrepresentation of V_1 . Since ρ_1 is irreducible, it follows that $\rho_0 \otimes \rho_1$ is irreducible.

For (2), note that $(\rho_0 \otimes \rho_1)(t)$ is conjugate to $(1 + J_{n_0})^{\oplus n_1}$. On the other hand, $\mathrm{ad}(\rho_0 \otimes \rho_1)(t)$ is conjugate to

$$\bigoplus_{i=1}^{n_0} (1 + J_{2i-1})^{\oplus n_1^2}.$$

Since n_0 is even and $1, 3, \dots, 2n_0 - 1$ are odd, $\rho_0 \otimes \rho_1$ is not a subquotient of $\mathrm{ad}(\rho_0 \otimes \rho_1)$ as $\mathrm{ad}(\rho_0 \otimes \rho_1)$ is semisimple.

The lemma is proved. \square

Corollary 8.2.5. *Let $n \geq 2$ be an integer and denote by n_0 and n_1 the unique even and odd numbers in $\{n, n+1\}$, respectively. Let A_0 and A_1 be two modular elliptic curves over F^+ such that $\mathrm{End}(A_{0\bar{F}}) = \mathrm{End}(A_{1\bar{F}}) = \mathbb{Z}$. Suppose that*

- (a) $A_{0\bar{F}}$ and $A_{1\bar{F}}$ are not isogenous to each other;
- (b) both $\mathrm{Sym}^{n_0-1} A_0$ and $\mathrm{Sym}^{n_1-1} A_1$ are modular; and
- (c) $[F^+ : \mathbb{Q}] > 1$ if $n \geq 3$.

If the (central critical) L -value $L(n, \mathrm{Sym}^{n_0-1} A_{0F} \times \mathrm{Sym}^{n_1-1} A_{1F})$ does not vanish, then we have

$$H_f^1(F, \mathrm{Sym}^{n_0-1} H_{\text{ét}}^1(A_{0\bar{F}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathrm{Sym}^{n_1-1} H_{\text{ét}}^1(A_{1\bar{F}}, \mathbb{Q}_\ell)(n)) = 0$$

for all but finitely many rational primes ℓ .

Proof. By (b) and [AC89], both $\mathrm{Sym}^{n_0-1} A_{0F}$ and $\mathrm{Sym}^{n_1-1} A_{1F}$ are modular. Thus, we may let Π_α be the (cuspidal) automorphic representation of $\mathrm{GL}_{n_\alpha}(\mathbb{A}_F)$ associated to $\mathrm{Sym}^{n_\alpha-1} A_{\alpha F}$ for $\alpha = 0, 1$, which is a relevant representation (Definition 1.1.3). We also have the identify

$$L(n+s, \mathrm{Sym}^{n_0-1} A_{0F} \times \mathrm{Sym}^{n_1-1} A_{1F}) = L(\frac{1}{2} + s, \Pi_0 \times \Pi_1)$$

of L -functions, and that the representation of Γ_F on $\mathrm{Sym}^{n_\alpha-1} H_{\text{ét}}^1(A_{\alpha\bar{F}}, \mathbb{Q}_\ell)$ is isomorphic to $\rho_{\Pi_\alpha, \ell}$ for $\alpha = 0, 1$. By Proposition 3.2.10 and (c), Hypothesis 3.2.9 is known in this case. Then the

corollary follows immediately from Theorem 8.2.1 and Proposition 8.1.5 (where we use (a) and (c)) with $E = \mathbb{Q}$. \square

Remark 8.2.6. In this remark, we summarize the current knowledge on the modularity of symmetric powers of elliptic curves, namely, condition (a) in Corollary 8.2.5. Let A be a modular elliptic curve over F^+ . We have

- $\text{Sym}^2 A$ is modular by [GJ76];
- $\text{Sym}^3 A$ is modular by [KS02];
- $\text{Sym}^4 A$ is modular by [Kim03];
- $\text{Sym}^5 A$ and $\text{Sym}^6 A$ are modular if F^+ is linearly disjoint from $\mathbb{Q}(\zeta_5)$ over \mathbb{Q} ;
- $\text{Sym}^7 A$ is modular if F^+ is linearly disjoint from $\mathbb{Q}(\zeta_{35})$ over \mathbb{Q} ;
- $\text{Sym}^8 A$ is modular if F^+ is linearly disjoint from $\mathbb{Q}(\zeta_7)$ over \mathbb{Q} ;

in which the last three cases are obtained in a series of recent work [CT14, CT15, CT17] of Clozel and Thorne.

8.3. Main theorem in the Selmer rank 1 case. We state the following weak version of the arithmetic Gan–Gross–Prasad conjecture.

Conjecture 8.3.1. *Suppose $L(\frac{1}{2}, \Pi_0 \times \Pi_1) = 0$ but $L'(\frac{1}{2}, \Pi_0 \times \Pi_1) \neq 0$. Then there exist*

- a standard indefinite hermitian space V_n of rank n over F , together with a self-dual $\prod_{v \notin \Sigma_\infty^+ \cup \Sigma_{\min}^+} \mathcal{O}_{F_v}$ -lattice Λ_n in $V_n \otimes_F \mathbb{A}_F^{\Sigma_\infty^+ \cup \Sigma_{\min}^+}$ (and put $V_{n+1} := (V_n)_\sharp$ and $\Lambda_{n+1} := (\Lambda_n)_\sharp$),
- an object $(K_n, K_{n+1}) \in \mathfrak{K}(V_n)_{\text{sp}}$ in which K_{n_α} is of the form

$$K_{n_\alpha} = \prod_{v \in \Sigma_{\min}^+} (K_{n_\alpha})_v \times \prod_{v \notin \Sigma_\infty^+ \cup \Sigma_{\min}^+} \text{U}(\Lambda_{n_\alpha})(\mathcal{O}_{F_v^+})$$

for $\alpha = 0, 1$,

such that for every prime λ of E , the graph $\Delta \text{Sh}(V_n, K_n)$ of the morphism $\text{sh}_\dagger: \text{Sh}(V_n, K_n) \rightarrow \text{Sh}(V_{n+1}, K_{n+1})$ (4.6) is nonvanishing in the quotient Chow group

$$\text{CH}^n(\text{Sh}(V_{n_0}, K_{n_0}) \times_{\text{Spec } F} \text{Sh}(V_{n_1}, K_{n_1}))_E / (\ker \phi_{\Pi_0}, \ker \phi_{\Pi_1}).$$

In the situation of the above conjecture, since both Π_0 and Π_1 are cuspidal, we have

$$\text{H}_{\text{ét}}^i((\text{Sh}(V_{n_0}, K_{n_0}) \times_{\text{Spec } F} \text{Sh}(V_{n_1}, K_{n_1}))_{\overline{F}}, E_\lambda) / (\ker \phi_{\Pi_0}, \ker \phi_{\Pi_1}) = 0$$

if $i \neq 2n - 1$. In particular, the Hochschild–Serre spectral sequence gives rise to a coboundary map

$$\begin{aligned} \text{AJ}_\lambda^{\Pi_0, \Pi_1}: Z^n(\text{Sh}(V_{n_0}, K_{n_0}) \times_{\text{Spec } F} \text{Sh}(V_{n_1}, K_{n_1})) \\ \rightarrow \text{H}^1(F, \text{H}_{\text{ét}}^{2n-1}((\text{Sh}(V_{n_0}, K_{n_0}) \times_{\text{Spec } F} \text{Sh}(V_{n_1}, K_{n_1}))_{\overline{F}}, E_\lambda(n)) / (\ker \phi_{\Pi_0}, \ker \phi_{\Pi_1})). \end{aligned}$$

Theorem 8.3.2. *Keep the setup in Subsection 7.1. Assume Hypothesis 3.2.9 for both n and $n + 1$. Let λ be a prime of E for which there exist*

- a standard indefinite hermitian space V_n of rank n over F , together with a self-dual $\prod_{v \notin \Sigma_\infty^+ \cup \Sigma_{\min}^+} \mathcal{O}_{F_v}$ -lattice Λ_n in $V_n \otimes_F \mathbb{A}_F^{\Sigma_\infty^+ \cup \Sigma_{\min}^+}$ (and put $V_{n+1} := (V_n)_\sharp$ and $\Lambda_{n+1} := (\Lambda_n)_\sharp$),
- an object $(K_n, K_{n+1}) \in \mathfrak{K}(V_n)_{\text{sp}}$ in which K_{n_α} is of the form

$$K_{n_\alpha} = \prod_{v \in \Sigma_{\min}^+} (K_{n_\alpha})_v \times \prod_{v \notin \Sigma_\infty^+ \cup \Sigma_{\min}^+} \text{U}(\Lambda_{n_\alpha})(\mathcal{O}_{F_v^+})$$

for $\alpha = 0, 1$, satisfying that $(K_{n_0})_v$ is a transferable open compact subgroup (Definition D.2.1) of $\text{U}(V_{n_0}^\circ)(F_v^+)$ for $v \in \Sigma_{\min}^+$,

such that

$$(8.9) \quad \text{AJ}_\lambda^{\Pi_0, \Pi_1}(\Delta \text{Sh}(V_n, K_n)) \neq 0.$$

If we further assume that either λ is admissible, or $n = 2$ and λ is weakly admissible and satisfies $H_{\text{ét}}^i(\text{Sh}(V_3, K_3)_{\overline{F}}, O_E/\lambda)/\ker \phi_{\Pi_1} = 0$ for $i \neq 2$, then we have

$$\dim_{E_\lambda} H_f^1(F, \rho_{\Pi_0, \lambda} \otimes_{E_\lambda} \rho_{\Pi_1, \lambda}(n)) = 1.$$

Remark 8.3.3. In fact, (8.9) already implies that the global epsilon factor of $\Pi_0 \times \Pi_1$ is -1 .

Proof of Theorem 8.3.2. We take a prime λ of E for which we may choose data $V_n, \Lambda_n, (K_n, K_{n+1})$ as in the statement of the theorem such that $\text{AJ}_\lambda^{\Pi_0, \Pi_1}(\Delta \text{Sh}(V_n, K_n)) \neq 0$. We assume that λ satisfies either (a) or (b) of Lemma 8.1.4. Lemma 8.1.3 and (L2) imply that $\text{AJ}_\lambda^{\Pi_0, \Pi_1}(\Delta \text{Sh}(V_n, K_n))$ belongs to the subspace

$$H_f^1(F, H_{\text{ét}}^{2n-1}((\text{Sh}(V_{n_0}, K_{n_0}) \times_{\text{Spec } F} \text{Sh}(V_{n_1}, K_{n_1}))_{\overline{F}}, E_\lambda(n))/(\ker \phi_{\Pi_0}, \ker \phi_{\Pi_1}))$$

hence to the submodule

$$H_f^1(F, H_{\text{ét}}^{2n-1}((\text{Sh}(V_{n_0}, K_{n_0}) \times_{\text{Spec } F} \text{Sh}(V_{n_1}, K_{n_1}))_{\overline{F}}, O_\lambda(n))/(\ker \phi_{\Pi_0}, \ker \phi_{\Pi_1}))$$

by Definition 2.4.2.

We adopt notation in Subsection 2.6 with the initial data in Definition 8.1.1. Define two nonnegative integers m_{per} and m_{lat} as follows.

- (1) By Hypothesis 3.2.9, we may choose a map

$$H_{\text{ét}}^{2n-1}((\text{Sh}(V_{n_0}, K_{n_0}) \times_{\text{Spec } F} \text{Sh}(V_{n_1}, K_{n_1}))_{\overline{F}}, O_\lambda(n))/(\ker \phi_{\Pi_0}, \ker \phi_{\Pi_1}) \rightarrow \mathbb{R}^c$$

of $O_\lambda[\Gamma_F]$ -modules such that the induced image of $\text{AJ}_\lambda^{\Pi_0, \Pi_1}(\Delta \text{Sh}(V_n, K_n))$ in $H_f^1(F, \mathbb{R}^c)$, denoted by s^c , is non-torsion. Let $s \in H_f^1(F, \mathbb{R})$ be the element corresponding to s^c under the isomorphism in Lemma 2.4.5. We put

$$m_{\text{per}} := \text{ord}_\lambda \left(s, H_f^1(F, \mathbb{R})/H_f^1(F, \mathbb{R})_{\text{tor}} \right)$$

(Definition 2.1.6), which is a nonnegative integer.

- (2) By Hypothesis 3.2.9, we have an isomorphism

$$H_{\text{ét}}^{2r_1}(\text{Sh}(V_{n_1}, K_{n_1})_{\overline{F}}, E_\lambda(r_1))/\ker \phi_{\Pi_1} \simeq (\mathbb{R}_1^c \otimes_{O_\lambda} E_\lambda)^{\oplus \mu_1}$$

of $E_\lambda[\Gamma_F]$ -modules for some integer $\mu_1 > 0$. We fix a map

$$H_{\text{ét}}^{2r_1}(\text{Sh}(V_{n_1}, K_{n_1})_{\overline{F}}, O_\lambda(r_1))/\ker \phi_{\Pi_1} \rightarrow (\mathbb{R}_1^c)^{\oplus \mu_1}$$

of $O_\lambda[\Gamma_F]$ -modules whose kernel and cokernel are both O_λ -torsion. Then we let m_{lat} be the smallest nonnegative integer such that both the kernel and the cokernel are annihilated by $\lambda^{m_{\text{lat}}}$.

Note that in (1), we obtain an element $s \in H_f^1(F, \mathbb{R})_{\mathbb{Q}} = H_f^1(F, \mathbb{R}_{\mathbb{Q}}) = H_f^1(F, \rho_{\Pi_0, \lambda} \otimes_{E_\lambda} \rho_{\Pi_1, \lambda}(n))$ that is nonzero. In particular, we have $\dim_{E_\lambda} H_f^1(F, \rho_{\Pi_0, \lambda} \otimes_{E_\lambda} \rho_{\Pi_1, \lambda}(n)) \geq 1$.

We start to prove the theorem by contradiction, hence assume

$$\dim_{E_\lambda} H_f^1(F, \rho_{\Pi_0, \lambda} \otimes_{E_\lambda} \rho_{\Pi_1, \lambda}(n)) \geq 2.$$

Take a sufficiently large positive integer m which will be determined later. We fix a uniformizer λ_0 of E_λ . By Lemma 8.1.3, we may apply Proposition 2.4.6 by taking Σ to be the set of places of F above $\Sigma_{\text{min}}^+ \cup \Sigma_\ell^+$. Then we obtain a submodule S of $H_{f, \mathbb{R}}^1(F, \mathbb{R}^{(m)})$ containing (the image of) $\lambda_0^{m\Sigma - m_{\text{per}}} s$ of order 0,²² that is free of rank 2 over $O_\lambda/\lambda^{m - m_\Sigma}$, and such that $\text{loc}_w|_S = 0$ for every

²²Here, $\lambda_0^{-m_{\text{per}}} s$ is any element in $H_f^1(F, \mathbb{R})$ satisfying $\lambda_0^{m_{\text{per}}} (\lambda_0^{-m_{\text{per}}} s) = s$.

nonarchimedean place $w \in \Sigma$ not above ℓ . Now we apply the discussion in Subsection 2.3 to the submodule $S \subseteq H^1(F, \bar{R}^{(m)})$. By (L5-1) and Lemma 2.3.6, we obtain an injective map

$$\theta_S: \text{Gal}(F_S/F_{\bar{\rho}^{(m)}}) \rightarrow \text{Hom}_{O_\lambda}(S, \bar{R}^{(m)})$$

whose image generates an O_λ -submodule containing $\lambda^{4\mathfrak{r}_R} \text{Hom}_{O_\lambda}(S, \bar{R}^{(m)})$, which further contains $\lambda^{4\mathfrak{r}_R} \text{Hom}_{O_\lambda}(S, \bar{R}^{(m)})$ by Lemma 2.3.3 and (L3). By (L5-2) and Lemma 2.6.1, we may choose an element $(\gamma_1, \gamma_2, \xi)$ in the image of $(\bar{\rho}_{1+}^{(m)}, \bar{\rho}_{2+}^{(m)}, \bar{\epsilon}_\ell^{(m)})|_{\text{Gal}(\bar{F}/F_{\text{rfix}}^+)}$ satisfying Lemma 2.6.1(2). It gives rise to an element $\gamma \in (\text{GL}_{n_0 n_1}(O_\lambda/\lambda^m) \times (O_\lambda/\lambda^m)^\times)\mathfrak{c}$ as in Notation 2.5.2 such that $(\bar{R}^{(m)})^{h_\gamma}$ is a free O_λ/λ^m -module of rank 1 by Lemma 2.6.2 and (2.4). Now we apply the discussion in Subsection 2.5. By Proposition 2.5.5 (with $m_0 = m_\Sigma$, $r_\gamma = 1$, $r_S = 2$), we may fix an (S, γ) -abundant pair $(\Psi_1, \Psi_2) \in G_{S, \gamma}^2$ (Definition 2.5.6). By the definition of a (S, γ) -abundant pair, we may choose a basis $\{s_1, s_2\}$ of S such that $\theta_S(\Psi_1)(s_2) = \theta_S(\Psi_2)(s_1) = 0$, and

$$(8.10) \quad \exp_\lambda \left(\theta_S(\Psi_j)(s_j), \bar{R}^{(m), h_\gamma} \right) \geq m - m_\Sigma - 4\mathfrak{r}_R$$

for $j = 1, 2$. Moreover, without lost of generality, we may assume $\lambda_0^{m_\Sigma - m_{\text{per}}} s = a_1 s_1 + a_2 s_2$ in which $a_1 \in O_\lambda^\times$.

First, we apply the discussion and notation in Subsection 7.3 to our situation with λ , m , $\Sigma_{\text{r}, \text{II}}^+ = \emptyset$, $\Sigma_{\text{II}}^+ = \Sigma_{\text{min}}^+$, (V_n, Λ_n) , K_n and (K_n, K_{n+1}) . By the Chebotarev density theorem, we can choose a γ -abundant place $w_{1+}^{(m)}$ of $F_+^{(m)}$ satisfying $\Psi_{w_{1+}^{(m)}} = \Psi_1$ and whose underlying prime \mathfrak{p}_1 of F^+ (and the underlying rational prime p_1) is a special inert prime satisfying (PII1)–(PII7) and **(PII8)**: the natural map

$$H_{\text{ét}}^{2r_1}(\text{Sh}(V_{n_1}, K_{n_1})_{\bar{F}}, O_\lambda(r_1)) / (\mathbb{T}_{n_1}^{\Sigma_{\text{II}}^+ \cup \Sigma_{p_1}^+} \cap \ker \phi_{\text{II}_1}) \rightarrow H_{\text{ét}}^{2r_1}(\text{Sh}(V_{n_1}, K_{n_1})_{\bar{F}}, O_\lambda(r_1)) / \ker \phi_{\text{II}_1}$$

is an isomorphism.

We also choose remaining data in the initial setup of Section 4 with $\mathbb{Q}_{p_1}^\Phi = \mathbb{Q}_{p_1^2}$, a definite uniformization datum $(V_{n_\alpha}^*, \mathfrak{i}_{n_\alpha}, \{\Lambda_{n_\alpha, q}^*\}_{q|p_1})$ for $\alpha = 0, 1$ as in Notation 4.4.7. By (2.4), (8.10), and our choice of S , we have

$$\exp_\lambda \left(s, H_f^1(F_{w_1}, \bar{R}^{(m)}) \right) \geq m - m_{\text{per}} - 4\mathfrak{r}_R,$$

which implies that

$$\exp_\lambda \left(\text{loc}_{\mathfrak{p}_1}([\Delta \text{Sh}(V_n, K_n)]), H_{\text{ét}}^{2n}((\text{Sh}(V_{n_0}, K_{n_0}) \times_{\text{Spec } F} \text{Sh}(V_{n_1}, K_{n_1}))_{F_{\mathfrak{p}_1}}, L(n)) / (\mathfrak{n}_0, \mathfrak{n}_1) \right) \geq m - m_{\text{per}} - 4\mathfrak{r}_R.$$

Here, we recall that

$$\mathfrak{n}_\alpha = \mathbb{T}_{n_\alpha}^{\Sigma_{\text{II}}^+ \cup \Sigma_{p_1}^+} \cap \ker \left(\mathbb{T}_{n_\alpha}^{\Sigma_{\text{min}}^+} \xrightarrow{\phi_{\text{II}_\alpha}} O_E \rightarrow O_E/\lambda^m \right)$$

for $\alpha = 0, 1$. Note that Assumption 7.3.2 for $\alpha = 0$ is satisfied by (L7) if $n \geq 3$ and by (L4) if $n = 2$; Assumption 7.3.2 for $\alpha = 1$ is satisfied by (L7) if $n \geq 3$ and by (PII8) if $n = 2$. Thus, we may apply Theorem 7.3.4 hence obtain

$$(8.11) \quad \exp_\lambda \left(\mathbb{1}_{\text{Sh}(V_n^*, K_{\text{sp}}^*)}, O_E[\text{Sh}(V_{n_0}^*, K_{n_0}^*) \times \text{Sh}(V_{n_1}^*, K_{n_1}^*)] / (\mathfrak{n}_0, \mathfrak{n}_1) \right) \geq m - m_{\text{per}} - 4\mathfrak{r}_R.$$

Second, we apply the discussion and notation in Subsection 7.2 to our situation with λ , m , $\Sigma_{\text{r}, \text{I}}^+ = \{\mathfrak{p}_1\}$, $\Sigma_{\text{I}}^+ = \Sigma_{\text{min}}^+ \cup \Sigma_{p_1}^+$, $V_n^\circ = V_n^*$, $K_n^\circ = K_n^*$ and $(K_{\text{sp}}^\circ, K_{n+1}^\circ) = (K_{\text{sp}}^*, K_{n+1}^*)$. By the Chebotarev density theorem, we can choose a γ -abundant place $w_{2+}^{(m)}$ of $F_+^{(m)}$ satisfying $\Psi_{w_{2+}^{(m)}} = \Psi_2$ and whose underlying prime \mathfrak{p}_2 of F^+ (and the underlying rational prime p_2) is a special inert prime satisfying (PI1)–(PI7), $p_2 \neq p_1$, and

(PI8): the natural map

$$H_{\text{ét}}^{2r_1}(\text{Sh}(V_{n_1}, K_{n_1})_{\overline{F}}, O_\lambda(r_1))/(\mathbb{T}_{n_1}^{\Sigma_1^+ \cup \Sigma_{p_2}^+} \cap \ker \phi_{\Pi_1}) \rightarrow H_{\text{ét}}^{2r_1}(\text{Sh}(V_{n_1}, K_{n_1})_{\overline{F}}, O_\lambda(r_1))/\ker \phi_{\Pi_1}$$

is an isomorphism.

We claim that there exists an element $c_2 \in H^1(F, \overline{R}^{(m_2)c})$ for some positive integer $m_2 \leq m$ satisfying

$$(8.12) \quad \exp_\lambda \left(\partial_{p_2} \text{loc}_{p_2}(c_2), H_{\text{sing}}^1(F_{p_2}, \overline{R}^{(m_2)c}) \right) \geq m - m_{\text{per}} - 4\mathfrak{t}_R - m_{\text{lat}};$$

and such that for every nonarchimedean place w of F not above $\Sigma^+ \cup \{\mathfrak{p}_1, \mathfrak{p}_2\}$,

$$(8.13) \quad \text{loc}_w(c_2) \in H_f^1(F_w, \overline{R}^{(m_2)c})$$

holds.

By Remark 4.3.8 and Remark 4.4.8, we know that there exists an isomorphism $U((V_{n_1}^\circ)^\infty) \simeq U(V_{n_1}^\infty)$ sending $K_{n_1}^\circ$ to K_{n_1} . Then the claim can be proved by the exactly same argument for the similar claim in the proof of Theorem 8.2.1, using (8.11) and the fact that $\bar{\rho}_{\Pi_0, \lambda, +}$ is rigid for $(\Sigma_{\text{min}}^+, \Sigma_{\text{lr}, 1}^+)$.²³

Now we deduce a contradiction. Replace s_2 by its image in $H_f^1(F, \overline{R}^{(m_2)})$. We also identify $\overline{R}^{(m_2)c}$ with $(\overline{R}^{(m_2)})^*$ via the polarization Ξ . Now we compute the local Tate pairing $\langle s_2, c_2 \rangle_w$ (2.1) for every nonarchimedean place w of F .

- Suppose that w is above Σ_{min}^+ . Then we have $\text{loc}_w(s_2) = 0$ by our choice of S . Thus, $\langle s_2, c_2 \rangle_w = 0$.
- Suppose that w is above Σ_ℓ^+ . Then by (L2), $R_{\mathbb{Q}}$ is crystalline with Hodge–Tate weights in $[1 - n, n]$. Thus, we have $\text{loc}_w(s_2) \in H_f^1(F_w, \overline{R}^{(m_2)})$ by Lemma 2.4.3(2) and (L1). By (8.13), Lemma 2.2.6 and (L1), we have $\lambda^{m_{\text{dif}}} \langle s_2, c_2 \rangle_w = 0$ where $\mathfrak{d}_\lambda = \lambda^{m_{\text{dif}}} \subseteq O_\lambda$ is the different ideal of $E_\lambda/\mathbb{Q}_\ell$.
- Suppose that w is not above $\Sigma_{\text{min}}^+ \cup \Sigma_\ell^+ \cup \{\mathfrak{p}_1, \mathfrak{p}_2\}$. Then by (L2), R is unramified. Thus, we have $\text{loc}_w(s_2) \in H_f^1(F_w, \overline{R}^{(m_2)})$ by Lemma 2.4.3(1). By (8.13) and Lemma 2.2.3, we have $\langle s_2, c_2 \rangle_w = 0$.
- Suppose that w is the unique place above \mathfrak{p}_1 . Then we have $\text{loc}_w(s_2) = 0$ by our choice of the basis $\{s_1, s_2\}$. Thus, we have $\langle s_2, c_2 \rangle_w = 0$.
- Suppose that w is the unique place above \mathfrak{p}_2 . Then we have

$$\exp_\lambda \left(\text{loc}_w(s_2), H_f^1(F_w, \overline{R}^{(m_2)}) \right) \geq m_2 - m_\Sigma - 4\mathfrak{t}_R.$$

By (8.12) and Lemma 2.2.3 again, we have

$$\exp_\lambda \left(\langle s_2, c_2 \rangle_w, O_\lambda/\lambda^{m_2} \right) \geq m - m_{\text{per}} - m_{\text{lat}} - m_\Sigma - 8\mathfrak{t}_R.$$

Therefore, as long as we take m such that $m > m_{\text{per}} + m_{\text{lat}} + m_\Sigma + 8\mathfrak{t}_R + m_{\text{dif}}$, we will have a contradiction to the relation

$$\sum_w \langle s_2, c_2 \rangle_w = 0,$$

where the sum is taken over all nonarchimedean places w of F . The theorem is proved. \square

²³In fact, one needs to use the additional fact that when $[F^+ : \mathbb{Q}] > 1$, both Shimura varieties Sh'_{n_0} and Sh'_{n_1} have proper smooth reduction at every place w of F above $\Sigma_{p_1}^+ \setminus \{\mathfrak{p}_1\}$. See Remark 5.1.8.

APPENDIX A. UNITARY DELIGNE–LUSZTIG VARIETIES

In this appendix, we study some unitary Deligne–Lusztig varieties in Subsections A.1 and A.2 for those used in Sections 4 and 5, respectively.

We fix a rational prime p . Let κ be a field containing \mathbb{F}_{p^2} . Recall from Subsection 1.3 that we denote by $\sigma: S \rightarrow S$ the absolute p -power Frobenius morphism for schemes S in characteristic p .

A.1. Unitary Deligne–Lusztig varieties in the smooth case. In this subsection, we introduce certain Deligne–Lusztig varieties that appear in the special fiber of the smooth integral model studied in Section 4.

Consider a pair $(\mathcal{V}, \{ , \})$ in which \mathcal{V} is a finite dimensional κ -linear space, and $\{ , \}: \mathcal{V} \times \mathcal{V} \rightarrow \kappa$ is a (not necessarily non-degenerate) pairing that is (κ, σ) -linear in the first variable and κ -linear in the second variable. For every κ -scheme S , put $\mathcal{V}_S := \mathcal{V} \otimes_{\kappa} \mathcal{O}_S$. Then there is a unique pairing $\{ , \}_S: \mathcal{V}_S \times \mathcal{V}_S \rightarrow \mathcal{O}_S$ extending $\{ , \}$ that is (\mathcal{O}_S, σ) -linear in the first variable and \mathcal{O}_S -linear in the second variable. For a subbundle $H \subseteq \mathcal{V}_S$, we denote by $H^\perp \subseteq \mathcal{V}_S$ its *right* orthogonal complement under $\{ , \}_S$.

Definition A.1.1. We say that a pair $(\mathcal{V}, \{ , \})$ is *admissible* if there exists an \mathbb{F}_{p^2} -linear subspace $\mathcal{V}_0 \subseteq \mathcal{V}_{\bar{\kappa}}$ such that the induced map $\mathcal{V}_0 \otimes_{\mathbb{F}_{p^2}} \bar{\kappa} \rightarrow \mathcal{V}_{\bar{\kappa}}$ is an isomorphism, and $\{x, y\} = -\{y, x\}^\sigma$ for every $x, y \in \mathcal{V}_0$.

Definition A.1.2. For a pair $(\mathcal{V}, \{ , \})$ and an integer h , we define a presheaf

$$\mathrm{DL}(\mathcal{V}, \{ , \}, h)$$

on Sch/κ such that for every $S \in \mathrm{Sch}/\kappa$, $\mathrm{DL}(\mathcal{V}, \{ , \}, h)(S)$ is the set of subbundles H of \mathcal{V}_S of rank h such that $H^\perp \subseteq H$. We call $\mathrm{DL}(\mathcal{V}, \{ , \}, h)$ the (*unitary*) *Deligne–Lusztig variety* (see Proposition A.1.3 below) attached to $(\mathcal{V}, \{ , \})$ of rank h .

Proposition A.1.3. *Consider an admissible pair $(\mathcal{V}, \{ , \})$. Put $N := \dim_{\kappa} \mathcal{V}$ and $d := \dim_{\kappa} \mathcal{V}^\perp$.*

- (1) *If $2h < N + d$ or $h > N$, then $\mathrm{DL}(\mathcal{V}, \{ , \}, h)$ is empty.*
- (2) *If $N + d \leq 2h \leq 2N$, then $\mathrm{DL}(\mathcal{V}, \{ , \}, h)$ is represented by a projective smooth scheme over κ of dimension $(2h - N - d)(N - h)$ with a canonical isomorphism for its tangent sheaf*

$$\mathcal{T}_{\mathrm{DL}(\mathcal{V}, \{ , \}, h)/\kappa} \simeq \mathrm{Hom} \left(\mathcal{H}/\mathcal{H}^\perp, \mathcal{V}_{\mathrm{DL}(\mathcal{V}, \{ , \}, h)}/\mathcal{H} \right)$$

where $\mathcal{H} \subseteq \mathcal{V}_{\mathrm{DL}(\mathcal{V}, \{ , \}, h)}$ is the universal subbundle.

- (3) *If $N + d < 2h \leq 2N$, then $\mathrm{DL}(\mathcal{V}, \{ , \}, h)$ is geometrically irreducible.*

Proof. Part (1) is obvious from the definitions.

For (2), $\mathrm{DL}(\mathcal{V}, \{ , \}, h)$ is a closed sub-presheaf of the Grassmannian scheme $\mathrm{Gr}(\mathcal{V}, h)$ classifying subbundles of \mathcal{V} of rank h , hence is represented by a projective scheme over κ . Now we compute the tangent sheaf. Consider a closed immersion $S \hookrightarrow \hat{S}$ in Sch/κ defined by an ideal sheaf \mathcal{I} with $\mathcal{I}^2 = 0$. Take an object $H \subseteq \mathcal{V}_S$ in $\mathrm{DL}(\mathcal{V}, \{ , \}, h)(S)$. Let D_H and G_H be the subset of $\mathrm{DL}(\mathcal{V}, \{ , \}, h)(\hat{S})$ and $\mathrm{Gr}(\mathcal{V}, h)(\hat{S})$ of elements that reduce to H , respectively. It is well-known that G_H is canonically a torsor over $\mathrm{Hom}_{\mathcal{O}_S}(H, (\mathcal{V}_S/H) \otimes_{\mathcal{O}_S} \mathcal{I})$. Since $\mathcal{I}^p = 0$, the right orthogonal complement \hat{H}^\perp depends only on H for every $\hat{H} \in G_H$. In particular, the subset D_H is canonically a torsor over the subgroup $\mathrm{Hom}_{\mathcal{O}_S}(H/H^\perp, (\mathcal{V}_S/H) \otimes_{\mathcal{O}_S} \mathcal{I})$ of $\mathrm{Hom}_{\mathcal{O}_S}(H, (\mathcal{V}_S/H) \otimes_{\mathcal{O}_S} \mathcal{I})$. Thus, $\mathrm{DL}(\mathcal{V}, \{ , \}, h)$ is smooth; and we have a canonical isomorphism for the tangent sheaf

$$\mathcal{T}_{\mathrm{DL}(\mathcal{V}, \{ , \}, h)/\kappa} \simeq \mathrm{Hom} \left(\mathcal{H}/\mathcal{H}^\perp, \mathcal{V}_{\mathrm{DL}(\mathcal{V}, \{ , \}, h)}/\mathcal{H} \right)$$

where \mathcal{H} is the universal subbundle. Note that this is a locally free $\mathcal{O}_{\mathrm{DL}(\mathcal{V}, \{ , \}, h)}$ -module of rank $(2h - N - d)(N - h)$.

For (3), we may assume that κ is algebraically closed. By Definitions A.1.1 and A.1.2, we have a canonical isomorphism $\mathrm{DL}(\mathcal{V}, \{ , \}, h) \simeq \mathrm{DL}(\mathcal{V}_0, \{ , \}_0, h) \otimes_{\mathbb{F}_{p^2}} \kappa$, where $\{ , \}_0$ denotes the restriction of $\{ , \}$ to \mathcal{V}_0 . Suppose $d = 0$. Then $\{ , \}_0$ is non-degenerate. By [BR06, Theorem 1], we know that $\mathrm{DL}(\mathcal{V}_0, \{ , \}_0, h)$ is geometrically irreducible. In general, we consider $\mathcal{V}'_0 := \mathcal{V}_0/\mathcal{V}'_0$ equipped with a pairing $\{ , \}'_0$ induced from $\{ , \}_0$. Then it is clear that the morphism $\mathrm{DL}(\mathcal{V}_0, \{ , \}_0, h) \rightarrow \mathrm{DL}(\mathcal{V}'_0, \{ , \}'_0, h)$ sending a point $H \in \mathrm{DL}(\mathcal{V}_0, \{ , \}_0, h)(S)$ to H/\mathcal{V}'_0 is an isomorphism. Thus, $\mathrm{DL}(\mathcal{V}_0, \{ , \}_0, h)$ is geometrically irreducible by the previous case. The proposition is proved. \square

Lemma A.1.4. *Consider a pair $(\mathcal{V}, \{ , \})$ with $\dim_{\kappa} \mathcal{V} = N \geq 2$ and $\dim_{\kappa} \mathcal{V}^{-1} = 0$, and a p -coprime coefficient ring L . Suppose that $p + 1$ is invertible in L .*

(1) *The subscheme $\mathrm{DL}(\mathcal{V}, \{ , \}, N - 1)$ is a hypersurface in $\mathbb{P}(\mathcal{V})$ of degree $p + 1$.*

(2) *The restriction map*

$$\mathrm{H}_{\text{ét}}^i(\mathbb{P}(\mathcal{V})_{\bar{\kappa}}, L) \rightarrow \mathrm{H}_{\text{ét}}^i(\mathrm{DL}(\mathcal{V}, \{ , \}, N - 1)_{\bar{\kappa}}, L)$$

induced by the obvious inclusion $\mathrm{DL}(\mathcal{V}, \{ , \}, N - 1) \rightarrow \mathbb{P}(\mathcal{V})$ is an isomorphism for $i \notin \{N - 2, 2N - 2\}$.

(3) *For every $i \in \mathbb{Z}$, $\mathrm{H}_{\text{ét}}^i(\mathrm{DL}(\mathcal{V}, \{ , \}, N - 1)_{\bar{\kappa}}, L)$ is a free L -module.*

(4) *When N is even, the action of $\mathrm{Gal}(\bar{\kappa}/\kappa)$ on $\mathrm{H}_{\text{ét}}^{N-2}(\mathrm{DL}(\mathcal{V}, \{ , \}, N - 1)_{\bar{\kappa}}, L(\frac{N-2}{2}))$ is trivial.*

Proof. The lemma is trivial if $N = 2$. Now we assume $N \geq 3$. Then $S := \mathrm{DL}(\mathcal{V}, \{ , \}, N - 1)$ is a geometrically connected smooth hypersurface in $\mathbb{P}(\mathcal{V})$ by Proposition A.1.3.

Part (1) follows since S is defined by a homogenous polynomial of degree $p + 1$, by its definition.

For (2), by the Lefschetz hyperplane theorem, the restriction map $\mathrm{H}_{\text{ét}}^i(\mathbb{P}(\mathcal{V})_{\bar{\kappa}}, L) \rightarrow \mathrm{H}_{\text{ét}}^i(S_{\bar{\kappa}}, L)$ is an isomorphism for $0 \leq i \leq N - 3$; and the Gysin map $\mathrm{H}_{\text{ét}}^i(S_{\bar{\kappa}}, L) \rightarrow \mathrm{H}_{\text{ét}}^{i+2}(\mathbb{P}(\mathcal{V})_{\bar{\kappa}}, L(1))$ is an isomorphism for $N - 1 \leq i \leq 2(N - 2)$. By (1), the composite map

$$\mathrm{H}_{\text{ét}}^i(\mathbb{P}(\mathcal{V})_{\bar{\kappa}}, L) \rightarrow \mathrm{H}_{\text{ét}}^i(S_{\bar{\kappa}}, L) \rightarrow \mathrm{H}_{\text{ét}}^{i+2}(\mathbb{P}(\mathcal{V})_{\bar{\kappa}}, L(1))$$

is given by the cup product with $c_1(\mathcal{O}_{\mathbb{P}(\mathcal{V})_{\bar{\kappa}}}(p + 1))$, which is an isomorphism for $i \neq 2N - 2$ since $p + 1$ is invertible in L . Thus, (2) follows.

Part (3) is an immediate consequence of (2).

For (4), it suffices to consider the case where $L = \mathbb{Q}_{\ell}$ for some $\ell \neq p$ by (3). Then it is well-known that $\mathrm{H}_{\text{ét}}^{N-2}(\mathrm{DL}(\mathcal{V}, \{ , \}, N - 1)_{\bar{\kappa}}, \mathbb{Q}_{\ell}(\frac{N-2}{2}))$ is spanned by Tate cycles over κ (see, for example, [HM78]). In particular, (4) follows. \square

Now we construct the special morphisms between Deligne–Lusztig varieties when rank increases.

Construction A.1.5. Let $(\mathcal{V}, \{ , \})$ be an admissible pair with $\dim_{\kappa} \mathcal{V} = n \geq 1$ satisfying $\dim \mathcal{V}^{-1} = n + 1 - 2\lfloor \frac{n+1}{2} \rfloor$. We put $\mathcal{V}_{\sharp} := \mathcal{V} \oplus \kappa 1$ and extend $\{ , \}$ to a pairing $\{ , \}_{\sharp}$ on \mathcal{V}_{\sharp} with $\{1, 1\}_{\sharp} = 0$. Suppose that we have another admissible pair $(\mathcal{V}_{\natural}, \{ , \}_{\natural})$ with $\dim_{\kappa} \mathcal{V}_{\natural} = n + 1$ satisfying $\dim \mathcal{V}_{\natural}^{-1} = n - 2\lfloor \frac{n}{2} \rfloor$, together with a κ -linear map $\delta: \mathcal{V}_{\sharp} \rightarrow \mathcal{V}_{\natural}$ of corank $\dim \mathcal{V}^{-1}$ such that $\{\delta(x), \delta(y)\}_{\natural} = \{x, y\}_{\sharp}$ for every $x, y \in \mathcal{V}_{\sharp}$. We construct a morphism

$$\delta_{\uparrow}: \mathrm{DL}(\mathcal{V}, \{ , \}, \lceil \frac{n+1}{2} \rceil) \rightarrow \mathrm{DL}(\mathcal{V}_{\natural}, \{ , \}_{\natural}, \lceil \frac{n+2}{2} \rceil)$$

by sending $H \in \mathrm{DL}(\mathcal{V}, \{ , \}, h^{\lceil \frac{n+1}{2} \rceil})(S)$ to $\delta(H \oplus \mathcal{O}_S)$. We call δ_{\uparrow} a *special morphism*.

Proposition A.1.6. *The morphism δ_{\uparrow} is well-defined, and is a regular embedding.*

Proof. When n is odd, δ is an isomorphism, which implies that δ_{\uparrow} is well-defined and is an isomorphism.

When n is even, δ is of corank 1. The identity $\{\delta(x), \delta(y)\}_{\natural} = \{x, y\}_{\sharp}$ for every $x, y \in \mathcal{V}_{\sharp}$ implies $\ker \delta \subset \mathcal{V}_{\sharp}^{-1} = \mathcal{V}_{\sharp} \oplus \kappa 1$. Take $S \in \mathrm{Sch}/_{\kappa}$. For $H \in \mathrm{DL}(\mathcal{V}, \{ , \}, \lceil \frac{n+1}{2} \rceil)(S)$, $H \oplus \mathcal{O}_S$ must contain

$\mathcal{V}_\#^{-1}$ hence $(\ker \delta)_S$. It follows that $\delta(H \oplus \mathcal{O}_S)$ has the same rank as H , which is $\lceil \frac{n+1}{2} \rceil = \lceil \frac{n+2}{2} \rceil$. Since $\{ , \}_\natural$ is nondegenerate, we have $\delta 1 \neq 0$ hence $(\delta \kappa 1)^\perp = \delta \mathcal{V}_\#$. In particular, $(\delta(H \oplus \mathcal{O}_S))^\perp$ is contained in $(\delta \mathcal{V}_\#)_S$, which implies that $(\delta(H \oplus \mathcal{O}_S))^\perp \subseteq \delta(H \oplus \mathcal{O}_S)$. In other words, δ_\dagger is well-defined. On the other hand, for $H_\natural \in \text{DL}(\mathcal{V}_\natural, \{ , \}_\natural, \lceil \frac{n+2}{2} \rceil)(S)$, whether $(\delta \kappa 1)_S \subseteq H \subseteq (\delta \mathcal{V}_\#)_S$ holds is a closed condition; and once it does, there is a unique $H \in \text{DL}(\mathcal{V}, \{ , \}, \lceil \frac{n+1}{2} \rceil)(S)$ such that $H_\natural = \delta(H \oplus \mathcal{O}_S)$. Thus, δ_\dagger is a regular embedding by Proposition A.1.3(2).

The proposition is proved. \square

A.2. Unitary Deligne–Lusztig varieties in the semistable case. In this subsection, we introduce certain Deligne–Lusztig varieties that appear in the special fiber of the semistable integral model studied in Section 5. We keep the notation from the previous subsection.

Definition A.2.1. For a pair $(\mathcal{V}, \{ , \})$ with $\dim_\kappa \mathcal{V} = N$, we define a presheaf

$$\text{DL}^\bullet(\mathcal{V}, \{ , \})$$

on Sch/κ such that for every $S \in \text{Sch}/\kappa$, $\text{DL}^\bullet(\mathcal{V}, \{ , \})(S)$ is the set of pairs (H_1, H_2) of subbundles of \mathcal{V}_S of ranks $\lceil \frac{N}{2} \rceil$ and $\lceil \frac{N}{2} \rceil - 1$, respectively, satisfying the following inclusion relations

$$\begin{array}{ccc} & H_1 & \\ & \subset & \subset \\ \mathcal{V}_S^{-1} & \subset & H_2 & & H_2^{-1} \\ & \subset & & & \subset \\ & H_1^{-1} & & & \end{array}$$

of subbundles of \mathcal{V}_S .

Proposition A.2.2. Consider an admissible pair $(\mathcal{V}, \{ , \})$. Put $N := \dim_\kappa \mathcal{V}$ and $d := \dim_\kappa \mathcal{V}^{-1}$.

- (1) If $d \geq \lceil \frac{N}{2} \rceil$, then $\text{DL}^\bullet(\mathcal{V}, \{ , \})$ is empty.
- (2) If $d \leq \lceil \frac{N}{2} \rceil - 1$, then $\text{DL}^\bullet(\mathcal{V}, \{ , \})$ is represented by a projective smooth scheme over κ , whose tangent sheaf fits canonically into a sequence

$$0 \rightarrow \mathcal{H}om(\mathcal{H}_1/\mathcal{H}_2, \mathcal{H}_2^{-1}/\mathcal{H}_1) \rightarrow \mathcal{T}_{\text{DL}^\bullet(\mathcal{V}, \{ , \})/\kappa} \rightarrow \mathcal{H}om(\mathcal{H}_2/\mathcal{V}_{\text{DL}^\bullet(\mathcal{V}, \{ , \})}^{-1}, \mathcal{H}_1^{-1}/\mathcal{H}_2) \rightarrow 0$$

where $\mathcal{V}_{\text{DL}^\bullet(\mathcal{V}, \{ , \})}^{-1} \subseteq \mathcal{H}_2 \subseteq \mathcal{H}_1 \subseteq \mathcal{V}_{\text{DL}^\bullet(\mathcal{V}, \{ , \})}$ are the universal subbundles.

- (3) If $N \geq 2$ and $d = N - 2\lfloor \frac{N}{2} \rfloor$, then $\text{DL}^\bullet(\mathcal{V}, \{ , \})$ is geometrically irreducible of dimension $\lfloor \frac{N}{2} \rfloor$.

Proof. Part (1) is obvious from the definitions.

For (2), let $\text{Gr}(\mathcal{V}, r)$ denote by the Grassmannian variety that classifies subspaces of \mathcal{V} of dimension r . Then $\text{DL}^\bullet(\mathcal{V}, \{ , \})$ is a closed sub-presheaf of $\text{Gr}(\mathcal{V}, \lceil \frac{N}{2} \rceil) \times \text{Gr}(\mathcal{V}, \lceil \frac{N}{2} \rceil - 1)$, hence it is represented by a projective scheme over κ . Now we prove that $\text{DL}^\bullet(\mathcal{V}, \{ , \})$ is smooth and compute its tangent sheaf. Consider a closed immersion $S \hookrightarrow \hat{S}$ in Sch/κ defined by an ideal sheaf \mathcal{I} with $\mathcal{I}^2 = 0$. Take an object $\mathcal{V}_S^{-1} \subseteq H_2 \subseteq H_1 \subseteq \mathcal{V}_S$ in $\text{DL}^\bullet(\mathcal{V}, \{ , \})(S)$. To lift (H_1, H_2) to a pair $(\hat{H}_1, \hat{H}_2) \in \text{DL}^\bullet(\mathcal{V}, \{ , \})(\hat{S})$, we first lift H_2 , which is canonically a torsor under the group $\text{Hom}_{\mathcal{O}_S}(H_2/\mathcal{V}_S^{-1}, (H_1^{-1}/H_2) \otimes_{\mathcal{O}_S} \mathcal{I})$ as \hat{H}_1^{-1} depends only on H_1^{-1} . Once such a lift \hat{H}_2 is given, the possible lifts of H_1 form a torsor under the group $\text{Hom}_{\mathcal{O}_S}(H_1/H_2, (H_2^{-1}/H_1) \otimes_{\mathcal{O}_S} \mathcal{I})$. In particular, Zariski locally, there is no obstruction to lifting (H_1, H_2) , hence $\text{DL}^\bullet(\mathcal{V}, \{ , \})$ is smooth. The statement on the tangent bundle of $\text{DL}^\bullet(\mathcal{V}, \{ , \})$ follows immediately from the discussion by considering the universal object on $\text{DL}^\bullet(\mathcal{V}, \{ , \})$.

For (3), similar to the argument for Proposition A.1.3(3), we may assume N even this time. Then the statement follows again by [BR06, Theorem 1]. \square

Construction A.2.3. Let $(\mathcal{V}, \{ , \})$ be an admissible pair with $\dim_\kappa \mathcal{V} = n \geq 2$ satisfying $\dim_\kappa \mathcal{V}^\perp = n - 2\lfloor \frac{n}{2} \rfloor$. We put $\mathcal{V}_\# := \mathcal{V} \oplus \kappa 1$ and extend $\{ , \}$ to a pairing $\{ , \}_\#$ on $\mathcal{V}_\#$ with $\{1, 1\}_\# = 0$. Suppose that we have another admissible pair $(\mathcal{V}_\natural, \{ , \}_\natural)$ with $\dim_\kappa \mathcal{V}_\natural = n + 1$ satisfying $\dim \mathcal{V}_\natural^\perp = n + 1 - 2\lfloor \frac{n+1}{2} \rfloor$, together with a κ -linear map $\delta: \mathcal{V}_\# \rightarrow \mathcal{V}_\natural$ of corank $\dim \mathcal{V}^\perp$ such that $\{\delta(x), \delta(y)\}_\natural = \{x, y\}_\#$ for every $x, y \in \mathcal{V}_\#$. Then similar to Construction A.1.5 and Proposition A.1.6, we have a morphism

$$\delta_\dagger: \mathrm{DL}^\bullet(\mathcal{V}, \{ , \}) \rightarrow \mathrm{DL}^\bullet(\mathcal{V}_\natural, \{ , \}_\natural)$$

by sending $(H_1, H_2) \in \mathrm{DL}^\bullet(\mathcal{V}, \{ , \})(S)$ to $(\delta(H_1 \oplus \mathcal{O}_S), \delta(H_2 \oplus \mathcal{O}_S)) \in \mathrm{DL}^\bullet(\mathcal{V}_\natural, \{ , \}_\natural)(S)$, which is a regular embedding.

Proposition A.2.4. *Suppose κ algebraically closed. Consider an admissible pair $(\mathcal{V}, \{ , \})$ over κ . Let $(\mathcal{H}_1, \mathcal{H}_2)$ be the universal object over $\mathrm{DL}^\bullet(\mathcal{V}, \{ , \})$.*

(1) *Suppose $\dim_\kappa \mathcal{V} = 2r + 1$ for some integer $r \geq 1$ and $\dim_\kappa \mathcal{V}^\perp = 1$. Then we have*

$$\int_{\mathrm{DL}^\bullet(\mathcal{V}, \{ , \})} c_r \left((\sigma^* \mathcal{H}_2) \otimes_{\mathcal{O}_{\mathrm{DL}^\bullet(\mathcal{V}, \{ , \})}} (\mathcal{H}_1^\perp / \mathcal{H}_2) \right) = \mathbf{d}_{r,p}^\bullet.$$

(2) *Suppose $\dim_\kappa \mathcal{V} = 2r$ for some integer $r \geq 1$ and $\dim_\kappa \mathcal{V}^\perp = 0$. Then we have*

$$\int_{\mathrm{DL}^\bullet(\mathcal{V}, \{ , \})} c_{r-1} \left((\sigma^* \mathcal{H}_2) \otimes_{\mathcal{O}_{\mathrm{DL}^\bullet(\mathcal{V}, \{ , \})}} (\mathcal{H}_1^\perp / \mathcal{H}_2) \right) \cdot c_1 (\mathcal{H}_1^\perp / \mathcal{H}_2) = \mathbf{d}_{r,p}^\bullet.$$

Here, $\mathbf{d}_{r,p}^\bullet$ is the number introduced in Notation 1.3.2.

Note that $\mathrm{DL}^\bullet(\mathcal{V}, \{ , \})$ is irreducible of dimension r , by Proposition A.2.2.

Proof. For (1), we let $\bar{\mathcal{V}}$ be the quotient space $\mathcal{V}/\mathcal{V}^\perp$, equipped with the induced pairing, which we still denote by $\{ , \}$. Then we have a canonical isomorphism $\mathrm{DL}^\bullet(\mathcal{V}, \{ , \}) \xrightarrow{\sim} \mathrm{DL}^\bullet(\bar{\mathcal{V}}, \{ , \})$ by sending a pair (H_1, H_2) to $(H_1/\mathcal{V}^\perp, H_2/\mathcal{V}^\perp)$. If we denote by $(\bar{\mathcal{H}}_1, \bar{\mathcal{H}}_2)$ the universal object over $\mathrm{DL}^\bullet(\bar{\mathcal{V}}, \{ , \})$. Then we have

$$c_r \left((\sigma^* \mathcal{H}_2) \otimes_{\mathcal{O}_{\mathrm{DL}^\bullet(\mathcal{V}, \{ , \})}} (\mathcal{H}_1^\perp / \mathcal{H}_2) \right) = c_{r-1} \left((\sigma^* \bar{\mathcal{H}}_2) \otimes_{\mathcal{O}_{\mathrm{DL}^\bullet(\bar{\mathcal{V}}, \{ , \})}} (\bar{\mathcal{H}}_1^\perp / \bar{\mathcal{H}}_2) \right) \cdot c_1 (\bar{\mathcal{H}}_1^\perp / \bar{\mathcal{H}}_2)$$

under the above isomorphism. Therefore, (1) follows from (2).

For (2), consider $\mathcal{V}_\# := \mathcal{V} \oplus \kappa 1$ and extend $\{ , \}$ to a pairing $\{ , \}_\#$ on $\mathcal{V}_\#$ with $\{1, 1\}_\# = 1$. Then we have Deligne–Lusztig varieties $\mathrm{DL}(\mathcal{V}_\#, \{ , \}_\#, h)$. In what follows, we only need to study the one with $h = r + 1$, and will simply write $\mathrm{DL}(\mathcal{V}_\#)$ for $\mathrm{DL}(\mathcal{V}_\#, \{ , \}_\#, r + 1)$. Since we will work with two spaces, we will denote by (\vdash, \dashv) for the (left, right) orthogonal complement for \mathcal{V} , and (\vDash, \vDash) for the (left, right) orthogonal complement for $\mathcal{V}_\#$.

We now define a correspondence

$$\mathrm{DL}(\mathcal{V}_\#) \xleftarrow{\pi} \widetilde{\mathrm{DL}}(\mathcal{V}) \xrightarrow{\pi^\bullet} \mathrm{DL}^\bullet(\mathcal{V})$$

of schemes over κ . For every κ -scheme S ,

- $\widetilde{\mathrm{DL}}(\mathcal{V})(S)$ is the set of pairs (H, H_2) where H is an element in $\mathrm{DL}(\mathcal{V}_\#)(S)$ and H_2 is a subbundle of H^\vDash of rank $r - 1$ that is contained in \mathcal{V}_S ;
- π sends $(H, H_2) \in \widetilde{\mathrm{DL}}(\mathcal{V})(S)$ to $H \in \mathrm{DL}(\mathcal{V})(S)$; and
- π^\bullet sends $(H, H_2) \in \widetilde{\mathrm{DL}}(\mathcal{V})(S)$ to $(H_1, H_2) \in \mathrm{DL}^\bullet(\mathcal{V})(S)$ where $H_1 := (H \cap \mathcal{V}_S)^\vdash$.

It needs to show that π^\bullet is well-defined, which amounts to the following four statements:

- H_1 is a subbundle of \mathcal{V}_S : It suffices to show that the composite map $H \rightarrow \mathcal{V}_\#^S \rightarrow \mathcal{O}_S$ is surjective, where the latter map is induced by the projection $\mathcal{V}_\# \rightarrow \kappa 1$. If not, then there exists a geometric point s of S such that H_s is contained in \mathcal{V}_s , which contradicts the inclusion $H_s^\natural \subseteq H_s$.
- $H_2 \subseteq H_1$: As $H^\natural \subseteq H$ by the definition of $\text{DL}(\mathcal{V}_\#)$, we have $H^\natural \subseteq H$ and $\{H^\natural, H\}_\# = 0$. Thus, $\{H^\natural \cap \mathcal{V}_S, H \cap \mathcal{V}_S\} = 0$, which implies $H_2 \subseteq H^\natural \cap \mathcal{V}_S \subseteq (H \cap \mathcal{V}_S)^\natural = H_1$.
- $H_1 \subseteq H_2^\natural$: As $H^\natural \subseteq H$, we have that $H_1^\natural = H \cap \mathcal{V}_S$ contains H_2 , which implies $H_1 = (H_1^\natural)^\natural \subseteq H_2^\natural$.
- $H_1 \subseteq H_2^\natural$: As $H^\natural \subseteq H$, we have $(H^\natural)^\natural \cap \mathcal{V}_S \subseteq H \cap \mathcal{V}_S$, which is equivalent to $(H^\natural \cap \mathcal{V}_S)^{\natural\natural} \subseteq H \cap \mathcal{V}_S$. As H_2 is contained in $H^\natural \cap \mathcal{V}_S$, we have $H_2^{\natural\natural} \subseteq H \cap \mathcal{V}_S = H_1^\natural$, which implies $H_1 \subseteq H_2^\natural$.

We denote by \mathcal{H} , $(\widetilde{\mathcal{H}}, \widetilde{\mathcal{H}}_2)$, and $(\mathcal{H}_1, \mathcal{H}_2)$ the universal objects over $\text{DL}(\mathcal{V}_\#)$, $\widetilde{\text{DL}}(\mathcal{V})$, and $\text{DL}^\bullet(\mathcal{V})$, respectively. By definition, we have $\widetilde{\mathcal{H}} = \pi^* \mathcal{H}$ and $\widetilde{\mathcal{H}}_2 = \pi^{\bullet\bullet} \mathcal{H}_2$.

We first study the morphism π . We say that a point $s \in \text{DL}(\mathcal{V}_\#)(\kappa)$ represented by H_s is *special* if H_s^\natural is a maximal isotropic subspace of \mathcal{V} satisfying $H_s^\natural = H_s^\natural$. Then there are exactly $(p+1)(p^3+1)\cdots(p^{2r-1}+1)$ special points. Let $\text{DL}(\mathcal{V}_\#)'$ be the locus of special points. It is clear that for every morphism $S \rightarrow \text{DL}(\mathcal{V}_\#) \setminus \text{DL}(\mathcal{V}_\#)'$, $\pi^{-1}(S)$ is a singleton; and for a special point s , we have $\pi^{-1}(s) = \mathbb{P}(H_s^\natural) \simeq \mathbb{P}_\kappa^{r-1}$. In particular, π is a blow-up along $\text{DL}(\mathcal{V}_\#)'$, for which we denote by $E \subseteq \widetilde{\text{DL}}(\mathcal{V})$ the exceptional divisor. In particular, π is projective. Moreover, E is exactly the zero locus of the canonical projection map

$$\widetilde{\mathcal{H}}^\natural / \widetilde{\mathcal{H}}_2 \rightarrow \mathcal{O}_{\widetilde{\text{DL}}(\mathcal{V})} \cdot 1 \subseteq \mathcal{O}_{\widetilde{\text{DL}}(\mathcal{V})} \otimes_\kappa \mathcal{V}_\#,$$

which implies

$$(A.1) \quad \widetilde{\mathcal{H}}^\natural / \widetilde{\mathcal{H}}_2 \simeq \mathcal{O}_{\widetilde{\text{DL}}(\mathcal{V})}(-E).$$

Next we study the morphism π^\bullet . We claim that π^\bullet is generically finite of degree $p+1$. Take a point $s \in \text{DL}^\bullet(\mathcal{V})(\kappa)$ represented by (H_{1s}, H_{2s}) . Then by construction, for every scheme S over $\{s\} \times_{\text{DL}^\bullet(\mathcal{V})} \widetilde{\text{DL}}(\mathcal{V})$, $\widetilde{\text{DL}}(\mathcal{V})(S)$ consists of subbundles $H \subseteq \mathcal{V}_\# \otimes_\kappa \mathcal{O}_S$ satisfying $H_{2s} \otimes_\kappa \mathcal{O}_S \subseteq H^\natural \subseteq H_{1s} \otimes_\kappa \mathcal{O}_S \oplus \mathcal{O}_S 1$ and $H^\natural \subseteq H$. Note that we have an induced pairing

$$\{ , \}_s : \frac{H_{1s} \oplus \kappa 1}{H_{2s}} \times \frac{H_{1s} \oplus \kappa 1}{H_{2s}} \rightarrow \kappa$$

that is σ -linear in the first variable and linear in the second variable. Then it is clear that when $\{ , \}_s$ is perfect, $\{s\} \times_{\text{DL}^\bullet(\mathcal{V})} \widetilde{\text{DL}}(\mathcal{V})$ is isomorphic to the union of $p+1$ copies of $\text{Spec } \kappa$. However, $\{ , \}_s$ fails to be perfect if and only if $H_1^\natural = H_1$. Thus, the locus where $\{ , \}_s$ fails to be perfect is a finite union of \mathbb{P}_κ^{r-1} . Therefore, π^\bullet is generically finite of degree $p+1$.

To proceed, we introduce two more bundles

$$\mathcal{E} := (\sigma^* \mathcal{H}^\natural) \otimes_{\text{DL}(\mathcal{V}_\#)} (\mathcal{H} / \mathcal{H}^\natural), \quad \mathcal{E}^\bullet := (\sigma^* \mathcal{H}_2) \otimes_{\text{DL}^\bullet(\mathcal{V})} (\mathcal{H}_1^\natural / \mathcal{H}_2)$$

on $\text{DL}(\mathcal{V}_\#)$ and $\text{DL}^\bullet(\mathcal{V})$ of ranks r and $r-1$, respectively.

We claim that

$$(A.2) \quad \mathcal{L} := \pi^{\bullet\bullet} (\mathcal{H}_1^\natural / \mathcal{H}_2) \simeq \mathcal{O}_{\widetilde{\text{DL}}(\mathcal{V})}(-E) \otimes_{\mathcal{O}_{\widetilde{\text{DL}}(\mathcal{V})}} (\widetilde{\mathcal{H}} / \widetilde{\mathcal{H}}^\natural).$$

In fact, we have

$$\mathcal{L} = (\widetilde{\mathcal{H}} \cap \mathcal{V}_{\widetilde{\text{DL}}(\mathcal{V})}) / \widetilde{\mathcal{H}}_2$$

by definition. Thus, the claim follows from the following injective map

$$\begin{array}{ccccccc}
0 & \longrightarrow & \widetilde{\mathcal{H}}_2 & \longrightarrow & \widetilde{\mathcal{H}}^\# & \longrightarrow & \mathcal{O}_{\widetilde{\text{DL}}(\mathcal{V})}(-E) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \widetilde{\mathcal{H}} \cap \mathcal{V}_{\widetilde{\text{DL}}(\mathcal{V})} & \longrightarrow & \widetilde{\mathcal{H}} & \longrightarrow & \mathcal{O}_{\widetilde{\text{DL}}(\mathcal{V})} \cdot 1 \longrightarrow 0
\end{array}$$

of short exact sequences of coherent sheaves on $\widetilde{\text{DL}}(\mathcal{V})$ by (A.1) and the Snake Lemma.

By (A.1) and (A.2), we have

$$\begin{aligned}
& \pi^*(c_r(\mathcal{E})) \\
&= c_r(\pi^*\mathcal{E}) \\
&= c_{r-1}\left(\left(\sigma^*\widetilde{\mathcal{H}}_2\right) \otimes_{\mathcal{O}_{\widetilde{\text{DL}}(\mathcal{V})}} \left(\mathcal{H}/\mathcal{H}^\#\right)\right) \cdot c_1\left(\mathcal{O}_{\widetilde{\text{DL}}(\mathcal{V})}(-pE) \otimes_{\mathcal{O}_{\widetilde{\text{DL}}(\mathcal{V})}} \left(\mathcal{H}/\mathcal{H}^\#\right)\right) \\
&= c_{r-1}\left(\left(\sigma^*\widetilde{\mathcal{H}}_2\right) \otimes_{\mathcal{O}_{\widetilde{\text{DL}}(\mathcal{V})}} \mathcal{L}(E)\right) \cdot c_1(\mathcal{L}((1-p)E)) \\
&= c_{r-1}\left(\pi^{\bullet\bullet}\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\widetilde{\text{DL}}(\mathcal{V})}} \mathcal{O}_{\widetilde{\text{DL}}(\mathcal{V})}(E)\right) \cdot c_1(\mathcal{L}((1-p)E)) \\
&= \left(c_{r-1}(\pi^{\bullet\bullet}\mathcal{E}^\bullet) + \sum_{i=1}^{r-1} c_1(E)^i c_{r-i-1}(\pi^{\bullet\bullet}\mathcal{E}^\bullet)\right) \cdot (c_1(\mathcal{L}) + (1-p)c_1(E)) \\
&= c_{r-1}(\pi^{\bullet\bullet}\mathcal{E}^\bullet) \cdot c_1(\mathcal{L}) + \sum_{i=1}^{r-1} c_1(E)^i c_1(\mathcal{L}) c_{r-i-1}(\pi^{\bullet\bullet}\mathcal{E}^\bullet) + (1-p) \sum_{i=1}^r c_1(E)^i c_{r-i}(\pi^{\bullet\bullet}\mathcal{E}^\bullet) \\
&= \pi^{\bullet\bullet}\left(c_{r-1}(\mathcal{E}^\bullet) \cdot c_1(\mathcal{H}_1^\#/\mathcal{H}_2)\right) + \sum_{i=1}^{r-1} c_1(E)^i c_1(\mathcal{L}) c_{r-i-1}(\pi^{\bullet\bullet}\mathcal{E}^\bullet) + (1-p) \sum_{i=1}^r c_1(E)^i c_{r-i}(\pi^{\bullet\bullet}\mathcal{E}^\bullet).
\end{aligned}$$

In particular, since π and π^\bullet are generically finite of degrees 1 and $p+1$, respectively, we have

$$\begin{aligned}
\text{(A.3)} \quad & (p+1) \int_{\text{DL}^\bullet(\mathcal{V})} c_{r-1}(\mathcal{E}) \cdot c_1(\mathcal{H}_1^\#/\mathcal{H}_2) - \int_{\text{DL}(\mathcal{V}_\#)} c_r(\mathcal{E}) \\
&= (p-1) \sum_{i=1}^r \int_{\widetilde{\text{DL}}(\mathcal{V})} c_1(E)^i c_{r-i}(\pi^{\bullet\bullet}\mathcal{E}^\bullet) - \sum_{i=1}^{r-1} \int_{\widetilde{\text{DL}}(\mathcal{V})} c_1(E)^i c_1(\mathcal{L}) c_{r-i-1}(\pi^{\bullet\bullet}\mathcal{E}^\bullet) \\
&= (p-1) \sum_{i=0}^{r-1} \int_E (-\eta)^i c_{r-i-1}(\pi^{\bullet\bullet}\mathcal{E}^\bullet|_E) - \sum_{i=0}^{r-2} \int_E (-\eta)^i c_1(\mathcal{L}|_E) c_{r-i-2}(\pi^{\bullet\bullet}\mathcal{E}^\bullet|_E)
\end{aligned}$$

where $\eta := c_1(\mathcal{O}_E(1))$. As $\widetilde{\mathcal{H}}/\widetilde{\mathcal{H}}^\# = \pi^*(\mathcal{H}/\mathcal{H}^\#)$, we have $\mathcal{L}|_E \simeq \mathcal{O}_E(-E) = \mathcal{O}_E(1)$. On the other hand, $\widetilde{\mathcal{H}}_2|_E$ is the tautological subbundle (of rank $r-1$), which satisfies the short exact sequence

$$0 \rightarrow \widetilde{\mathcal{H}}_2|_E \rightarrow \mathcal{O}_E^{\oplus r} \rightarrow \mathcal{O}_E(1) \rightarrow 0.$$

Thus, $\mathcal{F} := \pi^{\bullet\bullet}\mathcal{E}^\bullet|_E$, which equals $(\sigma^*\widetilde{\mathcal{H}}_2|_E) \otimes_{\mathcal{O}_E} (\mathcal{L}|_E)$, satisfies the short exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}_E(1)^{\oplus r} \rightarrow \mathcal{O}_E(p+1) \rightarrow 0.$$

the dual basis of $\rho_{N,\text{std}}^\vee$. For a subset $I \subseteq \{1, \dots, N\}$, we put $\langle I \rangle := \sum_{i \in I} i$, $I^\vee := \{N+1-i \mid i \in I\}$, $\varepsilon_I := \wedge_{i \in I} \varepsilon_i$ and $\varepsilon_I^\vee := \wedge_{i \in I} \varepsilon_i^\vee$ (in the increasing order of the indices). For $0 \leq \delta \leq r$, put

$$\rho_{N;\delta} := \left(\bigwedge^{\delta} \rho_{N,\text{std}} \right) \otimes \left(\bigwedge^{\delta} \rho_{N,\text{std}}^\vee \right),$$

which extends uniquely to a representation of GL_N^{ext} such that σ sends $\varepsilon_I \otimes \varepsilon_{J^\vee}$ to $(-1)^{\langle I \rangle + \langle J \rangle} \varepsilon_J \otimes \varepsilon_{I^\vee}$.

Remark B.1.1. In the next subsection, we will study the unramified unitary group $\text{U}(\text{V}_N)$ over nonarchimedean local fields. Then $\text{GL}_N^{\text{ext}}(\mathbb{C})$ is simply the Langlands dual group of $\text{U}(\text{V}_N)$, and we have $\mathbb{Z}[\mathbb{X}_N^*]^{\text{sym}} \simeq \mathbb{Z}[\widehat{\text{U}(\text{V}_N)}]^\sigma \text{W}_N$.

Lemma B.1.2. *We have*

$$\chi(\rho_{N;\delta}) = \begin{cases} \sum_{i=0}^{\delta} \binom{r-\delta+i}{\lfloor \frac{i}{2} \rfloor} \cdot \mathfrak{s}_{\delta-i}, & \text{if } N \text{ is odd;} \\ \sum_{j=0}^{\lfloor \frac{\delta}{2} \rfloor} \binom{r-\delta+2j}{j} \cdot \mathfrak{s}_{\delta-2j}, & \text{if } N \text{ is even.} \end{cases}$$

In particular, $\chi(\rho_{N;\delta})$ belongs to $\mathbb{Z}[\mathbb{X}_N^*]^{\text{sym}}$.

Proof. Note that for every $t \in \text{T}_N$, $t\sigma$ sends $\varepsilon_I \otimes \varepsilon_{J^\vee}$ to

$$(-1)^{\langle I \rangle + \langle J \rangle} \prod_{i \in I^\vee} \mu_i(t)^{-1} \prod_{j \in J} \mu_j(t) \cdot \varepsilon_J \otimes \varepsilon_{I^\vee}.$$

Thus, we have

$$\begin{aligned} \chi(\rho_{N;\delta})(t\sigma) &= \sum_{I \subseteq \{1, \dots, r\}, |I|=\delta} \prod_{i \in I^\vee} \mu_i(t)^{-1} \prod_{i \in I} \mu_i(t) \\ &= \sum_{I \subseteq \{1, \dots, r\}, |I|=\delta} \prod_{i \in I} \mu_i(t) \mu_{N+1-i}(t)^{-1}. \end{aligned}$$

To evaluate the above sum, we consider $i := |I \cap I^\vee|$, which has to be even when N is even. It is easy to see that for fixed $0 \leq i \leq \delta$ (that is even if N is even), the contribution from those subsets I to the above sum is

$$\binom{r-\delta+i}{\lfloor \frac{i}{2} \rfloor} \cdot \mathfrak{s}_{\delta-i}(t).$$

Thus, the lemma follows. □

Lemma B.1.3. *Suppose that $N = 2r$ is even.*

(1) *We have*

$$\prod_{i=1}^r (\lambda + \lambda^{-1} + \mu_i) = \chi(\rho_{N;r}) + \sum_{\delta=1}^r \chi(\rho_{N;r-\delta}) (\lambda^\delta + \lambda^{-\delta})$$

in $\mathbb{Z}[\mathbb{X}_N^*]^{\text{sym}} \otimes \mathbb{Z}[\lambda, \lambda^{-1}]$.

(2) *We have*

$$\sum_{j=1}^r \prod_{\substack{i=1 \\ i \neq j}}^r (\lambda + \lambda^{-1} + \mu_i) = \sum_{\delta=1}^r \delta \cdot \chi(\rho_{N;r-\delta}) \frac{\lambda^\delta - \lambda^{-\delta}}{\lambda - \lambda^{-1}}$$

in $\mathbb{Z}[\mathbb{X}_N^*]^{\text{sym}} \otimes \mathbb{Z}[\lambda, \lambda^{-1}]$.

Proof. Part (1) is follows from Lemma B.1.2 by comparing coefficients of powers of λ . Part (2) follows from (1) by taking derivative with respect to λ and dividing both sides of the resulted equality by $1 - \lambda^{-2}$. □

Lemma B.1.4. *Suppose that $N = 2r + 1$ is odd. We have*

$$\prod_{i=1}^r (\lambda + \lambda^{-1} + \boldsymbol{\mu}_i) = \sum_{\delta=0}^r \chi(\rho_{N;r-\delta}) \frac{\lambda^{\delta+1} + \lambda^{-\delta}}{\lambda + 1}$$

in $\mathbb{Z}[\mathbb{X}_N^*]^{\text{sym}} \otimes \mathbb{Z}[\lambda, \lambda^{-1}]$.

Proof. By Lemma B.1.2, the right-hand side of the desired identity equals

$$\sum_{\delta=0}^r \frac{\lambda^{\delta+1} + \lambda^{-\delta}}{\lambda + 1} \sum_{i=0}^{r-\delta} \binom{\delta+i}{\lfloor \frac{i}{2} \rfloor} \cdot \mathfrak{s}_{r-\delta-i},$$

which coincides with

$$\sum_{i=0}^r \left(\sum_{\delta=0}^{r-i} \frac{\lambda^{\delta+1} + \lambda^{-\delta}}{\lambda + 1} \binom{r-i}{\lfloor \frac{r-i-\delta}{2} \rfloor} \right) \mathfrak{s}_i$$

by substituting i by $r - \delta - 1$. Thus, it remains to show that

$$\sum_{\delta=0}^k \frac{\lambda^{\delta+1} + \lambda^{-\delta}}{\lambda + 1} \binom{k}{\lfloor \frac{k-\delta}{2} \rfloor} = (\lambda + \lambda^{-1})^k$$

for $0 \leq k \leq r$. However, we have

$$\begin{aligned} & \sum_{\delta=0}^k \frac{\lambda^{\delta+1} + \lambda^{-\delta}}{\lambda + 1} \binom{k}{\lfloor \frac{k-\delta}{2} \rfloor} \\ &= \binom{k}{0} \left(\frac{\lambda^{k+1} + \lambda^{-k}}{\lambda + 1} + \frac{\lambda^k + \lambda^{-(k-1)}}{\lambda + 1} \right) + \binom{k}{1} \left(\frac{\lambda^{k-1} + \lambda^{-(k-2)}}{\lambda + 1} + \frac{\lambda^{k-2} + \lambda^{-(k-3)}}{\lambda + 1} \right) + \dots \\ &= \binom{k}{0} (\lambda^k + \lambda^{-k}) + \binom{k}{1} (\lambda^{k-1} + \lambda^{-(k-1)}) + \dots \\ &= (\lambda + \lambda^{-1})^k. \end{aligned}$$

The lemma follows. \square

B.2. Two Hecke algebras. From now to the end of this section, we fix an unramified quadratic extension F/F^+ of nonarchimedean local fields. Let q be the residue cardinality of F^+ and \mathfrak{p} the maximal ideal of O_F .

Let $N \geq 1$ be an integer with $r := \lfloor \frac{N}{2} \rfloor$. Consider a hermitian space V_N over F (with respect to F/F^+) of rank N together with a basis $\{e_{-r}, \dots, e_r\}$ (with e_0 omitted if N is even) such that $(e_{-i}, e_j)_{V_N} = \delta_{ij}$ for $0 \leq i, j \leq r$. Via this basis, we identify $U(V_N)$ as a closed subgroup of $\text{Res}_{F/F^+} \text{GL}_N$. We study two lattices

$$(B.1) \quad \Lambda_N^\circ = O_F e_{-r} \oplus \dots \oplus O_F e_r, \quad \Lambda_N^\bullet = \mathfrak{p}^{-1} e_{-r} \oplus \dots \oplus \mathfrak{p}^{-1} e_{-1} \oplus O_F e_0 \oplus \dots \oplus O_F e_r$$

of V_N . We have $(\Lambda_N^\circ)^\vee = \Lambda_N^\circ$, $\mathfrak{p}\Lambda_N^\bullet \subseteq (\Lambda_N^\bullet)^\vee$, and that the O_F -module $(\Lambda_N^\bullet)^\vee / \mathfrak{p}\Lambda_N^\bullet$ has length $N - 2r$. Let K_N° and K_N^\bullet be the stabilizers of Λ_N° and Λ_N^\bullet , respectively, which are subgroups of $U(V_N)(F^+)$. It is clear that K_N° is hyperspecial maximal; K_N^\bullet is special maximal and is hyperspecial if and only if N is even. We have two *commutative* Hecke algebras

$$\mathbb{T}_N^\circ := \mathbb{Z}[K_N^\circ \backslash U(V_N)(F^+) / K_N^\circ], \quad \mathbb{T}_N^\bullet := \mathbb{Z}[K_N^\bullet \backslash U(V_N)(F^+) / K_N^\bullet].$$

Recall that by our convention in Subsection 1.3, the units in \mathbb{T}_N° and \mathbb{T}_N^\bullet are $\mathbb{1}_{K_N^\circ}$ and $\mathbb{1}_{K_N^\bullet}$, respectively. Let $A_N(F^+)$ (resp. $A_N(O_{F^+})$) be the subgroup of $U(V_N)(F^+)$ that acts on e_i by a scalar in F^+ (resp. O_{F^+}) for every $-r \leq i \leq r$.

Notation B.2.1. For each element $\mathbf{t} = (t_1, \dots, t_N) \in \mathbb{Z}^N$ satisfying $t_i + t_{N+1-i} = 0$ and $a \in F^\times$, we have an element $a^{\mathbf{t}} \in A_N(F^+)$ such that $a^{\mathbf{t}} \cdot e_{-i} = a^{t_{r+1-i}} e_{-i}$ for $1 \leq i \leq r$. For $0 \leq \delta \leq r$, put $\mathbf{t}_\delta := (1^\delta, 0^{N-2\delta}, (-1)^\delta)$. We let $\mathbb{T}_{N;\mathbf{t}}^\circ$ (resp. $\mathbb{T}_{N;\mathbf{t}}^\bullet$) be the element in \mathbb{T}_N° (resp. \mathbb{T}_N^\bullet) corresponding to the double coset $K_N^\circ \varpi^{\mathbf{t}} K_N^\circ$ (resp. $K_N^\bullet \varpi^{\mathbf{t}} K_N^\bullet$) for some uniformizer ϖ of F ; and simply write $\mathbb{T}_{N;\delta}^\circ$ (resp. $\mathbb{T}_{N;\delta}^\bullet$) for $\mathbb{T}_{N;\mathbf{t}_\delta}^\circ$ (resp. $\mathbb{T}_{N;\mathbf{t}_\delta}^\bullet$).

Remark B.2.2. The elements $\mathbb{T}_{N;\mathbf{t}}^\circ \in \mathbb{T}_N^\circ$ and $\mathbb{T}_{N;\mathbf{t}}^\bullet \in \mathbb{T}_N^\bullet$ do not depend on the choice of the basis $\{e_{-r}, \dots, e_r\}$ satisfying (B.1).

Now we recall Satake isomorphisms. Denote by W_N the Weyl group of $A_N(F^+)$ in $U(V_N)(F^+)$, which preserves $A_N(O_{F^+})$; and we have the two Satake isomorphisms²⁴

$$\begin{aligned} \text{Sat}_N^\circ: \mathbb{T}_N^\circ &\xrightarrow{\sim} \mathbb{Z}[A_N(F^+)/A_N(O_{F^+})]^{W_N}, \\ \text{Sat}_N^\bullet: \mathbb{T}_N^\bullet &\xrightarrow{\sim} \mathbb{Z}[A_N(F^+)/A_N(O_{F^+})]^{W_N}. \end{aligned}$$

In addition, we have an isomorphism

$$\mathbb{Z}[A_N(F^+)/A_N(O_{F^+})]^{W_N} \simeq \mathbb{Z}[\mathbb{X}_N^*]^{\text{sym}}$$

of rings under which \mathfrak{s}_δ corresponds to the characteristic function of the W_N -orbit of p^{t_δ} for every $0 \leq \delta \leq r$. In what follows, we will regard $\mathbb{Z}[\mathbb{X}_N^*]^{\text{sym}}$ as the target of both Satake isomorphisms Sat_N° and Sat_N^\bullet .

Notation B.2.3. Let $\mathbb{Z}[\mathbb{X}_N^*]'$ be the subring of $\mathbb{Z}[\mathbb{X}_N^*]$ generated by $\{\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_r\}$ over \mathbb{Z} , sending $\boldsymbol{\mu}_i$ to $\alpha_i + \alpha_i^{-1}$ for $1 \leq i \leq r$. For every ring L and every tuple $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N) \in L^N$ satisfying $\alpha_i \alpha_{N+1-i} = 1$, we have a homomorphism $\phi'_\alpha: \mathbb{Z}[\mathbb{X}_N^*]' \rightarrow L$, and denote

$$\begin{aligned} \phi_\alpha^\circ: \mathbb{T}_N^\circ &\xrightarrow{\text{Sat}_N^\circ} \mathbb{Z}[\mathbb{X}_N^*]^{\text{sym}} \subseteq \mathbb{Z}[\mathbb{X}_N^*]' \xrightarrow{\phi'_\alpha} L, \\ \phi_\alpha^\bullet: \mathbb{T}_N^\bullet &\xrightarrow{\text{Sat}_N^\bullet} \mathbb{Z}[\mathbb{X}_N^*]^{\text{sym}} \subseteq \mathbb{Z}[\mathbb{X}_N^*]' \xrightarrow{\phi'_\alpha} L, \end{aligned}$$

the composite homomorphisms.

Definition B.2.4. We denote

- Lat_N° the set of all self-dual lattices in V_N ;
- Lat_N^\bullet the set of all lattices L in V_N satisfying $\mathfrak{p}L \subseteq L^\vee$ and that $L^\vee/\mathfrak{p}L$ has length $N - 2\lfloor \frac{N}{2} \rfloor$;
- $\mathbb{T}_N^\circ \in \mathbb{Z}[K_N^\bullet \backslash U(V_N)(F^+)/K_N^\circ]$ the characteristic function of $K_N^\bullet K_N^\circ$; and
- $\mathbb{T}_N^\bullet \in \mathbb{Z}[K_N^\circ \backslash U(V_N)(F^+)/K_N^\bullet]$ the characteristic function of $K_N^\circ K_N^\bullet$.

Moreover, we define the *intertwining Hecke operator*

$$\mathbb{I}_N^\circ := \mathbb{T}_N^\circ \circ \mathbb{T}_N^{\bullet\circ} \in \mathbb{T}_N^\circ$$

where the composition is taken as composition of cosets.

Note that we have canonical injective homomorphisms

$$\mathbb{T}_N^\circ \rightarrow \text{End}_{\mathbb{Z}}(\mathbb{Z}[\text{Lat}_N^\circ]), \quad \mathbb{T}_N^\bullet \rightarrow \text{End}_{\mathbb{Z}}(\mathbb{Z}[\text{Lat}_N^\bullet])$$

sending $\mathbb{T}_{N;\mathbf{t}}^\circ$ to the endomorphism that takes $f \in \mathbb{Z}[\text{Lat}_N^\circ]$ to the function $\mathbb{T}_{N;\mathbf{t}}^\circ f$ satisfying $(\mathbb{T}_{N;\mathbf{t}}^\circ f)(L) = \sum f(L')$ where the sum is taken over all $L' \in \text{Lat}_N^\circ$ such that L' and L have relative position $\varpi^{\mathbf{t}}$ for $? = \circ, \bullet$.

²⁴Strictly speaking, we need to choose a square root of q^2 , which we take to be q .

Lemma B.2.5. *We have the identity*

$$\mathbb{I}_N^\circ = \begin{cases} \mathbb{T}_{N;r}^\circ + (q+1)\mathbb{T}_{N;r-1}^\circ + (q+1)(q^3+1)\mathbb{T}_{N;r-2}^\circ + \cdots + \prod_{i=1}^r (q^{2i-1}+1)\mathbb{T}_{N;0}^\circ, & \text{if } N = 2r; \\ \mathbb{T}_{N;r}^\circ + (q^3+1)\mathbb{T}_{N;r-1}^\circ + (q^3+1)(q^5+1)\mathbb{T}_{N;r-2}^\circ + \cdots + \prod_{i=1}^r (q^{2i+1}+1)\mathbb{T}_{N;0}^\circ, & \text{if } N = 2r+1 \end{cases}$$

in \mathbb{T}_N° .

Proof. To compute \mathbb{I}_N° , it suffices to compute its induced endomorphism on $\mathbb{Z}[\text{Lat}_N^\circ]$. Now we take an element $f \in \mathbb{Z}[\text{Lat}_N^\circ]$. Then

$$(\mathbb{T}_N^\circ(\mathbb{T}_N^\circ f))(\mathbb{L}_1^\circ) = \sum_{\substack{\mathbb{L}^\bullet \in \text{Lat}_N^\bullet \\ \mathbb{L}_1^\circ \subseteq \mathbb{L}^\bullet \subseteq \mathfrak{p}^{-1}\mathbb{L}_1^\circ}} (\mathbb{T}_N^\circ f)(\mathbb{L}^\bullet) = \sum_{\substack{\mathbb{L}^\bullet \in \text{Lat}_N^\bullet \\ \mathbb{L}_1^\circ \subseteq \mathbb{L}^\bullet \subseteq \mathfrak{p}^{-1}\mathbb{L}_1^\circ}} \sum_{\substack{\mathbb{L}_2^\circ \in \text{Lat}_N^\circ \\ \mathbb{L}_2^\circ \subseteq \mathbb{L}^\bullet \subseteq \mathfrak{p}^{-1}\mathbb{L}_2^\circ}} f(\mathbb{L}_2^\circ)$$

for every $\mathbb{L}_1^\circ \in \text{Lat}_N^\circ$. Note that for pairs $(\mathbb{L}_1^\circ, \mathbb{L}_2^\circ) \in (\text{Lat}_N^\circ)^2$ appearing in the formula above, we have $\mathfrak{p}\mathbb{L}_2^\circ \subseteq \mathbb{L}_1^\circ \subseteq \mathfrak{p}^{-1}\mathbb{L}_2^\circ$ and $[\mathbb{L}_1^\circ : \mathbb{L}_2^\circ] := [\mathbb{L}_1^\circ : \mathbb{L}_1^\circ \cap \mathbb{L}_2^\circ] + [\mathbb{L}_2^\circ : \mathbb{L}_1^\circ \cap \mathbb{L}_2^\circ] \in \{0, 2, \dots, 2r\}$.

Now for a pair $(\mathbb{L}_1^\circ, \mathbb{L}_2^\circ) \in (\text{Lat}_N^\circ)^2$ satisfying $\mathfrak{p}\mathbb{L}_2^\circ \subseteq \mathbb{L}_1^\circ \subseteq \mathfrak{p}^{-1}\mathbb{L}_2^\circ$, we consider the set

$$\text{Lat}_N^\bullet(\mathbb{L}_1^\circ, \mathbb{L}_2^\circ) := \{\mathbb{L}^\bullet \in \text{Lat}_N^\bullet \mid \mathbb{L}_1^\circ \subseteq \mathbb{L}^\bullet \subseteq \mathfrak{p}^{-1}\mathbb{L}_1^\circ, \mathbb{L}_2^\circ \subseteq \mathbb{L}^\bullet \subseteq \mathfrak{p}^{-1}\mathbb{L}_2^\circ\}.$$

It is easy to see that the cardinality of $\text{Lat}_N^\bullet(\mathbb{L}_1^\circ, \mathbb{L}_2^\circ)$ depends only on $[\mathbb{L}_1^\circ : \mathbb{L}_2^\circ]$. For $0 \leq \delta \leq r$, we denote by $c_{N,\delta}$ the cardinality of $\text{Lat}_N^\bullet(\mathbb{L}_1^\circ, \mathbb{L}_2^\circ)$ with $[\mathbb{L}_1^\circ : \mathbb{L}_2^\circ] = 2\delta$. Then the lemma is equivalent to showing that $c_{N,r} = 1$ and

$$c_{N,\delta} = \begin{cases} \prod_{i=1}^{r-\delta} (q^{2i-1} + 1), & 0 \leq \delta < r, \quad \text{when } N = 2r; \\ \prod_{i=1}^{r-\delta} (q^{2i+1} + 1), & 0 \leq \delta < r, \quad \text{when } N = 2r + 1. \end{cases}$$

Without lost of generality, we may assume $\mathbb{L}_1^\circ = \Lambda_N^\circ$ and

$$\mathbb{L}_2^\circ = \mathfrak{p}^{-1}e_{-r} \oplus \cdots \oplus \mathfrak{p}^{-1}e_{-r+\delta-1} \oplus O_F e_{-r+\delta} \oplus \cdots \oplus O_F e_{r-\delta} \oplus \mathfrak{p} O_F e_{r-\delta+1} \oplus \cdots \oplus \mathfrak{p} O_F e_r.$$

When $\delta = r$, Λ_N^\bullet is the only element in $\text{Lat}_N^\bullet(\mathbb{L}_1^\circ, \mathbb{L}_2^\circ)$. Thus, we have $c_{N,r} = 1$. For $0 \leq \delta < r$, we have $c_{N,\delta} = c_{N-2\delta,0}$. Thus, it suffices to show

$$c_{N,0} = \begin{cases} \prod_{i=1}^r (q^{2i-1} + 1) = (q+1) \cdots (q^{2r-1} + 1), & \text{when } N = 2r; \\ \prod_{i=1}^r (q^{2i+1} + 1) = (q^3+1) \cdots (q^{2r+1} + 1), & \text{when } N = 2r + 1. \end{cases}$$

However, $c_{N,0}$ is nothing but the number of maximal isotropic subspaces of the hermitian space $\Lambda_N^\circ \otimes_{O_F} O_F/\mathfrak{p}$ over O_F/\mathfrak{p} of dimension N , which is given by the above formula. Thus, the lemma is proved. \square

The following three lemmas will be used in later computation.

Lemma B.2.6. *We have the identity*

$$q^{\delta(N-\delta)} \chi(\rho_{N,\delta}) = \sum_{i=0}^{\delta} \begin{bmatrix} N-2i \\ \delta-i \end{bmatrix}_{-q} \text{Sat}_N^\circ(\mathbb{T}_{N;i}^\circ)$$

in $\mathbb{Z}[\mathbb{X}_N^*]^{\text{sym}}$ for $0 \leq \delta \leq r$.

Proof. This is [XZ, Lemma 9.2.4]. \square

Lemma B.2.7. *For every integer $k \geq 1$, we have*

$$\sum_{\delta=-k}^k q^{\delta^2} \begin{bmatrix} 2k \\ k - \delta \end{bmatrix}_{-q} = (q+1)(q^3+1) \cdots (q^{2k-1}+1).$$

Proof. For every integer $k \geq 1$, we have the Gauss polynomial identity

$$\sum_{\delta=0}^{2k} (-1)^\delta \begin{bmatrix} 2k \\ \delta \end{bmatrix}_\lambda = (1-\lambda)(1-\lambda^3) \cdots (1-\lambda^{2k-1})$$

in $\mathbb{Z}[\lambda]$.²⁵ Now we specialize the identity to $\lambda = -q^{-1}$. Then we get

$$\sum_{\delta=0}^{2k} (-1)^\delta (-q)^{-(2k-1)-(2k-3)-\cdots-(2k-2\delta+1)} \begin{bmatrix} 2k \\ \delta \end{bmatrix}_{-q} = q^{-k^2} (q+1)(q^3+1) \cdots (q^{2k-1}+1).$$

The lemma then follows by changing δ to $k - \delta$. □

Lemma B.2.8. *For every integer $k \geq 1$, we have*

$$\sum_{\delta=-k-1}^k (-1)^\delta \delta q^{\delta^2+\delta} \begin{bmatrix} 2k+1 \\ k - \delta \end{bmatrix}_{-q} - \sum_{\delta=-k}^k (-1)^\delta \delta q^{\delta^2+\delta} \begin{bmatrix} 2k \\ k - \delta \end{bmatrix}_{-q} = (-q)^k (q+1)(q^3+1) \cdots (q^{2k-1}+1).$$

Proof. In fact, we have

$$\begin{aligned} & \sum_{\delta=-k-1}^k (-1)^\delta \delta q^{\delta^2+\delta} \begin{bmatrix} 2k+1 \\ k - \delta \end{bmatrix}_{-q} - \sum_{\delta=-k}^k (-1)^\delta \delta q^{\delta^2+\delta} \begin{bmatrix} 2k \\ k - \delta \end{bmatrix}_{-q} \\ &= \sum_{\delta=-k-1}^k (-1)^\delta \delta q^{\delta^2+\delta} (-q)^{k+\delta+1} \begin{bmatrix} 2k \\ k - \delta - 1 \end{bmatrix}_{-q} \\ &= (-1)^{k+1} q^k \sum_{\delta=-k}^k (\delta - 1) q^{\delta^2} \begin{bmatrix} 2k \\ k - \delta \end{bmatrix}_{-q} \end{aligned}$$

which, by Lemma B.2.7, equals

$$(-q)^k (q+1)(q^3+1) \cdots (q^{2k-1}+1) + (-1)^{k+1} q^k \sum_{\delta=-k}^k \delta q^{\delta^2} \begin{bmatrix} 2k \\ k - \delta \end{bmatrix}_{-q}.$$

The lemma follows since

$$\sum_{\delta=-k}^k \delta q^{\delta^2} \begin{bmatrix} 2k \\ k - \delta \end{bmatrix}_{-q} = 0.$$

□

B.3. Enumeration of Hecke operators in the even rank case. In this subsection, we assume that $N = 2r$ is even.

Lemma B.3.1. *We have the identity*

$$q^{r^2} \prod_{i=1}^r (\mu_i + 2) = \text{Sat}_N^\circ(\mathbb{T}_{N;r}^\circ) + \sum_{\delta=1}^r (q+1)(q^3+1) \cdots (q^{2\delta-1}+1) \cdot \text{Sat}_N^\circ(\mathbb{T}_{N;r-\delta}^\circ)$$

in $\mathbb{Z}[\mathbb{X}_N^*]^{\text{sym}}$.

²⁵A proof can be found at <http://mathworld.wolfram.com/GaussPolynomialIdentity.html>.

Proof. By Lemma B.1.3(1) and Lemma B.2.6, we have

$$\begin{aligned} q^{r^2} \prod_{i=1}^r (\boldsymbol{\mu}_i + 2) &= q^{r^2} \chi(\rho_{N;r}) + q^{r^2} \sum_{\delta=1}^r 2\chi(\rho_{N;r-\delta}) \\ &= \sum_{i=0}^r \begin{bmatrix} 2r-2i \\ r-i \end{bmatrix}_{-q} \text{Sat}_N^\circ(\mathbb{T}_{N;i}^\circ) + \sum_{\delta=1}^r 2q^{\delta^2} \sum_{i=0}^{r-\delta} \begin{bmatrix} 2r-2i \\ r-\delta-i \end{bmatrix}_{-q} \text{Sat}_N^\circ(\mathbb{T}_{N;i}^\circ) \\ &= \sum_{i=0}^r \left(\sum_{\delta=-(r-i)}^{r-i} q^{\delta^2} \begin{bmatrix} 2r-2i \\ r-\delta-i \end{bmatrix}_{-q} \right) \text{Sat}_N^\circ(\mathbb{T}_{N;i}^\circ), \end{aligned}$$

which equals

$$\text{Sat}_N^\circ(\mathbb{T}_{N;r}^\circ) + \sum_{\delta=1}^r (q+1)(q^3+1) \cdots (q^{2\delta-1}+1) \cdot \text{Sat}_N^\circ(\mathbb{T}_{N;r-\delta}^\circ)$$

by Lemma B.2.7. The lemma is proved. \square

Lemma B.3.2. *We have the identity*

$$q^{r^2} \prod_{i=1}^r (\boldsymbol{\mu}_i - q - q^{-1}) = \text{Sat}_N^\circ(\mathbb{T}_{N;r}^\circ) + \sum_{\delta=1}^r (-q)^\delta (q+1)(q^3+1) \cdots (q^{2\delta-1}+1) \cdot \text{Sat}_N^\circ(\mathbb{T}_{N;r-\delta}^\circ)$$

in $\mathbb{Z}[\mathbb{X}_N^*]^{\text{sym}}$.

Proof. By Lemma B.1.3(1) and Lemma B.2.6, we have

$$\begin{aligned} q^{r^2} \prod_{i=1}^r (\boldsymbol{\mu}_i - q - q^{-1}) &= q^{r^2} \chi(\rho_{N;r}) + q^{r^2} \sum_{\delta=1}^r ((-q)^\delta + (-q)^{-\delta}) \chi(\rho_{N;r-\delta}) \\ &= \sum_{i=0}^r \begin{bmatrix} 2r-2i \\ r-i \end{bmatrix}_{-q} \text{Sat}_N^\circ(\mathbb{T}_{N;i}^\circ) + \sum_{\delta=1}^r \sum_{i=0}^{r-\delta} q^{\delta^2} ((-q)^\delta + (-q)^{-\delta}) \begin{bmatrix} 2r-2i \\ r-\delta-i \end{bmatrix}_{-q} \text{Sat}_N^\circ(\mathbb{T}_{N;i}^\circ) \\ &= \sum_{i=0}^r \left(\begin{bmatrix} 2r-2i \\ r-i \end{bmatrix}_{-q} + \sum_{\delta=1}^{r-i} (-1)^\delta (q^{\delta^2+\delta} + q^{\delta^2-\delta}) \begin{bmatrix} 2r-2i \\ r-\delta-i \end{bmatrix}_{-q} \right) \text{Sat}_N^\circ(\mathbb{T}_{N;i}^\circ) \\ &= \sum_{i=0}^r \left(\sum_{\delta=-(r-i)}^{r-i} (-1)^\delta q^{\delta^2+\delta} \begin{bmatrix} 2r-2i \\ r-\delta-i \end{bmatrix}_{-q} \right) \text{Sat}_N^\circ(\mathbb{T}_{N;i}^\circ). \end{aligned}$$

Thus, the lemma follows from Lemma B.3.3 below by comparing coefficients. \square

Lemma B.3.3. *For every integer $k \geq 1$, we have*

$$\sum_{\delta=-k}^k (-1)^\delta q^{\delta^2+\delta} \begin{bmatrix} 2k \\ k-\delta \end{bmatrix}_{-q} = (-q)^k (q+1)(q^3+1) \cdots (q^{2k-1}+1).$$

Proof. By Lemma B.2.7, the lemma is equivalent to the identity

$$(-q)^k \sum_{\delta=-k}^k q^{\delta^2} \begin{bmatrix} 2k \\ k-\delta \end{bmatrix}_{-q} = \sum_{\delta=-k}^k (-1)^\delta q^{\delta^2+\delta} \begin{bmatrix} 2k \\ k-\delta \end{bmatrix}_{-q}.$$

However, we have

$$\begin{aligned}
& (-q)^k \sum_{\delta=-k}^k q^{\delta^2} \begin{bmatrix} 2k \\ k-\delta \end{bmatrix}_{-q} - \sum_{\delta=-k}^k (-1)^\delta q^{\delta^2+\delta} \begin{bmatrix} 2k \\ k-\delta \end{bmatrix}_{-q} \\
&= \sum_{\delta=-k}^k (-1)^\delta q^{\delta^2+\delta} \left((-q)^{k-\delta} - 1 \right) \begin{bmatrix} 2k \\ k-\delta \end{bmatrix}_{-q} \\
&= \sum_{\delta=-k}^k (-1)^\delta q^{\delta^2+\delta} \left((-q)^{2k} - 1 \right) \begin{bmatrix} 2k-1 \\ k-\delta \end{bmatrix}_{-q} \\
&= \left((-q)^{2k} - 1 \right) \sum_{\delta=-k}^k (-1)^\delta q^{\delta^2+\delta} \begin{bmatrix} 2k-1 \\ k-\delta \end{bmatrix}_{-q}.
\end{aligned}$$

Note that in the last summation, the term of δ and the term of $1-\delta$ cancel with each other for $-k < \delta \leq k$; and the term of $-k$ is zero. Thus, the above summation is zero; and the lemma follows. \square

Lemma B.3.4. *We have the identity*

$$\begin{aligned}
& (q^{r^2+1} - q^{r^2-1}) \sum_{j=1}^r \prod_{\substack{i=1 \\ i \neq j}}^r (\mu_i - q - q^{-1}) \\
&= \sum_{\delta=1}^r \left((-q)^\delta (q+1)(q^3+1) \cdots (q^{2\delta-1}+1) - \sum_{i=0}^{\delta} (-1)^i (2i+1) q^{i^2+i} \begin{bmatrix} 2\delta+1 \\ \delta-i \end{bmatrix}_{-q} \right) \text{Sat}_N^\circ(\mathbb{T}_{N;r-\delta}^\circ)
\end{aligned}$$

in $\mathbb{Z}[\mathbb{X}_N^*]^{\text{sym}}$.

Proof. By Lemma B.1.3(2) and Lemma B.2.6, we have

$$\begin{aligned}
& (q^{r^2+1} - q^{r^2-1}) \sum_{j=1}^r \prod_{i \neq j} (\mu_i - q - q^{-1}) \\
&= q^{r^2} \sum_{\delta=1}^r (-1)^{\delta-1} \delta (q^\delta - q^{-\delta}) \cdot \chi(\rho_{N;r-\delta}) \\
&= \sum_{\delta=1}^r (-1)^{\delta-1} q^{\delta^2} (\delta q^\delta - \delta q^{-\delta}) \sum_{i=0}^{r-\delta} \begin{bmatrix} 2r-2i \\ r-\delta-i \end{bmatrix}_{-q} \text{Sat}_N^\circ(\mathbb{T}_{N;i}^\circ) \\
&= \sum_{i=0}^{r-1} \left(\sum_{\delta=1}^{r-i} (-1)^{\delta-1} q^{\delta^2} (\delta q^\delta - \delta q^{-\delta}) \begin{bmatrix} 2r-2i \\ r-\delta-i \end{bmatrix}_{-q} \right) \text{Sat}_N^\circ(\mathbb{T}_{N;i}^\circ).
\end{aligned}$$

Thus the lemma is equivalent to the identity

$$\begin{aligned}
& \sum_{\delta=0}^k (-1)^\delta (2\delta+1) q^{\delta^2+\delta} \begin{bmatrix} 2k+1 \\ k-\delta \end{bmatrix}_{-q} - \sum_{\delta=1}^k (-1)^\delta q^{\delta^2} (\delta q^\delta - \delta q^{-\delta}) \begin{bmatrix} 2k \\ k-\delta \end{bmatrix}_{-q} \\
&= (-q)^k (q+1)(q^3+1) \cdots (q^{2k-1}+1)
\end{aligned}$$

for every integer $k \geq 1$. In fact, we have

$$\begin{aligned} & \sum_{\delta=0}^k (-1)^\delta (2\delta+1) q^{\delta^2+\delta} \begin{bmatrix} 2k+1 \\ k-\delta \end{bmatrix}_{-q} - \sum_{\delta=1}^k (-1)^\delta q^{\delta^2} (\delta q^\delta - \delta q^{-\delta}) \begin{bmatrix} 2k \\ k-\delta \end{bmatrix}_{-q} \\ &= \sum_{\delta=-k-1}^k (-1)^\delta \delta q^{\delta^2+\delta} \begin{bmatrix} 2k+1 \\ k-\delta \end{bmatrix}_{-q} - \sum_{\delta=-k}^k (-1)^\delta q^{\delta^2} \delta q^\delta \begin{bmatrix} 2k \\ k-\delta \end{bmatrix}_{-q} \\ &= (-q)^k (q+1) (q^3+1) \cdots (q^{2k-1}+1) \end{aligned}$$

by Lemma B.2.8. The lemma follows. \square

Proposition B.3.5. *Let L be a ring. Consider an N -tuple $\alpha = (\alpha_1, \dots, \alpha_N) \in L^N$ satisfying $\alpha_i \alpha_{N+1-i} = 1$, which determines a homomorphism $\phi_\alpha^\circ: \mathbb{T}_N^\circ \rightarrow L$ as in Notation B.2.3.*

(1) *We have*

$$\phi_\alpha^\circ(\mathbb{T}_N^\circ) = q^{r^2} \prod_{i=1}^r \left(\alpha_i + \frac{1}{\alpha_i} + 2 \right).$$

(2) *We have*

$$\phi_\alpha^\circ((q+1)\mathbb{R}_N^\circ - \mathbb{T}_N^\circ) = -q^{r^2} \prod_{i=1}^r \left(\alpha_i + \frac{1}{\alpha_i} - q - \frac{1}{q} \right)$$

where

$$\mathbb{R}_N^\circ := \sum_{\delta=0}^{r-1} \frac{1 - (-q)^{r-\delta}}{q+1} (q+1)(q+3) \cdots (q^{2(r-\delta)-1} + 1) \cdot \mathbb{T}_{N;\delta}^\circ.$$

(3) *We have*

$$\phi_\alpha^\circ(\mathbb{R}_N^\circ + (q+1)\mathbb{T}_N^\circ) = - \left(q^{r^2+1} - q^{r^2-1} \right) \sum_{j=1}^r \prod_{\substack{i=1 \\ i \neq j}}^r \left(\alpha_i + \frac{1}{\alpha_i} - q - \frac{1}{q} \right)$$

where

$$\mathbb{T}_N^\circ := \sum_{\delta=0}^{r-1} \mathbf{d}_{r-\delta,q}^\bullet \cdot \mathbb{T}_{N;\delta}^\circ$$

in which the numbers $\mathbf{d}_{r-\delta,q}^\bullet$ are introduced in Notation 1.3.2.

Proof. Part (1) follows from Lemma B.2.5 and Lemma B.3.1. Part (2) follows from Lemma B.2.5 and Lemma B.3.2. Part (3) follows from Lemma B.3.4. \square

Lemma B.3.6. *We have*

$$\mathbb{T}_N^{\bullet\circ} \circ \mathbb{R}_N^\circ = \mathbb{R}_N^\bullet \circ \mathbb{T}_N^{\bullet\circ}, \quad \mathbb{T}_N^{\bullet\circ} \circ \mathbb{T}_N^\circ = \mathbb{T}_N^\bullet \circ \mathbb{T}_N^{\bullet\circ}$$

in $\mathbb{Z}[\mathbf{K}_N^\bullet \setminus \mathbf{U}(\mathbf{V}_N)(F^+)/\mathbf{K}_N^\circ]$, where \mathbb{R}_N° and \mathbb{T}_N° are defined in Proposition B.3.5 (2) and (3), respectively, and

$$\begin{cases} \mathbb{R}_N^\bullet := \sum_{\delta=0}^{r-1} \frac{1 - (-q)^{r-\delta}}{q+1} (q+1)(q+3) \cdots (q^{2(r-\delta)-1} + 1) \cdot \mathbb{T}_{N;\delta}^\bullet, \\ \mathbb{T}_N^\bullet := \sum_{\delta=0}^{r-1} \mathbf{d}_{r-\delta,q}^\bullet \cdot \mathbb{T}_{N;\delta}^\bullet. \end{cases}$$

Proof. In fact, by the same lattice counting argument as for Lemma B.2.5, we have

$$\mathbb{T}_N^{\bullet\circ} \circ \mathbb{T}_{N;\delta}^\circ = \mathbb{T}_{N;\delta}^\bullet \circ \mathbb{T}_N^{\bullet\circ}$$

for every $0 \leq \delta \leq r$. Then the lemma follows immediately. \square

B.4. Enumeration of Hecke operators in the odd rank case. In this subsection, we assume that $N = 2r + 1$ is odd.

Lemma B.4.1. *We have the identity*

$$q^{r^2+r} \prod_{i=1}^r (\mu_i + q + q^{-1}) = \text{Sat}_N^\circ(\mathbb{T}_{N;r}^\circ) + \sum_{\delta=1}^r (q^3 + 1)(q^5 + 1) \cdots (q^{2\delta+1} + 1) \cdot \text{Sat}_N^\circ(\mathbb{T}_{N;r-\delta}^\circ)$$

in $\mathbb{Z}[\mathbb{X}_N^*]^{\text{sym}}$.

Proof. By Lemma B.1.4 and Lemma B.2.6, we have

$$\begin{aligned} q^{r^2+r} \prod_{i=1}^r (\mu_i + q + q^{-1}) &= q^{r^2+r} \sum_{\delta=0}^r \frac{q^{\delta+1} + q^{-\delta}}{q+1} \cdot \chi(\rho_{N;r-\delta}) \\ &= q^{r^2+r} \sum_{\delta=0}^r \frac{q^{\delta+1} + q^{-\delta}}{q+1} \cdot q^{-(r-\delta)(r+1+\delta)} \sum_{i=0}^{r-\delta} \begin{bmatrix} 2r+1-2i \\ r-\delta-i \end{bmatrix}_{-q} \text{Sat}_N^\circ(\mathbb{T}_{N;i}^\circ) \\ &= \frac{1}{q+1} \sum_{i=0}^r \left(\sum_{\delta=0}^{r-i} (q^{2\delta+1} + 1) q^{\delta^2} \begin{bmatrix} 2(r-i)+1 \\ r-i-\delta \end{bmatrix}_{-q} \right) \text{Sat}_N^\circ(\mathbb{T}_{N;i}^\circ) \\ &= \frac{1}{q+1} \sum_{i=0}^r \left(\sum_{\delta=-(r-i)-1}^{r-i} q^{\delta^2} \begin{bmatrix} 2(r-i)+1 \\ r-i-\delta \end{bmatrix}_{-q} \right) \text{Sat}_N^\circ(\mathbb{T}_{N;i}^\circ). \end{aligned}$$

Thus the lemma is equivalent to the identity

$$\sum_{\delta=-k-1}^k q^{\delta^2} \begin{bmatrix} 2k+1 \\ k-\delta \end{bmatrix}_{-q} = (q+1)(q^3+1) \cdots (q^{2k+1}+1)$$

for every integer $k \geq 0$. By Lemma B.2.7, we have

$$\sum_{\delta=-k-1}^{k+1} q^{\delta^2} \begin{bmatrix} 2k+2 \\ k+1-\delta \end{bmatrix}_{-q} = (q+1)(q^3+1) \cdots (q^{2k+1}+1).$$

Thus, it remains to show

$$\sum_{\delta=-k-1}^{k+1} q^{\delta^2} \begin{bmatrix} 2k+2 \\ k+1-\delta \end{bmatrix}_{-q} = \sum_{\delta=-k-1}^k q^{\delta^2} \begin{bmatrix} 2k+1 \\ k-\delta \end{bmatrix}_{-q}.$$

However, the difference equals

$$\begin{aligned} \sum_{\delta=-k-1}^{k+1} q^{a^2} \left(\begin{bmatrix} 2k+2 \\ k+1-\delta \end{bmatrix}_{-q} - \begin{bmatrix} 2k+1 \\ k-\delta \end{bmatrix}_{-q} \right) &= \sum_{\delta=-k-1}^{k+1} q^{a^2} (-q)^{k+1-a} \begin{bmatrix} 2k+1 \\ k+1-\delta \end{bmatrix}_{-q} \\ &= (-q)^{k+1} \sum_{\delta=-k-1}^k (-1)^\delta q^{\delta^2+\delta} \begin{bmatrix} 2k+1 \\ k+1-\delta \end{bmatrix}_{-q} \end{aligned}$$

which equals zero as the term of δ and the term of $-\delta-1$ cancel each other. The lemma follows. \square

Lemma B.4.2. *We have the identity*

$$q^{r^2+r} \prod_{i=1}^r (\mu_i - 2) = \sum_{\delta=0}^r \mathbf{d}_{\delta,q} \cdot \text{Sat}_N^\circ(\mathbb{T}_{N;r-\delta}^\circ)$$

in $\mathbb{Z}[\mathbb{X}_N^*]^{\text{sym}}$, in which the numbers $\mathbf{d}_{\delta,q}$ are introduced in Notation 1.3.2.

Proof. By Lemma B.1.4 and Lemma B.2.6, we have

$$\begin{aligned}
q^{r^2+r} \prod_{i=1}^r (\mu_i - 2) &= q^{r^2+r} \sum_{\delta=0}^r (-1)^\delta (2\delta + 1) \cdot \chi(\rho_{N;r-\delta}) \\
&= q^{r^2+r} \sum_{\delta=0}^r (-1)^\delta (2\delta + 1) \cdot q^{-(r-\delta)(r+1+\delta)} \sum_{i=0}^{r-\delta} \begin{bmatrix} 2r+1-2i \\ r-\delta-i \end{bmatrix}_{-q} \text{Sat}_N^\circ(\mathbb{T}_{N;i}^\circ) \\
&= \sum_{i=0}^r \left(\sum_{\delta=0}^{r-i} (-1)^\delta (2\delta + 1) q^{\delta(\delta+1)} \sum_{i=0}^{r-\delta} \begin{bmatrix} 2(r-i)+1 \\ r-i-\delta \end{bmatrix}_{-q} \right) \text{Sat}_N^\circ(\mathbb{T}_{N;i}^\circ) \\
&= \sum_{\delta=0}^r \mathbf{d}_{\delta,q} \cdot \text{Sat}_N^\circ(\mathbb{T}_{N;r-\delta}^\circ).
\end{aligned}$$

The lemma is proved. \square

Proposition B.4.3. *Let L be a ring. Consider an N -tuple $\alpha = (\alpha_1, \dots, \alpha_N) \in L^N$ satisfying $\alpha_i \alpha_{N+1-i} = 1$, which determines a homomorphism $\phi_\alpha^\circ: \mathbb{T}_N^\circ \rightarrow L$ as in Notation B.2.3.*

(1) *We have*

$$\phi_\alpha^\circ(\mathbb{T}_N^\circ) = q^{r^2+r} \prod_{i=1}^r \left(\alpha_i + \frac{1}{\alpha_i} + q + \frac{1}{q} \right).$$

(2) *We have*

$$\phi_\alpha^\circ(\mathbb{T}_N^\circ) = q^{r^2+r} \prod_{i=1}^r \left(\alpha_i + \frac{1}{\alpha_i} - 2 \right)$$

where

$$\mathbb{T}_N^\circ := \sum_{\delta=0}^r \mathbf{d}_{r-\delta,q} \cdot \mathbb{T}_{N;\delta}^\circ$$

in which the numbers $\mathbf{d}_{r-\delta,q}$ are introduced in Notation 1.3.2.

Proof. Part (1) follows from Lemma B.2.5 and Lemma B.4.1. Part (2) follows from Lemma B.4.2. \square

Lemma B.4.4. *We have*

$$\mathbb{T}_N^{\bullet\circ} \circ \mathbb{T}_N^\circ = \left((q+1)^2 \mathbb{T}_N^{\bullet\bullet} + \mathbb{T}_N^{\bullet\circ} \circ \mathbb{T}_N^{\circ\bullet} \right) \circ \mathbb{T}_N^{\circ\circ}$$

in $\mathbb{Z}[\mathbb{K}_N^\bullet \setminus \mathbb{U}(\mathbb{V}_N)(F^+) / \mathbb{K}_N^\circ]$, where \mathbb{T}_N° is defined in Proposition B.4.3(2), and

$$\mathbb{T}_N^{\bullet\bullet} := \sum_{\delta=0}^{r-1} \mathbf{d}_{r-\delta,q}^{\bullet\bullet} \cdot \mathbb{T}_{N;\delta}^{\bullet\bullet}$$

This lemma is a hard exercise in combinatorics. In fact, our proof below is by brutal force; it would be interesting to find a conceptual proof.

Proof. It suffices to show that for every element $f \in \mathbb{Z}[\text{Lat}_N^\circ]$, we have

$$(B.2) \quad \left((q+1)^2 \mathbb{T}_N^{\bullet\bullet} + \mathbb{T}_N^{\bullet\circ} \circ \mathbb{T}_N^{\circ\bullet} \right) (\mathbb{T}_N^{\bullet\circ}(f)) = \mathbb{T}_N^{\bullet\circ}(\mathbb{T}_N^\circ(f))$$

in $\mathbb{Z}[\text{Lat}_N^\bullet]$. Without loss of generality, we may just consider their values on Λ_N^\bullet .

For every $L \in \text{Lat}_N^\circ$ and $0 \leq \delta \leq r$, we denote

- $\circ c_\delta^\bullet(L)$ the number of $L^\bullet \in \text{Lat}_N^\bullet$ satisfying $L \subseteq L^\bullet$ and $(L^\bullet + \Lambda_N^\bullet) / \Lambda_N^\bullet \simeq (O_F/\mathfrak{p})^{\oplus \delta}$; and
- $\circ c_\delta^\circ(L)$ the number of $L^\circ \in \text{Lat}_N^\circ$ satisfying $L^\circ \subseteq \Lambda_N^\bullet$ and $L / (L \cap L^\circ) \simeq (O_F/\mathfrak{p})^{\oplus \delta}$.

We then have

$$\begin{aligned} (\mathbf{T}_{N;\delta}^\bullet(\mathbf{T}_N^{\circ\circ}(f)))(\Lambda_N^\bullet) &= \sum_{L \in \text{Lat}_N^\circ} c_\delta^\bullet(L) \cdot f(L), \\ (\mathbf{T}_N^{\circ\circ}(\mathbf{T}_{N;\delta}^\circ(f)))(\Lambda_N^\bullet) &= \sum_{L \in \text{Lat}_N^\circ} c_\delta^\circ(L) \cdot f(L). \end{aligned}$$

We claim the following identities

$$(B.3) \quad c_\delta^\bullet(L) = \begin{cases} q^{(\delta-\gamma)(\delta-\gamma+2)} \begin{bmatrix} r-\gamma \\ \delta-\gamma \end{bmatrix}_{q^2}, & \text{if } (L + \Lambda_N^\bullet)/\Lambda_N^\bullet \simeq (O_F/\mathfrak{p})^{\oplus\gamma} \text{ for some } 0 \leq \gamma \leq \delta; \\ 0, & \text{otherwise;} \end{cases}$$

$$(B.4) \quad c_\delta^\circ(L) = \begin{cases} q^{(\delta-\gamma)^2} \begin{bmatrix} r-\gamma \\ \delta-\gamma \end{bmatrix}_{q^2}, & \text{if } (L + \Lambda_N^\bullet)/\Lambda_N^\bullet \simeq (O_F/\mathfrak{p})^{\oplus\gamma} \text{ for some } 0 \leq \gamma \leq \delta; \\ 0, & \text{otherwise.} \end{cases}$$

For (B.3), we must have $(L + \Lambda_N^\bullet)/\Lambda_N^\bullet \subseteq (L^\bullet + \Lambda_N^\bullet)/\Lambda_N^\bullet \simeq (O_F/\mathfrak{p})^{\oplus\delta}$. Thus, the otherwise case is confirmed. Suppose $(L + \Lambda_N^\bullet)/\Lambda_N^\bullet \simeq (O_F/\mathfrak{p})^{\oplus\gamma}$ for some $0 \leq \gamma \leq \delta$. Then $(\mathfrak{p}\Lambda_N^\bullet + L)/L$ is an isotropic subspace of $\mathfrak{p}^{-1}L/L$ of dimension γ . Moreover, $c_\delta^\bullet(L)$ is the same as the number of maximal isotropic subspaces of $((\mathfrak{p}\Lambda_N^\bullet + L)/L)^\perp/((\mathfrak{p}\Lambda_N^\bullet + L)/L)$ whose intersection with (the image of) $(\mathfrak{p}^{-1}L \cap \Lambda_N^\bullet + L)/L$, which itself is a maximal isotropic subspace, has dimension $r - \delta$. Thus, we obtain (B.3) by Lemma B.4.5 below since $((\mathfrak{p}\Lambda_N^\bullet + L)/L)^\perp/((\mathfrak{p}\Lambda_N^\bullet + L)/L)$ has dimension $2r + 1 - 2\gamma$.

For (B.4), we must have $(L + \Lambda_N^\bullet)/\Lambda_N^\bullet \simeq L/(L \cap \Lambda_N^\bullet)$ which is a quotient of $L/(L \cap L^\circ) \simeq (O_F/\mathfrak{p})^{\oplus\delta}$. Thus, the otherwise case is confirmed. Suppose $(L + \Lambda_N^\bullet)/\Lambda_N^\bullet \simeq (O_F/\mathfrak{p})^{\oplus\gamma}$ for some $0 \leq \gamma \leq \delta$. Then $(L + \Lambda_N^\bullet)/\Lambda_N^\bullet$ is an isotropic subspace of $\mathfrak{p}^{-1}\Lambda_N^\bullet/\Lambda_N^\bullet$ of dimension γ . Moreover, $c_\delta^\circ(L)$ is the same as the number of maximal isotropic subspaces of $((L + \Lambda_N^\bullet)/\Lambda_N^\bullet)^\perp/((L + \Lambda_N^\bullet)/\Lambda_N^\bullet)$ whose intersection with (the image of) $(\mathfrak{p}^{-1}\Lambda_N^\bullet \cap \mathfrak{p}^{-1}L + \Lambda_N^\bullet)/\Lambda_N^\bullet$, which itself is a maximal isotropic subspace, has dimension $r - \delta$. Thus, we obtain (B.4) by Lemma B.4.5 since $((L + \Lambda_N^\bullet)/\Lambda_N^\bullet)^\perp/((L + \Lambda_N^\bullet)/\Lambda_N^\bullet)$ has dimension $2r - 2\gamma$.

Now we come back to the values of (B.2) on Λ_N^\bullet . By a similar proof of Lemma B.2.5, we have

$$\mathbf{T}_N^{\circ\circ} \circ \mathbf{T}_N^{\circ\circ} = \mathbf{T}_{N;r}^\bullet + (q+1)\mathbf{T}_{N;r-1}^\bullet + (q+1)(q^3+1)\mathbf{T}_{N;r-2}^\bullet + \cdots + \prod_{i=1}^r (q^{2i-1}+1)\mathbf{T}_{N;0}^\bullet$$

in \mathbf{T}_N^\bullet . Then under Notation 1.3.2, we have

$$(B.5) \quad \begin{aligned} & \left((q+1)^2 \mathbf{T}_N^\bullet + \mathbf{T}_N^{\circ\circ} \circ \mathbf{T}_N^{\circ\circ} \right) \circ \mathbf{T}_N^{\circ\circ} \\ &= \mathbf{T}_{N;r}^\bullet \circ \mathbf{T}_N^{\circ\circ} + \sum_{\delta=0}^{r-1} \left((q+1)\mathbf{d}_{r-\delta,q} + (-q)^{r-\delta+1}(q+1)(q^3+1)\cdots(q^{2(r-\delta)-1}+1) \right) \mathbf{T}_{N;\delta}^\bullet \circ \mathbf{T}_N^{\circ\circ}. \end{aligned}$$

By (B.3), (B.4) and (B.5), the lemma is equivalent to that for every integer $k \geq 0$, we have

$$\sum_{\delta=0}^k \mathbf{d}_{k-\delta,q} q^{\delta^2} \begin{bmatrix} k \\ \delta \end{bmatrix}_{q^2} = q^{k(k+2)} + \sum_{\delta=0}^{k-1} \left((q+1)\mathbf{d}_{k-\delta,q} + (-q)^{k-\delta+1}(q+1)(q^3+1)\cdots(q^{2(k-\delta)-1}+1) \right) q^{\delta(\delta+2)} \begin{bmatrix} k \\ \delta \end{bmatrix}_{q^2},$$

or equivalently,

$$(B.6) \quad \sum_{\delta=0}^k \mathbf{d}_{\delta,q} q^{(k-\delta)^2} \begin{bmatrix} k \\ \delta \end{bmatrix}_{q^2} = q^{k(k+2)} + \sum_{\delta=1}^k \left((q+1)\mathbf{d}_{\delta,q} + (-q)^{\delta+1}(q+1)(q^3+1)\cdots(q^{2\delta-1}+1) \right) q^{(k-\delta)(k-\delta+2)} \begin{bmatrix} k \\ \delta \end{bmatrix}_{q^2}.$$

By Lemma B.2.8, we have

$$\begin{aligned}
& (-q)^{\delta+1}(q+1)(q^3+1)\cdots(q^{2\delta-1}+1) \\
&= -q \sum_{j=-\delta-1}^{\delta} (-1)^j j q^{j^2+j} \begin{bmatrix} 2\delta+1 \\ \delta-j \end{bmatrix}_{-q} + q \sum_{j=-\delta}^{\delta} (-1)^j j q^{j^2+j} \begin{bmatrix} 2\delta \\ \delta-j \end{bmatrix}_{-q} \\
&= -q \mathbf{d}_{\delta,q} + q \sum_{j=-\delta}^{\delta} (-1)^j j q^{j^2+j} \begin{bmatrix} 2\delta \\ \delta-j \end{bmatrix}_{-q}.
\end{aligned}$$

Thus, (B.6) is equivalent to

$$\sum_{\delta=0}^k \mathbf{d}_{\delta,q} q^{(k-\delta)^2} \begin{bmatrix} k \\ \delta \end{bmatrix}_{q^2} = \sum_{\delta=0}^k \left(\mathbf{d}_{\delta,q} + q \sum_{j=-\delta}^{\delta} (-1)^j j q^{j^2+j} \begin{bmatrix} 2\delta \\ \delta-j \end{bmatrix}_{-q} \right) q^{(k-\delta)(k-\delta+2)} \begin{bmatrix} k \\ \delta \end{bmatrix}_{q^2},$$

or equivalently,

$$\text{(B.7)} \quad \sum_{\delta=0}^k \mathbf{d}_{\delta,q} q^{(k-\delta)^2} (q^{2(k-\delta)} - 1) \begin{bmatrix} k \\ \delta \end{bmatrix}_{q^2} = - \sum_{\delta=0}^k \sum_{j=-\delta}^{\delta} (-1)^j j q^{j^2+j} \begin{bmatrix} 2\delta \\ \delta-j \end{bmatrix}_{-q} q^{(k-\delta+1)^2} \begin{bmatrix} k \\ \delta \end{bmatrix}_{q^2}.$$

However, we have

$$\begin{aligned}
& \sum_{\delta=0}^k \mathbf{d}_{\delta,q} q^{(k-\delta)^2} (q^{2(k-\delta)} - 1) \begin{bmatrix} k \\ \delta \end{bmatrix}_{q^2} \\
&= \sum_{\delta=0}^{k-1} \mathbf{d}_{\delta,q} q^{(k-\delta)^2} (q^{2(k-\delta)} - 1) \begin{bmatrix} k \\ \delta \end{bmatrix}_{q^2} \\
&= \sum_{\delta=0}^{k-1} \sum_{j=-\delta-1}^{\delta} (-1)^j j q^{j^2+j} \begin{bmatrix} 2\delta+1 \\ \delta-j \end{bmatrix}_{-q} q^{(k-\delta)^2} (q^{2(k-\delta)} - 1) \begin{bmatrix} k \\ \delta \end{bmatrix}_{q^2} \\
&= \sum_{\delta=0}^{k-1} \sum_{j=-\delta-1}^{\delta} (-1)^j j q^{j^2+j} \begin{bmatrix} 2\delta+1 \\ \delta-j \end{bmatrix}_{-q} q^{(k-\delta)^2} (q^{2\delta+2} - 1) \begin{bmatrix} k \\ \delta+1 \end{bmatrix}_{q^2} \\
&= \sum_{\delta=0}^{k-1} \sum_{j=-\delta-1}^{\delta} (-1)^j j q^{(k-\delta)^2+j^2+j} ((-q)^{2\delta+2} - 1) \begin{bmatrix} 2\delta+1 \\ \delta-j \end{bmatrix}_{-q} \begin{bmatrix} k \\ \delta+1 \end{bmatrix}_{q^2} \\
&= \sum_{\delta=0}^{k-1} \sum_{j=-\delta-1}^{\delta} (-1)^j j q^{(k-\delta)^2+j^2+j} ((-q)^{\delta-j+1} - 1) \begin{bmatrix} 2\delta+2 \\ \delta-j+1 \end{bmatrix}_{-q} \begin{bmatrix} k \\ \delta+1 \end{bmatrix}_{q^2} \\
&= \sum_{\delta=1}^k \sum_{j=-\delta}^{\delta-1} (-1)^j j q^{(k+1-\delta)^2+j^2+j} ((-q)^{\delta-j} - 1) \begin{bmatrix} 2\delta \\ \delta-j \end{bmatrix}_{-q} \begin{bmatrix} k \\ \delta \end{bmatrix}_{q^2} \\
&= \sum_{\delta=0}^k \sum_{j=-\delta}^{\delta} (-1)^j j q^{(k+1-\delta)^2+j^2+j} ((-q)^{\delta-j} - 1) \begin{bmatrix} 2\delta \\ \delta-j \end{bmatrix}_{-q} \begin{bmatrix} k \\ \delta \end{bmatrix}_{q^2}.
\end{aligned}$$

Thus, (B.7) is equivalent to

$$\sum_{\delta=0}^k \sum_{j=-\delta}^{\delta} (-1)^j j q^{(k+1-\delta)^2+j^2+j} (-q)^{\delta-j} \begin{bmatrix} 2\delta \\ \delta-j \end{bmatrix}_{-q} \begin{bmatrix} k \\ \delta \end{bmatrix}_{q^2} = 0,$$

which is obvious since

$$\sum_{j=-\delta}^{\delta} j q^{j^2} \begin{bmatrix} 2\delta \\ \delta-j \end{bmatrix}_{-q} = 0.$$

The lemma is finally proved. \square

Lemma B.4.5. *Let V be a (nondegenerate) hermitian space over O_F/\mathfrak{p} of dimension $m \geq 1$ with $r = \lfloor \frac{m}{2} \rfloor$, and $Y_0 \subseteq V$ a maximal isotropic subspace. Then the number of maximal isotropic subspaces $Y \subseteq V$ satisfying $\dim_{O_F/\mathfrak{p}}(Y \cap Y_0) = r - s$ with $0 \leq s \leq r$ is given by*

$$\begin{cases} q^{s(s+2)} \begin{bmatrix} r \\ s \end{bmatrix}_{q^2}, & \text{if } m = 2r + 1; \\ q^{s^2} \begin{bmatrix} r \\ s \end{bmatrix}_{q^2}, & \text{if } m = 2r. \end{cases}$$

Proof. We will prove the case for m odd and leave the similar case for m even to readers. We fix an integer $0 \leq s \leq r$. It is easy to see that the number of choices of the intersection $Y \cap Y_0$ (of dimension $r - s$) is

$$\frac{(q^{2r} - 1)(q^{2(r-1)} - 1) \cdots (q^{2(r-s+1)} - 1)}{(q^{2s} - 1)(q^{2(s-1)} - 1) \cdots (q^2 - 1)} = \begin{bmatrix} r \\ s \end{bmatrix}_{q^2}.$$

Then we count the number of Y with $Y \cap Y_0$ fixed. We take a basis $\{e_{-r}, \dots, e_r\}$ of V such that $(e_{-i}, e_j)_V = \delta_{i,j}$ for $0 \leq i, j \leq r$; Y_0 is spanned by $\{e_{-r}, \dots, e_{-1}\}$; and $Y \cap Y_0$ is spanned by $\{e_{-r}, \dots, e_{-s-1}\}$. Let $\{f_1, \dots, f_s\}$ be an element in Y^s such that $\{e_{-r}, \dots, e_{-s-1}, f_1, \dots, f_s\}$ form a basis of Y . Then since Y is isotropic, the coefficients on $\{e_{s+1}, \dots, e_r\}$ of each f_i have to be zero. In particular, there is unique such element $\{f_1, \dots, f_s\} \in Y^s$ that is of the form

$$(f_1, \dots, f_s) = (e_1, \dots, e_s) + (e_{-s}, \dots, e_{-1}, e_0) \begin{pmatrix} A \\ v \end{pmatrix}$$

with (uniquely determined) $A \in M_{s,s}(O_F/\mathfrak{p})$ and $v \in M_{1,s}(O_F/\mathfrak{p})$. Moreover, the isotropic condition on Y is equivalent to that $A + A' = 0$ where A' is the q -th Frobenius of A (and no condition on v). It follows that the number for such Y with given $Y \cap Y_0$ (of dimension $r - s$) is $q^{s(s+2)}$. Thus, the lemma follows. \square

APPENDIX C. SOME REPRESENTATION THEORY FOR UNITARY GROUPS

In this section, we prove several results for representations of unitary groups. Unless specified otherwise, all representations will have coefficients in \mathbb{C} . In Subsection C.1, we recall some general facts about the local base change for unitary groups. In Subsection 6.2, we study the representation appeared in the cohomology of Fermat hypersurfaces, and also compute the local base change of some semistable representations. In Subsection C.3, we collect everything we need from the endoscopic classification for unitary groups in Proposition C.3.1 and derive two corollaries from it.

C.1. Local base change for unitary groups. In this subsection, we fix an unramified quadratic extension F/F^+ of nonarchimedean local fields.

Consider a hermitian space V over F (with respect to F/F^+) of rank N . Put $G := U(V)$. For an irreducible admissible representation π of $G(F^+)$, we denote by $\text{BC}(\pi)$ its base change, which is an irreducible admissible representation of $\text{GL}_N(F)$. Such local base change is defined by [Rog90] when $N \leq 3$ and by [Mok15, KMSW] for general N .

We review the construction of $\text{BC}(\pi)$ in certain special cases. For a parabolic subgroup P of G and an admissible representation σ of $P(F^+)$, we denote by $I_P^G(\sigma)$ the normalized parabolic induction, which is an admissible representation of $G(F^+)$. Fix a minimal parabolic subgroup P_{\min} of G .

We first review Langlands classification of irreducible admissible representations of $G(F^+)$ (see, for example, [Kon03]). For an irreducible admissible representation π of $G(F^+)$, there is a unique

parabolic subgroup P of G containing P_{\min} with Levi quotient M_P , a unique tempered representation τ of $M_P(F^+)$, and a unique strictly positive (unramified) character χ of $P_\pi(F^+)$, such that π is isomorphic to the unique irreducible quotient of $I_P^G(\tau\chi)$, which we denote by $J_P^G(\tau\chi)$, known as the Langlands quotient.

We then review the construction of tempered representations from discrete series representations (see, for example, [Jan14]). Let τ be an irreducible admissible tempered representation of $G(F^+)$. Then there is a unique parabolic subgroup P of G containing P_{\min} , and a discrete series representation σ of $M_P(F^+)$ such that τ is a direct summand of $I_P^G(\sigma)$. In fact, $I_P^G(\sigma)$ is a direct sum of finitely many tempered representations of multiplicity one.

Now suppose that $\pi \simeq J_P^G(\tau\chi)$ is a Langlands quotient. Then we may write

$$M_P = G_0 \times \operatorname{Res}_{F/F^+} \operatorname{GL}_{r_1} \times \cdots \times \operatorname{Res}_{F/F^+} \operatorname{GL}_{r_t}$$

with G_0 the unitary factor, under which

$$\chi = \mathbf{1} \boxtimes |\det_{r_1}|_F^{-s_1} \boxtimes \cdots \boxtimes |\det_{r_t}|_F^{-s_t}$$

for unique real numbers $0 < s_1 < \cdots < s_t$, where \det_r denotes the determinant on $\operatorname{GL}_r(F)$. Suppose $\tau = \tau_0 \boxtimes \tau_1 \boxtimes \cdots \boxtimes \tau_t$ under the above decomposition. Consider a standard parabolic subgroup \tilde{P} of GL_N whose Levi is $\operatorname{GL}_{r_t} \times \cdots \times \operatorname{GL}_{r_1} \times \operatorname{GL}_{N_0} \times \operatorname{GL}_{r_1} \times \cdots \times \operatorname{GL}_{r_t}$. Then we have

$$\operatorname{BC}(\pi) \simeq J_{\tilde{P}}^{\operatorname{GL}_N} \left(\tau_t^{\vee c} |\det_{r_t}|_F^{s_t} \boxtimes \cdots \boxtimes \tau_1^{\vee c} |\det_{r_1}|_F^{s_1} \boxtimes \operatorname{BC}(\tau_0) \boxtimes \tau_1 |\det_{r_1}|_F^{-s_1} \boxtimes \cdots \boxtimes \tau_t |\det_{r_t}|_F^{-s_t} \right)$$

which is a Langlands quotient of $\operatorname{GL}_N(F)$. Here, τ^c stands for $\tau \circ c$.

Now suppose that τ is an irreducible admissible tempered representation of $G(F^+)$, which is a direct summand of $I_P^G(\sigma)$ for some square-integrable representation σ of $P(F^+)$. Write $\sigma = \sigma_0 \boxtimes \sigma_1 \boxtimes \cdots \boxtimes \sigma_t$, similar to the previous case. Then under the same notation, we have

$$\operatorname{BC}(\tau) \simeq I_{\tilde{P}}^{\operatorname{GL}_N} (\sigma_t^{\vee c} \boxtimes \cdots \boxtimes \sigma_1^{\vee c} \boxtimes \operatorname{BC}(\sigma_0) \boxtimes \sigma_1 \boxtimes \cdots \boxtimes \sigma_t)$$

which is an irreducible admissible representation of $\operatorname{GL}_N(F)$.

Finally, if π is an irreducible admissible representation of $G(F^+)$ that is a constituent of an unramified principal series, then $\operatorname{BC}(\pi)$ is a constituent of an unramified principal series of $\operatorname{GL}_N(F)$. Thus, it makes sense to talk about the Satake parameter of $\operatorname{BC}(\pi)$, denoted by $\boldsymbol{\alpha}(\operatorname{BC}(\pi))$.

In what follows, we will suppress the parabolic subgroup \tilde{P} of GL_N when it is clear. We denote by St_N the Steinberg representation of $\operatorname{GL}_N(F)$.

C.2. Tate–Thompson representations. In this subsection, we fix an unramified quadratic extension F/F^+ of nonarchimedean local fields, with residue field extension κ/κ^+ . Let q be the residue cardinality of F^+ and \mathfrak{p} the maximal ideal of O_F .

Let $N \geq 2$ be an integer with $r := \lfloor \frac{N}{2} \rfloor$. Consider a hermitian space V_N over F of rank N together with a self-dual lattice Λ_N . Put $\bar{U}_N := U(V_N)$, and let K_N be the stabilizer of Λ_N which is a hyperspecial maximal subgroup of $U_N(F^+)$. Put $\bar{\Lambda}_N := \Lambda_N \otimes_{O_{F^+}} \kappa^+$ and $\bar{U}_N := U(\bar{\Lambda}_N)$. Then we have the reduction homomorphism $K_N \rightarrow \bar{U}_N(\kappa^+)$.

Let $\operatorname{Iso}(\bar{\Lambda}_N) \subseteq \mathbb{P}(\bar{\Lambda}_N)$ be the isotropic locus, that is, it parameterizes hyperplanes H of $\bar{\Lambda}_N$ satisfying $H^\perp \subseteq H$. Then $\operatorname{Iso}(\bar{\Lambda}_N)$ is a smooth hypersurface in $\mathbb{P}(\bar{\Lambda}_N)$. In particular, $\operatorname{Iso}(\bar{\Lambda}_N)$ has dimension $N - 2$ and admits a natural action by $\bar{U}_N(\kappa^+)$. For a rational prime ℓ that is invertible in κ , put

$$\operatorname{H}^{\operatorname{prim}}(\operatorname{Iso}(\bar{\Lambda}_N)_{\bar{\kappa}}, \bar{\mathbb{Q}}_\ell) := \ker \left(\cup_{c_1}(\mathcal{O}_{\mathbb{P}(\bar{\Lambda}_N)}(1)) : \operatorname{H}_{\text{ét}}^{N-2}(\operatorname{Iso}(\bar{\Lambda}_N)_{\bar{\kappa}}, \bar{\mathbb{Q}}_\ell) \rightarrow \operatorname{H}_{\text{ét}}^N(\operatorname{Iso}(\bar{\Lambda}_N)_{\bar{\kappa}}, \bar{\mathbb{Q}}_\ell(1)) \right).$$

It is well-known by Tate–Thompson that (see, for example, [HM78]) there is a unique irreducible representation Ω_N of $\bar{U}_N(\kappa^+)$ such that Ω_N is isomorphic to $\iota_\ell^{-1} \mathbf{H}^{\text{prim}}(\text{Iso}(\bar{\Lambda}_N)_{\bar{\kappa}}, \bar{\mathbb{Q}}_\ell)$ as representations of $\bar{U}_N(\kappa^+)$ for every isomorphism $\iota_\ell: \mathbb{C} \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell$. We call Ω_N the *Tate–Thompson representation*. We often regard Ω_N as a representation of K_N by inflation according to the context.

To describe Ω_N , we first recall some notation from parabolic induction of finite reductive groups. For every N , we fix a Borel subgroup P_N of \bar{U}_N . For positive integers r_1, \dots, r_t satisfying $r_1 + \dots + r_t \leq r$, we obtain a parabolic subgroup $P_N^{(r_1, \dots, r_t)}$ of \bar{U}_N containing P_N , whose Levi quotient $M_N^{(r_1, \dots, r_t)}$ is isomorphic to $\bar{U}_{N-2(r_1+\dots+r_t)} \times \text{Res}_{\kappa/\kappa^+} \text{GL}_{r_1} \times \dots \times \text{Res}_{\kappa/\kappa^+} \text{GL}_{r_t}$. For example, we have $P_N^{(1^r)} = P_N$. Given a representation σ of $M_N^{(r_1, \dots, r_t)}(\kappa^+)$, we denote by $\text{Ind}_{P_N^{(r_1, \dots, r_t)}}^{\bar{U}_N} \sigma$ the parabolic induction, which is a representation of $\bar{U}_N(\kappa^+)$.

Now we suppose $N = 2r$ even. The irreducible constituents of $\text{Ind}_{P_N}^{\bar{U}_N} \mathbf{1}$ is parameterized by irreducible representations of the Weyl group $W_N \simeq \{\pm 1\}^r \rtimes \mathfrak{S}_r$. For every irreducible representation ϵ of W_N , we denote by $\text{PS}(\epsilon)$ the corresponding irreducible representation of $\bar{U}_N(\kappa^+)$. We now specify a character $\epsilon_N^{\text{TT}}: W_N \rightarrow \{\pm 1\}$ as the extension of the product homomorphism $\{\pm 1\}^r \rightarrow \{\pm 1\}$, which is invariant under the \mathfrak{S}_r -action, to W_N that is trivial on $\{+1\}^r \rtimes \mathfrak{S}_r$.

Proposition C.2.1. *We have*

- (1) *When $N = 2r$ is even, the representation Ω_N is isomorphic to $\text{PS}(\epsilon_N^{\text{TT}})$.*
- (2) *When $N = 2r$ is even, Ω_N is the unique nontrivial irreducible representation of $\bar{U}_N(\kappa^+)$ satisfying $\dim \Omega_N^{P_N(\kappa^+)} = \dim \Omega_N^{P_N^{(r)}(\kappa^+)} = 1$.*
- (3) *The representation Ω_3 is the (unique) cuspidal unipotent representation of $\bar{U}_3(\kappa^+)$.*
- (4) *When $N = 2r + 1$ is odd with $r \geq 1$, the representation Ω_N is a multiplicity free constituent of $\text{Ind}_{P_N^{(1^{r-1})}}^{\bar{U}_N} \Omega_3 \boxtimes \mathbf{1}^{\boxtimes r-1}$.*

Proof. We recall some notion of Deligne–Lusztig characters. Let \mathfrak{S}_N be the group of N -permutations, and \mathfrak{P}_N its conjugacy classes which is canonically identified with the set of partitions of N . For every $\pi \in \mathfrak{P}_N$, we let R_π be the Deligne–Lusztig character (of $\bar{U}_N(\kappa^+)$) [DL76, Corollary 4.3] associated to the trivial representation of the maximal torus corresponding to π . Let R_N be the character of the representation Ω_N . Then by [HM78, Theorem 1], we have

$$(C.1) \quad R_N = (-1)^{N+1} \sum_{\pi \in \mathfrak{P}_N} \frac{\chi_N(\pi)}{z_\pi} R_\pi$$

where χ_N is the character function (on \mathfrak{P}_N) of the unique nontrivial subrepresentation of the standard representation of \mathfrak{S}_N ; and $N!/z_\pi$ is the cardinality of the conjugacy class π . By [DL76, Theorem 6.8], we have the following orthogonality relation

$$(C.2) \quad \langle R_\pi, R_{\pi'} \rangle = \begin{cases} 0, & \text{if } \pi \neq \pi'; \\ z_\pi, & \text{if } \pi = \pi'. \end{cases}$$

We are ready to prove the proposition.

For (1), note that ϵ_N^{TT} is the unique nontrivial character of W_N that is trivial on $\{+1\}^r \rtimes \mathfrak{S}_r$. Thus, (1) follows from (2) by [Cur79, Theorem 4.4.5].

For (2), it suffices to show that $\dim \Omega_N^{P_N(\kappa^+)} = 1$ and $\Omega_N^{P_N^{(r)}(\kappa^+)} \neq 0$. Let R'_{2r} be the character of $\text{Ind}_{P_{2r}}^{\bar{U}_{2r}} \mathbf{1}$. Then by [DL76, Proposition 8.2], we have $R'_{2r} = R_{(2^r)}$. By (C.1) and (C.2), we have

$$\langle R_{2r}, R'_{2r} \rangle = \left\langle - \sum_{\pi \in \mathfrak{P}_{2r}} \frac{\chi_{2r}(\pi)}{z_\pi} R_\pi, R_{(2^r)} \right\rangle = -\chi_{2r}((2^r)) = -(-1) = 1,$$

which implies $\dim \Omega_N^{\mathbb{P}_N(\kappa^+)} = 1$. Let $Y_N \subseteq \bar{\Lambda}_N$ be the maximal isotropic subspace stabilized by $\mathbb{P}_N^{(r)}$. Then $\mathbb{P}(Y_N)$ is contained in $\text{Iso}(\bar{\Lambda}_N)$, which gives rise to an element in $\text{CH}^{r-1}(\text{Iso}(\bar{\Lambda}_N))$. It is well-known that its cohomology class subtracted by $c_1(\mathcal{O}_{\mathbb{P}(\bar{\Lambda}_N)}(1))$ is a nonzero element in $\text{H}^{\text{prim}}(\text{Iso}(\bar{\Lambda}_N), \overline{\mathbb{Q}}_\ell)(r-1)$, which is fixed by $\mathbb{P}_N^{(r)}(\kappa^+)$ by construction. Thus, we have $\Omega_N^{\mathbb{P}_N(\kappa^+)} \neq 0$; and (1) and (2) follow.

For (3), we have $R_3 = \frac{1}{3}(R_{(1^3)} - R_{(3)})$ by (C.1). Then as computed in [Pra, Example 6.2], Ω_3 is the unique cuspidal unipotent representation of $\bar{U}_3(\kappa^+)$.

For (4), let R'_{2r+1} be the character of $\text{Ind}_{\mathbb{P}_{2r+1}^{(1^{r-1})}}^{\bar{U}_{2r+1}}(\Omega_3 \boxtimes \mathbf{1}^{\boxtimes r-1})$. Then by [DL76, Proposition 8.2], we have

$$R'_{2r+1} = \frac{1}{3} \left(R_{(2^{r-1}, 1^3)} - R_{(2^{r-1}, 3)} \right).$$

By (C.1) and (C.2), we have

$$\begin{aligned} \langle R_{2r+1}, R'_{2r+1} \rangle &= \left\langle \sum_{\pi \in \mathfrak{P}_{2r+1}} \frac{\chi_{2r+1}(\pi)}{z_\pi} R_\pi, \frac{1}{3} \left(R_{(2^{r-1}, 1^3)} - R_{(2^{r-1}, 3)} \right) \right\rangle \\ &= \frac{1}{3} \left(\chi_{2r+1}((2^{r-1}, 1^3)) - \chi_{2r+1}((2^{r-1}, 3)) \right) = \frac{1}{3}(2 - (-1)) = 1. \end{aligned}$$

Thus, (4) follows. \square

From now on, we assume that $N = 2r$ is even. Let V'_{2r} be another hermitian space over F together with a lattice Λ'_{2r} satisfying $\Lambda'_{2r} \subseteq (\Lambda'_{2r})^\vee$ and $(\Lambda'_{2r})^\vee / \Lambda'_{2r} \simeq \kappa$. Put $U'_{2r} := \text{U}(V'_{2r})$, and let K'_{2r} be the stabilizer of Λ'_{2r} which is a special maximal subgroup of $U'_{2r}(F^+)$. The following proposition exhibits an example of the local Jacquet–Langlands correspondence.

Proposition C.2.2. *Define*

- \mathcal{S} to be the set of isomorphism classes of irreducible admissible representations π of $U_{2r}(F^+)$ such that $\pi|_{K_{2r}}$ contains Ω_{2r} and that the Satake parameter of $\text{BC}(\pi)$ contains $\{q, q^{-1}\}$ with multiplicity one but does not contain $\{-1, -1\}$ (Remark 3.1.6);
- \mathcal{S}' to be the set of isomorphism classes of irreducible admissible representations π' of $U'_{2r}(F^+)$ such that $\pi'|_{K'_{2r}}$ contains the trivial representation and that the Satake parameter of $\text{BC}(\pi')$ contains $\{q, q^{-1}\}$ with multiplicity one but does not contain $\{-1, -1\}$.

Then there is a unique bijection between \mathcal{S} and \mathcal{S}' such that π and π' correspond if and only if $\text{BC}(\pi) \simeq \text{BC}(\pi')$.

Proof. We first note that both $\text{BC}(\pi)$ and $\text{BC}(\pi')$ are constituents of unramified principal series. We define a correspondence between \mathcal{S} and \mathcal{S}' via the condition that the two Satake parameters $\alpha(\text{BC}(\pi))$ and $\alpha(\text{BC}(\pi'))$ coincide, which is clearly a bijection. By Lemma C.2.3(2) and Lemma C.2.4 below, we have $\text{BC}(\pi) \simeq \text{BC}(\pi')$ if π and π' correspond. The proposition is proved. \square

Lemma C.2.3. *For every irreducible admissible representation π of $U_{2r}(F^+)$ such that $\pi|_{K_{2r}}$ contains Ω_{2r} (hence π is a constituent of an unramified principal series).*

- (1) *If the Satake parameter of $\text{BC}(\pi)$ does not contain $\{q, q^{-1}\}$ and does not contain $\{-1, -1\}$, then π is unramified.*
- (2) *If the Satake parameter of $\text{BC}(\pi)$ contains $\{q, q^{-1}\}$ with multiplicity one, then there exists a unique multi-subset $\{s_2, \dots, s_r\} \subseteq \mathbb{C}/(\log q)^{-1}\pi i$ with $\text{Re } s_i \geq 0$ such that if we arrange them so that $0 \leq \text{Re } s_2 \leq \dots \leq \text{Re } s_r$ holds, then $\text{BC}(\pi)$ is isomorphic to the unique irreducible quotient of*

$$\text{I}^{\text{GL}_{2r}} \left(| \cdot |_F^{s_r} \boxtimes \dots \boxtimes | \cdot |_F^{s_2} \boxtimes \text{St}_2 \boxtimes | \cdot |_F^{s_2} \boxtimes \dots \boxtimes | \cdot |_F^{-s_r} \right).$$

Proof. We fix a decomposition

$$\Lambda_{2r} = O_F e_{-r} \oplus \cdots \oplus O_F e_{-1} \oplus O_F e_1 \oplus \cdots \oplus O_F e_r$$

in which $(e_{-i}, e_j) = \delta_{ij}$ for $1 \leq i, j \leq r$. For $0 \leq i \leq r$, put

$$V_{2i} := F e_{-i} \oplus \cdots \oplus F e_{-1} \oplus F e_1 \oplus \cdots \oplus F e_i$$

which is a hermitian subspace of V_{2r} . We take the minimal parabolic (Borel) subgroup P_{\min} of $G := U_{2r}$ to be the stabilizer of the flag $F e_{-r} \subseteq \cdots \subseteq F e_{-r} \oplus \cdots \oplus F e_{-1}$. We also fix a Levi subgroup of P_{\min} to be $\text{Res}_{F/F^+} \text{GL}(F e_1) \times \cdots \times \text{Res}_{F/F^+} \text{GL}(F e_r)$.

Put $K := K_{2r}$, which is a hyperspecial maximal subgroup of $G(F^+)$. Let I be the subgroup of K of elements whose reduction modulo \mathfrak{p} stabilizes the flag $\kappa e_{-r} \subseteq \cdots \subseteq \kappa e_{-r} \oplus \cdots \oplus \kappa e_{-1}$, which is an Iwahori subgroup of $G(F^+)$. Let J be the subgroup of K of elements whose reduction modulo \mathfrak{p} stabilizes the subspace $\kappa e_{-r} \oplus \cdots \oplus \kappa e_{-1}$, which is a parahoric subgroup of $G(F^+)$. We clearly have $I \subseteq J \subseteq K$. Now we realize the Weyl group $W_{2r} \simeq \{\pm 1\}^r \rtimes \mathfrak{S}_r$ explicitly as a subgroup of K . For $1 \leq i \leq r$, we let i -th -1 in W_{2r} correspond to the element that only switches e_{-i} and e_i , denoted by w_i . For every $\sigma \in \mathfrak{S}_r$, we let $(1^r, \sigma) \in W_{2r}$ correspond to the element that sends $e_{\pm i}$ to $e_{\pm \sigma(i)}$, denoted by $w'_\sigma \in J$. Then $\{w_1, w'_{(1,2)}, \dots, w'_{(r-1,r)}\}$ is a set of distinguished generators of W_{2r} . We recall the Bruhat decompositions

$$K = \coprod_{w \in W_{2r}} I w I, \quad K = \prod_{i=0}^r J w_1 \cdots w_i J.$$

For $w \in W$, we let $0 \leq i(w) \leq r$ be the unique integer such that $w \in J w_1 \cdots w_{i(w)} J$.

By Proposition C.2.1(2), we have a K -equivariant embedding $\Omega_{2r} \hookrightarrow \mathbb{C}[I \backslash K]$, unique up to scalar, hence obtain a distinguished subspace $\Omega_{2r}^I \subseteq \mathbb{C}[I \backslash K / I]$ of dimension one. We would like to find a generator of Ω_{2r}^I . Now we compute the character of the $\mathbb{C}[I \backslash K / I]$ -module Ω_{2r}^I . By Proposition C.2.1(1), the element $\mathbb{1}_{I w_1 I}$ acts on Ω_{2r}^I by -1 ; and by Proposition C.2.1(2), which implies that Ω_{2r}^I is contained in $\mathbb{C}[J \backslash K / J]$, the element $\mathbb{1}_{I w'_{(i,i+1)} I}$ acts on Ω_{2r}^I by $[I w'_{(i,i+1)} I : I] = q^2$ for every $1 \leq i \leq r - 1$. It follows that Ω_{2r}^I is spanned by the following function:

$$f := \sum_{w \in W} (-1)^{i(w)} q^{r-i(w)} \cdot \mathbb{1}_{I w I} \in \mathbb{C}[I \backslash K / I].$$

Take an irreducible admissible representation π of $U_{2r}(F^+)$ such that $\pi|_K$ contains Ω_{2r} . Then it is a constituent of an unramified principal series. In other words, there exists a unique multi-subset $\{s_1, \dots, s_r\} \subseteq \mathbb{C}/(\log q)^{-1} \pi i$ with $\text{Re } s_i \geq 0$ such that π is a constituent of

$$I_{P_{\min}}^G \left(| \cdot |_F^{-s_1} \boxtimes \cdots \boxtimes | \cdot |_F^{-s_r} \right).$$

We recall the projection map

$$\mathcal{P}: \mathbb{C}[I \backslash K / I] \rightarrow I_{P_{\min}}^G \left(| \cdot |_F^{-s_1} \boxtimes \cdots \boxtimes | \cdot |_F^{-s_r} \right)^I$$

defined at the beginning of [Cas80, Section 2], which is $\mathbb{C}[I \backslash K / I]$ -equivariant. Put $\phi := \mathcal{P} f$.

Now we separate the discussion.

Suppose that we are in the situation of (1). We may assume $0 \leq \text{Re } s_1 \leq \cdots \leq \text{Re } s_r$. Then there exist a unique nonnegative integer r_0 and unique positive integers r_1, \dots, r_t satisfying $r_0 + \cdots + r_t = r$, such that

$$0 = \text{Re } s_1 = \cdots = \text{Re } s_{r_0} < \text{Re } s_{r_0+1} = \cdots = \text{Re } s_{r_0+r_1} < \cdots < \text{Re } s_{r_0+\cdots+r_{t-1}+1} = \cdots = \text{Re } s_r$$

holds. For every $1 \leq i \leq t$, put

$$\tau_i := I^{\text{GL}_{r_i}} \left(| \cdot |_F^{-s_{r_0+\cdots+r_{i-1}+1}} \boxtimes \cdots \boxtimes | \cdot |_F^{-s_{r_0+\cdots+r_i}} \right) \otimes |\det_{r_i}|_F^{\text{Re } s_{r_0+\cdots+r_i}}$$

which is an irreducible tempered representation of $\mathrm{GL}_{r_i}(F)$. Put $G_0 := \mathrm{U}(\mathrm{V}_{2r_0})$ and $P_{0\min} := G_0 \cap P_{\min}$. As $|\cdot|_F^{-s_1} \boxtimes \cdots \boxtimes |\cdot|_F^{-s_{r_0}}$ is a discrete series representation of $P_{0\min}(F^+)$, the parabolic induction

$$\tau_0 := \mathrm{I}_{P_{0\min}}^{G_0} \left(|\cdot|_F^{-s_1} \boxtimes \cdots \boxtimes |\cdot|_F^{-s_{r_0}} \right)$$

is a finite direct sum of irreducible tempered representations of $G_0(F^+)$. As $\{s_1, \dots, s_{r_0}\}$ does not contain $(2 \log q)^{-1} \pi i$, τ_0 is actually irreducible by [Gol95, Theorem 1.4 & Theorem 3.4]. In particular, we obtain a Langlands quotient

$$\mathrm{J}_P^G \left(\tau_0 \boxtimes \left(\boxtimes_{i=1}^t \tau_i |\det_{r_i}|_F^{-\mathrm{Re} s_{r_0+\cdots+r_i}} \right) \right),$$

where P is the parabolic subgroup of G containing P_0 whose Levi quotient is isomorphic to $G_0 \times \mathrm{Res}_{F/F^+} \mathrm{GL}_{r_1} \times \cdots \times \mathrm{Res}_{F/F^+} \mathrm{GL}_{r_t}$. We claim

$$(C.3) \quad \phi \neq 0 \in \mathrm{J}_P^G \left(\tau_0 \boxtimes \left(\boxtimes_{i=1}^t \tau_i |\det_{r_i}|_F^{-\mathrm{Re} s_{r_0+\cdots+r_i}} \right) \right).$$

Assuming this claim, then π is isomorphic to $\mathrm{J}_P^G \left(\tau_0 \boxtimes \left(\boxtimes_{i=1}^t \tau_i |\det_{r_i}|_F^{-\mathrm{Re} s_{r_0+\cdots+r_i}} \right) \right)$, the unique irreducible quotient of $\mathrm{I}_{P_{\min}}^G \left(|\cdot|_F^{-s_1} \boxtimes \cdots \boxtimes |\cdot|_F^{-s_r} \right)$, which is unramified. Thus, (1) follows.

Now we prove (C.3). Let $w \in \mathrm{W}_{2r}$ be the element acting trivially on V_{2r_0} and switching $e_{-(r_0+\cdots+r_{i-1}+j)}$ with $e_{r_0+\cdots+r_i+1-j}$ for every $1 \leq j \leq r_i$ and then every $1 \leq i \leq t$. By [Kon03, Corollary 3.2], (C.3) is equivalent to

$$(C.4) \quad \int_{N(F^+)} \phi(w_n) dn \neq 0$$

in which N is the unipotent radical of P and the integral is absolutely convergent. By [Cas80, Theorem 3.4], we have for $r_0 + 1 \leq i \leq r$ that

$$T_{w_i}(\phi) = \frac{q^{1-2s_i} - 1}{q(1 - q^{-2s_i})} \cdot \phi,$$

which is nonzero as $\mathrm{Re} s_i > 0$ and $s_i \neq \frac{1}{2}$. Here, the operator T_w is defined at the beginning of [Cas80, Section 3]. From this we obtain (C.4) hence (C.3).

Suppose that we are in the situation of (2). Then we may assume $s_1 = \frac{1}{2}$ and $s_i \neq \frac{1}{2}$ for $2 \leq i \leq r$. Let Q be the parabolic subgroup of G stabilizing the flag $F e_{-r} \subseteq \cdots \subseteq F e_{-r} \oplus \cdots \oplus F e_{-2}$, whose Levi quotient is $\mathrm{U}(\mathrm{V}_2) \times \mathrm{Res}_{F/F^+} \mathrm{GL}(F e_2) \times \cdots \times \mathrm{Res}_{F/F^+} \mathrm{GL}(F e_r)$. Then we have a canonical inclusion

$$\mathrm{I}_Q^G \left(\mathrm{Sp}_2 \boxtimes |\cdot|_F^{-s_2} \boxtimes \cdots \boxtimes |\cdot|_F^{-s_r} \right) \subseteq \mathrm{I}_{P_{\min}}^G \left(|\cdot|_F^{-1/2} \boxtimes |\cdot|_F^{-s_2} \boxtimes \cdots \boxtimes |\cdot|_F^{-s_r} \right)$$

where Sp_2 denotes the Steinberg representation of $\mathrm{U}(\mathrm{V}_2)(F^+)$. As $\mathbb{1}_{I_{w_1 I}}$ acts by -1 on ϕ , we have

$$\phi \in \mathrm{I}_Q^G \left(\mathrm{Sp}_2 \boxtimes |\cdot|_F^{-s_2} \boxtimes \cdots \boxtimes |\cdot|_F^{-s_r} \right).$$

In particular, it follows that π is a constituent of $\mathrm{I}_Q^G \left(\mathrm{Sp}_2 \boxtimes |\cdot|_F^{-s_2} \boxtimes \cdots \boxtimes |\cdot|_F^{-s_r} \right)$.

To proceed, we may assume $0 \leq \mathrm{Re} s_2 \leq \cdots \leq \mathrm{Re} s_r$. Then there exist unique positive integers r_0, \dots, r_t satisfying $r_0 + \cdots + r_t = r$, such that

$$0 = \mathrm{Re} s_2 = \cdots = \mathrm{Re} s_{r_0} < \mathrm{Re} s_{r_0+1} = \cdots = \mathrm{Re} s_{r_0+r_1} < \cdots < \mathrm{Re} s_{r_0+\cdots+r_{t-1}+1} = \cdots = \mathrm{Re} s_r$$

holds. For every $1 \leq i \leq t$, put

$$\tau_i := \mathrm{I}^{\mathrm{GL}_{r_i}} \left(|\cdot|_F^{-s_{r_0+\cdots+r_{i-1}+1}} \boxtimes \cdots \boxtimes |\cdot|_F^{-s_{r_0+\cdots+r_i}} \right) \otimes |\det_{r_i}|_F^{\mathrm{Re} s_{r_0+\cdots+r_i}}$$

which is an irreducible tempered representation of $\mathrm{GL}_{r_i}(F)$. Put $G_0 := \mathrm{U}(\mathrm{V}_{2r_0})$ and $Q_0 := G_0 \cap Q$. As $\mathrm{Sp}_2 \boxtimes |\cdot|_F^{-s_2} \boxtimes \cdots \boxtimes |\cdot|_F^{-s_{r_0}}$ is a discrete series representation of $Q_0(F^+)$, the parabolic induction

$$\mathrm{I}_{Q_0}^{G_0} \left(\mathrm{Sp}_2 \boxtimes |\cdot|_F^{-s_2} \boxtimes \cdots \boxtimes |\cdot|_F^{-s_{r_0}} \right)$$

is a finite direct sum of irreducible tempered representations of $G_0(F^+)$. Let τ_0 be the unique direct summand such that ϕ is contained in the subspace

$$I_P^G \left(\tau_0 \boxtimes \left(\boxtimes_{i=1}^t \tau_i | \det_{r_i} |_F^{-\operatorname{Re} s_{r_0 + \dots + r_i}} \right) \right) \subseteq I_Q^G \left(\operatorname{Sp}_2 \boxtimes | \cdot |_F^{-s_2} \boxtimes \dots \boxtimes | \cdot |_F^{-s_r} \right),$$

where P is the parabolic subgroup of G containing P_0 whose Levi quotient is isomorphic to $G_0 \times \operatorname{Res}_{F/F^+} \operatorname{GL}_{r_1} \times \dots \times \operatorname{Res}_{F/F^+} \operatorname{GL}_{r_t}$. In particular, we obtain a Langlands quotient

$$J_P^G \left(\tau_0 \boxtimes \left(\boxtimes_{i=1}^t \tau_i | \det_{r_i} |_F^{-\operatorname{Re} s_{r_0 + \dots + r_i}} \right) \right).$$

By the same proof of (C.3), we obtain

$$\phi \neq 0 \in J_P^G \left(\tau_0 \boxtimes \left(\boxtimes_{i=1}^t \tau_i | \det_{r_i} |_F^{-\operatorname{Re} s_{r_0 + \dots + r_i}} \right) \right).$$

Then $\operatorname{BC}(\pi)$ is isomorphic to the unique irreducible quotient of

$$I^{\operatorname{GL}_{2r}} \left(\left(\boxtimes_{i=1}^t \tau_i^{\vee c} | \det_{r_i} |_F^{\operatorname{Re} s_{r_0 + \dots + r_i}} \right) \boxtimes \operatorname{BC}(\tau_0) \boxtimes \left(\boxtimes_{i=1}^t \tau_i | \det_{r_i} |_F^{-\operatorname{Re} s_{r_0 + \dots + r_i}} \right) \right).$$

However, $\operatorname{BC}(\tau_0)$ is isomorphic to

$$\begin{aligned} & I^{\operatorname{GL}_{2r_0}} \left(| \cdot |_F^{s_{r_0}} \boxtimes \dots \boxtimes | \cdot |_F^{s_2} \boxtimes \operatorname{BC}(\operatorname{Sp}_2) \boxtimes | \cdot |_F^{-s_2} \boxtimes \dots \boxtimes | \cdot |_F^{-s_{r_0}} \right) \\ & \simeq I^{\operatorname{GL}_{2r_0}} \left(| \cdot |_F^{s_{r_0}} \boxtimes \dots \boxtimes | \cdot |_F^{s_2} \boxtimes \operatorname{St}_2 \boxtimes | \cdot |_F^{-s_2} \boxtimes \dots \boxtimes | \cdot |_F^{-s_{r_0}} \right) \end{aligned}$$

which is irreducible. Thus, (2) follows.

The lemma is proved. \square

Lemma C.2.4. *For every irreducible admissible representation π' of $U'_{2r}(F^+)$ such that $(\pi')^{K'_{2r}} \neq \{0\}$, there exists a unique multi-subset $\{s_2, \dots, s_r\} \subseteq \mathbb{C}/(\log q)^{-1}\pi i$ with $\operatorname{Re} s_i \geq 0$ such that if we arrange them so that $0 \leq \operatorname{Re} s_2 \leq \dots \leq \operatorname{Re} s_r$ holds, then $\operatorname{BC}(\pi')$ is isomorphic to the unique irreducible quotient of*

$$I^{\operatorname{GL}_{2r}} \left(| \cdot |_F^{s_r} \boxtimes \dots \boxtimes | \cdot |_F^{s_2} \boxtimes \operatorname{St}_2 \boxtimes | \cdot |_F^{-s_2} \boxtimes \dots \boxtimes | \cdot |_F^{-s_r} \right).$$

Proof. We fix a decomposition

$$\Lambda'_{2r} = O_F e_{-r} \oplus \dots \oplus O_F e_{-2} \oplus \Lambda'_2 \oplus O_F e_2 \oplus \dots \oplus O_F e_r$$

in which $(e_{-i}, e_j) = \delta_{ij}$ for $2 \leq i, j \leq r$. For $1 \leq i \leq r$, put

$$V'_{2i} := F e_{-i} \oplus \dots \oplus F e_{-2} \oplus \Lambda'_2 \otimes_{O_F} F \oplus F e_2 \oplus \dots \oplus F e_i$$

which is a hermitian subspace of V'_{2r} . We take the minimal parabolic subgroup P_{\min} of $G := U'_{2r}$ to be the stabilizer of the flag $F e_{-r} \subseteq \dots \subseteq F e_{-r} \oplus \dots \oplus F e_{-2}$. We also fix a Levi subgroup of P_{\min} to be $U(V'_2) \times \operatorname{Res}_{F/F^+} \operatorname{GL}(F e_2) \times \dots \times \operatorname{Res}_{F/F^+} \operatorname{GL}(F e_r)$.

Take an irreducible admissible representation π' of $G(F^+)$ such that $(\pi')^{K'_{2r}} \neq 0$. Then it is a constituent of an unramified principal series. In other words, there exists a unique multi-subset $\{s_2, \dots, s_r\} \subseteq \mathbb{C}/(\log q)^{-1}\pi i$ with $\operatorname{Re} s_i \geq 0$ such that π' is a constituent of

$$I_{P_{\min}}^G \left(\mathbf{1}'_2 \boxtimes | \cdot |_F^{-s_2} \boxtimes \dots \boxtimes | \cdot |_F^{-s_r} \right),$$

where $\mathbf{1}'_2$ denotes the trivial representation of $U(V'_2)(F^+)$.

To proceed, we may assume $0 \leq \operatorname{Re} s_2 \leq \dots \leq \operatorname{Re} s_r$. Then there exist unique positive integers r_0, \dots, r_t satisfying $r_0 + \dots + r_t = r$, such that

$$0 = \operatorname{Re} s_2 = \dots = \operatorname{Re} s_{r_0} < \operatorname{Re} s_{r_0+1} = \dots = \operatorname{Re} s_{r_0+r_1} < \dots < \operatorname{Re} s_{r_0+\dots+r_{t-1}+1} = \dots = \operatorname{Re} s_r$$

holds. For every $1 \leq i \leq t$, put

$$\tau_i := I^{\operatorname{GL}_{r_i}} \left(| \cdot |_F^{-s_{r_0+\dots+r_{i-1}+1}} \boxtimes \dots \boxtimes | \cdot |_F^{-s_{r_0+\dots+r_i}} \right) \otimes | \det_{r_i} |_F^{\operatorname{Re} s_{r_0+\dots+r_i}}$$

which is an irreducible tempered representation of $\mathrm{GL}_{r_i}(F)$. Put $G_0 := \mathrm{U}(V'_{2r_0})$ and $P_{0\min} := G_0 \cap P_{\min}$. As $\mathbf{1}'_2 \boxtimes |_{F^{-s_2}} \boxtimes \cdots \boxtimes |_{F^{-s_{r_0}}}$ is a discrete series representation of $P_{0\min}(F^+)$, the parabolic induction

$$\mathrm{I}_{P_{0\min}}^{G_0} \left(\mathbf{1}'_2 \boxtimes |_{F^{-s_2}} \boxtimes \cdots \boxtimes |_{F^{-s_{r_0}}} \right)$$

is a finite direct sum of irreducible tempered representations of $G_0(F^+)$. Let τ_0 be the unique direct summand with nonzero invariants under $\mathrm{K}'_{2r} \cap G_0(F^+)$. In particular, we obtain a Langlands quotient

$$\mathrm{J}_P^G \left(\tau_0 \boxtimes \left(\boxtimes_{i=1}^t \tau_i |_{\det_{r_i} F}^{-\mathrm{Re} s_{r_0 + \cdots + r_i}} \right) \right)$$

where P is the parabolic subgroup of G containing P_0 whose Levi quotient is isomorphic to $G_0 \times \mathrm{Res}_{F/F^+} \mathrm{GL}_{r_1} \times \cdots \times \mathrm{Res}_{F/F^+} \mathrm{GL}_{r_t}$. We claim

$$(C.5) \quad \mathrm{J}_P^G \left(\tau_0 \boxtimes \left(\boxtimes_{i=1}^t \tau_i |_{\det_{r_i} F}^{-\mathrm{Re} s_{r_0 + \cdots + r_i}} \right) \right)^{\mathrm{K}'_{2r}} \neq \{0\}.$$

Assuming this claim, then $\mathrm{BC}(\pi')$ is isomorphic to the unique irreducible quotient of

$$\mathrm{I}^{\mathrm{GL}_{2r}} \left(\left(\boxtimes_{i=1}^t \tau_i^{\vee c} |_{\det_{r_i} F}^{\mathrm{Re} s_{r_0 + \cdots + r_i}} \right) \boxtimes \mathrm{BC}(\tau_0) \boxtimes \left(\boxtimes_{i=1}^t \tau_i |_{\det_{r_i} F}^{-\mathrm{Re} s_{r_0 + \cdots + r_i}} \right) \right).$$

However, $\mathrm{BC}(\tau_0)$ is isomorphic to

$$\begin{aligned} & \mathrm{I}^{\mathrm{GL}_{2r_0}} \left(|_{F^{s_{r_0}}} \boxtimes \cdots \boxtimes |_{F^{s_2}} \boxtimes \mathrm{BC}(\mathbf{1}'_2) \boxtimes |_{F^{-s_2}} \boxtimes \cdots \boxtimes |_{F^{-s_{r_0}}} \right) \\ & \simeq \mathrm{I}^{\mathrm{GL}_{2r_0}} \left(|_{F^{s_{r_0}}} \boxtimes \cdots \boxtimes |_{F^{s_2}} \boxtimes \mathrm{St}_2 \boxtimes |_{F^{-s_2}} \boxtimes \cdots \boxtimes |_{F^{-s_{r_0}}} \right) \end{aligned}$$

which is irreducible. The lemma follows.

Now we prove (C.5). Let $w \in G(F^+)$ be the element acting trivially on V'_{2r_0} and switching $e_{-(r_0 + \cdots + r_{i-1} + j)}$ with $e_{r_0 + \cdots + r_i + 1 - j}$ for every $1 \leq j \leq r_i$ and then every $1 \leq i \leq t$. Note that we have a canonical inclusion

$$\mathrm{I}_P^G \left(\tau_0 \boxtimes \left(\boxtimes_{i=1}^t \tau_i |_{\det_{r_i} F}^{-\mathrm{Re} s_{r_0 + \cdots + r_i}} \right) \right) \subseteq \mathrm{I}_{P_{\min}}^G \left(\mathbf{1}'_2 \boxtimes |_{F^{-s_2}} \boxtimes \cdots \boxtimes |_{F^{-s_r}} \right)$$

which is $G(F^+)$ -equivariant. Let ϕ be the unique element in the latter space, realized as functions on $G(F^+)$, that takes value 1 on K'_{2r} , which belongs to the former space by our choice of τ_0 . By [Kon03, Corollary 3.2], (C.5) is equivalent to

$$(C.6) \quad \int_{N(F^+)} \phi(wn) dn \neq 0$$

in which N is the unipotent radical of P and the integral is absolutely convergent. By [Cas80, Theorem 3.1], we have for $r_0 + 1 \leq i \leq r$ that

$$T_{w_i}(\phi) = \frac{1 - q^{-1-2s_i}}{1 - q^{-2s_i}} \cdot \phi$$

which is nonzero as $\mathrm{Re} s_i > 0$. Here, the operator T_w is defined at the beginning of [Cas80, Section 3]. From this we obtain (C.6) hence (C.5). \square

Remark C.2.5. In fact, for $\pi \in \mathcal{S}$ and $\pi' \in \mathcal{S}'$ in Proposition C.2.2 that correspond to each other, they also correspond under the local theta correspondence with respect to the unramified additive character and the trivial splitting character.

C.3. Results from endoscopic classification. Let F/F^+ be a totally imaginary quadratic extension of a totally real number field in the main text. We state the following proposition, which summarises all we need from Arthur's endoscopic classification for unitary groups in this article. In particular, we will use the notion of local base change for unitary groups defined over F_v^+ for every place v of F^+ , denoted by BC as well, for which we have discussed some special cases when v is inert in F in Subsection C.1.

Proposition C.3.1. *Take a relevant representation (Definition 1.1.3) Π of $\mathrm{GL}_N(\mathbb{A}_F)$. Let V be a standard definite or indefinite hermitian space over F of rank N and $\pi = \otimes_v \pi_v$ an irreducible admissible representation of $\mathrm{U}(V)(\mathbb{A}_{F^+})$. We have*

- (1) *If $\mathrm{BC}(\pi_v) \simeq \Pi_v$ for every place v of F^+ , then the discrete automorphic multiplicity of π is 1. In particular, (V, π) is a relevant pair (Definition 3.2.7).*
- (2) *If (V, π) is a relevant pair such that $\mathrm{BC}(\pi) \simeq \Pi$,²⁶ then we have $\mathrm{BC}(\pi_v) \simeq \Pi_v$ for every place v of F^+ . In particular, the discrete automorphic multiplicity of π is 1 by (1).*
- (3) *If v is archimedean but not $\underline{\tau}_\infty$, then $\mathrm{BC}(\pi_v) \simeq \Pi_v$ if and only if π_v is the trivial representation.*
- (4) *If $v = \underline{\tau}_\infty$, then $\mathrm{BC}(\pi_v) \simeq \Pi_v$ if and only if π_v is the trivial representation (resp. is one of the N discrete series representations with the Harish-Chandra parameter $\{\frac{1-N}{2}, \frac{3-N}{2}, \dots, \frac{N-3}{2}, \frac{N-1}{2}\}$) when V is definite (resp. indefinite).*

Proof. Parts (1) and (2) are consequences of [KMSW, Theorem 1.7.1] for generic packets. Parts (3) and (4) follow from (1), (2), and the definition of relevant representations. \square

The above proposition has the following two immediate corollaries as two examples of the global Jacquet–Langlands correspondence.

Corollary C.3.2. *Take a finite place \mathfrak{p} of F^+ inert in F . Let V and V' be a standard definite and a standard indefinite hermitian space over F , respectively, of even rank $N = 2r$, satisfying $V_v \simeq V'_v$ (for which we fix) for every place v of F^+ other than $\underline{\tau}_\infty$ and \mathfrak{p} . Let π be a stable automorphic representation of $\mathrm{U}(V)(\mathbb{A}_{F^+})$ (Definition 3.2.3) such that π_∞ is trivial and $\pi_{\mathfrak{p}}$ belongs to the set \mathcal{S} in Proposition C.2.2 (in particular, $V \otimes_{F^+} F_{\mathfrak{p}}^+$ admits a self-dual lattice). Consider the representation $\pi' := \pi'_{\underline{\tau}_\infty} \otimes \pi'_{\mathfrak{p}} \otimes \pi'^{\underline{\tau}_\infty, \mathfrak{p}}$ of $\mathrm{U}(V')(\mathbb{A}_{F^+})$ where*

- \circ $\pi'_{\underline{\tau}_\infty}$ *is a discrete series representation of $\mathrm{U}(V')(F_{\underline{\tau}_\infty}^+)$ with the Harish-Chandra parameter $\{r - \frac{1}{2}, r - \frac{3}{2}, \dots, \frac{3}{2} - r, \frac{1}{2} - r\}$; and*
- \circ $\pi'_{\mathfrak{p}} \in \mathcal{S}'$ *is the representation of $\mathrm{U}(V')(F_{\mathfrak{p}}^+)$ corresponding to $\pi_{\mathfrak{p}}$ as in Proposition C.2.2.*

Then the discrete automorphic multiplicity of π' is 1.

Proof. Put $\Pi := \mathrm{BC}(\pi)$. By Proposition C.3.1 and Proposition C.2.2, we have $\mathrm{BC}(\pi'_v) \simeq \Pi_v$ for every place v of F^+ . The corollary follows by Proposition C.3.1(1). \square

Corollary C.3.3. *Take a finite place \mathfrak{p} of F^+ inert in F . Let V and V' be a standard definite and a standard indefinite hermitian space over F , respectively, of odd rank $N = 2r + 1$, satisfying $V_v \simeq V'_v$ (for which we fix) for every place v of F^+ other than $\underline{\tau}_\infty$ and \mathfrak{p} . Let π' be a stable automorphic representation of $\mathrm{U}(V')(\mathbb{A}_{F^+})$ such that $\pi'_{\underline{\tau}_\infty}$ is a discrete series representation of $\mathrm{U}(V')(F_{\underline{\tau}_\infty}^+)$ (Definition 3.2.3) with the Harish-Chandra parameter $\{r, r - 1, \dots, 1 - r, -r\}$; $\pi'_{\underline{\tau}}$ is trivial for every archimedean place $\underline{\tau} \neq \underline{\tau}_\infty$; and $\pi'_{\mathfrak{p}}$ is unramified. Consider the representation $\pi := \pi_{\underline{\tau}_\infty} \otimes \pi_{\mathfrak{p}} \otimes (\pi')^{\underline{\tau}_\infty, \mathfrak{p}}$ of $\mathrm{U}(V)(\mathbb{A}_{F^+})$ where*

- \circ $\pi_{\underline{\tau}_\infty}$ *is trivial; and*
- \circ $\pi_{\mathfrak{p}}$ *is unramified satisfying $\mathrm{BC}(\pi_{\mathfrak{p}}) \simeq \mathrm{BC}(\pi'_{\mathfrak{p}})$.*

²⁶Note that $\mathrm{BC}(\pi)$ exists by Proposition 3.2.8.

Then the discrete automorphic multiplicity of π is 1.

Proof. Put $\Pi' := \text{BC}(\pi')$. By Proposition C.3.1 and Proposition C.2.2, we have $\text{BC}(\pi_v) \simeq \Pi'_v$ for every place v of F^+ . The corollary follows by Proposition C.3.1(1). \square

APPENDIX D. SOME TRACE FORMULAE ARGUMENT

This appendix has two goals. In Subsection D.1, we remove some conditions in a theorem of Caraiani and Scholze [CS17]. In Subsection D.2, we prove a formula computing the dimension of old forms in an L -packet for unitary groups. These two subsections are independent on a logical level; we collect them together in one appendix mainly because the argument we use are similar, namely, trace formulae.

We keep the setup in the main text.

D.1. Vanishing of cohomology off middle degree.

Definition D.1.1. Let $N \geq 1$ be an integer, and Σ^+ a finite set of nonarchimedean places of F^+ containing Σ_{ram}^+ . Consider a homomorphism $\phi: \mathbb{T}_N^{\Sigma^+} \rightarrow \kappa$ with κ a field. We say that ϕ is *cohomologically generic* if

$$H_{\text{ét}}^i(\text{Sh}(V, K), \kappa)_{\mathbb{T}_N^{\Sigma^+} \cap \ker \phi} = 0$$

holds for

- every finite set $\Sigma^{+'}$ of nonarchimedean places of F^+ containing Σ^+ ,
- every integer $i \neq N - 1$, and
- every standard indefinite hermitian space V over F and every object $K \in \mathfrak{K}(V)$ of the form $K_{\Sigma^{+'}} \times \prod_{v \notin \Sigma_{\infty}^+ \cup \Sigma^{+'}} \text{U}(\Lambda)(O_{F_v^+})$ for a self-dual $\prod_{v \notin \Sigma_{\infty}^+ \cup \Sigma^{+'}} O_{F_v}$ -lattice Λ in $V \otimes_F \mathbb{A}_F^{\Sigma^+ \cup \Sigma^{+'}}$.

The following definition is essentially [CS17, Definition 1.9].

Definition D.1.2. Let $\phi: \mathbb{T}_N^{\Sigma^+} \rightarrow \kappa$ be a homomorphism with κ a field. For a place w of F^+ not in Σ^+ that splits in F , we say that ϕ is *decomposed generic at w* if $\phi(H_w) \in \kappa[T]$ has distinct (nonzero) roots in which there is no pair with ratio equal to $\|w\|$.²⁷ Here, $H_w \in \mathbb{T}_{N,w}[\mathbb{T}]$ is the Hecke polynomial.

Theorem D.1.3. Let $N \geq 1$ be an integer, and Σ^+ a finite set of nonarchimedean places of F^+ containing Σ_{ram}^+ . Let V be a standard indefinite hermitian space over F . Let $\phi: \mathbb{T}_N^{\Sigma^+} \rightarrow \overline{\mathbb{F}}_{\ell}$ be a homomorphism. Suppose $[F^+ : \mathbb{Q}] > 1$. Suppose that there exists a place w of F^+ not in $\Sigma^+ \cup \Sigma_{\ell}^+$ that splits in F , such that ϕ is decomposed generic at w . Then we have

$$H_{\text{ét}}^i(\text{Sh}(V, K), \overline{\mathbb{F}}_{\ell})_{\ker \phi} = 0$$

for every integer $i \neq N - 1$, and every object $K \in \mathfrak{K}(V)$ of the form $K_{\Sigma^+} \times \prod_{v \notin \Sigma_{\infty}^+ \cup \Sigma^+} \text{U}(\Lambda)(O_{F_v^+})$ for a self-dual $\prod_{v \notin \Sigma_{\infty}^+ \cup \Sigma^+} O_{F_v}$ -lattice Λ in $V \otimes_F \mathbb{A}_F^{\Sigma^+ \cup \Sigma^+}$.

Proof. When F contains an imaginary quadratic field and every place in Σ^+ splits in F (which implies $[F^+ : \mathbb{Q}] > 1$), the statement of the theorem can be deduced from the analogous statement for the unitary similitude group, namely Case 2 of [CS17, Theorem 6.3.1(2)]. In this subsection, we will explain how to remove these restrictions.

In the statement of the theorem, let w_0 be the underlying rational prime of w . We fix an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_{w_0}$ that induces the place w of F . Put $G := \text{Res}_{F^+/\mathbb{Q}} \text{U}(V)$. We have the Deligne homomorphism $h: \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{G}_m \rightarrow G \otimes_{\mathbb{Q}} \mathbb{R}$ as in Section 3.2. Put $K_{w_0,0} := \prod_{v|w_0} \text{U}(\Lambda)(O_{F_v^+})$, which is a hyperspecial maximal subgroup of $G(\mathbb{Q}_{w_0})$. We fix a character $\varpi: F^{\times} \backslash \mathbb{A}_F^{\times} \rightarrow \mathbb{C}^{\times}$ that

²⁷In fact, as pointed out in [CS, Remark 1.4], there is no need to assume that the roots are distinct.

is unramified outside Σ^+ such that $\varpi|_{\mathbb{A}_{F^+}^\times}$ is the quadratic character η_{F/F^+} associated to F/F^+ . Put $\Sigma := \{p \mid \Sigma_p^+ \cap \Sigma^+ \neq \emptyset\}$.

We define a subtorus $T \subseteq \text{Res}_{F/\mathbb{Q}} \mathbf{G}_m$ such that for every commutative \mathbb{Q} -algebra R ,

$$T(R) = \{a \in F \otimes_{\mathbb{Q}} R \mid \text{Nm}_{F/F^+} a \in R^\times\}.$$

We fix a CM type Φ containing τ_∞ , and a sufficiently small open compact subgroup $K_T \subseteq T(\mathbb{A}^\infty)$ such that $(K_T)_p$ is maximal for every $p \notin \Sigma$. Then Φ induces a Deligne homomorphism $h_\Phi: \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{G}_m \rightarrow T \otimes_{\mathbb{Q}} \mathbb{R}$. We also put $\mathfrak{T} := T(\mathbb{A}^{\infty, w_0})/T(\mathbb{Z}_{(w_0)})K_T^{w_0}$ similar to Definition 3.5.5.

Put $\tilde{G} := G \times T$ and $\tilde{h} := h \times h_\Phi$. Then we have the Shimura datum (\tilde{G}, \tilde{h}) , which is of Hodge type. Its reflex field is the composition $F.F_\Phi \subseteq \mathbb{C}$. Therefore, for every sufficiently small open compact subgroup $K \subseteq G(\mathbb{A}^\infty)$, we have the Shimura variety $\text{Sh}(\tilde{G}, \tilde{h})_{K \times K_T}$, which is smooth projective (as $[F^+ : \mathbb{Q}] > 1$) over $F.F_\Phi$ of dimension $N-1$. When K is of the form $K^{w_0}K_{w_0,0}$, it has a canonical smooth projective model $\mathcal{S}(\tilde{G}, \tilde{h})_{K^{w_0}}$ over $W(\overline{\mathbb{F}}_{w_0})$ which admits a moduli interpretation similar to the one introduced in Section 4.1. Note that $F.F_\Phi$ is contained in $W(\overline{\mathbb{F}}_{w_0})_{\mathbb{Q}}$ under the isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_{w_0}$.

The discussion in [CS17], except in Section 5, is valid for all proper Shimura varieties of Hodge type including the above one. Thus, we need to modify the argument in [CS17, Section 5] for our case.

Let μ and $\tilde{\mu}$ be the Hodge cocharacters corresponding to h and \tilde{h} , respectively. We have the natural projection map $B(\tilde{G}, \tilde{\mu}) \rightarrow B(G, \mu)$ of Kottwitz sets, which is a bijection. For every $b \in B(G, \mu)$, we have the corresponding Kottwitz groups \tilde{J}_b and J_b , with a canonical isomorphism $\tilde{J}_b \simeq J_b \times T$. For every (sufficiently small) open compact subgroup $K^{w_0} \subseteq G(\mathbb{A}^{\infty, w_0})$ and positive integer m , we have the Igusa variety $\mathcal{S}_{\text{Mant}, K^{w_0}, m}^b$ for the integral model $\mathcal{S}(\tilde{G}, \tilde{h})_{K^{w_0}}$, which is a \mathfrak{T} -scheme over $\overline{\mathbb{F}}_{w_0}$. Define

$$[\text{H}_{\mathfrak{T}, c}(\mathcal{S}_{\text{Mant}}^b, \overline{\mathbb{Q}}_\ell)] := \bigoplus_i (-1)^i \varinjlim_{K^{w_0}, m} \text{H}_{\mathfrak{T}, c}^i(\mathcal{S}_{\text{Mant}, K^{w_0}, m}^b, \overline{\mathbb{Q}}_\ell),$$

which is virtual representation of $G(\mathbb{A}^{\infty, w_0}) \times J_b(\mathbb{Q}_{w_0})$. The crucial point is that our G is the honest unitary group, rather than the unitary similitude group. Then [CS17, Theorem 5.2.3] is modified as

$$\text{tr}(\phi \mid \text{H}_{\mathfrak{T}, c}(\mathcal{S}_{\text{Mant}}^b, \overline{\mathbb{Q}}_\ell)) = \sum_{(H, s, \eta)} \iota(G, H) \text{ST}_e^H(\phi^H)$$

where the sum is taken over equivalent classes of elliptic endoscopic triples (H, s, η) of G ; and we use the character ϖ for the Langlands–Shelstad transfer. This formula can be proved in the same way as for [Shi10, Theorem 7.2] since our Shimura variety has a similar moduli interpretation as seen in Subsection 4.1, although the Shimura datum (\tilde{G}, \tilde{h}) is not of PEL type in the sense of Kottwitz. We can fix the representatives of the triples (H, s, η) as in [CS17, Page 734] but without the similitude factor. In particular, [CS17, Corollary 5.2.5] is modified as

$$\text{tr}(\phi \mid \text{H}_{\mathfrak{T}, c}(\mathcal{S}_{\text{Mant}}^b, \overline{\mathbb{Q}}_\ell)) = \sum_{G_{\vec{n}}} \iota(G, G_{\vec{n}}) \text{ST}_e^{G_{\vec{n}}}(\phi^{\vec{n}}).$$

The next statement [CS17, Proposition 5.3.1] or rather [Shi11, Corollary 4.7], namely,

$$I_{\text{geom}}^{G_{\vec{n}}\theta}(f^{\vec{n}}\theta) = \tau(G_{\vec{n}})^{-1} \text{ST}_e^{G_{\vec{n}}}(\phi^{\vec{n}})$$

holds as long as $f^{\vec{n}}$ and $\phi^{\vec{n}}$ are associated in the sense of [Lab99, 3.2]. Here, $G_{\vec{n}}$ is the group $\text{Res}_{F/\mathbb{Q}} \text{GL}_{\vec{n}} \rtimes \{1, \theta\}$. Note that, for rational primes in Σ , we do not have explicit local base change transfer. However, we will see shortly that there are enough associated pairs at these primes to make the remaining argument work, following an idea in [Shi].

For the test function $\phi \in C_c^\infty(G(\mathbb{A}^{\infty, w_0}) \times J_b(\mathbb{Q}_{w_0}))$ in [CS17, Theorem 5.3.2], if we assume $\phi = \phi_\Sigma \otimes \phi^\Sigma$ in which ϕ_Σ is the characteristic function of some open compact subgroup $K_\Sigma \subseteq G(\mathbb{Q}_\Sigma)$, then for every $G_{\bar{n}}$, $\phi^{\bar{n}}$ is associated to some function $f^{\bar{n}}$ in the sense above. This is shown in the claim in the proof of [Shi, Proposition 1.4]. In particular, for such ϕ , we have

$$\mathrm{tr} \left(\phi \mid H_{\mathfrak{I}, c}(\mathcal{S}_{\mathrm{Mant}}^b, \overline{\mathbb{Q}}_\ell) \right) = \sum_{G_{\bar{n}}} \iota(G, G_{\bar{n}}) I_{\mathrm{spec}}^{G_{\bar{n}} \theta}(f^{\bar{n}} \theta)$$

in view the above identities and [CS17, (5.3.2)]. The remaining argument toward [CS17, Theorem 5.5.7] is same as it is on the GL-side, for which it suffices to use the above test functions ϕ . In fact, our case is slightly easier as we do not have the similitude factor.

The argument towards Theorem D.1.3 or [CS17, Theorem 6.3.1(2)] only uses [CS17, Theorem 5.5.7]. Therefore, Theorem D.1.3 holds. \square

Corollary D.1.4. *Let the situation be as in Subsection 6.1. Suppose $[F^+ : \mathbb{Q}] > 1$. Then for all but finitely many primes λ of E , the composite homomorphism*

$$(D.1) \quad \mathbb{T}_N^{\Sigma^+} \xrightarrow{\phi_\Pi} O_E \rightarrow O_E/\lambda$$

is cohomologically generic (Definition D.1.1).

Proof. As pointed out in the proof of [CH13, Proposition 3.2.5], we can choose a nonarchimedean place w of F such that Π_w is unramified with distinct Satake parameters. Let $\{\alpha_1, \dots, \alpha_N\}$ be the Satake parameter of Π_w , which are algebraic integers. Since Π_w is generic, we have $\alpha_i/\alpha_j \notin \{1, \|w\|\}$ for $i \neq j$. Thus, for every sufficiently large rational prime ℓ , we have $\alpha_i/\alpha_j \notin \{1, \|w\|\}$ for $i \neq j$ even in $\overline{\mathbb{F}}_\ell$. Let λ be a prime of E above such a rational prime ℓ . Applying the Chebotarev density theorem to any residual Galois representation $\bar{\rho}_{\Pi, \lambda}$ of $\rho_{\Pi, \lambda}$, we conclude that there are infinitely many nonarchimedean places w of F^+ not in $\Sigma^+ \cup \Sigma_\ell^+$ that splits in F , such that (D.1) is decomposed generic at w (Definition D.1.2). Thus, (D.1) is cohomologically generic by Theorem D.1.3. The corollary follows. \square

D.2. Dimension of old forms. Let $N = 2r$ be an even positive integer. We consider

- a relevant representation Π of $\mathrm{GL}_N(\mathbb{A}_F)$,
- two disjoint finite sets Σ_{min}^+ and Σ_{lr}^+ of nonarchimedean places of F^+ such that Σ_{min}^+ contains Σ_{ram}^+ ; $\Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}}^+$ contains Σ_Π^+ (Notation 3.1.4); and every place in Σ_{lr}^+ is inert in F ,
- a finite set Σ^+ of nonarchimedean places of F^+ containing $\Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}}^+$,
- a standard definite or indefinite hermitian space V over F of rank N such that V_v is not split for $v \in \Sigma_{\mathrm{lr}}^+$,
- a self-dual $\prod_{v \notin \Sigma_\infty^+ \cup \Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}}^+} O_{F_v}$ -lattice Λ in $V \otimes_F \mathbb{A}_F^{\Sigma_\infty^+ \cup \Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}}^+}$,
- an object $K \in \mathfrak{K}(V)$ of the form

$$K = \prod_{v \in \Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}}^+} K_v \times \prod_{v \notin \Sigma_\infty^+ \cup \Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}}^+} \mathrm{U}(\Lambda)(O_{F_v^+}),$$

satisfying that K_v is special maximal for $v \in \Sigma_{\mathrm{lr}}^+$.

We have the homomorphism

$$\phi_\Pi : \mathbb{T}_N^{\Sigma^+} \rightarrow \overline{\mathbb{Q}}_\ell$$

given by Π . Fix an isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$.

Definition D.2.1. Let v be a nonarchimedean place of F^+ . We say that an open compact subgroup K_v of $\mathrm{U}(V)(F_v^+)$ is *transferable* if the following two conditions are satisfied.

- (1) For every endoscopic group H of $U(V_v)$, if we let $f_{K_v}^H$ be the endoscopic transfer of $\mathbb{1}_{K_v}$, then there exists a compactly supported smooth function $\phi_{K_v}^H$ on $H(F_v)$ such that $\mathbb{1}_{K_v}$ and $\phi_{K_v}^H$ are associated in the sense of [Lab99, 3.2].
- (2) When H is the quasi-split unitary group of rank N , we can take $\phi_{K_v}^H$ to be supported on $GL_N(O_{F_v})$ once we identify $H(F_v)$ with $GL_N(F_v)$.

We call the function $\phi_{K_v}^H$ in (2) an *inertial transfer* of K_v if K_v is transferable, and will drop the superscript H in practice.

Lemma D.2.2. *Let v be a nonarchimedean place of F^+ .*

- (1) *If v splits in F , then every open compact subgroup K_v is transferable.*
- (2) *If v is not in $\Sigma_\infty^+ \cup \Sigma_{\min}^+ \cup \Sigma_{\text{lr}}^+$, then the hyperspecial maximal subgroup $U(\Lambda)(O_{F_v^+})$ is transferable which admits $\mathbb{1}_{GL_N(O_{F_v})}$ as an inertial transfer.*
- (3) *If v is in $\Sigma_{\min}^+ \cup \Sigma_{\text{lr}}^+$, then every sufficiently small open compact subgroup K_v is transferable.*

Proof. Part (1) is trivial. Part (2) is the combination of the endoscopic fundamental lemma and the base change fundamental lemma.

For (3), for sufficiently small K_v , condition (1) in Definition D.2.1 is proved in [Mor10, Lemma 8.4.1(1)]; and condition (2) can be achieved by [Lab99, Proposition 3.1.7(2)] (see the proof of [Lab99, Proposition 3.3.2]). \square

Proposition D.2.3. *Suppose that K_v is transferable for $v \in \Sigma_{\min}^+$. For every $v \in \Sigma_{\text{lr}}^+$, let c_v be equal to 1 (resp. 0) if one can (resp. cannot) find purely imaginary complex numbers s_2, \dots, s_r such that Π_v is isomorphic to the induction*

$$I^{GL_{2r}} \left(| \cdot |_F^{s_r} \boxtimes \cdots \boxtimes | \cdot |_F^{s_2} \boxtimes \text{St}_2 \boxtimes | \cdot |_F^{-s_2} \boxtimes \cdots \boxtimes | \cdot |_F^{-s_r} \right)$$

(see Subsection C.1 for the notation of induced representations). Then we have the identities

$$\dim \overline{\mathbb{Q}}_\ell[\text{Sh}(V, K)][\iota_\ell \phi_\Pi] = \left| \prod_{v \in \Sigma_{\min}^+} \text{tr}(\Pi_v(\phi_{K_v}) \circ A_{\Pi_v}) \prod_{v \in \Sigma_{\text{lr}}^+} c_v \right|,$$

$$\dim H_{\text{ét}}^{N-1}(\text{Sh}(V, K)_{\overline{F}}, \overline{\mathbb{Q}}_\ell)[\iota_\ell \phi_\Pi] = N \left| \prod_{v \in \Sigma_{\min}^+} \text{tr}(\Pi_v(\phi_{K_v}) \circ A_{\Pi_v}) \prod_{v \in \Sigma_{\text{lr}}^+} c_v \right|,$$

when V is definite and indefinite, respectively, for any inertial transfer ϕ_{K_v} for K_v and any normalized intertwining operator A_{Π_v} for Π_v [Shi11, Section 4.1], for $v \in \Sigma_{\min}^+$.

Proof. We only prove the case where V is indefinite, and leave the case where V is definite (which is slightly easier) to the readers.

By Proposition 3.2.4(1), we know that Π is tempered everywhere. Moreover, every discrete automorphic representation of $U(V)(\mathbb{A}_{F^+})$ whose global base change is isomorphic to Π has to be cuspidal as well. Thus, we have $H_{\text{ét}}^i(\text{Sh}(V, K)_{\overline{F}}, \overline{\mathbb{Q}}_\ell)[\iota_\ell \phi_\Pi] = 0$ for $i \neq N - 1$.

If there exists $v \in \Sigma_{\text{lr}}^+$ such that $c_v = 0$, then by Lemma C.2.4 and the above fact that Π_v is tempered, we have $H_{\text{ét}}^{N-1}(\text{Sh}(V, K)_{\overline{F}}, \overline{\mathbb{Q}}_\ell)[\iota_\ell \phi_\Pi] = 0$. Thus, the proposition follows. In what follows, we assume $c_v = 1$ for every $v \in \Sigma_{\text{lr}}^+$.

By Proposition C.3.1 and Lemma C.2.4, we have

$$\dim H_{\text{ét}}^{N-1}(\text{Sh}(V, K)_{\overline{F}}, \overline{\mathbb{Q}}_\ell)[\iota_\ell \phi_\Pi] = N \prod_{v \in \Sigma_{\min}^+} \sum_{\text{BC}(\pi_v) \simeq \Pi_v} \dim(\pi_v)^{K_v},$$

where the sum is taken over isomorphism classes of irreducible admissible representations π_v of $U(V)(F_v^+)$ such that $BC(\pi_v) \simeq \Pi_v$ (for $v \in \Sigma_{\min}^+$). Thus, our goal is to show

$$(D.2) \quad \prod_{v \in \Sigma_{\min}^+} \sum_{BC(\pi_v) \simeq \Pi_v} \dim(\pi_v)^{K_v} = \left| \prod_{v \in \Sigma_{\min}^+} \operatorname{tr}(\Pi_v(\phi_{K_v}) \circ A_{\Pi_v}) \right|.$$

Now for $v \in \Sigma_{\text{lr}}^+$, we replace K_v by a smaller subgroup (with the same notation) that is transferable by Lemma D.2.2(3).²⁸ Let $K' \in \mathfrak{K}(V)$ be the correspondent product. So we have

$$\dim H_{\text{ét}}^{N-1}(\operatorname{Sh}(V, K')_{\overline{F}}, \overline{\mathbb{Q}}_\ell)[\iota_\ell \phi_\Pi] = N \prod_{v \in \Sigma_{\min}^+ \cup \Sigma_{\text{lr}}^+} \sum_{BC(\pi_v) \simeq \Pi_v} \dim(\pi_v)^{K_v}.$$

On the other hand, by [Shi, (1.8) & (1.9)]²⁹ with $j = 0$ and ξ the trivial representation, we have

$$\dim H_{\text{ét}}^{N-1}(\operatorname{Sh}(V, K)_{\overline{F}}, \overline{\mathbb{Q}}_\ell)[\iota_\ell \phi_\Pi] = N \left| \prod_{v \in \Sigma_{\min}^+ \cup \Sigma_{\text{lr}}^+} \operatorname{tr}(\Pi_v(\phi_{K_v}) \circ A_{\Pi_v}) \right|,$$

where ϕ_{K_v} is any inertial transfer for K_v and A_{Π_v} is any normalized intertwining operator for Π_v , for $v \in \Sigma_{\text{lr}}^+$. Since $\sum_{BC(\pi_v) \simeq \Pi_v} \dim(\pi_v)^{K_v} \geq 1$ for every $v \in \Sigma_{\text{lr}}^+$, (D.2) will be implied by

$$(D.3) \quad \prod_{v \in \Sigma_{\text{lr}}^+} \sum_{BC(\pi_v) \simeq \Pi_v} \dim(\pi_v)^{K_v} = \left| \prod_{v \in \Sigma_{\text{lr}}^+} \operatorname{tr}(\Pi_v(\phi_{K_v}) \circ A_{\Pi_v}) \right|.$$

For this, we choose an imaginary quadratic number field $E \subseteq \mathbb{C}$ satisfying

- E is not included in F ;
- if a rational prime p underlies Σ_{\min}^+ , then p splits in E ;
- if a rational prime p underlies Σ_{lr}^+ , then p is inert in E ;
- the quadratic base change of Π to $\check{F} := F.E$, denoted by $\check{\Pi}$, remains cuspidal (hence relevant).

Let $\check{F}^+ \subseteq \check{F}$ be the maximal totally real subfield; let $\check{\Sigma}_{\text{lr}}^+$ be the set of places of \check{F}^+ above Σ_{lr}^+ ; and let $\check{\Sigma}^+$ the (finite set) of nonarchimedean places \check{v} of \check{F}^+ not in $\check{\Sigma}_{\text{lr}}^+$ such that $\check{\Pi}_{\check{v}}$ is ramified. By our choice of E , \check{F}/\check{F}^+ is everywhere unramified; every place in $\check{\Sigma}^+$ splits in \check{F} ; and every place in Σ_{lr}^+ splits into two places of \check{F}^+ both inert in \check{F} . Let \check{V} be the standard definite hermitian space over \check{F} of rank N such that $\check{V}_{\check{v}}$ (for a nonarchimedean place \check{v}) is not split if and only if $\check{v} \in \check{\Sigma}_{\text{lr}}^+$. Take an object $\check{K} \in \mathfrak{K}(\check{V})$ of the form $\check{K} = \prod \check{K}_{\check{v}}$ satisfying

- $\check{K}_{\check{v}}$ is hyperspecial maximal if \check{v} is inert in \check{F} and not in $\check{\Sigma}_{\text{lr}}^+$;
- $\check{K}_{\check{v}}$ is given by K_v for $\check{v} \in \check{\Sigma}_{\text{lr}}^+$, where $v \in \Sigma_{\text{lr}}^+$ underlies \check{v} ;
- $\check{\pi}_{\check{v}}$ has nonzero $\check{K}_{\check{v}}$ invariants if \check{v} splits in \check{F} , where $\check{\pi}_{\check{v}}$ is the descent of $\check{\Pi}_{\check{v}}$ to an irreducible admissible representation of $U(\check{V})(\check{F}_{\check{v}}^+)$.

Then we have

$$(D.4) \quad \dim \overline{\mathbb{Q}}_\ell[\operatorname{Sh}(\check{V}, \check{K})][\iota_\ell \phi_{\check{\Pi}}] = \prod_{\check{v} \in \check{\Sigma}^+ \cup \check{\Sigma}_{\text{lr}}^+} \sum_{BC(\check{\pi}_{\check{v}}) \simeq \check{\Pi}_{\check{v}}} \dim(\check{\pi}_{\check{v}})^{\check{K}_{\check{v}}}.$$

²⁸We expect that the special maximal subgroup K_v is already transferable; but we do not need to address this issue.

²⁹Strictly speaking, Shin's formulae are stated for unitary similitude groups and assuming F containing an imaginary quadratic subfield. However, we can modify his argument to honest unitary groups and without the constrain on F by our moduli interpretation as we did in the proof of Theorem D.1.3.

On the other hand, by [Shi, (1.8) & (1.9)], we have

$$(D.5) \quad \dim \overline{\mathbb{Q}}_\ell[\mathrm{Sh}(\check{V}, \check{K})][\iota_\ell \phi_{\check{\Pi}}] = \left| \prod_{\check{v} \in \check{\Sigma}^+ \cup \check{\Sigma}_{\mathrm{r}}^+} \mathrm{tr}(\check{\Pi}_{\check{v}}(\phi_{\check{K}_{\check{v}}}) \circ A_{\check{\Pi}_{\check{v}}}) \right|.$$

Here, for $\check{v} \in \check{\Sigma}^+$, we take $\phi_{\check{K}_{\check{v}}}$ to be $\mathbb{1}_{\check{K}_{\check{v}}} \otimes \mathbb{1}_{\check{K}_{\check{v}}}$; and for $\check{v} \in \check{\Sigma}_{\mathrm{r}}^+$, we take $\phi_{\check{K}_{\check{v}}}$ to be ϕ_{K_v} where v is place of F^+ underlying \check{v} . Then it is easy to see that for $\check{v} \in \check{\Sigma}^+$, we have

$$(D.6) \quad \left| \mathrm{tr}(\check{\Pi}_{\check{v}}(\phi_{\check{K}_{\check{v}}}) \circ A_{\check{\Pi}_{\check{v}}}) \right| = \sum_{\mathrm{BC}(\check{\pi}_{\check{v}}) \simeq \check{\Pi}_{\check{v}}} \dim(\check{\pi}_{\check{v}})^{\check{K}_{\check{v}}} \geq 1$$

(in fact, the sum is taken over a singleton). Combining (D.4), (D.5), and (D.6), we obtain

$$\prod_{\check{v} \in \check{\Sigma}_{\mathrm{r}}^+} \sum_{\mathrm{BC}(\check{\pi}_{\check{v}}) \simeq \check{\Pi}_{\check{v}}} \dim(\check{\pi}_{\check{v}})^{\check{K}_{\check{v}}} = \left| \prod_{\check{v} \in \check{\Sigma}_{\mathrm{r}}^+} \mathrm{tr}(\check{\Pi}_{\check{v}}(\phi_{\check{K}_{\check{v}}}) \circ A_{\check{\Pi}_{\check{v}}}) \right|,$$

which is nothing but

$$\left(\prod_{v \in \Sigma_{\mathrm{r}}^+} \sum_{\mathrm{BC}(\pi_v) \simeq \Pi_v} \dim(\pi_v)^{K_v} \right)^2 = \left| \prod_{v \in \Sigma_{\mathrm{r}}^+} \mathrm{tr}(\Pi_v(\phi_{K_v}) \circ A_{\Pi_v}) \right|^2.$$

Thus, (D.3) follows. The proposition is proved. □

APPENDIX E. DEFORMATION OF GALOIS REPRESENTATIONS

We consider a subfield $F \subseteq \mathbb{C}$ that is a CM number field. We adopt the notation concerning ground fields in Subsection 1.3; and we put $\eta := \eta_{F/F^+}$ for short. The main objective of this appendix is to generalize results in [CHT08] and [Tho12] concerning the relation between the Galois deformation algebra and the Hecke algebra, informally known as R=T theorems. In Subsection E.1, we collect some facts concerning essentially conjugate self-dual representations, which is also frequently used in the main text. In Subsection E.2, we recall the notion and facts of lifting and deformation of Galois representations. In Subsection E.3, we study Fontaine–Laffaille local deformations. In Subsection E.4, we study representations of tame groups, which will be used in the next two subsections. In Subsection E.5, we study minimally ramified local deformations. In Subsection E.6, we study local deformations related to the level raising. In Subsection E.7, we state and prove our R=T theorem for both unitary Shimura sets and unitary Shimura varieties. In Subsection E.8, we study the rigidity property for reduction of automorphic Galois representations, in the sense of Definition E.7.1.

In this appendix, we shall slightly change our notation system from Section 2 to fit the one used in [CHT08]. We fix an odd prime ℓ and an isomorphism $\iota_\ell: \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$. Consider a finite extension E_λ of \mathbb{Q}_ℓ inside $\overline{\mathbb{Q}}_\ell$. Let \mathcal{O} (rather than O_λ in Section 2) be the ring of integers of E_λ , λ the maximal ideal of \mathcal{O} , and $k := \mathcal{O}/\lambda$ the residue field. Following [CHT08], we denote by $\mathcal{C}_\mathcal{O}^f$ the category of commutative local Artinian \mathcal{O} -algebras with residue field k , and let $\mathcal{C}_\mathcal{O}$ be the category of topological local \mathcal{O} -algebras whose objects are inverse limits of objects of $\mathcal{C}_\mathcal{O}^f$. For an object R of $\mathcal{C}_\mathcal{O}$, we shall denote by \mathfrak{m}_R its maximal ideal. For an \mathcal{O} -valued character, we will use the same notation for its induced R -valued character for every object R of $\mathcal{C}_\mathcal{O}$.

Take an integer $N \geq 1$.

E.1. Extension of essentially conjugate self-dual representations. In this subsection, we collect some notion and facts on the extension of essentially conjugate self-dual representations.

Notation E.1.1. We recall the group scheme \mathcal{G}_N from [CHT08, Section 1]. Put

$$\mathcal{G}_N := (\mathrm{GL}_N \times \mathrm{GL}_1) \rtimes \{1, \mathbf{c}\}$$

with $\mathbf{c}^2 = 1$ and

$$\mathbf{c}(g, \mu)\mathbf{c} = (\mu {}^t g^{-1}, \mu)$$

for $(g, \mu) \in \mathrm{GL}_N \times \mathrm{GL}_1$. In what follows, we will often regard GL_N as a subgroup of \mathcal{G}_N via the embedding $g \mapsto (g, 1, 1)$. Denote by $\nu: \mathcal{G}_N \rightarrow \mathrm{GL}_1$ the homomorphism such that $\nu|_{\mathrm{GL}_N \times \mathrm{GL}_1}$ is the projection to the factor GL_1 and that $\nu(\mathbf{c}) = -1$. We have the adjoint action ad of \mathcal{G}_N on M_N , given by

$$\mathrm{ad}(g, \mu)(x) = gxg^{-1}, \quad \mathrm{ad}(\mathbf{c})(x) = -{}^t x$$

for $x \in M_N$ and $(g, \mu) \in \mathrm{GL}_N \times \mathrm{GL}_1$.

Let $\tilde{\Gamma}$ be a topological group, and $\Gamma \subseteq \tilde{\Gamma}$ a subgroup of index at most two.

Notation E.1.2. Let R (rather than L in Section 2) be a commutative topological \mathbb{Z}_ℓ -algebra. For a continuous homomorphism

$$r: \tilde{\Gamma} \rightarrow \mathcal{G}_N(R)$$

such that the image of $r|_\Gamma$ is contained in $\mathrm{GL}_N(R) \times R^\times$, we denote

$$r^\natural: \Gamma \rightarrow \mathrm{GL}_N(R) \times R^\times \rightarrow \mathrm{GL}_N(R)$$

the composition of $r|_\Gamma$ with the projection to $\mathrm{GL}_N(R)$.

Lemma E.1.3. *Suppose $[\tilde{\Gamma} : \Gamma] = 2$. Let R be a commutative topological \mathbb{Z}_ℓ -algebra and $\chi: \tilde{\Gamma} \rightarrow R^\times$ a continuous character. We have*

- (1) *If $r: \tilde{\Gamma} \rightarrow \mathcal{G}_N(R)$ is a continuous homomorphism satisfying $r^{-1}(\mathrm{GL}_N(R) \times R^\times) = \Gamma$ and $\nu \circ r = \chi$, then for every $\gamma \in \tilde{\Gamma} \setminus \Gamma$, we have*

$$r^{\natural, \gamma} = B \circ \chi r^{\natural, \nu} \circ B^{-1},$$

where A is given from $r(\gamma) = (B, -\chi(\gamma), \mathbf{c})$.

- (2) *Let $\rho: \Gamma \rightarrow \mathrm{GL}_N(R)$ be a continuous homomorphism, γ an element in $\tilde{\Gamma} \setminus \Gamma$, and $B \in \mathrm{GL}_N(R)$ such that $\rho^\gamma = B \circ \chi \rho^\nu \circ B^{-1}$ and $B {}^t B^{-1} = \mu_B \chi(\gamma)^{-1} \rho(\gamma^2)$ for some $\mu_B \in \{\pm 1\}$. Then there exists a unique continuous homomorphism*

$$r: \tilde{\Gamma} \rightarrow \mathcal{G}_N(R)$$

satisfying $r|_\Gamma = (\rho, \chi|_\Gamma, 1)$ and $r(\gamma) = (B, \mu_B \chi(\gamma), \mathbf{c})$.

- (3) *Suppose in (2) that R is a field and ρ is absolutely irreducible. If ρ^γ and $\chi \rho^\nu$ are conjugate, then ρ induces a homomorphism $r: \tilde{\Gamma} \rightarrow \mathcal{G}_N(R)$ satisfying $r|_\Gamma = (\rho, \chi)$, unique up to changing the $\mathrm{GL}_N(R)$ -component of $r(\gamma)$ by a scalar in R^\times .*

Proof. Part (1) is a special case of [CHT08, Lemma 2.1.1].

For (2), we check that

$$r(\gamma^2) = (B, \mu_B \chi(\gamma), \mathbf{c}) \cdot (B, \mu_B \chi(\gamma), \mathbf{c}) = (\mu_B \chi(\gamma) B {}^t B^{-1}, \chi(\gamma^2), 1) = (\rho(\gamma^2), \chi(\gamma^2), 1).$$

Since $\tilde{\Gamma}$ is generated by Γ and γ , we obtain a unique continuous homomorphism $r: \tilde{\Gamma} \rightarrow \mathcal{G}_N(R)$ as in (2).

For (3), by Schur's lemma, the element B is unique up to scalar in R^\times , which implies the existence and also the uniqueness of μ_B . Thus, (3) follows immediately. \square

E.2. Deformation problems. In this subsection, we introduce the notion of deformation problems. Let $\tilde{\Gamma}$ be a topological group, and $\Gamma \subseteq \tilde{\Gamma}$ a subgroup of index at most two.

Notation E.2.1. We consider a pair (\bar{r}, χ) , where

○ $\bar{r}: \tilde{\Gamma} \rightarrow \mathcal{G}_N(k)$ is a homomorphism,

○ $\chi: \tilde{\Gamma} \rightarrow \mathcal{O}^\times$ a continuous homomorphism, known as the *similitude character*,

subject to the relation $\bar{r}^{-1}(\mathrm{GL}_N(k) \times k^\times) = \Gamma$ and $\nu \circ \bar{r} = \chi$.

The following definition slightly generalizes [CHT08, Definition 2.2.1].

Definition E.2.2. A *lifting* of \bar{r} to an object R of \mathcal{C}_θ is a continuous homomorphism $r: \tilde{\Gamma} \rightarrow \mathcal{G}_N(R)$ satisfying $r \bmod \mathfrak{m}_R = \bar{r}$ and $\nu \circ r = \chi$. We say that two liftings are equivalent if they are conjugate by an element in $1 + \mathrm{M}_N(\mathfrak{m}_R) \subset \mathrm{GL}_N(R) \subset \mathcal{G}_N(R)$. By a *deformation* of \bar{r} , we mean an equivalence class of liftings of \bar{r} .³⁰

Now suppose that Γ is topologically finitely generated. Then there exists a universal lifting

$$r^{\mathrm{univ}}: \tilde{\Gamma} \rightarrow \mathcal{G}_N(\mathbf{R}_{\bar{r}}^{\mathrm{loc}})$$

of \bar{r} to an object $\mathbf{R}_{\bar{r}}^{\mathrm{loc}}$ of \mathcal{C}_θ such that, for every object R of \mathcal{C}_θ , the set of liftings of \bar{r} to R is in natural bijection with $\mathrm{Hom}_{\mathcal{C}_\theta}(\mathbf{R}_{\bar{r}}^{\mathrm{loc}}, R)$. Since Γ is topologically finitely generated, it is well-known that $\mathbf{R}_{\bar{r}}^{\mathrm{loc}}$ is Noetherian; and there exists natural isomorphisms

$$\mathrm{Hom}_k(\mathfrak{m}_{\mathbf{R}_{\bar{r}}^{\mathrm{loc}}} / (\mathfrak{m}_{\mathbf{R}_{\bar{r}}^{\mathrm{loc}}}^2, \lambda), k) \simeq \mathrm{Hom}_{\mathcal{C}_\theta}(\mathbf{R}_{\bar{r}}^{\mathrm{loc}}, k[\varepsilon]/(\varepsilon^2)) \simeq Z^1(\tilde{\Gamma}, \mathrm{ad} \bar{r}),$$

where $Z^1(\tilde{\Gamma}, \mathrm{ad} \bar{r})$ denotes the group of 1-cocycles of $\tilde{\Gamma}$ with values in the adjoint representation $(\mathrm{ad} \bar{r}, \mathrm{M}_N(k))$. Explicitly, for $\phi \in Z^1(\tilde{\Gamma}, \mathrm{ad} \bar{r})$, the corresponding lifting of \bar{r} to $k[\varepsilon]/(\varepsilon^2)$ is given by

$$r_\phi(g) = (1 + \varepsilon\phi(g))\bar{r}(g)$$

for every $g \in \tilde{\Gamma}$. For two cocycles $\phi_1, \phi_2 \in Z^1(\tilde{\Gamma}, \mathrm{ad} \bar{r})$, the corresponding liftings r_{ϕ_1} and r_{ϕ_2} are equivalent if and only if there exists an element $x \in \mathrm{M}_N(k)$ such that

$$\phi_1(g) - \phi_2(g) = (1 - \mathrm{ad} \bar{r}(g))(x)$$

for every $g \in \tilde{\Gamma}$. Thus, the equivalence classes of liftings of \bar{r} to $k[\varepsilon]/(\varepsilon^2)$ is in natural bijection with $H^1(\tilde{\Gamma}, \mathrm{ad} \bar{r})$.

Definition E.2.3. A *local deformation problem* of \bar{r} is a closed formal subscheme \mathcal{D} of $\mathrm{Spf} \mathbf{R}_{\bar{r}}^{\mathrm{loc}}$ that is invariant under the conjugate action by $1 + \mathrm{M}_N(\mathfrak{m}_{\mathbf{R}_{\bar{r}}^{\mathrm{loc}}})$.

By the moduli interpretation of $\mathbf{R}_{\bar{r}}^{\mathrm{loc}}$, giving a local deformation problem of \bar{r} is equivalent to giving a collection of liftings of \bar{r} to objects in \mathcal{C}_θ satisfying certain conditions (see [CHT08, Definition 2.2.2 & Lemma 2.2.3]).

Definition E.2.4. For a local deformation problem \mathcal{D} of \bar{r} , we define the *tangent space* of \mathcal{D} , denoted by $L(\mathcal{D})$, to be the image of the subspace

$$L^1(\mathcal{D}) := \mathrm{Hom}_k(\mathfrak{m}_{\mathbf{R}_{\bar{r}}^{\mathrm{loc}}} / (\mathfrak{m}_{\mathbf{R}_{\bar{r}}^{\mathrm{loc}}}^2, \mathcal{I}, \lambda), k) \subseteq Z^1(\tilde{\Gamma}, \mathrm{ad} \bar{r})$$

under the natural map $Z^1(\tilde{\Gamma}, \mathrm{ad} \bar{r}) \rightarrow H^1(\tilde{\Gamma}, \mathrm{ad} \bar{r})$, where $\mathcal{I} \subseteq \mathbf{R}_{\bar{r}}^{\mathrm{loc}}$ is the closed ideal defining \mathcal{D} .

Note that we have the identity

$$(E.1) \quad \dim_k L^1(\mathcal{D}) = N^2 + \dim_k L(\mathcal{D}) - \dim_k H^0(\tilde{\Gamma}, \mathrm{ad} \bar{r}).$$

³⁰Strictly speaking, a lifting or a deformation of \bar{r} depends on the similitude character χ . But we choose to follow the terminology in [CHT08] by not spelling the characters out, as the relevance on the similitude character is always clear from the context.

Remark E.2.5. Later, when we consider a nonarchimedean place v of F^+ and take $\tilde{\Gamma} = \Gamma_{F_v^+}$, the subgroup Γ we implicitly take is always $\Gamma_{F_v^+} \cap \Gamma_F$.

Now we apply Notation E.2.1 and Definition E.2.2 to the case where $\tilde{\Gamma} = \Gamma_{F^+}$ and $\Gamma = \Gamma_F$.

Definition E.2.6. A *global deformation problem* is a tuple $(\bar{r}, \chi, S, \{\mathcal{D}_v\}_{v \in S})$, where

- (\bar{r}, χ) is a pair as in Notation E.2.1;
- S is a finite set of nonarchimedean places of F^+ containing all ℓ -adic places and those places v such that \bar{r}_v is ramified;
- \mathcal{D}_v is a local deformation problem of \bar{r}_v (Remark E.2.5) for each $v \in S$.

We take a global deformation problem $\mathcal{S} := (\bar{r}, \chi, S, \{\mathcal{D}_v\}_{v \in S})$. For $v \in S$, we denote by \mathcal{I}_v the closed ideal of $R_{\bar{r}_v}^{\text{loc}}$ defining \mathcal{D}_v . For a subset $T \subseteq S$, put

$$(E.2) \quad R_{\mathcal{S}, T}^{\text{loc}} := \widehat{\bigotimes}_{v \in T} R_{\bar{r}_v}^{\text{loc}} / \mathcal{I}_v,$$

where the completed tensor product is taken over \mathcal{O} . Recall from [CHT08, Definition 2.2.1] that a T -framed lifting of \bar{r} to an object R of $\mathcal{C}_{\mathcal{O}}$ is a tuple $(r; \{\beta_v\}_{v \in T})$, where r is a lifting of \bar{r} to R (Definition E.2.2), and $\beta_v \in 1 + M_N(\mathfrak{m}_R)$ for $v \in T$. Two T -framed liftings $(r; \{\beta_v\}_{v \in T})$ and $(r'; \{\beta'_v\}_{v \in T})$ of \bar{r} to R are said equivalent, if there exists $x \in 1 + M_N(\mathfrak{m}_R)$ such that $r' = x^{-1} \circ r \circ x$ and $\beta'_v = x^{-1} \beta_v$ for every $v \in T$. A T -framed deformation of \bar{r} is an equivalence class of T -framed liftings of \bar{r} . We say that a T -framed lifting $(r; \{\beta_v\}_{v \in T})$ is of type \mathcal{S} if r_v belongs to \mathcal{D}_v for every $v \in S$, and is unramified for every $v \notin S$. Note that being of type \mathcal{S} is a property invariant under the conjugate action by $1 + M_N(\mathfrak{m}_R)$. Thus it makes sense to speak of T -framed deformation of type \mathcal{S} . Let $\text{Def}_{\mathcal{S}}^{\square T} : \mathcal{C}_{\mathcal{O}} \rightarrow \mathbf{Set}$ be the functor that sends an object R to the set of T -framed deformations of \bar{r} to R of type \mathcal{S} .

Let $\Gamma_{F^+, S}$ be the Galois group of the maximal subextension of \bar{F}/F^+ that is unramified outside S . Recall the cohomology group $H_{\mathcal{S}, T}^i(\Gamma_{F^+, S}, \text{ad } \bar{r})$ for $i \geq 0$ introduced after [CHT08, Definition 2.2.7]. By [CHT08, Lemma 2.3.4], these are finite dimensional k -vector spaces, and satisfy $H_{\mathcal{S}, T}^i(\Gamma_{F^+, S}, \text{ad } \bar{r}) = 0$ for $i > 3$.

Proposition E.2.7. *Assume that $\bar{r}|_{\Gamma_F}$ is absolutely irreducible. Then for every subset $T \subseteq S$, the functor $\text{Def}_{\mathcal{S}}^{\square T}$ is represented by a Noetherian \mathcal{O} -algebra $R_{\mathcal{S}}^{\square T}$ in $\mathcal{C}_{\mathcal{O}}$. Put $R_{\mathcal{S}}^{\text{univ}} := R_{\mathcal{S}}^{\square \emptyset}$. We further have*

- (1) *There is a canonical isomorphism*

$$\text{Hom}_k \left(\mathfrak{m}_{R_{\mathcal{S}}^{\square T}} / (\mathfrak{m}_{R_{\mathcal{S}}^{\square T}}^2, \lambda, \mathfrak{m}_{R_{\mathcal{S}}^{\text{loc}, T}}), k \right) \simeq H_{\mathcal{S}, T}^1(\Gamma_{F^+, S}, \text{ad } \bar{r}),$$

where we regard $\mathfrak{m}_{R_{\mathcal{S}}^{\text{loc}, T}}$ as its image under the tautological homomorphism $R_{\mathcal{S}}^{\text{loc}, T} \rightarrow R_{\mathcal{S}}^{\square T}$. Moreover, if $H_{\mathcal{S}, T}^2(\Gamma_{F^+, S}, \text{ad } \bar{r}) = 0$ and for $v \in S \setminus T$, \mathcal{D}_v is formally smooth over \mathcal{O} , then $R_{\mathcal{S}}^{\square T}$ is a power series ring over $R_{\mathcal{S}}^{\text{loc}, T}$ in $\dim_k H_{\mathcal{S}, T}^1(\Gamma_{F^+, S}, \text{ad } \bar{r})$ variables.

- (2) *The choice of a lifting $r_{\mathcal{S}}^{\text{univ}} : \Gamma_{F^+} \rightarrow \mathcal{G}_N(R_{\mathcal{S}}^{\text{univ}})$ in the universal deformation determines an extension of the tautological homomorphism $R_{\mathcal{S}}^{\text{univ}} \rightarrow R_{\mathcal{S}}^{\square T}$ to an isomorphism*

$$R_{\mathcal{S}}^{\text{univ}}[[X_{v; i, j}]_{v \in T; 1 \leq i, j \leq N}] \xrightarrow{\sim} R_{\mathcal{S}}^{\square T}$$

such that, for every $v \in T$, the universal frame at v is given by $\beta_v = 1 + (X_{v; i, j})_{1 \leq i, j \leq N}$.

Proof. These are exactly [CHT08, Proposition 2.2.9 & Corollary 2.2.13] except that they consider only local deformation problems at split places (that is, they assume that all places in S are split in F). However, the same argument can be applied to the general case without change. \square

E.3. Fontaine–Laffaille deformations. In this subsection, we study Fontaine–Laffaille deformations at ℓ -adic places. We take a place v of F^+ above ℓ ; and let w be the place of F above v induced by the inclusion $F \subseteq \overline{F}_v^+$. We assume that ℓ is unramified in F , and denote by $\sigma \in \text{Gal}(F_w/\mathbb{Q}_\ell)$ the absolute Frobenius element.

Definition E.3.1. We say that E_λ is F -inclusive if E_λ contains the image of all embeddings of F into $\overline{\mathbb{Q}_\ell}$.

We first suppose that E_λ is F -inclusive, and put $\Sigma_w := \text{Hom}_{\mathbb{Z}_\ell}(O_{F_w}, \mathcal{O})$. Following [CHT08], we use a covariant version of the Fontaine–Laffaille theory [FL82]. Let $\mathcal{MF}_{\mathcal{O},w}$ be the category of $O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}$ -modules M of finite length equipped with

- a decreasing filtration $\{\text{Fil}^i M\}_{i \in \mathbb{Z}}$ by $O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}$ -submodules that are O_{F_w} -direct summands, satisfying $\text{Fil}^0 M = M$ and $\text{Fil}^{\ell-1} M = 0$, and
- a Frobenius structure, that is, $\sigma \otimes 1$ -linear maps $\Phi^i: \text{Fil}^i M \rightarrow M$ for $i \in \mathbb{Z}$, satisfying the relations $\Phi^i|_{\text{Fil}^{i+1} M} = \ell \Phi^{i+1}$ and $\sum_{i \in \mathbb{Z}} \Phi^i \text{Fil}^i M = M$.

Let $\mathcal{MF}_{k,w}$ be the full subcategory of $\mathcal{MF}_{\mathcal{O},w}$ of objects that are annihilated by λ .

For an object M of $\mathcal{MF}_{\mathcal{O},w}$, there is canonical decomposition

$$M = \bigoplus_{\tau \in \Sigma_w} M_\tau,$$

where $M_\tau := M \otimes_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}, \tau \otimes 1} \mathcal{O}$. Then we have $\text{Fil}^i M = \bigoplus_{\tau \in \Sigma_w} \text{Fil}^i M_\tau$ with $\text{Fil}^i M_\tau = M_\tau \cap \text{Fil}^i M$, and that Φ^i induces \mathcal{O} -linear maps

$$\Phi_\tau^i: \text{Fil}^i M_\tau \rightarrow M_{\tau \circ \sigma^{-1}}.$$

We put

$$\text{gr}^i M_\tau := \text{Fil}^i M_\tau / \text{Fil}^{i+1} M_\tau, \quad \text{gr}^\bullet M_\tau := \bigoplus_i \text{gr}^i M_\tau, \quad \text{gr}^\bullet M := \bigoplus_{\tau \in \Sigma_w} \text{gr}^\bullet M_\tau.$$

We define the set of τ -Fontaine–Laffaille weights of M to be

$$\text{HT}_\tau(M) := \{i \in \mathbb{Z} \mid \text{gr}^i M_\tau \neq 0\}.$$

We say that M has *regular Fontaine–Laffaille weights* if $\text{gr}^i M_\tau$ is generated over \mathcal{O} by at most one element for every $\tau \in \Sigma_w$ and every $i \in \mathbb{Z}$.

For every integer a satisfying $0 \leq a \leq \ell - 2$, let $\mathcal{MF}_{\mathcal{O},w}^{[0,a]}$ be the full subcategory of $\mathcal{MF}_{\mathcal{O},w}$ consisting of objects M satisfying $\text{Fil}^{a+1} M = 0$. In particular, we have $\mathcal{MF}_{\mathcal{O},w}^{[0,\ell-2]} = \mathcal{MF}_{\mathcal{O},w}$ by definition. There is a duality functor $D^{[0,a]}$ on the category $\mathcal{MF}_{\mathcal{O},w}^{[0,a]}$ such that for every object M of $\mathcal{MF}_{\mathcal{O},w}^{[0,a]}$, the object $D^{[0,a]}(M)$ is defined as follows:

- the underlying $O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}$ -module of $D^{[0,a]}(M)$ is $\text{Hom}_{O_{F_w}}(M, F_w/O_{F_w})$;
- $\text{Fil}^i D^{[0,a]}(M) = \text{Hom}_{O_{F_w}}(M/\text{Fil}^{a+1-i} M, F_w/O_{F_w})$;
- for $f \in \text{Fil}^i D^{[0,a]}(M)$ and $m \in \text{Fil}^j M$, we have

$$\Phi^i(f)(\Phi^j(m)) = \begin{cases} \ell^{a-i-j} f(m)^\sigma & \text{if } i+j \leq a; \\ 0 & \text{if } i+j > a. \end{cases}$$

It is easy to see that $D^{[0,a]}(M)$ is a well-defined object of $\mathcal{MF}_{\mathcal{O},w}$ (see [CHT08, Page 34]), and that $D^{[0,a]}(D^{[0,a]}(M)) = M$.

Let $\mathcal{O}[\Gamma_{F_w}]^{\text{f.l.}}$ be the category of \mathcal{O} -modules of finite length equipped with a continuous action of Γ_{F_w} . In [CHT08, 2.4.1], the authors defined an exact fully faithful, covariant \mathcal{O} -linear functor

$$\mathbf{G}_w: \mathcal{MF}_{\mathcal{O},w} \rightarrow \mathcal{O}[\Gamma_{F_w}]^{\text{f.l.}}$$

whose essential image is closed under taking sub-objects and quotient objects.³¹ The length of an object M in $\mathcal{MF}_{\mathcal{O},w}$ as an \mathcal{O} -module equals $[F_w : \mathbb{Q}_\ell]$ times the length of $\mathbf{G}_w(M)$ as an \mathcal{O} -module. If M belongs to $\mathcal{MF}_{\mathcal{O},w}^{[0,a]}$, then we have

$$\mathbf{G}_w(D^{[0,a]}(M)) = \mathbf{G}_w(M)^\vee(-a),$$

where $\mathbf{G}_w(M)^\vee := \mathrm{Hom}_{\mathcal{O}}(\mathbf{G}_w(M), E_\lambda/\mathcal{O})$ with the dual Galois action. For two objects M_1, M_2 of $\mathcal{MF}_{\mathcal{O},w}$, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{MF}_{\mathcal{O},w}}(M_1, M_2) \xrightarrow{\sim} H^0(F_w, \mathrm{Hom}_{\mathcal{O}}(\mathbf{G}_w(M_1), \mathbf{G}_w(M_2)))$$

and a canonical injective map

$$\mathrm{Ext}_{\mathcal{MF}_{\mathcal{O},w}}^1(M_1, M_2) \hookrightarrow \mathrm{Ext}_{\mathcal{O}[\Gamma_{F_w}]^{\mathrm{f.l.}}}^1(\mathbf{G}_w(M_1), \mathbf{G}_w(M_2)),$$

where the target is canonically isomorphic to $H^1(F_w, \mathrm{Hom}_k(\mathbf{G}_w(M_1), \mathbf{G}_w(M_2)))$ if M_1, M_2 are both objects of $\mathcal{MF}_{k,w}$.

Example E.3.2. For an integer a satisfying $0 \leq a \leq \ell - 2$ and an object R of $\mathcal{C}_{\mathcal{O}}^f$, we have an object $R\{a\}$ of $\mathcal{MF}_{\mathcal{O},w}$ defined as follows: the underlying $O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}$ -module is simply $((F_w/O_{F_w}) \otimes R)e_a$, with the filtration given by

$$\mathrm{Fil}^i R\{a\} = \begin{cases} ((F_w/O_{F_w}) \otimes R)e_a & \text{if } i \leq a; \\ 0 & \text{if } i > a. \end{cases}$$

Finally, the Frobenius structure is determined by $\Phi^a(e_a) = e_a$. Then we have

$$\mathbf{G}_w(R\{a\}) \simeq R(-a)$$

as $\mathcal{O}[\Gamma_{F_w}]$ -modules, where $(-a)$ denotes the Tate twist, and $D^{[0,b]}(R\{a\}) \simeq R\{b-a\}$ for every integer b satisfying $a \leq b \leq \ell - 2$.

Construction E.3.3. We construct a functor $\mathbf{-}^\sigma : \mathcal{MF}_{\mathcal{O},w} \rightarrow \mathcal{MF}_{\mathcal{O},w}$ as follows: for an object M of $\mathcal{MF}_{\mathcal{O},w}$, the underlying $O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}$ -module of M^σ is $O_{F_w} \otimes_{O_{F_w}, \sigma} M$ with the induced filtration and Frobenius structure. Then we have $M_\tau^\sigma = M_{\tau \circ \sigma^{-1}}$ for every $\tau \in \Sigma_w$, and that $\mathbf{G}_w(M^\sigma)$ is isomorphic to $\mathbf{G}_w(M)$ but with the action of Γ_{F_w} twisted by the absolute Frobenius of F_w : if we denote by ρ and ρ_σ the actions of Γ_{F_w} on $\mathbf{G}_w(M)$ and $\mathbf{G}_w(M^\sigma)$, respectively, then they satisfy

$$\rho_\sigma(g) = \rho(\tilde{\sigma}^{-1}g\tilde{\sigma}),$$

where $\tilde{\sigma} \in \mathrm{Gal}(\overline{F}_v^+/\mathbb{Q}_\ell)$ is a lift of the absolute Frobenius.

We now let $\mathbf{-}^c : \mathcal{MF}_{\mathcal{O},w} \rightarrow \mathcal{MF}_{\mathcal{O},w}$ be the $[F_v^+ : \mathbb{Q}_\ell]$ -th iteration of the functor $\mathbf{-}^\sigma$ constructed above. Take an object M of $\mathcal{MF}_{\mathcal{O},w}$. Suppose that M is finite free over $O_{F_w} \otimes R$ for some object R of $\mathcal{C}_{\mathcal{O}}^f$. Then giving an isomorphism $M \simeq D^{[0,a]}(M^c)$ is equivalent to giving a perfect pairing

$$\langle \cdot, \cdot \rangle : M^c \times M \rightarrow R\{a\}$$

in the category $\mathcal{MF}_{\mathcal{O},w}$, where $R\{a\}$ is the object in Example E.3.2. The latter is equivalent to giving, for each $\tau \in \Sigma_w$, an R -bilinear perfect pairing $\langle \cdot, \cdot \rangle_\tau : M_{\tau^c} \times M_\tau \rightarrow (\mathbb{Q}_\ell/\mathbb{Z}_\ell) \otimes R$ satisfying that

- (1) for every $i, j \in \mathbb{Z}$ and every $x \in \mathrm{Fil}^i M_\tau$ and $y \in \mathrm{Fil}^j M_\tau$, $\langle \Phi_{\tau^c}^i x, \Phi_\tau^j y \rangle_\tau$ equals $\ell^{a-i-j} \langle x, y \rangle_\tau$ (resp. 0) if $i + j \leq a$ (resp. $i + j > a$); and
- (2) for every $i \in \mathbb{Z}$, the annihilator of $\mathrm{Fil}^i M_\tau$ under $\langle \cdot, \cdot \rangle_\tau$ is $\mathrm{Fil}^{a+1-i} M_{\tau^c}$; in particular, $\langle \cdot, \cdot \rangle_\tau$ induces an R -linear isomorphism $\mathrm{gr}^i M_\tau \simeq \mathrm{Hom}_R(\mathrm{gr}^{a-i} M_{\tau^c}, (\mathbb{Q}_\ell/\mathbb{Z}_\ell) \otimes R)$.

³¹In fact, if \mathbf{U}_S is the contravariant functor from $\mathcal{MF}_{\mathcal{O},w}$ to $\mathcal{O}[\Gamma_{F_w}]^{\mathrm{f.l.}}$ defined in [FL82], then we have $\mathbf{G}_w(M) \simeq \mathbf{U}_S(D^{[0,\ell-2]}(M))(2-\ell)$.

Definition E.3.4. Let R be an object of \mathcal{C}_θ^f , and $\rho: \Gamma_{F_w} \rightarrow \mathrm{GL}_n(R)$ a continuous representation. We say ρ is *crystalline with Fontaine–Laffaille weights in $[0, a]$* for some $0 \leq a \leq \ell - 2$ if (R^n, ρ) lies in the essential image of the functor $\mathbf{G}_w: \mathcal{MF}_{\theta, w}^{[0, a]} \rightarrow \mathcal{O}[\Gamma_{F_w}]^{\mathrm{f.l.}}$; in this case, we say that ρ has *regular Fontaine–Laffaille weights* if so does $\mathbf{G}_w^{-1}(\rho)$.

Now we consider a pair (\bar{r}, χ) from Notation E.2.1 with $\tilde{\Gamma} = \Gamma_{F_v^+}$ and $\Gamma = \Gamma_{F_v^+} \cap \Gamma_F = \Gamma_{F_w}$. We do not assume that E_λ is F -inclusive. We choose an F -inclusive finite unramified extension E'_λ of E_λ , with the ring of integers \mathcal{O}' and the residue field k' .

Assumption E.3.5. There exist an integer a satisfying $0 \leq a \leq \ell - 2$ and an element $b \in \mathbb{Z}/2\mathbb{Z}$ such that

- (1) $\chi = \eta_v^b \epsilon_{\ell, v}^{-a}$; and
- (2) $\bar{r}^\natural \otimes_k k'$ is crystalline with regular Fontaine–Laffaille weights in $[0, a]$.

Definition E.3.6. Under Assumption E.3.5, we define $\mathcal{D}^{\mathrm{FL}}$ to be the local deformation problem of \bar{r} that classifies the liftings $r: \Gamma_{F_v^+} \rightarrow \mathcal{G}_N(R)$ of \bar{r} to objects R of \mathcal{C}_θ such that for every Artinian quotient R' of $R \otimes_\theta \mathcal{O}'$, $r^\natural \otimes_{R'} R'$ is crystalline with Fontaine–Laffaille weights in $[0, a]$ (Definition E.3.4).

Remark E.3.7. It is straightforward to check that Assumption E.3.5 and Definition E.3.6 do not depend on the choice of E'_λ .

Lemma E.3.8. *Suppose $\ell > N$ and assume Assumption E.3.5. We have*

$$\dim_k \mathrm{L}(\mathcal{D}^{\mathrm{FL}}) - \dim_k \mathrm{H}^0(F_v^+, \mathrm{ad} \bar{r}) = [F_v^+ : \mathbb{Q}_\ell] \cdot \frac{N(N-1)}{2}.$$

Proof. We may assume $E_\lambda = E'_\lambda$.

Suppose first that v is split in F . Then we have $F_w = F_v^+$, and that a lifting r in $\mathcal{D}^{\mathrm{FL}}(R)$ of \bar{r} is of the form $r = (\rho, \epsilon_{\ell, v}^{-a}): \Gamma_{F_w} \rightarrow \mathrm{GL}_N(R) \times R^\times$ such that for every Artinian quotient R' of R , $\rho \otimes_{R'} R'$ lies in the essential image of the functor \mathbf{G}_w . Then the lemma is exactly [CHT08, Corollary 2.4.3].

Suppose now that v is inert in F ; and denote by $\Gamma_{w/v}$ the Galois group of the quadratic extension F_w/F_v^+ . Then the restriction map induces an isomorphism

$$\mathrm{H}^1(F_v^+, \mathrm{ad} \bar{r}) \xrightarrow{\sim} \mathrm{H}^1(F_w, \mathrm{ad} \bar{r})^{\Gamma_{w/v}}.$$

Put $M := \mathbf{G}_w^{-1}(\bar{r}^\natural)$. Then the deformations of \bar{r} to $k[\varepsilon]/(\varepsilon^2)$ that lie in the essential image of \mathbf{G}_w are classified by $\mathrm{Ext}_{\mathcal{MF}_{k, w}}^1(M, M)$, which is canonically a $\Gamma_{w/v}$ -stable subspace of $\mathrm{H}^1(F_w, \mathrm{ad} \bar{r})$. Therefore, we have

$$\mathrm{L}(\mathcal{D}^{\mathrm{FL}}) = \mathrm{Ext}_{\mathcal{MF}_{k, w}}^1(M, M) \cap \mathrm{H}^1(F_w, \mathrm{ad} \bar{r})^{\Gamma_{w/v}} = \mathrm{Ext}_{\mathcal{MF}_{k, w}}^1(M, M)^{\Gamma_{w/v}}.$$

In fact, the induced action of $\Gamma_{w/v}$ on $\mathrm{Ext}_{\mathcal{MF}_{k, w}}^1(M, M)$ can be described as follows. Recall the functor $\mathbf{-}^c$ in Construction E.3.3. Then $\mathbf{G}_w(M^c)$ is isomorphic to $\bar{r}^\natural|_{\Gamma_{F_w}}$. Since \bar{r}^\natural, c and $\bar{r}^\natural, v(-a)$ are conjugate, we have $M \simeq \mathrm{D}^{[0, a]}(M^c)$. We fix such an isomorphism $M \simeq \mathrm{D}^{[0, a]}(M^c)$ hence obtain a pairing $\langle \cdot, \cdot \rangle$ (with $R = k$) as in Construction E.3.3. Then for an element $[E] \in \mathrm{Ext}_{\mathcal{MF}_{k, w}}^1(M, M)$ represented by an extension $0 \rightarrow M \rightarrow E \rightarrow M \rightarrow 0$, the image of $[E]$ under the action of the (unique) non-trivial element in $\Gamma_{w/v}$ is obtained by applying the functor $\mathrm{D}^{[0, a]}(\mathbf{-}^c)$ to $0 \rightarrow M \rightarrow E \rightarrow M \rightarrow 0$.

To compute $\mathrm{Ext}_{\mathcal{MF}_{k, w}}^1(M, M)^{\Gamma_{w/v}}$, we recall first the following long exact sequence in [CHT08, Lemma 2.4.2]:

(E.3)

$$0 \longrightarrow \mathrm{End}_{\mathcal{MF}_{k, w}}(M) \longrightarrow \mathrm{Fil}^0 \mathrm{Hom}_{\mathcal{O}_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}}(M, M) \xrightarrow{\alpha} \mathrm{Hom}_{\mathcal{O}_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}, \sigma \otimes 1}(\mathrm{gr}^\bullet M, M) \xrightarrow{\beta} \mathrm{Ext}_{\mathcal{MF}_{k, w}}^1(M, M) \longrightarrow 0,$$

where

- $\text{Fil}^0 \text{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}}(M, M)$ denotes the $O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}$ -submodule of $\text{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}}(M, M)$ of endomorphisms that preserve the filtration;
- the map α takes an element $f \in \text{Fil}^0 \text{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}}(M, M)$ to $(f\Phi^i - \Phi^i f)_{i \in \mathbb{Z}}$; and
- the map β is defined as follows: if $\varphi = (\varphi^i)_{i \in \mathbb{Z}}$ is a $\sigma \otimes 1$ -linear map from $\text{gr}^\bullet M$ to M , then $\beta(\varphi)$ is given by the extension class of $E = M \oplus M$ with the filtration $\text{Fil}^i E = \text{Fil}^i M \oplus \text{Fil}^i M$ and the Frobenius structure

$$\Phi_E^i := \begin{pmatrix} \Phi^i & \varphi^i \\ 0 & \Phi^i \end{pmatrix}.$$

To prove the lemma, we need to derive an analogous long exact sequence similar to (E.3) but with the last term $\text{Ext}_{\mathcal{M}_{\mathcal{F}_{k,w}}}^1(M, M)^{\Gamma_{w/v}}$. For the first term, note that we have a canonical isomorphism $\text{End}_{\mathcal{M}_{\mathcal{F}_{k,w}}}(M) \simeq \text{H}^0(F_w, \text{ad } \bar{r})$, which contains $\text{H}^0(F_v^+, \text{ad } \bar{r})$ as a submodule. For the second term, let $\text{Fil}^0 \text{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}}(M, M)^+$ be the submodule of $\text{Fil}^0 \text{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}}(M, M)$ consisting of elements $f = (f_\tau)_{\tau \in \Sigma_w}$ such that $-f_{\tau^c}$ is the adjoint of f_τ under the pairing $\langle \cdot, \cdot \rangle_\tau$ for every $\tau \in \Sigma_w$. For the third term, let $\text{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}, \sigma \otimes 1}(\text{gr}^\bullet M, M)^+$ denote by the submodule of $\text{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}, \sigma \otimes 1}(\text{gr}^\bullet M, M)$ consisting of $\varphi = (\varphi^i)_{i \in \mathbb{Z}}$ such that

$$(E.4) \quad \langle \Phi_{\tau^c}^i(x), \varphi_\tau^{\alpha-i}(y) \rangle_\tau + \langle \varphi_{\tau^c}^i(x), \Phi_\tau^{\alpha-i}(y) \rangle_\tau = 0$$

is satisfies for every $x \in \text{gr}^i M_{\tau^c}$ and $y \in \text{gr}^{\alpha-i} M_\tau$.

Then (E.3) induces an exact sequence

$$0 \longrightarrow \text{H}^0(F_v^+, \text{ad } \bar{r}) \longrightarrow \text{Fil}^0 \text{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}}(M, M)^+ \xrightarrow{\alpha} \text{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}, \sigma \otimes 1}(\text{gr}^\bullet M, M)^+ \xrightarrow{\beta} \text{Ext}_{\mathcal{M}_{\mathcal{F}_{k,w}}}^1(M, M)^{\Gamma_{w/v}} \longrightarrow 0$$

of k -vector spaces. We now compute the dimension of the middle two terms. From the description of $\text{Fil}^0 \text{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}}(M, M)^+$, it is clear that f_{τ^c} is determined by f_τ for every $\tau \in \Sigma_w$. On the other hand, for each fixed τ , all the possible choices of f_τ form a k -vector space of dimension $\frac{N(N+1)}{2}$. Thus, we have

$$\dim_k \text{Fil}^0 \text{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}}(M, M)^+ = [F_v^+ : \mathbb{Q}_\ell] \cdot \frac{N(N+1)}{2}.$$

For $\text{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}, \sigma \otimes 1}(\text{gr}^\bullet M, M)^+$, we first note that the map

$$\bigoplus_i \Phi_\tau^i : \text{gr}^\bullet M_\tau \rightarrow M_{\tau \circ \sigma^{-1}}$$

is an isomorphism for every $\tau \in \Sigma_w$. It follows from (E.4) that $\varphi_{\tau^c} := \bigoplus_i \varphi_\tau^i$ is determined by $\varphi_\tau := \bigoplus_i \varphi_\tau^i$. On the other hand, for each fixed τ , all the possible choices of $\varphi_\tau : \text{gr}^\bullet M_\tau \rightarrow M_{\tau \circ \sigma^{-1}}$ form a k -vector space of dimension N^2 . Thus, we have

$$\dim_k \text{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathcal{O}, \sigma \otimes 1}(\text{gr}^\bullet M, M)^+ = [F_v^+ : \mathbb{Q}_\ell] \cdot N^2.$$

The Lemma follows immediately. \square

Proposition E.3.9. *Suppose $\ell > N$ and assume Assumption E.3.5. The local deformation problem \mathcal{D}^{FL} is formally smooth over $\text{Spf } \mathcal{O}$ of pure relative dimension $N^2 + [F_v^+ : \mathbb{Q}_\ell] \cdot \frac{N(N-1)}{2}$.*

Proof. By Lemma E.3.8, it suffices to show that \mathcal{D}^{FL} is formally smooth over \mathcal{O} . We may again assume $E_\lambda = E'_\lambda$. When v is split in F/F^+ , the proposition has been proved in [CHT08, Lemma 2.4.1].

Now we suppose that v is inert in F . Fix a subset $\Sigma_w^+ \subset \Sigma_w$ so that $\Sigma_w = \Sigma_w^+ \amalg \Sigma_w^{+,c}$. Let R be an object of $\mathcal{C}_\mathcal{O}^f$ and $I \subset R$ an ideal satisfying $\mathfrak{m}_R I = (0)$. Let r be a lifting of \bar{r} to R/I , and put

$M := \mathbf{G}_w^{-1}(r^{\natural})$. Note that M is an object of $\mathcal{MF}_{\mathcal{O},w}^{[0,a]}$, and that for every $\tau \in \Sigma_w$, M_τ is free of rank N over R/I and $\text{Fil}^i M_\tau$ is a R/I -direct summand of M_τ .

Recall the functor $\mathbf{-}^c$ in Construction E.3.3. Then $\mathbf{G}_w(M^c)$ is isomorphic to $\bar{r}^{\natural,c}|_{\Gamma_{F_w}}$. Since $\bar{r}^{\natural,c}$ and $\bar{r}^{\natural,\vee}(-a)$ are conjugate, we have $M \simeq \mathbf{D}^{[0,a]}(M^c)$. We fix such an isomorphism $M \simeq \mathbf{D}^{[0,a]}(M^c)$ hence obtain a pairing $\langle \cdot, \cdot \rangle$ (with $R = R/I$) as in Construction E.3.3. In view of Assumption E.3.5(2), let $m_{\tau,1} < \dots < m_{\tau,N}$ be the τ -Hodge–Tate weights of M for every $\tau \in \Sigma_w$. Then there exists a basis $e_{\tau,1}, \dots, e_{\tau,N}$ of M_τ over R/I satisfying $\text{Fil}^{m_{\tau,N+1-i}} M_\tau = \bigoplus_{j=1}^i (R/I)e_{\tau,j}$ for every $1 \leq i \leq N$. By duality, we have $m_{\tau^c,i} + m_{\tau,N+1-i} = a$. Then we may choose the basis $(e_{\tau,i})$ such that $\langle e_{\tau^c,i}, e_{\tau,j} \rangle_\tau = \delta_{i,N+1-j}$ for every $\tau \in \Sigma_w^+$ and every $1 \leq i, j \leq N$.

We now define an object $\widetilde{M} = \bigoplus_{\tau \in \Sigma_w} \widetilde{M}_\tau$ of $\mathcal{MF}_{\mathcal{O},w}$ that reduces to M , whose underlying $O_{F_w} \otimes \mathcal{O}$ -module is free over $O_{F_w} \otimes R$, and an isomorphism $\widetilde{M} \simeq \mathbf{D}^{[0,a]}(\widetilde{M}^c)$ that reduces to the previous isomorphism $M \simeq \mathbf{D}^{[0,a]}(M^c)$, as follows. As an R -module, we take $\widetilde{M}_\tau = R^{\oplus N}$ with the basis $(\tilde{e}_{\tau,i})$ that lifts the basis $(e_{\tau,i})$ of M_τ . We lift $\langle \cdot, \cdot \rangle$ to an R -bilinear perfect pairing $\widetilde{M}_{\tau^c} \times \widetilde{M}_\tau \rightarrow R$ such that $\langle \tilde{e}_{\tau^c,i}, \tilde{e}_{\tau,j} \rangle_\tau = \delta_{i,N+1-j}$ still holds for every $\tau \in \Sigma_w^+$ and every $1 \leq i, j \leq N$. For the filtration, we put $\text{Fil}^m \widetilde{M}_\tau := \bigoplus_{j=1}^i R\tilde{e}_{\tau,j}$ for m satisfying $m_{\tau,N-i} < m \leq m_{\tau,N+1-i}$. Then $\widetilde{M} \otimes_R R/I$ is isomorphic to M as filtered $O_{F_w} \otimes R/I$ -modules; and the condition (2) in Construction E.3.3 holds for \widetilde{M} as well. For the Frobenius structure on \widetilde{M} , we first define maps $\tilde{\Phi}_\tau^{m_{\tau,i}} : \text{Fil}^{m_{\tau,i}} \widetilde{M}_\tau \rightarrow \widetilde{M}_{\tau \circ \sigma^{-1}}$ for $\tau \in \Sigma_w^+$ by the recursive induction on i . For $i = N$, we take $\tilde{\Phi}_\tau^{m_{\tau,N}}$ to be an arbitrary lift of $\Phi_\tau^{m_{\tau,N}} : \text{Fil}^{m_{\tau,N}} M_\tau \rightarrow M_{\tau \circ \sigma^{-1}}$ for $\tau \in \Sigma_w^+$. For $i \leq N - 1$, we take $\tilde{\Phi}_\tau^{m_{\tau,i}}$ to be a lift of $\Phi_\tau^{m_{\tau,i}} : \text{Fil}^{m_{\tau,i}} M_\tau \rightarrow M_{\tau \circ \sigma^{-1}}$ that restricts to $\ell^{m_{\tau,i} - m_{\tau,i+1}} \tilde{\Phi}_\tau^{m_{\tau,i+1}}$ on $\text{Fil}^{m_{\tau,i+1}} \widetilde{M}_\tau$. By Nakayama's lemma, we have

$$\widetilde{M}_{\tau \circ \sigma^{-1}} = \sum_i \tilde{\Phi}_\tau^{m_{\tau,i}}(\text{Fil}^{m_{\tau,i}} \widetilde{M}_\tau)$$

for every $\tau \in \Sigma_w^+$. Finally, we define $\tilde{\Phi}_{\tau^c}^i : \widetilde{M}_{\tau^c} \rightarrow \widetilde{M}_{\tau^c \circ \sigma^{-1}}$ for $\tau \in \Sigma_w^+$ to be the unique R -linear map satisfying the condition (1) in Construction E.3.3 for \widetilde{M} . This finishes the construction of \widetilde{M} and the isomorphism $\widetilde{M} \simeq \mathbf{D}^{[0,a]}(\widetilde{M}^c)$, which together give rise to a lifting \tilde{r} of \bar{r} to R that reduces to r . Thus, \mathcal{D}^{FL} is formally smooth over \mathcal{O} .

The proposition is proved. \square

E.4. Representations of the tame group. In this subsection, we will study conjugate self-dual representations of the tame group, and define the notion of minimally ramified deformations of such representations.

Definition E.4.1. Let $q \geq 1$ be a positive integer coprime to ℓ . We define the q -tame group, denoted by T_q , to be the semidirect product topological group $t^{\mathbb{Z}\ell} \rtimes \phi_q^{\widehat{\mathbb{Z}}}$ where ϕ_q maps t to t^q , that is, $\phi_q t \phi_q^{-1} = t^q$. For every integer $b \geq 1$, We identify T_{q^a} as a subgroup of T_q topologically generated by t and $\phi_{q^b} = \phi_q^b$.

We consider a reductive group G over \mathcal{O} , together with a surjective homomorphism $\nu : G \rightarrow H$ over \mathcal{O} , where H is an algebraic group over \mathcal{O} of multiplicative type. Consider a pair $(\bar{\varrho}, \mu)$ in which $\bar{\varrho} : T_q \rightarrow G(k)$ and $\mu : T_q \rightarrow H(\mathcal{O})$ are continuous homomorphisms satisfying $\nu \circ \bar{\varrho} = \bar{\mu}$ and $\mu(t) = 1$. Similar to the case in Subsection E.2, let $\mathbf{R}_\varrho^{\text{loc}}$ be the \mathcal{O} -algebra in $\mathcal{C}_\mathcal{O}$ that parameterizes liftings ϱ of $\bar{\varrho}$ satisfying $\nu \circ \varrho = \mu$.³² The following proposition generalizes the tame case of [Sho18, Theorem 2.5].

³²Here, once again we omit the similitude character μ in the ring $\mathbf{R}_\varrho^{\text{loc}}$, in order to be consistent with the previous convention.

Proposition E.4.2. *The ring $R_{\mathcal{O}}^{\text{loc}}$ is a local complete intersection, flat and of pure relative dimension d over \mathcal{O} , where d is the relative dimension of the kernel of ν over \mathcal{O} .*

Proof. We follow the same line as in the proof of [Sho18, Theorem 2.5]. Let G_0 and G_1 be the fibers at 1 and $\mu(\phi_q)$ of the homomorphism ν , respectively. Define the subscheme $\mathcal{M}(G, q)$ of $G_0 \times_{\text{Spec } \mathcal{O}} G_1$ such that for every object R of $\mathcal{C}_{\mathcal{O}}$, $\mathcal{M}(G, q)(R)$ consists of pairs $(\mathbf{A}, \mathbf{B}) \in G_0(R) \times G_1(R)$ satisfying

$$(E.5) \quad \mathbf{B} \mathbf{A} \mathbf{B}^{-1} = \mathbf{A}^q.$$

It suffices to show that $\mathcal{M}(G, q)$ is a local complete intersection, flat and of pure relative dimension d over \mathcal{O} , since $R_{\mathcal{O}}^{\text{loc}}$ is the completion of $\mathcal{M}(G, q)$ at the k -point $(\bar{\varrho}(t), \bar{\varrho}(\phi_q))$.

First, we show that every geometric fiber of $\mathcal{M}(G, q) \rightarrow \text{Spec } \mathcal{O}$ is of pure dimension d . Consider the natural projection

$$p: \mathcal{M}(G, q) \rightarrow G_0$$

to the first factor. Take a geometric point $\text{Spec } K \rightarrow \text{Spec } \mathcal{O}$. For a point $\mathbf{A}_0 \in G_0(K)$ in the image of $p(K)$, let $Z(\mathbf{A}_0)$ be the centralizer of \mathbf{A}_0 in $G_{0,K}$ as a closed subscheme of $G_{0,K}$, and $C(\mathbf{A}_0)$ the conjugacy class of \mathbf{A}_0 , which is a locally closed subscheme of $G_{0,K}$ isomorphic to $G_{0,K}/Z(\mathbf{A}_0)$. Then $C(\mathbf{A}_0)$ lies in the image of p_K . For every point $\mathbf{A} \in C(\mathbf{A}_0)(K)$, the fiber $p_K^{-1}(\mathbf{A})$ is a torsor under the group $Z(\mathbf{A})$, which is conjugate to $Z(\mathbf{A}_0)$. Thus, $p_K^{-1}(C(\mathbf{A}_0))$ is irreducible of dimension

$$\dim p_K^{-1}(C(\mathbf{A}_0)) = \dim C(\mathbf{A}_0) + \dim Z(\mathbf{A}_0) = \dim G_{0,K} = d.$$

To continue, we choose an embedding $e: G_K \rightarrow \text{GL}_{m,K}$ of algebraic groups over K for some integer $m \geq 1$. By (E.5), the image of $e(K) \circ p(K)$ consists only of matrices whose generalized eigenvalues are $(q^{m^l} - 1)$ -th roots of unity, hence finitely many conjugacy classes in $\text{GL}_m(K)$. We claim that the image of $p(K)$ consists of finitely many conjugacy classes in $G_0(K)$ as well, which implies that $\mathcal{M}(G, q)_K$ is of pure dimension d . In fact, we have the following commutative diagram

$$\begin{array}{ccc} G_0(K) // G_0(K) & \longrightarrow & \text{GL}_m(K) // \text{GL}_m(K) \\ \downarrow & & \downarrow \\ (G_{0,K} // G_{0,K})(K) & \longrightarrow & (\text{GL}_{m,K} // \text{GL}_{m,K})(K) \end{array}$$

of sets, in which the bottom map is finite since the morphism $G_{0,K} // G_{0,K} \rightarrow \text{GL}_{m,K} // \text{GL}_{m,K}$ is; and the left map is also finite due to the finiteness of unipotent conjugacy classes [Lus76]; it follows that the upper map is finite as well.

The above discussion shows that the morphism $\mathcal{M}(G, q) \rightarrow \text{Spec } \mathcal{O}$ is of pure relative dimension d . Now we take a closed point $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ of $\mathcal{M}(G, q)$, which induces a homomorphism

$$\mathcal{O}_{\mathcal{M}(G,q),(\bar{\mathbf{A}},\bar{\mathbf{B}})} \rightarrow \mathcal{O}_{G_0,\bar{\mathbf{A}}} \hat{\otimes}_{\mathcal{O}} \mathcal{O}_{G_1,\bar{\mathbf{B}}}$$

of corresponding complete local rings. As both G_0 and G_1 are smooth over \mathcal{O} of pure relative dimension d , both $\mathcal{O}_{G_0,\bar{\mathbf{A}}}$ and $\mathcal{O}_{G_1,\bar{\mathbf{B}}}$ are power series rings over \mathcal{O} in d variables. The relation (E.5), or equivalently, the relation $\mathbf{A} = \mathbf{B}^{-1} \mathbf{A}^q \mathbf{B}$, is defined by d equations in $\mathcal{O}_{G_0,\bar{\mathbf{A}}} \hat{\otimes}_{\mathcal{O}} \mathcal{O}_{G_1,\bar{\mathbf{B}}}$. In other words, $\mathcal{M}(G, q)$ is a local complete intersection hence Cohen-Macaulay. Therefore, $\mathcal{M}(G, q)$ is flat over \mathcal{O} . The proposition is proved. \square

Take an integer $n \geq 1$. Now we apply the above discussion to the homomorphism $\nu: \mathcal{G}_n \rightarrow \text{GL}_1$ in Notation E.1.1. Consider a pair $(\bar{\varrho}, \mu)$ from Notation E.2.1 with $\tilde{\Gamma} = \text{T}_q$ and $\Gamma = \text{T}_{q^2}$, such that $\mu(t) = 1$. In particular, $\bar{\varrho}: \text{T}_q \rightarrow \mathcal{G}_n(k)$ is a homomorphism and $\mu: \text{T}_q \rightarrow \mathcal{O}^\times$ is a (continuous) similitude character. Write

$$(E.6) \quad \bar{\varrho}(t) = \bar{\mathbf{A}} = (\bar{A}, 1, 1), \quad \bar{\varrho}(\phi_q) = \bar{\mathbf{B}} = (\bar{B}, -\mu(\phi_q), \mathbf{c})$$

for $\bar{A}, \bar{B} \in \mathrm{GL}_n(k)$. For a lifting ϱ of $\bar{\varrho}$ to an object R of $\mathcal{C}_\mathcal{O}$, we write $\varrho(t) = \mathbf{A} = (A, 1, 1)$ and $\varrho(\phi_q) = \mathbf{B} = (B, -\mu(\phi_q), \mathbf{c})$. Then the pair (A, B) reduce to (\bar{A}, \bar{B}) , and satisfy the relation

$$(E.7) \quad B^t A^{-1} B^{-1} = A^q.$$

Corollary E.4.3. *The ring $\mathbb{R}_{\bar{\varrho}}^{\mathrm{loc}}$ is a local complete intersection, flat and of pure relative dimension n^2 over \mathcal{O} .*

Proof. This follows immediately from Proposition E.4.2 since the kernel of $\nu: \mathcal{G}_n \rightarrow \mathrm{GL}_1$ is of dimension n^2 . \square

From now till the end of this subsection, we assume $l \geq n$. Denote by \mathcal{N}_n (resp. \mathcal{U}_n) the closed subscheme of M_n (resp. GL_n) defined by the equation $X^n = 0$ (resp. $(A - 1)^n = 0$). For every object R of $\mathcal{C}_\mathcal{O}$, we have the *truncated exponential map* $\exp: \mathcal{N}_n(R) \rightarrow \mathcal{U}_n(R)$ defined by the formula

$$\exp X = 1 + X + \cdots + \frac{X^{n-1}}{(n-1)!},$$

which is an bijection. Its inverse is given by the *truncated logarithm map* $\log: \mathcal{U}_n(R) \rightarrow \mathcal{N}_n(R)$ defined by the formula

$$\log A = \sum_{i=1}^{n-1} (-1)^{i-1} \frac{(A-1)^i}{i}.$$

Let \mathfrak{P}_n be the set of partitions of n . By the classification of nilpotent orbits in GL_n , for $K = k, E_\lambda$, we have canonical surjective maps $\pi: \mathcal{N}_n(K) \rightarrow \mathfrak{P}_n$ such that the fibers of π are exactly the orbits in $\mathcal{N}_n(K)$ under the conjugate action of $\mathrm{GL}_n(K)$.

By the continuity of $\bar{\varrho}$, we know that \bar{A} in (E.6) is unipotent, which implies $\bar{A} \in \mathcal{U}_n(k)$. Put $\bar{X} := \log \bar{A} \in \mathcal{N}_n(k)$. Following [Boo19, Definition 3.9], we define the functor $\mathrm{Nil}_{\bar{X}}: \mathcal{C}_\mathcal{O} \rightarrow \mathbf{Set}$ that sends an object R of $\mathcal{C}_\mathcal{O}$ to the set of elements $X \in \mathcal{N}_n(R)$ that reduce to \bar{X} and are of the form CX_0C^{-1} , where X_0 is an element in $\mathcal{N}_n(\mathcal{O})$ satisfying $\pi(X_0) = \pi(\bar{X})$ and $C \in \mathrm{GL}_n(R)$, where we regard X_0 as an element in $\mathcal{N}_n(E_\lambda)$ in the notation $\pi(X_0)$.

Definition E.4.4. We say that a lifting ϱ of $\bar{\varrho}$ to an object R of $\mathcal{C}_\mathcal{O}$ is *minimally ramified* if there exists an element $X \in \mathrm{Nil}_{\bar{X}}(R)$ such that $\varrho^{\natural}(t) = \exp X$.

Let $\mathcal{D}_{\bar{\varrho}}^{\mathrm{min}}$ be the local deformation problem of $\bar{\varrho}$ (Definition E.2.3) that classifies minimally ramified liftings of $\bar{\varrho}$.

Proposition E.4.5. *The local deformation problem $\mathcal{D}_{\bar{\varrho}}^{\mathrm{min}}$ is formally smooth over $\mathrm{Spf} \mathcal{O}$ of pure relative dimension n^2 .*

Proof. We follow the approach of [Boo19, Proposition 5.6], where a similar result for symplectic or orthogonal representations was proved.

Consider the morphism $\alpha: \mathcal{D}_{\bar{\varrho}}^{\mathrm{min}} \rightarrow \mathrm{Nil}_{\bar{X}}$ that sends a lifting ϱ to $\log A$ if $\varrho^{\natural}(t) = A$. In the definition of $\mathrm{Nil}_{\bar{X}}$, we may fix the nilpotent element $X_0 \in \mathcal{N}_n(\mathcal{O})$. Moreover, up to conjugation in $\mathrm{GL}_n(\mathcal{O})$, we may assume

$$X_0 = \begin{pmatrix} J_{n_1} & & \\ & \ddots & \\ & & J_{n_r} \end{pmatrix},$$

where $n = n_1 + \cdots + n_r$, and J_{n_i} is the Jordan block of size n_i -by- n_i as in Subsection 1.3. Let $Z_n(X_0)$ be the centralizer of X_0 in $\mathrm{GL}_{n,\mathcal{O}}$, which is a closed subscheme of $\mathrm{GL}_{n,\mathcal{O}}$. By [Boo19, Remark 4.18], $Z_n(X_0)$ is smooth over \mathcal{O} . By [Boo19, Lemma 3.11], $\mathrm{Nil}_{\bar{X}}$ is represented by a formal power series ring over \mathcal{O} in $n^2 - \dim_{\mathcal{O}} Z_n(X_0)$ variables, where $\dim_{\mathcal{O}} Z_n(X_0)$ denotes the relative dimension

of $Z_n(X_0)$ over \mathcal{O} . Thus, it suffices to show that α is represented by a formal scheme formally smooth of pure relative dimension $\dim_{\mathcal{O}} Z_n(X_0)$ over $\text{Nil}_{\bar{X}}$.

Take a lifting ϱ of $\bar{\varrho}$ to an object R of $\mathcal{C}_{\mathcal{O}}$. Then $\varrho(\phi_q)$ has the form $(B, -\mu(\phi_q), \mathbf{c})$ with $B \in \text{GL}_n(R)$ that reduces to \bar{B} and satisfies

$$(E.8) \quad B {}^t X B^{-1} = -qX$$

by (E.7). For each given $X \in \text{Nil}_{\bar{X}}$, if there exists $B \in \text{GL}_n(R)$ that reduces to \bar{B} and satisfies (E.8), then the set of all elements B form a torsor under the group

$$\widehat{Z}_n(X)(R) := \{g \in 1 + M_n(\mathfrak{m}_R) \mid gXg^{-1} = X\},$$

which is isomorphic to the group of R -valued points of the formal completion of the group scheme $Z_n(X_0)$ along the unit section. Thus, to finish the proof, it suffices to show that the equation (E.8) admits at least one solution for B that reduces to \bar{B} .

Assume first $X = X_0$ in $\mathcal{N}_n(R)$. Then

$$B_0 := \begin{pmatrix} A_{n_1} & & & \\ & \ddots & & \\ & & A_{n_r} & \\ & & & \end{pmatrix}, \text{ where } A_{n_i} := \begin{pmatrix} & & & (-q)^{n_i-1} \\ & & \ddots & \\ & & & -q \\ 1 & & & \end{pmatrix},$$

is a solution to (E.8). In the general case, we write $X = CX_0C^{-1}$ for some $C \in \text{GL}_n(R)$. Then $B := CB_0 {}^t C$ satisfies the equation (E.8). Up to multiplying C by an element in $Z_n(X_0)(R)$ from the right, we can make $B \in \text{GL}_n(R)$ to reduce to \bar{B} . This finishes the proof of the proposition. \square

Recall from Definition E.2.4 that $L(\mathcal{D}_{\bar{\varrho}}^{\min}) \subseteq H^1(T_q, \text{ad } \bar{\varrho})$ is tangent space of the local deformation problem $\mathcal{D}_{\bar{\varrho}}^{\min}$.

Corollary E.4.6. *We have $\dim_k L(\mathcal{D}_{\bar{\varrho}}^{\min}) = \dim_k H^0(T_q, \text{ad } \bar{\varrho})$.*

Proof. Suppose $\mathcal{D}_{\bar{\varrho}}^{\min} = \text{Spf } \mathbf{R}_{\bar{\varrho}}^{\min}$. By (E.1), we have

$$\dim_k \mathfrak{m}_{\mathbf{R}_{\bar{\varrho}}^{\min}} / (\lambda, \mathfrak{m}_{\mathbf{R}_{\bar{\varrho}}^{\min}}^2) = \dim_k L(\mathcal{D}_{\bar{\varrho}}^{\min}) + n^2 - \dim_k H^0(T_q, \text{ad } \bar{\varrho}).$$

From this, the corollary follows immediately from Proposition E.4.5. \square

To end this subsection, we record the following lemma concerning decomposition of representations of the q -tame group, in which part (1) will be used later and part (2) is only for complement.

Lemma E.4.7. *Let $(\bar{\rho}, \bar{M})$ be an unramified representation of $T_q = t^{\mathbb{Z}_\ell} \times \phi_q^{\widehat{\mathbb{Z}}}$ over k of dimension N . Suppose that \bar{M} admits a decomposition*

$$\bar{M} = \bar{M}_1 \oplus \cdots \oplus \bar{M}_s$$

stable under the action of $\bar{\rho}(\phi_q)$ such that the characteristic polynomials of $\bar{\rho}(\phi_q)$ on M_i are mutually coprime for $1 \leq i \leq s$. Let (ρ, M) be a lifting of $(\bar{\rho}, \bar{M})$ to an object R of $\mathcal{C}_{\mathcal{O}}$. Then we have

(1) *There is a unique decomposition*

$$M = M_1 \oplus \cdots \oplus M_s$$

of free R -modules, such that M_i is stable under the action of $\rho(\phi_q)$ and it is a lifting of \bar{M}_i as a ϕ_q -module.

(2) *Write $\rho(t) = (\rho(t)_{i,j})$ with $\rho(t)_{i,j} \in \text{Hom}_R(M_j, M_i)$. Suppose that q is not an eigenvalue for the canonical action of ϕ_q on $\text{Hom}_k(\bar{M}_j, \bar{M}_i)$ for all $i \neq j$. Then we have $\rho(t)_{i,j} = 0$ for all $i \neq j$; in other words, the decomposition in (1) is stable under the whole group T_q .*

Proof. For (1), let $P(T) \in R[T]$ be the characteristic polynomial of $\rho(\phi_q)$ on M . By Hensel's lemma, $P(T)$ admits a unique decomposition

$$P(T) = \prod_{i=1}^s P_i(T)$$

such that $P_i(T) \bmod \mathfrak{m}_R$ is the characteristic polynomial of $\bar{\rho}(\phi_q)$ on \bar{M}_i for $1 \leq i \leq s$. We put $Q_i(T) = \prod_{j \neq i} P_j(T)$ for $1 \leq i \leq s$. We put $M_i := Q_i(\phi_q)M$ and $N_i := P_i(\phi_q)M$. Then both M_i and N_i are both stable under ϕ_q ; M_i is annihilated by $P_i(\phi_q)$; N_i is annihilated by $Q_i(\phi_q)$; hence $M_i \cap N_i = \{0\}$. Using Nakayama's lemma, it is easy to see that $(P_i(T), Q_i(T)) = R[T]$. Thus, there exist polynomials $F_i, G_i \in R[T]$ such that $F_i P_i + G_i Q_i = 1$ in $R[T]$. We then obtain

$$F_i(\phi_q)N_i + G_i(\phi_q)M_i = M,$$

hence $M = M_i \oplus N_i$. To complete the proof of (1), it suffices to show that $N_i = \bigoplus_{j \neq i} M_j$. By definition, it is clear that $M_j \subseteq N_i$ for every $j \neq i$, hence $\bigoplus_{j \neq i} M_j \subseteq N_i$. The inverse inclusion follows from the fact that the ideal of $R[T]$ generated by Q_j for $j \neq i$ is same as the ideal generated by P_i .

For (2), we choose a basis of M over R adapted to the decomposition of M in (1). We identify $\rho(t)$ and $\rho(\phi_q)$ with their matrices under this basis. We have $\rho(\phi_q)_{i,j} = 0$ for $i \neq j$ since each M_i is stable under $\rho(\phi_q)$. Let $J \subset R$ be the ideal generated by the coefficients of $\rho(t)_{i,j}$ for $i \neq j$. We have to show that $J = 0$. By Nakayama's lemma, it suffices to show that $J = \mathfrak{m}_R J$. As

$$\rho(t)^q = (1 + (\rho(t) - 1))^q = 1 + q(\rho(t) - 1) + \sum_{a \geq 2} \binom{q}{a} (\rho(t) - 1)^a,$$

and $\rho(t) \equiv 1 \pmod{\mathfrak{m}_R}$, we have

$$(\rho(t)^q)_{i,j} \equiv q\rho(t)_{i,j} \pmod{\mathfrak{m}_R J}$$

for $i \neq j$. The relation $\phi_q t = t^q \phi_q$ implies that

$$\rho(\phi_q)_{i,i} \rho(t)_{i,j} = (\rho(t)^q)_{i,j} \rho(\phi_q)_{j,j} \equiv q\rho(t)_{i,j} \rho(\phi_q)_{j,j} \pmod{\mathfrak{m}_R J}.$$

It follows that

$$\rho(t)_{i,j} P_i(q\rho(\phi_q)_{j,j}) \equiv P_i(\rho(\phi_q)_{i,i}) \rho(t)_{i,j} \equiv 0 \pmod{\mathfrak{m}_R J}$$

for $i \neq j$. By assumption, if $\bar{\alpha}$ is an eigenvalue of $\bar{\rho}(\phi_q)_{i,i}$, then $q^{-1}\bar{\alpha}$ is not an eigenvalue of $\bar{\rho}(\phi_q)_{j,j}$. It follows that $P_i(q\rho(\phi_q)_{j,j})$ is invertible, hence $\rho(t)_{i,j} \equiv 0 \pmod{\mathfrak{m}_R J}$.

The lemma is proved. \square

E.5. Minimally ramified deformations. In this subsection, we define and study the minimally ramified deformations at places coprime to ℓ . Thus, we take a nonarchimedean place v of F^+ that is not above ℓ .

When v is split in F , the problem has been studied in [CHT08, 2.4.4]. So we assume that v is nonsplit in F . Let w be the unique prime of F above v . Let $I_{F_v^+} \subseteq \Gamma_{F_v^+}$ be the inertia subgroup, and $P_{F_v^+}$ the maximal closed subgroup of $I_{F_v^+}$ of pro-order coprime to ℓ . Put $T_v := \Gamma_{F_v^+}/P_{F_v^+}$. Similarly, we have Γ_{F_w} , I_{F_w} , P_{F_w} , and T_w . Finally, put $T_w^+ := \Gamma_{F_v^+}/P_{F_w}$.

Remark E.5.1. The group T_v is a $\|v\|$ -tame group (Definition E.4.1). When w is unramified over v , we have $P_{F_w} = P_{F_v^+}$, $T_w^+ = T_v$, and that the subgroup T_w of T_v is a $\|v\|^2$ -tame group. When w is ramified over v , we have $P_{F_v^+}/P_{F_w} \simeq \mathbb{Z}/2\mathbb{Z}$, that the natural map $T_w \rightarrow T_v$ is an isomorphism, and a canonically split short exact sequence

$$1 \rightarrow P_{F_v^+}/P_{F_w} \rightarrow T_w^+ \rightarrow T_v \rightarrow 1.$$

We recall some facts about extensions of representations of P_{F_w} from [CHT08]. For an irreducible representation τ of P_{F_w} with coefficients in k , we put

$$\Gamma_\tau := \{\sigma \in \Gamma_{F_w} \mid \tau^\sigma \simeq \tau\},$$

where we recall from Subsection 1.3 that τ^σ denotes the representation given by $\tau^\sigma(g) = \tau(\sigma g \sigma^{-1})$ for $g \in P_{F_w}$. Let T_τ be the image of Γ_τ in $T_w = \Gamma_{F_w}/P_{F_w}$. As P_{F_w} is normal in $\Gamma_{F_v^+}$, we may similarly define

$$\Gamma_\tau^+ := \{\sigma \in \Gamma_{F_v^+} \mid \tau^\sigma \simeq \tau\},$$

and denote by T_τ^+ its image in T_w^+ .

Lemma E.5.2. *We have the following properties for τ :*

- (1) *the dimension of τ is coprime to ℓ ; and τ has a unique deformation to a representation $\tilde{\tau}$ of P_{F_w} over \mathcal{O} ;*
- (2) *$\tilde{\tau}$ in (1) admits a unique extension to a representation of $\Gamma_\tau \cap I_{F_w}$ over \mathcal{O} whose determinant has order coprime to ℓ ;*
- (3) *there exists an extension of $\tilde{\tau}$ in (2) to a representation of Γ_τ over \mathcal{O} .*

Proof. This is [CHT08, Lemma 2.4.11]. □

Now we consider a pair (\bar{r}, χ) from Notation E.2.1 with $\tilde{\Gamma} = \Gamma_{F_v^+}$ and $\Gamma = \Gamma_{F_v^+} \cap \Gamma_F = \Gamma_{F_w}$. Our first goal is to define the notion of *minimally ramified liftings* of \bar{r} (Definition E.5.8).

Recall from Notation E.1.2 that we have the induced homomorphism $\bar{r}^\natural: \Gamma_{F_w} \rightarrow \mathrm{GL}_N(k)$. For an irreducible representation τ of P_{F_w} with coefficients in k , we put

$$M_\tau(\bar{r}) := \mathrm{Hom}_{k[P_{F_w}]}(\tau, \bar{r}^\natural).$$

Then $\tau \otimes_k M_\tau(\bar{r})$ is canonically the τ -isotypic component of \bar{r}^\natural . As τ extends to a representation of Γ_τ , the k -vector space $M_\tau(\bar{r})$ is equipped with a natural action by T_τ ; and $\tau \otimes_k M_\tau(\bar{r})$ is equipped with a natural action by Γ_τ .

We denote by $\mathfrak{T} = \mathfrak{T}(\bar{r})$ the set of isomorphism classes of irreducible representations τ of P_{F_w} such that $M_\tau(\bar{r}) \neq 0$. Then Γ_{F_w} acts on \mathfrak{T} by conjugation, whose orbits we denote by $\mathfrak{T}/\Gamma_{F_w}$. For $\tau \in \mathfrak{T}$, we write $[\tau]$ for its orbit in $\mathfrak{T}/\Gamma_{F_w}$.

Definition E.5.3. We say that E_λ is \bar{r} -*inclusive* if every $\tau \in \mathfrak{T}(\bar{r})$ is absolutely irreducible.

We first suppose that E_λ is \bar{r} -inclusive. Let \tilde{E}_λ be an unramified quadratic extension of E_λ , with the ring of integers $\tilde{\mathcal{O}}$ and the residue field \tilde{k} .

Choose an element $\gamma \in \Gamma_{F_v^+} \setminus \Gamma_{F_w}$. By Lemma E.1.3, the homomorphism \bar{r} is determined by an element $\bar{\Psi} \in \mathrm{GL}_N(k)$ satisfying

$$\bar{r}^{\natural, \gamma} = \bar{\Psi} \circ \chi \bar{r}^{\natural, \vee} \circ \bar{\Psi}^{-1}, \quad \bar{\Psi}^t \bar{\Psi}^{-1} = -\chi(\gamma)^{-1} \bar{r}^\natural(\gamma^2).$$

In what follows, we will adopt the following simplified notation: for a representation τ of a subgroup of $\Gamma_{F_v^+}$, we write τ^* for $\chi \tau^\vee$. Due to the existence of $\bar{\Psi}$, we know that if $\tau \in \mathfrak{T}$, then $\tau^{\gamma, *}$ $\in \mathfrak{T}$ as well. As $\gamma^2 \in \Gamma_{F_w}$, the assignment $\tau \mapsto \tau^{\gamma, *}$ induces an involution on the set $\mathfrak{T}/\Gamma_{F_w}$, which does not depend on the choice of γ .

Construction E.5.4. We now would like to construct a Γ_{F_w} -stable partition $\mathfrak{T} = \mathfrak{T}_1 \sqcup \mathfrak{T}_2 \sqcup \mathfrak{T}_3$. For each subset \mathfrak{T}_i , we will specify, for every $\tau \in \mathfrak{T}_i$, an extension of $\tilde{\tau}$ in Lemma E.5.2(2) to a representation of Γ_τ with coefficients in \mathcal{O} ($\tilde{\mathcal{O}}$ if $i = 3$) in a compatible way, specified below.

We start from the following observation. Suppose $[\tau] = [\tau^{\gamma, *}]$ in $\mathfrak{T}/\Gamma_{F_w}$. Then there exists an element $h \in \Gamma_{F_w}$, unique up to left multiplication by an element in Γ_τ , such that $\tau^{\gamma, *} \simeq \tau^{h^{-1}}$, or equivalently, $\tau^{h\gamma} \simeq \tau^*$. Then we have $(h\gamma)^2 \in \Gamma_\tau$ but $h\gamma \notin \Gamma_\tau$. Denote by $\tilde{\Gamma}_\tau$ the subgroup of $\Gamma_{F_v^+}$

generated by Γ_τ and $h\gamma$, which contains Γ_τ as a subgroup of index two. Let $\tilde{\Gamma}_\tau$ be the image of Γ_τ in T_w^+ , which contains T_τ as a subgroup of index two.

- (1) We define \mathfrak{T}_1 to be the subset of \mathfrak{T} consisting of τ such that $[\tau] \neq [\tau^{\gamma,*}]$. We choose a subset $\mathfrak{T}_1^\heartsuit \subseteq \mathfrak{T}_1$ such that $\{\tau, \tau^{\gamma,*} \mid \tau \in \mathfrak{T}_1^\heartsuit\}$ is a set of representatives for the Γ_τ -action on \mathfrak{T}_1 . For each element $\tau \in \mathfrak{T}_1^\heartsuit$, we choose an extension of $\tilde{\tau}$ in Lemma E.5.2(2) to a representation of Γ_τ with coefficients in \mathcal{O} , which we still denote by $\tilde{\tau}$. For a general element $\tau \in \mathfrak{T}_1$, there are two cases. If $\tau \simeq \tau_1^h$ for (unique) $\tau_1 \in \mathfrak{T}_1^\heartsuit$ and some $h \in \Gamma_{F_w}$, then we choose $\tilde{\tau}$ to be $\tilde{\tau}_1^h$, as the extension to $\Gamma_\tau = h^{-1}\Gamma_{\tau_1}h$. If $\tau \simeq (\tau_1^h)^{\gamma,*}$ for (unique) $\tau_1 \in \mathfrak{T}_1^\heartsuit$ and some $h \in \Gamma_{F_w}$, then we choose $\tilde{\tau}$ to be $(\tilde{\tau}_1^h)^{\gamma,*}$, as the extension to $\Gamma_\tau = \gamma^{-1}h^{-1}\Gamma_{\tau_1}h\gamma$.
- (2) We define \mathfrak{T}_2 to be the subset of \mathfrak{T} consisting of τ such that $[\tau] = [\tau^{\gamma,*}]$, and that the images of Γ_τ and $\tilde{\Gamma}_\tau$ in $\Gamma_{F_v^+}/I_{F_v^+}$ are different. We choose a subset $\mathfrak{T}_2^\heartsuit \subseteq \mathfrak{T}_2$ of representatives for the Γ_τ -action on \mathfrak{T}_2 . For each element $\tau \in \mathfrak{T}_2^\heartsuit$, we choose an extension $\tilde{\tau}$ from Lemma E.5.6(1) below to a representation of Γ_τ with coefficients in \mathcal{O} . For $\tau \in \mathfrak{T}_2$ in general, we have $\tau \simeq \tau_2^h$ for (unique) $\tau_2 \in \mathfrak{T}_2^\heartsuit$ and some $h \in \Gamma_{F_w}$; and we choose $\tilde{\tau}$ to be $\tilde{\tau}_2^h$, as the extension to $\Gamma_\tau = h^{-1}\Gamma_{\tau_2}h$.
- (3) We define \mathfrak{T}_3 to be the subset of \mathfrak{T} consisting of τ such that $[\tau] = [\tau^{\gamma,*}]$, and that the images of Γ_τ and $\tilde{\Gamma}_\tau$ in $\Gamma_{F_v^+}/I_{F_v^+}$ are the same. We choose a subset $\mathfrak{T}_3^\heartsuit \subseteq \mathfrak{T}_3$ of representatives for the Γ_τ -action on \mathfrak{T}_3 . For each element $\tau \in \mathfrak{T}_3^\heartsuit$, we choose an extension $\tilde{\tau}$ from Lemma E.5.6(2) below to a representation of Γ_τ with coefficients in $\tilde{\mathcal{O}}$. For $\tau \in \mathfrak{T}_3$ in general, we have $\tau \simeq \tau_3^h$ for (unique) $\tau_3 \in \mathfrak{T}_3^\heartsuit$ and some $h \in \Gamma_{F_w}$; and we choose $\tilde{\tau}$ to be $\tilde{\tau}_3^h$, as the extension to $\Gamma_\tau = h^{-1}\Gamma_{\tau_3}h$.

In addition, we put $\mathfrak{T}^\heartsuit := \mathfrak{T}_1^\heartsuit \sqcup \mathfrak{T}_2^\heartsuit \sqcup \mathfrak{T}_3^\heartsuit$.

Remark E.5.5. The partition $\mathfrak{T} = \mathfrak{T}_1 \sqcup \mathfrak{T}_2 \sqcup \mathfrak{T}_3$ does not depend on the choice of γ . Moreover, if \mathfrak{T}_3 is nonempty, then w is ramified over v .

Lemma E.5.6. *Let $\tau \in \mathfrak{T}$ be an element of dimension d .*

- (1) *If $\tau \in \mathfrak{T}_2$, then the representation $\tilde{\tau}$ in Lemma E.5.2(2) extends to a representation of Γ_τ with coefficients in \mathcal{O} such that $\tilde{\tau}^{\gamma'} \simeq \tilde{\tau}^*$ still holds for every $\gamma' \in \tilde{\Gamma}_\tau \setminus \Gamma_\tau$.*
- (2) *If $\tau \in \mathfrak{T}_3$, then the representation $\tilde{\tau}$ in Lemma E.5.2(2) extends to a representation of Γ_τ with coefficients in $\tilde{\mathcal{O}}$ such that $\tilde{\tau}^{\gamma'} \simeq \tilde{\tau}^*$ still holds for every $\gamma' \in \tilde{\Gamma}_\tau \setminus \Gamma_\tau$.*

Proof. We fix a splitting $\Gamma_{F_v^+} \simeq P_{F_v^+} \rtimes T_v$ and an isomorphism $T_v \simeq T_q = t^{\mathbb{Z}\ell} \rtimes \phi_q^{\widehat{\mathbb{Z}}}$ with the q -tame group (Definition E.4.1) where $q = \|v\|$. Then we have the induced splitting $\Gamma_\tau \simeq P_{F_w} \rtimes T_\tau$, where $T_\tau = t_\tau^{\mathbb{Z}\ell} \rtimes \phi_\tau^{\widehat{\mathbb{Z}}}$ is a subgroup of T_q , with $t_\tau = t^\ell$ and $\phi_\tau = \phi_q^b$ for unique integers $a \geq 0$ and $b > 0$. To extend $\tilde{\tau}$ in Lemma E.5.2(2) to a representation of Γ_τ , it suffices to specify $\tilde{\tau}(\phi_\tau)$.

For (1), there are two cases.

First, we suppose that w is unramified over v . Then b is even; and $\tilde{\Gamma}_\tau$ is the image of $\tilde{\Gamma}_\tau$ in T_v . Then $\tilde{\Gamma}_\tau$ is generated by T_τ and an element $\gamma' \in T_v$ of the form $(\tilde{t}_\tau, \phi_q^{b/2})$ such that $\gamma'^2 = (\tilde{t}_\tau^{q^{b/2}+1}, \phi_q^b)$ lies in Γ_τ . As $[\tau] = [\tau^{\gamma,*}]$, we have $\tau^{\gamma'} \simeq \tau^*$. We choose a basis of τ hence regard τ as a homomorphism $\tau: P_{F_w} \rightarrow \mathrm{GL}_d(k)$. By Lemma E.5.2(1,2), we have a continuous homomorphism $\tilde{\tau}: \Gamma_\tau \cap I_{F_w} \rightarrow \mathrm{GL}_d(\mathcal{O})$ such that $\tilde{\tau}|_{P_{F_w}}$ is a lifting of τ , unique up to conjugation in $1 + M_d(\lambda)$. In particular, there is an element $A \in \mathrm{GL}_d(\mathcal{O})$, unique up to scalar in \mathcal{O}^\times , such that

$\tilde{\tau}^{\gamma'}(g) = B\tilde{\tau}^*(g)B^{-1}$ for every $g \in \Gamma_\tau \cap \mathbf{I}_{F_w}$. Since $\phi_\tau = \tilde{t}_\tau^{-q^{b/2}-1}\gamma'^2$, we have

$$\begin{aligned} \tilde{\tau}(\phi_\tau g \phi_\tau^{-1}) &= \tilde{\tau}(\tilde{t}_\tau^{-q^{b/2}-1}\gamma'^2 g \gamma'^{-2} \tilde{t}_\tau^{q^{b/2}+1}) = \tilde{\tau}(\tilde{t}_\tau^{-q^{b/2}-1})\tilde{\tau}(\gamma'^2 g \gamma'^{-2})\tilde{\tau}(\tilde{t}_\tau^{q^{b/2}+1}) \\ &= \tilde{\tau}(\tilde{t}_\tau^{-q^{b/2}-1})B\tilde{\tau}^*(\gamma'^{-1} g \gamma')B^{-1}\tilde{\tau}(\tilde{t}_\tau^{q^{b/2}+1}) \\ &= \tilde{\tau}(\tilde{t}_\tau^{-q^{b/2}-1})(B^t B^{-1})\tilde{\tau}(g)({}^t B B^{-1})\tilde{\tau}(\tilde{t}_\tau^{q^{b/2}+1}) \end{aligned}$$

for every $g \in \Gamma_\tau \cap \mathbf{I}_{F_w}$. We put $\tilde{\tau}(\phi_\tau) := -\chi(\phi_q^{b/2})\tilde{\tau}(\tilde{t}_\tau^{-q^{b/2}-1})(B^t B^{-1})$. Then we obtain the desired extension as in (1).

Second, we suppose that w is ramified over v . By the definition of \mathfrak{Z}_2 , the image of $\tilde{\Gamma}_\tau$ in $\Gamma_{F_v^+}/\mathbf{I}_{F_v^+}$ contains $\phi_\tau^{\widehat{\mathbb{Z}}}$ as a subgroup of index two. Thus, there exists an element $\gamma' \in \tilde{\Gamma}_\tau \setminus \Gamma_\tau$ such that $\gamma^2 = h\phi_\tau$ for some $h \in \Gamma_\tau \cap \mathbf{I}_{F_w}$. The remaining argument is same to the above case.

For (2), by the definition of \mathfrak{Z}_3 , the image of $\tilde{\Gamma}_\tau$ in $\Gamma_{F_v^+}/\mathbf{I}_{F_v^+}$ coincides with $\phi_\tau^{\widehat{\mathbb{Z}}}$. In particular, we can find an element $\gamma' \in \tilde{\Gamma}_\tau \setminus \Gamma_\tau$ contained in $\mathbf{I}_{F_v^+} \setminus \mathbf{I}_{F_w}$. By Lemma E.5.2(1,2), we have a continuous homomorphism $\tilde{\tau}: \Gamma_\tau \cap \mathbf{I}_{F_w} \rightarrow \mathrm{GL}_d(\mathcal{O})$ such that $\tilde{\tau}|_{\mathbf{P}_{F_w}}$ is a lifting of τ , unique up to conjugation in $1 + \mathrm{M}_d(\lambda)$. As we have $\tau^{\gamma'} \simeq \tau^*$ and $\tau^{\phi_\tau} \simeq \tau$, there are elements $A, B \in \mathrm{GL}_d(\mathcal{O})$ such that

$$(E.9) \quad \tilde{\tau}^{\gamma'}(g) = A\tilde{\tau}^*(g)A^{-1},$$

$$(E.10) \quad \tilde{\tau}^{\phi_\tau}(g) = B\tilde{\tau}(g)B^{-1},$$

for every $g \in \Gamma_\tau \cap \mathbf{I}_{F_w}$. It follows from (E.9) that the desired element $\tilde{\tau}(\phi_\tau) \in \mathrm{GL}_d(\tilde{\mathcal{O}})$ has to satisfy the equation

$$(E.11) \quad \chi(\phi_\tau)A^t \tilde{\tau}(\phi_\tau)^{-1}A^{-1} = \tilde{\tau}(\gamma'\phi_\tau\gamma'^{-1}) = \tilde{\tau}(\gamma'\phi_\tau\gamma'^{-1}\phi_\tau^{-1})\tilde{\tau}(\phi_\tau),$$

where we note that $\gamma'\phi_\tau\gamma'^{-1}\phi_\tau^{-1} \in \Gamma_\tau \cap \mathbf{I}_{F_w}$. However, by (E.10), we have

$$\tilde{\tau}^{\gamma'\phi_\tau\gamma'^{-1}}(g) = (\tilde{\tau}(\gamma'\phi_\tau\gamma'^{-1}\phi_\tau^{-1})B)\tilde{\tau}(g)(\tilde{\tau}(\gamma'\phi_\tau\gamma'^{-1}\phi_\tau^{-1})B)^{-1}$$

for every $g \in \Gamma_\tau \cap \mathbf{I}_{F_w}$. On the other hand, by (E.9) and (E.10), we have

$$\tilde{\tau}^{\gamma'^{-1}\phi_\tau\gamma'}(g) = (A^t B^{-1}A^{-1})\tilde{\tau}(g)(A^t B^{-1}A^{-1})^{-1}$$

for every $g \in \Gamma_\tau \cap \mathbf{I}_{F_w}$. Since τ is absolutely irreducible, it follows that there exists $\beta \in \mathcal{O}^\times$ such that

$$A^t B^{-1}A^{-1} = \beta \cdot \tilde{\tau}(\gamma'\phi_\tau\gamma'^{-1}\phi_\tau^{-1})B.$$

Take an element $\alpha \in \tilde{\mathcal{O}}^\times$ such that $\alpha^2 = \beta\chi(\phi_\tau)$. Then it is clear that $\tilde{\tau}(\phi_\tau) = \alpha B \in \mathrm{GL}_d(\tilde{\mathcal{O}})$ is a solution to (E.11).

The lemma is proved. \square

Using Construction E.5.4, we now discuss the structure of liftings of \bar{r} . We replace E_λ by \tilde{E}_λ so that every representations $\tilde{\tau}$ in Construction E.5.4 have coefficients in \mathcal{O} . Let $r: \Gamma_{F_v^+} \rightarrow \mathcal{G}_N(R)$ be a lifting of \bar{r} to an object R of $\mathcal{C}_\mathcal{O}$. By Lemma E.1.3, to give such a lifting r is equivalent to giving an element $\Psi \in \mathrm{GL}_N(R)$ that reduces to $\bar{\Psi}$ and satisfies

$$r^{\natural, \gamma} = \Psi \circ \chi r^{\natural, \vee} \circ \Psi^{-1}, \quad \Psi^t \Psi^{-1} = -\chi(\gamma)^{-1} r^{\natural}(\gamma^2).$$

For every $\tau \in \mathfrak{Z}$, put

$$M_\tau(r) := \mathrm{Hom}_{R[\mathbf{P}_{F_w}]}(\tilde{\tau} \otimes_{\mathcal{O}} R, r^{\natural}),$$

which is a finite free R -module equipped with the induced continuous action by T_τ . Denote by $m_\tau \geq 1$ the rank of $M_\tau(r)$. Let $\tau' \in \mathfrak{Z}$ be the unique element such that $\tau^{\gamma'} \simeq \tau'^*$. Choose an

isomorphism $\iota_\tau: \tau^\gamma \xrightarrow{\sim} \tau'^*$, which, by construction E.5.4, lifts to an isomorphism $\iota_{\tilde{\tau}}: \tilde{\tau}^\gamma \xrightarrow{\sim} \tilde{\tau}'^*$ of representations of $\Gamma_{\tau'}$. Then we have isomorphisms

$$M_\tau(r)^\gamma \xrightarrow{\sim} \text{Hom}_{R[\mathbb{P}_{F_w}]}(\tilde{\tau}^\gamma \otimes_{\mathcal{O}} R, r^{\natural, \gamma}) \xrightarrow{\sim} \text{Hom}_{R[\mathbb{P}_{F_w}]}(\tilde{\tau}'^* \otimes_{\mathcal{O}} R, r^{\natural, *}) \xrightarrow{\sim} \text{Hom}_{R[\mathbb{P}_{F_w}]}(r^{\natural}, \tilde{\tau}' \otimes_{\mathcal{O}} R),$$

where the second isomorphism is induced by $\iota_{\tilde{\tau}}$ and Ψ . As τ' is absolutely irreducible, we obtain a perfect R -bilinear pairing

$$M_\tau(r)^\gamma \times M_{\tau'}(r) \rightarrow \text{End}_{R[\mathbb{P}_{F_w}]}(\tilde{\tau}' \otimes_{\mathcal{O}} R) = R,$$

which induces an isomorphism

$$\theta_{\tilde{\tau}, r}: M_\tau(r)^\gamma \xrightarrow{\sim} M_{\tau'}(r)^\vee := \text{Hom}_R(M_{\tau'}(r), R)$$

of $R[\mathbb{T}_{\tau'}]$ -modules. In particular, we have

(E.12)

$$r^{\natural} \simeq \left(\bigoplus_{\tau \in \mathfrak{I}_1^\heartsuit} \left(\text{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}} (\tilde{\tau} \otimes_{\mathcal{O}} M_\tau(r)) \oplus \text{Ind}_{\Gamma_{\tau^\gamma}}^{\Gamma_{F_w}} (\tilde{\tau}^{\gamma, *}) \otimes_{\mathcal{O}} M_\tau(r)^{\gamma, \vee} \right) \right) \oplus \left(\bigoplus_{\tau \in \mathfrak{I}_2^\heartsuit \sqcup \mathfrak{I}_3^\heartsuit} \text{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}} (\tilde{\tau} \otimes_{\mathcal{O}} M_\tau(r)) \right)$$

as representations of Γ_{F_w} .

Now for every τ , we fix an isomorphism $b_\tau: M_\tau(\bar{r}) \xrightarrow{\sim} k^{\oplus m_\tau}$ of k -vector spaces, and let $\bar{\varrho}_\tau: \mathbb{T}_\tau \rightarrow \text{GL}_{m_\tau}(k)$ be the induced homomorphism. There are two cases.

- (a) Suppose $\tau \in \mathfrak{I}_1$. Then $M_{\tau'}(r)$ is determined by $M_\tau(r)$. If we choose an isomorphism $M_\tau(r) \simeq R^{\oplus m_\tau}$ of R -modules that reduces to b_τ , then we obtain a continuous homomorphism

$$\varrho_\tau: \mathbb{T}_\tau \rightarrow \text{GL}_{m_\tau}(R)$$

that reduces to $\bar{\varrho}_\tau$.

- (b) Suppose $\tau \in \mathfrak{I}_2 \sqcup \mathfrak{I}_3$. Let h be element from Construction E.5.4. Then $\theta_{\tilde{\tau}, r}$ induces an isomorphism $M_\tau(r)^{h\gamma} \xrightarrow{\sim} M_\tau(\tau)^\vee$ of $R[\mathbb{T}_\tau]$ -modules. Applying Lemma E.1.3 to $r = \bar{r}$, we obtain a homomorphism

$$\bar{\varrho}_\tau: \tilde{\mathbb{T}}_\tau \rightarrow \mathcal{G}_{m_\tau}(k)$$

satisfying $\bar{\varrho}_\tau^{-1}(\text{GL}_{m_\tau}(k) \times k^\times) = \mathbb{T}_\tau$ and $\nu \circ \bar{\varrho}_\tau = \eta_v^{\mu_\tau}$ for some $\mu_\tau \in \mathbb{Z}/2\mathbb{Z}$ determined by $\tilde{\tau}$.³³ In general, if we choose an isomorphism $M_\tau(r) \simeq R^{\oplus m_\tau}$ of R -modules that reduces b_τ , then we obtain a continuous homomorphism

$$\varrho_\tau: \tilde{\mathbb{T}}_\tau \rightarrow \mathcal{G}_{m_\tau}(R)$$

that reduces to $\bar{\varrho}_\tau$ and satisfies $\nu \circ \varrho_\tau = \eta_v^{\mu_\tau}$.

The following proposition is the counterpart of [CHT08, Corollary 2.4.13] when v is nonsplit in F .

Proposition E.5.7. *We keep the choices of $\gamma \in \Gamma_{F_v^+} \setminus \Gamma_{F_w}$, those in Construction E.5.4, ι_τ , and b_τ . We also recall that E_λ now is an unramified quadratic extension of a \bar{r} -inclusive extension of \mathbb{Q}_ℓ . For every object R of $\mathcal{C}_{\mathcal{O}}$, the assignment*

$$r \mapsto (\varrho_\tau)_{\tau \in \mathfrak{I}^\heartsuit}$$

establishes a bijection between deformations of \bar{r} to R and equivalence classes of tuples $(\varrho_\tau)_{\tau \in \mathfrak{I}^\heartsuit}$ where

- (a) for $\tau \in \mathfrak{I}_1^\heartsuit$, $\varrho_\tau: \mathbb{T}_\tau \rightarrow \text{GL}_{m_\tau}(R)$ is a continuous homomorphism that reduces to $\bar{\varrho}_\tau$;
(b) for $\tau \in \mathfrak{I}_2^\heartsuit \sqcup \mathfrak{I}_3^\heartsuit$, $\varrho_\tau: \tilde{\mathbb{T}}_\tau \rightarrow \mathcal{G}_{m_\tau}(R)$ is a continuous homomorphism that reduces to $\bar{\varrho}_\tau$ and satisfies $\nu \circ \varrho_\tau = \eta_v^{\mu_\tau}$.

³³In fact, when $\tau \in \mathfrak{I}_2$, one can always modify $\tilde{\tau}$ to make $\mu_\tau = 0$; but when $\tau \in \mathfrak{I}_3$, μ_τ is determined by τ .

Here, two tuples $(\varrho_\tau)_{\tau \in \mathfrak{S}^\heartsuit}$ and $(\varrho'_\tau)_{\tau \in \mathfrak{S}^\heartsuit}$ are said to be equivalent if ϱ_τ and ϱ'_τ are conjugate by elements in $1 + \mathbf{M}_{m_\tau}(\mathfrak{m}_R)$ for every $\tau \in \mathfrak{S}^\heartsuit$.

Proof. We now attach to every tuple $(\varrho_\tau)_{\tau \in \mathfrak{S}^\heartsuit}$ as in the statement a lifting r explicitly. Denote by M_τ the $R[\mathbb{T}_\tau]$ -module corresponding to ϱ_τ . Consider

$$M := \left(\bigoplus_{\tau \in \mathfrak{S}_1^\heartsuit} \left(\text{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}} (\tilde{\tau} \otimes_{\mathcal{O}} M_\tau) \oplus \text{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}} (\tilde{\tau}^{\gamma,*} \otimes_{\mathcal{O}} M_\tau^{\gamma,\vee}) \right) \right) \oplus \left(\bigoplus_{\tau \in \mathfrak{S}_2^\heartsuit \sqcup \mathfrak{S}_3^\heartsuit} \text{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}} (\tilde{\tau} \otimes_{\mathcal{O}} M_\tau) \right),$$

which is a free R -module of rank N , equipped with a continuous action by Γ_{F_w} . Moreover, we have $M \otimes_R R/\mathfrak{m}_R \simeq \bar{r}^\natural$ as representations of Γ_{F_w} by (E.12). Thus, we may fix an isomorphism $M \simeq R^{\otimes N}$ such that the induced continuous homomorphism $\rho = \rho_M: \Gamma_{F_w} \rightarrow \text{GL}_N(R)$ reduces to \bar{r}^\natural . Thus, by Lemma E.1.3, to construct the desired lifting r from ρ , it amounts to finding an element $\Psi \in \text{GL}_N(R)$ satisfying

$$(E.13) \quad \rho^\gamma = \Psi \circ \chi \rho^\vee \circ \Psi^{-1}, \quad \Psi {}^t\Psi^{-1} = -\chi(\gamma)^{-1} \rho(\gamma^2).$$

We will construct Ψ as a direct sum of Ψ_τ for $\tau \in \mathfrak{S}^\heartsuit$.

For $\tau \in \mathfrak{S}_1^\heartsuit$, we note that $\tilde{\tau}(\gamma^{-2}) \otimes \varrho_\tau(\gamma^{-2})$ induces an isomorphism

$$\text{Ind}_{\Gamma_{\tau\gamma^2}}^{\Gamma_{F_w}} (\tilde{\tau}^{\gamma^2} \otimes_{\mathcal{O}} M_\tau^{\gamma^2}) \simeq \left(\text{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}} (\tilde{\tau}^{\gamma,*} \otimes_{\mathcal{O}} M_\tau^{\gamma,\vee}) \right)^{\gamma,*} \xrightarrow{\sim} \text{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}} (\tilde{\tau} \otimes_{\mathcal{O}} M_\tau).$$

Thus, we obtain an isomorphism

$$(E.14) \quad \left(\text{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}} (\tilde{\tau} \otimes_{\mathcal{O}} M_\tau) \oplus \text{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}} (\tilde{\tau}^{\gamma,*} \otimes_{\mathcal{O}} M_\tau^{\gamma,\vee}) \right)^{\gamma,*} \xrightarrow{\sim} \text{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}} (\tilde{\tau} \otimes_{\mathcal{O}} M_\tau) \oplus \text{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}} (\tilde{\tau}^{\gamma,*} \otimes_{\mathcal{O}} M_\tau^{\gamma,\vee})$$

as the composition of the canonical isomorphism

$$\left(\text{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}} (\tilde{\tau} \otimes_{\mathcal{O}} M_\tau) \oplus \text{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}} (\tilde{\tau}^{\gamma,*} \otimes_{\mathcal{O}} M_\tau^{\gamma,\vee}) \right)^{\gamma,*} \xrightarrow{\sim} \text{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}} (\tilde{\tau}^{\gamma,*} \otimes_{\mathcal{O}} M_\tau^{\gamma,\vee}) \oplus \left(\text{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}} (\tilde{\tau}^{\gamma,*} \otimes_{\mathcal{O}} M_\tau^{\gamma,\vee}) \right)^{\gamma,*},$$

and the isomorphism

$$\text{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}} (\tilde{\tau}^{\gamma,*} \otimes_{\mathcal{O}} M_\tau^{\gamma,\vee}) \oplus \left(\text{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}} (\tilde{\tau}^{\gamma,*} \otimes_{\mathcal{O}} M_\tau^{\gamma,\vee}) \right)^{\gamma,*} \xrightarrow{\sim} \text{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}} (\tilde{\tau} \otimes_{\mathcal{O}} M_\tau) \oplus \text{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}} (\tilde{\tau}^{\gamma,*} \otimes_{\mathcal{O}} M_\tau^{\gamma,\vee})$$

given by the matrix $\begin{pmatrix} 0 & -\chi(\gamma) \text{Ind}^{\gamma^{-2}} \\ 1 & 0 \end{pmatrix}$. We now let Ψ_τ be the matrix representing the isomorphism

$$\left(\text{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}} (\tilde{\tau} \otimes_{\mathcal{O}} M_\tau) \oplus \text{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}} (\tilde{\tau}^{\gamma,*} \otimes_{\mathcal{O}} M_\tau^{\gamma,\vee}) \right)^* \xrightarrow{\sim} \left(\text{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}} (\tilde{\tau} \otimes_{\mathcal{O}} M_\tau) \oplus \text{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}} (\tilde{\tau}^{\gamma,*} \otimes_{\mathcal{O}} M_\tau^{\gamma,\vee}) \right)^\gamma$$

induced from (E.14) by duality.

For $\tau \in \mathfrak{S}_2^\heartsuit \sqcup \mathfrak{S}_3^\heartsuit$, let h be the element in Construction E.5.4. Put $\gamma' := h\gamma$, which is an element in $\tilde{\Gamma}_\tau \setminus \Gamma_\tau$. The homomorphism $\varrho_\tau: \tilde{\Gamma}_\tau \rightarrow \mathcal{G}_{m_\tau}(R)$ induces an isomorphism $M_\tau^{\gamma'} \xrightarrow{\sim} M_\tau^\vee$ by Lemma E.1.3(1), which induces an isomorphism $M_\tau^{\gamma,\vee} \xrightarrow{\sim} M_\tau^{h^{-1}}$. On the other hand, by Lemma E.5.6, we have an isomorphism $\tilde{\tau}^{\gamma,*} \simeq \tilde{\tau}^{h^{-1}}$. Thus, we obtain an isomorphism

$$(E.15) \quad \left(\text{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}} (\tilde{\tau} \otimes_{\mathcal{O}} M_\tau) \right)^{\gamma,*} \xrightarrow{\sim} \text{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}} (\tilde{\tau} \otimes_{\mathcal{O}} M_\tau)$$

as the composition of the canonical isomorphism

$$\left(\text{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}} (\tilde{\tau} \otimes_{\mathcal{O}} M_\tau) \right)^{\gamma,*} \xrightarrow{\sim} \text{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}} (\tilde{\tau}^{\gamma,*} \otimes_{\mathcal{O}} M_\tau^{\gamma,\vee}),$$

the isomorphism

$$\text{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}} (\tilde{\tau}^{\gamma,*} \otimes_{\mathcal{O}} M_\tau^{\gamma,\vee}) \xrightarrow{\sim} \text{Ind}_{\Gamma_{\tau h^{-1}}}^{\Gamma_{F_w}} (\tilde{\tau}^{h^{-1}} \otimes_{\mathcal{O}} M_\tau^{h^{-1}})$$

specified above, and the isomorphism

$$\mathrm{Ind}_{\Gamma_{\tau^{h^{-1}}}}^{\Gamma_{F_w}} (\tilde{\tau}^{h^{-1}} \otimes_{\mathcal{O}} M_{\tau}^{h^{-1}}) \xrightarrow{\sim} \mathrm{Ind}_{\Gamma_{\tau}}^{\Gamma_{F_w}} (\tilde{\tau} \otimes_{\mathcal{O}} M_{\tau})$$

given by the action of h^{-1} . We now let Ψ_{τ} be the matrix representing the isomorphism

$$\left(\mathrm{Ind}_{\Gamma_{\tau}}^{\Gamma_{F_w}} (\tilde{\tau} \otimes_{\mathcal{O}} M_{\tau}) \right)^* \xrightarrow{\sim} \left(\mathrm{Ind}_{\Gamma_{\tau}}^{\Gamma_{F_w}} (\tilde{\tau} \otimes_{\mathcal{O}} M_{\tau}) \right)^{\gamma}$$

induced from (E.15) by duality.

Finally, we put $\Psi := \bigoplus_{\tau \in \mathfrak{S}^{\heartsuit}} \Psi_{\tau}$. Then (E.13) follows by construction. In other words, we have assigned a lifting r from the tuple $(\varrho_{\tau})_{\tau \in \mathfrak{S}^{\heartsuit}}$. It is straightforward to check that such assignment is inverse to the assignment in the proposition. The proposition follows. \square

From now till the end of this subsection, we assume $l \geq N$. Using Proposition E.5.7, we can define minimally ramified liftings of \bar{r} . We now do *not* assume that E_{λ} is \bar{r} -inclusive. We choose an unramified quadratic extension E'_{λ} of an \bar{r} -inclusive unramified extension of E_{λ} , with the ring of integers \mathcal{O}' and the residue field k' . We also keep the choices of $\gamma \in \Gamma_{F_v^+} \setminus \Gamma_{F_w}$, those in Construction E.5.4, ι_{τ} , and b_{τ} , as in Proposition E.5.7 (with respect to E'_{λ}).

Definition E.5.8. We say that a lifting r of \bar{r} to some object R of $\mathcal{C}_{\mathcal{O}}$ is *minimally ramified* if in the tuple $(\varrho_{\tau})_{\tau \in \mathfrak{S}^{\heartsuit}}$ corresponding to the composition $\Gamma_{F_v^+} \xrightarrow{r} \mathcal{G}_N(R) \rightarrow \mathcal{G}_N(R \otimes_{\mathcal{O}} \mathcal{O}')$, every homomorphism ϱ_{τ} is a minimally ramified lifting of $\bar{\varrho}_{\tau}$ in the following sense.

- (1) For $\tau \in \mathfrak{S}_1^{\heartsuit}$, minimally ramified liftings of $\bar{\varrho}_{\tau}$ is defined in the sense of [CHT08, Definition 2.4.14].
- (2) For $\tau \in \mathfrak{S}_2^{\heartsuit}$, note that $\tilde{\mathrm{T}}_{\tau}$ is isomorphic to the q_{τ} -tame group for some power q_{τ} of $\|v\|$ under which the subgroup T_{τ} is the q_{τ}^2 -tame group. Thus, we may define minimally ramified liftings of $\bar{\varrho}_{\tau}$ using Definition E.4.4 (with respect to the similitude character $\eta_v^{\mu_{\tau}}$, which is trivial on T_{τ});
- (3) For $\tau \in \mathfrak{S}_3^{\heartsuit}$, note that $\tilde{\mathrm{T}}_{\tau} \simeq \mathrm{T}_{\tau} \times \mathbb{Z}/2\mathbb{Z}$. Then, by Lemma E.1.3, we may regard the homomorphism ϱ_{τ} as a continuous homomorphism $\varrho_{\tau}: \mathrm{T}_{\tau} \rightarrow G(R)$, where G is a symplectic (resp. orthogonal) group of rank m_{τ} if μ_{τ} is 0 (resp. 1). Thus, we may define minimally ramified liftings of $\bar{\varrho}_{\tau}$ using [Boo19, Definition 5.4].

Remark E.5.9. It is straightforward to check that Definition E.5.8 do not depend on the choice of E'_{λ} , $\gamma \in \Gamma_{F_v^+} \setminus \Gamma_{F_w}$, those in Construction E.5.4, ι_{τ} , and b_{τ} .

Now we allow v to be a nonarchimedean place of F^+ that is not above ℓ , but not necessarily nonsplit in F . Again, we consider a pair (\bar{r}, χ) from Notation E.2.1 with $\tilde{\Gamma} = \Gamma_{F_v^+}$ and $\Gamma = \Gamma_{F_v^+} \cap \Gamma_F$.

Definition E.5.10. We define \mathcal{D}^{\min} to be the local deformation problem of \bar{r} that classifies minimally ramified liftings in the sense of Definition E.5.8 (resp. [CHT08, Definition 2.4.14]) when v is nonsplit (resp. split) in F .

Proposition E.5.11. *We have*

- (1) The ring $\mathbf{R}_{\bar{r}}^{\mathrm{loc}}$ is a reduced local complete intersection, flat and of pure relative dimension N^2 over \mathcal{O} .
- (2) Every irreducible component of $\mathrm{Spf} \mathbf{R}_{\bar{r}}^{\mathrm{loc}}$ is a local deformation problem (Definition E.2.3).
- (3) If $\ell \geq N$, then \mathcal{D}^{\min} is an irreducible component of $\mathrm{Spf} \mathbf{R}_{\bar{r}}^{\mathrm{loc}}$ and is formally smooth over $\mathrm{Spf} \mathcal{O}$ of pure relative dimension N^2 .

Proof. We may assume $E_{\lambda} = E'_{\lambda}$.

For (1), when v splits in F , this is [Sho18, Theorem 2.5]. Thus, we may assume that v is nonsplit in F . By Proposition E.5.7, $\mathbf{R}_{\bar{r}}^{\text{loc}}$ is a power series ring over

$$\widehat{\bigotimes}_{\tau \in \mathfrak{T}^\heartsuit} \mathbf{R}_{\bar{\varrho}_\tau}^{\text{loc}}.$$

We now claim that for every $\tau \in \mathfrak{T}^\heartsuit$, $\mathbf{R}_{\bar{\varrho}_\tau}^{\text{loc}}$ is a local complete intersection, flat and equidimensional. Indeed, for $\tau \in \mathfrak{T}_1^\heartsuit$, this is [Sho18, Theorem 2.5]; for $\tau \in \mathfrak{T}_2^\heartsuit$, this is Corollary E.4.3; for $\tau \in \mathfrak{T}_3^\heartsuit$, this is Proposition E.4.2 for G a symplectic or orthogonal group with the trivial similitude character. On the other hand, by [BG19, Theorem 3.3.2] or [BP, Theorem 1], we know that $\mathbf{R}_{\bar{r}}^{\text{loc}}[1/\ell]$ is reduced and of pure dimension $\dim \mathcal{G}_N = N^2$. Thus, $\mathbf{R}_{\bar{r}}^{\text{loc}}$ is a local complete intersection, flat and of pure relative dimension N^2 over \mathcal{O} . Since $\mathbf{R}_{\bar{r}}^{\text{loc}}$ is generically reduced and Cohen–Macaulay, it is reduced. (1) is proved.

For (2), take an irreducible component \mathcal{D} of $\text{Spf } \mathbf{R}_{\bar{r}}^{\text{loc}}$, and let \mathcal{L}_N be the formal completion of $\text{GL}_{N,\mathcal{O}}$ along the unit section. Then the conjugation action induces a homomorphism $\mathcal{L}_N \times_{\text{Spf } \mathcal{O}} \mathcal{D} \rightarrow \text{Spf } \mathbf{R}_{\bar{r}}^{\text{loc}}$ whose image contains \mathcal{D} . Since \mathcal{L}_N is irreducible, the image is irreducible hence has to be \mathcal{D} . In other words, \mathcal{D} is a local deformation problem.

For (3), since \mathcal{D}^{min} is Zariski closed in $\text{Spf } \mathbf{R}_{\bar{r}}^{\text{loc}}$ from its definition, it suffices to show that \mathcal{D}^{min} is formally smooth over $\text{Spf } \mathcal{O}$ of pure relative dimension N^2 . When v splits in F , this is [CHT08, Corollary 2.4.21]. Thus, we may assume that v is nonsplit in F . For $\tau \in \mathfrak{T}^\heartsuit$, let $\mathcal{D}_{\bar{\varrho}_\tau}^{\text{min}}$ be the local deformation problem of $\bar{\varrho}_\tau$ classifying minimally ramified liftings of $\bar{\varrho}_\tau$ in various cases in Definition E.5.8. By Proposition E.5.7 and Definition E.5.8, \mathcal{D}^{min} is formally smooth over

$$\prod_{\tau \in \mathfrak{T}^\heartsuit} \mathcal{D}_{\bar{\varrho}_\tau}^{\text{min}}.$$

We claim that for every $\tau \in \mathfrak{T}^\heartsuit$, $\mathcal{D}_{\bar{\varrho}_\tau}^{\text{min}}$ is formally smooth over $\text{Spf } \mathcal{O}$. Indeed, for $\tau \in \mathfrak{T}_1^\heartsuit$, this is [CHT08, Lemma 2.4.19]; for $\tau \in \mathfrak{T}_2^\heartsuit$, this is Proposition E.4.5; for $\tau \in \mathfrak{T}_3^\heartsuit$, this is a part of [Boo19, Theorem 1.1]. Thus, \mathcal{D}^{min} is formally smooth over $\text{Spf } \mathcal{O}$.

It remains to compute the dimension. By (E.1), it suffices to show that

$$(E.16) \quad \dim_k \mathbf{L}(\mathcal{D}^{\text{min}}) = \dim_k \mathbf{H}^0(F_v^+, \text{ad } \bar{r}).$$

For every $\tau \in \mathfrak{T}^\heartsuit$, let $\mathbf{L}(\mathcal{D}_{\bar{\varrho}_\tau}^{\text{min}})$ be the tangent space of the deformation problem $\mathcal{D}_{\bar{\varrho}_\tau}^{\text{min}}$, which is a subspace of $\mathbf{H}^1(\mathbf{T}_\tau, \text{ad } \bar{\varrho})$ (resp. $\mathbf{H}^1(\tilde{\mathbf{T}}_\tau, \text{ad } \bar{\varrho})$) if $\tau \in \mathfrak{T}_1^\heartsuit$ (resp. $\tau \in \mathfrak{T}_2^\heartsuit \sqcup \mathfrak{T}_3^\heartsuit$). By Proposition E.5.7, we have

$$(E.17) \quad \dim_k \mathbf{L}(\mathcal{D}^{\text{min}}) = \sum_{\tau \in \mathfrak{T}^\heartsuit} \dim_k \mathbf{L}(\mathcal{D}_{\bar{\varrho}_\tau}^{\text{min}}).$$

We claim that

$$(E.18) \quad \dim_k \mathbf{L}(\mathcal{D}_{\bar{\varrho}_\tau}^{\text{min}}) = \begin{cases} \dim_k \mathbf{H}^0(\mathbf{T}_\tau, \text{ad } \bar{\varrho}_\tau) & \text{if } \tau \in \mathfrak{T}_1^\heartsuit; \\ \dim_k \mathbf{H}^0(\tilde{\mathbf{T}}_\tau, \text{ad } \bar{\varrho}_\tau) & \text{if } \tau \in \mathfrak{T}_2^\heartsuit \sqcup \mathfrak{T}_3^\heartsuit. \end{cases}$$

Indeed, for $\tau \in \mathfrak{T}_1^\heartsuit$, this is [CHT08, Corollary 2.4.20]; for $\tau \in \mathfrak{T}_2^\heartsuit$, this is Corollary E.4.6; for $\tau \in \mathfrak{T}_3^\heartsuit$, this is a part of [Boo19, Theorem 1.1] as $\dim_k \mathbf{H}^0(\tilde{\mathbf{T}}_\tau, \text{ad } \bar{\varrho}_\tau) = \dim_k \mathbf{H}^0(\mathbf{T}_\tau, \text{ad } \bar{\varrho}_\tau)$. From (E.12) for \bar{r} , we have

$$\mathbf{H}^0(F_v^+, \text{ad } \bar{r}) \simeq \left(\bigoplus_{\tau \in \mathfrak{T}_1^\heartsuit} \mathbf{H}^0(\mathbf{T}_\tau, \text{ad } \bar{\varrho}_\tau) \right) \oplus \left(\bigoplus_{\tau \in \mathfrak{T}_2^\heartsuit \sqcup \mathfrak{T}_3^\heartsuit} \mathbf{H}^0(\tilde{\mathbf{T}}_\tau, \text{ad } \bar{\varrho}_\tau) \right).$$

Together with (E.17) and (E.18), we obtain (E.16).

The proposition is proved. \square

Now we discuss one example of minimally ramified liftings, which is used in Proposition 8.1.5. Consider an elliptic curve A over F_v^+ . For every rational prime ℓ , we fix an isomorphism $H_{\text{ét}}^1(A_{\alpha\bar{F}}, \mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell^{\oplus 2}$, hence obtain a continuous homomorphism $\rho_{A,\ell}: \Gamma_{F_v^+} \rightarrow \text{GL}_2(\mathbb{Z}_\ell)$. Suppose $N \geq 2$. We obtain a continuous homomorphism

$$r_{A,\ell}: \Gamma_{F_v^+} \rightarrow \mathcal{G}_N(\mathbb{Z}_\ell) = (\text{GL}_N(\mathbb{Z}_\ell) \times \mathbb{Z}_\ell^\times) \rtimes \{1, \mathbf{c}\}$$

by the formula $r_{A,\ell}(\gamma) = (\text{Sym}^{N-1} \rho_{A,\ell}(\gamma), \eta_v^{N-1} \epsilon_{\ell,v}^{1-N}(\gamma), \mathbf{c}(\gamma))$, where $\mathbf{c}(\gamma) = \mathbf{c}$ if and only if $\gamma \in \Gamma_{F_v^+} \setminus \Gamma_F$. Denote by $\bar{r}_{A,\ell}$ the composition of $r_{A,\ell}$ and the projection $\mathcal{G}_N(\mathbb{Z}_\ell) \rightarrow \mathcal{G}_N(\mathbb{F}_\ell)$.

Proposition E.5.12. *For all but finitely many rational primes $\ell \geq N$, every lifting of $\bar{r}_{A,\ell}$ to an object R of $\mathcal{C}_{\mathbb{Z}_\ell}$ (with respect to the similitude character $\eta_v^N \epsilon_{\ell,v}^{1-N}$) is minimally ramified in the sense of Definition E.5.8.*

Proof. For simplicity, we only prove the proposition for v nonsplit in F . The split case is similar and easier, which we leave to readers. Thus, let w be the unique place of F above v . Fix a finite extension F'_w of F_w in \bar{F}_v^+ so that $A' := A \otimes_{F_v^+} F'_w$ has either good or split multiplicative reduction. We further request that F'_w/F_w is totally ramified if A' has good reduction. Let T'_w be the image of the $\text{Gal}(\bar{F}_v^+/F'_w)$ of Γ_{F_w} in $\text{T}_w = \Gamma_{F_w}/\text{P}_{F_w}$. We fix an isomorphism $\text{T}_w \simeq \text{T}_q = t^{\mathbb{Z}_\ell} \times \phi_q^{\widehat{\mathbb{Z}}}$ with the q -tame group, where $q = \|w\|$. We now assume $\ell > [F'_w : F_w]$. Then T'_w is generated by t and ϕ_q^b for a unique integer $b > 0$. We then also assume $\ell \geq q^{b \cdot N!} (> N)$. Let $\mathfrak{T} = \mathfrak{T}(\bar{r}_{A,\ell})$ be the set of isomorphism classes of absolutely irreducible representations of P_{F_w} appearing in $\bar{r}_{A,\ell}^{\natural}$ as before.

We first consider the case where A' has split multiplicative reduction. Let u be the valuation of the j -invariant $j(A)$ in F'_w , which is a negative integer. Assume further that ℓ is coprime to u . Then $\rho_{A',\ell}(t)$ is conjugate to $(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix})$ in $\text{GL}_2(\mathbb{Z}_\ell)$, which implies that $\text{Sym}^{N-1} \rho_{A',\ell}(t)$ is conjugate to $1 + J_N$ in $\text{GL}_N(\mathbb{Z}_\ell)$. It follows that \mathfrak{T} is a singleton, say $\{\tau\}$; and every lifting ϱ_τ of $\bar{\varrho}_\tau$ is minimally ramified. Thus, every lifting r of $\bar{r}_{A,\ell}$ is minimally ramified.

We then consider the case where A' has good reduction. Then $\text{T}'_w = \text{T}_w$ hence $b = 1$. Let $\alpha, \beta \in \bar{\mathbb{Q}}_\ell$ be the two eigenvalues of $\rho_{A',\ell}(\phi_q)$. Then α, β are Weil $q^{-1/2}$ -numbers in $\bar{\mathbb{Q}}$, which depend only on A' , not on ℓ . We further assume that ℓ satisfies that $\alpha, \beta \in \bar{\mathbb{Z}}_\ell^\times$, and that the image of the set $\{(\alpha/\beta)^{N-1}, (\alpha/\beta)^{N-3}, \dots, (\alpha/\beta)^{3-N}, (\alpha/\beta)^{1-N}\}$ in $\bar{\mathbb{F}}_\ell^\times$ does not contain q . It follows that for every $\tau \in \mathfrak{T}$, every lifting ϱ_τ of $\bar{\varrho}_\tau$ is actually unramified, hence minimally ramified. Thus, every lifting r of $\bar{r}_{A,\ell}$ is minimally ramified.

Since in both cases, we only exclude finitely many rational primes ℓ , the proposition follows. \square

More generally, we would like to propose the following conjecture. As in the initial setup of Subsection 6.1, let Π be a relevant representation of $\text{GL}_N(\mathbb{A}_F)$ (Definition 1.1.3), and $E \subseteq \mathbb{C}$ a strong coefficient field of Π (Definition 3.2.5). Then for every prime λ of E , we have a continuous homomorphism $\rho_{\Pi,\lambda}: \Gamma_F \rightarrow \text{GL}_N(E_\lambda)$.

E.6. Level-raising deformations. In this subsection, we discuss level-raising deformations. Assume $\ell \geq N \geq 2$. We take a nonarchimedean place v of F^+ that is inert in F and not above ℓ . Let w be the unique place of F above v . Recall that we have $\text{T}_v = \Gamma_{F_v^+}/\text{P}_{F_v^+}$ and $\text{T}_w = \Gamma_{F_w}/\text{P}_{F_w}$. Then T_v is isomorphic to the q -tame group and the subgroup T_w is the q^2 -tame group (Definition E.4.1), where $q = \|v\|$.

We consider a pair (\bar{r}, χ) from Notation E.2.1 with $\tilde{\Gamma} = \Gamma_{F_v^+}$ and $\Gamma = \Gamma_{F_v^+} \cap \Gamma_F = \Gamma_{F_w}$, such that \bar{r} is unramified and $\chi = \eta_v^\mu \epsilon_{\ell,v}^{1-N}$ for some $\mu \in \mathbb{Z}/2\mathbb{Z}$. Then by Lemma E.5.2(1), every lifting r of \bar{r} to an object R of \mathcal{C}_θ factors through T_v . In particular, we may apply the discussion in Subsection E.4 to the pair (\bar{r}, χ) .

Now assume $\ell \nmid (q^2 - 1)$ and that the generalized eigenvalues of $\bar{r}^{\natural}(\phi_w)$ in $\bar{\mathbb{F}}_\ell$ contain the pair $\{q^{-N}, q^{-N+2}\}$ exactly once. By Lemma E.4.7(1), for every lifting r of \bar{r} to an object R of \mathcal{C}_θ , we

have a canonical decomposition

$$(E.19) \quad R^{\oplus N} = M_0 \oplus M_1$$

of free R -modules such that if we write $P_0(T)$ for the characteristic polynomial for $r^{\natural}(\phi_w)$, then $P_0(T) \equiv (T - q^{-N})(T - q^{-N+2}) \pmod{\mathfrak{m}_R}$.

Definition E.6.1. Let (\bar{r}, χ) be as above. We define \mathcal{D}^{mix} to be the local deformation problem of \bar{r} (Definition E.2.3) that classifies liftings r to an object R of $\mathcal{C}_{\mathcal{O}}$ such that

- the decomposition (E.19) is stable under the action of $r^{\natural}(I_{F_w})$;
- the action of $r^{\natural}(I_{F_w})$ on M_1 is trivial;
- for every $t \in I_{F_w}$, the characteristic polynomial of $r^{\natural}(t)$ on M_0 is $(T - 1)^2$.

We define

- (1) \mathcal{D}^{unr} to be the local deformation problem contained in \mathcal{D}^{mix} so that the action of $r^{\natural}(I_{F_w})$ on M_0 is also trivial;
- (2) \mathcal{D}^{ram} to be the local deformation problem contained in \mathcal{D}^{mix} so that $P_0(T) = (T - q^{-N})(T - q^{-N+2})$ in $R[T]$.

It is clear that \mathcal{D}^{unr} coincides with \mathcal{D}^{min} from Definition E.5.10.

Proposition E.6.2. *Suppose $\ell \nmid (q^2 - 1)$ and that the generalized eigenvalues of $\bar{r}^{\natural}(\phi_w)$ in $\overline{\mathbb{F}}_{\ell}$ contain the pair $\{q^{-N}, q^{-N+2}\}$ exactly once. Then the formal scheme \mathcal{D}^{mix} is formally smooth over $\text{Spf } \mathcal{O}[[x_0, x_1]]/(x_0x_1)$ of pure relative dimension $N^2 - 1$ such that the irreducible components defined by $x_0 = 0$ and $x_1 = 0$ are \mathcal{D}^{unr} and \mathcal{D}^{ram} , respectively. In particular, \mathcal{D}^{ram} is formally smooth over $\text{Spf } \mathcal{O}$ of pure relative dimension N^2 .*

Proof. We fix an isomorphism $T_v \simeq T_q = t^{\mathbb{Z}\ell} \rtimes \phi_q^{\widehat{\mathbb{Z}}}$ so that $\phi_w = \phi_q^2$. We write $k^{\oplus N} = \bar{M}_0 \oplus \bar{M}_1$ so that $\bar{r}^{\natural}(\phi_q^2)$ has eigenvalues q^{-N} and q^{-N+2} on \bar{M}_0 . Without loss of generality, we may assume that \bar{M}_0 is spanned by the first two factors and \bar{M}_1 is spanned by the last $N - 2$ factors. Thus, we obtain two unramified homomorphisms $\bar{r}_0: T_q \rightarrow \mathcal{G}_2(k)$ and $\bar{r}_1: T_q \rightarrow \mathcal{G}_{N-2}(k)$. Let \mathcal{D}_0 be the local deformation problem of \bar{r}_0 classifying liftings r_0 of \bar{r}_0 so that the characteristic polynomial of $r_0^{\natural}(t)$ is $(T - 1)^2$. Let \mathcal{D}_1 be the local deformation problem of \bar{r}_1 classifying unramified liftings.

Suppose $N \geq 3$. We say that lifting r of \bar{r} to an object R of $\mathcal{C}_{\mathcal{O}}$ is *standard* if

$$r^{\natural}(t) = \begin{pmatrix} A_0 & 0 \\ 0 & 1_{N-2} \end{pmatrix}, \quad r(\phi_q) = \left(\begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix}, (-1)^{\mu+1} q^{1-N}, \mathbf{c} \right)$$

for some $A_0, B_0 \in \text{GL}_2(R)$ and $B_1 \in \text{GL}_{N-2}(R)$. Let $\mathcal{D}_{0,1}^{\text{mix}} \subseteq \mathcal{D}^{\text{mix}}$ be the locus of standard liftings. Then we have a natural isomorphism

$$\mathcal{D}_{0,1}^{\text{mix}} \simeq \mathcal{D}_0 \times_{\text{Spf } \mathcal{O}} \mathcal{D}_1$$

of formal schemes over $\text{Spf } \mathcal{O}$.

For $n \geq 1$, denote by \mathcal{L}_n the formal completion of $\text{GL}_{n,\mathcal{O}}$ along the unit section. Then \mathcal{L}_N acts on \mathcal{D}^{mix} by conjugation. We claim that $\mathcal{D}_{0,1}^{\text{mix}}$ generates \mathcal{D}^{mix} under the action of \mathcal{L}_N . For this, it suffices to show that for every lifting r of \bar{r} to an object R of $\mathcal{C}_{\mathcal{O}}$, the maps

$$B: M_0 \rightarrow R^{\oplus N} \rightarrow M_1, \quad B: M_1 \rightarrow R^{\oplus N} \rightarrow M_0$$

induced by B from Lemma E.1.3(1) for $\gamma = \phi_q$, are both zero. Since the two maps intertwine the actions r^{\natural} and $r^{\natural,\vee} \otimes \epsilon_{\ell}^{1-N}$ of T_{q^2} , it suffices to show that the generalized eigenvalues of $r_0^{\natural,\vee} \otimes \epsilon_{\ell}^{1-N}(\phi_q^2)$ and the generalized eigenvalues of $r_1^{\natural}(\phi_q^2)$ are disjoint. However, this follows from the condition that the generalized eigenvalues of $\bar{r}^{\natural}(\phi_w)$ in $\overline{\mathbb{F}}_{\ell}$ contain the pair $\{q^{-N}, q^{-N+2}\}$ exactly once.

The above claim induces a canonical isomorphism

$$\mathcal{D}_{0,1}^{\text{mix}} \times_{\text{Spf } \mathcal{O}} (\mathcal{L}_2 \times_{\text{Spf } \mathcal{O}} \mathcal{L}_{N-2} \setminus \mathcal{L}_N) \xrightarrow{\sim} \mathcal{D}^{\text{mix}}.$$

By Proposition E.4.5, \mathcal{D}_1 is formally smooth over $\text{Spf } \mathcal{O}$ of pure relative dimensions $(N-2)^2$. Since $\mathcal{L}_2 \times_{\text{Spf } \mathcal{O}} \mathcal{L}_{N-2} \setminus \mathcal{L}_N$ is formally smooth over $\text{Spf } \mathcal{O}$ of pure relative dimension $N^2 - (N-2)^2 - 4$, it suffices to prove the proposition for $N = 2$.

Now we assume $N = 2$. After changing a basis, we may assume

$$\bar{r}(\phi_q)(\bar{B}, (-1)^{\mu+1}q^{1-N}, \mathfrak{c}), \quad \bar{B} = \begin{pmatrix} 0 & (-1)^{\mu+1} \\ q & 0 \end{pmatrix}.$$

Then we have

$$\bar{r}^{\natural}(\phi_q^2) = (-1)^{\mu+1}q^{1-N}\bar{B}^t\bar{B}^{-1} = \begin{pmatrix} q^{-N} & 0 \\ 0 & q^{-N+2} \end{pmatrix}.$$

For every object R of $\mathcal{C}_{\mathcal{O}}$, the set $\mathcal{D}^{\text{mix}}(R)$ is bijective to the set of pairs (B, X) where $B \in \text{GL}_2(R)$ and $X \in \text{M}_2(R)$ satisfying $B \equiv \bar{B} \pmod{\mathfrak{m}_R}$, $X \equiv 0 \pmod{\mathfrak{m}_R}$, that the characteristic polynomial of X is T^2 , and the relation

$$(E.20) \quad B^t X B^{-1} = -qX.$$

Indeed, the bijection is given by $r(\phi_q) = (B, (-1)^{\mu+1}q^{1-N}, \mathfrak{c})$ and $r^{\natural}(t) = 1_2 + X$. We let $\mathcal{D}_0^{\text{mix}}$ be the subscheme of \mathcal{D}^{mix} defined by the condition that $r^{\natural}(\phi_q^2) = (-1)^{\mu+1}q^{1-N}B^tB^{-1}$ is a diagonal matrix. Take a lifting $r \in \mathcal{D}_0^{\text{mix}}(R)$ corresponding to the pair (B, X) ; we must have

$$B = \begin{pmatrix} 0 & (-1)^{\mu+1}(1+x) \\ q(1+y) & 0 \end{pmatrix}, \quad r^{\natural}(\phi_q^2) = \begin{pmatrix} q^{-N}(1+x)(1+y)^{-1} & 0 \\ 0 & q^{-N+2}(1+y)(1+x)^{-1} \end{pmatrix}$$

for some $x, y \in \mathfrak{m}_R$. Then by (E.20), $X = \begin{pmatrix} 0 & 0 \\ x_0 & 0 \end{pmatrix}$ for some $x_0 \in \mathfrak{m}_R$ satisfying $(x-y)x_0 = 0$. Put $x_1 := x - y$. Then we obtain an isomorphism

$$\mathcal{D}_0^{\text{mix}} \simeq \text{Spf } \mathcal{O}[[x_0, x_1, y]]/(x_0x_1)$$

such that

- $x_0 = 0$ if and only if r is unramified;
- $x_1 = 0$ if and only if $P_0(T) = (T - q^{-N})(T - q^{-N+2})$, where P_0 is the characteristic polynomial of $r^{\natural}(\phi_w) = r^{\natural}(\phi_q^2)$.

Finally, note that \mathcal{L}_2 acts on \mathcal{D}^{mix} by conjugation, which induces a canonical isomorphism

$$\mathcal{D}_0^{\text{mix}} \times_{\text{Spf } \mathcal{O}} (\mathcal{L}_1 \times_{\text{Spf } \mathcal{O}} \mathcal{L}_1 \setminus \mathcal{L}_2) \xrightarrow{\sim} \mathcal{D}^{\text{mix}}.$$

The proposition (for $N = 2$) follows as $\mathcal{L}_1 \times_{\text{Spf } \mathcal{O}} \mathcal{L}_1 \setminus \mathcal{L}_2$ is formally smooth over $\text{Spf } \mathcal{O}$ of pure relative dimension 2. The entire proposition is now proved. \square

E.7. An almost minimal R=T theorem. In this subsection, we prove a version of the R=T theorem for a global Galois representation. Assume $\ell \geq N \geq 2$ and that ℓ is unramified in F .

We consider a pair (\bar{r}, χ) from Notation E.2.1 with $\tilde{\Gamma} = \Gamma_{F^+}$ and $\Gamma = \Gamma_F$, in which $\chi = \eta^\mu \epsilon_\ell^{1-N}$ for some $\mu \in \mathbb{Z}/2\mathbb{Z}$. We take two finite sets Σ_{min}^+ and Σ_{lr}^+ of nonarchimedean places of F^+ such that

- Σ_{min}^+ , Σ_{lr}^+ , and Σ_ℓ^+ are mutually disjoint;
- Σ_{min}^+ contains Σ_{ram}^+ ;
- every place $v \in F^+$ is inert in F and satisfies $\ell \nmid (\|v\|^2 - 1)$.

Definition E.7.1. We say that \bar{r} is *rigid for* $(\Sigma_{\text{min}}^+, \Sigma_{\text{lr}}^+)$ if the following are satisfied:

- (1) For v in Σ_{min}^+ , every lifting of \bar{r}_v is minimally ramified.

- (2) For v in Σ_{lr}^+ , the generalized eigenvalues of $\bar{r}_v^\natural(\phi_w)$ in $\bar{\mathbb{F}}_\ell$ contain the pair $\{\|v\|^{-N}, \|v\|^{-N+2}\}$ exactly once, where w is the unique place of F above v .
- (3) For v in Σ_ℓ^+ , \bar{r}_v^\natural is crystalline with regular Fontaine–Laffaille weights in $[0, N - 1]$ (Definition E.3.4).
- (4) For a nonarchimedean place v of F^+ not in $\Sigma_{\text{min}}^+ \cup \Sigma_{\text{lr}}^+ \cup \Sigma_\ell^+$, the homomorphism \bar{r}_v is unramified.

Suppose now that \bar{r} is rigid for $(\Sigma_{\text{min}}^+, \Sigma_{\text{lr}}^+)$. Consider a global deformation problem (Definition E.2.6)

$$\mathcal{S} := (\bar{r}, \eta^\mu \epsilon_\ell^{1-N}, \Sigma_{\text{min}}^+ \cup \Sigma_{\text{lr}}^+ \cup \Sigma_\ell^+, \{\mathcal{D}_v\}_{v \in \Sigma_{\text{min}}^+ \cup \Sigma_{\text{lr}}^+ \cup \Sigma_\ell^+})$$

where

- for $v \in \Sigma_{\text{min}}^+$, \mathcal{D}_v is the local deformation problem classifying all liftings of \bar{r}_v ;
- for $v \in \Sigma_{\text{lr}}^+$, \mathcal{D}_v is the local deformation problem $\mathcal{D}_v^{\text{ram}}$ of \bar{r}_v from Definition E.6.1;
- for $v \in \Sigma_\ell^+$, \mathcal{D}_v is the local deformation problem $\mathcal{D}_v^{\text{FL}}$ of \bar{r}_v from Definition E.3.6.

Then we have the *global universal deformation ring* $R_{\mathcal{S}}^{\text{univ}}$ from Proposition E.2.7.

Remark E.7.2. It is possible that \bar{r} is rigid for two pairs $(\Sigma_{\text{min}}^+, \Sigma_{\text{lr}}^+)$ and $(\Sigma_{\text{min}}^{+'}, \Sigma_{\text{lr}}^{+'})$. Then $R_{\mathcal{S}}^{\text{univ}}$ and $R_{\mathcal{S}' }^{\text{univ}}$ are different in general, where \mathcal{S}' denotes the corresponding global deformation problem for $(\Sigma_{\text{min}}^{+'}, \Sigma_{\text{lr}}^{+'})$.

Now we state an R=T theorem. Let V be a standard definite or indefinite hermitian space (Definition 3.2.1) over F of rank N , such that V_v is not split for $v \in \Sigma_{\text{lr}}^+$. We fix a self-dual $\prod_{v \notin \Sigma_{\text{min}}^+ \cup \Sigma_{\text{lr}}^+ \cup \Sigma_\ell^+} O_{F_v}$ -lattice Λ in $V \otimes_F \mathbb{A}_F^{\Sigma_{\text{min}}^+ \cup \Sigma_{\text{lr}}^+ \cup \Sigma_\ell^+}$, and an element $K \in \mathfrak{K}(V)$ (Definition 3.1.11) of the form

$$K = \prod_{v \in \Sigma_{\text{min}}^+ \cup \Sigma_{\text{lr}}^+} K_v \times \prod_{v \notin \Sigma_{\text{min}}^+ \cup \Sigma_{\text{lr}}^+ \cup \Sigma_\ell^+} U(\Lambda)(O_{F_v^+})$$

in which K_v is special maximal for $v \in \Sigma_{\text{lr}}^+$.

Let Σ^+ be a finite set of nonarchimedean places of F^+ containing $\Sigma_{\text{min}}^+ \cup \Sigma_{\text{lr}}^+$; so we have the abstract unitary Hecke algebra $\mathbb{T}_N^{\Sigma^+}$ (Definition 3.1.9). Let $\phi: \mathbb{T}_N^{\Sigma^+} \rightarrow k$ be a homomorphism such that

- for every nonarchimedean place v of F^+ not in $\Sigma^+ \cup \Sigma_\ell^+$ that induces one place w of F , we have $\phi|_{\mathbb{T}_{N,v}} = \phi_\alpha$ (Construction 3.1.8) where $\alpha = (\alpha_1, \dots, \alpha_N)$ is the unitary abstract Hecke parameter at v (Definition 3.1.3) satisfying that $\{\alpha_1 \|v\|^{N-1}, \dots, \alpha_N \|v\|^{N-1}\}$ are the generalized eigenvalues of $\bar{r}_v^\natural(\phi_w^{-1})$ in $\bar{\mathbb{F}}_\ell$;
- for every nonarchimedean place v of F^+ not in $\Sigma^+ \cup \Sigma_\ell^+$ that splits into two places w_1 and w_2 of F , we have $\phi|_{\mathbb{T}_{N,v}} = \phi_\alpha$ (Construction 3.1.8) where $\alpha = ((\alpha_{1,1}, \dots, \alpha_{1,N}); (\alpha_{2,1}, \dots, \alpha_{2,N}))$ is the unitary abstract Hecke parameter at v (Definition 3.1.3) satisfying that for $i = 1, 2$, $\{\alpha_{i,1} \sqrt{\|v\|}^{-N-1}, \dots, \alpha_{i,N} \sqrt{\|v\|}^{-N-1}\}^{34}$ are the generalized eigenvalues of $\bar{r}_v^\natural(\phi_{w_i}^{-1})$ in $\bar{\mathbb{F}}_\ell$.

We write \mathfrak{m} for the kernel of ϕ .

Theorem E.7.3. *Suppose $\Sigma_{\text{lr}}^+ = \emptyset$ if N is odd. Under the above setup, we further assume*

- (D1): ℓ is unramified in F , and $\ell \geq 2(N + 1)$;
- (D2): $\bar{r}^\natural|_{\text{Gal}(\bar{F}/F(\zeta_\ell))}$ is absolutely irreducible;
- (D3): \bar{r} is rigid for $(\Sigma_{\text{min}}^+, \Sigma_{\text{lr}}^+)$ (Definition E.7.1);
- (D4): ϕ is cohomologically generic (Definition D.1.1) when V is indefinite.

³⁴Recall that we have fixed an isomorphism $\iota_\ell: \mathbb{C} \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell$ at the beginning of this section. Thus, for every positive integer q coprime to ℓ , \sqrt{q} is a well-defined element in $\bar{\mathbb{Z}}_\ell$ hence in $\bar{\mathbb{F}}_\ell$.

Let T be the image of $\mathbb{T}_N^{\Sigma^+}$ in $\text{End}_{\mathcal{O}}(\mathbb{H}_{\text{ét}}^{N-1}(\text{Sh}(\mathbb{V}, \mathbb{K})_{\overline{F}}, \mathcal{O}))$ (resp. $\text{End}_{\mathcal{O}}(\mathcal{O}[\text{Sh}(\mathbb{V}, \mathbb{K})])$) when \mathbb{V} is indefinite (resp. definite). If $\mathsf{T}_m \neq 0$, then we have the following:

- (1) There is a canonical isomorphism $\mathbb{R}_{\mathcal{S}}^{\text{univ}} \xrightarrow{\sim} \mathsf{T}_m$ of local complete intersection \mathcal{O} -algebras.
- (2) When \mathbb{V} is indefinite (resp. definite), $\mathbb{H}_{\text{ét}}^{N-1}(\text{Sh}(\mathbb{V}, \mathbb{K})_{\overline{F}}, \mathcal{O})_m$ (resp. $\mathcal{O}[\text{Sh}(\mathbb{V}, \mathbb{K})]_m$) is a finite free T_m -module.
- (3) We have $\mu \equiv N \pmod{2}$.

The rest of this subsection is devoted to the proof of the theorem. We will use the Taylor–Wiles patching argument following [CHT08] and [Tho12]. Put $\mathsf{S} := \Sigma_{\text{min}}^+ \cup \Sigma_{\text{ir}}^+ \cup \Sigma_{\ell}^+$. To prove the theorem, we may replace E_{λ} by a finite unramified extension. Thus, we may assume that k contains all eigenvalues of matrices in $r^{\natural}(\Gamma_F)$.

Remark E.7.4. By (D1), we know that F is not contained in $F^+(\zeta_{\ell})$. Thus, by [Tho12, Theorem A.9], (D1) and (D2) imply that $\bar{r}(\text{Gal}(\overline{F}/F^+(\zeta_{\ell})))$ is adequate in the sense of [Tho12, Definition 2.3].

Recall that a prime v of F^+ is called a *Taylor–Wiles prime* for the global deformation problem \mathcal{S} if

- $v \notin \mathsf{S}$; v splits in F ; and $\|v\| \equiv 1 \pmod{\ell}$;
- \bar{r}_v is unramified;
- $\bar{r}_v^{\natural}(\phi_w)$ is not a scalar and admits an eigenvalue $\bar{\alpha}_v \in k$, called *special eigenvalue*, such that $\bar{r}_v^{\natural}(\phi_w)$ acts semisimply on the generalized eigenspace for $\bar{\alpha}_v$, where w is the place of F above v induced by the inclusion $F \subseteq \overline{F}_v^+$.

A *Taylor–Wiles system* is a tuple $(\mathsf{Q}, \{\bar{\alpha}_v\}_{v \in \mathsf{Q}})$ where Q is a finite set of Taylor–Wiles primes, and $\bar{\alpha}_v$ is a special eigenvalue for every $v \in \mathsf{Q}$. For such a system, we write $r_v^{\natural} = r_v^{\bullet} \oplus r_v^{\circ}$ for every $v \in \mathsf{Q}$, where r_v^{\bullet} (resp. r_v°) is the generalized eigenspace for $\bar{\alpha}_v$ (resp. for generalized eigenvalues other than $\bar{\alpha}_v$). Then we have another global deformation problem (see [Tho12, Definition 4.1])

$$\mathcal{S}(\mathsf{Q}) := (\bar{r}, \eta^{\mu} \epsilon_{\ell}^{1-N}, \mathsf{S} \cup \mathsf{Q}, \{\mathcal{D}_v\}_{v \in \mathsf{S} \cup \mathsf{Q}})$$

where \mathcal{D}_v is the same as in \mathcal{S} for $v \in \mathsf{S}$; and for $v \in \mathsf{Q}$, \mathcal{D}_v is the local deformation problem of \bar{r}_v that classifies liftings r_v so that r_v^{\natural} is of the form $r_v^{\bullet} \oplus r_v^{\circ}$ in which r_v^{\bullet} is a lifting of \bar{r}_v^{\bullet} on which \mathbb{I}_{F_w} acts by scalars, and r_v° is an unramified lifting of \bar{r}_v° .

We now discuss the existence of Taylor–Wiles systems. For each $v \in \mathsf{S}$, we have the tangent space $L(\mathcal{D}_v) \subseteq H^1(F_v^+, \text{ad } \bar{r})$ from Definition E.2.4. Let $L(\mathcal{D}_v)^{\perp} \subseteq H^1(F_v^+, \text{ad } \bar{r}(1))$ be the annihilator of $L(\mathcal{D}_v)$ under the local Tate duality induced by the perfect pairing $\text{ad } \bar{r} \times \text{ad } \bar{r}(1) \rightarrow k(1)$ sending (x, y) to $\text{tr}(xy)$. Recall that $\Gamma_{F^+, \mathsf{S}}$ is the Galois group of the maximal subextension of \overline{F}/F^+ that is unramified outside S . For every subset $\mathsf{T} \subseteq \mathsf{S}$, we define $H_{\mathcal{S}^{\perp}, \mathsf{T}}^1(\Gamma_{F^+, \mathsf{S}}, \text{ad } \bar{r}(1))$ to be the kernel of the natural map

$$H^1(\Gamma_{F^+, \mathsf{S}}, \text{ad } \bar{r}(1)) \rightarrow \bigoplus_{v \in \mathsf{S} \setminus \mathsf{T}} H^1(F_v^+, \text{ad } \bar{r}(1))/L(\mathcal{D}_v)^{\perp}.$$

Recall the \mathcal{O} -algebras $\mathbb{R}_{\mathcal{S}, \mathsf{T}}^{\text{loc}}$ (E.2) and $\mathbb{R}_{\mathcal{S}(\mathsf{Q})}^{\square_{\mathsf{T}}}$ from Proposition E.2.7. Moreover, $\mathbb{R}_{\mathcal{S}(\mathsf{Q})}^{\square_{\mathsf{T}}}$ is naturally an algebra over $\mathbb{R}_{\mathcal{S}, \mathsf{T}}^{\text{loc}}$.

Lemma E.7.5. *Let the situation be as in Theorem E.7.3. Let T be a subset of S . For every integer $b \geq \dim_k H_{\mathcal{S}^{\perp}, \mathsf{T}}^1(\Gamma_{F^+, \mathsf{S}}, \text{ad } \bar{r}(1))$ and every integer $n \geq 1$, there is a Taylor–Wiles system $(\mathsf{Q}_n, \{\bar{\alpha}_v\}_{v \in \mathsf{Q}_n})$ satisfying*

- (1) $|\mathsf{Q}_n| = b$;
- (2) $\|v\| \equiv 1 \pmod{\ell^n}$;

(3) $\mathbf{R}_{\mathcal{S}(\mathbb{Q}_n)}^{\square_{\mathbb{T}}}$ can be topologically generated over $\mathbf{R}_{\mathcal{S},\mathbb{T}}^{\text{loc}}$ by

$$g_{b,\mathbb{T}} := b - \sum_{v \in \mathbb{T} \cap \Sigma_{\ell}^+} [F_v^+ : \mathbb{Q}_{\ell}] \frac{N(N-1)}{2} - N[F^+ : \mathbb{Q}] \frac{1 + (-1)^{\mu+1-N}}{2}$$

elements.

Proof. By (E.1), Proposition E.3.9(3), Proposition E.5.11, and Proposition E.6.2, we have for every $v \in \mathcal{S}$ that

$$\dim_k L(\mathcal{D}_v) - \dim_k H^0(F_v^+, \text{ad } \bar{r}(1)) = \begin{cases} [F_v^+ : \mathbb{Q}_{\ell}] \frac{N(N-1)}{2} & \text{if } v \mid \ell; \\ 0 & \text{if } v \nmid \ell. \end{cases}$$

Then the lemma follows from [Tho12, Proposition 4.4]³⁵ in view of Remark E.7.4. \square

Now we take a Taylor–Wiles system $(\mathbb{Q}_n, \{\bar{\alpha}_v\}_{v \in \mathbb{Q}_n})$ as in the above lemma. For each $v \in \mathbb{Q}_n$, we

- put $d_v := \dim_k \bar{r}_v^{\bullet}$;
- let $P_{d_v} \subseteq \text{GL}_N$ be the standard upper-triangular parabolic subgroup corresponding to the partition $(N - d_v, d_v)$;
- let κ_v be the residue field of F_v^+ , and Δ_v the maximal quotient of κ_v^{\times} of ℓ -power order;
- fix an isomorphism $K_v \simeq \text{GL}_N(O_{F_v^+})$ and denote by $K_{v,0} \subseteq K_v$ the parahoric subgroup corresponding to P_{d_v} ;
- let $K_{v,1}$ be the kernel of the canonical map

$$K_{v,0} \rightarrow P_{d_v}(\kappa_v) \rightarrow \text{GL}_{d_v}(\kappa_v) \xrightarrow{\det} \kappa_v^{\times} \rightarrow \Delta_v.$$

We then

- put $\Delta_{\mathbb{Q}_n} := \prod_{v \in \mathbb{Q}_n} \Delta_v$; and let $\mathfrak{a}_{\mathbb{Q}_n}$ be the augmentation ideal of $\mathcal{O}[\Delta_{\mathbb{Q}_n}]$;
- write $\mathfrak{m}_{\mathbb{Q}_n}$ for the kernel of the composite homomorphism $\mathbb{T}_N^{\Sigma^+ \cup \mathbb{Q}_n} \rightarrow \mathbb{T}_N^{\Sigma^+} \xrightarrow{\phi} k$;
- or $i = 0, 1$, put

$$K_i(\mathbb{Q}_n) = \prod_{v \notin \mathbb{Q}_n} K_v \times \prod_{v \in \mathbb{Q}_n} K_{v,i},$$

which are subgroups of K .

In particular, $K_1(\mathbb{Q}_n)$ is a normal subgroup of $K_0(\mathbb{Q}_n)$; and we have a canonical isomorphism

$$(E.21) \quad K_0(\mathbb{Q}_n)/K_1(\mathbb{Q}_n) \xrightarrow{\sim} \Delta_{\mathbb{Q}_n}.$$

Now we introduce some patching module from automorphic input. For every open compact subgroup $\mathfrak{k} \in \{K, K_0(\mathbb{Q}_n), K_1(\mathbb{Q}_n)\}$, we put

$$H_{\mathfrak{k}} := \begin{cases} \mathcal{O}[\text{Sh}(\mathbb{V}, \mathfrak{k})] & \text{if } \mathbb{V} \text{ is definite;} \\ H_{N-1}(\text{Sh}(\mathbb{V}, \mathfrak{k})(\mathbb{C}), \mathcal{O}) & \text{if } \mathbb{V} \text{ is indefinite.} \end{cases}$$

Here, H_{N-1} stands for the singular homology for complex manifolds. By (E.21), $H_{K_1(\mathbb{Q}_n)}$ is canonically a module over $\mathcal{O}[\Delta_{\mathbb{Q}_n}]$.

Lemma E.7.6. *Let the situation be as in Theorem E.7.3. The $\mathcal{O}[\Delta_{\mathbb{Q}_n}]$ -module $H_{K_1(\mathbb{Q}_n), \mathfrak{m}_{\mathbb{Q}_n}}$ is a finite and free. Moreover, the canonical map*

$$H_{K_1(\mathbb{Q}_n), \mathfrak{m}_{\mathbb{Q}_n}} / \mathfrak{a}_{\mathbb{Q}_n} \rightarrow H_{K_0(\mathbb{Q}_n), \mathfrak{m}_{\mathbb{Q}_n}}$$

is an isomorphism.

³⁵Strictly speaking, the set S in [Tho12, Proposition 4.4] consists of only places split in F . But the same argument works in our case as well.

Proof. Note that by Definition 3.1.11, every open compact subgroup $k \in \{K, K_0(Q_n), K_1(Q_n)\}$ is neat; in particular, $t^{-1}U(V)(F^+)t \cap k$ has no torsion elements for every $t \in U(V)(\mathbb{A}_{F^+}^\infty)$.

When V is definite, the lemma (even without localization at \mathfrak{m}_{Q_n}) follows by the same argument for [Tho12, Lemma 6.4].

Suppose now that V is indefinite. For an abelian group A , we let $C_\bullet(\mathrm{Sh}(V, k)(\mathbb{C}), A)$ be the complex of singular chains for the complex manifold $\mathrm{Sh}(V, k)(\mathbb{C})$ with coefficients in A . By Artin's comparison theorem between the singular cohomology and the étale cohomology, the dual complex $\mathrm{Hom}_A(C_\bullet(\mathrm{Sh}(V, k)(\mathbb{C}), A), A)$ calculates $H_{\text{ét}}^\bullet(\mathrm{Sh}(V, k)_{\mathbb{C}}, A)$. Now take $A = \mathcal{O}/\lambda^m$ for integers $m \geq 1$. By (D4), we know that

$$H_i(\mathrm{Sh}(V, k)(\mathbb{C}), \mathcal{O}/\lambda^m)_{\mathfrak{m}_{Q_n}} = H_i(C_\bullet(\mathrm{Sh}(V, k)(\mathbb{C}), \mathcal{O}/\lambda^m)_{\mathfrak{m}_{Q_n}}) = 0$$

for every $i \neq N - 1$. On the other hand, by [KT17, Lemma 6.9], for every $m \geq 1$, $C_\bullet(\mathrm{Sh}(V, K_1(Q_n))(\mathbb{C}), \mathcal{O}/\lambda^m)$ is a perfect complex of free $\mathcal{O}/\lambda^m[\Delta_{Q_n}]$ -modules; and there is a canonical isomorphism

$$C_\bullet(\mathrm{Sh}(V, K_1(Q_n))(\mathbb{C}), \mathcal{O}/\lambda^m) \otimes_{\mathcal{O}[\Delta_{Q_n}]} \mathcal{O}[\Delta_{Q_n}]/\mathfrak{a}_{Q_n} \simeq C_\bullet(\mathrm{Sh}(V, K_0(Q_n))(\mathbb{C}), \mathcal{O}/\lambda^m).$$

Then the lemma follows easily by taking the homology group and passing to the limit for $m \rightarrow \infty$. \square

Proof of Theorem E.7.3. When V is indefinite, by (D4) and Artin's comparison theorem between the singular cohomology and the étale cohomology, we have a canonical isomorphism $H_{K, m} \simeq \mathrm{Hom}_{\mathcal{O}}(H_{\text{ét}}^{N-1}(\mathrm{Sh}(V, K)_{\overline{F}}, \mathcal{O})_{\mathfrak{m}}, \mathcal{O})$, under which $\mathbb{T}_{\mathfrak{m}}$ is identified with the image of $\mathbb{T}_N^{\Sigma^+}$ in $\mathrm{End}_{\mathcal{O}}(H_{K, m})$. Thus, in both cases, $H_{K, m}$ is a finite free \mathcal{O} -module.

First, we need to construct a canonical homomorphism $\mathbb{R}_{\mathcal{S}}^{\mathrm{univ}} \rightarrow \mathbb{T}_{\mathfrak{m}}$. It is well-known that $\mathbb{T}_{\mathfrak{m}}[1/\ell]$ is a reduced finite E_λ -algebra. As $H_{K, m}$ is a finite free \mathcal{O} -module, $\mathbb{T}_{\mathfrak{m}}$ is a reduced finite flat \mathcal{O} -algebra. Every point $x \in \mathrm{Spec} \mathbb{T}_{\mathfrak{m}}[1/\ell]$ corresponds to a relevant representation Π_x of $\mathrm{GL}_N(\mathbb{A}_F)$ (Definition 1.1.3) such that

- the associated Galois representation $\rho_{\Pi_x, \ell}$ from Proposition 3.2.4(2) is residually isomorphic to $\bar{r}^{\natural} \otimes_k \overline{\mathbb{F}}_\ell$ (hence residually absolutely irreducible by (D2));
- there exists a cuspidal automorphic representation of $U(V)(\mathbb{A}_{F^+})$ satisfying $\mathrm{BC}(\pi) \simeq \Pi$ and $\pi^K \neq \{0\}$.

In fact, $\rho_{\Pi_x, \ell}$ comes from a continuous homomorphism

$$\rho_x : \Gamma_F \rightarrow \mathrm{GL}_N(\mathbb{T}_x),$$

which is a lifting of \bar{r}^{\natural} . By a theorem of Carayol [Car94, Théorème 2], the product homomorphism

$$\prod_{x \in \mathrm{Spec} \mathbb{T}_{\mathfrak{m}}[1/\ell]} \rho_x : \Gamma_F \rightarrow \mathrm{GL}_N(\prod_x \mathbb{T}_x)$$

is conjugate to some continuous homomorphism $\rho_{\mathfrak{m}} : \Gamma_F \rightarrow \mathrm{GL}_N(\mathbb{T}_{\mathfrak{m}})$ that is a lifting of \bar{r}^{\natural} . Moreover, by Proposition 3.2.4(2), we know that $\rho_{\mathfrak{m}}$ is $(1 - N)$ -polarizable (Definition 2.4.7). Thus, by Lemma E.1.3, we obtain a continuous homomorphism

$$r_{\mathfrak{m}} : \Gamma_{F^+} \rightarrow \mathcal{G}_N(\mathbb{T}_{\mathfrak{m}})$$

satisfying $r_{\mathfrak{m}}^{\natural} = \rho_{\mathfrak{m}}$, which is a lifting of \bar{r} . We claim that $r_{\mathfrak{m}}$ satisfies the global deformation problem \mathcal{S} . Indeed, since $\Pi_{x, w}$ is unramified for nonarchimedean places w of F not above $\Sigma_{\mathrm{min}}^+ \cup \Sigma_{\mathrm{lr}}^+$, we know that $\bar{r}_{\mathfrak{m}, v}$ belongs to $\mathcal{D}_v^{\mathrm{FL}}$ for $v \in \Sigma_\ell^+$ by [CH13, Theorem 3.2.3(b,c)]; and $\bar{r}_{\mathfrak{m}, v}$ is unramified for $v \notin S$ by Proposition 3.2.4(2). By Lemma C.2.4 and Proposition 3.2.4(2), $\bar{r}_{\mathfrak{m}, v}$ belongs to $\mathcal{D}_v^{\mathrm{ram}}$

for $v \in \Sigma_{\text{lr}}^+$.³⁶ Therefore, by the universal property of $\mathbf{R}_{\mathcal{F}}^{\text{univ}}$, we obtain a canonical homomorphism

$$(E.22) \quad \varphi: \mathbf{R}_{\mathcal{F}}^{\text{univ}} \rightarrow \mathbf{T}_{\mathfrak{m}}$$

of \mathcal{O} -algebras. Moreover, it is clear that our homomorphism $r_{\mathfrak{m}}$ satisfies [CHT08, Proposition 3.4.4](2,3) as well, which implies that φ is surjective. Thus, it remains to show that φ is injective.

We follow the strategy for [Tho12, Theorem 6.8]. We take an integer $n \geq 1$, and a Taylor–Wiles system $(Q_n, \{\bar{\alpha}_v\}_{v \in Q_n})$ from Lemma E.7.5. For each $v \in Q_n$, we

- let $\text{Art}_v: F_v^\times \rightarrow \Gamma_{F_v^+}^{\text{ab}}$ be the local Artin map;
- let $\varpi_v \in F_v^+$ be the uniformizer such that $\text{Art}_v(\varpi_v)$ coincides with the image of ϕ_v^{-1} in $\Gamma_{F_v^+}^{\text{ab}}$;
- let pr_{ϖ_v} be the commuting projection defined in [Tho12, Propositions 5.9 & 5.12];
- for every $\alpha \in O_{F_v^+}^\times$, let $\mathbf{V}_v^\alpha \in \mathbb{Z}[K_{v,1} \setminus K_{v,0}/K_{v,1}]$ be the characteristic function of the double coset

$$K_{v,1} \begin{pmatrix} 1_{N-1} & 0 \\ 0 & \alpha \end{pmatrix} K_{v,1}$$

For $i = 0, 1$, we put

$$M_{i,Q_n} := \left(\prod_{v \in Q_n} \text{pr}_{\varpi_v} \right) H_{K_i(Q_n), \mathfrak{m}_{Q_n}},$$

and let \mathbf{T}_{i,Q_n} be the image of $\mathbb{T}_N^{\Sigma^+ \cup Q_n}$ in $\text{End}_{\mathcal{O}}(M_{i,Q_n})$. We also put

$$M := H_{K, \mathfrak{m}}.$$

Then the canonical map $M \rightarrow H_{K, \mathfrak{m}_{Q_n}}$ is an isomorphism, hence we obtain canonical surjective homomorphisms

$$\mathbf{T}_{1,Q_n} \twoheadrightarrow \mathbf{T}_{0,Q_n} \twoheadrightarrow \mathbf{T}_{\mathfrak{m}}$$

of \mathcal{O} -algebras. Similar to $\mathbf{T}_{\mathfrak{m}}$, we obtain a continuous homomorphism

$$r_{i,Q_n}: \Gamma_{F^+} \rightarrow \mathcal{G}_N(\mathbf{T}_{i,Q_n}),$$

which is a lifting of \bar{r} , for $i = 0, 1$. We have the following two claims:

- (1) For every $v \in Q_n$, there is a continuous character $\mathbf{v}_v: O_{F_v^+}^\times \rightarrow \mathbf{T}_{1,Q_n}^\times$ such that
 - (a) for every $\alpha \in O_{F_v^+}^\times$, the actions of \mathbf{V}_v^α and $\mathbf{v}_v(\alpha)$ on M_{1,Q_n} coincide;
 - (b) $r_{1,Q_n,v}^\natural$ has a (unique) decomposition $r_{1,Q_n,v}^\bullet \oplus r_{1,Q_n,v}^\circ$ such that $r_{1,Q_n,v}^\bullet$ is a lifting of \bar{r}_v^\bullet on which $I_{F_v^+}$ acts via the character $\mathbf{v}_v \circ \text{Art}_v^{-1}$, and $r_{1,Q_n,v}^\circ$ is an unramified lifting of \bar{r}_v° .
- (2) The composite map

$$M = H_{K, \mathfrak{m}_{Q_n}} \rightarrow H_{K_0(Q_n), \mathfrak{m}_{Q_n}} \xrightarrow{\prod_{v \in Q_n} \text{pr}_{\varpi_v}} M_{0,Q_n}$$

is an isomorphism. In particular, the canonical homomorphism $\mathbf{T}_{0,Q_n} \rightarrow \mathbf{T}_{\mathfrak{m}}$ is an isomorphism; and r_{0,Q_n} and $r_{\mathfrak{m}}$ are equivalent liftings of \bar{r} .

³⁶This is not correct if N is odd, which is the only reason that we suppose $\Sigma_{\text{lr}}^+ = \emptyset$ if N is odd in the statement of the theorem.

Indeed, these claims follow easily from [Tho12, Propositions 5.9 & 5.12].

It follows from (1) that r_{1, Q_n} satisfies the global deformation problem $\mathcal{S}(Q_n)$, which induces a canonical surjective homomorphism

$$\varphi_n : \mathbf{R}_{\mathcal{S}(Q_n)}^{\text{univ}} \rightarrow \mathbf{T}_{1, Q_n}$$

of \mathcal{O} -algebras. Now we claim that φ_n is naturally a homomorphism of $\mathcal{O}[\Delta_{Q_n}]$ -algebras. Indeed, take a universal lifting $r_{\mathcal{S}(Q_n)}^{\text{univ}}$ for \bar{r} over $\mathbf{R}_{\mathcal{S}(Q_n)}^{\text{univ}}$. Then for each $v \in Q_n$, there is a unique character $v_v^{\text{univ}} : \Delta_v \rightarrow (\mathbf{R}_{\mathcal{S}(Q_n)}^{\text{univ}})^{\times}$ such that $I_{F_v^+}$ acts on $r_{\mathcal{S}(Q_n), v}^{\text{univ}, \bullet}$ via the character

$$I_{F_v^+} \xrightarrow{\text{Art}_v^{-1}} \mathcal{O}_{F_v^+}^{\times} \rightarrow \kappa_v^{\times} \rightarrow \Delta_v \xrightarrow{v_v^{\text{univ}}} (\mathbf{R}_{\mathcal{S}(Q_n)}^{\text{univ}})^{\times}.$$

Then $\mathbf{R}_{\mathcal{S}(Q_n)}^{\text{univ}}$ becomes an $\mathcal{O}[\Delta_{Q_n}]$ -algebra via the character $\prod_{v \in Q_n} v_v^{\text{univ}} : \Delta_{Q_n} \rightarrow (\mathbf{R}_{\mathcal{S}(Q_n)}^{\text{univ}})^{\times}$. Moreover, φ_n is a homomorphism of $\mathcal{O}[\Delta_{Q_n}]$ -algebras. By (2) and Lemma E.7.6, we obtain a canonical commutative diagram

$$\begin{array}{ccc} \mathbf{R}_{\mathcal{S}(Q_n)}^{\text{univ}} / \mathfrak{a}_{Q_n} & \xrightarrow{\sim} & \mathbf{R}_{\mathcal{S}}^{\text{univ}} \\ \varphi_n / \mathfrak{a}_{Q_n} \downarrow & & \downarrow \varphi \\ \mathbf{T}_{1, Q_n} / \mathfrak{a}_{Q_n} & \xrightarrow{\sim} & \mathbf{T}_{0, Q_n} \xrightarrow{\sim} \mathbf{T}_m \end{array}$$

of \mathcal{O} -algebras where all horizontal arrows are isomorphisms.

Now we fix a subset $T \subseteq S$ of cardinality t . Choose universal liftings

$$r_{\mathcal{S}}^{\text{univ}} : \Gamma_{F^+} \rightarrow \mathcal{G}_N(\mathbf{R}_{\mathcal{S}}^{\text{univ}}), \quad r_{\mathcal{S}(Q_n)}^{\text{univ}} : \Gamma_{F^+} \rightarrow \mathcal{G}_N(\mathbf{R}_{\mathcal{S}(Q_n)}^{\text{univ}})$$

for \bar{r} over $\mathbf{R}_{\mathcal{S}}^{\text{univ}}$ and $\mathbf{R}_{\mathcal{S}(Q_n)}^{\text{univ}}$, respectively, such that $r_{\mathcal{S}}^{\text{univ}} = r_{\mathcal{S}(Q_n)}^{\text{univ}} \bmod \mathfrak{a}_{Q_n}$. By Proposition E.2.7(2), we obtain isomorphisms

$$\mathbf{R}_{\mathcal{S}}^{\text{univ}} [[X_{v; i, j}]_{v \in T; 1 \leq i, j \leq N}] \xrightarrow{\sim} \mathbf{R}_{\mathcal{S}}^{\square_T}, \quad \mathbf{R}_{\mathcal{S}(Q_n)}^{\text{univ}} [[X_{v; i, j}]_{v \in T; 1 \leq i, j \leq N}] \xrightarrow{\sim} \mathbf{R}_{\mathcal{S}(Q_n)}^{\square_T}$$

of \mathcal{O} -algebras. In particular, we have a surjective homomorphism $\mathbf{R}_{\mathcal{S}}^{\square_T} \rightarrow \mathbf{R}_{\mathcal{S}}^{\text{univ}}$, which makes $\mathbf{R}_{\mathcal{S}}^{\text{univ}}$ an algebra over $\mathbf{R}_{\mathcal{S}, T}^{\text{loc}}$.

We put

$$S_{\infty} := \mathcal{O}[[X_{v; i, j}]_{v \in T; 1 \leq i, j \leq N} [[Y_1, \dots, Y_b]]];$$

and let $\mathfrak{a}_{\infty} \subseteq S_{\infty}$ be the augmentation ideal. Put

$$R_{\infty} := \mathbf{R}_{\mathcal{S}, T}^{\text{loc}} [[Z_1, \dots, Z_{g_{b, T}}]]$$

where $g_{b, T}$ is the number appeared in Lemma E.7.5. Applying the usual patching lemma (see the proof of [BLGG11, Theorem 3.6.1], or [Tho12, Lemma 6.10]), we have the following:

- There exists a homomorphism $S_{\infty} \rightarrow R_{\infty}$ of \mathcal{O} -algebras so that we have an isomorphism $R_{\infty} / \mathfrak{a}_{\infty} R_{\infty} \simeq \mathbf{R}_{\mathcal{S}}^{\text{univ}}$ of $\mathbf{R}_{\mathcal{S}, T}^{\text{loc}}$ -algebras.
- There exist an R_{∞} -module M_{∞} and an isomorphism $M_{\infty} / \mathfrak{a}_{\infty} M_{\infty} \simeq M$ of $\mathbf{R}_{\mathcal{S}}^{\text{univ}}$ -modules.
- As an S_{∞} -module, M_{∞} is finite and free.

In particular, we have

$$\text{depth}_{R_{\infty}}(M_{\infty}) \geq \dim S_{\infty} = 1 + |T|N^2 + b.$$

On the other hand, by Proposition E.3.9, Proposition E.5.11(3), and Proposition E.6.2, we know that $\mathbf{R}_{\mathcal{S}, T}^{\text{loc}}$ is a formal power series ring over \mathcal{O} in

$$|T|N^2 + \sum_{v \in T \cap \Sigma_{\ell}^+} [F_v^+ : \mathbb{Q}_{\ell}] \frac{N(N-1)}{2}$$

variables. It follows that R_∞ is a regular local ring of dimension

$$1 + |\mathbb{T}|N^2 + \sum_{v \in \mathbb{T} \cap \Sigma^+} [F_v^+ : \mathbb{Q}_\ell] \frac{N(N-1)}{2} + g_{b,\mathbb{T}} = 1 + |\mathbb{T}|N^2 + b - N[F^+ : \mathbb{Q}] \frac{1 + (-1)^{\mu+1-N}}{2}.$$

As $\dim R_\infty \geq \text{depth}_{R_\infty}(M_\infty)$, we obtain Theorem E.7.3(3). By the Auslander–Buchsbaum theorem, M_∞ is a finite free R_∞ -module. Thus, M is a finite free $\mathbf{R}_{\mathcal{F}}^{\text{univ}}$ -module. In particular, the surjective homomorphism φ (E.22) is injective hence an isomorphism. Theorem E.7.3(1,2) are proved. \square

E.8. Rigidity of automorphic Galois representations. In this subsection, we study the rigidity property for reduction of automorphic Galois representations. Let us take the initial setup of Subsection 6.1, hence let Π be a relevant representation of $\text{GL}_N(\mathbb{A}_F)$ (Definition 1.1.3) for $N \geq 2$, and $E \subseteq \mathbb{C}$ a strong coefficient field of Π (Definition 3.2.5). Then for every prime λ of E , we have a continuous homomorphism $\rho_{\Pi,\lambda} : \Gamma_F \rightarrow \text{GL}_N(E_\lambda)$.

Conjecture E.8.1. *Let Π and E be as above. Fix a finite set Σ^+ of nonarchimedean place of F^+ containing Σ_Π^+ (Notation 3.1.4). Then for all but finitely many primes λ of E , we have*

- (1) $\rho_{\Pi,\lambda}$ is residually absolutely irreducible (so we have the residual homomorphism $\bar{\rho}_{\Pi,\lambda}$ and may put $\bar{r}_{\Pi,\lambda} := \bar{\rho}_{\Pi,\lambda,+}$ from Remark 6.1.6), with the similitude character $\chi_\lambda := \eta_{F/F^+}^N \epsilon_\ell^{1-N}$;
- (2) $\bar{\rho}_{\Pi,\lambda}|_{\text{Gal}(\bar{F}/F(\zeta_\ell))}$ is absolutely irreducible, where ℓ is the underlying rational prime of λ ;
- (3) $\bar{r}_{\Pi,\lambda}$ is rigid for $(\Sigma_\Pi^+, \emptyset)$ (Definition E.7.1).

Remark E.8.2. When $N = 2$, Conjecture E.8.1 is not hard to verify. In fact, if the coefficient field of Π is \mathbb{Q} , then it follows from Proposition E.5.12 and Serre’s theorem on the image of residual Galois representations of elliptic curves [Ser72].

Concerning Conjecture E.8.1(1), we have the following lemma.

Lemma E.8.3. *Let Π and E be as above. Suppose that there exists a nonarchimedean place w of F such that Π_w is supercuspidal. Then there exists a finite set Λ_1 of primes of E depending on Π_w only, such that for every $\lambda \notin \Lambda_1$, $\rho_{\Pi,\lambda}$ is residually absolutely irreducible.*

Proof. Let W_{F_w} be the Weil group of F_w . Since Π_w is supercuspidal, we have the induced continuous representation $\rho_{\Pi_w} : W_{F_w} \rightarrow \text{GL}_N(\mathbb{C})$ via the local Langlands correspondence, which is irreducible. Fix an arithmetic Frobenius element ϕ_w in W_{F_w} . We have a natural quotient map $W_{F_w} \rightarrow \mathbb{Z}$ sending ϕ_w to 1. For every integer $b \geq 1$, let $W_{F_w}^b$ be the inverse image of $b\mathbb{Z}$. Then there exist an absolutely irreducible representation τ of I_{F_w} and a character χ of $W_{F_w}^b$, such that the underlying representation of ρ_{Π_w} is isomorphic to $\text{Ind}_{W_{F_w}^b}^{W_{F_w}} \tau \otimes \chi$, where b is the smallest positive integer satisfying $\tau^{\phi_w^b} \simeq \tau$. We may choose a finite extension E' of E inside \mathbb{C} , and a finite set Λ' of primes of E' , such that both τ and χ are defined over $O_{E',(\Lambda')}$. In particular, the image of ρ_{Π_w} is contained in $\text{GL}_N(O_{E',(\Lambda')})$, up to conjugation. We claim that there exists a finite set Λ'_1 of primes of E' containing Λ' , such that the composite map

$$\rho_{\Pi_w} : W_{F_w} \rightarrow \text{GL}_N(O_{E',(\Lambda')}) \rightarrow \text{GL}_N(O_{E'}/\lambda')$$

is absolutely irreducible for $\lambda' \notin \Lambda'_1$. Now let Λ_1 be the set of primes of E underlying Λ'_1 . Then the lemma follows by Proposition 3.2.4(2).

It remains to show the claim. In fact, let Λ'_1 be the smallest set of primes of E' containing Λ' such that every $\lambda' \notin \Lambda'_1$ satisfies

- the underlying rational prime does not divide $b|I_{F_w}/\ker \rho_{\Pi_w}|$;
- $\bar{\tau}_{\lambda'} := \tau \otimes_{O_{E',(\Lambda')}} O_{E'}/\lambda'$ remains absolutely irreducible;

○ b remains the smallest positive integer that satisfies $\bar{\tau}_{\lambda'}^{\phi_w^b} \simeq \bar{\tau}_{\lambda'}$.

Then Λ'_1 is a finite set, meeting the requirement in the claim. The lemma is proved. □

Concerning the entire Conjecture E.8.1, we have the following theorem.

Theorem E.8.4. *Let Π and E be as above. Suppose that there exist two nonarchimedean places w_c and w_s of F such that Π_{w_c} is supercuspidal and Π_{w_s} is a twist of the Steinberg representation, respectively, in which w_s is not above Σ_{ram}^+ . Then Conjecture E.8.1 holds for Π and E .*

The rest of this subsection is devoted to the proof of the theorem. Let v_c and v_s be the primes of F^+ underlying w_c and w_s , respectively, both of which are in Σ_{Π}^+ . Without loss of generality, we may assume that w_c and w_s are induced by the embedding $F \hookrightarrow \bar{F}_{v_c}^+$ and $F \hookrightarrow \bar{F}_{v_s}^+$, respectively. Let p_s be the underlying rational prime of w_s , which is then unramified in F .

Lemma E.8.5. *Under the situation of Theorem E.8.4, there exists a finite set Λ_2 of primes of E containing Λ_1 such that for every λ not in Λ_2 , the restriction $\bar{\rho}_{\Pi, \lambda}|_{\text{Gal}(\bar{F}/F(\zeta_\ell))}$ remains absolutely irreducible, where ℓ is the underlying rational prime of λ .*

Proof. By twisting Π with a Dirichlet character, we may assume that Π_{w_s} is just the Steinberg representation.

We first use the integral model of certain Shimura variety to compute the reduction of the monodromy operator at w_s . Choose a finite extension $F \subseteq \check{F} \subseteq \mathbb{C}$ satisfying

- \check{F} is a CM field, with $\check{F}^+ := \check{F}^{c=1}$;
- \check{F}/F is Galois and soluble;
- \check{F} contains an imaginary quadratic field;
- $[\check{F}^+ : \mathbb{Q}]$ is even;
- p_s is unramified in \check{F} ;
- the extension \check{F}/\check{F}^+ is unramified at all nonarchimedean places;
- every place of \check{F}^+ above p splits in \check{F} if p underlies Σ_{Π}^+ ;
- w_c splits completely in \check{F} ;
- for every prime w of F not above v_c , the local base change of Π_w to $\check{F}_{\check{w}}$ has nonzero Iwahori fixed vectors for every prime \check{w} of \check{F} above w .

Let \check{w}_s be the prime of \check{F} induced by the embedding $\check{F} \hookrightarrow \bar{F}_{v_s}^+$ which is above w_s , and \check{v}_s the prime of \check{F}^+ underlying \check{w}_s . Let $\check{\Pi}$ be the base change of Π to \check{F} by [AC89], which is again a relevant representation of $\text{GL}_N(\mathbb{A}_{\check{F}})$. Moreover, we have $\rho_{\check{\Pi}, \lambda} = \rho_{\Pi, \lambda}|_{\text{Gal}(\bar{F}/\check{F})}$ for every prime λ of E , and that $\rho_{\check{\Pi}, \lambda}$ is absolutely irreducible for every λ not in Λ_1 from Lemma E.8.3 (with $w = w_c$).

We fix following data

- a central division algebra B over \check{F} of dimension N^2 as in [HT01, Section I.7], whose invariants at \check{w}_s and \check{w}_s^c are $1/N$ and $-1/N$, respectively, and that splits at all other places of \check{F} ;
- an element $\beta \in B^\times$ as in [HT01, Lemma I.7.1] with τ the default archimedean place, which gives rise to a reductive group $G := G_\beta$ over \mathbb{Q} ;
- a cuspidal automorphic representation π of $G(\mathbb{A})$ (as in [HT01, Theorem VI.2.1]) satisfying
 - for every rational prime p not underlying Σ_{Π}^+ , π_p is unramified;
 - $\text{BC}(\pi) = (\psi, \check{\Pi}_B)$, where $\check{\Pi}_B$ is the Jacquet–Langlands transfer of $\check{\Pi}$ to $(B^{\text{op}})^\times$, and ψ is a Dirichlet character that is unramified at p_s ;
 - π^∞ is defined over E (by possibly replacing E by a finite extension in \mathbb{C});
- a decomposable open compact subgroup U of $G(\mathbb{A}^\infty)$ satisfying
 - U is neat;

- $\pi^U \neq \{0\}$;
- U_p is hyperspecial maximal for p not underlying $\Sigma_{\mathbb{H}}^+$;
- U_{p_s} is of the form $\mathbb{Z}_{p_s}^\times \times U_{\check{v}_s} \times U_{p_s}^{\check{v}_s}$ in which $U_{\check{v}_s}$ is the maximal open compact subgroup of $B_{\check{w}_s}^\times$.

Then we obtain a Shimura variety X_U over \check{F} as in [HT01, Section III.1], which is projective and smooth of relative dimension $N - 1$. Moreover, $X_U \otimes_{\check{F}} \check{F}_{\check{w}_s}$ admits a natural projective integral model \mathbf{X}_U over $O_{\check{F}_{\check{w}_s}}$, which is strictly semistable, whose strata on the special fiber are strata of the reduced subscheme of Drinfeld’s formal upper half space of relative dimension $N - 1$ (see, for example, [Tho14, Theorem 6.2]). For every prime λ of E , we have the associated weight spectral sequence $E_{s,\lambda}^{p,q}$ converging to $H_{\text{ét}}^{p+q}(X_U \otimes_{\check{F}} \overline{F}, O_\lambda)$, together with a monodromy map $\mu_\lambda: E_{1,\lambda}^{\bullet,\bullet} \rightarrow E_{1,\lambda}^{\bullet+2,\bullet-2}$. Now since strata of the reduced subscheme of Drinfeld’s formal upper half space satisfy [Ito05, Assumption 2.1] by the proof of [Ito05, Theorem 1.1], we have a canonical finite bicomplex $E_1^{p,q}$ of finite free O_E -modules with only horizontal differentials together with a monodromy map $\mu: E_1^{\bullet,\bullet} \rightarrow E_1^{\bullet+2,\bullet-2}$, such that for every prime λ of E , $E_{1,O_E}^{p,q} \otimes_{O_E} O_\lambda$ is canonically isomorphic to $E_{1,O_\lambda}^{p,q}$ as bicomplexes in $\text{Mod}(O_\lambda)$, under which $\mu \otimes_{O_E} O_\lambda$ coincides with μ_λ . Let $E_2^{p,q}$ be the cohomology of $E_1^{p,q}$ (under horizontal differentials). Let Λ_{mot} be the support of the finite O_E -module $\bigoplus_{p,q} (E_2^{p,q})_{\text{tor}}$, which is a finite set of primes of E . Then for every prime $\lambda \notin \Lambda_{\text{mot}}$, the spectral sequence $E_{s,\lambda}^{p,q}$ degenerates at the second page by the incompatibility of weights, similarly to the proof of [Tho14, Proposition 6.5]. Moreover, if we put

$$H_{\text{mot}}^n(X_U, O_E) := \bigoplus_{p+q=n} E_2^{p,q}$$

with the induced monodromy map $\mu_{\text{mot}}: H_{\text{mot}}^n(X_U, O_E) \rightarrow H_{\text{mot}}^n(X_U, O_E)$, then for every $\lambda \notin \Lambda_{\text{mot}}$, we have a canonical isomorphism $H_{\text{mot}}^n(X_U, O_E) \otimes_{O_E} O_\lambda \simeq H_{\text{ét}}^n(X_U \otimes_{\check{F}} \overline{F}, O_\lambda)$ such that there exists a topological generator t_λ in the \mathbb{Z}_ℓ -quotient of $I_{\check{F}_{\check{w}_s}}$ whose action on $H_{\text{ét}}^n(X_U \otimes_{\check{F}} \overline{F}, O_\lambda)$ is given by $1 + \mu_{\text{mot}}$.

Put $H_{\text{mot}}^n(X_U, \mathbb{C}) := H_{\text{mot}}^n(X_U, O_E) \otimes_{O_E} \mathbb{C}$. Note that the Hecke algebra $\mathbb{Z}[U^p \backslash G(\mathbb{A}^{\infty,p})/U^p]$ acts on $E_1^{p,q}$ hence on $H_{\text{mot}}^n(X_U, \mathbb{C})$. Let $H_{\text{mot}}^n(X_U, \mathbb{C})[\pi]$ be the $\pi^{\infty,p}$ -isotypic part of $H_{\text{mot}}^n(X_U, \mathbb{C})$, and put $H_{\text{mot}}^n(X_U, O_E)[\pi] := H_{\text{mot}}^n(X_U, \mathbb{C})[\pi] \cap H_{\text{mot}}^n(X_U, O_E)$. For every $\lambda \notin \Lambda_{\text{mot}}$, we put $H_{\text{mot}}^n(X_U, O_\lambda)[\pi] := H_{\text{mot}}^n(X_U, O_E)[\pi] \otimes_{O_E} O_\lambda$, which is a subspace of $H_{\text{ét}}^n(X_U \otimes_{\check{F}} \overline{F}, O_\lambda)$ stable under the action of $\text{Gal}(\overline{F}/\check{F})$, and is nonzero if and only if $n = N - 1$. Now we characterize $H_{\text{mot}}^n(X_U, O_\lambda)[\pi]$ as an $O_\lambda[\text{Gal}(\overline{F}/\check{F})]$ -module for $\lambda \notin \Lambda_1 \cup \Lambda_{\text{mot}}$. For $\lambda \notin \Lambda_1 \cup \Lambda_{\text{mot}}$, $\rho_{\mathbb{H},\lambda}$ is residually absolutely irreducible, for which we may fix a $\text{Gal}(\overline{F}/\check{F})$ -stable lattice R_λ , which is a free O_λ -module of rank N . Now we let R_λ^ψ be the $O_\lambda[\text{Gal}(\overline{F}/\check{F})]$ -module on which $\text{Gal}(\overline{F}/\check{F})$ acts by $\rho_{\mathbb{H},\lambda}$ twisted by the character $\text{res}(\psi)|_{\text{Gal}(\overline{F}/\check{F})}^{-1}$ as in [HT01, Corollary VII.1.10], which is again residually absolutely irreducible. By [HT01, Corollary VII.1.10], we obtain an isomorphism

$$(E.23) \quad H_{\text{mot}}^n(X_U, O_\lambda)[\pi] \otimes_{O_\lambda} E_\lambda \simeq (R_\lambda^\psi \otimes_{O_\lambda} E_\lambda)^{\oplus m}$$

of $E_\lambda[\text{Gal}(\overline{F}/\check{F})]$ -modules, for some positive integer m independent of λ . Since $\check{\Pi}_{\check{w}_s}$ is the Steinberg representation and $\text{res}(\psi)|_{\text{Gal}(\overline{F}/\check{F})}^{-1}$ is unramified at p_s , by Proposition 3.2.4(2), the restriction endomorphism $1 + \mu_{\text{mot}}$ of $H_{\text{mot}}^n(X_U, O_E)[\pi]$ is conjugate to $(1 + J_N)^{\oplus m}$. Thus, there exists a finite set Λ_2 of primes of E containing $\Lambda_1 \cup \Lambda_{\text{mot}}$ and all those λ whose underlying rational prime ramifies in \check{F} , such that for every $\lambda \notin \Lambda_2$, the reduction of $1 + \mu_{\text{mot}}$ as, an endomorphism of $H_{\text{mot}}^n(X_U, O_E)[\pi] \otimes_{O_E} O_E/\lambda$, is again conjugate to $(1 + J_N)^{\oplus m}$. By (E.23), \bar{R}_λ^ψ is an $O_\lambda/\lambda[\text{Gal}(\overline{F}/\check{F})]$ -submodule of $H_{\text{mot}}^n(X_U, O_E)[\pi] \otimes_{O_E} O_E/\lambda$. Thus, the image of $\text{Gal}(\overline{F}/\check{F}(\zeta_\ell))$ in $\text{End}(\bar{R}_\lambda^\psi)$, hence $\bar{\rho}_{\mathbb{H},\lambda}(\text{Gal}(\overline{F}/\check{F}(\zeta_\ell)))$, contain a unipotent matrix that is conjugate to $1 + J_N$. In

particular, $\bar{\rho}_{\Pi, \lambda}|_{\text{Gal}(\bar{F}/\check{F}(\zeta_\ell))}$ is absolutely indecomposable. On the other hand, since $\text{Gal}(\bar{F}/\check{F}(\zeta_\ell))$ is a normal subgroup of $\text{Gal}(\bar{F}/\check{F})$ of finite index, $\bar{\rho}_{\Pi, \lambda}|_{\text{Gal}(\bar{F}/\check{F}(\zeta_\ell))}$ is absolutely semisimple, hence has to be absolutely irreducible. Thus, $\bar{\rho}_{\Pi, \lambda}|_{\text{Gal}(\bar{F}/F(\zeta_\ell))}$ remains absolutely irreducible for $\lambda \notin \Lambda_2$. \square

Proof of Theorem E.8.4. Let Λ_2 be the set in Lemma E.8.5. It suffices to study (3) in Theorem E.8.4. We take a prime λ of E not in Λ_2 , whose underlying rational prime ℓ does not underlie Σ^+ , and satisfies $\ell \geq 2(N+1)$. In particular, we have

- (a) ℓ is unramified in F ;
- (b) Π_w is unramified for every place w of F above ℓ ;
- (c) $\bar{\rho}_{\Pi, \lambda}|_{\text{Gal}(\bar{F}/F(\zeta_\ell))}$ is absolutely irreducible, which implies that $\bar{r}_{\Pi, \lambda}(\text{Gal}(\bar{F}/F^+(\zeta_\ell)))$ is adequate by Remark E.7.4;
- (d) Proposition E.3.9 holds for the local deformation problem \mathcal{D}^{FL} of $\bar{r}_{\Pi, \lambda, v}$ for every $v \in \Sigma_\ell^+$;
- (e) Proposition E.5.11 holds for $\bar{r}_{\Pi, \lambda, v}$ for every $v \in \Sigma^+$.

For a collection $\mathcal{D}_{\Sigma^+} = \{\mathcal{D}_v \mid v \in \Sigma^+\}$ in which \mathcal{D}_v is an irreducible component of $\text{Spf } R_{\bar{r}_{\Pi, \lambda, v}}^{\text{loc}}$ for $v \in \Sigma^+$, we define a global deformation problem (Definition E.2.6)

$$\mathcal{S}(\mathcal{D}_{\Sigma^+}) := (\bar{r}_{\Pi, \lambda}, \eta_{F/F^+}^\mu \epsilon_\ell^{1-N}, \Sigma^+ \cup \Sigma_\ell^+, \{\mathcal{D}_v\}_{v \in \Sigma^+ \cup \Sigma_\ell^+})$$

where for $v \in \Sigma^+$, \mathcal{D}_v is the prescribed irreducible component (which is a local deformation problem by Proposition E.5.11(2)) in \mathcal{D}_{Σ^+} ; and for $v \in \Sigma_\ell^+$, \mathcal{D}_v is the local deformation problem \mathcal{D}^{FL} of $\bar{r}_{\Pi, \lambda, v}$ from Definition E.3.6. Now by (a–e), and the same proof of [Tho12, Theorem 10.1] (which assumes that $\Sigma^+ \cup \Sigma_\ell^+$ consists only of places split in F), we know that the global universal deformation ring $R_{\mathcal{S}(\mathcal{D}_{\Sigma^+})}^{\text{univ}}$ is a finite \mathcal{O} -module. Moreover, we have $\mu \equiv N \pmod{2}$. By (d,e), and the same proof of [Gee11, Lemma 5.1.3] (which assumes that $\Sigma^+ \cup \Sigma_\ell^+$ consists only of places split in F), we know that the Krull dimension of $R_{\mathcal{S}(\mathcal{D}_{\Sigma^+})}^{\text{univ}}$ is at least one. Thus, $R_{\mathcal{S}(\mathcal{D}_{\Sigma^+})}^{\text{univ}}[1/\ell]$ is nonzero. By choosing a $\bar{\mathbb{Q}}_\ell$ -point of $\text{Spec } R_{\mathcal{S}(\mathcal{D}_{\Sigma^+})}^{\text{univ}}[1/\ell]$, we obtain a relevant representation $\Pi(\mathcal{D}_{\Sigma^+})$ of $\text{GL}_N(\mathbb{A}_F)$ satisfying

- $\Pi(\mathcal{D}_{\Sigma^+})$ is unramified outside Σ^+ ;
- for every place w of F above Σ^+ , there is an open compact subgroup U_w of $\text{GL}_N(F_w)$ depending only on Π_w , such that $\Pi(\mathcal{D}_{\Sigma^+})_{U_w} \neq \{0\}$;
- $\rho_{\Pi(\mathcal{D}_{\Sigma^+}), \ell}$ and $\rho_{\Pi, \lambda} \otimes_{E_\lambda} \bar{\mathbb{Q}}_\ell$ are residually isomorphic.

In fact, the second property is a consequence of Proposition E.5.7. Note that there are only finitely relevant representations of $\text{GL}_N(\mathbb{A}_F)$ up to isomorphism, satisfying the first two properties. By the strong multiplicity one property for GL_N [PS79], we know that for ℓ large enough, Π is the only relevant representation of $\text{GL}_N(\mathbb{A}_F)$ up to isomorphism, satisfying all the three properties.

Now we claim that for two different collections \mathcal{D}_{Σ^+} and \mathcal{D}'_{Σ^+} , the relevant representations $\Pi(\mathcal{D}_{\Sigma^+})$ and $\Pi(\mathcal{D}'_{\Sigma^+})$ are not isomorphic. Assuming this claim, then for ℓ large enough, we have only one collection, which is $\{\mathcal{D}_v^{\text{min}} \mid v \in \Sigma^+\}$, that is, $\bar{r}_{\Pi, \lambda}$ is rigid for $(\Sigma_\Pi^+, \emptyset)$. The theorem is proved.

For the claim itself, we take a place $v \in \Sigma^+$. Then the local components of $\Pi(\mathcal{D}_{\Sigma^+})$ above v give rise to a continuous homomorphism $r: \Gamma_{F_v^+} \rightarrow \mathcal{G}_N(\bar{\mathbb{Q}}_\ell)$, which corresponds to a $\bar{\mathbb{Q}}_\ell$ -point x_r in $\text{Spec } R_{\bar{r}_{\Pi, \lambda, v}}^{\text{loc}}[1/\ell]$. Now the dimension of the tangent space of $\text{Spec } R_{\bar{r}_{\Pi, \lambda, v}}^{\text{loc}}[1/\ell]$ at x_r is equal to

$$\begin{aligned} & N^2 + \dim_{\bar{\mathbb{Q}}_\ell} H^1(F_v^+, \text{ad } r) - \dim_{\bar{\mathbb{Q}}_\ell} H^0(F_v^+, \text{ad } r) = N^2 + \dim_{\bar{\mathbb{Q}}_\ell} H^2(F_v^+, \text{ad } r) \\ & = N^2 + \dim_{\bar{\mathbb{Q}}_\ell} H^0(F_v^+, (\text{ad } r)(1)) \leq N^2 + \dim_{\bar{\mathbb{Q}}_\ell} H^0(F_w, (\text{ad } r^\natural)(1)), \end{aligned}$$

where w is the place of F induced by the embedding $F \hookrightarrow \bar{F}_v^+$. However, since $\Pi(\mathcal{D}_{\Sigma^+})_w$ is generic, we have $\dim_{\bar{\mathbb{Q}}_\ell} H^0(F_w, (\text{ad } r^\natural)(1)) = 0$ by [BLGGT14, Lemma 1.3.2(1)]. Thus, by Proposition

E.5.11(1), $\text{Spec } \mathbf{R}_{r_{\Pi,\lambda,v}}^{\text{loc}}[1/\ell]$ is smooth at x_r , which implies that x_r can not lie on two irreducible components. The claim then follows. \square

Remark E.8.6. In fact, using the same proof, one can obtain Theorem E.8.4 for Π satisfying a weaker condition, namely, by asking Π to be regular algebraic rather than Definition 1.1.3(3).

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