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# ON SUBEXPONENTIAL CONVERGENCE TO EQUILIBRIUM OF MARKOV PROCESSES

ARMAND BERNOU

ABSTRACT. Studying the subexponential convergence towards equilibrium of a strong Markov process, we exhibit an intermediate Lyapunov condition equivalent to the control of some moment of a hitting time. This provides a link, similar (although more intricate) to the one existing in the exponential case, between the coupling method and the approach based on the existence of a Lyapunov function for the generator, in the context of the subexponential rates found by [FR05], [DFG09] and [Hai16].

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## INTRODUCTION AND MAIN RESULT

The study of the convergence towards an invariant measure of continuous-time Markov processes has generated a large literature devoted to the geometric case (also referred to as the exponential case). Meyn and Tweedie and coauthors [MT93c, MT93b, DMT95], developed stability concepts for continuous-time Markov processes along with simple criteria for non-explosion, Harris-recurrence, positive Harris-recurrence, ergodicity and geometric convergence to equilibrium. When applying those stability concepts, the key question of the existence of verifiable conditions emerges. In the discrete-time context, development of Foster-Lyapunov-type conditions on the transition kernel has provided such criteria. In the continuous-time context, Foster-Lyapunov inequalities applied to the (extended) generator of the process play the same role. One of the key results of this theory is the equivalence of two conditions, both implying an exponential convergence towards equilibrium: the control of the moment of the hitting time of a set with appropriate properties, which can be seen as the conditions necessary to apply a coupling method, and the existence of some test function satisfying a Foster-Lyapunov inequality with respect to the generator. Loosely speaking, considering a topological space  $E$  and a  $E$ -valued strong Markov process  $(X_t)_{t \geq 0}$ , with semigroup  $(\mathcal{P}_t)_{t \geq 0}$ , invariant probability distribution  $\pi$  and with appropriate properties (irreducibility, non-explosion and aperiodicity, see Section 1 for precise definitions), we have the following result. Roughly, a set  $C \in \mathcal{B}(E)$  is said to be *petite* if there is a probability measure  $a$  on  $\mathcal{B}(\mathbb{R}_+)$  and a non-trivial measure  $\nu$  on  $\mathcal{B}(E)$  such that  $\forall x \in C, \int_0^\infty \mathcal{P}_t(x, \cdot) a(dt) \geq \nu(\cdot)$ .

**Theorem 1** (Exponential case, [MT93a]). *Assume that  $(X_t)_{t \geq 0}$  is non-explosive, irreducible, and aperiodic. Then the following conditions are equivalent.*

1. *There exist a closed petite set  $C \in \mathcal{B}(E)$  and some constants  $\delta > 0$  and  $\kappa > 1$  such that, setting*

$$\tau_C(\delta) = \inf\{t > \delta, X_t \in C\},$$

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we have

$$(1) \quad \sup_{x \in C} \mathbb{E}_x[\kappa^{\tau_C(\delta)}] < \infty.$$

2. There exist a closed petite set  $C \in \mathcal{B}(E)$ , some constants  $b > 0$ ,  $\beta > 0$  and  $V : E \rightarrow [1, \infty]$  finite at some  $x_0 \in E$  such that, in the sense of Notation 5,

$$(2) \quad \mathcal{L}V \leq -\beta V + b\mathbf{1}_C.$$

Any of those conditions implies that the set  $S_V = \{x : V(x) < \infty\}$  is absorbing and full (see Section 1 for the precise definitions) for any  $V$  solution of (2), and that there exists  $\rho < 1$  and  $d > 0$  such that for all  $x \in E$ ,

$$\|\mathcal{P}_t(x, \cdot) - \pi(\cdot)\|_{TV} \leq dV(x)\rho^t.$$

In the study of subgeometric rates, the situation is quite different. While a moment condition of some hitting time set similar to (1) can be found, as well as a Lyapunov condition similar to (2), there is no equivalence between them. In this note, we identify an intermediate Lyapunov condition, equivalent to the moment condition for subgeometric convergence rates, and prove the following result, with the same notations as above.

**Theorem 2.** Assume that  $(X_t)_{t \geq 0}$  is non-explosive, irreducible and aperiodic. Let  $\phi : [1, \infty) \rightarrow \mathbb{R}_+^*$   $C^1$ , strictly increasing, strictly concave with  $\phi(x) \leq x$  for all  $x \geq 1$  and  $\frac{\phi(x)}{x} \downarrow 0$ ,  $\phi(x) - x\phi'(x) \uparrow$  when  $x \rightarrow \infty$ . Define the function  $H_\phi(\cdot)$  on  $[1, \infty)$  by

$$H_\phi(u) = \int_1^u \frac{ds}{\phi(s)},$$

and let  $H_\phi^{-1} : [0, \infty) \rightarrow [1, \infty)$  be its inverse function. Consider the three following conditions.

1. There exists a compact petite subset  $C$  of  $E$  and some  $r > 0$  such that, for  $\tilde{\tau}_C^r$  defined by

$$(3) \quad \tilde{\tau}_C^r = \inf \left\{ t > 0, \int_0^t \mathbf{1}_C(X_s) ds \geq \frac{T}{r} \right\},$$

where  $T$  is an exponential random variable with parameter 1 independent of everything else, we have

$$(4) \quad \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}_C^r)] < \infty \quad \text{for all } x \in E \quad \text{and} \quad \sup_{x \in C} \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}_C^r)] < \infty.$$

2. There exists a compact petite subset  $C$  of  $E$ , two constants  $\kappa, \eta > 0$  and a function  $\psi$  on  $\mathbb{R}_+ \times E$  with values in  $[1, \infty)$ , continuous and non-decreasing in its first argument, continuous in its second argument, such that for all  $t \geq 0$ ,  $x \in E$ ,

$$H_\phi^{-1}(t) \leq \psi(t, x) \quad \text{and} \quad (\partial_t + \mathcal{L})\psi(t, x) \leq \kappa H_\phi^{-1}(t)\mathbf{1}_C(x) - \phi(H_\phi^{-1}(t)),$$

with moreover  $\psi(0, x) \leq \kappa$  for all  $x \in C$  and for all  $x \in E$ ,  $\mathcal{L}\psi(0, x) \leq \kappa\mathbf{1}_C(x) - \eta$ .

3. There exists a compact petite subset  $C$  of  $E$ , a constant  $K > 0$  and  $V : E \rightarrow [1, \infty)$  continuous with precompact sublevel sets such that for all  $x \in E$ ,

$$(5) \quad \mathcal{L}V(x) \leq -\phi(V(x)) + K\mathbf{1}_C(x).$$

Conditions 1. and 2. are equivalent, and both are implied by Condition 3. Moreover, in those three cases, there exists an invariant probability measure  $\pi$  for  $(\mathcal{P}_t)_{t \geq 0}$  on  $E$  and for all  $x \in E$ ,

$$\lim_{t \rightarrow \infty} \phi(H_\phi^{-1}(t)) \|\mathcal{P}_t(x, \cdot) - \pi(\cdot)\|_{TV} = 0.$$

The fact that (5) implies the convergence was proved by Douc, Fort and Guillin [DFG09], see also [FR05] for the polynomial case, and was simplified, with stronger hypothesis and for the case of the total variation distance, by Hairer [Hai16]. The papers [FR05] and [DFG09] also identify a moment condition similar to (4), however they do not provide an equivalence result between the two conditions.

The remaining part of this note is organized as follows. In Section 1, we recall the main definitions of the theory of convergence for continuous-time strong Markov processes, and define our notion of extended generator, following [Dav18]. In Section 2, we prove the new results of Theorem 2 above.

## 1. SETTING, DEFINITIONS AND PRELIMINARY RESULTS

**1.1. Setting and definitions.** Let  $X = (X_t)_{t \geq 0}$  be a continuous-time strong Markov process with values in a Polish space  $E$ . For  $x \in E$ , we write  $\mathbb{P}_x$  for the probability measure such that  $\mathbb{P}_x(X_0 = x) = 1$ ,  $\mathbb{E}_x$  the corresponding expectation. We denote by  $(\mathcal{P}_t)_{t \geq 0}$  the corresponding semigroup: for all functions  $f$  in  $\mathcal{B}_b(E)$  with  $\mathcal{B}_b(E) = \{f : E \rightarrow \mathbb{R}, f \text{ measurable and bounded}\}$ , for all  $x \in E$ , we have  $\mathcal{P}_t f(x) = \mathbb{E}_x[f(X_t)]$ . We set, for  $f \in \mathcal{B}_b(E)$ ,  $x \in E$ ,  $\hat{\mathcal{L}}f(x) = \frac{d}{dt} \mathbb{E}_x[f(X_t)]|_{t=0}$  provided this object exists. We call  $\hat{\mathcal{L}}$  the (strong) generator and  $\mathcal{D}(\hat{\mathcal{L}})$  its domain given by

$$\mathcal{D}(\hat{\mathcal{L}}) = \left\{ f : E \rightarrow \mathbb{R}, \forall x \in E, \lim_{t \rightarrow 0} \frac{\mathcal{P}_t f(x) - f(x)}{t} \text{ exists} \right\}.$$

Let us recall some more definitions. We say that a continuous-time Markov process  $(X_t)_{t \geq 0}$  with values in  $E$  is non-explosive if there exists a family of pre-compact open sets  $(O_n)_{n \geq 0}$  such that  $O_n \rightarrow E$  as  $n \rightarrow \infty$ , and such that, setting for all  $m \geq 0$ ,  $T_m = \inf\{t > 0, X_t \notin O_m\}$ , for all  $x \in E$ ,

$$\mathbb{P}_x \left( \lim_{m \rightarrow \infty} T_m = \infty \right) = 1.$$

We say that  $(X_t)_{t \geq 0}$  is  $\varphi$ -irreducible for some  $\sigma$ -finite measure  $\varphi$  if  $\varphi(B) > 0$  implies that for all  $B \in \mathcal{B}(E)$ , for all  $x \in E$ ,  $\mathbb{E}_x[\int_0^\infty \mathbf{1}_B(X_s) ds] > 0$ . A  $\varphi$ -irreducible process admits a maximal irreducibility measure  $\psi$  such that  $\mu$  is absolutely continuous with respect to  $\psi$  for any other irreducibility measure  $\mu$  [Num84]. A set  $A \in \mathcal{B}(E)$  such that  $\psi(A) > 0$  for some maximal irreducibility measure  $\psi$  is then said to be accessible, and full is  $\psi(A^c) = 0$ . A set  $A \in \mathcal{B}(E)$  is said to be absorbing if  $\mathbb{P}_x(X_t \in A) = 1$  for all  $x \in A$ ,  $t \geq 0$ . We simply say that  $(X_t)_{t \geq 0}$  is irreducible if it is  $\varphi$ -irreducible for some  $\sigma$ -finite measure  $\varphi$ .

A non-empty measurable set  $C$  is said to be petite if there exists a probability measure  $a$  on  $\mathcal{B}(\mathbb{R}_+)$  and a non-trivial  $\sigma$ -finite measure  $\nu$  on  $\mathcal{B}(E)$  such that

$$\forall x \in C, \int_0^\infty \mathcal{P}_t(x, \cdot) a(dt) \geq \nu(\cdot).$$

We say that a process  $(X_t)_{t \geq 0}$  with associated semigroup  $(\mathcal{P}_t)_{t \geq 0}$  is aperiodic if there exists an  $m > 0$  such that, denoting by  $\delta_m$  the Dirac mass at  $m$ , there exists an accessible  $\delta_m$ -petite set  $C$  (i.e. petite with measure  $a = \delta_m$  on  $\mathbb{R}_+$ ) and some  $t_0 \geq 0$  such that for all  $x \in C$ ,  $t \geq t_0$ ,  $\mathcal{P}_t(x, C) > 0$ .

We assume furthermore that our process is Feller, in the sense that for all  $t > 0$ , all continuous bounded function  $f : E \rightarrow \mathbb{R}$ , the function  $\mathcal{P}_t f : E \rightarrow \mathbb{R}$  is also continuous.

The (weak) Feller property implies that  $(X_s)_{s \geq 0}$  has a càdlàg modification, which we will always consider from now on, see for instance [RY91, Theorem 2.7]. In particular, the hitting times of closed sets are stopping times.

We have the following result on  $\mathcal{D}(\hat{\mathcal{L}})$ .

**Proposition 3.** [Dav18, Propositions 14.10 and 14.13] For  $f \in \mathcal{D}(\hat{\mathcal{L}})$ , for all  $x \in E$ , all  $t \geq 0$ , we have  $\int_0^t |\hat{\mathcal{L}}f(X_s)| ds < \infty$   $\mathbb{P}_x$ -a.s. Moreover, defining the real-valued process  $(C_t^f)_{t \geq 0}$  by

$$C_t^f = f(X_t) - f(X_0) - \int_0^t \hat{\mathcal{L}}f(X_s) ds,$$

the process  $(C_t^f)_{t \geq 0}$  is a  $\mathbb{P}_x$ -local martingale for any  $x \in E$ .

Following Davis [Dav18], we define an extension of the generator  $\hat{\mathcal{L}}$  in the following way.

**Definition 4.** Let  $\mathcal{D}(\mathcal{L})$  denote the set of measurable functions  $f : E \rightarrow \mathbb{R}$  with the following property: there exists a measurable function  $h : E \rightarrow \mathbb{R}$  such that for all  $x \in E$ , there holds  $\mathbb{P}_x(\forall t \geq 0, \int_0^t |h(X_s)| ds < \infty) = 1$ , and the process

$$C_t^f = f(X_t) - f(X_0) - \int_0^t h(X_s) ds,$$

is a  $\mathbb{P}_x$ -local martingale. In this case, we set  $\mathcal{L}f := h$ . We call  $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$  the extended generator of  $(X_t)_{t \geq 0}$ .

The extended generator is indeed an extension: we have  $\mathcal{D}(\hat{\mathcal{L}}) \subset \mathcal{D}(\mathcal{L})$  and  $\mathcal{L}$  and  $\hat{\mathcal{L}}$  coincide on  $\mathcal{D}(\hat{\mathcal{L}})$ . Following [Dav18] again, we introduce the following notation.

**Notation 5.** For  $f : E \rightarrow \mathbb{R}$ , for  $g : E \rightarrow \mathbb{R}$  measurable such that  $\int_0^t |g(X_s)| ds < \infty$  for all  $t \geq 0$ ,  $\mathbb{P}_x$ -almost surely for all  $x \in E$ , we write

$$\mathcal{L}f \leq g$$

if the process

$$f(X_t) - f(x) - \int_0^t g(X_s) ds$$

is a  $\mathbb{P}_x$ -local supermartingale for all  $x \in E$ .

**Remark 6** ([Hai16]). It is possible to have  $\mathcal{L}f \leq g$  even in situations where  $f$  does not belong to the extended domain of  $\mathcal{L}$ . For instance, take  $f(x) = -|x|$  when  $(X_t)_{t \geq 0}$  is a Brownian motion. In this case, one has  $\mathcal{L}f \leq 0$ , but  $f \notin \mathcal{D}(\mathcal{L})$ , and a fortiori  $f \notin \mathcal{D}(\hat{\mathcal{L}})$ .

Similarly, we introduce

**Notation 7.** If  $j : \mathbb{R}_+ \times E \rightarrow \mathbb{R}$  is  $C^1$  in its first argument, for  $k : \mathbb{R}_+ \times E \rightarrow \mathbb{R}$  measurable such that for all  $t \geq 0$ , we have  $\int_0^t |k(s, X_s)| ds < \infty$   $\mathbb{P}_x$ -a.s. for all  $x \in E$ , we write

$$(\partial_t + \mathcal{L})j \leq k$$

$$\text{if } M_t := j(t, X_t) - j(0, x) - \int_0^t k(s, X_s) ds$$

is a  $\mathbb{P}_x$ -local supermartingale for all  $x \in E$ .

In this note, we use the following definition of the total variation distance: for two probability measures  $\mu, \nu$  on  $E$ , we set

$$\|\mu - \nu\|_{TV} = \frac{1}{2} \sup_{A \in \mathcal{B}(E)} |\mu(A) - \nu(A)|.$$

As a consequence, we have

$$\|\mu - \nu\|_{TV} = \inf_{Z \sim \mu, Z' \sim \nu} \mathbb{P}(Z \neq Z'),$$

where the infimum is taken over all couples of random variables such that  $Z$  has law  $\mu$  and  $Z'$  has law  $\nu$ .

**1.2. Extended generator and local martingales.** As we are working in an abstract framework, we heavily use the extended generator (see Notations 5 and 7) and the inequalities of the form

$$\mathcal{L}f \leq g \quad \text{and} \quad (\partial_t + \mathcal{L})\psi \leq \psi_2.$$

For this reason, we will use several preliminary results from [Hai16] that we detail below.

**Proposition 8.** *Let  $(y_t)_{t \geq 0}$  be a real-valued càdlàg semimartingale and let  $\varphi : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$  be a function that is  $C^1$  in its first argument, and  $C^2$  and concave in its second argument. Then, the process*

$$\varphi(t, y_t) - \int_0^t \partial_x \varphi(s, y_{s-}) dy_s - \int_0^t \partial_t \varphi(s, y_{s-}) ds$$

is non-increasing.

*Proof.* As  $(y_t)_{t \geq 0}$  is a semimartingale, we can write it as  $y_t = A_t + M_t$ , where  $(A_t)_{t \geq 0}$  is a process of finite variation and  $(M_t)_{t \geq 0}$  is a local martingale. From Itô's formula for càdlàg processes, see for instance [JS87, Theorem 4.57], we then have

$$\begin{aligned} \varphi(t, y_t) &= \varphi(0, y_0) + \int_0^t \partial_x \varphi(s, y_{s-}) dy_s + \int_0^t \partial_t \varphi(s, y_{s-}) ds \\ &\quad + \int_0^t \partial_x^2 \varphi(s, y_{s-}) d\langle M \rangle_s^c + \sum_{s \in [0, t]} \left( \varphi(s, y_s) - \varphi(s, y_{s-}) - \partial_x \varphi(s, y_{s-}) \Delta y_s \right), \end{aligned}$$

where  $\langle M \rangle_t^c$  denotes the quadratic variation of the continuous part of  $M$  at time  $t$ , with  $\Delta y_s$  defined by  $\Delta y_s = y_s - y_{s-}$ . Since  $\langle M \rangle_t^c$  is an increasing process, and  $\partial_x^2 \varphi(\cdot, \cdot) \leq 0$  by hypothesis, the claim follows.  $\square$

Recall that we write  $\mathcal{L}$  for the extended generator of our  $E$ -valued Markov process  $(X_t)_{t \geq 0}$ .

**Corollary 9.** *Let  $F, G : E \rightarrow \mathbb{R}$  such that*

$$\mathcal{L}F \leq G$$

in the sense of Notation 5. Then, if  $\varphi : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$  is a function that is  $C^1$  in its first argument, and  $C^2$  and concave in its second argument with additionally  $\partial_x \varphi \geq 0$ , then for all  $t \geq 0$ , all  $x \in E$ ,

$$(\partial_t + \mathcal{L})\varphi(t, F(x)) \leq \partial_t \varphi(t, F(x)) + \partial_x \varphi(t, F(x))G(x),$$

in the sense of Notation 7.

*Proof.* Set  $y_t = F(X_t)$  for all  $t \geq 0$ . We have

$$dy_t = G(X_t)dt + dN_t + dM_t,$$

with  $M$  a càdlàg local martingale such that  $M_0 = 0$  and  $N$  a non-increasing process. By Proposition 8, there is a non-increasing process  $(R_t)_{t \geq 0}$  such that

$$d\varphi(t, y_t) = \partial_x \varphi(t, y_{t-}) dy_t + \partial_t \varphi(t, y_{t-}) dt + dR_t,$$

so that

$$d\varphi(t, y_t) = \partial_x \varphi(t, y_{t-})(G(X_t)dt + dN_t + dM_t) + \partial_t \varphi(t, y_{t-})dt + dR_t.$$

Since  $\partial_x \varphi$  is non-negative, the process

$$\varphi(t, y_t) - \varphi(0, y_0) - \int_0^t \left( \partial_t \varphi(s, y_{s-}) + \partial_x \varphi(s, y_{s-})G(X_s) \right) ds$$

is indeed a local supermartingale (as sum of a local martingale and of a non-increasing process).  $\square$

**1.3. Properties of  $\phi$  and  $H_\phi^{-1}$ .** We recall that  $\phi : [1, \infty) \rightarrow \mathbb{R}_+$  is  $C^1$ , strictly increasing, strictly concave such that  $\phi(1) > 0$ ,  $\phi(x) \leq x$  for all  $x \geq 1$ ,  $\frac{\phi(x)}{x} \downarrow 0$  and  $\phi(x) - x\phi'(x) \uparrow \infty$  when  $x \rightarrow \infty$ . The function  $H_\phi$  is defined, for all  $u \geq 1$  by

$$H_\phi(u) = \int_1^u \frac{ds}{\phi(s)},$$

and we consider the corresponding inverse function  $H_\phi^{-1} : [0, \infty) \rightarrow [1, \infty)$ .

**Lemma 10.** *The following inequality holds:*

$$(6) \quad H_\phi^{-1}(s+t) \leq H_\phi^{-1}(s)H_\phi^{-1}(t) \quad \text{for all } s, t \geq 0.$$

*Proof.* Set  $g(\cdot) := (\ln \circ H_\phi^{-1})(\cdot)$ , and consider the function given, for all  $s, t \geq 0$ , by

$$h(s, t) := g(s+t) - g(s) - g(t).$$

For all  $s \geq 0$ ,  $h(s, 0) = 0$  since  $H_\phi^{-1}(0) = 1$ . Moreover, using that  $(H_\phi^{-1})'(u) = (\phi \circ H_\phi^{-1})(u)$  for all  $u \geq 0$ ,

$$\partial_t h(s, t) = \frac{\phi(H_\phi^{-1}(t+s))}{H_\phi^{-1}(t+s)} - \frac{\phi(H_\phi^{-1}(t))}{H_\phi^{-1}(t)} \leq 0,$$

using that  $\frac{\phi(x)}{x} \downarrow 0$  as  $x \rightarrow \infty$ . Hence  $h(s, t) \leq 0$  for all  $s, t \geq 0$  and the conclusion follows by taking the exponential.  $\square$

An immediate study also shows that

$$(7) \quad \phi(\kappa x) \leq \kappa \phi(x) \quad \text{for all } x \geq 0, \text{ all } \kappa \geq 1.$$

We will use several times the following remark, based on the definition of  $\tilde{\tau}_C^r$ , see (3).

**Remark 11.** *For all  $x \in E$  and all non-decreasing  $C^1$  function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $f(0) = 0$ ,*

$$\mathbb{E}_x[f(\tilde{\tau}_C^r)] = \mathbb{E}_x \left[ \int_0^\infty e^{-r \int_0^s \mathbf{1}_C(X_u) du} f'(s) ds \right].$$

*Indeed, it suffices to use that  $\mathbb{E}_x[f(\tilde{\tau}_C^r)] = \int_0^\infty \mathbb{P}_x(\tilde{\tau}_C^r \geq s) f'(s) ds$  and that*

$$\mathbb{P}_x(\tilde{\tau}_C^r \geq s) = \mathbb{P}_x \left( T \geq r \int_0^s \mathbf{1}_C(X_u) du \right) = \mathbb{E}_x \left[ e^{-r \int_0^s \mathbf{1}_C(X_u) du} \right].$$

## 2. PROOF OF THEOREM 2

In this section, we give the proofs of the results stated in Theorem 2.

**2.1. Proof that Condition 3 implies Condition 2.** We introduce  $\psi_0 : \mathbb{R}_+ \times [1, \infty) \rightarrow [1, \infty)$  defined by  $\psi_0(t, x) = H_\phi^{-1}(H_\phi(x) + t)$ . It is  $C^1$  in its first argument  $t$  and  $C^2$  in its second argument. Moreover, for all  $t \geq 0$ , all  $x \geq 1$ ,

$$\partial_x \psi_0(t, x) = H_\phi'(x)(H_\phi^{-1})'(H_\phi(x) + t) = \frac{\phi(H_\phi^{-1}(H_\phi(x) + t))}{\phi(x)} \geq 0.$$

Next,

$$\begin{aligned}\partial_x^2 \psi_0(t, x) &= \frac{\phi' \left( H_\phi^{-1}(H_\phi(x) + t) \right) \phi \left( H_\phi^{-1}(H_\phi(x) + t) \right) - \phi'(x) \phi \left( H_\phi^{-1}(H_\phi(x) + t) \right)}{\phi^2(x)} \\ &= \frac{\phi \left( H_\phi^{-1}(H_\phi(x) + t) \right)}{\phi^2(x)} \left( \phi' \left( H_\phi^{-1}(H_\phi(x) + t) \right) - \phi'(x) \right) \leq 0,\end{aligned}$$

since the first factor is positive, while the second one is negative because  $\phi'$  is decreasing and  $x \leq H_\phi^{-1}(H_\phi(x) + t)$ . We conclude that  $\psi_0$  satisfies the assumption of Corollary 9. We set  $\psi(t, x) = 2\psi_0(t, V(x)) - H_\phi^{-1}(t)$ . On the one hand

$$H_\phi^{-1}(t) = 2H_\phi^{-1}(t) - H_\phi^{-1}(t) \leq 2\psi_0(t, V(x)) - H_\phi^{-1}(t) = \psi(t, x)$$

for all  $t \geq 0$ , all  $x \in E$ , and, using Corollary 9 and that  $(H_\phi^{-1})' = \phi \circ H_\phi^{-1}$ , one has

$$\begin{aligned}(\partial_t + \mathcal{L})\psi(t, x) &\leq 2\partial_t \psi_0(t, V(x)) + 2\partial_x \psi_0(t, V(x))\mathcal{L}V(x) - \phi(H_\phi^{-1}(t)) \\ &= 2\phi \left( H_\phi^{-1}(H_\phi(V(x)) + t) \right) + 2 \frac{\phi \left( H_\phi^{-1}(H_\phi(V(x)) + t) \right)}{\phi(V(x))} \mathcal{L}V(x) - \phi(H_\phi^{-1}(t)) \\ &\leq 2\phi \left( H_\phi^{-1}(H_\phi(V(x)) + t) \right) + 2 \frac{\phi \left( H_\phi^{-1}(H_\phi(V(x)) + t) \right)}{\phi(V(x))} (-\phi(V(x)) + K\mathbf{1}_C(x)) - \phi(H_\phi^{-1}(t)) \\ &\leq 2K \frac{\phi \left( H_\phi^{-1}(H_\phi(V(x)) + t) \right)}{\phi(V(x))} \mathbf{1}_C(x) - \phi(H_\phi^{-1}(t)),\end{aligned}$$

where we used the bound on  $\mathcal{L}V$  from Condition 3. Using now (6) and (7) (recall that  $H_\phi^{-1}(t) \geq 1$ ), we conclude that

$$(\partial_t + \mathcal{L})\psi(t, x) \leq 2K \frac{\phi \left( H_\phi^{-1}(t)V(x) \right)}{\phi(V(x))} \mathbf{1}_C(x) - \phi(H_\phi^{-1}(t)) \leq 2KH_\phi^{-1}(t)\mathbf{1}_C(x) - \phi(H_\phi^{-1}(t)).$$

We also have  $\psi(0, x) = 2V(x) - 1$ , so that indeed  $\sup_{x \in C} \psi(0, x) < \infty$  (because  $C$  is compact and  $V$  has precompact sublevel sets), and, using that  $\mathcal{L}\mathbf{1} = 0$ , recalling Condition 3, that  $V \geq 1$  and that  $\phi$  is non-decreasing,

$$\mathcal{L}\psi(0, x) = 2K\mathbf{1}_C(x) - 2\phi(V(x)) \leq 2K\mathbf{1}_C(x) - 2\phi(1),$$

which completes the proof.

**2.2. Proof that Condition 2 implies Condition 1.** Let  $x \in E$  and set, for all  $t \geq 0$ ,

$$M_t = \psi(t, X_t) - \psi(0, x) - \kappa \int_0^t \mathbf{1}_C(X_s) H_\phi^{-1}(s) ds + \int_0^t \phi(H_\phi^{-1}(s)) ds,$$

then by Condition 2,  $(M_t)_{t \geq 0}$  is a  $\mathbb{P}_x$ -local supermartingale starting at 0. Hence there exists an increasing to infinity sequence  $(\sigma_i)_{i \geq 1}$  of stopping times such that for all  $i \geq 1$ ,  $(M_{t \wedge \sigma_i})_{t \geq 0}$  is a bounded supermartingale.

**Step 1.** We introduce the stopping time

$$\tilde{\tau}^1 = \inf \left\{ t \geq 0, \int_0^t \mathbf{1}_C(X_u) du \geq \frac{1}{2\kappa} \right\},$$

and note that  $X_{\tilde{\tau}^1} \in C$  almost surely. In this step, we show that for all  $x \in E$ ,

$$\mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1)] \leq 2\psi(0, x).$$

For all  $i \geq 1$ , using that  $H_\phi^{-1}$  is non-decreasing and then that  $\tilde{\tau}^1 \wedge \sigma_i \leq \tilde{\tau}^1$ ,

$$\begin{aligned} \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1 \wedge \sigma_i)] &\leq \mathbb{E}_x[\psi(\tilde{\tau}^1 \wedge \sigma_i, X_{\tilde{\tau}^1 \wedge \sigma_i})] \\ &= \mathbb{E}_x\left[\psi(0, x) + \kappa \int_0^{\tilde{\tau}^1 \wedge \sigma_i} \mathbf{1}_C(X_u) H_\phi^{-1}(u) du - \int_0^{\tilde{\tau}^1 \wedge \sigma_i} \phi(H_\phi^{-1}(s)) ds + M_{\tilde{\tau}^1 \wedge \sigma_i}\right] \\ &\leq \psi(0, x) + \kappa \mathbb{E}_x\left[\int_0^{\tilde{\tau}^1 \wedge \sigma_i} \mathbf{1}_C(X_u) H_\phi^{-1}(u) du\right] \\ &\leq \psi(0, x) + \kappa \mathbb{E}_x\left[H_\phi^{-1}(\tilde{\tau}^1 \wedge \sigma_i) \int_0^{\tilde{\tau}^1} \mathbf{1}_C(X_u) du\right] \\ &= \psi(0, x) + \frac{1}{2} \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1 \wedge \sigma_i)]. \end{aligned}$$

We obtain that for all  $i \geq 1$ ,

$$\mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1 \wedge \sigma_i)] \leq 2\psi(0, x),$$

and an application of the monotone convergence theorem allows us to conclude.

**Step 2.** We consider the quantity defined for all  $x \in E$ , for  $\rho \geq 0$  and  $r > 0$  by

$$A_{x,\rho,r} := \mathbb{E}_x\left[\int_0^\infty e^{-r \int_0^s \mathbf{1}_C(X_u) du} (H_\phi^{-1})'(s) e^{-\rho s^2} ds\right]$$

which is finite because  $(H_\phi^{-1})'(s) = \phi(H_\phi^{-1}(s)) \leq H_\phi^{-1}(s)$ , whence  $H_\phi^{-1}(s) \leq H_\phi^{-1}(0)e^s = e^s$ . We have

$$\begin{aligned} A_{x,\rho,r} &= \mathbb{E}_x\left[\int_0^{\tilde{\tau}^1} e^{-r \int_0^s \mathbf{1}_C(X_u) du} (H_\phi^{-1})'(s) e^{-\rho s^2} ds\right] + \mathbb{E}_x\left[\int_{\tilde{\tau}^1}^\infty e^{-r \int_0^s \mathbf{1}_C(X_u) du} (H_\phi^{-1})'(s) e^{-\rho s^2} ds\right] \\ &\leq \mathbb{E}_x\left[\int_0^{\tilde{\tau}^1} (H_\phi^{-1})'(s) ds\right] + \mathbb{E}_x\left[\int_{\tilde{\tau}^1}^\infty e^{-r \int_0^{\tilde{\tau}^1} \mathbf{1}_C(X_u) du} e^{-r \int_{\tilde{\tau}^1}^s \mathbf{1}_C(X_u) du} (H_\phi^{-1})'(s) e^{-\rho s^2} ds\right] \\ &\leq \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1)] + \mathbb{E}_x\left[e^{-r \int_0^{\tilde{\tau}^1} \mathbf{1}_C(X_u) du} \int_{\tilde{\tau}^1}^\infty e^{-r \int_{\tilde{\tau}^1}^s \mathbf{1}_C(X_u) du} (H_\phi^{-1})'(s) e^{-\rho s^2} ds\right]. \end{aligned}$$

Using the strong Markov property

$$\begin{aligned} A_{x,\rho,r} &\leq \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1)] \\ &\quad + \mathbb{E}_x\left[e^{-r \int_0^{\tilde{\tau}^1} \mathbf{1}_C(X_u) du} \mathbb{E}_{X_{\tilde{\tau}^1}}\left[\int_0^\infty e^{-r \int_0^s \mathbf{1}_C(X_u) du} (H_\phi^{-1})'(\tilde{\tau}^1 + s) e^{-\rho(s+\tilde{\tau}^1)^2} ds\right]\right] \\ &\leq \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1)] + \mathbb{E}_x\left[e^{-r \int_0^{\tilde{\tau}^1} \mathbf{1}_C(X_u) du} H_\phi^{-1}(\tilde{\tau}^1) \mathbb{E}_{X_{\tilde{\tau}^1}}\left[\int_0^\infty e^{-r \int_0^s \mathbf{1}_C(X_u) du} (H_\phi^{-1})'(s) e^{-\rho s^2} ds\right]\right] \end{aligned}$$

because  $(H_\phi^{-1})'(\tilde{\tau}^1 + s) = \phi(H_\phi^{-1}(\tilde{\tau}^1 + s)) \leq \phi(H_\phi^{-1}(\tilde{\tau}^1) H_\phi^{-1}(s)) \leq H_\phi^{-1}(\tilde{\tau}^1) \phi(H_\phi^{-1}(s))$  by (6) and (7). Using the definition of  $A_{x,\rho,r}$  and the fact that  $X_{\tilde{\tau}^1} \in C$ , we conclude that

$$(8) \quad A_{x,\rho,r} \leq \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1)] + \mathbb{E}_x\left[e^{-r \int_0^{\tilde{\tau}^1} \mathbf{1}_C(X_u) du} H_\phi^{-1}(\tilde{\tau}^1)\right] \sup_{y \in C} A_{y,\rho,r}.$$

**Step 3.** We now prove that there is  $r_0 > 0$  (large) such that

$$\sup_{x \in C} \mathbb{E}_x \left[ e^{-r_0 \int_0^{\tilde{\tau}^1} \mathbf{1}_C(X_u) du} H_\phi^{-1}(\tilde{\tau}^1) \right] \leq \frac{1}{2}.$$

By definition of  $\tilde{\tau}^1$ ,  $\int_0^{\tilde{\tau}^1} \mathbf{1}_C(X_u) du = \frac{1}{2\kappa}$ . Hence, for all  $x \in E$ ,

$$\mathbb{E}_x \left[ e^{-r \int_0^{\tilde{\tau}^1} \mathbf{1}_C(X_u) du} H_\phi^{-1}(\tilde{\tau}^1) \right] = \mathbb{E}_x \left[ e^{-\frac{r}{2\kappa}} H_\phi^{-1}(\tilde{\tau}^1) \right] \leq 2e^{-\frac{r}{2\kappa}} \psi(0, x)$$

by Step 1. Since  $\kappa = \sup_{x \in C} \psi(0, x) < \infty$  by assumption, the conclusion follows.

**Step 4.** Coming back to (8), choosing  $r = r_0$  and taking the supremum over  $x \in C$  on both sides and using Step 3, we find

$$\sup_{x \in C} A_{x, \rho, r_0} \leq \sup_{x \in C} \mathbb{E}_x [H_\phi^{-1}(\tilde{\tau}^1)] + \frac{1}{2} \sup_{x \in C} A_{x, \rho, r_0},$$

so that, using Step 1 and that  $\psi(0, \cdot) \leq \kappa$  on  $C$ ,

$$\sup_{x \in C} A_{x, \rho, r_0} \leq 4\kappa.$$

We now apply Fatou's lemma and Remark 11,

$$\sup_{x \in C} \mathbb{E}_x [H_\phi^{-1}(\tilde{\tau}_C^{r_0})] = \sup_{x \in C} \mathbb{E}_x \left[ \int_0^\infty e^{-r_0 \int_0^s \mathbf{1}_C(X_u) du} (H_\phi^{-1})'(s) ds \right] \leq \sup_{x \in C} \liminf_{\rho \rightarrow 0} A_{x, \rho, r_0} \leq 4\kappa.$$

**Conclusion** We come back to (8) using the results of Step 1 and Step 4. For all  $x \in E$ ,

$$\begin{aligned} A_{x, \rho, r_0} &\leq \mathbb{E}_x [H_\phi^{-1}(\tilde{\tau}^1)] + \mathbb{E}_x \left[ e^{-r_0 \int_0^{\tilde{\tau}^1} \mathbf{1}_C(X_u) du} H_\phi^{-1}(\tilde{\tau}^1) \right] \sup_{x \in C} A_{x, \rho, r_0} \\ &\leq \mathbb{E}_x [H_\phi^{-1}(\tilde{\tau}^1)] (1 + 4\kappa) \\ &\leq 2\psi(0, x) (1 + 4\kappa). \end{aligned}$$

Hence, as in Step 4,

$$\mathbb{E}_x [H_\phi^{-1}(\tilde{\tau}_C^{r_0})] = \mathbb{E}_x \left[ \int_0^\infty e^{-r_0 \int_0^s \mathbf{1}_C(X_u) du} (H_\phi^{-1})'(s) ds \right] \leq \liminf_{\rho \rightarrow 0} A_{x, \rho, r_0} \leq 2\psi(0, x) (1 + 4\kappa).$$

**2.3. Proof that Condition 1 implies Condition 2.** We fix  $r > 0$  so that Condition 1 holds and recall that the randomized hitting time is given by

$$\tilde{\tau}_C^r = \inf \left\{ t > 0, \int_0^t \mathbf{1}_C(X_s) ds > \frac{T}{r} \right\},$$

where  $T$  is a random variable with exponential law of parameter 1 independent of everything else. For the sake of simplicity we will omit the superscript  $r$  in what follows and write  $\tilde{\tau}_C = \tilde{\tau}_C^r$ . Our goal is to show that

$$\psi(t, x) = \mathbb{E}_x \left[ H_\phi^{-1}(\tilde{\tau}_C + t) \right] = \mathbb{E}_x \left[ \int_0^\infty e^{-r \int_0^s \mathbf{1}_C(X_u) du} (H_\phi^{-1})'(s + t) ds \right]$$

satisfies Condition 2. The second equality follows from Remark 11.

We of course have  $\psi(t, x) \geq H_\phi^{-1}(t)$  for all  $t \geq 0$ , all  $x \in E$ , and  $\kappa = \sup_{x \in C} \psi(0, x)$  is finite by assumption.

Consider a sequence  $(\varphi_\epsilon)_{\epsilon>0}$  of continuous functions such that  $\varphi_\epsilon(x) \downarrow \mathbf{1}_C(x)$  and  $\epsilon \leq \varphi_\epsilon(x) \leq 1$  for all  $x \in E$ . This is possible because  $C$  is compact. We set, for all  $\epsilon > 0$ ,

$$\psi_\epsilon(t, x) = \mathbb{E}_x \left[ \int_0^\infty e^{-r \int_0^s \varphi_\epsilon(X_u) du} (H_\phi^{-1})'(s+t) ds \right].$$

**Step 1: Computation of  $(\partial_t + \mathcal{L})\psi_\epsilon(t, x)$ .** We first have, for  $(t, x) \in \mathbb{R}_+ \times E$ ,

$$\partial_t \psi_\epsilon(t, x) = \mathbb{E}_x \left[ \int_0^\infty e^{-r \int_0^s \varphi_\epsilon(X_u) du} (H_\phi^{-1})''(s+t) ds \right].$$

This is easily justified, using that  $\varphi_\epsilon \geq \mathbf{1}_C(x)$  and that

$$\mathbb{E}_x \left[ \int_0^\infty e^{-r \int_0^s \mathbf{1}_C(X_u) du} (H_\phi^{-1})''(s+t) ds \right] = \mathbb{E}_x [(H_\phi^{-1})'(\tilde{\tau}_C + t)] \leq \phi(1) H_\phi^{-1}(t) \mathbb{E}_x [H_\phi^{-1}(\tilde{\tau}_C)] < \infty$$

by assumption. We used that  $(H_\phi^{-1})'(s+t) = \phi(H_\phi^{-1}(s+t)) \leq \phi(1) H_\phi^{-1}(s+t) \leq \phi(1) H_\phi^{-1}(s) H_\phi^{-1}(t)$  by (6).

We use the strong generator. We fix  $t \geq 0$  and recall that

$$\mathcal{L}\psi_\epsilon(t, x) = \lim_{v \rightarrow 0} \frac{1}{v} (\mathbb{E}_x [\psi_\epsilon(t, X_v)] - \psi_\epsilon(t, x)).$$

For  $v > 0$ , we have

$$\begin{aligned} \mathbb{E}_x [\psi_\epsilon(t, X_v)] &= \mathbb{E}_x \left( \mathbb{E}_{X_v} \left[ \int_0^\infty e^{-r \int_0^s \varphi_\epsilon(X_u) du} (H_\phi^{-1})'(s+t) ds \right] \right) \\ &= \mathbb{E}_x \left[ \int_0^\infty e^{-r \int_0^s \varphi_\epsilon(X_{u+v}) du} (H_\phi^{-1})'(s+t) ds \right] \\ &= \mathbb{E}_x \left[ \int_0^\infty e^{-r \int_v^{s+v} \varphi_\epsilon(X_u) du} (H_\phi^{-1})'(s+t) ds \right]. \end{aligned}$$

Noting that

$$\int_v^{s+v} \varphi_\epsilon(X_u) du = \int_0^s \varphi_\epsilon(X_u) du - \int_0^v \varphi_\epsilon(X_u) du + \int_s^{s+v} \varphi_\epsilon(X_u) du,$$

we find

$$\begin{aligned} &\mathbb{E}_x [\psi_\epsilon(t, X_v)] - \psi_\epsilon(t, x) \\ &= \mathbb{E}_x \left[ \int_0^\infty e^{-r \int_0^s \varphi_\epsilon(X_u) du} (H_\phi^{-1})'(s+t) \left( e^{r \int_0^v \varphi_\epsilon(X_u) du} e^{-r \int_s^{s+v} \varphi_\epsilon(X_u) du} - 1 \right) ds \right]. \end{aligned}$$

Note that since  $(X_t)_{t \geq 0}$  is càdlàg and  $\varphi_\epsilon$  is smooth, it holds that  $\lim_{v \rightarrow 0} \frac{1}{v} \int_0^v \varphi_\epsilon(X_u) du = \varphi_\epsilon(X_0)$  and  $\lim_{v \rightarrow 0} \frac{1}{v} \int_s^{s+v} \varphi_\epsilon(X_u) du = \varphi_\epsilon(X_s)$  a.s. We easily conclude by dominated convergence, using that  $\mathbf{1}_C \leq \varphi_\epsilon \leq 1$  and that

$$\mathbb{E}_x \left[ \int_0^\infty e^{-r \int_0^s \varphi_\epsilon(X_u) du} (H_\phi^{-1})'(s+t) ds \right] \leq \mathbb{E}_x [H_\phi^{-1}(\tilde{\tau}_C + t)] \leq H_\phi^{-1}(t) \mathbb{E}_x [H_\phi^{-1}(\tilde{\tau}_C)] < \infty,$$

that

$$\begin{aligned}\mathcal{L}\psi_\epsilon(t, x) &= \lim_{v \rightarrow 0} \frac{1}{v} \left( \mathbb{E}_x[\psi_\epsilon(t, X_v)] - \psi_\epsilon(t, x) \right) \\ &= r\mathbb{E}_x \left[ \int_0^\infty e^{-r \int_0^s \varphi_\epsilon(X_u) du} (H_\phi^{-1})'(s+t) (\varphi_\epsilon(x) - \varphi_\epsilon(X_s)) ds \right] \\ &= r\varphi_\epsilon(x)\psi_\epsilon(t, x) - r\mathbb{E}_x \left[ \int_0^\infty e^{-r \int_0^s \varphi_\epsilon(X_u) du} (H_\phi^{-1})'(s+t) \varphi_\epsilon(X_s) ds \right].\end{aligned}$$

Note that  $\partial_s(e^{-r \int_0^s \varphi_\epsilon(X_u) du}) = -r\varphi_\epsilon(X_s)e^{-r \int_0^s \varphi_\epsilon(X_u) du}$  a.s., so that, by integration by parts,

$$\begin{aligned}r\mathbb{E}_x \left[ \int_0^\infty e^{-r \int_0^s \varphi_\epsilon(X_u) du} (H_\phi^{-1})'(s+t) \varphi_\epsilon(X_s) ds \right] &= \mathbb{E}_x \left[ \left[ -e^{-r \int_0^s \varphi_\epsilon(X_u) du} (H_\phi^{-1})'(s+t) \right]_0^\infty \right] \\ &\quad + \mathbb{E}_x \left[ \int_0^\infty e^{-r \int_0^s \varphi_\epsilon(X_u) du} (H_\phi^{-1})''(s+t) ds \right].\end{aligned}$$

Using that  $\varphi_\epsilon \geq \epsilon$  and the properties of  $\phi$  ( $(H_\phi^{-1})'$  is subexponential), one can check that

$$\lim_{s \rightarrow \infty} \mathbb{E}_x \left[ e^{-r \int_0^s \varphi_\epsilon(X_u) du} (H_\phi^{-1})'(s+t) \right] = 0,$$

from which we conclude that

$$\mathbb{E}_x \left[ \left[ -e^{-r \int_0^s \varphi_\epsilon(X_u) du} (H_\phi^{-1})'(s+t) \right]_0^\infty \right] = (H_\phi^{-1})'(t) = \phi(H_\phi^{-1}(t)).$$

We have proved that, in the sense of the strong generator (which a fortiori implies the result for the weak generator),

$$\begin{aligned}(9) \quad (\partial_t + \mathcal{L})\psi_\epsilon(t, x) &= r\varphi_\epsilon(x)\psi_\epsilon(t, x) - \phi(H_\phi^{-1}(t)) \\ &\leq r\varphi_\epsilon(x)H_\phi^{-1}(t)\psi_\epsilon(0, x) - \phi(H_\phi^{-1}(t)).\end{aligned}$$

We finally used that  $\psi_\epsilon(t, x) \leq H_\phi^{-1}(t)\psi_\epsilon(0, x)$ , because  $H_\phi^{-1}(t+s) \leq H_\phi^{-1}(t)H_\phi^{-1}(s)$ .

**Step 2: limit as  $\epsilon \rightarrow 0$ .** By (9), we know that

$$M_t^\epsilon = \psi_\epsilon(t, X_t) - \psi_\epsilon(0, x) - r \int_0^t \varphi_\epsilon(X_s) H_\phi^{-1}(s) \psi_\epsilon(0, X_s) ds + \int_0^t \phi(H_\phi^{-1}(s)) ds$$

is a local supermartingale for each  $\epsilon > 0$ , and we want to check that

$$M_t = \psi(t, X_t) - \psi(0, x) - r \int_0^t \mathbf{1}_C(X_s) H_\phi^{-1}(s) \psi(0, X_s) ds + \int_0^t \phi(H_\phi^{-1}(s)) ds$$

is also a local supermartingale.

It classically suffices to check that for all  $T > 0$ ,  $\sup_{[0, T]} |M_t^\epsilon - M_t| \rightarrow 0$  a.s. as  $\epsilon \rightarrow 0$ . To this aim, the only issue is to verify that for all  $T > 0$ , all compact subset  $K \subset E$ ,

$$(10) \quad \sup_{[0, T] \times K} |\psi_\epsilon(t, x) - \psi(t, x)| \rightarrow 0.$$

Recalling that  $\varphi_\epsilon \geq \mathbf{1}_C$  and that  $(H_\phi^{-1})'$  is non-decreasing, we observe that by definition of  $\psi_\epsilon$  and  $\psi$ , it holds that

$$\sup_{[0, T]} |\psi_\epsilon(t, x) - \psi(t, x)| = \psi(T, x) - \psi_\epsilon(T, x).$$

Since now  $\varphi_\epsilon \downarrow \mathbf{1}_C$  pointwise, we deduce from the monotone convergence theorem that for each  $x \in E$ ,

$$\begin{aligned} \psi_\epsilon(T, x) &= \mathbb{E}_x \left[ \int_0^\infty e^{-\int_0^s \varphi_\epsilon(X_u) du} (H_\phi^{-1})'(T+s) ds \right] \stackrel{\epsilon \rightarrow 0}{\uparrow} \mathbb{E}_x \left[ \int_0^\infty e^{-\int_0^s \mathbf{1}_C(X_u) du} (H_\phi^{-1})'(T+s) ds \right] \\ &= \mathbb{E}_x [H_\phi^{-1}(\tilde{\tau}_C + T)] = \psi(t, x). \end{aligned}$$

By [Kal02, Theorem 17.25], it follows from the Feller property that when  $y \rightarrow x$ , the process  $(X_t^y)_{t \geq 0}$  with semigroup  $(\mathcal{P}_t)_{t \geq 0}$  and  $X_0^y = y$  converges in distribution, in the Skorokhod space  $\mathbb{D}([0, \infty), E)$ , towards the process  $(X_t^x)_{t \geq 0}$  with semigroup  $(\mathcal{P}_t)_{t \geq 0}$  and  $X_0^x = x$ . We easily deduce the continuity in  $x$  of  $\psi_\epsilon(T, x)$  and  $\psi(T, x)$ . We then may use Dini's theorem to conclude that, as desired,

$$\sup_{x \in K} [\psi(T, x) - \psi_\epsilon(T, x)] \rightarrow 0$$

as  $\epsilon \rightarrow 0$ , for any compact  $K$  of  $E$ .

**Step 3 : Conclusion.** It remains to verify that

$$\mathcal{L}\psi(0, x) \leq \kappa \mathbf{1}_C(x) - \eta.$$

Using Step 1 with  $t = 0$ , we have

$$\begin{aligned} \mathcal{L}\psi_\epsilon(0, x) &= r\varphi_\epsilon(x)\psi_\epsilon(0, x) - \mathbb{E}_x \left[ \int_0^\infty e^{-r \int_0^s \varphi_\epsilon(X_u) du} (H_\phi^{-1})''(s) ds \right] - \phi(H_\phi^{-1}(0)) \\ &\leq r\varphi_\epsilon(x)\psi_\epsilon(0, x) - \phi(1). \end{aligned}$$

We throwed away the non-negative expectation and used that  $H_\phi^{-1}(0) = 1$ . Using the same limit procedure as in Step 2 (through local supermartingales), we conclude that

$$\mathcal{L}\psi(0, x) \leq r\mathbf{1}_C(x)\psi(0, x) - \phi(1)$$

and conclude using that  $\psi(0, x)$  is bounded on  $C$ .

#### 2.4. Proof of the result from Condition 2.

**Existence of an invariant measure.** According to [MT93a, Theorems 5 and 6], an invariant probability measure  $\pi$  exists as soon as there exist a petite set  $C$ , a constant  $b > 0$  and a continuous function  $W : E \rightarrow [0, \infty)$  such that

$$\mathcal{L}W(x) \leq -1 + b\mathbf{1}_C(x).$$

It directly follows from Condition 2 that  $W(x) := \frac{\psi(0, x)}{\eta}$  is convenient. Moreover, by [MT93a, Theorem 7], for all  $x \in E$ ,

$$(11) \quad \|\mathcal{P}_t(x, \cdot) - \pi(\cdot)\|_{TV} \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

**Convergence result** By [FR05, Theorem 1], with  $f_* = 1$  and  $r_*(s) = \phi(H_\phi^{-1}(s))$  for all  $s \geq 0$ ,  $\Psi_1(u) = u$  and  $\Psi_2(v) = 1$ , it suffices to verify the following three conditions.

(a)  $r_*$  is a rate function in the sense of [FR05], i.e.  $\lim_{s \rightarrow \infty} \frac{1}{s} \log(r_*(s)) = 0$ . Indeed, setting  $g(s) = \ln(H_\phi^{-1}(s))$ ,  $g'(s) = \frac{\phi(H_\phi^{-1}(s))}{H_\phi^{-1}(s)} \rightarrow 0$  as  $s \rightarrow \infty$ , by hypothesis on  $\phi$ . Therefore  $\frac{g(s)}{s} \rightarrow 0$  as  $s \rightarrow \infty$ . Since  $\phi(H_\phi^{-1}(s)) \leq \phi(1)H_\phi^{-1}(s)$ , the conclusion follows.

(b) There is  $t_0 > 0$  such that the Markov chain with matrix  $\mathcal{P}_{t_0}$  is irreducible. This follows from (11) and [MT93b, Theorem 6.1].

(c) There is  $\delta > 0$  such that, with the petite set  $C$  of Condition 2 and recalling that

$$\tau_C(\delta) = \inf\{t \geq \delta, X_t \in C\},$$

$$\sup_{x \in C} \mathbb{E}_x \left[ \int_0^{\tau_C(\delta)} 1 ds \right] + \sup_{x \in C} \mathbb{E}_x \left[ \int_0^{\tau_C(\delta)} \phi(H_\phi^{-1}(s)) ds \right] < \infty.$$

Since  $\phi$  is bounded from below, it suffices to study the second term. By the usual supermartingale argument, recalling the condition on  $\psi$ , we have

$$\mathbb{E}_x[\psi(\tau_C(\delta), X_{\tau_C(\delta)})] \leq \psi(0, x) + \kappa \mathbb{E}_x \left[ \int_0^{\tau_C(\delta)} \mathbf{1}_C(X_s) H_\phi^{-1}(s) ds \right] - \mathbb{E}_x \left[ \int_0^{\tau_C(\delta)} \phi(H_\phi^{-1}(s)) ds \right].$$

Since now  $\int_0^{\tau_C(\delta)} H_\phi^{-1}(s) \mathbf{1}_C(X_s) ds = \int_0^\delta H_\phi^{-1}(s) \mathbf{1}_C(X_s) ds \leq H_\phi^{-1}(\delta) \delta$ , we conclude that

$$\mathbb{E}_x \left[ \int_0^{\tau_C(\delta)} \phi(H_\phi^{-1}(s)) ds \right] \leq \psi(0, x) + \kappa \delta H_\phi^{-1}(\delta).$$

Since  $\psi(0, \cdot)$  is bounded on  $C$  by assumption, we conclude with e.g.  $\delta = 1$ .

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