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# Multi-Site Fishing Models.

Pierre Auger<sup>1</sup>, Olivier Pironneau<sup>2</sup>

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## Abstract

Mathematical models are used to predict the long term effect of fishing on the fish biomass. The simplest models involve ordinary differential equations and functions of time only. Some more complex models have an array of functions of time only whose index represent the spatial position of each site. In this article we build on these and propose extensions using partial differential equations to account for the monodimensionality of the coast or the bidimensionality of the sea. We investigate the capacity of these models to validate non-fishing zones to fight over fishing.

**French title:** *Modèle de pêche multi-sites.*

Pour la pêche les modèles mathématiques servent à prédire la biomasse des poissons sur des temps longs. Les modèles les plus simples utilisent des équations différentielles pour des fonctions du temps seulement. Quelques modèles plus complexes généralisent les modèles précédents à un ensemble fini de sites avec des équations différentielles pour un vecteur de fonctions du temps où les indices sont associés aux sites. Nous proposons de généraliser cette approche en passant à la limite sur le nombre de sites afin d'obtenir des équations aux dérivées partielles à une variable d'espace si l'on considère seulement la côte ou 2 variables d'espaces si l'on considère la mer. Nous nous intéressons à la capacité de ces modèles à valider les zones de non-pêches pour combattre la surpêche.

*Keywords:* Fishery modelling, Marine Protected Area, Partial differential equations.

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## 1. Introduction

The logistic model introduced by Verhulst [7] for population dynamics and controlling the fish biomass  $B(t)$  has been extended by Schaefer [6] to fisheries with an effort function  $E(t)$  – conveniently representing the number of fishing boats at sea – and a catchability coefficient  $q$ . In [2] an extension relating the fishing effort to the market price  $p$  of fish is analysed. It applies to a coastal area with a single fishing zone; the fish biomass is a function of the fish reproduction rate  $r$  (which includes natural death) and the fish mortality due to fishing; in absence of fishing, there is a natural limit to  $B(t_\infty)$  which we denote by  $k$  (the carrying capacity of the site). Mortality due to fishing is proportional to fish biomass and fishing effort:

$$\frac{1}{B} \frac{dB}{dt} = r \left(1 - \frac{B}{k}\right) - qE \quad (1)$$

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The fishing effort is a function of the price of fish at time  $t$ ,  $p(t)$ :

$$\frac{1}{E} \frac{dE}{dt} = pqB - c. \quad (2)$$

This says that the rate of the fishing effort is proportional to the difference between the revenue and the cost  $c$  of fishing per unit of fishing effort.

When the market is liquid the price adjusts daily to balance supply  $qBE$  and demand  $D(p)$ . The demand is a monotone decreasing function of price, for example  $D(p) = \frac{A}{1+\beta p}$ . Hence  $p = \max\{p_{min}, (A - qBE)/(\beta qBE)\}$ ; it is necessary to introduce  $p_{min}$  beyond which selling of fish cannot happen, but it may be convenient to take  $p_{min} = 0$ .

This model has 2 nonlinear ODE; the long limit solution can either be  $B(t_\infty) = 0$  or  $B_\infty > 0$  depending on the parameters and the initial conditions  $B_0, E_0$ .

Extension of this model to multiple adjacent sites have been proposed in [2] among others. It uses the following notations

*Notations.*

- $B_i(t)$  the biomass at site  $i$  at time  $t$ .
- $E_i(t)$  the fishing effort at site  $i$  at time  $t$ .
- $\beta_j^i$  is the rate of migration of boats from site  $i$  to site  $j$ . Assume  $\beta_j^i = \beta_i^j$ .
- $k_i$  the carrying capacity of zone  $i$ .
- $m_j^i$  is the rate of migration of fish from site  $i$  to site  $j$ . Assume :  $m_j^i = m_i^j$ .
- $q$  is the catchability.
- $\epsilon, \phi$  time scales for population dynamics of fish and price adjustment respectively.
- $r$  is the fish population growth rate.
- $c$  is the cost per unit fishing effort.
- $p$  market price of fish.
- $A, \beta$  coefficients in the demand curve  $D(p)$ .
- $\mathbb{I}_i = 1$  if  $i$  is a fishing site and 0 otherwise.

The coast is divided in adjacent sites where fishing is allowed or forbidden.  $I_i$  is defined accordingly. The generalisation of the previous model, proposed in [2], is

$$\begin{aligned} \frac{dB_i}{dt} & - \epsilon r B_i \left(1 - \frac{B_i}{k_i} - \mathbb{I}_i \frac{q}{r} E_i\right) = m_{i-1}^i B_{i-1} + m_{i+1}^i B_{i+1} - (m_{i-1}^i + m_{i+1}^i) B_i, \\ \frac{dE_i}{dt} & - \epsilon (pqB_i - c_i) E_i = \beta_{i-1}^i E_{i-1} + \beta_{i+1}^i E_{i+1} - (\beta_{i-1}^i + \beta_{i+1}^i) E_i, \\ \frac{dp}{dt} & = \left(\frac{A}{1 + \beta p} - \sum_i q_i \mathbb{I}_i B_i E_i\right) \phi, \quad i = 1.., I. \end{aligned} \quad (3)$$

In this article we look at these models from 4 angles:

1. What are the partial differential equations, limit of the model when the number of sites tends to infinity?
2. Can they assert the efficiency of marine protected zones?
3. Can it be generalised to two dimensions, i.e. not just the coast?
4. Conversely, starting from the PDE can we recover the above discrete model?

We will take these four questions one by one and simulate numerically all the derived models and make interesting conclusions at the end.

## 2. A one dimensional continuous model for coastal fishing

Given a small mesh size  $\delta x$ , for any function  $x \rightarrow y(x)$  let  $y_i$  denote the value at  $x_i = i\delta x$  of  $y(x)$ . Let us interpret  $m_i^j$  as  $m(j\delta x, i\delta x)$  and similarly for  $\beta$ . The system above can be rewritten as

$$\begin{aligned} \frac{dB_i}{dt} &= -\epsilon r B_i \left(1 - \frac{B_i}{k_i} - \mathbb{I}_i \frac{q}{r} E_i\right) \\ &= (m^i B)_{i+1} + (m^i B)_{i-1} - 2(m^i B)_i - B_i(m_{i+1}^i + m_{i-1}^i - 2m_i^i) \\ &= \delta_{xx}(m(x)B) - B\delta_{xx}m(x) = \delta_x(m\delta_x B) + \delta_x B\delta_x m \end{aligned} \tag{4}$$

$$\begin{aligned} \frac{dE_i}{dt} &= -\epsilon(pqB_i - c_i)E_i \\ &= \delta_{xx}(\beta^i E) - E\delta_{xx}\beta^i = \delta_x(\beta\delta_x E) + \delta_x E\delta_x \beta \end{aligned} \tag{5}$$

$$\frac{dp}{dt} = \left(\frac{A}{1 + \beta p} - \sum_i q_i \mathbb{I}_i B_i E_i\right)\phi \tag{6}$$

where  $\delta_x$  and  $\delta_{xx}$  are finite difference operators:

$$(\delta_x y)_i = \frac{y_{i+1} - y_{i-1}}{2\delta x}, \quad (\delta_{xx} y)_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{(\delta x)^2}.$$

Let  $\delta x \rightarrow 0$ , then the system tends to

$$\begin{aligned} \partial_t B - \partial_x(m\partial_x B) &= \partial_x B\partial_x m + \epsilon r B \left(1 - \frac{B}{k} - \mathbb{I}(x)\frac{q}{r}E\right) \\ \partial_t E - \partial_{xx}(\beta(x)E) &= \partial_x E\partial_x \beta + \epsilon(pqB - c)E, \text{ for all } x \text{ such that } \mathbb{I}(x) = 1 \\ \partial_t p &= \left(\frac{A}{1 + \beta p} - \int_a^b \mathbb{I}(x)qBE dx\right)\phi \end{aligned}$$

If the price adjusts instantaneously to the market then

$$p = \frac{1}{\beta} \left(\frac{A}{\int_a^b \mathbb{I}(x)qnE dx} - 1\right)$$

and, implicitly applying  $\partial_x(\beta E) = 0$  at interfaces of fishing to non fishing zones. We summarize the results in the following proposition.

**Proposition 1.** When  $\delta x \rightarrow 0$  and fish price adjusts instantaneously, the fish biomass  $B$  and the fishing effort  $E$  are solution of the following integro-partial-differential system:

$$\begin{aligned} \partial_t B - \partial_x(m\partial_x B) &= \partial_x B \partial_x m + \epsilon B r \left(1 - \frac{B}{k} - \mathbb{I} \frac{q}{r} E\right) \\ \partial_t E - \partial_x(\mathbb{I} \partial_x(\beta E)) &= \mathbb{I}(x) \left( \partial_x E \partial_x \beta + \epsilon E \left( \frac{Bq}{\beta} \left( \frac{A}{\int_a^b \mathbb{I} q B E dx} - 1 \right) - c \right) \right) \end{aligned} \quad (7)$$

where  $\mathbb{I}(x)$  is one if  $x$  is in a fishing zone and zero if  $x$  is in a marine protected area.

2.1. Stationary solution when  $m$  and  $\beta$  are small

$$\frac{B}{k} + \mathbb{I} \frac{q}{r} E = 1, \quad \mathbb{I} \frac{Bq}{\beta} \left( \frac{A}{\int_a^b \mathbb{I} B q E dx} - 1 \right) = c \mathbb{I}$$

or equivalently

$$Bq\mathbb{I} = c\mathbb{I}\beta \frac{\lambda}{A - \lambda}, \quad \mathbb{I}E = \frac{r}{q} \left(1 - \frac{B}{k}\right), \quad \lambda = \int_a^b Bq\mathbb{I}E = \int_a^b r \left(B - \frac{B^2}{k}\right)$$

There is an autonomous equation for  $\lambda$ :

$$\begin{aligned} \lambda &= \int_a^b \mathbb{I} \frac{rc\beta}{q} \frac{\lambda}{A - \lambda} \left(1 - \frac{c\beta}{kq} \frac{\lambda}{A - \lambda}\right) = \int_a^b \mathbb{I} \frac{rc\beta}{q} \frac{\lambda}{A - \lambda} \left(1 + \frac{c\beta}{kq} - \frac{c\beta A}{kq} \frac{1}{A - \lambda}\right) \Rightarrow \\ 1 &= \frac{1}{A - \lambda} \int_a^b \mathbb{I} \frac{rc\beta}{q} \left(1 + \frac{c\beta}{kq}\right) - \frac{1}{(A - \lambda)^2} \int_a^b \mathbb{I} \frac{rA}{k} \left(\frac{c\beta}{q}\right)^2 = \frac{\beta Y + \beta^2 X}{A - \lambda} - \frac{A\beta^2 X}{(A - \lambda)^2} \\ \text{with } X &= \int_a^b \mathbb{I} \frac{rc^2}{kq^2} \text{ and } Y = \int_a^b \mathbb{I} \frac{rc}{q}. \end{aligned} \quad (8)$$

So, equation  $a'z^2 - b'z + 1 = 0$  gives  $z = 1/(A - \lambda)$  with  $b' = \beta Y + \beta^2 X$  and  $a' = A\beta^2 X$ . If the discriminant  $b'^2 - 4a' > 0$ , i.e.  $(Y + \beta X)^2 > 4AX$ , there is a positive root,

$$z = \frac{1}{2a'}(b' + \sqrt{b'^2 - 4a'}), \quad \lambda = A - \frac{1}{z}, \quad B_\infty = \frac{c\beta}{q} \frac{\lambda}{A - \lambda}, \quad E_\infty = \frac{r}{q} \left(1 - \frac{B_\infty}{k}\right).$$

If (7) is stable, its limit when  $t \rightarrow \infty$  is  $B_\infty, E_\infty$ . If  $B_\infty = 0$ , the fish disappear; it is possible only if  $A = c\beta(b - a)E_\infty$  and  $E_\infty q = r$ , i.e.  $Aq = c\beta(b - a)r$ .

2.2. Numerical Simulations

A semi-implicite finite difference scheme is used:

$$\begin{aligned} \frac{1}{\delta t}(B^{j+1} - B^j) - \partial_x(m\partial_x B^{j+1}) + B^{j+1}\epsilon r \left(\frac{B^j}{k} + \mathbb{I} \frac{q}{r} E^j\right) &= \partial_x B^j \partial_x m + B^j \epsilon r \\ \frac{1}{\delta t}(E^{j+1} - E^j) - \partial_x(\mathbb{I} \partial_x(\beta E^{j+1})) + E^{j+1}\epsilon \mathbb{I} \left(\frac{B^j q}{\beta} + c\right) &= \mathbb{I}(x) \left( \partial_x E^j \partial_x \beta + \epsilon E^j \left( \frac{AB^j q}{\beta \int_a^b \mathbb{I} q B^j E^j dx} \right) \right). \end{aligned} \quad (9)$$

Then this time-discrete system is discretized in space by a finite element method of degree 1.

### 2.2.1. Numerical Example

When all parameters are equal to 1 and  $L = \int_a^b \mathbb{I} > 1$  the second degree equation analysed above has two roots:  $X = Y = L$ ,  $(Y + \beta X)^2 = 4L^2 > 4AX = 4L$ ,  $b' = L + L^2$ ,  $a' = L$ ,  

$$b' + \sqrt{b'^2 - 4a'} = L + L^2 + \sqrt{(L + L^2)^2 - 4L} > 2a'A^{-1} = 2L.$$

In the following simulation the coast has length  $L = 3$ ; it is divided into a fishing zone  $(0, \frac{L}{2})$  and a marine protected zone  $(\frac{L}{2}, L)$ .

The other parameters are  $m = 0.1$  and  $\beta = 0.1$ . At time  $t = 0$  we assume that  $B = 2B_\infty$  and  $E = E_\infty/2$ . At  $x = 0$  we enforce  $n = 2B_\infty$  and at  $x = L$ ,  $E = 0.01$ .

The results are shown on Figures 1 and 2, with a catchability is  $q = 1$  and a fish and boat migration  $m = 0.1, \beta = 0.1$ . The biomass is decreasing in the middle of the fishing zone but reach an equilibrium in the no-fishing zone.

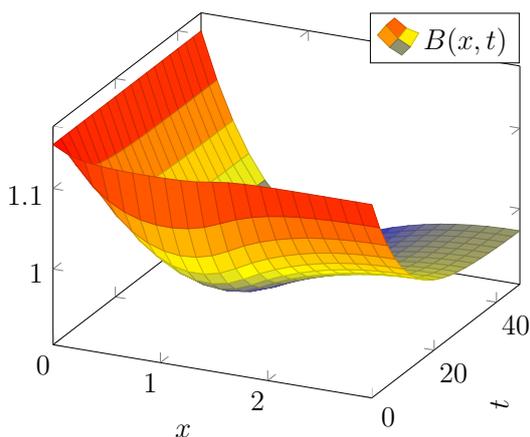


Figure 1: The fish biomass  $B(x, t)$  computed by the continuous model (7) versus costal position  $x \in (0, 3)$  at time (depth on the figure)  $t \in (0, 50)$ ;  $x > 1.5$  is a no fishing zone. The catchability is  $q = 1$ . The fish and the boat migration are  $m = 0.1, \beta = 0.1$ .

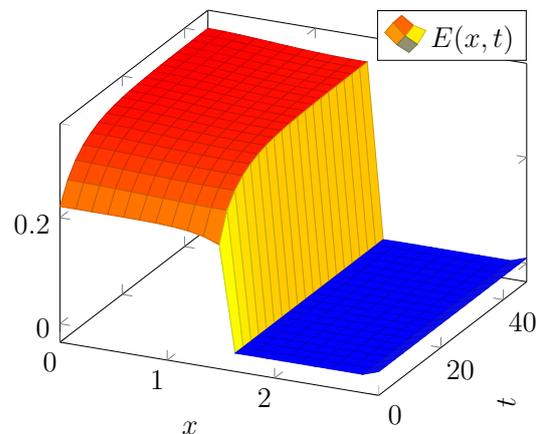


Figure 2: Fishing effort  $E$  near the coast  $x \in (0, 3)$  at time (depth on the figure)  $t \in (0, 50)$ ;  $x > 1.5$  is a no fishing zone. The catchability is  $q = 1$ . The fish and the boat migration are  $m = 0.1, \beta = 0.1$ .

To study the influence of the fish migration we ran another case with fish migration into the no-fishing zone  $m = 0.1 + 0.1x^2$ ; Results are shown on Figure 3 and Figure 4. The migration stabilises the biomass because it diminishes the catch in the fishing zone; yet this diminution does not affect  $E$  very much.

To study the influence of boat migration we ran another case with  $\beta = 0.1 + 0.1x^2$ ; Results are shown on Figure 5 and Figure 6.

To study the influence of the catchability we ran a case with  $q = 1 + \frac{x}{2}$ . Results are shown on Figure 7 and Figure 8.

These simulations show mild influence of migration coefficients and strong influence of the catchability. Furthermore while  $B$  is affected most,  $E$  does not change much, indicating that the coupling between  $B$  and  $E$  is not strong enough.

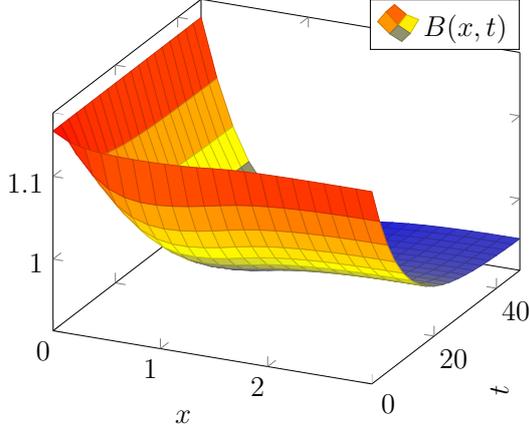


Figure 3: Number of fish  $n$  near the coast  $x \in (0, 3)$  at time (depth on the figure)  $t \in (0, 50)$ ;  $x > 1.5$  is a no fishing zone. The catchability is  $q = 1$ . The fish and the boat migration are  $m = 0.1 + 0.1x^2$ ,  $\beta = 0.1$  respectively.

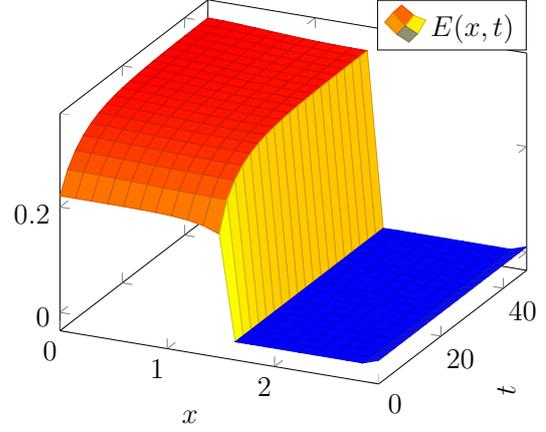


Figure 4: Fishing effort  $E$  near the coast  $x \in (0, 3)$  at time (depth on the figure)  $t \in (0, 50)$ ;  $x > 1.5$  is a no fishing zone. The catchability is  $q = 1$ . The fish and the boat migration are  $m = 0.1 + 0.1x^2$ ,  $\beta = 0.1$  respectively.

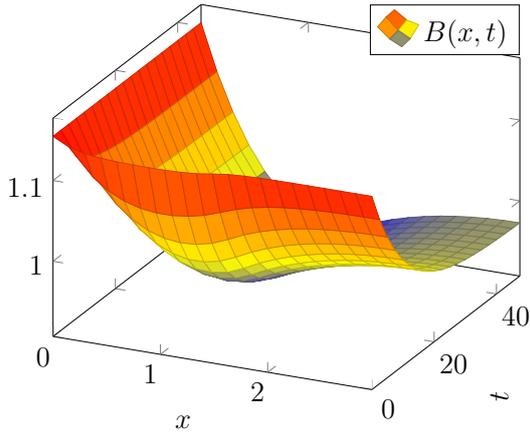


Figure 5: Number of fish  $n$  near the coast  $x \in (0, 3)$  at time (depth on the figure)  $t \in (0, 50)$ ;  $x > 1.5$  is a no fishing zone. The catchability is  $q = 1$ . The fish and the boat migration are  $m = 0.1, \beta = 0.1 + 0.1x^2$ .

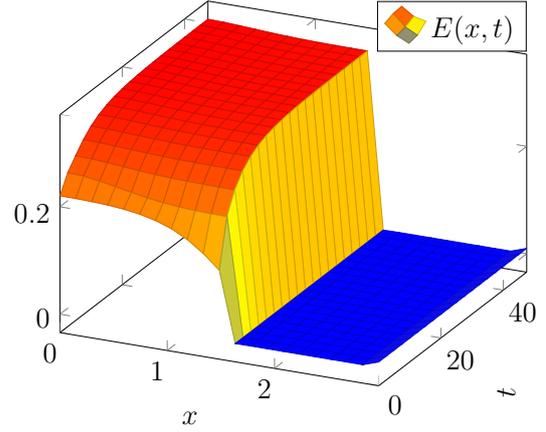


Figure 6: Number of boats  $E$  near the coast  $x \in (0, 3)$  at time (depth on the figure)  $t \in (0, 50)$ ;  $x > 1.5$  is a no fishing zone. The catchability is  $q = 1$ . The fish and the boat migration are  $m = 0.1, \beta = 0.1 + 0.1x^2$ .

### 3. Extension to two dimensional domains: fishing in the open sea

Consider a domain  $\Omega$  with a subdomain  $\mathcal{O}$  where fishing is forbidden. As before let  $\mathbb{I}$  be the fishing zone indicator. By analogy with the one dimensional case, it is reasonable to consider , for all  $\mathbf{x} \in \Omega$  and all  $T > 0$ :

$$\begin{aligned} \partial_t B - \nabla \cdot (m \nabla B) &= \nabla B \cdot \nabla m + B \epsilon r \left( 1 - \frac{B}{k} - \mathbb{I} \frac{q}{r} E \right) \\ \partial_t E - \nabla \cdot (\beta \nabla E) &= \mathbb{I}(x) \left( \nabla E \cdot \nabla \beta + \epsilon E \left( \frac{Bq}{\beta} \left( \frac{A}{\int_a^b \mathbb{I} q B E dx} - 1 \right) - c \right) \right) \end{aligned} \quad (10)$$

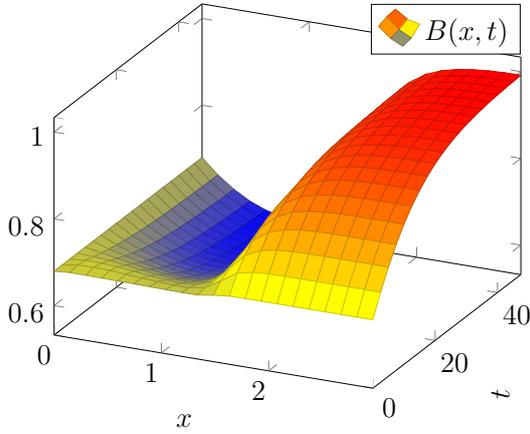


Figure 7: Number of fish  $n$  near the coast  $x \in (0, 3)$  at time (depth on the figure)  $t \in (0, 50)$ ;  $x > 1.5$  is a no fishing zone. The catchability is  $q = 1 + \frac{x}{2}$ . The fish and the boat migration are  $m = 0.1, \beta = 0.1$ .

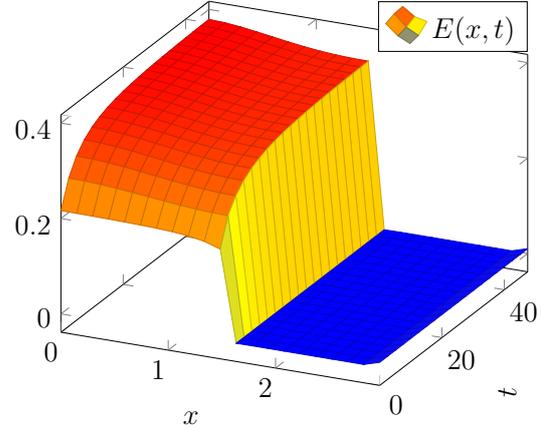


Figure 8: Number of boats  $E$  near the coast  $x \in (0, 3)$  at time (depth on the figure)  $t \in (0, 50)$ ;  $x > 1.5$  is a no fishing zone. The catchability is  $q = 1 + \frac{x}{2}$ . The fish and the boat migration are  $m = 0.1, \beta = 0.1$ .

### 3.1. A numerical test

The domain is a rectangle  $(-L, L) \times (0, H)$  with a rectangular no fishing zone  $(-L_1, L_1) \times (H_1, H)$ .

Initially the number of fish is  $2B_\infty$  on the left (i.e.  $x < 0$ ) and  $B_\infty$  on the right. The number of boats is  $E_\infty$  on the left and  $2E_\infty$  on the right.

On the left vertical boundary we assume that  $n = 2B_\infty$ , i.e. many fish. On the right vertical boundary  $n = B_\infty$ . On the top boundary of the no fishing rectangle we impose  $E = 0.01$  and for the second test  $m = 2B_\infty$  and nothing for the first test.

All parameters are equal to 1, except  $\beta = 0.1$  and  $m = 0.1$  for the first test and  $m = 0.5$  for the second. The system is simulated up to time  $t = 50$  with 50 time steps. The mesh has 1483 vertices. and 2884 triangles. Results are shown on Figures 9 and 10. It shows clearly how fishermen migrate from the right (where  $B$  is low initially) to the left and making the left zone harder to fish in. In the end there are no fish in the protected zone as well when the fish migration coefficient is small:  $m = 0.1$ ; when  $m = 0.5$  the left zone and the no fishing zone at the center are no longer empty at the end of the computation

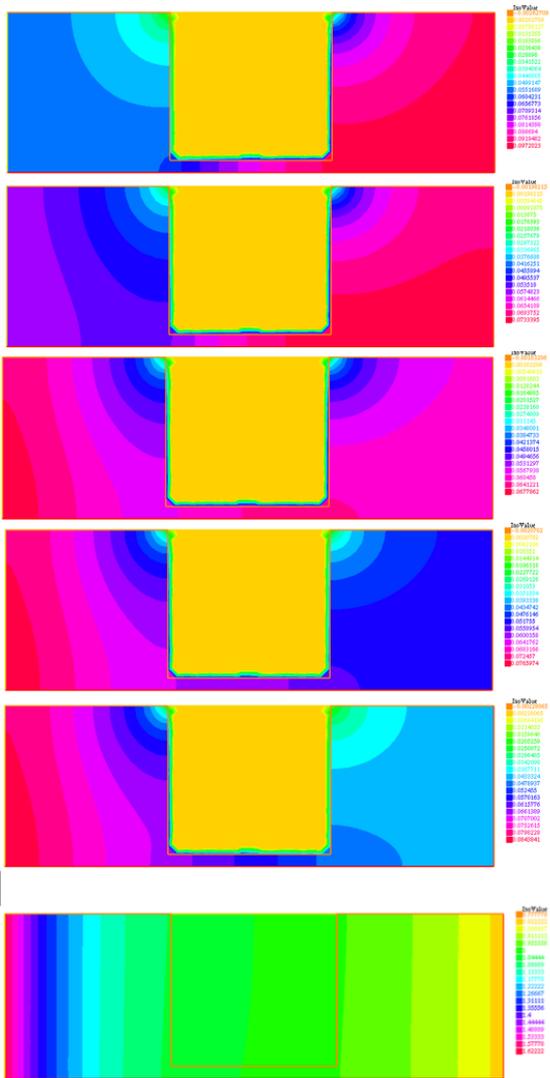


Figure 9: Case 1. Top 5 pictures: Evolution (one picture ever 10 time steps) of the fishing effort. The fish density is not prescribed at the top of the no-fishing zone and  $m = 0.1$ . The last picture is the fish density at the final time.

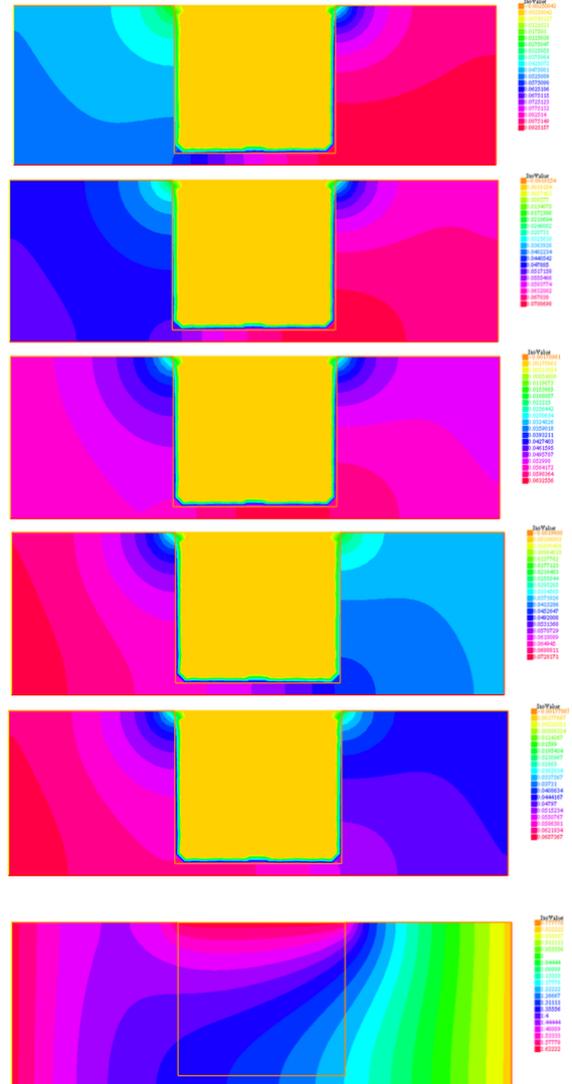


Figure 10: Case 2. Top 5 pictures: Evolution (one picture ever 10 time steps) of the fishing effort. The fish density is prescribed at the top of the no-fishing zone and  $m = 0.5$ . The last picture is the fish density at the final time.

#### 4. A continuous model for Plankton, Fish and Fishermen

The biggest difficulty of the previous model is the mobility function  $m$  which is probably very hard to guess in practice.

In this section we will see that if we can give a more solid ground to the previous two dimensional model. To this end we need to introduce plankton as an incentive for the motion of the fish. The notations are as follow:

##### 4.1. Notations

###### Variables

- $P(x, t)$  plankton density at point  $x$  and time  $t$ .

- $B(x, t)$  fish population at  $x$  and time  $t$ .
- $E_i(t)$  fishing effort by fishermen  $i$  at time  $t$
- $X_i(t)$  position of fisherman  $i$  at time  $t$ .
- $\pi(x, t)$  probability density of fishing boats at  $x$  at time  $t$ .
- $p(t)$  price of fish at time  $t$ .

#### Parameters

- $\Omega$  the domain occupied by the sea.  $(0, T)$ , the time interval for the study.
- $\epsilon$  time scale of the dynamics of plankton
- $v(x, t)$  sea current at point  $x$  and time  $t$ .
- $\mu, \nu, \kappa$  diffusion constant of plankton, fish position, boat position
- $a$  reproduction rate of plankton
- $b$  rate at which the plankton is eaten by fish
- $\alpha, \beta, \gamma, K = \frac{\alpha \bar{T}^2}{\beta + \gamma}$  behavioral parameters of a fisherman
- $r$  rate of reproduction of fish
- $d$  rate of death of fish
- $q$  catchability of fish
- $c$  cost of operation of a boat
- $A, B$  parameters of the demande for fish for a given price.

#### 4.2. Plankton

The plankton density  $P$  is regenerated at rate  $aP$  and eaten by the fish at rate  $b$  times  $B$ , the fish density. The logistic equation for  $P$  is:

$$\epsilon \partial_t P + v \cdot \nabla P - \mu \Delta P = aP(1 - P) - bBP, \quad P(0) = P^0 \text{ in } \Omega \times (0, T) \quad (11)$$

where  $P^0(x)$  is the probability density function of plankton at initial time. The model assumes that in absence of fish the long time limit of  $P$  is one. Here  $v$  is the sea current velocity. The term  $-aP^2$  models the natural death of the plankton. Having the rate of change of plankton density proportional to the concentration is convenient because of the following remark. More realistic models can be found in [1].

**Remark 1.** The right hand side is also  $aP(1 - \frac{b}{a}B - P)$ ; if  $B < \frac{a}{b}$ , then  $P$  is positive and bounded by  $1 - \frac{b}{a}B$  if it is so at time zero. Otherwise  $P$  may become negative and the model is no longer meaningful.

**Remark 2.** When  $c' := 1 - \frac{b}{a}B$  is constant and  $v = 0$  and  $\mu = 0$ , the solution of  $\dot{P} = aP(c' - P)$  is  $P = c' e^{ac' \frac{t}{\epsilon}} / (1 + e^{ac' \frac{t}{\epsilon}})$ , tends to  $c'$  when  $t \rightarrow \infty$ . When  $\mu > 0$  and  $\Omega$  is bounded there is no non-zero limit  $t \rightarrow \infty$  of

$$\epsilon \partial_t P - \mu \Delta P = aP(c' - P), \quad P(0) \text{ given in } (0, c') \text{ and } P|_{\partial\Omega} = 0. \quad (12)$$

### 4.3. Modelling of a fishing boat

A single fisherman will want to maximize its catch while keeping his cost minimum. Its catch is proportional to the fish population  $B(x, t)$  and its cost is proportional to the square of the modulus of the velocity of the boat to operate the engine; he may also prefer not to go too faraway so  $X(\bar{T})$  has a cost. So over a time interval  $(0, \bar{T})$ ,

$$\max_U J(U) := \int_0^{\bar{T}} \alpha B(X(t), t) dt - \int_0^{\bar{T}} \frac{\beta}{2} |U|^2 dt - \frac{\gamma}{2} |X(\bar{T})|^2 : \dot{X} = U(X(t)), \quad X(0) = 0,$$

where  $\alpha$  is a function of the price of the fish and  $\beta, \gamma$  include the price of fuel; they are constant over  $(0, \bar{T})$  but vary from one day to the next.

The solution is found by a calculus of variations. Let  $\delta U$  be a deviation from the optimal solution, then  $\delta \dot{X} = \delta U$ ,  $\delta X(0) = 0$ ,

$$\begin{aligned} \delta J &= \int_0^{\bar{T}} [\alpha \nabla B \delta X - \beta U \cdot \delta U] - \gamma X(\bar{T}) \cdot \delta X(\bar{T}) \\ &= \left[ \int_0^{\bar{T}} \alpha \nabla B d\tau - \gamma X(\bar{T}) \right] \cdot \delta X(\bar{T}) - \int_0^{\bar{T}} \left[ \left( \int_0^t \alpha \nabla B d\tau \right) \delta \dot{X} + \beta U \cdot \delta U \right] \end{aligned} \quad (13)$$

We will have  $\delta J = 0$  if

$$\int_0^t \alpha \nabla B d\tau + \beta U = \int_0^{\bar{T}} \alpha \nabla B d\tau - \gamma X(\bar{T})$$

which implies

$$\begin{aligned} \ddot{X}(t) &= -\frac{\alpha}{2\beta} \nabla B(X(t), t), \quad X(0) = 0, \quad \beta \dot{X}(\bar{T}) + \gamma X(\bar{T}) = 0. \\ U &= \frac{\alpha}{\beta} \int_t^{\bar{T}} \nabla B d\tau + \frac{\alpha\gamma}{\beta + \gamma} \int_0^{\bar{T}} \left( \int_0^t \nabla B(X(\tau), \tau) d\tau - \nabla B(X(t), t) \right) dt \end{aligned} \quad (14)$$

As a first approximation, we assume that  $\nabla B$  is constant; it gives

$$U = \frac{\alpha \bar{T}^2}{\beta + \gamma} \nabla B$$

*Interpretation.* In this formula  $\bar{T}$  is the length of time over which the fisherman takes a decision; it is not the whole day because only a rough knowledge of the local value of  $\nabla B$  near his boat is available to the fisherman. We retain from the above analysis that the velocity of the boat when it moves is proportional to  $\nabla B$ . The coefficient of proportionality is a consequence of the psychology of the fisherman; it can only be measured.

### 4.4. Fish

Fish move so as to optimize their food intake for a given energy expenditure. The problem is similar to that of the fisherman who, in the previous paragraph optimized his catch for a given effort level. Hence by the same optimisation argument the optimal swimming velocity of the fish is proportional to the gradient of the plankton density.

Fish die naturally at rate  $d$  and reproduce at a rate  $r$  proportional, function of the food supply  $P$ .

Fishing depletes the fish population; let  $N_m(t)$  be the number of fish caught during the time interval  $(0, T)$ . Let  $n_m(t) = \dot{N}_m(t)$ . Then the fish population  $B$  is driven by

$$\partial_t B + \nabla \cdot (B \nabla P) - \nu \Delta B = r(P)B - Bd - \sum_1^M n_m(t) \delta_{X_m(t)} \quad B(\cdot, 0) = B^0(\cdot) \quad \text{given,} \quad (15)$$

where  $\delta_x$  denote the Dirac mass at  $x$ . In the sequel we assume that  $r(P) = rP$ .

#### 4.5. Modelling of the number of fish caught by a fisherman

In an ideal situation one would state that  $n_m(t) = qB(X_m(t), t)$ , the coefficient of proportionality  $q$  being the *catchability*. However the fishermen have an operating cost  $c$  which must be kept less than the profit  $pqB$  where  $p$  is the market price of fish.

In [2],  $n_m(t) = B(X_m(t), t)qE_m(t)$ , i.e. the coefficient of proportionality between  $n_m$  and  $B(X_m(t), t)$  has a new coefficient, the ‘‘fishing effort’’  $E_m$  of the fisherman  $m$ ; it is given by

$$\frac{1}{E} \frac{dE}{dt} = pqB - c \quad (16)$$

The fishing effort is defined on the trajectory of the fisherman’s boat, given by

$$\dot{X}_m = U = K \nabla B(X_m(t), t), \quad X(0) \text{ given.} \quad (17)$$

The price of fish is given by balancing demand  $\frac{A}{1 + Bp}$  and supply  $q\overline{EB}$  where  $\overline{EB} = \sum_1^M E_m(t)B(X_m(t), t)$  is the sum of all fishing efforts:  $p = \max\{0, \frac{1}{B}(\frac{A}{q\overline{EB}} - 1)\}$ . So (15) becomes

$$\partial_t B + \nabla \cdot (B \nabla P) - \nu \Delta B = r(P)B - Bd - \sum_1^M qE_m(t)B(X_m(t), t) \quad B(\cdot, 0) = B^0(\cdot) \quad (18)$$

with (17) and

$$\dot{E}_m(t) = E_m(t) \max\left\{0, \left[ \left( \frac{A}{\sum_1^M E_m B(X_m, t)} - q \right) \frac{B(X_m(t), t)}{B} - c \right] \right\}. \quad (19)$$

**Proposition 2.** *A continuous model for  $P, B, \{E_m\}_1^M$ , the plankton density, the fish biomass and the effort function of each fisherman, is given by (11), (18), (17), (19).*

#### 4.6. A large number of fishing boats

Let us assume that at  $t = 0$  we know the distribution of fishing boats at sea. Let us add some randomness, with a 2D Brownian motion  $W$ ,  $\mathbf{E}[W_1 W_2] = 0$ , to the trajectories of each boat:

$$dX_t = K \nabla B(X_t, t) dt + \kappa dW_t. \quad (20)$$

The PDF of this process is given by the Kolmogorov equation

$$\partial_t \pi + \nabla \cdot (\pi K \nabla B) - \kappa \Delta \pi = 0, \quad \pi(0) \text{ given in } \mathbf{R}^2. \quad (21)$$

So  $V := \mathbf{E}[EB] = \int_{\mathbf{R}^2} EB\pi dx$  with  $E$  given by (16) with  $p = \frac{1}{B}(\frac{A}{qV} - 1)$  and  $B$  given by

$$\partial_t B + \nabla \cdot (B \nabla P) - \nu \Delta B = r(P)B - Bd - qEB \quad B(\cdot, 0) = B^0(\cdot) \quad (22)$$

Note that this is an approximation because it is not true that  $\sum_1^M B(X_m(t), t)qE_m(t)\delta_{X_m(t)}$  converges to  $qEB$  when  $X_m$  is random and  $M \rightarrow \infty$ .

**Definition 1.** *When the number of fishing boats  $M$  is large, a continuous model for  $P, B, E, \pi$ , the plankton density, the fish biomass the global effort function and the PDF of the boat density, is given by (11), (22), (16), (21).*

#### 4.7. Long time solution

*Fishing allowed everywhere.* Let us assume that the time scale of the dynamics of  $P$  is small:  $\epsilon \ll 1$ . Then  $\bar{P}_\infty := \lim_{t \rightarrow \infty} w = 1 - \frac{b}{a}B$ . Let  $\bar{B}_\infty = \lim_{t \rightarrow \infty} \int_{\Omega} B dx$  and similarly for  $\bar{E}_\infty$ . Integrating the equation of  $B$  over  $\Omega$  and the equation of  $E$  gives

$$\bar{B}_\infty(r\bar{P}_\infty - d - q\bar{E}_\infty) = 0, \quad pq\bar{B}_\infty - c = 0, \quad Bp = \frac{A}{q\bar{E}_\infty\bar{B}_\infty} - 1.$$

This leads to an equation for  $\bar{B}_\infty$ ; with  $B' = B/q$  and  $b' = b/a$  it is

$$r'b'\bar{B}_\infty^2 - (r - d - B'cb')\bar{B}_\infty - (r - d)B'c + A = 0. \quad (23)$$

*With a marine protected zone.* The interesting case is when parameters are such that the above equation has no solution (fish going extinct); in the no fishing zone the equations for  $P$  and  $B$  give an stationary average  $\bar{B}_\infty = (1 - \frac{d}{r})\frac{a}{b}$ ,  $\bar{P}_\infty = \frac{d}{r}$ .

#### 4.8. Numerical Simulations

##### 4.8.1. Plankton-Fish and no fishing boats

$$\begin{aligned} \partial_t P - \mu \Delta P &= aP(1 - P) - bBP, \quad P(0) = P^0(x) \text{ given, positive, bounded.} \\ \partial_t B + \nabla \cdot (B \nabla P) - \nu \Delta B &= r(P)B - dB \quad B(\cdot, 0) = B^0(\cdot) \text{ given.} \end{aligned} \quad (24)$$

The long time limit is  $\bar{B}_\infty = (1 - \frac{d}{r})\frac{a}{b}$ ,  $\bar{P}_\infty = \frac{d}{r}$ . Hence it is important that  $d < r$ . The domain is a rectangle  $\Omega = (-L, L) \times (0, H)$  with  $L = 3, H = 2$ . Define  $II = \Omega \setminus (-L_1, L_1) \times (H_1, H)$  with  $L_1 = 1, H_1 = 0.15$ . Starting from  $P^0 = \mathbf{1}_{x < L_1}$  and  $B^0 = 0.01 + 0.5\mathbf{1}_{-L_1 < x < L_1}$ , we obtained figure 11 at  $T = 2$ , when  $r = 1, a = 2, b = 1, d = 0.5$ . All diffusions,  $\mu, \kappa, \nu$  are set to 0.1. Results are shown on figure 11. The computation shows that most fish swim slightly to the left because that is where the plankton is. The long time limit is  $\bar{P}_\infty = 0.5, \bar{B}_\infty = 1$ .

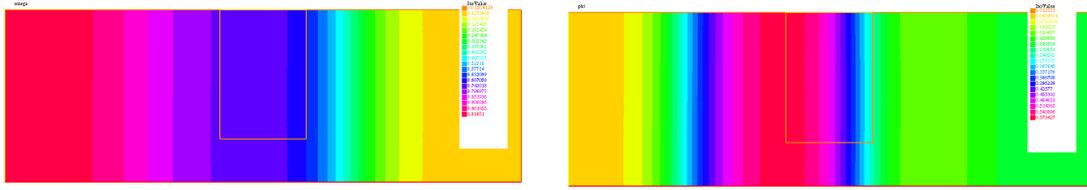


Figure 11:  $P$

$B$

#### 4.8.2. Plankton-Fish-Fishing boats

On Figure 12 we present the result of a simulation of the system of Proposition 2 (11),(18),(20),(16) with  $K = 80$ ,  $q = 20$ ,  $A = 10$ ,  $B = 1$ ,  $c = 1$ ;  $P^0$  and  $B^0$  as above and 50 boats randomly distributed with a Gaussian distribution centered at  $x = -1$  on the coast.

Three Simulations are performed. One with fishing restriction in a rectangle in the middle of the domain, not touching its base, in which it is seen that the fish population does not go to zero near the fishing zone.

In the second simulation fishing is allowed everywhere. Then it is seen that the fish population tends rapidly to zero everywhere.

In the third simulation  $d = 2$ , forcing  $\bar{P}_\infty = 0$ . Even with a no fishing zone the fish population goes to zero everywhere.

The fish do as above and a strip swims to the left; the fishermen hunt them and fish almost all the fish nearer to the coast. Finally figure 13 shows the fish price variation for the first 2 cases.

#### 4.8.3. Plankton - Fish - Fishing Effort - Fishermen density

We assume that there is a density of fishermen spread uniformly on the coast initially and spreading according to (21). As above when fishing is allowed everywhere, the fish density tends to zero everywhere. If fishing is not allowed in a rectangle then the fish density does not go to zero in the rectangle. The long time limit is  $\bar{B}_\infty = 0$  because (23) is  $\bar{B}_\infty^2 - \frac{79}{80}\bar{B}_\infty + 9.925 = 0$  which has no real root. Results are shown on figure 14. The conclusion of this section is that the two models are capable of asserting the benefit of a marine protected area. The model of Proposition 2 is more rigorous but harder to simulate. When the number of fishermen is large enough one may use the model defined in Definition 1, with some caution.

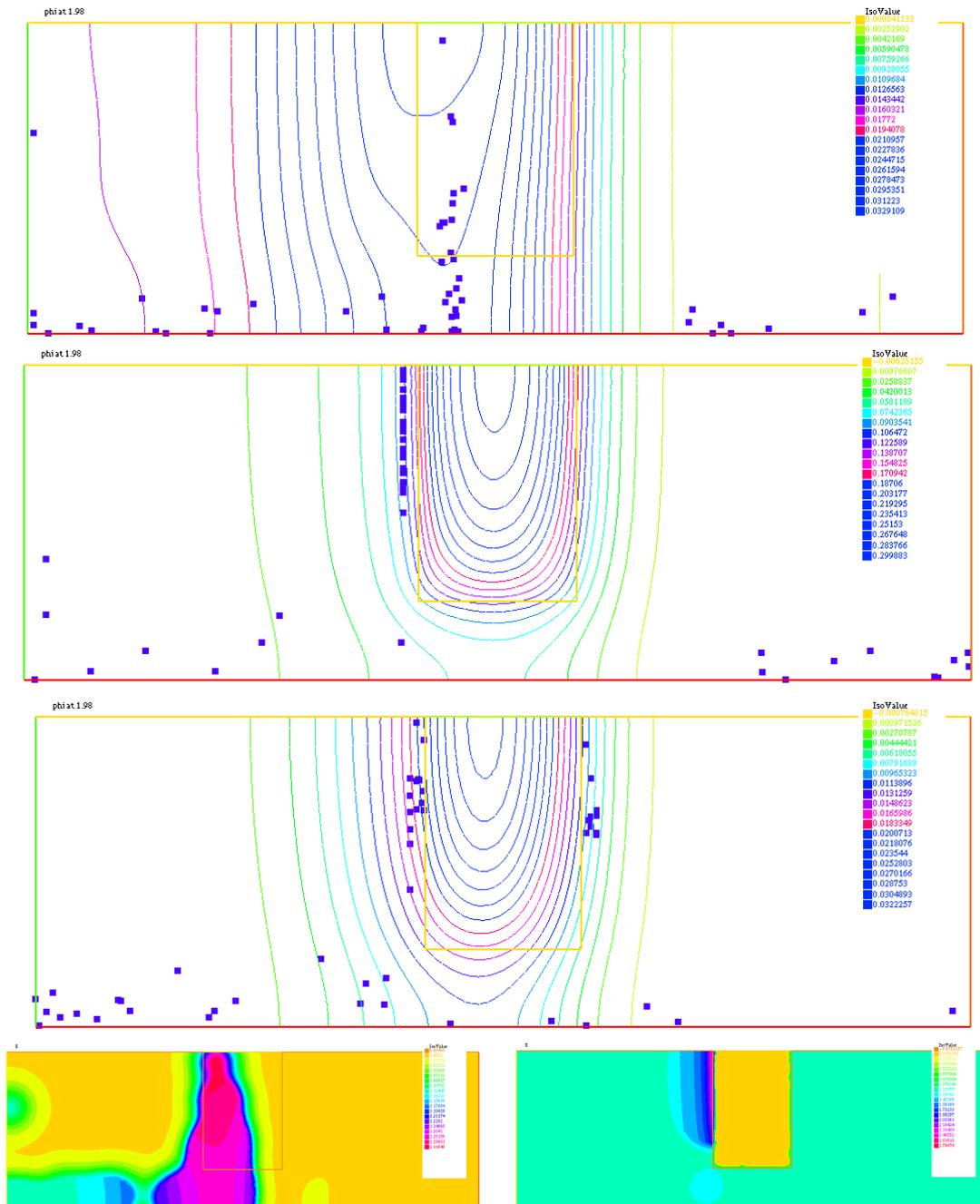


Figure 12: Top 2 figures: Level lines of the fish population function and the boats positions at  $T = 2$ . Top: fishing is allowed everywhere. Below, fishing is not allowed in the rectangle. On the top image all fish have disappeared. On the second image the dynamics in the rectangle is sufficient to keep the fish population from extinction. The third image corresponds to  $d = 0$ ; even with a no fishing zone fish disappear everywhere. Bottom: color map of the fishing effort without (left) and with (right) a non fishing zone

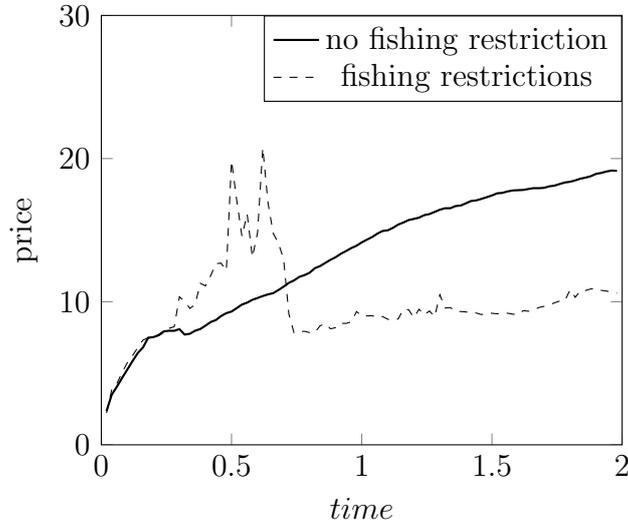


Figure 13: Variation of fish price versus time. The crisis at  $t = 0.7$  in the case of a no fishing zone is due to the fact that fishermen who were glued on the right side of the no fishing zone randomly pass on the left side of it. It happens when the gradient of  $B$  is large.

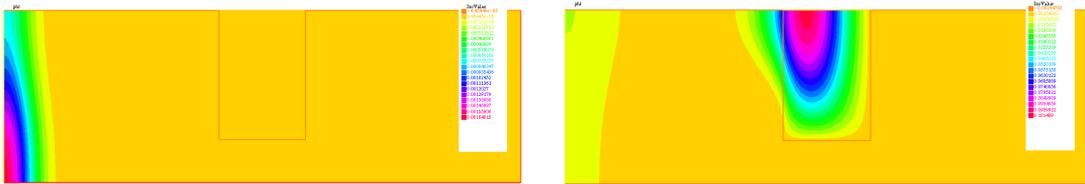


Figure 14: Left Fish density at  $T=2$ , when fishing is allowed everywhere. Right, fishing is not allowed in the rectangle

## 5. Very coarse discretization

In this section we take the opposite path: going from the continuous system to a discrete system with a few fishing or no fishing zones.

### 5.1. Three zones

To connect this analysis with the discrete systems proposed earlier in [2]; we consider 2 zones and an interface or, equivalently, 3 zones: fishing is allowed on the left, but may be forbidden on the right. The PDEs are discretized using finite differences on a very coarse mesh made of  $5 \times 3$  points (see figure 15). On the boundary of  $\Omega = (-L, L) \times (0, H)$  normal derivatives of  $P$  and  $B$  are set to zero; referring to figure 15 this means that all quantities at vertices  $\mathbf{q}_n$ ,  $n = 8, 7, 6$  and  $\mathbf{q}_n$ ,  $n = 11, 12, 13$  are equal to the same quantities at  $\mathbf{q}_n$ ,  $n = 1, 2, 3$  respectively. Similarly quantities at  $\mathbf{q}_n$ ,  $n = 0, 4$  are equal to the same quantities at  $\mathbf{q}_n$ ,  $n = 1, 3$  respectively. Quantities at  $\mathbf{q}_n$ ,  $n = 5, 9, 10, 14$  do not appear in the discretisation. It implies that there are only 3 points of interest, of coordinates

$$\mathbf{q}_1 = \left\{-\frac{L}{2}, \frac{H}{2}\right\}, \quad \mathbf{q}_2 = \left\{0, \frac{H}{2}\right\}, \quad \mathbf{q}_3 = \left\{\frac{L}{2}, \frac{H}{2}\right\}.$$

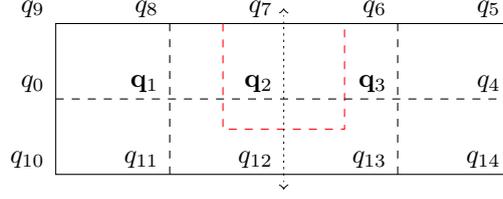


Figure 15: *Finite difference grid for the discretization of the domain  $\Omega$  which can be viewed either as 2 zones and an interface in the center or 3 zones, left, right and the zone in red. On the right of the vertical arrowed line in the center we may impose no fishing.*

The unknowns are  $\{P_n, B_n, E_n\}_{n=1,2,3}$ . Let  $h = L/4$  be the mesh size, then a finite element method of degree one gives

$$\begin{aligned}
\epsilon \dot{P}_1 - \frac{\mu}{h^2}(P_2 - P_1) &= aP_1(1 - P_1) - bB_1P_1 \\
\epsilon \dot{P}_2 - \frac{\mu}{h^2}(P_3 + P_1 - 2P_2) &= aP_2(1 - P_2) - bB_2P_2 \\
\epsilon \dot{P}_3 - \frac{\mu}{h^2}(P_2 - P_3) &= aP_3(1 - P_3) - bB_3P_3 \\
\dot{B}_1 - \frac{\mu}{h^2}(B_2 - B_1) - \frac{K}{2h} [(B_2 + B_1)(P_2 - P_1)] &= rB_1P_1 - dB_1 - qB_1E_1 \\
\dot{B}_2 - \frac{\mu}{h^2}(B_3 + B_1 - 2B_2) + \frac{K}{2h} [(B_2 + B_1)(P_2 - P_1) - (B_2 + B_3)(P_3 - P_2)] &= rB_2P_2 - dB_2 - qB_2E_2 \\
\dot{B}_3 - \frac{\mu}{h^2}(B_2 - B_3) + \frac{K}{2h} [(B_2 + B_3)(P_3 - P_2)] &= rB_3P_3 - dB_3 - qB_3E_3 \\
\dot{E}_n = E_n \max \left\{ 0, \left[ \left( \frac{A}{E_1B_1 + E_2B_2 + E_3B_3} - q \right) \frac{B_n}{B} - c \right] \right\}, \quad n = 1, 2, 3. & \quad (25)
\end{aligned}$$

Additionally  $P_n(0), B_n(0), E_n(0), n = 1, 2, 3$  must be given. Note that if  $E_i(0) = 0$  then  $E_i(t) = 0$  at all times  $t > 0$ . Hence a no-fishing zone is specified simply by putting  $E_n(0) = 0$  in the zone.

*Parameters.*

```

N = 150      # number of time step
T = 10      # final time
h = 7       # grid size
mu = 0.1    # diffusion
a = 2       # growth rate of plankton
b=0.5      # rate of plankton eaten by fish
r = 1.5    # fish reproduction rate
d = 1      # fish death rate
q = 1      # catchability
A,B = 10,1 # price constants
c = 1      # cost for fisherman
K = 3      # moving speed of fish to plankton

```

*Initial conditions.*

$$w_1, w_2, w_3 = 0.5, 0.5, 0.0$$

$$\phi_1, \phi_2, \phi_3 = 1, 1, 1$$

$$E_1, E_2, E_3 = 0.01, 0., 0.$$

**Results:** see Figure 16. The results are shown on figure 16 and show fish depletion on

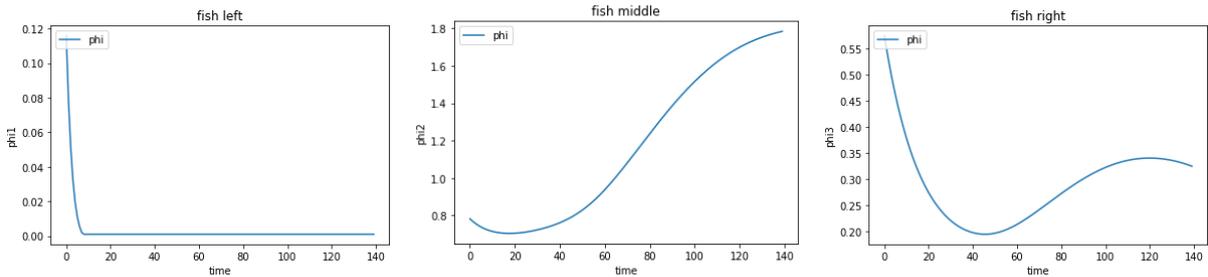


Figure 16: Evolution in time of the fish population at points  $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ , fishing is allowed everywhere

the left where fishing occurs, and non zero limit of fish biomass in the center and on the right.

## 6. A Marine Protected Area surrounded by 2 fishing zones

The previous simulation has shown that a no-fishing zone surrounded by 2 fishing zones seems capable of keeping the fish biomass non-zero in the MPA. In this paragraph we further validate the concept with an asymptotic analysis.

We consider a large fishing area in the middle of which a protected area where fishing is prohibited has been installed. We therefore consider a central zone 2 which is a Marine Protected Area (MPA) surrounded on both sides by fishing zones respectively 1 and 3. The model takes into account the dynamics of the plankton population as well as that of the fish that feed on it and are caught. Let  $P_1(t)$ ,  $P_2(t)$  and  $P_3(t)$  be the plankton biomass in zone 1, 2 and 3 at time  $t$ . Similarly, let  $B_1(t)$ ,  $B_2(t)$  and  $B_3(t)$  be the fish biomass in zone 1, 2 and 3 at time  $t$  in Megatons. Let  $E(t)$  be the total fishing effort at time  $t$ .  $r$  is the growth rate of the plankton species and  $K$  its carrying capacity assumed to be the same in the three zones. Fish feed on plankton by predation.  $a$  is a predation rate.  $e$  is the conversion coefficient from plankton biomass to fish biomass.  $d$  is the fish mortality rate. Fishing boats catch fish.  $q$  is the catchability. The complete model reads:

$$\frac{dP_1}{d\tau} = \epsilon(kP_2 - kP_1) + rP_1\left(1 - \frac{P_1}{K}\right) - aP_1B_1 \quad (26)$$

$$\frac{dP_2}{d\tau} = \epsilon(kP_1 + kP_3 - 2kP_2) + rP_2\left(1 - \frac{P_2}{K}\right) - aP_2B_2 \quad (27)$$

$$\frac{dP_3}{d\tau} = \epsilon(kP_2 - kP_3) + rP_3\left(1 - \frac{P_3}{K}\right) - aP_3B_3 \quad (28)$$

$$\frac{dB_1}{d\tau} = \epsilon(mB_2 - mB_1) + eaP_1B_1 - dB_1 - qB_1E \quad (29)$$

$$\frac{dB_2}{d\tau} = \epsilon(mB_1 + mB_3 - 2mB_2) + eaP_2B_2 - dB_2 \quad (30)$$

$$\frac{dB_3}{d\tau} = \epsilon(mB_2 - mB_3) + eaP_3B_3 - dB_3 - qB_3E \quad (31)$$

$$\frac{dE}{d\tau} = \frac{A}{\beta} - cE - \frac{q}{\beta}B_1E - \frac{q}{\beta}B_3E \quad (32)$$

Where  $\tau$  is the fast time,  $t = \varepsilon\tau$  is the slow time and where  $\varepsilon \ll 1$  is a small dimensionless parameter.

**Remark 3.** *This model is identical to (25) except that notations have changed and the transfer rate  $K$  is zero.*

Hence this complete model assumes that the displacement of the plankton and the fish between the zones is by diffusion. We assume that the three areas are large and that the migratory flows between the zones remain small. Thus, the diffusion of plankton and fish takes place on a slow time scale while population dynamics in each patch is fast in comparison. It is also assumed that the price varies according to the difference between the demand and the supply. The demand  $D(p) = \frac{A}{(1+\beta p)}$  is assumed to be a monotone decreasing function of price. The supply is the instantaneous catch in the two fishing areas. We further assume that the price is fixed very rapidly.

We will now investigate whether it is possible to obtain an equilibrium corresponding to an overfishing situation leading to the extinction of the fish stock in the two fishing zones while the fish succeeds in surviving in the marine protected area (MPA). Therefore, let us look for existence of a fast equilibrium with no fish in fishing zones and fish in MPA,  $(K, P_2^*, K, 0, B_2^*, 0, E^*)$ . It is simple to check that such an equilibrium exists in the positive quadrant with:

$$P_2^* = \frac{d}{ea}, \quad B_2^* = \frac{r}{a}\left(1 - \frac{d}{eaK}\right), \quad E^* = \frac{A}{c\beta} \quad (33)$$

Furthermore it is locally asymptotically stable. Now let's look at the dynamics on the slow timescale. The reduced model is obtained by replacing the previous fast equilibrium in the complete model and using the slow time. The slow model is a diffusion model with plankton and fish exchanges between the zones:

$$\frac{dP_1}{dt} = (kP_2^* - kK), \quad \frac{dP_2}{dt} = (2kK - 2kP_2^*), \quad \frac{dP_3}{dt} = (kP_2^* - kK) \quad (34)$$

$$\frac{dB_1}{dt} = mB_2^*, \quad \frac{dB_2}{dt} = -2mB_2^*, \quad \frac{dB_3}{dt} = mB_2^* \quad (35)$$

However, these flows between the different zones are small. Fish move from the central area to the two fishing zones and are harvested rapidly. As a consequence, there is no fish in the fishing areas and the plankton tends to its carrying capacity. This simple model shows that it is possible to obtain sites with fish and sites without fish in meta-population type models. For this, it is necessary to have a small flow of plankton and fish between the different sites. Cases of fast diffusion or going on at the same time scale as local population dynamics do not allow it.

## 7. Conclusion

Starting from the multi-site model of [2] we have derived its continuous integro-differential limit. Numerical simulations have asserted the marine protected areas as an efficient tool for keeping a minimum fish biomass at sea, at least in simple cases. The continuous model can be extended in 2D by analogy, but without rigorous justification.

In an attempt to build similar 2D models with more mathematical justification we have turned to the optimisation of utility functions for the fish and the fishermen. Fish change place to gather more food and as the food density grows its feeding effort becomes less. It has been shown that the fish velocity is proportional to the gradient of its food density  $P$ . This requires the modelling of plankton (or smaller fish), which, in turn, moves with the sea currents. Similarly it is in the interest of the fishermen to move proportionally to the gradient of the fish biomass  $B$ . Such behavioural remarks lead to a model for  $P, B$  and the fishing effort  $E^m$  of each fisherman  $m$ . The simulations have shown that it is again possible to design marine protected areas which will prevent the disappearance of fish. Unfortunately as the number of fishermen grows one would like to work with the probability density of the fishermen presence at point  $x$  at time  $t$ , but the consequent continuous model requires more mathematical studies to be asserted rigorously.

Finally, in the last section, we have discretised the continuous model on a very coarse mesh in order to derive simpler multi-site 2D models consistent with the continuous models and for which all parameters have a clear meaning. Again they show that it is possible to design a marine protected zone in which the fish population has a non-zero asymptotic limit.

Let us recall that any model needs to be confronted to data for validity, a huge task for us for the future. Then all parameters must be guessed from the data, but we have shown in [3] that the techniques of AI are quite efficient for such a task.

## Acknowledgement

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## References

- [1] Peter J.S. Franks: NPZ models of Planton dynamics. *J. of Oceanography*, vol 58, pp 379-387, 2002.
- [2] T. Brochier, P. Auger, D. Thiao, A. Bah, S. Ly, T. Nguyen Huu, P. Brehmer. Can overexploited fisheries recover by self-organization? Reallocation of the fishing effort as an emergent form of governance. *Marine Policy*, 95 (2018) 46-56.
- [3] P. Auger and O. Pironneau: Parameter Identification by Statistical Learning of a Stochastic Dynamical System Modelling a Fishery with price variation. *Comptes rendus de l'académie des sciences*. May 2020.
- [4] F. Hecht (2012): New development in FreeFem++, *J. Numer. Math.*, 20, pp. 251-265. (see also [www.freefem.org](http://www.freefem.org).)
- [5] A. Moussaoui, M. Bensenane, P. Auger, A. Bah, On the optimal size and number of reserves in a multi-site fishery model, *Journal of Biological Systems*. Vol. 23, No. 01, pp. 31-47 (2015)
- [6] M. B. Schaefer: Some aspects of the dynamics of populations important to the management of commercial marine fisheries. *Inter-American Tropical Tuna Commission*, p. 2526, (1954).
- [7] Pierre-François Verhulst: Notice sur la loi que la population poursuit dans son accroissement. *Correspondance mathématique et physique*, No. 10, (1838), p. 113-121.