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# THE FRISCH-PARISI CONJECTURE I: PRESCRIBED MULTIFRACTAL BEHAVIOR, AND A PARTIAL SOLUTION

JULIEN BARRAL AND STÉPHANE SEURET

ABSTRACT. In this work and its companion [12], we construct Baire function spaces in which typical elements share the same prescribed multifractal behavior and obey a multifractal formalism, providing a solution to the so-called Frisch-Parisi conjecture for functions, an inverse problem raised by S. Jaffard. In this first part, a family  $\mathcal{E}_d$  of almost-doubling fully supported capacities on  $\mathbb{R}^d$  with prescribed singularity spectra is constructed. With each  $\mu \in \mathcal{E}_d$  we associate a Baire function space  $\mathbf{B}^\mu(\mathbb{R}^d)$  (a generalisation of Hölder-Zygmund spaces) in which typical functions share the same singularity spectrum as  $\mu$ . This yields a partial solution to the conjecture. In [12], we introduce and study a family  $\mathbf{B} = \{\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)\}_{\mu \in \mathcal{E}_d, (p,q) \in [1,+\infty]^2}$  of *heterogeneous* Besov spaces that contains  $\{\mathbf{B}^\mu(\mathbb{R}^d)\}_{\mu \in \mathcal{E}_d}$  and generalises in a natural direction the family of standard Besov spaces, and we solve the inverse problem exhaustively inside  $\mathbf{B}$ .

## 1. INTRODUCTION

This paper and its companion [12] deal with multifractal analysis of functions, which originates from the first geometric quantification of the Hölder singularities structure in fully developed turbulence [48, 47, 25]. This subject is an instance of the natural concept of multifractality, which comes into play as soon as, given a mapping  $h : X \rightarrow A$  between a metric space  $(X, d)$  and a set  $A$ , one wants to describe geometrically the level sets of  $h$  by considering the mapping  $\sigma : \alpha \in A \mapsto \dim h^{-1}(\{\alpha\})$ , where  $\dim$  stands for the Hausdorff dimension. Indeed, in many situations, the level sets of  $h$  form an uncountable family of (disjoint) fractal sets, and  $\sigma$  is sometimes called *multifractal spectrum*. This spectrum provides a hierarchy between these level sets, according to their size measured by their Hausdorff dimension. Such spectra occur in many mathematical fields, such as harmonic and functional analysis (in the description of fine properties of Fourier series [33, 14] or typical elements in function spaces [16, 36]), probability theory (to describe fine properties of Brownian motion or SLE curves [53, 54, 44, 26]), multiplicative chaos and Gaussian free field, random covering problems [9, 32, 58, 5]), ergodic theory, dynamical and iterated function systems (to analyse Gibbs/harmonic measures on conformal repellers, Birkhoff averages, and self-similar measures [56, 46, 23, 24, 60]), metric number theory (Diophantine approximation and ubiquity theory [41, 30, 11]), shrinking targets problems and dynamical covering problems [29, 22]), the previous references being far from exhaustive.

In the multifractal analysis of a real valued function  $f \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ , the function  $h$  of interest is the pointwise Hölder exponent function  $h_f$ , which is defined as follows. Given  $x_0 \in \mathbb{R}^d$ , and  $H \in \mathbb{R}_+$ ,  $f$  is said to belong to  $\mathcal{C}^H(x_0)$  if there exist a polynomial  $P$  of degree at most  $\lfloor H \rfloor$ , a constant  $C > 0$ , and a neighborhood  $V$  of  $x_0$  such that

$$\forall x \in V, \quad |f(x) - P(x - x_0)| \leq C|x - x_0|^H.$$

The *pointwise Hölder exponent* of  $f \in L_{\text{loc}}^{\infty}(\mathbb{R}^d)$  at  $x_0$  is

$$(1.1) \quad h_f(x_0) = \sup \left\{ H \in \mathbb{R}_+ : f \in \mathcal{C}^H(x_0) \right\},$$

and  $f$  is said to have a Hölder singularity of order  $h_f(x_0)$  at  $x_0$ . The associated multifractal spectrum, also called *singularity spectrum* of  $f$ , is the mapping

$$\sigma_f : H \in \mathbb{R} \cup \{\infty\} \mapsto \dim E_f(H) \in [0, d] \cup \{-\infty\}, \text{ where } E_f(H) := h_f^{-1}(\{H\})$$

(note that  $E_f(H) = \emptyset$  for  $H < 0$ ). Again,  $\dim$  stands for the Hausdorff dimension, with the convention  $\dim \emptyset = -\infty$ . The function  $f$  is said to be *multifractal* when  $E_f(H) \neq \emptyset$  for at least two values of  $H$ .

The idea of considering this spectrum goes back to the physicists U. Frisch and G. Parisi [25], who aimed at quantifying geometrically the local variations of the velocity field of a turbulent fluid, and introduced the term *multifractal*. Another fundamental idea pointed out by Frisch and Parisi consisted in coupling the singularity spectrum with a large deviations approach, in order to statistically describe the Hölder singularities distribution (similar to Mandelbrot's approach for measures [48]). This led to the notion of *multifractal formalisms* for functions. In such a formalism, the singularity spectrum  $\sigma_f$  of a Hölder continuous function  $f$  is always dominated by (and in good cases, coincides with) the Legendre-Fenchel transform

$$\zeta_f^*(H) := \inf_{q \in \mathbb{R}} Hq - \zeta_f(q)$$

of a mapping  $\zeta_f : \mathbb{R} \rightarrow \mathbb{R}$ , called the *scaling function* or the  $L^q$ -*spectrum* of  $f$ : for every  $h \geq 0$ ,  $\sigma_f(h) \leq \zeta_f^*(h)$ . This mapping  $\zeta_f$  is a kind of free energy function encapsulating the asymptotic statistical distribution of the Hölder singularities as the observation scale tends to 0, and it can be numerically estimated [38]. For instance, in their seminal article, Frisch and Parisi used for  $\zeta_f$  the scaling exponent of the moments of the increments of  $f$ , informally defined as

$$|h|^{-d} \int_{\Omega} |f(x+h) - f(x)|^q dx \sim |h|^{\zeta_f(q)} \quad \text{as } h \rightarrow 0,$$

where  $\Omega$  is a fixed bounded domain on which  $f$  is supposed to be fully supported. The heuristics developed in [25] lead to seek for the largest classes of functions for which the equality

$$(1.2) \quad \sigma_f(H) = \zeta_f^*(H)$$

holds at any  $H$  such that  $\zeta_f^*(H) \geq 0$ . In such a situation, one says that *the multifractal formalism holds* for  $f$ , or that  $f$  satisfies the multifractal formalism. Then, the spectrum  $\sigma_f$  is a continuous concave mapping with support included in  $(0, +\infty)$ , and assuming that the topological support of  $f$  is full, one necessarily has  $\sigma_f(H) = d = -\zeta_f(0)$  for some  $H \geq 0$  (for instance the level set  $E_f(H)$  may have a positive Lebesgue measure).

We will come back to rigorous definitions of multifractal formalisms for measures in Section 2.2, and for functions in the second paper [12]. The concept of multifractal formalism motivated many works in geometric measure theory [15, 51, 43, 45], dynamical systems in connection with the thermodynamic formalism [55], and analysis [34, 36, 37]. It provides a powerful framework to describe the fine geometric structure of invariant measures of various dynamical systems [19, 57, 55] and self-similar and self-affine measures [42, 51, 52, 43, 24, 6], self-similar functions [34], as well as limit measures or functions in multiplicative chaos theory [31, 9, 8]. The singularity spectrum and its suitable

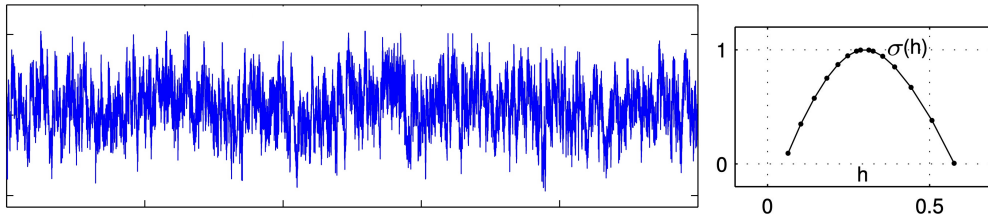


FIGURE 1. Estimated singularity spectrum (right) for the 1D velocity of a turbulent flow (left) - Credit to P. Abry, H. Wendt

extensions to non-bounded functions have also been used to describe the regularity properties of celebrated functions like Riemann’s and Brjuno’s functions [33, 59, 39], stochastic processes like Lévy processes and general classes of Markov processes [35, 7, 62], as well as Lévy processes in multifractal time [10]. Coming back to turbulence, it is worth mentioning that recently Banica and Vega established a remarkable connection between multifractal functions obeying some multifractal formalism, namely Riemann’s function  $\sum_{n \geq 1} \frac{\sin(n^2 x)}{n^2}$  and some of its variants, and specific solutions to the binormal curvature flow, a geometric PDE used to model the evolution of vortex filaments [2].

Multifractal formalisms are also relevant in many other applications, due to the existence of stable algorithms that precisely estimate scaling functions  $\zeta_f$  of numerical data. Then, a key observation is that for most of real-life data associated with intermittent phenomena, their estimated singularity spectra  $\zeta_f^*$  have a characteristic strictly concave bell shape (see [1] and Figure 1). This is also the case for the singularity spectra of important classes of functions possessing scaling properties [34, 10, 8]. This behavior is in striking contrast to the results established for typical functions in the classical function spaces, where “typical” is meant in the sense of Baire categories<sup>1</sup>. Indeed, it has been proved that typical increasing real functions (Buczolich&Nagy [16]), typical functions in Sobolev and Besov spaces (Jaffard [36], Jaffard&Meyer [40]), and typical measures (Buczolich&Seuret, Bayart [17, 13]) satisfy a multifractal formalism and have an affine increasing singularity spectrum. One concludes that, from the multifractal standpoint, realistic behaviors are not reproduced by typical elements in the standard function spaces. Precise statements regarding typical singularity spectrum and multifractal formalisms in Besov spaces are discussed in [12].

On the other hand, the previous genericity results show that many multifractal functions do satisfy some multifractal formalism without assuming any scale invariance properties. In [36], Jaffard seeks for Baire topological spaces of functions in which typical functions have a prescribed singularity spectrum, and do obey some multifractal formalism. He names this inverse problem “Frisch-Parisi conjecture”, and provides a partial solution to it: some intersections of homogeneous Besov spaces are Baire topological spaces in which typical functions possess an increasing compactly supported singularity spectrum, with a prescribed concave part, and another affine part; moreover, typical elements partially obey some multifractal formalism (see [12] for a detailed description of Jaffard’s result). Again, no scale invariance is assumed.

<sup>1</sup>Recall that in a Baire topological space  $E$ , a property  $\mathcal{P}$  is called typical, or generic, when the set  $\{f \in E : f \text{ satisfies } \mathcal{P}\}$  is of second category in  $E$ , or equivalently contains a dense  $G_\delta$ -set, that is the intersection of a countable family of dense and open sets. One says that typical elements in  $E$  satisfy  $\mathcal{P}$  when  $\mathcal{P}$  is typical in  $E$ .

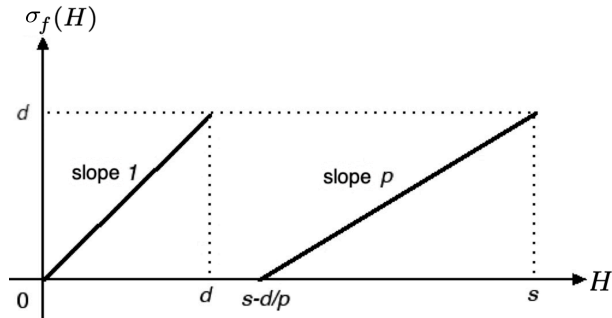


FIGURE 2. Typical multifractal spectrum of probability measures (left) or functions in  $B_q^{s,p}(\mathbb{R}^d)$  when  $s > d/p$  (right).

The *Frisch-Parisi conjecture* considered by Jaffard is formulated as follows:

**Conjecture 1** (Frisch-Parisi conjecture). *Let  $\mathcal{S}_d$  be the set of functions  $\sigma : \mathbb{R} \rightarrow [0, d] \cup \{-\infty\}$  such that  $\sigma$  is concave, continuous, with compact support included in  $(0, +\infty)$  and whose maximum equals  $d$ . For every  $\sigma \in \mathcal{S}_d$ , there exists a Baire function space of functions defined on  $\mathbb{R}^d$  in which any typical element  $f$  obeys some multifractal formalism and satisfies  $\sigma_f = \sigma$ .*

Note that the set  $\mathcal{S}_d$  consists of those mappings  $\sigma$  which are admissible to be the singularity spectrum of some Hölder continuous function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  whose pointwise Hölder exponents range in a compact subinterval of  $(0, +\infty)$ , such that  $\dim E_f(H) = d$  for at least one exponent  $H$ , and which obeys a multifractal formalism.

In the present paper, we first prove the existence of almost-doubling and  $\mathbb{Z}^d$ -invariant Radon measures fully supported on  $\mathbb{R}^d$  with prescribed singularity spectrum, and which satisfy the standard multifractal formalisms for measures developed in [15, 51] (Theorem 4). Such a measure  $\nu$  possesses scaling-like properties. Also, it possesses the homogeneous property that its restrictions to any non-empty closed ball of positive measure shares the same multifractal properties as  $\nu$ . Up to now, such a property for measures obeying the multifractal formalism and with prescribed singularity spectrum was only known for measures supported on a totally disconnected set [3] (see also [18] for results on the prescription of the singularity spectrum for measures). Then, we introduce a family  $\{\mathbf{B}^\mu(\mathbb{R}^d)\}_{\mu \in \mathcal{E}_d}$  of Baire functions spaces indexed by the set  $\mathcal{E}_d$  of capacities  $\mu$  obtained as positive powers of one of the measures  $\nu$  built before (i.e. there exists  $s > 0$  such that  $\mu(E) = \nu(E)^s$  for all  $E \in \mathcal{B}(\mathbb{R}^d)$ ). The definition of  $\mathbf{B}^\mu(\mathbb{R}^d)$  is based on wavelets; an equivalent definition based on moduli of smoothness is achieved in [12]. We prove that Baire typical functions in  $\mathbf{B}^\mu(\mathbb{R}^d)$  have a multifractal spectrum equal to that of  $\mu$ . Since prescribing the multifractal spectrum of measures makes it possible to prescribe that of capacities, this yields the following result:

**Theorem 2.** *For every  $\sigma \in \mathcal{S}_d$ , there exists a Baire space of functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  in which any Baire typical element  $f$  satisfies  $\sigma_f = \sigma$ .*

This does not solve completely the Frisch-Parisi conjecture, since the multifractal formalism is not investigated in this paper. This is achieved in the second paper [12], where a much wider family  $\{\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)\}_{\mu \in \mathcal{E}_d, (p,q) \in [1, +\infty]^2}$  of Baire function spaces, extending classical Besov spaces and referred to as Besov spaces in *multifractal environment*, is introduced and

studied ( $\mathbf{B}^\mu(\mathbb{R}^d)$ ) corresponds to the special case  $p = q = \infty$ ). Using the spaces  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  with suitable parameters  $\mu$ ,  $p$  and  $q$  yields various solutions to the original inverse problem:

**Theorem 3** ([12]). *Conjecture 1 is true.*

Next section describes the steps leading to Theorem 2.

## 2. STATEMENTS OF THE MAIN RESULTS

**2.1. Some notations and definitions.** The Lebesgue measure on  $\mathbb{R}^d$  is denoted  $\mathcal{L}^d$ .

If  $E$  is a Borel subset of  $\mathbb{R}^d$ , the Borel  $\sigma$ -algebra of  $E$  is denoted  $\mathcal{B}(E)$ .  $|E|$  stands for the Euclidean diameter of  $E$ .

Given  $x \in \mathbb{R}^d$  and  $r \in \mathbb{R}_+$ , the closed Euclidean ball centered at  $x$  with radius  $r$  is denoted  $B(x, r)$ .

For  $j \in \mathbb{Z}$ ,  $\mathcal{D}_j$  stands for the collection of closed dyadic cubes of generation  $j$ , i.e. the cubes  $\lambda_{j,k} = 2^{-j}k + 2^{-j}[0, 1]^d$ , where  $k \in \mathbb{Z}^d$ . We also set  $\mathcal{D} = \bigcup_{j \in \mathbb{Z}} \mathcal{D}_j$ , and if  $\lambda = \lambda_{j,k} \in \mathcal{D}_j$  we denote  $x_\lambda = 2^{-j}k$ .

For  $x \in \mathbb{R}^d$ ,  $\lambda_j(x)$  stands for the closure of the unique dyadic cube of generation  $j$ , product of semi-open to the right dyadic intervals, which contains  $x$ .

For  $j \in \mathbb{Z}$ ,  $\lambda \in \mathcal{D}_j$ , and  $N \in \mathbb{N}^*$ ,  $N\lambda$  denotes the cube with same center as  $\lambda$  and radius equal to  $N \cdot 2^{-j-1}$  in  $(\mathbb{R}^d, \|\cdot\|_\infty)$ . For instance,  $3\lambda$  is the union of those  $\lambda' \in \mathcal{D}_j$  such that  $\partial\lambda \cap \partial\lambda' \neq \emptyset$  ( $\partial\lambda$  stands for the frontier of the cube  $\lambda$ ).

The domain of a function  $g : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$  is defined as  $g^{-1}(\mathbb{R})$ , and denoted by  $\text{dom}(g)$ . When  $g$  is concave and finite, one sets  $g'(+\infty) = \lim_{t \rightarrow +\infty} g'(t^+)$  and  $g'(-\infty) = \lim_{t \rightarrow -\infty} g'(t^+)$ .

**Definition 2.1.** *The set of Hölder set functions on  $\mathcal{B}(\mathbb{R}^d)$  is defined as*

$$(2.1) \quad \mathcal{H}(\mathbb{R}^d) = \left\{ \mu : \mathcal{B}(\mathbb{R}^d) \rightarrow \mathbb{R}_+ \cup \{\infty\} : \exists C, s > 0, \forall E \subset \mathbb{R}^d, \mu(E) \leq C|E|^s \right\}.$$

*Then, the set of Hölder capacities is defined as*

$$(2.2) \quad \mathcal{C}(\mathbb{R}^d) = \left\{ \mu \in \mathcal{H}(\mathbb{R}^d) : \forall E, F \in \mathcal{B}(\mathbb{R}^d), E \subset F \Rightarrow \mu(E) \leq \mu(F) \right\}.$$

*and the set of Hölder Radon measures is defined as*

$$(2.3) \quad \mathcal{M}(\mathbb{R}^d) = \left\{ \mu \in \mathcal{C}(\mathbb{R}^d) : \mu \text{ is a Radon measure} \right\}.$$

*The topological support  $\text{supp}(\mu)$  of  $\mu \in \mathcal{H}(\mathbb{R}^d)$  is the set of points  $x \in \mathbb{R}^d$  for which  $\mu(B(x, r)) > 0$  for every  $r > 0$ . A capacity  $\mu$  is fully supported when  $\text{supp}(\mu) = \mathbb{R}^d$ .*

*Similarly, one defines the sets  $\mathcal{H}([0, 1]^d)$ ,  $\mathcal{C}([0, 1]^d)$  and  $\mathcal{M}([0, 1]^d)$  by replacing  $\mathbb{R}^d$  by  $[0, 1]^d$  in the above definitions.*

**Definition 2.2.** *For  $s > 0$ , a set function  $\mu \in \mathcal{H}(\mathbb{R}^d)$  is  $s$ -Hölder when there exists  $C > 0$  such that  $\mu(E) \leq C|E|^s$  for all  $E \in \mathcal{B}(\mathbb{R}^d)$ .*

*Then, for  $\mu \in \mathcal{H}(\mathbb{R}^d)$ ,  $s > 0$ , and  $E \in \mathbb{R}^d$ , define*

$$\mu^s(E) = \mu(E)^s \quad \text{and} \quad \mu^{(+s)}(E) = \mu(E)|E|^s,$$

*and if  $\mu$  is  $s_0$ -Hölder, then for all  $s \in (0, s_0)$ , define*

$$\mu^{(-s)}(E) = \begin{cases} 0 & \text{if } |E| = 0, \\ \mu(E)|E|^{-s} & \text{if } 0 < |E| < +\infty, \\ \infty & \text{otherwise.} \end{cases}$$

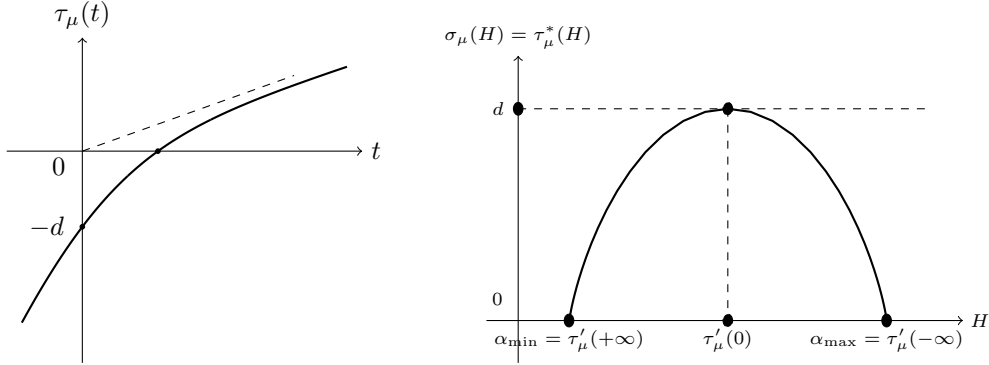


FIGURE 3. **Left:** Free energy function of  $\mu \in \mathcal{C}([0, 1]^d)$  satisfying the MF. **Right:** The singularity spectrum of  $\mu$ .

Starting from  $\mu \in \mathcal{H}(\mathbb{R}^d)$ ,  $\mu^s$  and  $\mu^{(+s)}$  as defined above still belong to  $\mathcal{H}(\mathbb{R}^d)$ . If in addition  $\mu$  is  $(s + \varepsilon)$ -Hölder, then  $\mu^{(-s)}$  also belongs to  $\mathcal{H}(\mathbb{R}^d)$ .

**2.2. Almost-doubling measures and capacities with prescribed multifractal behavior.** Multifractal formalisms for measures find their origin in works by physicists who proposed to characterize “strange sets” by considering, for any invariant probability measure  $\mu$  on such a set  $S$ , the partition of  $S$  into iso-Hölder sets of  $\mu$ . They further estimated the “fractal” dimensions of these sets using the Legendre transform of some free energy function, the  $L^q$ -spectrum, closely related to the Renyi generalized dimensions [28, 27]. Their ideas were later rigorously formalized by mathematicians (see, e.g. [15, 43, 51]).

The local behavior of elements of  $\mathcal{H}([0, 1]^d)$  is described via their pointwise Hölder exponents, also called local dimensions in the case of measures.

**Definition 2.3.** Let  $\mu \in \mathcal{H}([0, 1]^d)$ . For  $x \in \text{supp}(\mu)$ , the lower and upper pointwise Hölder exponents of  $\mu$  at  $x$  are respectively defined by

$$\underline{h}_\mu(x) = \liminf_{j \rightarrow +\infty} \frac{\log_2 \mu(\lambda_j(x))}{-j} \quad \text{and} \quad \bar{h}_\mu(x) = \limsup_{j \rightarrow \infty} \frac{\log_2 \mu(\lambda_j(x))}{-j}.$$

Whenever  $\underline{h}_\mu(x) = \bar{h}_\mu(x)$ , the common limit is called  $h_\mu(x)$ . Then, for  $\alpha \in \mathbb{R}$ ,

$$\underline{E}_\mu(\alpha) = \{x \in \text{supp}(\mu) : \underline{h}_\mu(x) = \alpha\}, \quad \bar{E}_\mu(\alpha) = \{x \in \text{supp}(\mu) : \bar{h}_\mu(x) = \alpha\},$$

and  $E_\mu(\alpha) = \underline{E}_\mu(\alpha) \cap \bar{E}_\mu(\alpha)$ .

The singularity (or multifractal) spectrum of  $\mu$  is then the mapping

$$\sigma_\mu : \alpha \in \mathbb{R} \mapsto \dim \underline{E}_\mu(\alpha).$$

**Definition 2.4.** The  $L^q$ -spectrum of  $\mu \in \mathcal{H}([0, 1]^d)$  with  $\text{supp}(\mu) \neq \emptyset$  is defined by

$$\tau_\mu : q \in \mathbb{R} \mapsto \liminf_{j \rightarrow +\infty} -\frac{1}{j} \log_2 \sum_{\substack{\lambda \in \mathcal{D}_j, \lambda \subset [0, 1]^d, \\ \mu(\lambda) > 0}} \mu(\lambda)^q.$$

Then, one always has (see [15, 45])

$$(2.4) \quad \sigma_\mu(\alpha) \leq \tau_\mu^*(\alpha) := \inf_{q \in \mathbb{R}} q\alpha - \tau_\mu(q).$$

**Definition 2.5.** A set function  $\mu \in \mathcal{H}([0, 1]^d)$  with  $\text{supp}(\mu) \neq \emptyset$  is said to obey the multifractal formalism (MF) over an interval  $I \subset \mathbb{R}$  when for all  $\alpha \in I$ ,

$$(2.5) \quad \sigma_\mu(\alpha) = \tau_\mu^*(\alpha).$$

It is said to obey the strong multifractal formalism (SMF) over  $I$  if for all  $\alpha \in I$ , in addition to (2.5) one has  $\dim E_\mu(\alpha) = \tau_\mu^*(\alpha)$ .

When  $I = \mathbb{R}$ , one simply says that the MF or the SMF holds for  $\mu$ .

**Remark 2.6.** Note that the Hölder exponents are sometimes defined as

$$\begin{aligned} \underline{h}_\mu(x) &= \liminf_{r \rightarrow 0^+} \frac{\log \mu(B(x, r))}{\log(r)} \quad \text{and} \quad \bar{h}_\mu(x) = \limsup_{r \rightarrow 0^+} \frac{\log \mu(B(x, r))}{\log(r)}, \\ \text{or} \quad \underline{h}_\mu(x) &= \liminf_{j \rightarrow \infty} \frac{\log_2 \mu(3\lambda_j(x))}{-j} \quad \text{and} \quad \bar{h}_\mu(x) = \limsup_{j \rightarrow \infty} \frac{\log_2 \mu(3\lambda_j(x))}{-j}. \end{aligned}$$

In this case,  $\mu(\lambda)$  is replaced by  $\mu(3\lambda)$  in the definition of the  $L^q$ -spectrum. However, in this paper we mainly consider doubling or almost doubling capacities (see Definition 2.8) for which the previous notions of exponents, level sets, singularity spectrum and  $L^q$ -spectrum do not depend on whether dyadic cubes or centered balls are considered.

When  $\mu \in \mathcal{M}([0, 1]^d)$  is positive, it is known [43, 3] that  $\tau'_\mu(-\infty)$  exists and is finite if and only if  $\tau_\mu$  is finite in a neighborhood of  $0^-$ , and in this case  $\tau_\mu : \mathbb{R} \rightarrow \mathbb{R}$  is a non-decreasing, concave map with  $\tau_\mu(1) = 0$ ; moreover, in this case

$$\text{dom}(\tau_\mu^*) = [\tau'_\mu(+\infty), \tau'_\mu(-\infty)] = \{\alpha \in \mathbb{R} : \tau_\mu^*(\alpha) \geq 0\} \subset [0, \infty).$$

One also has obviously that  $\tau_\mu(1) = 0$ . Due to (2.4), this yields  $\sigma_\mu \leq \text{Id}_\mathbb{R}$ . Moreover,  $\tau_\mu^*(\alpha) = \alpha$  if and only if  $\alpha \in [\tau'_\mu(1^+), \tau'_\mu(1^-)]$ . Thus, when  $\mu$  obeys the MF, there must exist  $D$  such that  $\sigma_\mu(D) = D$  [50]. Also, if  $\mu$  has full support in  $[0, 1]^d$ , then  $\tau_\mu(0) = -d$ , and  $\tau_\mu^*$  reaches its maximum, equal to  $d$ , exactly over the interval  $[\tau'_\mu(0)^-, \tau'_\mu(0)^+]$ .

**Definition 2.7.** Let  $\mathcal{T}_{d, \mathcal{M}}$  be the set of concave increasing functions  $\tau : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\tau(1) = 0$ ,  $\tau(0) = -d$  and  $\text{dom}(\tau^*)$  is a compact subset of  $(0, +\infty)$ .

Let  $\mathcal{S}_{d, \mathcal{M}}$  be the set of functions  $\sigma : \mathbb{R} \rightarrow [0, d] \cup \{-\infty\}$  such that  $\sigma$  is compactly supported with support included in  $(0, +\infty)$ , concave, continuous,  $\sigma \leq \text{Id}_\mathbb{R}$  and there exist two exponents  $D, D' > 0$  such that  $\sigma(D) = D$  and  $\sigma(D') = d$ .

One easily checks that these two sets  $\mathcal{S}_{d, \mathcal{M}}$  and  $\mathcal{T}_{d, \mathcal{M}}$  are Legendre transforms of each other. The set  $\mathcal{S}_{d, \mathcal{M}}$  is the class of admissible singularity spectra for measures strongly obeying the MF with a singularity spectrum compactly supported on  $(0, \infty)$ , and  $\mathcal{T}_{d, \mathcal{M}}$  is the class of corresponding admissible  $L^q$ -spectra.

Note that if  $\mu \in \mathcal{H}([0, 1]^d)$  has a non-empty support and  $s > 0$  one has  $\sigma_{\mu^s} = \sigma_\mu(\cdot/s)$ , and (see the proof of Theorem 2), the set  $\mathcal{S}_d$  defined in Conjecture 1 is related to  $\mathcal{S}_{d, \mathcal{M}}$  by the formula

$$(2.6) \quad \mathcal{S}_d = \{\sigma(s \cdot) : \sigma \in \mathcal{S}_{d, \mathcal{M}}, s > 0\}.$$

Consequently, in view of the comments made before the statement of Theorem 2, to establish this theorem we are going to construct, for every  $\sigma \in \mathcal{S}_{d, \mathcal{M}}$ , a fully supported  $\mu \in \mathcal{M}([0, 1]^d)$  obeying the SMF and satisfying  $\sigma_\mu = \sigma$ . The measure we obtain will possess additional properties introduced now.



**Definition 2.8.** Let  $\Phi$  be the set of non decreasing functions  $\phi : \mathbb{N} \rightarrow \mathbb{R}_+$  such that  $\lim_{j \rightarrow +\infty} \frac{\phi(j)}{j} = 0$ . A capacity  $\mu \in \mathcal{C}(\mathbb{R}^d)$  is almost doubling when there is  $\phi \in \Phi$  such that

$$(2.7) \quad \text{for all } x \in \text{supp}(\mu) \text{ and } j \in \mathbb{N}, \quad \mu(3\lambda_j(x)) \leq e^{\phi(j)} \mu(\lambda_j(x)).$$

Equivalently, there is a mapping  $\phi : (0, 1] \rightarrow \mathbb{R}_+$  such that  $\lim_{r \rightarrow 0^+} \frac{\phi(r)}{\log(r)} = 0$  and for all  $x \in \text{supp}(\mu)$  and  $r \in (0, 1]$  one has

$$\mu(B(x, 2r)) \leq e^{\phi(r)} \mu(B(x, r)).$$

When  $\phi$  is constant, the capacity  $\mu$  is doubling in the usual sense.

**Definition 2.9.** A set function  $\mu \in \mathcal{H}(\mathbb{R}^d)$  satisfies property (P) if there exist  $C, s_1, s_2 > 0$  and  $\phi \in \Phi$  such that:

(P<sub>1</sub>) for all  $j \in \mathbb{N}$  and  $\lambda \in \mathcal{D}_j$ ,

$$(2.8) \quad C^{-1}2^{-js_2} \leq \mu(\lambda) \leq C2^{-js_1}.$$

(P<sub>2</sub>) for all  $j, j' \in \mathbb{N}$  with  $j' \geq j$ , for all  $\lambda, \tilde{\lambda} \in \mathcal{D}_j$  such that  $\partial\lambda \cap \partial\tilde{\lambda} \neq \emptyset$ , and  $\lambda' \in \mathcal{D}_{j'}$  such that  $\lambda' \subset \lambda$ :

$$(2.9) \quad C^{-1}2^{-\phi(j)}2^{(j'-j)s_1} \mu(\lambda') \leq \mu(\tilde{\lambda}) \leq C2^{\phi(j)}2^{(j'-j)s_2} \mu(\lambda').$$

For  $\mu \in \mathcal{H}(\mathbb{R}^d)$ , (P<sub>1</sub>) is a uniform Hölder control, from above and below, of  $\mu$ , and (P<sub>2</sub>) is a rescaled version of (P<sub>1</sub>), which implies the almost doubling property. Our result on prescription of multifractal behavior for measures is the following.

**Theorem 4.** There exists a family of measures  $\mathcal{M}_d$  in  $\mathcal{M}(\mathbb{R}^d)$  such that :

- (1) Every  $\mu \in \mathcal{M}_d$  is  $\mathbb{Z}^d$ -invariant, fully supported on  $\mathbb{R}^d$ , satisfies property (P), and  $\mu|_{[0,1]^d}$  obeys the SMF.
- (2)  $\mathcal{S}_{d,\mathcal{M}} = \{\sigma_{\mu|_{[0,1]^d}} : \mu \in \mathcal{M}_d\}$ .

The family  $\mathcal{M}_d \subset \mathcal{M}(\mathbb{R}^d)$  is built in Section 3, by constructing, for every  $\sigma \in \mathcal{S}_{d,\mathcal{M}}$ , one fully supported Borel probability measure  $\mu$  on  $[0, 1]^d$ , which obeys the SMF, and such that  $\sigma_\mu = \sigma$ . Then  $\mathcal{M}_d$  is obtained by periodisation of such measures  $\mu$ .

To prove Theorem 2, and the Frisch-Parisi conjecture in [12], the following class of capacities is needed.

**Definition 2.10.** The set  $\mathcal{E}_d \subset \mathcal{C}(\mathbb{R}^d)$  is defined as the set of positive powers of measures  $\mu \in \mathcal{M}_d$ , i.e.

$$(2.10) \quad \mathcal{E}_d = \{\mu^s : \mu \in \mathcal{M}_d, s > 0\}.$$

An element of  $\mathcal{E}_d$  is called a multifractal environment.

Theorem 4 and (2.6) have the following immediate consequence.

**Corollary 5.**

- (1) Every  $\mu \in \mathcal{E}_d$  is  $\mathbb{Z}^d$ -invariant, fully supported on  $\mathbb{R}^d$ , satisfies property (P), and  $\mu|_{[0,1]^d}$  obeys the SMF.
- (2)  $\mathcal{S}_d = \{\sigma_{\mu|_{[0,1]^d}} : \mu \in \mathcal{E}_d\}$ .

**Remark 2.11.** *As important examples of fully supported  $\mathbb{Z}^d$ -invariant doubling multifractal capacities satisfying property (P) and whose restriction to  $[0, 1]^d$  obey the multifractal formalism, let us mention the Gibbs capacities obtained as  $\nu^s$ , where  $s > 0$  and  $\nu$  is a Gibbs measure associated with a  $\mathbb{Z}^d$ -invariant real valued Hölder potential on  $\mathbb{R}^d$  (see [12, Remark 1.15] for more details). The  $L^q$ -spectrum of  $\sigma_{\nu^s}|_{[0,1]^d}$  is then analytic.*

*Let us mention an interesting open question: By Theorem 4, each  $\sigma \in \mathcal{S}_d$  is the multifractal spectrum of a capacity satisfying property (P). Which mappings  $\sigma \in \mathcal{S}_d$  can be obtained as singularity spectrum  $\sigma_\mu$  associated with a doubling capacity  $\mu$ ?*

**2.3. A function space in which typical functions have a prescribed multifractal spectrum.** Let  $\{\phi, \{\psi^{(i)}\}_{i=1, \dots, 2^d-1}\}$  be a family of wavelets defining a multi-resolution analysis with reconstruction in  $L^2(\mathbb{R}^d)$  (see [49, Ch. 2 and 3] for a general construction).

Let  $\Lambda = \bigcup_{j \in \mathbb{Z}} \Lambda_j$ , where for a given  $j \in \mathbb{Z}$

$$\Lambda_j = \{(i, j, k) : i \in \{1, \dots, 2^d - 1\}, k \in \mathbb{Z}^d\}.$$

For every  $\lambda = (i, j, k) \in \Lambda$ , denote by  $\psi_\lambda$  the function  $x \mapsto \psi^{(i)}(2^j x - k)$ .

By the standard results of [49], the two family of functions  $\{2^{dj/2} \psi_\lambda\}_{j \in \mathbb{Z}, \lambda \in \Lambda_j}$  and  $\{\phi(\cdot - k)\}_{k \in \mathbb{Z}} \cup \{2^{dj/2} \psi_\lambda\}_{j \in \mathbb{N}, \lambda \in \Lambda_j}$  form two orthonormal bases of  $L^2(\mathbb{R}^d)$ , and every  $f \in L^2(\mathbb{R}^d)$  can be expanded as

$$f = \sum_{k \in \mathbb{Z}^d} \beta(k) \phi(\cdot - k) + \sum_{j \in \mathbb{N}} \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda = \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda,$$

where

$$(2.11) \quad \beta(k) = \int_{\mathbb{R}^d} f(x) \phi(x - k) dx \quad \text{and} \quad c_\lambda = \int_{\mathbb{R}^d} 2^{dj} \psi_\lambda(x) f(x) dx \quad (k \in \mathbb{Z}^d, \lambda \in \Lambda).$$

Pay attention to the  $L^\infty$  normalisation used to define the wavelet coefficients  $(c_\lambda)_{\lambda \in \Lambda}$ . The two decompositions of  $f$  will be used, but the first one is the most important one in the following.

It is known since the seminal works by Jaffard (see [37] for a survey) that wavelet coefficients are useful to analyze the pointwise regularity and to understand the multifractal behavior of functions. They are also key in this paper, as well as in [12].

**Definition 2.12.** *For every  $r \in \mathbb{N}$ , call  $\mathcal{F}_r$  the set of those  $\{\phi, \{\psi^{(i)}\}_{i=1, \dots, 2^d-1}\}$  which define a multi-resolution analysis with reconstruction in  $L^2(\mathbb{R}^d)$ , and such that  $\phi$  and the  $\psi^{(i)}$  are compactly supported,  $r$  times continuously differentiable functions, and every  $\psi^{(i)}$  has  $r$  vanishing moments, that is for every multi-index  $\alpha \in \mathbb{N}^d$  of length smaller than or equal to  $r$ ,  $\int_{\mathbb{R}^d} x_1^{\alpha_1} \cdots x_d^{\alpha_d} \psi^{(i)}(x) dx = 0$ .*

It is standard that  $\mathcal{F}_r \neq \emptyset$  for all  $r \in \mathbb{N}$  (see [49, Prop. 4, section 3.7] for instance).

Fix  $r \in \mathbb{N}^*$  and  $\Psi \in \mathcal{F}_r$ , and for any  $f \in L^\infty(\mathbb{R}^d)$ , define the sequences  $(\beta(k))_{k \in \mathbb{Z}^d}$  and  $(c_\lambda)_{\lambda \in \Lambda}$  as in (2.11). Recall that the Hölder-Zygmund spaces  $\{\mathcal{C}^s(\mathbb{R}^d)\}_{s>0}$  are subspaces of  $L^\infty(\mathbb{R}^d)$ , which have the following wavelet characterisation (see [49, Ch. 6] or [61]): If  $0 < s < r$ , then

$$(2.12) \quad f \in \mathcal{C}^s(\mathbb{R}^d) \iff \begin{cases} \beta \in \ell^\infty(\mathbb{Z}^d), \\ (\varepsilon_j)_{j \in \mathbb{N}} \in \ell^\infty(\mathbb{N}), \text{ where } \varepsilon_j = \left\| (2^{js} c_\lambda)_{\lambda \in \Lambda_j} \right\|_{\ell^\infty(\Lambda_j)} \end{cases}$$

and in this case the decomposition  $f = \sum_{k \in \mathbb{Z}^d} \beta(k) \phi(\cdot - k) + \sum_{j \in \mathbb{N}} \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda$  holds.

Let us now introduce two functions spaces  $B_{\Psi}^{\mu}(\mathbb{R}^d)$  and  $\mathbf{B}_{\Psi}^{\mu}(\mathbb{R}^d)$  associated with any  $\mu \in \mathcal{H}(\mathbb{R}^d)$  satisfying property (P<sub>1</sub>) of Definition 2.9 and any enough regular  $\Psi$ . Recall Definition 2.2, and observe that as soon as  $\mu \in \mathcal{H}(\mathbb{R}^d)$  satisfies property (P<sub>1</sub>), the set functions  $\mu^{(+s)}$ , and  $\mu^{(-s)}$  of Definition 2.2 satisfy (P<sub>1</sub>) as well (when  $s$  is small enough in the case of  $\mu^{(-s)}$ ), and the same is true for (P).

**Definition 2.13.** Write  $\mu(\lambda) = \mu(\lambda_{j,k})$  for every  $\lambda = (i, j, k)$ .

Let  $\mu \in \mathcal{H}(\mathbb{R}^d)$  satisfy property (P<sub>1</sub>) of Definition 2.9 with exponents  $0 < s_1 \leq s_2$ , and consider an integer  $n \geq \lfloor s_2 \rfloor + 1$ .

Fix a wavelet  $\Psi \in \mathcal{F}_n$ , and consider for a function  $f \in L^p(\mathbb{R}^d)$  the quantity

$$(2.13) \quad |f|_{\mu, \Psi} = \|(\varepsilon_j^{\mu})_{j \in \mathbb{N}}\|_{\ell^{\infty}(\mathbb{N})}, \text{ where } \varepsilon_j^{\mu} = \left\| \left( \frac{c_{\lambda}}{\mu(\lambda)} \right)_{\lambda \in \Lambda_j} \right\|_{\ell^{\infty}(\Lambda_j)}.$$

Then, define

$$(2.14) \quad \begin{aligned} B_{\Psi}^{\mu}(\mathbb{R}^d) &= \{f \in L^{\infty}(\mathbb{R}^d) : |f|_{\mu, \Psi} < +\infty\} \\ \text{and } \mathbf{B}_{\Psi}^{\mu}(\mathbb{R}^d) &= \bigcap_{0 < \varepsilon < \min(s_1, 1)} B_{\Psi}^{\mu^{(-\varepsilon)}}(\mathbb{R}^d). \end{aligned}$$

If  $B_{\Psi}^{\mu}(\mathbb{R}^d)$  (resp.  $B_{\Psi}^{\mu}(\mathbb{R}^d)$ ) does not depend on  $\Psi$ , we simply denote it by  $\mathbf{B}^{\mu}(\mathbb{R}^d)$  (resp.  $B^{\mu}(\mathbb{R}^d)$ ).

The space  $(B_{\Psi}^{\mu}(\mathbb{R}^d), \| \cdot \|_{L^{\infty}(\mathbb{R}^d)} + | \cdot |_{\mu, \Psi})$  is a Banach space, and since  $B_{\Psi}^{\mu^{(-\varepsilon)}}(\mathbb{R}^d) \hookrightarrow B_{\Psi}^{\mu^{(-\varepsilon')}}(\mathbb{R}^d)$  for all  $0 < \varepsilon < \varepsilon' < \min(1, s_1)$ ,  $\mathbf{B}_{\Psi}^{\mu}(\mathbb{R}^d)$  can be endowed with a Fréchet space structure (Section 5.2 exhibits a basis of neighborhoods of the origin).

Note that  $B_{\Psi}^{\mu}(\mathbb{R}^d) \hookrightarrow \mathcal{C}^{s_1}(\mathbb{R}^d)$ , and if  $\mu = (\mathcal{L}^d)^{s/d}$ , then  $B_{\Psi}^{\mu}(\mathbb{R}^d) = \mathcal{C}^s(\mathbb{R}^d)$ , so we recover the independence with respect to the choice of  $\Psi$  stated in (2.12). In general, we can get such a property for the larger space  $\mathbf{B}_{\Psi}^{\mu}(\mathbb{R}^d)$  only. Indeed, the following result is proved in [12], where we investigate a generalisation of the classical Besov spaces, of which Hölder-Zygmund spaces are special cases.

**Theorem 6** ([12]). Assume that  $\mu$  is an almost-doubling capacity satisfying property (P). The Fréchet space  $\mathbf{B}_{\Psi}^{\mu}(\mathbb{R}^d)$  is independent of the choice of the wavelet  $\Psi \in \mathcal{F}_n$  used to define the spaces  $B_{\Psi}^{\mu^{(-\varepsilon)}}(\mathbb{R}^d)$ . The same holds for  $B_{\Psi}^{\mu}(\mathbb{R}^d)$  if  $\mu$  is doubling and property (P) holds for  $\mu$  with  $\phi$  equal to 0.

The multifractal properties of functions in  $\mathbf{B}^{\mu}(\mathbb{R}^d)$  are now stated when  $\mu \in \mathcal{E}_d$  (recall that in this case Property (P) holds).

**Theorem 7.** Let  $\mu \in \mathcal{E}_d$  be a multifractal environment.

(1) For all  $f \in \mathbf{B}^{\mu}(\mathbb{R}^d)$ ,

$$(2.15) \quad \sigma_f(H) \leq \begin{cases} \sigma_{\mu}(H) = \tau_{\mu}^*(H) & \text{if } H \leq \tau'_{\mu}(0^-) \\ d & \text{if } H > \tau'_{\mu}(0^-). \end{cases}$$

(2) For typical functions  $f \in \mathbf{B}^{\mu}(\mathbb{R}^d)$ , one has  $\sigma_f = \sigma_{\mu}$ .

Note that the same holds in  $B_{\Psi}^{\mu}(\mathbb{R}^d)$ . Combining the previous results, we can prove Theorem 2.

*Proof of Theorem 2.* Let  $\sigma \in \mathcal{S}_d$ . Let  $\sigma_{\mathcal{M}} = \sigma(\cdot/s)$ , where  $s$  is the unique positive real number such that  $\sigma(\cdot/s) \leq \text{Id}_{\mathbb{R}}$  and there exists at least one  $H$  such that  $\sigma(H/s) = H$ . In other words,  $s$  is the unique real number such that  $\sigma^*(s) = 0$ . In particular,  $\sigma_{\mathcal{M}} \in \mathcal{S}_{d,\mathcal{M}}$ . By Theorem 4, there exists  $\mu \in \mathcal{M}_d$  such that  $\sigma_{\mu} = \sigma_{\mathcal{M}}$ . Moreover, both  $\mu$  and  $\mu^s$  obey the SMF.

Now, we apply Theorems 7 with the capacity  $\mu^s \in \mathcal{E}_d$ : in the Baire space  $\mathbf{B}^{\mu^s}(\mathbb{R}^d)$ , typical functions have  $\sigma$  as singularity spectrum.  $\square$

Section 3 is dedicated to the construction of the class of measures  $\mathcal{M}_d$  (Definitions 3.8 and 3.14) with prescribed multifractal behavior as described in Theorem 4.

In section 4, the upper bound for the singularity spectrum of all functions in  $\mathbf{B}^{\mu}(\mathbb{R}^d)$  is obtained (part (1) of Theorem 7). Part (2) of Theorem 7 is shown in Section 5. It consists first in exhibiting a specific function  $g_{\mu}$  whose singularity spectrum equals  $\sigma_{\mu}$ , and then, by perturbation of the wavelet coefficients of  $g_{\mu}$ , constructing a dense  $G_{\delta}$ -set in  $\mathbf{B}^{\mu}(\mathbb{R}^d)$  in which all functions share the same multifractal spectrum as  $g_{\mu}$ .

Finally, in Section 6, some properties relating capacities (and associated auxiliary measures) belonging to  $\mathcal{E}_d$  with the dyadic approximation in  $\mathbb{R}^d$  are proved. These results are not necessary in this paper to obtain our main results, but play a major role in the solution to the Frisch-Parisi conjecture obtained in [12].

### 3. MEASURES WITH PRESCRIBED MULTIFRACTAL BEHAVIOR

In Section 3.1, additional general properties associated with multifractal formalism for capacities are recalled. Section 2.2 is a preparation to the construction of the measures satisfying the requirements of Theorem 4. The construction is achieved when  $d = 1$  in Section 3.3. Then, in Sections 3.4 to 3.6 the conclusions of Theorem 4 are obtained. The construction is extended to the case  $d \geq 2$  in Section 3.7 and we get the desired family of measures  $\mathcal{M}_d$ . Finally, in Sections 6.1 and 6.2 we investigate some connections between the elements of  $\mathcal{E}_d$  and metric number theory: a ubiquity theorem associated with  $\mu \in \mathcal{E}_d$  and the family of dyadic vectors is established, and it is proved that auxiliary measures associated with  $\mu \in \mathcal{E}_d$  are supported on the set of points which are badly approximated by dyadic vectors. These properties are necessary to achieve the multifractal analysis of typical elements of the Baire spaces considered in [12].

**3.1. Additional notions related to the multifractal formalism for capacities.** Let us introduce the following notations for  $\varepsilon > 0$ ,  $\alpha \in \mathbb{R}$ , and  $I = [a, b]$  an interval:

$$(3.1) \quad \alpha \pm \varepsilon = [\alpha - \varepsilon, \alpha + \varepsilon] \quad \text{and} \quad I \pm \varepsilon = [a - \varepsilon, b + \varepsilon].$$

Also, the convention  $\log(0) = -\infty$  is adopted.

Next propositions complete the properties mentioned in Section 2.2 about multifractal analysis of capacities (some of these properties will be used only in [12]). Recall that the Legendre spectrum  $\alpha \mapsto \tau_{\mu}^*(\alpha)$  is increasing on the interval  $\alpha \leq \tau'_{\mu}(0^-)$ , and is decreasing on  $\alpha \geq \tau'_{\mu}(0^+)$ . The following Propositions 3.1 and 3.3 are easily deduced from any of the following sources [3, 15, 51, 45, 4].

**Proposition 3.1.** *Let  $\mu \in \mathcal{C}([0, 1]^d)$  such that  $\text{supp}(\mu) \neq \emptyset$ . For  $\alpha \in \mathbb{R}$ , let*

$$\underline{E}_{\mu}^{\leq}(\alpha) = \{x \in \text{supp}(\mu) : \underline{h}_{\mu}(x) \leq \alpha\} \quad \text{and} \quad \overline{E}_{\mu}^{\geq}(\alpha) = \{x \in \text{supp}(\mu) : \overline{h}_{\mu}(x) \geq \alpha\}.$$

One has:

- (1) For every  $\alpha \leq \tau'_{\mu}(0^-)$ ,  $\dim \underline{E}_{\mu}^{\leq}(\alpha) \leq \tau_{\mu}^*(\alpha)$ .

(2) For every  $\alpha \geq \tau'_\mu(0^+)$ ,  $\dim \overline{E}_\mu^{\geq}(\alpha) \leq \tau_\mu^*(\alpha)$ .

The distribution of a capacity at small scales can be described through its large deviations spectrum.

**Definition 3.2.** Let  $\mu \in \mathcal{C}([0, 1]^d)$  with  $\text{supp}(\mu) \neq \emptyset$ . For  $I \subset \mathbb{R}$  and  $j \in \mathbb{N}^*$  define

$$\mathcal{D}_\mu(j, I) = \left\{ \lambda \in [0, 1]^d, \lambda \in \mathcal{D}_j : \frac{\log_2 \mu(\lambda)}{-j} \in I \right\}.$$

The lower and upper large deviations spectra of  $\mu$  are defined respectively as

$$\begin{aligned} \underline{\sigma}_\mu^{\text{LD}} : \alpha \in \mathbb{R} &\mapsto \lim_{\varepsilon \rightarrow 0} \liminf_{j \rightarrow \infty} \frac{\log_2 \#\mathcal{D}_\mu(j, \alpha \pm \varepsilon)}{j} \\ \text{and } \overline{\sigma}_\mu^{\text{LD}} : \alpha \in \mathbb{R} &\mapsto \lim_{\varepsilon \rightarrow 0} \limsup_{j \rightarrow +\infty} \frac{\log_2 \#\mathcal{D}_\mu(j, \alpha \pm \varepsilon)}{j}. \end{aligned}$$

**Proposition 3.3.** Let  $\mu \in \mathcal{C}([0, 1]^d)$  with  $\text{supp}(\mu) \neq \emptyset$ , such that  $\mu$  obeys the SMF (Definition 2.5). One has  $\text{dom}(\tau_\mu^*) = \{\alpha \in \mathbb{R} : \tau_\mu^*(\alpha) \geq 0\}$ , and:

(1) For every  $\alpha \in \mathbb{R}$ , one has

$$\sigma_\mu(\alpha) = \dim E_\mu(\alpha) = \dim \underline{E}_\mu(\alpha) = \dim \overline{E}_\mu(\alpha) = \underline{\sigma}_\mu^{\text{LD}}(\alpha) = \overline{\sigma}_\mu^{\text{LD}}(\alpha) = \tau_\mu^*(\alpha).$$

(2) For every  $\alpha \leq \tau'_\mu(0^-)$ ,  $\dim \underline{E}_\mu^{\leq}(\alpha) = \tau_\mu^*(\alpha)$ .

(3) For every  $\alpha \geq \tau'_\mu(0^+)$ ,  $\dim \overline{E}_\mu^{\geq}(\alpha) = \tau_\mu^*(\alpha)$ .

(4) For every  $\eta > 0$  and every interval  $I \subset \text{dom}(\tau_\mu^*)$ , there exists  $\varepsilon_0 > 0$  and  $J_0 \in \mathbb{N}$  such that for every  $\varepsilon \in (0, \varepsilon_0)$  and  $j \geq J_0$ , for  $\tilde{I} \in \{I, I \pm \varepsilon\}$ ,

$$\left| \frac{\log_2 \#\mathcal{D}_\mu(j, \tilde{I})}{j} - \sup_{\alpha \in I} \tau_\mu^*(\alpha) \right| \leq \eta.$$

(5) If  $\text{dom}(\tau_\mu^*)$  is compact, then  $\text{dom}(\tau_\mu^*) = [\tau'_\mu(+\infty), \tau'_\mu(-\infty)]$  and there exists a positive decreasing sequence  $(\varepsilon_j)_{j \geq 0}$  tending to 0 when  $j \rightarrow +\infty$ , such that for all  $j \in \mathbb{N}$  and  $\lambda \in \mathcal{D}_j$ ,

$$\tau'_\mu(+\infty) - \varepsilon_j \leq \frac{\log_2 \mu(\lambda)}{-j} \leq \tau'_\mu(-\infty) + \varepsilon_j.$$

We now prove Theorem 4 in the case  $d = 1$ .

**3.2. A family of probability vectors associated with  $\sigma \in \mathcal{S}_{1, \mathcal{M}}$ .** Fix  $\sigma \in \mathcal{S}_{1, \mathcal{M}}$  (recall Definition 2.7). In this section, a sequence of probability vectors  $(p_N)_{N \geq 1}$  (where  $p_N \in [0, 1]^{2^N}$ ), which constitutes the core of the construction of a measure  $\mu$  satisfying both (P) and the MF with  $\tau_\mu^* = \sigma$ , is defined. For this, write  $\text{dom}(\sigma) = [\alpha_{\min}, \alpha_{\max}]$ . When  $\alpha_{\min} = 1 = \alpha_{\max}$ , the Lebesgue measure on  $[0, 1]$  yields a solution to the inverse problem studied in this Section 3. So it is assumed from now on that  $\alpha_{\min} < \alpha_{\max}$ .

Let us start by introducing two parameters  $D$  and  $D'$  defined as follows:

- if  $\sigma(1) = 1$ , set  $D = D' = 1$ .
- if  $\sigma(1) \neq 1$ , let  $0 < D < 1 < D'$  be such that  $\sigma(D) = D$  and  $\sigma(D') = 1$ .

The existence of such exponents  $D$  and  $D'$  is justified by the discussion preceding Definition 2.7.

Then, fix an integer  $N_0$  large enough so that for all  $N \geq N_0$ , setting  $\varepsilon_N = 2 \log_2(N)/N$ , there exists a finite subset  $A_N = \{\alpha_{N,i} : i = 1, \dots, 2m_N\}$  of  $[\alpha_{\min}, \alpha_{\max}]$  satisfying:

- $\varepsilon_{N_0} \leq \alpha_{\min}/4$ ;
- $m_N \leq 2N(\alpha_{\max} - \alpha_{\min})$ ;
- $D, D' \in A_N$ ;
- for every  $i \in \{1, \dots, m_N - 1\}$ ,  $(4N)^{-1} < \alpha_{N,i+1} - \alpha_{N,i} < N^{-1}$ ;
- the following inclusions hold:

$$(3.2) \quad A_N \subset \sigma^{-1}\left(\left[\frac{1}{N} + \varepsilon_N, 1\right]\right) \subset \bigcup_{i=1}^{m_N} \left[\alpha_{N,i} - \frac{1}{N}, \alpha_{N,i} + \frac{1}{N}\right];$$

- for every  $i \in \{m_N + 1, \dots, 2m_N\}$ ,  $\alpha_{N,i} = \alpha_{N,2m_N-i+1}$ ;
- if  $\sigma(\alpha_{\min}) > 0$ , then  $\alpha_{N,1} = \alpha_{\min}$ .

The continuity of  $\sigma$  is used to get (3.2), and when  $D \neq D'$  the above conditions impose that  $|D - D'| \geq (4N)^{-1}$ .

Denote by  $i_N$  (resp.  $i'_N$ ) the index in  $[1, m_N]$  such that  $D = \alpha_{N,i_N}$  (resp.  $D' = \alpha_{N,i'_N}$ ). Note that  $i_N = i'_N$  if and only if  $D = D' = 1$ .

For each  $1 \leq i \leq m_N$  such that  $i \notin \{i_N, i'_N\}$ , set

$$(3.3) \quad R_{N,i} = \lfloor 2^{N(\sigma(\alpha_{N,i}) - \varepsilon_N) - 1} \rfloor,$$

which implies that for every  $i$ ,  $1 \leq R_{N,i} \leq 2^{N-1}N^{-2}$ .

When  $D = D'$ , one has  $i_N = i'_N$  and one sets

$$R_{N,i_N} = 2^{N-1} - \sum_{i=1, i \neq i_N}^{m_N} R_{N,i}.$$

When  $D < D'$ , one has  $i_N < i'_N$  and in this case one sets

$$(3.4) \quad R_{N,i_N} = \lfloor 2^{N\sigma(\alpha_{N,i_N}) - 1} \rfloor = \lfloor 2^{ND-1} \rfloor \quad \text{and} \quad R_{N,i'_N} = 2^{N-1} - \sum_{i=1, i \neq i'_N}^{m_N} R_{N,i}.$$

In all cases, by construction

$$\sum_{i=1, i \neq i'_N}^{m_N} R_{N,i} \leq m_N 2^{N-1}N^{-2} + \mathbf{1}_{\{D \neq D'\}} 2^{ND-1} = o(2^{N-1}) \quad \text{as } N \rightarrow \infty,$$

since the term  $\mathbf{1}_{\{D \neq D'\}} 2^{ND-1}$  appears if and only if  $D < 1$ . This also implies that

$$(3.5) \quad R_{N,i'_N} = 2^{N-1}(1 + o(1)).$$

Without restriction, we choose  $N_0$  large enough so that

$$(3.6) \quad \text{for all } N \geq N_0, \quad \sum_{i=1, i \neq i'_N}^{m_N} R_{N,i} \leq 2^{N-2}.$$

Finally, for  $N \geq N_0$  and  $m_N < i \leq 2m_N$ , set  $R_{N,i} = R_{N,2m_N-i+1}$ , so that

$$\sum_{i=1}^{2m_N} R_{N,i} = 2^N.$$

**Definition 3.4.** *The collection of exponents  $(\beta_{N,i})_{0 \leq i \leq 2N-1}$  is defined as follows:*

$$(3.7) \quad \text{for all } 1 \leq j \leq 2m_N, \quad \beta_{N,i} = \alpha_{N,j} \quad \text{if } \sum_{k=1}^{j-1} R_{N,k} \leq i < \sum_{k=1}^j R_{N,k}.$$

In other words,  $(\beta_{N,i})_{0 \leq i < 2^N - 1}$  is obtained by repeating  $R_{N,1}$  times the value  $\alpha_{N,1}$ ,  $R_{N,2}$  times  $\alpha_{N,2}$ , and so on, until repeating  $R_{N,2^{m_N}}$  times  $\alpha_{N,2^{m_N}} = \alpha_{N,1}$ .

**Lemma 3.5.** *Let  $p_N = (p_{N,i})_{0 \leq i < 2^N - 1}$  be the probability vector defined by*

$$p_{N,i} = \frac{2^{-N\beta_{N,i}}}{\sum_{j=0}^{2^N-1} 2^{-N\beta_{N,j}}}.$$

One has  $p_{N,0} = p_{N,2^N-1}$ , and if  $|i - i'| \leq 1$ , then

$$(3.8) \quad \frac{p_{N,i}}{p_{N,i'}} \in [2^{-1}, 2].$$

In addition, for  $N$  large enough,

$$(3.9) \quad p_{N,i} 2^{N\beta_{N,i}} = 1 + \varepsilon_{N,i},$$

where  $\varepsilon_{N,i} = O(N^{-1})$  uniformly in  $0 \leq i < 2^N$ .

*Proof.* By definition,

$$p_{N,i} 2^{N\beta_{N,i}} = \frac{1}{2 \sum_{j=1}^{m_N} 2^{-N\alpha_{N,j}} R_{N,j}}.$$

In order to estimate  $p_{N,i} 2^{N\beta_{N,i}}$  uniformly in  $i$ , recall that  $\sigma \leq \text{Id}_{\mathbb{R}}$ , so that using the definition of  $R_{N,i}$  and  $\varepsilon_N$ , one gets

$$\begin{aligned} \lfloor 2^{ND-1} \rfloor 2^{-ND} &= R_{N,i_N} 2^{-N\alpha_{N,i_N}} \leq \sum_{i=1}^{m_N} 2^{-N\alpha_{N,i}} R_{N,i} \\ &\leq \sum_{1 \leq i \neq i_N, i' \neq i'_N \leq m_N} 2^{N(\sigma(\alpha_{N,i}) - \alpha_{N,i} - \varepsilon_N)} + R_{N,i_N} 2^{-ND} + \mathbf{1}_{D \neq D'} R_{N,i'_N} 2^{-ND'} \\ &\leq m_N N^{-2} + 2^{-ND} \lfloor 2^{ND-1} \rfloor + \mathbf{1}_{D \neq D'} 2^{N(1-D')}. \end{aligned}$$

Also, recall that when  $D \neq D'$ ,  $D < 1$  and  $D' > 1$ . Consequently, since  $\lfloor 2^{ND-1} \rfloor 2^{-ND} = 1/2 + o(1)$ , the previous sequence of inequalities gives (3.9).

The fact that (3.8) holds when  $0 \leq i, i' \leq 2^N - 1$  and  $|i - i'| \leq 1$  follows from the equality  $p_{N,i}/p_{N,i'} = 2^{-N(\beta_{N,i} - \beta_{N,i'})}$  and the fact that  $|\beta_{N,i'} - \beta_{N,i}| \leq N^{-1}$ .

Finally,  $p_{N,0} = p_{N,2^N-1}$  by definition of these parameters.  $\square$

**3.3. Construction of the measure  $\mu_\sigma$  associated with  $\sigma \in \mathcal{S}_{1,\mathcal{M}}$ .** A Moran measure  $\mu_\sigma$  is iteratively constructed as concatenation of pieces of Bernoulli product measures associated with the probability vectors  $(p_N)_{N \geq N_0}$ . The sequence  $(p_N)_{N \geq N_0}$  has been built so that when  $N \rightarrow +\infty$ , the singularity spectrum of the Bernoulli product measure associated with  $p_N$  pointwise converges to  $\sigma$ . Indeed, each  $p_N$  has been chosen so that, heuristically, there are  $2^{N\sigma(\alpha_{N,i})}$  weights of order  $2^{-N\alpha_{N,i}}$  and the  $\alpha_{N,i}$  tend to be more or less uniformly distributed in the domain of  $\sigma$ .

Further ingredients are introduced:

- For  $N \geq N_0$ , an integer  $\ell_N \geq N^2$  is fixed, such that  $(\ell_N)_{N \geq N_0}$  forms an increasing sequence;
- consider the product space

$$\Sigma = \prod_{N=N_0}^{\infty} \{0, \dots, 2^N - 1\}^{\ell_N};$$

- for  $N \geq N_0$ , if  $g = \ell + \sum_{n=N_0}^{N-1} \ell_n$  with  $1 \leq \ell \leq \ell_N$ , a word of generation (or length)  $g$

$$(J_{N_0}, J_{N_0+1}, \dots, J_{N-1}, J_N) \in \Sigma_g := \left( \prod_{n=N_0}^{N-1} \{0, \dots, 2^n - 1\}^{\ell_n} \right) \times \{0, \dots, 2^N - 1\}^\ell$$

is also denoted  $J_{N_0} \cdot J_{N_0+1} \cdots J_N$ ; then the cylinder consisting of those elements in  $\Sigma$  with common prefix  $J_{N_0} \cdot J_{N_0+1} \cdots J_N$  is denoted  $[J_{N_0} \cdot J_{N_0+1} \cdots J_N]$ , and the set of such cylinders of generation  $g$  is denoted  $\mathcal{C}_g$ ;

- the space  $\Sigma$  is endowed with the  $\sigma$ -field  $\mathcal{B}$  generated by the cylinders.

**Definition 3.6.** *The probability measure  $\nu_\sigma$  on  $(\Sigma, \mathcal{B})$  is defined as follows. For all  $N \geq N_0$ , for all  $1 \leq \ell \leq \ell_N$ , for  $g = \ell + \sum_{n=N_0}^{N-1} \ell_n$  and  $[J_{N_0} \cdot J_{N_0+1} \cdots J_N] \in \mathcal{C}_g$ , set*

$$(3.10) \quad \nu_\sigma([J_{N_0} \cdot J_{N_0+1} \cdots J_N]) = \left( \prod_{n=N_0}^{N-1} \prod_{k=1}^{\ell_n} p_{n,j_{n,k}} \right) \prod_{k=1}^{\ell} p_{N,k},$$

where :

- for every  $n \geq N_0$ , for every  $i \in \{1, \dots, \ell_n\}$ ,  $j_{n,i} \in \{0, \dots, 2^n - 1\}$ ,
- for  $N_0 \leq n \leq N-1$ ,  $J_n = j_{n,1} \cdots j_{n,\ell_n} \in \{0, \dots, 2^n - 1\}^{\ell_n}$ ,
- $J_N = j_{N,1} \cdots j_{N,\ell} \in \{0, \dots, 2^N - 1\}^\ell$ .

**Remark 3.7.** *Formula (3.10) could be written*

$$\nu_\sigma([J_{N_0} \cdot J_{N_0+1} \cdots J_N]) = \prod_{n=N_0}^N \mu_n(J_n),$$

where  $\mu_n$  is the Bernoulli measure associated with the parameters  $p_n = (p_{n,i})_{i=0, \dots, 2^n - 1}$ .

It is immediate to check that (3.10) is consistent, in the sense that  $\nu_\sigma(\Sigma) = 1$  and for every integers  $g' > g \geq 1$ , for every cylinder  $J \in \mathcal{C}_g$ ,  $\nu_\sigma(J) = \sum_{J' \in \mathcal{C}_{g'}, J' \subset J} \nu_\sigma(J')$ .

Using (3.9), one sees that there exists  $C > 0$  such that for each  $N \geq N_0$  and  $(J_n)_{N_0 \leq n \leq N} \in \prod_{n=N_0}^N \{0, \dots, 2^n - 1\}^{\ell_n}$ ,

$$\nu_\sigma([J_{N_0} \cdot J_{N_0+1} \cdots J_N]) \leq \prod_{n=N_0}^N \left( (1 + C/n) 2^{-n\alpha_{\min}} \right)^{\ell_n}.$$

Hence  $\nu_\sigma$  is atomless since the right-hand side tends to 0 as  $N$  tends to infinity.

Every  $g \in \mathbb{N}^*$  is decomposed in a unique way under the form  $g = \ell + \sum_{n=N_0}^{N-1} \ell_n$  with  $N \geq N_0$  and  $1 \leq \ell \leq \ell_N$  (when  $N = N_0$ , the sum  $\sum_{n=N_0}^{N-1} \ell_n$  is 0). Using this decomposition, one can define the mapping

$$(3.11) \quad \gamma : \begin{cases} \mathbb{N}^* \rightarrow \mathbb{N}^* \\ g \mapsto \gamma(g) := N\ell + \sum_{n=N_0}^{N-1} n\ell_n. \end{cases}$$

The space  $\Sigma$  provides a natural coding of  $[0, 1]$ . Indeed, considering the coding map

$$(3.12) \quad \pi : x = \left( (x_{N,k})_{k=1}^{\ell_N} \right)_{N \geq N_0} \in \Sigma \mapsto \sum_{N=N_0}^{\infty} 2^{-\sum_{n=N_0}^{N-1} n\ell_n} \sum_{k=1}^{\ell_N} x_{N,k} 2^{-kN} \in [0, 1],$$

for each  $g \in \mathbb{N}^*$ ,  $\pi$  maps bijectively the elements of  $\mathcal{C}_g$  onto the set of closed dyadic subintervals of generation  $\gamma(g)$  of  $[0, 1]$ .



**Definition 3.8.** For every  $\sigma \in \mathcal{S}_{1,\mathcal{M}}$ , consider the Borel probability measure on  $[0, 1]$

$$\tilde{\mu}_\sigma = \nu_\sigma \circ \pi^{-1},$$

where  $\nu_\sigma$  is the measure constructed above in (3.10). Then,  $\mu_\sigma$  is defined as the natural periodized version of  $\tilde{\mu}_\sigma$ , i.e. the  $\mathbb{Z}$ -invariant measure

$$\mu_\sigma : B \in \mathcal{B}(\mathbb{R}) \mapsto \sum_{k \in \mathbb{Z}} \mu((B \cap [k, k+1)) - k).$$

Finally, set

$$\mathcal{M}_1 = \{\mu_\sigma : \sigma \in \mathcal{S}_{1,\mathcal{M}}\} \subset \mathcal{M}(\mathbb{R}).$$

The measures  $\mu_\sigma$  and  $\tilde{\mu}_\sigma$  are said to be associated with  $\sigma \in \mathcal{S}_{1,\mathcal{M}}$ .

**Proposition 3.9.** Every  $\mu \in \mathcal{M}_1$  satisfies the property (P) of Definition 2.9.

Moreover, if  $\mu$  is associated with  $\sigma \in \mathcal{S}_{1,\mathcal{M}}$ , then  $\mu|_{[0,1]}$  has  $\sigma$  as multifractal spectrum, and it obeys the SMF on  $\mathbb{R}_+$ .

Observe that since  $\nu_\sigma$  is atomless and  $\pi$  is 1-to-1 outside a countable set of points, for any closed dyadic subinterval  $\lambda$  of  $[0, 1]$  of generation  $n \in \gamma(\mathbb{N}^*)$ , one has  $\mu_\sigma(\lambda) = \nu_\sigma([w])$ , where  $[w]$  is the unique cylinder of generation  $\gamma^{-1}(n)$  such that  $\pi([w]) = \lambda$ .

Next sections are devoted to the proofs of the various properties of  $\mu_\sigma$ , which, in particular, yield Proposition 3.9.

For the rest of this section,  $\sigma \in \mathcal{S}_{1,\mathcal{M}}$  is fixed, and we simply denote by  $\mu$  and  $\nu$  the measures  $\mu_\sigma$  and  $\nu_\sigma$  associated with  $\sigma$ .

### 3.4. The measure $\mu$ satisfies property (P).

**Lemma 3.10.** The measure  $\mu$  is almost doubling.

*Proof.* Let  $g \in \mathbb{N}^*$  and write it under the form  $g = \ell + \sum_{n=N_0}^{N-1} \ell_n \in \mathbb{N}$  with  $N \geq N_0$  and  $1 \leq \ell \leq \ell_N$ .

First, note that if  $g$ , hence  $N$ , is large enough, the term  $1 + \varepsilon_{N,i}$  in (3.9) is greater than  $1/2$  and smaller than  $3/2$ . Hence, for any  $1 \leq i \leq 2m_N$ ,

$$(3.13) \quad 2^{-N(\alpha_{\max} + \tilde{\varepsilon}_N)} \leq p_{N,i} \leq 2^{-N(\alpha_{\min} - \tilde{\varepsilon}_N)},$$

where  $(\tilde{\varepsilon}_N)_{N \geq 1}$  is a non-increasing sequence (independent of  $i$ ) converging to 0.

We start by dealing with the dyadic intervals of generation  $\gamma(g)$ .

Consider two closed dyadic subintervals  $\lambda$  and  $\hat{\lambda}$  of  $[0, 1]$  of generation  $\gamma(g)$  such that  $\lambda$  is the left neighbor of  $\hat{\lambda}$ . By construction,  $\lambda$  and  $\hat{\lambda}$  are the images by  $\pi$  of two cylinders  $[J]$  and  $[\tilde{J}]$  in  $\mathcal{C}_g$  such that, denoting by  $u$  the longest common prefix of the words  $J$  and  $\tilde{J}$ , there exist  $N_1 \geq N_0$  and  $0 \leq j < 2^{N_1} - 2$  such that  $J = u \cdot j \cdot v$  and  $\tilde{J} = u \cdot (j+1) \cdot \tilde{v}$ , where:

- (1) there is  $1 \leq \tilde{\ell}_1 \leq \ell_{N_1}$  such that  $u \in \prod_{N=N_0}^{N_1-1} \{0, \dots, 2^n - 1\}^{\ell_n} \cdot \{0, \dots, 2^{N_1} - 1\}^{\tilde{\ell}_1}$ ;
- (2)  $v$  and  $\tilde{v}$  belong to  $\{0, \dots, 2^{N_1} - 1\}^{\ell_{N_1} - 1 - \tilde{\ell}_1} \cdot \prod_{N=N_1}^{N-1} \{0, \dots, 2^n - 1\}^{\ell_n} \cdot \{0, \dots, 2^N - 1\}^\ell$ , and :
  - (a) either  $v$  and  $\tilde{v}$  are empty words,
  - (b) or all letters of  $\tilde{v}$  are 0 and all letters of  $v$  are as large as possible.

From (3.8), (3.10) and the fact that  $p_{n,0} = p_{n,2^n-1}$  for every  $n \geq N_0$ , one deduces

$$(3.14) \quad 2^{-1} \leq \frac{\mu(\lambda)}{\mu(\widehat{\lambda})} \leq 2.$$

Consider now two neighboring intervals  $\lambda$  and  $\widehat{\lambda}$  of generation  $j$ , where  $\gamma(g) < j \leq \gamma(g+1)$ . Let  $\lambda'$  and  $\widehat{\lambda}'$  be the elements of  $\mathcal{D}_{\gamma(g)}$  which contain  $\lambda$  and  $\widehat{\lambda}$  respectively. These intervals are either equal or neighbors. By construction, when  $N$  is large enough, one has by (3.8) and (3.10)

$$(3.15) \quad 2^{-N(\alpha_{\max}+\varepsilon)} \leq \frac{\mu(\lambda)}{\mu(\lambda')} \leq 1,$$

and if  $j = \gamma(g+1)$ , then

$$(3.16) \quad 2^{-N(\alpha_{\max}+\varepsilon)} \leq \frac{\mu(\lambda)}{\mu(\lambda')} \leq 2^{-N(\alpha_{\min}-\varepsilon)},$$

where  $\varepsilon \leq \alpha_{\min}/2$  for instance (this is due to the choice of  $N_0$  with  $\varepsilon_{N_0} \leq \alpha_{\min}/4$ ).

**Remark 3.11.** *Observe that by construction, when  $g$  gets larger, the  $\varepsilon$  in (3.16) can be taken very small and converges to zero as  $g \rightarrow +\infty$ , again because of the construction of  $p_N$ . This remark is used in Remark 3.15 below.*

The same property as (3.15) holds true for  $\widehat{\lambda}$  and  $\widehat{\lambda}'$ , hence

$$(3.17) \quad \frac{\mu(\lambda')}{\mu(\widehat{\lambda}')} 2^{-(\alpha_{\max}+\varepsilon)N} \leq \frac{\mu(\lambda)}{\mu(\widehat{\lambda})} = \frac{\mu(\lambda)}{\mu(\lambda')} \frac{\mu(\lambda')}{\mu(\widehat{\lambda}')} \frac{\mu(\widehat{\lambda}')}{\mu(\widehat{\lambda})} \leq \frac{\mu(\lambda')}{\mu(\widehat{\lambda}')} 2^{(\alpha_{\max}+\varepsilon)N}.$$

Let

$$(3.18) \quad \phi(j) = \begin{cases} 0 & \text{if } 0 \leq j < N_0 \ell_{N_0} \\ (1 + (\alpha_{\max} + \varepsilon))N & \text{if } \gamma(g) \leq j < \gamma(g+1). \end{cases}$$

Note that  $\phi(j)/j \leq (1 + (\alpha_{\max} + \varepsilon))N/\gamma(g)$  which tends to 0 as  $j \rightarrow \infty$ . This follows from the fact that

$$(3.19) \quad \gamma(g) \geq \sum_{n=N_0}^{N-1} n \ell_n \gg N^2$$

as  $N \rightarrow \infty$  since  $\ell_n \geq n^2$  for all  $n \geq N_0$ . Hence  $\phi \in \Phi$ .

Applying (3.14) to  $\lambda'$  and  $\widehat{\lambda}'$ , and using (3.17), there exists  $\tilde{J}$  such that for  $j \geq \tilde{J}$ ,

$$(3.20) \quad 2^{-\phi(j)} \leq \frac{\mu(\lambda)}{\mu(\widehat{\lambda})} \leq 2^{\phi(j)},$$

Upon adding a constant to  $\phi$  (to take into account the small generations  $j \leq \tilde{J}$ ), one concludes that  $\mu|_{[0,1]}$  is almost doubling in the sense of Definition 2.8.

To prove that  $\mu$  is almost doubling on  $\mathbb{R}$ , it is enough to observe that by symmetry of the coefficients ( $p_{n,0} = p_{n,2^n-1}$ ), for any  $g \in \mathbb{N}$   $\mu|_{[1-2^{-\gamma(g)},1]}(\cdot + 1 - 2^{-\gamma(g)}) = \mu|_{[0,2^{-\gamma(g)]}$ , and then to use the periodicity of  $\mu$ .  $\square$

**Lemma 3.12.** *The measure  $\mu$  satisfies (P).*

*Proof.* First, consider subintervals of  $[0, 1]$ .

Let  $\varepsilon > 0$ . For  $N \geq N_0$  and  $g = \ell + \sum_{n=N_0}^{N-1} \ell_n$  with  $1 \leq \ell \leq \ell_N$ , any dyadic interval  $\lambda \in \mathcal{D}_j$  with  $\gamma(g) \leq j < \gamma(g+1)$  satisfies, if  $N$  is large enough

$$2^{-(\gamma(g)+N)(\alpha_{\max}+\varepsilon/2)} \leq \mu(\lambda) \leq 2^{-\gamma(g)(\alpha_{\min}-\varepsilon/2)},$$

(use (3.13) for instance). By our choice for  $\ell_N$  and (3.19), for  $\gamma(g) \leq j < \gamma(g+1)$ ,  $N/j$  converges to 0 as  $j \rightarrow +\infty$ . Hence, for  $j$  large enough

$$(3.21) \quad 2^{-j(\alpha_{\max}+\varepsilon)} \leq \mu(\lambda) \leq 2^{-j(\alpha_{\min}-\varepsilon)}.$$

So, (2.8) is satisfied with  $s_2 = \alpha_{\max} + \varepsilon$  and  $s_1 = \alpha_{\min} - \varepsilon$ , and some constant  $C > 0$ . This yields property (P<sub>1</sub>).

Let us move to (P<sub>2</sub>).

Let  $g, g' \in \mathbb{N}^*$ ,  $j, j' \in \mathbb{N}^*$  with  $j' > j$  and  $N' \geq N \geq N_0$  such that:

- $g = \ell + \sum_{n=N_0}^{N-1} \ell_n$  with  $1 \leq \ell \leq \ell_N$ , and  $\gamma(g) \leq j < \gamma(g+1)$ ,
- $g' = \ell' + \sum_{n=N_0}^{N'-1} \ell_n$  with  $1 \leq \ell' \leq \ell_{N'}$ , and  $\gamma(g') \leq j' < \gamma(g'+1)$ .

Consider two neighboring dyadic intervals  $\lambda, \hat{\lambda} \in \mathcal{D}_j$ , and an interval  $\lambda' \in \mathcal{D}_{j'}$  such that  $\lambda' \subset \lambda$ .

Due to the doubling property of  $\mu$  applied to  $\lambda$  and  $\hat{\lambda}$ , we have

$$(3.22) \quad 2^{-\phi(j)} \frac{\mu(\lambda)}{\mu(\lambda')} \leq \frac{\mu(\hat{\lambda})}{\mu(\lambda')} = \frac{\mu(\hat{\lambda})}{\mu(\lambda)} \frac{\mu(\lambda)}{\mu(\lambda')} \leq 2^{\phi(j)} \frac{\mu(\lambda)}{\mu(\lambda')}.$$

For  $J \leq j$ , denote by  $\lambda_{|J}$  the unique element of  $\mathcal{D}_J$  which contains  $\lambda$ , and for  $j < J \leq j'$  denote by  $\tilde{\lambda}_{|J}$  the unique element  $\tilde{\lambda}$  of  $\mathcal{D}_J$  such that  $\lambda' \subset \tilde{\lambda} \subset \lambda$ . We have

$$\frac{\mu(\lambda_{|\gamma(g)+N})}{\mu(\lambda'_{|\gamma(g')})} \leq \frac{\mu(\lambda)}{\mu(\lambda')} \leq \frac{\mu(\lambda_{|\gamma(g)})}{\mu(\lambda'_{|\gamma(g')})}.$$

It is easily seen that  $N + N' = o(j) + o(j' - j)$  as  $j, j' \rightarrow +\infty$ . Consequently, using the multiplicative structure of  $\mu$  and (3.13) yields a function  $\tilde{\phi} \in \Phi$ , as well as a constant  $C \geq 1$ , depending on  $\mu$  only, such that

$$(3.23) \quad C^{-1} 2^{-j\tilde{\phi}(j)} 2^{(j'-j)(\alpha_{\min}-\varepsilon)} \leq \frac{\mu(\lambda)}{\mu(\lambda')} \leq C 2^{j\tilde{\phi}(j)} 2^{(j'-j)(\alpha_{\max}+\varepsilon)}.$$

Incorporating (3.23) in (3.22) shows that (P<sub>2</sub>) holds with the same exponents  $s_1$  and  $s_2$  as in (P<sub>1</sub>).

Finally, the same arguments as those developed at the end of the proof of Lemma 3.10 ensure that the property true on  $[0, 1]$  extends to  $\mathbb{R}$ .  $\square$

**3.5. The  $L^q$ -spectrum of  $\mu|_{[0,1]}$  equals  $\sigma^*$ .** Let  $\tau = \sigma^*$ . Since  $\sigma \in \mathcal{S}_{1,\mathcal{M}}$ , one has  $\tau \in \mathcal{T}_{1,\mathcal{M}}$ .

For simplification, denote  $\mu|_{[0,1]}$  by  $\mu$ . For all  $j \in \mathbb{N}$ , let

$$\mathcal{D}_j^0 = \{\lambda \in \mathcal{D}_j : \lambda \subset [0, 1]^d\}.$$

Fix  $t \in \mathbb{R}$  and  $g = \ell + \sum_{n=N_0}^{N-1} \ell_n$  with  $N \geq N_0$  and  $1 \leq \ell \leq \ell_N$ . Assume that  $g$  is so large that (3.15) holds for every  $j \geq \gamma(g)$ .

First, remark that, for the integers  $j$  such that  $\gamma(g) \leq j < \gamma(g+1)$ , for every  $\lambda \in \mathcal{D}_{\gamma(g)}^0$ , by (3.15) one has

$$2^{(j-\gamma(g))} 2^{-N|t|(\alpha_{\max}+\varepsilon)} \leq \sum_{\lambda' \in \mathcal{D}_j, \lambda' \subset \lambda} \frac{\mu(\lambda')^t}{\mu(\lambda)^t} \leq 2^{(j-\gamma(g))} 2^{N|t|(\alpha_{\max}+\varepsilon)}.$$

Since  $N + (j - \gamma(g)) = o(\gamma(g))$  as  $g \rightarrow +\infty$ , one deduces that

$$(3.24) \quad \sum_{\lambda \in \mathcal{D}_j^0} \mu(\lambda)^t = 2^{o(\gamma(g))} \sum_{\lambda \in \mathcal{D}_{\gamma(g)}^0} \mu(\lambda)^t.$$

This shows that it is enough to study  $\liminf_{g \rightarrow +\infty} \frac{1}{-\gamma(g)} \log_2 \sum_{I \in \mathcal{D}_{\gamma(g)}^0} \mu(I)^t$  to find the value  $\tau_\mu(t)$  (actually,  $\tau_\mu$  will be proved to be a limit, not only a liminf).

- Let us start with the lower bound for  $\tau_\mu(t)$ .

The multiplicative structure defining  $\nu$  and  $\mu$  using concatenation of pieces of Bernoulli product measures yields

$$(3.25) \quad \sum_{\lambda \in \mathcal{D}_{\gamma(g)}^0} \mu(\lambda)^t = \left( \prod_{n=N_0}^{N-1} \left( \sum_{i=0}^{2^n-1} p_{n,i}^t \right)^{\ell_n} \right) \cdot \left( \sum_{i=0}^{2^N-1} p_{N,i}^t \right)^\ell.$$

For each  $n \geq N_0$ , using (3.9), one has

$$(3.26) \quad C_{n,t}^{-1} 2^{-nt\beta_{n,i}} \leq p_{n,i}^t \leq 2^{-nt\beta_{n,i}} C_{n,t},$$

where  $C_{n,t}$  tends to 1 when  $n \rightarrow +\infty$  (and does not depend on  $i \in \{0, \dots, 2^n - 1\}$ ). Hence, using (3.7), the definition of the  $R_{n,i}$  and the inequality  $2R_{n,i} \leq 2^{n\sigma(\alpha_{n,i})}$  which follows from (3.3), one gets

$$\begin{aligned} \sum_{i=0}^{2^n-1} p_{n,i}^t &\leq C_{n,t} \sum_{i=0}^{2^n-1} 2^{-tn\beta_{n,i}} \leq C_{n,t} \sum_{i=0}^{m_n} 2R_{n,i} 2^{-tn\alpha_{n,i}} \leq C_{n,t} \sum_{i=1}^{m_n} 2^{n(\sigma(\alpha_{n,i})-t\alpha_{n,i})} \\ &\leq C_{n,t} m_n 2^{-n \inf\{t\alpha - \sigma(\alpha) : \alpha \in \text{dom}(\sigma)\}} = C_{n,t} m_n 2^{-n\tau(t)}. \end{aligned}$$

Consequently,

$$\sum_{\lambda \in \mathcal{D}_{\gamma(g)}^0} \mu(\lambda)^t \leq 2^{-\gamma(g)\tau(t)} \cdot \left( \prod_{n=N_0}^{N-1} (C_{n,t} m_n)^{\ell_n} \right) \cdot (C_{N,t} m_N)^\ell.$$

Since  $\log(m_n) = o(n)$ , one sees that  $\ell \log(m_N) + \sum_{n=N_0}^{N-1} \ell_n \log m_n = o(\gamma(g))$ . Combining this with the fact that  $(C_{n,t})_{n \geq N_0}$  converges to 1 as  $n$  tends to infinity, one deduces that  $\left( \prod_{n=N_0}^{N-1} (C_{n,t} m_n)^{\ell_n} \right) \cdot (C_{N,t} m_N)^\ell = 2^{o(\gamma(g))}$  and

$$\tau_\mu(t) = \liminf_{g \rightarrow +\infty} \frac{-1}{\gamma(g)} \log_2 \sum_{\lambda \in \mathcal{D}_{\gamma(g)}^0} \mu(\lambda)^t \geq \tau(t).$$

- Let us now estimate  $\limsup_{g \rightarrow +\infty} \frac{-1}{\gamma(g)} \log_2 \sum_{\lambda \in \mathcal{D}_{\gamma(g)}^0}$ .

Suppose first that  $\sigma(\tau'(t^+)) > 0$ . By construction, one can fix  $N'_0 \geq N_0$  such that for all  $n \geq N'_0$ , there exists an integer  $1 \leq i_{n,t} \leq m_n$ , such that  $|\alpha_{n,i_{n,t}} - \tau'(t^+)| \leq 1/n$ ,  $i_{n,t} \neq i_n$  and  $i_{n,t} \neq i'_n$ . The Legendre transform  $\sigma = \tau^*$  implies then that  $t\tau'(t^+) - \tau(t) = \sigma(\tau'(t^+))$ .

In addition, by continuity of  $\sigma$ ,  $\lim_{n \rightarrow +\infty} \eta_n = 0$ , where  $\eta_n = \sigma(\alpha_{n,i_{n,t}}) - t\alpha_{n,i_{n,t}} + \tau(t)$ . Bounding from below the sums in (3.25) by the sum only over those integers  $j$  such that  $\beta_{n,j} = \alpha_{n,i_{n,t}}$  (see (3.7)), and recalling (3.26) and the definition (3.3) of  $R_{n,i}$ ,

$$\begin{aligned} \sum_{\lambda \in \mathcal{D}_{\gamma(g)}^0} \mu(\lambda)^t &\geq \left( \prod_{n=N_0}^{N'_0-1} \left( \sum_{i=0}^{2^n-1} p_{n,i}^t \right)^{\ell_n} \right) \cdot \left( \prod_{n=N'_0}^{N-1} \left( C_{n,t}^{-1} \lfloor 2^{n(\sigma(\alpha_{n,i_{n,t}}) - \varepsilon_n)} \rfloor 2^{-tn\alpha_{n,i_{n,t}}} \right)^{\ell_n} \right) \\ &\quad \cdot \left( C_{N,t}^{-1} \lfloor 2^{N(\sigma(\alpha_{N,i_{N,t}}) - \varepsilon_N)} \rfloor 2^{-tN\alpha_{N,i_{N,t}}} \right)^\ell. \end{aligned}$$

Recalling now that  $\varepsilon_n = \frac{2 \log_2 n}{n}$ , and setting  $C_t = \prod_{n=N_0}^{N'_0-1} \left( \sum_{i=0}^{2^n-1} p_{n,i}^t \right)^{\ell_n}$ , one gets

$$\begin{aligned} \sum_{\lambda \in \mathcal{D}_{\gamma(g)}^0} \mu(\lambda)^t &\geq C_t \left( \prod_{n=N'_0}^{N-1} \left( C_{n,t}^{-1} \frac{2^{n(\sigma(\alpha_{n,i_{n,t}}) - t\alpha_{n,i_{n,t}})}}{4n^2} \right)^{\ell_n} \right) \cdot \left( C_{N,t}^{-1} \frac{2^{N(\sigma(\alpha_{N,i_{N,t}}) - t\alpha_{N,i_{N,t}})}}{4N^2} \right)^\ell \\ &= C_t 2^{-\gamma(g)\tau(t)} \left( \prod_{n=N'_0}^{N-1} \left( C_{n,t}^{-1} \frac{2^{n\eta_n}}{4n^2} \right)^{\ell_n} \right) \cdot \left( C_{N,t}^{-1} \frac{2^{N\eta_N}}{4N^2} \right)^\ell = 2^{-\gamma(g)(\tau(t)+o(1))} \end{aligned}$$

as  $g \rightarrow +\infty$ , where we used that  $\log(C_{n,t}) + n\eta_n + \log(4n^2) = o(n)$  (recall that  $C_{n,t} \rightarrow 1$  when  $n \rightarrow +\infty$  uniformly in  $t$ ). The last lines imply that

$$\limsup_{g \rightarrow +\infty} \frac{-1}{\gamma(g)} \log_2 \sum_{\lambda \in \mathcal{D}_{\gamma(g)}^0} \mu(\lambda)^t \leq \tau(t).$$

This equation and the lower bound already obtained for  $\tau_\mu(t)$  show that  $\tau_\mu(t) = \tau(t)$ .

It remains us to consider the extremal case  $\sigma(\tau'(t^+)) = 0$ , which may happen only if  $\tau'(t^+) \in \{\alpha_{\min}, \alpha_{\max}\}$ .

Suppose that  $\tau'(t^+) = \alpha_{\min}$  and  $\sigma(\alpha_{\min}) = 0$ . One has  $0 = \sigma(\alpha_{\min}) = \tau^*(\alpha_{\min}) = t^+ \tau'(t^+) - \tau(t)$ , so  $\tau(t) = t\alpha_{\min}$ , and  $t_0 = \min\{t \in \mathbb{R} : \tau(t) = \alpha_{\min}t\} < +\infty$ . In addition,  $t_0 > 0$  since  $\tau(0) < 0$ . Also, for  $t \in [0, t_0]$ ,  $\sigma(\tau'(t^+)) \in (0, 1]$  and we know from the first part of this proof that  $\tau_\mu(t) = \tau(t)$  on this interval  $[0, t_0]$ . To conclude, it is thus enough to show that this last equality holds over the whole interval  $[t_0, +\infty)$  as well.

At first, for all  $t \geq t_0$ ,  $\varepsilon \in (0, t_0)$  and  $n \in \mathbb{N}$ , by subadditivity of  $x \geq 0 \mapsto x^{t/(t_0-\varepsilon)}$ ,

$$\sum_{\lambda \in \mathcal{D}_{\gamma(g)}^0} \mu(\lambda)^t \leq \left( \sum_{\lambda \in \mathcal{D}_{\gamma(g)}^0} \mu(\lambda)^{t_0-\varepsilon} \right)^{t/(t_0-\varepsilon)},$$

so

$$(3.27) \quad \tau_\mu(t) = \liminf_{g \rightarrow +\infty} -\frac{1}{\gamma(g)} \log_2 \sum_{\lambda \in \mathcal{D}_{\gamma(g)}^0} \mu(\lambda)^t \geq \frac{t}{t_0 - \varepsilon} \tau(t_0 - \varepsilon).$$

On the other hand, consider the interval  $[0, 2^{-\gamma(g)}]$  in  $\mathcal{D}_{\gamma(g)}$ . Its  $\mu$ -mass is by construction  $2^{-\gamma(g)(\alpha_{\min}+o(1))}$  as  $g \rightarrow +\infty$ , so

$$\limsup_{g \rightarrow +\infty} -\frac{1}{\gamma(g)} \log_2 \sum_{\lambda \in \mathcal{D}_{\gamma(g)}^0} \mu(\lambda)^t \leq \limsup_{g \rightarrow +\infty} -\frac{1}{\gamma(g)} \log_2 2^{-t\gamma(g)(\alpha_{\min}+o(1))} = \alpha_{\min}t.$$

Letting  $\varepsilon \rightarrow 0$  in (3.27) and using that  $\alpha_{\min} = \tau(t_0)/t_0$ , yields  $\tau_\mu(t) = \alpha_{\min}t = \tau(t)$ .

The case  $\tau'(t^+) = \alpha_{\max}$  and  $\sigma(\alpha_{\max}) = 0$  works similarly by considering  $t_0 = \max\{t \in \mathbb{R} : \tau(t) = \alpha_{\max}t\} \in (-\infty, 0)$ , and the element of  $\mathcal{D}_{\gamma(g)}^0$  whose  $\mu$ -mass is minimal, i.e. equal to  $2^{-\gamma(g)(\alpha_{\max}+o(1))}$ .

**3.6. The SMF holds for  $\mu$  with  $\sigma_\mu = \sigma$ .** The facts that  $E_\mu(\alpha) = \emptyset$  for  $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$  and  $\dim E_\mu(\alpha) \leq \sigma(\alpha)$  for  $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ , follow from Proposition 3.1 and Section 3.5 where it is proved that  $\tau_\mu^* = \sigma$  (so  $\tau_\mu^*(\alpha) = -\infty$  if  $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$ ).

Further, it follows from the construction and the choice of the weights  $p_{n,i}$  that there exist real numbers  $x$  at which  $h_\mu(x) = \alpha_{\min}$ , and other real numbers  $x$  at which  $h_\mu(x) = \alpha_{\max}$ . Hence,  $\sigma_\mu(\alpha_{\min}) \geq 0$  and  $\sigma_\mu(\alpha_{\max}) \geq 0$ .

In particular, if  $\sigma(\alpha_{\min}) = 0$  (resp.  $\sigma(\alpha_{\max}) = 0$ ), then  $\sigma_\mu(\alpha_{\min}) = 0$  (resp.  $\sigma_\mu(\alpha_{\max}) = 0$ ) and the SMF holds at  $\alpha_{\min}$  (resp.  $\alpha_{\max}$ ).

Now, fix  $\alpha \in [\alpha_{\min}, \alpha_{\max}]$  such that  $\sigma(\alpha) > 0$ . For each  $N \geq N_0$ , let

$$(3.28) \quad \mathcal{J}_{N,\alpha} = \{j \in \{0, \dots, 2^N - 1\} : j \text{ is odd and } |\beta_{N,j} - \alpha| \leq N^{-1}\}.$$

Let  $\varepsilon > 0$ . Recalling the definitions of Section 3.2 we first observe that the exponents  $\beta_{N,j}$  considered in the definition of  $\mathcal{J}_{N,\alpha}$  correspond to at most nine distinct exponents  $\alpha_{N,i}$  (since  $\alpha_{N,i} - \alpha_{N,i-1} \leq (4N)^{-1}$ ). This observation, together with the continuity of  $\sigma$  and the definition of the numbers  $R_{N,i}$  imply that for  $N$  large enough,

$$(3.29) \quad 2^{N(\sigma(\alpha)-\varepsilon)} \leq \#\mathcal{J}_{N,\alpha} \leq 2^{N(\sigma(\alpha)+\varepsilon)}.$$

Consider the measure  $\nu_\alpha$  supported on

$$\Sigma_\alpha = \prod_{n=N_0}^{\infty} \mathcal{J}_{n,\alpha}^{\ell_n} \subset \Sigma$$

defined by setting, for each  $N \geq N_0$ ,  $0 \leq \ell < \ell_N$  and for every word  $J_{N_0} \cdot J_{N_0+1} \cdots J_N \in \left(\prod_{n=N_0}^{N-1} \{0, \dots, 2^n - 1\}^{\ell_n}\right) \times \{0, \dots, 2^N - 1\}^{\ell}$ :

$$\nu_\alpha([J_{N_0} \cdots J_N]) = \begin{cases} (\#\mathcal{J}_{N,\alpha})^{-\ell} \prod_{n=N_0}^{N-1} (\#\mathcal{J}_{n,\alpha})^{-\ell_n} & \text{if } [J_{N_0} \cdots J_N] \cap \Sigma_\alpha \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

This last formula is consistent, and the measure  $\nu_\alpha$  is well-defined and atomless.

**Proposition 3.13.** *The measure  $\mu_\alpha = \nu_\alpha \circ \pi^{-1}$  is defined as the push-forward measure of  $\nu_\alpha$  on the interval  $[0, 1]$  (recall (3.12)). This measure is supported by  $\pi(\Sigma_\alpha)$ , and for every  $x \in \pi(\Sigma_\alpha)$ ,  $h_\mu(x) = \alpha$  and  $h_{\mu_\alpha}(x) = \sigma(\alpha)$ .*

*Proof.* For all  $\omega \in \Sigma$ , denote by  $[\omega|_g]$  the cylinder of generation  $g \in \mathbb{N}$  which contains  $\omega$ . From the definition of  $\mathcal{J}_{N,\alpha}$ , for every  $\omega \in \Sigma_\alpha$  one has

$$\alpha - \varepsilon \leq \liminf_{g \rightarrow +\infty} -\frac{1}{\gamma(g)} \log(\mu(\pi([\omega|_g]))) \leq \limsup_{g \rightarrow +\infty} -\frac{1}{\gamma(g)} \log(\mu(\pi([\omega|_g]))) \leq \alpha + \varepsilon.$$

Since this holds for every choice of  $\varepsilon > 0$ ,

$$\lim_{g \rightarrow +\infty} -\frac{1}{\gamma(g)} \log(\mu(\pi([\omega|_g]))) = \alpha.$$

Moreover,  $\lim_{g \rightarrow +\infty} \frac{\gamma(g+1)}{\gamma(g)} = 1$  and  $\mu$  is almost doubling, so  $\pi(\Sigma_\alpha) \subset E_\mu(\alpha)$ .

On the other hand, from (3.29) one deduces that for every  $\omega \in \Sigma_\alpha$

$$\sigma(\alpha) - \varepsilon \leq \liminf_{g \rightarrow +\infty} \frac{-1}{\gamma(g)} \log \left( \mu_\alpha(\pi([\omega|_g])) \right) \leq \limsup_{g \rightarrow +\infty} \frac{-1}{\gamma(g)} \log \left( \mu_\alpha(\pi([\omega|_g])) \right) \leq \sigma(\alpha) + \varepsilon.$$

Again, this holds for every choice of  $\varepsilon > 0$ , hence

$$\lim_{g \rightarrow +\infty} -\frac{1}{\gamma(g)} \log \left( \mu_\alpha(\pi([\omega|_g])) \right) = \sigma(\alpha).$$

Since  $\lim_{g \rightarrow +\infty} \frac{\gamma(g+1)}{\gamma(g)} = 1$ , the measure  $\mu_\alpha$ , which is supported by  $\pi(\Sigma_\alpha)$ , is exact dimensional with dimension  $\sigma(\alpha)$ , so  $\dim(\Sigma_\alpha) \geq \sigma(\alpha)$ . The combination of the last two facts imply that  $\sigma_\mu(\alpha) = \dim E_\mu(\alpha) \geq \sigma(\alpha)$ . Since the converse inequality holds true by the multifractal formalism, the proof is complete.  $\square$

**3.7. The case  $d \geq 2$ .** If  $\sigma \in \mathcal{S}_{d,\mathcal{M}}$ , then the map  $\tilde{\sigma} : \alpha \in \mathbb{R} \mapsto d^{-1}\sigma(d \cdot \alpha)$  belongs to  $\mathcal{S}_{1,\mathcal{M}}$ . Let  $\tilde{\mu}_\sigma$  be the measure associated with  $\tilde{\sigma}$  as built in the previous sections in dimension 1. It is easily checked that the tensor product measure  $\mu = (\tilde{\mu}_\sigma)^{\otimes d}$  possesses all the required properties so that  $\sigma = \sigma_\mu$  and  $\mu$  obeys the SMF. In addition, for all  $\alpha \in \text{dom}(\sigma)$ , if  $\tilde{\nu}_{d^{-1}\alpha}$  and  $\tilde{\mu}_{d^{-1}\alpha}$  are the measures built in Section 3.6 associated with the exponent  $d^{-1}\alpha$ , then the measure  $\mu_\alpha := ((\tilde{\mu}_{d^{-1}\alpha})^{\otimes d})$  satisfies the same properties as  $\mu_\alpha$  (see Proposition 3.13).

**Definition 3.14.** Set  $\mathcal{M}_d = \{\mu^{\otimes d} : \mu \in \mathcal{M}_1\}$ .

**Remark 3.15.** By construction, for any  $\mu \in \mathcal{M}_d$ , writing  $\text{dom}(\sigma_\mu) = [\alpha_{\min}, \alpha_{\max}]$ , the inequality (3.21) holds, and the estimates made in the previous sections show that for every  $\varepsilon > 0$ , there exists  $j_\varepsilon \in \mathbb{N}$  such that for all  $j' \geq j \geq j_\varepsilon$ , for all  $\lambda, \tilde{\lambda} \in \mathcal{D}_j$  such that  $\partial\lambda \cap \partial\tilde{\lambda} \neq \emptyset$ , and all  $\lambda' \in \mathcal{D}_{j'}$  such that  $\lambda' \subset \lambda$ , one has

$$(3.30) \quad \mu(\lambda') \leq \mu(\tilde{\lambda}) 2^{j\varepsilon} 2^{-(j'-j)(\alpha_{\min}-\varepsilon)}.$$

Also, from the construction of  $\mu$ , for all integers  $j, j' \geq 0$  and  $\lambda \in \mathcal{D}_j$ , one has

$$(3.31) \quad \mu(\lambda \cdot [0, 2^{-j'}]^d) = \mu(\lambda) 2^{-\phi_\lambda} 2^{-j'\alpha_{\min} + \tilde{\phi}_\lambda(j')},$$

where:

- $\lambda \cdot [0, 2^{-j'}]^d$  is the concatenation of  $\lambda$  and  $[0, 2^{-j'}]^d$ , meaning that  $\lambda \cdot [0, 2^{-j'}]^d$  is the image of  $[0, 2^{-j'}]^d$  by the canonical similarity which maps  $[0, 1]^d$  onto  $\lambda$ ,
- $\phi_\lambda \in \mathbb{R}$  and  $\tilde{\phi}_\lambda \in \Phi$  are uniform  $o(j)$  in the sense that

$$(3.32) \quad \lim_{j \rightarrow +\infty} \sup \left\{ \frac{|\phi_\lambda|}{j} : \lambda \in \mathcal{D}_j \right\} = \lim_{j' \rightarrow +\infty} \sup \left\{ \frac{|\tilde{\phi}_\lambda(j')|}{j'} : \lambda \in \bigcup_{j \in \mathbb{N}} \mathcal{D}_j \right\} = 0.$$

These properties are key to prove the optimal upper bound for the singularity spectrum of typical functions in [12].

#### 4. UPPER BOUND FOR THE SINGULARITY SPECTRUM IN $\mathbf{B}^\mu(\mathbb{R}^d)$ WHEN $\mu \in \mathcal{E}_d$

For  $\lambda = (i, j, k) \in \Lambda_j$ , we identify  $\lambda$  with  $\lambda_{j,k} \in \mathcal{D}_j$ , and write  $\mu(\lambda)$  for  $\mu(\lambda_{j,k})$  and  $\lambda \subset E$  for  $\lambda_{j,k} \subset E$ .

Recall the definition (2.14) of  $\mathbf{B}^\mu(\mathbb{R}^d)$ . To prove Theorem 7(1) and Theorem 7(2), the notion of wavelet leaders is key.

**Definition 4.1** (Wavelet leaders). *Given  $\Psi \in \bigcup_{r \in \mathbb{N}} \mathcal{F}_r$  and  $f \in L^p_{\text{loc}}(\mathbb{R}^d)$  for  $p \in [1, +\infty]$ , denoting the wavelet coefficients of  $f$  associated with  $\Psi$  by  $(c_\lambda)_{\lambda \in \Lambda}$ , the wavelet leader of  $f$  associated with  $\lambda \in \mathcal{D}$  (see Section 2.1 for the notations) is defined as:*

$$(4.1) \quad L_\lambda^f = \sup\{|c_{\lambda'}| : \lambda' = (i, j, k) \in \Lambda, \lambda'_{j,k} \subset 3\lambda\}.$$

Pointwise Hölder exponents of Hölder continuous functions (recall (1.1)) are related to the wavelet leaders as follows (see [37, Corollary 1]).

**Proposition 4.2.** *Let  $r \in \mathbb{N}^*$  and  $\Psi \in \mathcal{F}_r$ . If  $f \in \mathcal{C}^\varepsilon(\mathbb{R}^d)$  for some  $\varepsilon > 0$ , then for every  $x_0 \in \mathbb{R}^d$ ,  $h_f(x_0) < r$  if and only  $\liminf_{j \rightarrow \infty} \frac{\log L_{\lambda_j^f(x)}^f}{\log(2^{-j})} < r$ , and in this case*

$$(4.2) \quad h_f(x_0) = \liminf_{j \rightarrow \infty} \frac{\log L_{\lambda_j^f(x)}^f}{\log(2^{-j})}.$$

Hence, as observed by Jaffard, and rephrased in the language of the present paper, if the support of  $\sigma_f$  is bounded and sufficiently smooth wavelets  $\Psi$  are used, then the singularity spectrum  $\sigma_f$  of  $f$  coincides with the singularity spectrum of the capacity  $\nu \in \mathcal{C}(\mathbb{R}^d)$  defined by  $\nu(B) = \sup\{L_\lambda^f : \lambda \in \mathcal{D}, \lambda \subset B\}$  for all  $B \in \mathcal{B}(\mathbb{R}^d)$ .

*Proof of Theorem 7(1).* Fix some integer  $r > \max(\text{supp}(\sigma_\mu))$  and  $\Psi \in \mathcal{F}_r$ . By definition (2.13) and Theorem 6,  $f \in \mathbf{B}^\mu(\mathbb{R}^d)$  implies that for every  $\varepsilon$ , there exists a constant  $C_f \geq 0$  such that for every  $j \geq 0$ , for every  $\lambda \in \Lambda_j$ ,  $|c_\lambda| \leq C_f \mu(\lambda) 2^{j\varepsilon}$ . Without loss of generality, we assume that  $C_f = 1$ .

Let us observe that the hierarchal structure of the capacity  $\mu$  and inequality (2.9) impose that for every dyadic cube  $\lambda \in \Lambda_j$ , for every cube  $\lambda' \in \Lambda_{j'}$  with  $j' \geq j$  and  $\lambda' \subset 3\lambda$ ,

$$|c_{\lambda'}| \leq \mu(\lambda') 2^{j'\varepsilon} \leq \mu(\lambda) 2^{\phi(j)} 2^{j'\varepsilon} 2^{-(j'-j)s_1}.$$

Let us choose  $\varepsilon < s_1/2$  and  $j$  so large that  $2^{\phi(j)} \leq 2^{j\varepsilon}$ . Then

$$|c_{\lambda'}| \leq \mu(\lambda) 2^{2j\varepsilon} 2^{-(j'-j)(s_1-\varepsilon)} \leq \mu(\lambda) 2^{2j\varepsilon}.$$

In particular,

$$(4.3) \quad L_\lambda^f \leq \mu(\lambda) 2^{2j\varepsilon}.$$

Thus, due to (4.2), one has  $h_f(x) \geq \underline{h}_\mu(x)$  for all  $x \in \mathbb{R}^d$ . Then, if  $H \leq \tau'_\mu(0^-)$ , due to Proposition 3.3(1), the fact that  $E_f(H) \cap [0, 1]^d \subset \underline{E}_{\mu|_{[0,1]^d}}(H)$  implies that  $\dim(E_f(H) \cap [0, 1]^d) \leq \tau_\mu^*(H) = \sigma_\mu(H)$ , and the  $\mathbb{Z}^d$ -invariance of  $\underline{h}_\mu$  yields  $\sigma_f(H) = \dim E_f(H) \leq \tau_\mu^*(H)$ . For  $H > \tau'_\mu(0^-)$ , the desired inequality  $\sigma_f(H) \leq d$  is trivial.  $\square$

## 5. TYPICAL SINGULARITY SPECTRUM IN $\mathbf{B}^\mu(\mathbb{R}^d)$

The environment  $\mu \in \mathcal{E}_d$  and  $\Psi \in \mathcal{F}_r$  for some integer  $r > \max(\text{supp}(\sigma_\mu))$  are fixed.

**5.1. A saturation function.** Let us set

$$(5.1) \quad g_\mu(x) = \sum_{j \geq 0} \sum_{\lambda \in \Lambda_j} c_\lambda^{g_\mu} \psi_\lambda(x), \quad \text{where } c_\lambda^{g_\mu} = \mu(\lambda).$$

Obviously,  $g_\mu \in B_\Psi^\mu(\mathbb{R}^d) \subset \mathbf{B}^\mu(\mathbb{R}^d)$  (recall (2.14)). This function  $g_\mu$  is referred to as a saturation function, since it has the largest possible wavelet coefficients in  $\mathbf{B}^\mu(\mathbb{R}^d)$ .



Also, since  $\mu$  is a capacity, every wavelet leader of  $g_\mu$  is explicit: for every  $\lambda \in \Lambda_j$ ,  $L_\lambda^g = \max(|c_{\lambda'}^g| : \lambda' \subset 3\lambda) = \max(\mu(\lambda') : \lambda' \in \Lambda_j, \lambda' \subset 3\lambda)$ . Then, since  $\mu$  is almost doubling, the following lemma is immediate.

**Lemma 5.1.** *For every  $x \in \mathbb{R}$ , one has  $h_{g_\mu}(x) = \underline{h}_\mu(x)$  and  $\limsup_{j \rightarrow \infty} \frac{\log L_j^{g_\mu}(x)}{\log 2^{-j}} = \bar{h}_\mu(x)$ . In particular,  $\sigma_{g_\mu} = \sigma_\mu = \tau_\mu^*$ .*

We now compute the singularity spectrum of functions whose wavelet coefficients are of same order as those of  $g_\mu$  at an infinite number of generations  $j \in \mathbb{Z}$ .

**Proposition 5.2.** *Let  $f \in \mathbf{B}^\mu(\mathbb{R}^d)$  such that for any  $L \in \mathbb{Z}^d$ , there exists an increasing sequence of integers  $(j_n)_{n \in \mathbb{N}}$ , and a positive sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  converging to 0 such that for all  $n \geq 1$  and  $\lambda = (i, j_n, k) \in \Lambda_{j_n}$  such that  $\lambda_{j_n, k} \subset L + 3[0, 1]^d$  the inequality  $2^{-j_n \varepsilon_n} c_\lambda^{g_\mu} \leq |c_\lambda^f|$  holds. Then  $\sigma_f = \sigma_{g_\mu} = \tau_\mu^*$ .*

*Proof.* Fix  $(j_n)_{n \in \mathbb{N}}$  and  $(\varepsilon_n)_{n \in \mathbb{N}}$  as in the statement. Due to the  $\mathbb{Z}^d$ -invariance of  $\mu$ , it is enough to prove that  $\dim(E_f(H) \cap [0, 1]^d) = \tau_\mu^*(H)$  for all  $H \in \mathbb{R}$ .

We already observed in the previous section that any  $f \in \mathbf{B}^\mu(\mathbb{R}^d)$ , for all  $x \in [0, 1]^d$ ,  $h_f(x) \geq \underline{h}_\mu(x)$ . In addition, by the assumption made in this proposition, for every  $\varepsilon > 0$ , for every  $x \in [0, 1]^d$ ,

$$(5.2) \quad \bar{h}_\mu(x) = \limsup_{j \rightarrow +\infty} \frac{\log L_j^g(x)}{\log 2^{-j}} \geq \liminf_{n \rightarrow +\infty} \frac{\log L_{j_n}^g(x)}{\log 2^{-j_n}} \geq \liminf_{n \rightarrow +\infty} \frac{\log L_{j_n}^f(x) 2^{j_n \varepsilon}}{\log 2^{-j_n}} \geq h_f(x) - \varepsilon.$$

This implies that:

- for all  $H > \tau_\mu'(0^-)$ ,  $E_f(H) \cap [0, 1]^d \subset \bar{E}_{\mu|_{[0, 1]^d}}(H)$ , hence  $\dim(E_f(H) \cap [0, 1]^d) \leq \tau_\mu^*(H)$  by Proposition 3.3(2).
- for all  $H \in [\alpha_{\min}, \alpha_{\max}]$ , one has  $E_{\mu|_{[0, 1]^d}}(H) \subset E_f(H) \cap [0, 1]^d$ . So  $\dim(E_f(H) \cap [0, 1]^d) \geq \tau_\mu^*(H)$ .

The combination of the last two facts concludes the proof.  $\square$

**5.2. Typical multifractal behavior in  $\mathbf{B}^\mu(\mathbb{R}^d)$ : proof of Theorem 2(2).** Recall property (P) (equations (2.8) and (2.9)) satisfied by  $\mu$ .

Fix  $\Psi \in \mathcal{F}_r$  with some  $r \geq s_2 + 1$ . Recall that the family of Banach spaces

$$\left\{ \left( B_\Psi^{\mu^{(-\varepsilon)}}(\mathbb{R}^d), \|\cdot\|_{L^p(\mathbb{R}^d)} + |\cdot|_{\mu^{(-\varepsilon)}, \Psi} \right) \right\}_{0 < \varepsilon \leq \min(1, s_1)}$$

is such that  $B_\Psi^{\mu^{(-\varepsilon)}}(\mathbb{R}^d) \hookrightarrow B_\Psi^{\mu^{(-\varepsilon')}}(\mathbb{R}^d)$  for all  $0 < \varepsilon < \varepsilon' \leq \min(s_1, 1)$ . As a consequence,  $\mathbf{B}^\mu(\mathbb{R}^d)$  can be endowed with a Fréchet space structure, with a countable basis of neighborhoods of the origin given by

$$(5.3) \quad \left\{ \mathcal{N}_m = \left\{ f \in \mathbf{B}^\mu(\mathbb{R}^d) : \|f\|_{L^\infty(\mathbb{R}^d)} + |f|_{\mu^{(-\frac{1}{m}), \Psi}} < \frac{1}{m} \right\} \right\}_{\substack{m \in \mathbb{N}, \\ m > \max(1, s_1^{-1})}}.$$

For every integer  $m > m_0 = \lfloor \max(1, s_1^{-1}) \rfloor + 1$ , set

$$V_m = \left\{ f \in \mathbf{B}^\mu(\mathbb{R}^d) : \forall j \geq 0, \forall \lambda \in \Lambda_j, \frac{|c_\lambda^f|}{c_\lambda^{g_\mu}} \in m^{-1} \{1, \dots, m^2\} \right\}.$$

Then let

$$(5.4) \quad \mathcal{G} = \limsup_{m \rightarrow \infty} (V_m + \mathcal{N}_{2^{\lceil m \log(m) \rceil}}).$$

Each  $\bigcup_{\ell \geq m} V_\ell$ ,  $m \geq m_0$ , is dense in  $\mathbf{B}^\mu(\mathbb{R}^d)$ , so  $\mathcal{G}$  is a dense  $G_\delta$  set.

When  $f \in \mathcal{G}$ , there exists an increasing sequence  $(j_n)_{n \geq 0}$  such that  $f \in V_{j_n} + \mathcal{N}_{2^{\lceil j_n \log(j_n) \rceil}}$  for all  $n \geq 0$ . Fix  $L \in \mathbb{Z}^d$ . For every  $n$ , for every  $\lambda \in \Lambda_{j_n}$  such that  $\lambda \subset L + 3[0, 1]^d$ , by definition of  $V_{j_n}$  and  $\mathcal{N}_{2^{\lceil j_n \log(j_n) \rceil}}$ , one has

$$|c_\lambda^f| \geq j_n^{-1} c_\lambda^{g_\mu} - 2^{-\lceil j_n \log(j_n) \rceil} c_\lambda^{g_\mu} 2^{j_n 2^{-j_n \log(j_n)}} \geq c_\lambda^{g_\mu} 2^{-j_n \varepsilon_n},$$

where  $\varepsilon_n$  tends to 0 as  $n$  tends to infinity. By Proposition 5.2, this shows that  $\sigma_f = \sigma_{g_\mu}$ . As a conclusion, the set of those functions  $f \in \mathbf{B}^\mu(\mathbb{R}^d)$  satisfying  $\sigma_f = \sigma_{g_\mu}$  is Baire generic.

## 6. TWO ADDITIONAL PROPERTIES OF THE CAPACITIES $\mu \in \mathcal{E}_d$

In the next subsections, two finer properties satisfied by the capacities  $\mu \in \mathcal{E}_d$  are proved. They may look a bit disconnected from the rest of the paper and our purpose, but are key to get the full solution to the Frisch-Parisi conjecture in [12].

### 6.1. A conditioned ubiquity property associated with the elements of $\mathcal{E}_d$ .

Let  $\mu \in \mathcal{E}_d$ . In this section, we measure the size of the set of those points  $x \in \mathbb{R}^d$  which are infinitely often close to dyadic vectors  $2^{-j}k \in \mathbb{R}^d$  such that the order of magnitude of  $\mu(\lambda_{j,k})$  is  $2^{-j\alpha_{\min}}$ .

**Definition 6.1.** A dyadic vector  $2^{-j}k$ ,  $j \in \mathbb{N}$ ,  $k \in \mathbb{Z}^d$ , is irreducible when  $k \in \mathbb{Z}^d \setminus (2\mathbb{Z})^d$ .

The irreducible representation of a dyadic vector  $2^{-j}k$  with  $j \in \mathbb{N}$  and  $k \in \mathbb{Z}^d$  is the unique irreducible dyadic vector  $2^{-\bar{j}}\bar{k}$  such that  $2^{-j}k = 2^{-\bar{j}}\bar{k}$ .

If  $\lambda = 2^{-j}(k + [0, 1]^d) \in \mathcal{D}_j$ , then its associated irreducible cube is  $\bar{\lambda} := 2^{-\bar{j}}(\bar{k} + [0, 1]^d) \in \mathcal{D}_{\bar{j}}$ , where  $2^{-\bar{j}}\bar{k}$  is the irreducible representation of  $2^{-j}k$ .

Observe that  $\lambda$  is the dyadic cube of generation  $j$  located at the “bottom-left” corner of  $\bar{\lambda}$ . We can write  $\lambda = \bar{\lambda} \cdot [0, 2^{-(j-\bar{j})}]^d$ , with the notations defined in Remark 3.15.

**Definition 6.2.** For  $\delta > 1$  and  $j \geq 1$ , let  $(j)_\delta$  be the largest integer in  $\gamma(\mathbb{N}) \cap [0, j/\delta]$  (recall the definition (3.11) of the mapping  $\gamma$ ).

For any positive sequence  $\eta = (\eta_j)_{j \geq 1}$ , let us define the set

$$X_j(\delta, \eta) = \left\{ 2^{-(j)_\delta} k \in [0, 1]^d : \begin{cases} k \in \mathbb{Z}^d \setminus 2\mathbb{Z}^d, \\ \mu(2^{-(j)_\delta}(k + [0, 1]^d)) \geq 2^{-(j)_\delta(\alpha_{\min} + \eta_{(j)_\delta})}, \\ \mu(2^{-(j)_\delta} k + 2^{-j}[0, 1]^d) \geq 2^{-j(\alpha_{\min} + \eta_j)} \end{cases} \right\}.$$

Recall that by construction and (3.21),

$$\begin{aligned} \mu(2^{-(j)_\delta}(k + [0, 1]^d)) &\leq 2^{-(j)_\delta(\alpha_{\min} - \varepsilon)} \\ \mu(2^{-(j)_\delta} k + 2^{-j}[0, 1]^d) &\leq 2^{-j(\alpha_{\min} - \varepsilon)}, \end{aligned}$$

which are complementary to the inequalities used to defined  $X_j(\delta, \eta)$ . Hence,  $X_j(\delta, \eta)$  contains irreducible dyadic vectors of generation  $(j)_\delta$  whose  $\mu$ -mass is controlled both at generation  $(j)_\delta$  and at generation  $j$  by the exponent  $\alpha_{\min}$  (note that  $(j)_\delta \sim j/\delta$ ).

**Definition 6.3.** For any positive sequence  $\eta = (\eta_j)_{j \geq 1}$  and any increasing sequence of integers  $(j_n)_{n \geq 1}$ , set

$$S(\delta, \eta, (j_n)_{n \geq 1}) = \bigcap_{N \geq 1} \bigcup_{n \geq N} \bigcup_{2^{-(j_n)\delta} k \in X_{j_n}(\delta, \eta)} (2^{-(j_n)\delta} k + 2^{-j_n} [0, 1]^d).$$

An element  $y \in S(\delta, \eta, (j_n)_{n \geq 1})$  satisfies  $|y - 2^{-(j_n)\delta} k| \leq 2^{-j_n} \sim 2^{-\delta \cdot (j_n)\delta}$  for infinitely many dyadic vectors of the form  $2^{-(j_n)\delta} k \in X_{j_n}(\delta, \eta)$ : we say that  $y$  is approximated at rate  $\delta$  by the elements of the sets  $X_{j_n}(\delta, \eta)$ ,  $n \geq 1$  (around which  $\mu$ -mass is locally controlled by  $\alpha_{\min}$  at generations  $j_n$  and  $j_n(\delta)$ ).

Recall that the lower Hausdorff dimension of a Borel probability measure  $\nu$  on  $\mathbb{R}^d$  is the infimum of the Hausdorff dimension of the Borel sets of positive  $\nu$ -measure (see [21] for instance):  $\dim \nu = \inf\{\dim E : E \subset \mathbb{R}^d \text{ and } \nu(E) > 0\}$ .

**Proposition 6.4.** Let  $\mu \in \mathcal{E}_d$ , and suppose that  $\sigma_\mu(\alpha_{\min}) > 0$ .

There is a positive sequence  $\eta = (\eta_j)_{j \geq 1}$  converging to 0 when  $j \rightarrow +\infty$  such that for any  $\delta > 1$ , for any increasing sequence of integers  $(j_n)_{n \geq 1}$ , there exists a Borel probability measure  $\nu$  on  $\mathbb{R}^d$  of lower Hausdorff dimension larger than or equal to  $\sigma_\mu(\alpha_{\min})/\delta$ , and such that  $\nu(S(\delta, \eta, (j_n)_{n \geq 1})) = 1$ .

In particular,  $\dim S(\delta, \eta, (j_n)_{n \geq 1}) \geq \sigma_\mu(\alpha_{\min})/\delta$ .

Note that when  $\mu = (\mathcal{L})^s$  for some  $s > 0$ , Proposition 6.4 is already obtained in [36].

*Proof.* We treat the case  $d = 1$ ; the case  $d \geq 2$  follows from an easy adaptation. Without loss of generality, assume that  $\mu \in \mathcal{M}_1$  instead of  $\mu \in \mathcal{E}_1$ : this is obviously equivalent up to some positive power of  $\mu$ . For simplicity,  $\sigma_\mu$  is denoted by  $\sigma$ .

**Preliminary observation.** Recall the construction of  $\mu$  and Section 3.2.

**Definition 6.5.** Let  $g = \ell + \sum_{n=N_0}^{N-1} \ell_n \in \mathbb{N}^*$ , with  $N \geq N_0$  and  $1 \leq \ell \leq \ell_N$ . A real number  $x \in [0, 1]$  satisfies property  $P(\alpha_{\min}, g)$  when there exists a word  $w \in \Sigma_g$  such that  $x \in \pi([w])$  and writing  $w = J_{N_0} \cdot J_{N_0+1} \cdots J_{N-1} \cdot J$  with  $J_n = j_{n,1} \cdots j_{n,\ell_n} \in \{0, \dots, 2^n - 1\}^{\ell_n}$  for  $n \in \{N_0, \dots, N-1\}$  and  $J_N = j_{N,1} \cdots j_{N,\ell} \in \{0, \dots, 2^N - 1\}^\ell$ , then all the  $j_{n,i}$  are such that  $\beta_{n,j_{n,i}} = \alpha_{\min}$ .

It is direct to see that there exists a sequence  $(\eta_j)_{j \geq 1}$  such that for all  $x \in [0, 1]$ , for all  $g \geq 1$ , if  $x$  satisfies property  $P(\alpha_{\min}, g)$ , then for all  $1 \leq j \leq \gamma(g)$ , one has

$$\mu(\lambda_j(x)) \geq 2^{-j(\alpha_{\min} + \eta_j)}.$$

Fix such a sequence  $\eta = (\eta_j)_{j \geq 1}$ .

Fix  $\delta > 1$  and an increasing sequence of integers  $(j_n)_{n \geq 1}$ . We are going to construct a Cantor subset  $\mathcal{K}$  included in  $S(\delta, \eta, (j_n)_{n \geq 1})$  and a Borel probability measure  $\nu$  supported on  $\mathcal{K}$  such that for all closed dyadic subcubes  $\lambda$  of  $[0, 1]^d$  of generation  $j \geq 0$ , one has  $\nu(\lambda) \leq 2^{-j(\delta^{-1}\sigma(\alpha_{\min}) - \psi(j))}$ , where the function  $\psi : \mathbb{N} \rightarrow (0, +\infty)$  tends to 0 as  $n \rightarrow \infty$ . The mass distribution principle (see [20]) allows then to conclude that  $\dim S(\delta, \eta, (j_n)_{n \geq 1}) \geq \sigma(\alpha_{\min})/\delta$ .

We proceed in three steps. Notations and definitions of Section 3.6 are adopted.

**Step 1:** Construction of a family of measures  $(\nu^\lambda)_{\lambda \in \mathcal{D}}$ .

A family of auxiliary measures indexed by the closed dyadic subintervals of  $[0, 1]$  is built in a very similar way as  $\mu_{\alpha_{\min}}$  in Section 3.6.

Let us introduce a notation: for  $j \in \mathbb{N}^*$ , set

$$N(j) = \begin{cases} N_0 & \text{if } 1 \leq j \leq \ell_{N_0} N_0, \\ N & \text{if } j > \ell_{N_0} N_0 \text{ and } \gamma(\sum_{n=N_0}^{N-1} \ell_n) < j \leq \gamma(\sum_{n=N_0}^N \ell_n). \end{cases}$$

Observe that

$$(6.1) \quad \lim_{j \rightarrow +\infty} \frac{N(j)}{j} = 0.$$

Let  $N \geq N_0 + 1$ ,  $1 \leq \ell \leq \ell_N$ , and  $g = \ell + \sum_{n=N_0}^{N-1} \ell_n$ . Let  $J$  be an integer such that  $\gamma(g-1) < J \leq \gamma(g)$ . Note that  $J \geq j_0 := \ell_{N_0} N_0 + 1$ .

Fix  $\lambda \in \mathcal{D}_J$ , and construct a measure  $\nu^\lambda$  supported on  $\lambda$  as follows.

For each  $n \geq N = N(J)$ , consider

$$(6.2) \quad \mathcal{J}_{n, \alpha_{\min}} = \{j \in \{0, \dots, 2^n - 1\} : j \text{ is odd and } \beta_{n,j} = \alpha_{\min}\}.$$

Using (3.3) and (3.4), one sees that for an  $n \geq N$ ,

$$(6.3) \quad \#\mathcal{J}_{n, \alpha_{\min}} \geq 2^{n(\sigma(\alpha_{\min}) - 2\varepsilon_n)}.$$

Writing  $\lambda = K2^{-J} + 2^{-J}[0, 1]$ , denote by  $\lambda_g \subset \lambda$  the dyadic subinterval  $K2^{-J} + 2^{-\gamma(g)}[0, 1]$  and  $[w_{\lambda_g}]$  the unique cylinder such that  $\pi([w_{\lambda_g}]) = \lambda_g$ . Observe that  $[w_{\lambda_g}] \in \mathcal{C}_g$ , the set of cylinders of generation  $g$  in  $\Sigma$ . Then, consider the set

$$\Sigma^\lambda = \{w_{\lambda_g}\} \times (\mathcal{J}_{N, \alpha_{\min}})^{\ell_{N-\ell}} \times \prod_{n=N+1}^{\infty} (\mathcal{J}_{n, \alpha_{\min}})^{\ell_n} \subset \Sigma,$$

and for each  $n \geq N$  and  $w \in \Sigma_g \times \{0, \dots, 2^N - 1\}^{\ell_{N-\ell}} \times \prod_{k=N+1}^n \{0, \dots, 2^k - 1\}^{\ell_k}$  set

$$\rho^\lambda([w]) = \begin{cases} (\#\mathcal{J}_{N, \alpha_{\min}})^{-\ell_{N+\ell}} \prod_{k=N+1}^n (\#\mathcal{J}_{k, \alpha_{\min}})^{-\ell_k} & \text{if } [w] \cap \Sigma^\lambda \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

This yields an atomless measure  $\rho^\lambda$  whose support is  $\Sigma^\lambda$ . Finally, the measure  $\nu^\lambda = \rho^\lambda \circ \pi^{-1}$  is a probability measure supported on  $\lambda_g \subset [0, 1]$ .

By construction of  $\nu^\lambda$ , using (6.3), for  $g' \geq g$  and  $\lambda' \in \mathcal{D}_{\gamma(g')}$ , one has either  $\nu^\lambda(\lambda') = 0$ , or  $\lambda' \cap \pi(\Sigma^\lambda) \neq \emptyset$  and

$$\nu^\lambda(\lambda') \leq 2^{-(\gamma(g') - \gamma(g))(\sigma(\alpha_{\min}) - 2\varepsilon_{N(J)})} \leq 2^{-(\gamma(g') - J)(\sigma(\alpha_{\min}) - 2\varepsilon_{N(J)})} 2^{N(J)\sigma(\alpha_{\min})}.$$

Consequently, for every  $g' \geq g$  and  $\gamma(g') < j \leq \gamma(g' + 1)$ , for  $\lambda' \in \mathcal{D}_j$  one has

$$(6.4) \quad \nu^\lambda(\lambda') \leq 2^{-(j-J)(\sigma(\alpha_{\min}) - 2\varepsilon_{N(J)})} 2^{2N(J)\sigma(\alpha_{\min})}.$$

This inequality extends easily to all integers  $j$  such that  $J \leq j \leq \gamma(g)$  and  $\lambda' \in \mathcal{D}_j$ .

**Remark 6.6.** *By construction, since only odd integers  $j$  are considered in the definition of the sets  $\mathcal{J}_{n, \alpha_{\min}}$ , if  $\hat{\lambda} \subsetneq \lambda$  and  $\nu^\lambda(\hat{\lambda}) > 0$ , then  $\hat{\lambda} = \lambda_{\hat{j}, \hat{k}}$  with  $\hat{k}2^{-\hat{j}}$  irreducible. Moreover, writing  $\gamma(\hat{g}) < \hat{j} \leq \gamma(\hat{g} + 1)$ , if property  $P(\alpha_{\min}, g)$  of Definition 6.5 holds for all  $x \in \lambda$ , then  $P(\alpha_{\min}, \hat{g})$  holds for all  $x \in \hat{\lambda}$ .*

We finally set  $\nu^\lambda = \nu^{[0, 2^{-j_0}]}$  if  $\lambda \in \bigcup_{j=1}^{j_0-1} \mathcal{D}_j$  and  $\lambda \subset [0, 1]$ .

**Step 2:** Construction of a Cantor set  $\mathcal{K} \subset S(\delta, (\eta_j)_{j \geq 1}, (j_n)_{n \geq 1})$  and a Borel probability measure  $\nu$  supported on  $\mathcal{K}$ .

Recall that  $j_0 = N_0 \ell_{N_0} + 1$ . Define  $n_1 = 0$ ,  $\mathcal{G}_1 = \{[0, 2^{-j_0}]\}$  and a set function  $\nu$  on  $\mathcal{G}_1$  by  $\nu([0, 2^{-j_0}]) = 1$ . Note that  $\gamma(\ell_{N_0}) < j_0 \leq \gamma(\ell_{N_0} + 1)$ , and that for all  $x \in [0, 2^{-j_0}]$ , property  $P(\alpha_{\min}, \ell_{N_0})$  holds (recall Definition 6.5).

Let  $p$  be a positive integer. Suppose that  $p$  families  $\mathcal{G}_1, \dots, \mathcal{G}_p$  of closed dyadic intervals, as well as  $p$  integers  $0 = n_1 < n_2 < \dots < n_p$  are constructed such that:

- (a) for every  $k \in \{2, \dots, p\}$ ,  $(j_{n_k})_\delta \geq j_0$ ;
- (b) for every  $k \in \{2, \dots, p\}$ ,  $\mathcal{G}_k \subset \{x + 2^{-j_{n_k}}[0, 1]^d : x \in X_{j_{n_k}}(\delta, \eta)\} \subset \mathcal{D}_{j_{n_k}}$ ;
- (c) for every  $k \in \{1, \dots, p\}$ , writing  $\gamma(g_k) < j_{n_k} \leq \gamma(g_k + 1)$  for some integer  $g_k$ , every  $x \in \mathcal{G}_k$  satisfies property  $P(\alpha_{\min}, g_k)$ ;
- (d) for every  $k \in \{2, \dots, p\}$ , the irreducible intervals  $\{\bar{\lambda} : \lambda \in \mathcal{G}_k\}$  are pairwise disjoint;
- (e) for every  $k \in \{2, \dots, p\}$  and every element of  $\lambda \in \mathcal{G}_k$ , there is a unique  $\lambda^\dagger \in \mathcal{G}_{k-1}$  such that  $\lambda \subset \bar{\lambda} \subset \lambda^\dagger$ ;
- (f) the measure  $\nu$  is defined on the  $\sigma$ -algebra generated by the elements of  $\bigcup_{k=1}^p \mathcal{G}_k$  by the following formula: for all  $2 \leq k \leq p$  and  $\lambda \in \mathcal{G}_k$ ,

$$\nu(\lambda) := \nu(\lambda^\dagger) \nu^{\lambda^\dagger}(\bar{\lambda});$$

- (g) for all  $2 \leq k \leq p$  and  $\lambda \in \mathcal{G}_k$ ,

$$(6.5) \quad \nu(\lambda) \leq 2^{-j_{n_k}} \left( \delta^{-1} \sigma(\alpha_{\min}) - 3\varepsilon_N(j_{n_{k-1}}) \right).$$

Let us explain how to build  $n_{p+1}$  and  $\mathcal{G}_{p+1}$ .

Write  $\gamma(g_k) < j_{n_p} \leq \gamma(g_k + 1)$ , where  $g_k = \ell + \sum_{n=N_0}^{N-1} \ell_n \in \mathbb{N}$  with  $N \geq N_0$  and  $1 \leq \ell \leq \ell_N$ . Fix  $n_{p+1}$  so that  $\gamma(g_k + 1) \leq (j_{n_{p+1}})_\delta$  (other constraints on  $n_{p+1}$  will be given a few lines below).

Consider  $\lambda^\dagger \in \mathcal{G}_p$ . For every  $\hat{\lambda} \in \mathcal{D}_{(j_{n_{p+1}})_\delta}$  with  $\hat{\lambda} \subset \lambda^\dagger$  and  $\nu^{\lambda^\dagger}(\hat{\lambda}) > 0$ , (6.4) gives

$$\nu(\lambda^\dagger) \nu^{\lambda^\dagger}(\hat{\lambda}) \leq \nu(\lambda^\dagger) 2^{-((j_{n_{p+1}})_\delta - j_{n_p}) \sigma(\alpha_{\min}) - 2\varepsilon_N(j_{n_p})} 2^{2N((j_{n_{p+1}})_\delta) \sigma(\alpha_{\min})}.$$

By (6.5) applied to  $\nu(\lambda^\dagger)$ , and then (6.1), choosing  $n_{p+1}$  large enough yields that  $\nu(\lambda^\dagger) \nu^{\lambda^\dagger}(\hat{\lambda}) \leq 2^{-j_{n_{p+1}} (\delta^{-1} \sigma(\alpha_{\min}) - 3\varepsilon_N(j_{n_p}))}$  (the equivalence  $(j_{n_{p+1}})_\delta \sim j_{n_{p+1}}/\delta$  was used).

Further, one sets

$$(6.6) \quad \mathcal{G}_{p+1} = \bigcup_{\lambda^\dagger \in \mathcal{G}_p} \left\{ k 2^{-(j_{n_{p+1}})_\delta} + 2^{-j_{n_{p+1}}} [0, 1] : \left\{ \begin{array}{l} \hat{\lambda} = k 2^{-(j_{n_{p+1}})_\delta} + 2^{-(j_{n_{p+1}})_\delta} [0, 1] \subset \lambda^\dagger \\ \nu^{\lambda^\dagger}(\hat{\lambda}) > 0 \end{array} \right\} \right\}.$$

By construction,  $\mathcal{G}_{p+1} \subset \mathcal{D}_{j_{n_{p+1}}}$ , and each interval  $\lambda \in \mathcal{G}_{p+1}$  is the left-most interval inside the corresponding interval  $\hat{\lambda} \in \mathcal{D}_{(j_{n_{p+1}})_\delta}$ . It follows from this, (c) and Remark 6.6 that property (c) holds at generation  $p+1$  as well.

Next, for every  $\lambda \in \mathcal{G}_{p+1}$  associated with  $\hat{\lambda} \in \mathcal{D}_{(j_{n_{p+1}})_\delta}$  and  $\lambda^\dagger \in \mathcal{G}_p$ , one finally sets  $\nu(\lambda) = \nu(\lambda^\dagger) \nu^{\lambda^\dagger}(\hat{\lambda})$ .

The previous construction and the above remarks show that all the items (a)-(g) above hold with  $p+1$  as well. Finally, we define

$$\mathcal{K} = \bigcap_{p \geq 1} \bigcup_{\lambda \in \mathcal{G}_p} \lambda,$$

and the set function  $\nu$  defined on the elements of  $\bigcup_{p \geq 1} \mathcal{G}_p$  extends to a Borel probability measure on  $[0, 1]$ , whose topological support is  $\mathcal{K}$ . It is direct to check that  $\nu$  is atomless, and that due to property (d) and the preliminary observation,  $\mathcal{K} \subset S(\delta, \eta, (j_n)_{n \geq 1})$ .

**Step 3:** Let us study the Hölder properties of  $\nu$  to get a lower bound for its lower Hausdorff dimension.

Fix a closed dyadic subinterval  $\lambda$  in  $[0, 1]$  of generation  $j \geq j_{n_2}$  such that the interior of  $\lambda$  intersects  $\mathcal{K}$ . Let  $p \geq 2$  be the smallest integer such that the interior of  $\lambda$  intersects at least two elements of  $\mathcal{G}_p$ . Necessarily,  $j \leq j_{n_p}$ .

Let  $\lambda^\uparrow$  be the unique element of  $\mathcal{G}_{p-1}$  such that the interior of  $\lambda$  intersects  $\lambda^\uparrow$ . Since  $\nu$  is atomless,  $\nu(\lambda) \leq \nu(\lambda^\uparrow)$ . In addition,  $\nu(\lambda) = \nu(\lambda^\uparrow)\nu^{\lambda^\uparrow}(\widehat{\lambda})$  where  $\widehat{\lambda}$  is associated with  $\lambda$  as in (6.6). Consequently, for every  $p$ , denoting  $\varepsilon_{N(j_{n_p})}$  simply by  $\tilde{\varepsilon}_p$ , if  $j \leq j_{n_{p-1}}$  then

$$\nu(\lambda) \leq \nu(\lambda^\uparrow) \leq 2^{-j_{n_{p-1}}(\delta^{-1}\sigma(\alpha_{\min})-3\tilde{\varepsilon}_{p-2})} \leq 2^{-j(\delta^{-1}\sigma(\alpha_{\min})-3\tilde{\varepsilon}_{p-2})},$$

and if  $j > j_{n_{p-1}}$ , then by (6.4) and (6.5), one has

$$\begin{aligned} \nu(\lambda) &= \nu(\lambda^\uparrow)\nu^{\lambda^\uparrow}(\widehat{\lambda}) \\ &\leq 2^{-j_{n_{p-1}}(\delta^{-1}\sigma(\alpha_{\min})-3\tilde{\varepsilon}_{p-2})} 2^{-(j-j_{n_{p-1}})(\sigma(\alpha_{\min})-2\tilde{\varepsilon}_{p-1})} 2^{2N(j)\sigma(\alpha_{\min})} \\ &= 2^{-j(\delta^{-1}\sigma(\alpha_{\min})-\varphi(\lambda))}, \end{aligned}$$

where  $\varphi(\lambda) = 3\tilde{\varepsilon}_{p-2} + \frac{(j-j_{n_{p-1}})(\sigma(\alpha_{\min})(\delta^{-1}-1) + 3\tilde{\varepsilon}_{p-2} - 2\tilde{\varepsilon}_{p-1}) + 2N(j)\sigma(\alpha_{\min})}{j}$ .

Pay attention to the fact that in the formula above,  $p$  depends a priori on  $\lambda$  and  $j$ . However, this dependence can be uniformly controlled. Indeed, observe that  $\varphi(\lambda) \leq 6\tilde{\varepsilon}_{p-2} + \frac{2N(j)\sigma(\alpha_{\min})}{j}$ , and that when  $j$  tends to  $+\infty$ ,

$\min\{p \geq 2 : \exists \lambda \in \mathcal{D}_j \text{ such that the interior of } \lambda \text{ intersects at least 2 elements of } \mathcal{G}_p\}$

also tends to  $+\infty$ . Consequently,  $\tilde{\varepsilon}_{p-2}$  converges uniformly to 0 over  $\{\lambda \in \mathcal{D}_j, \text{Int}(\lambda) \cap \mathcal{K} \neq \emptyset\}$  as  $j \rightarrow +\infty$ . Thus, remembering that (6.1) holds as well, one concludes that there exists a function  $\psi : \mathbb{N} \rightarrow (0, +\infty)$  such that  $\lim_{j \rightarrow +\infty} \psi(j) = 0$  and

$$\text{for every } \lambda \in \mathcal{D}_j^0, \quad \nu(\lambda) \leq 2^{-j(\delta^{-1}\sigma(\alpha_{\min})-\psi(j))}.$$

In particular the lower Hausdorff dimension of  $\nu$  is greater than  $\sigma(\alpha_{\min})/\delta$ . Since  $\mathcal{K} \subset S(\delta, \eta, (j_n)_{n \geq 1})$ ,  $\nu(\mathcal{K}) = 1$ , we get  $\dim S(\delta, \eta, (j_n)_{n \geq 1}) \geq \delta^{-1}\sigma(\alpha_{\min})$ , and the conclusions of Proposition 6.4 holds in dimension 1.  $\square$

## 6.2. The set of badly approximated points supports the auxiliary measures $\mu_\alpha$ .

The measures  $\mu_\alpha$  described in Proposition 3.13 are supported on the set of points which are badly approximated by dyadic vectors, as stated by the following lemma. This property is key for the study of typical singularity spectra in [12].

**Proposition 6.7.** *Let  $\mu \in \mathcal{E}_d$ . For every  $x \in [0, 1]^d$ , call  $\overline{\lambda_j(x)} \in \mathcal{D}_{\frac{1}{j(x)}}$  the irreducible representation of  $\lambda_j(x)$ . For every  $\alpha \in [\alpha_{\min}, \alpha_{\max}]$  such that  $\tau_\mu^*(\alpha) > 0$ , for  $\mu_\alpha$ -almost every  $x$ , one has  $\lim_{n \rightarrow +\infty} \frac{j_n(x)}{j_n} = 1$ .*

*Proof.* Fix  $\alpha \in [\alpha_{\min}, \alpha_{\max}]$  and  $\delta > 1$ . For  $j \in \mathbb{N}^*$ , let  $E_\mu(\alpha, \delta, j) = \{x \in E_\mu(\alpha) : \frac{j(x)}{j} \leq \delta^{-1}\}$  and

$$E_\mu(\alpha, \delta) := \left\{x \in E_\mu(\alpha) : \liminf_{j \rightarrow +\infty} \frac{j(x)}{j} \leq \delta^{-1}\right\} = \limsup_{j \rightarrow +\infty} E_\mu(\alpha, \delta, j).$$

For  $\varepsilon > 0$ , let

$$F_\mu(\alpha, j, \varepsilon) = \{x \in [0, 1]^d : \forall j' \geq j, 2^{-j'(\alpha+\varepsilon)} \leq \mu(\lambda_{j'}(x)) \leq 2^{-j'(\alpha-\varepsilon)}\}.$$

Setting  $j_\delta = \lfloor j/\delta \rfloor$ , the following inclusion holds :

$$E_\mu(\alpha, \delta) \subset \bigcap_{\varepsilon > 0} \bigcap_{J \geq 1} \bigcup_{j \geq J} \bigcup_{\substack{\lambda_{j_\delta, k} \in \mathcal{D}_{j_\delta} : \\ \lambda_{j_\delta, k} \cap F_\mu(\alpha, j_\delta, \varepsilon) \neq \emptyset}} B(2^{-j_\delta} k, 2^{-j}).$$

Using Proposition 3.3(1) or (4), for every fixed  $\varepsilon > 0$ , one sees that the cardinality of  $\{\lambda_{j_\delta, k} \in \mathcal{D}_{j_\delta} : \lambda_{j_\delta, k} \cap F_\mu(\alpha, j_\delta, \varepsilon) \neq \emptyset\}$  is less than  $2^{j_\delta(\tau_\mu^*(\alpha)+\varepsilon)}$  when  $j$  is large.

Combining this with the previous embedding, coverings of  $E_\mu(\alpha, \delta)$  are obtained using sets of the form  $\bigcup_{j \geq J} \bigcup_{\lambda_{j_\delta, k} \in \mathcal{D}_{j_\delta} : \lambda_{j_\delta, k} \cap F_\mu(\alpha, j_\delta, \varepsilon) \neq \emptyset} B(2^{-j_\delta} k, 2^{-j})$ , and it is easily seen that  $\dim E_\mu(\alpha, \delta) \leq \tau_\mu^*(\alpha)/\delta$ . This implies that  $\mu_\alpha(E_\mu(\alpha, \delta)) = 0$ , again because  $\mu_\alpha$  may give a positive mass to a set  $E$  only when  $\dim E \geq \tau_\mu^*(\alpha)$ . Since this holds for all  $\delta > 1$ ,  $\liminf_{j \rightarrow \infty} \frac{j(x)}{j} = 1$  for  $\mu_\alpha$ -almost every  $x$ , and in particular  $\lim_{n \rightarrow \infty} \frac{j_n(x)}{j_n} = 1$ .  $\square$

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