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# BESOV SPACES IN MULTIFRACTAL ENVIRONMENT AND THE FRISCH-PARISI CONJECTURE

JULIEN BARRAL AND STÉPHANE SEURET

ABSTRACT. We give a solution to the so-called Frisch-Parisi conjecture by constructing a Baire functional space in which typical functions satisfy a multifractal formalism, with a prescribed singularity spectrum. This achievement combines three ingredients developed in this paper. First we prove the existence of almost-doubling fully supported Radon measure on  $\mathbb{R}^d$  with a prescribed multifractal spectrum. Second we define new *heterogeneous* Besov like spaces possessing a wavelet characterization; this uses the previous doubling measures. Finally, we fully describe the multifractal nature of typical functions in these functional spaces.

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## 1. INTRODUCTION

This paper deals with multifractal analysis of functions, which originates from the first geometric quantification of the Hölder singularities structure in fully developed turbulence [44, 43, 23]. This subject is an instance of the natural concept of multifractality, which comes into play as soon as given a mapping  $h : X \rightarrow A$  between a metric space  $(X, d)$  and a set  $A$ , one describes geometrically the level sets of  $h$  by considering the mapping  $\sigma : \alpha \in A \mapsto \dim h^{-1}(\{\alpha\})$ , where  $\dim$  stands for the Hausdorff dimension. Indeed, in many interesting situations, the non empty level sets of  $h$  form an uncountable family of fractal sets, and  $\sigma$  is sometimes called *multifractal spectrum*. When non constant, this spectrum provides a hierarchy between these level sets, according to their sizes measured by their Hausdorff dimensions. Such spectra have been considered in many mathematical fields, such as harmonic and functional analysis (in the description of fine properties of Fourier series [31, 12] or typical elements in functional spaces [14, 34]), probability theory (to describe fine properties of Brownian motion or SLE curves [49, 51, 59, 24]), multiplicative chaos and Gaussian free field, random covering problems [8, 30, 55, 4]), ergodic theory, dynamical and iterated function systems (in the multifractal analysis of Gibbs measures such as the harmonic measure on conformal repellers, Birkhoff averages, and self-similar measures [53, 42, 21, 22, 57]), metric number theory (Diophantine approximation and ubiquity theory [38, 28, 10]), shrinking targets problems and dynamical covering problems [27, 20]), the previous references being far from exhaustive.

In the multifractal analysis of a real valued function  $f \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ , the function  $h$  of interest is the pointwise Hölder exponent function  $h_f$ , which is defined as follows. Given  $x_0 \in \mathbb{R}^d$ , and  $H \in \mathbb{R}_+$ ,  $f$  is said to belong to  $\mathcal{C}^H(x_0)$  if there exist a polynomial  $P$  of degree at most  $\lfloor H \rfloor$ , a constant  $C > 0$ , and a neighborhood  $V$  of  $x_0$  such that

$$\forall x \in V, \quad |f(x) - P(x - x_0)| \leq C|x - x_0|^H.$$

The *pointwise Hölder exponent* of  $f \in L_{\text{loc}}^\infty(\mathbb{R}^d)$  at  $x_0$  is

$$(1) \quad h_f(x_0) = \sup \{H \in \mathbb{R}_+ : f \in \mathcal{C}^H(x_0)\},$$

and  $f$  is said to have a Hölder singularity of order  $h_f(x_0)$  at  $x_0$ .

The associated spectrum, called *singularity spectrum* of  $f$ , is the mapping

$$\sigma_f : H \in \mathbb{R} \cup \{\infty\} \mapsto \dim E_f(H) \in [0, d] \cup \{-\infty\}, \text{ where } E_f(H) := h_f^{-1}(\{H\})$$

(note that  $E_f(H) = \emptyset$  for  $H < 0$ ). Again,  $\dim$  stands for the Hausdorff dimension, with the convention  $\dim \emptyset = -\infty$ . The function  $f$  is said to be *multifractal* when  $E_f(H) \neq \emptyset$  for at least two values of  $H$ .

The idea of considering this spectrum is due to the physicists Frisch and Parisi [23], who aimed at quantifying geometrically the local variations of the velocity field of a turbulent fluid, and introduced the term *multifractal*. Another fundamental idea pointed out by Frisch and Parisi consisted in coupling with the singularity spectrum a large deviations approach, in order to statistically describe the Hölder singularities distribution (in Mandelbrot's spirit for measures [44]). This led to the so-called multifractal formalisms for functions. Since defining rigorously such a formalism is a little involved and will be done later in Section 2, let us say at the moment that schematically, in such a formalism, the singularity spectrum  $\sigma_f$  of a Hölder continuous function  $f$  is always dominated by (and in good cases, coincides with) the Legendre-Fenchel transform

$$\zeta_f^*(H) := \inf_{q \in \mathbb{R}} Hq - \zeta_f(q)$$

of a function  $\zeta_f : \mathbb{R} \rightarrow \mathbb{R}$ , called the *scaling function* or the  $L^q$ -*spectrum* of  $f$ :  $\sigma_f \leq \zeta_f^*$ . The mapping  $\zeta_f$  is a kind of free energy function encapsulating the asymptotic statistical distribution of the Hölder singularities as the observation scale tends to 0, and it can be numerically estimated. For instance, in their seminal article, Frisch and Parisi used for  $\zeta_f$  the scaling exponent of the moments of the increments of  $f$ , informally defined as

$$|h|^{-d} \int_{\Omega} |f(x+h) - f(x)|^q dx \sim |h|^{\zeta_f(q)} \quad \text{as } h \rightarrow 0,$$

where  $\Omega$  is a fixed bounded domain on which  $f$  is supposed to be fully supported. The heuristics developed in [23] lead to seek for the largest as possible classes of functions for which the equality

$$(2) \quad \sigma_f(H) = \zeta_f^*(H)$$

holds at any  $H$  such that  $\zeta_f^*(H) \geq 0$ . In such a situation, one says that *the multifractal formalism holds* for  $f$ , or that  $f$  satisfies the multifractal formalism. Then, the spectrum  $\sigma_f$  is a continuous concave map with support included in  $(0, \infty)$ , and assuming that the topological support of  $f$  is full, one necessarily has  $\sigma_f(H) = d = -\zeta_f(0)$  for some  $H \geq 0$  (for instance the level set  $E_f(H)$  may have a positive Lebesgue measure).

We will come back to rigorous definitions of multifractal formalisms for functions and measures in Sections 2.5 and 3. The concept of multifractal formalism motivated many works in geometric measure theory [13, 47, 40, 41], dynamical systems

in connection with the thermodynamic formalism [52], and analysis [32, 34, 35]. It provides a powerful framework to describe the fine geometric structure of invariant measures of some dynamical systems [18, 54, 52] and the closely related self-similar and self-affine measures [39, 47, 48, 40, 22, 5], self-similar functions [32], as well as limit measures or functions in multiplicative chaos theory [29, 8, 7]. The singularity spectrum and its suitable extensions to non bounded functions have also been used to describe the geometry of celebrated functions like Riemann’s and Brjuno’s functions [31, 56, 36], stochastic processes like Lévy processes and general classes of Markov processes [33, 6, 60], as well as Lévy processes in multifractal time [9].

Multifractal formalisms are also relevant in some applications, due to the existence of stable algorithms that precisely estimate scaling functions  $\zeta_f$  of numerical data. Then, a key observation is that for most of real-life data associated to intermittent phenomena, the associated estimated singularity spectra  $\zeta_f^*$  have a characteristic strictly concave bell shape (see [1] and Figure 1). This is also the case for the singularity spectra of important classes of functions possessing scaling properties [32, 9, 7]. This behavior is in striking contrast to the results established for typical functions in some classical functional spaces, where “typical” is meant in the sense of Baire categories<sup>1</sup>. Indeed, it has been proved that typical increasing real functions (Buczolich&Nagy [14]), typical functions in some Sobolev and Besov spaces (Jaffard [34], Jaffard&Meyer [37]), and typical measures (Buczolich&Seuret, Bayart [15, 11]) satisfy a multifractal formalism but possess an affine increasing singularity spectrum. One can conclude that, from the view point of multifractals, classical function spaces do not provide “realistic” typical elements. A precise statement regarding the typical spectrum in Besov spaces is recalled in Sections 2.4 (Theorem 2.18), while the validity of some multifractal formalism in these spaces is discussed in Section 2.5 (see also Figure 2).

On the other hand, the previous genericity results show that many multifractal functions do satisfy some multifractal formalism without assuming any scale invariance properties. In [34], Jaffard seeks for Baire topological spaces of functions in which typical functions have a prescribed singularity spectrum, and do obey some multifractal formalism. He gives this inverse problem the name “Frisch-Parisi conjecture”, and provides a partial solution to it: he considers intersections of homogeneous Besov spaces and gets Baire topological spaces in which typical functions possess an increasing compactly supported singularity spectrum, with a prescribed concave part, and another part which is necessarily linear; moreover, typical elements partially obey some multifractal formalism (see Section 2.6 for a detailed description of Jaffard’s result). Again, no scale invariance is assumed.

In order to give a flavour of our results, we need to formulate more precisely the inverse problem in what consists *Frisch-Parisi conjecture* as considered by Jaffard:

**Conjecture 1.1** (Frisch-Parisi conjecture). *Let  $\mathcal{S}_d$  be the set of functions  $\sigma : \mathbb{R} \rightarrow [0, d] \cup \{-\infty\}$  such that  $\sigma$  is concave, continuous, with compact support included in*

<sup>1</sup>Recall that in a Baire topological space  $E$ , a property  $\mathcal{P}$  is called typical, or generic, when the set  $\{f \in E : f \text{ satisfies } \mathcal{P}\}$  is of second category in  $E$ , or equivalently is a dense  $G_\delta$ -set, that is the intersection of a countable family of dense and open sets. One says that typical elements in  $E$  satisfy  $\mathcal{P}$  when  $\mathcal{P}$  is typical in  $E$ .

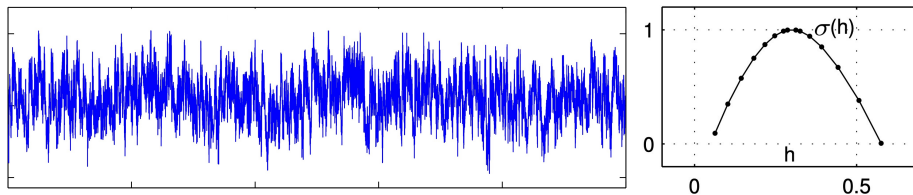


FIGURE 1. Estimated multifractal spectrum (right) for the 1D velocity of a turbulent flow (left) - Credit to P. Abry, H. Wendt

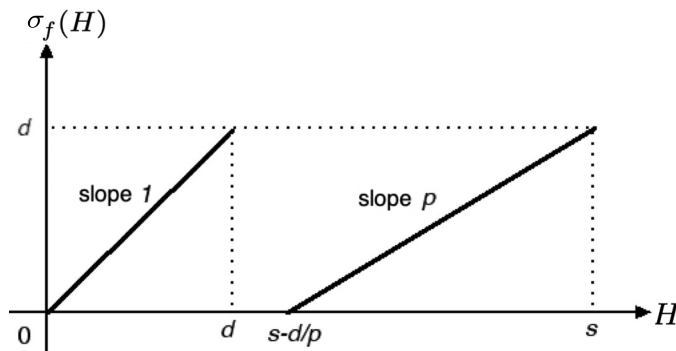


FIGURE 2. Typical multifractal spectrum of probability measures (left) or functions in  $B_q^{s,p}(\mathbb{R}^d)$  when  $s > d/p$  (right).

$(0, \infty)$  and whose maximum equals  $d$ . For every  $\sigma \in \mathcal{S}_d$ , there exists a Baire functional space of functions defined on  $\mathbb{R}^d$  in which any element  $f$  in a residual set satisfies the following properties: (i)  $\sigma_f = \sigma$ ; (ii)  $f$  obeys some multifractal formalism.

Note that the set  $\mathcal{S}_d$  consists of those mappings  $\sigma$  which are admissible to be the singularity spectrum of some Hölder continuous function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  whose pointwise Hölder exponents range in a compact subinterval of  $(0, \infty)$ , such that  $\dim E_h(H) = d$  for at least one exponent  $H$ , and which satisfies some multifractal formalism. The multifractal formalism for functions adopted in this paper will be specified in Section 2.5. It is based on the multifractal formalism associated with the so-called wavelet leaders, and developed by Jaffard in particular in [35].

In the present paper, we introduce Baire function spaces in which typical functions have the expected bell-shape singularity spectrum, and satisfy the multifractal formalism mentioned above. This construction follows from three ingredients developed in this paper, each of them having its own interest.

First we prove the existence of almost-doubling and  $\mathbb{Z}^d$ -invariant Radon measures fully supported on  $\mathbb{R}^d$  with prescribed singularity spectrum, and which satisfy the multifractal formalisms for measures developed in [13, 47] (Theorem 2.10 and Corollary 2.10). Up to now, such a result was only known for measures supported on a totally disconnected set [2] (see also [16] for results on the prescription of the singularity spectrum for measures). These measures possess scaling like properties.

Second, we introduce new functional spaces  $B_q^{\mu,p}(\mathbb{R}^d)$  that we call Besov spaces in *multifractal environment*, whose definition is based on a modification of the usual notion of  $L^p$ -moduli of smoothness. These spaces depend on an almost-doubling capacity  $\mu$ , that we call *environment*. Then, we study the wavelet decomposition of functions belonging to  $B_q^{\mu,p}(\mathbb{R}^d)$ , and prove that the intersection of suitable perturbations of the space  $B_q^{\mu,p}(\mathbb{R}^d)$  define a Fréchet space  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  very nicely characterized in terms of wavelet coefficients (see Definition 2.14 and Theorem 2.16).

Finally, thanks to the previous wavelet characterization, we perform the multifractal analysis of typical functions in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ , when the environment  $\mu$  is a positive power of one of the almost doubling measures we built before.

As a by-product of the previous results, using the spaces  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  with suitable parameters  $\mu$ ,  $p$  and  $q$ , we obtain the following theorem:

**Theorem 1.2.** *Conjecture 1.1 is true.*

It is worth noting that scaling like properties play a role via  $\mu$  in this solution, but that typical functions do not possess such properties, though they inherit their multifractal structure from  $\mu$ .

We describe precisely our three main results in the next section.

## 2. STATEMENTS OF THE MAIN RESULTS

**2.1. Some notations and definitions.** The set of non negative (resp. positive) integers is denoted by  $\mathbb{N}$  (resp.  $\mathbb{N}^*$ ), and the set of non negative real numbers and positive (resp. negative) real numbers are respectively denoted by  $\mathbb{R}_+$  and  $\mathbb{R}_+^*$  (resp.  $\mathbb{R}_-^*$ ).

If  $E$  is a Borel subset of  $\mathbb{R}^d$ , the Borel  $\sigma$ -algebra of  $E$  is denoted  $\mathcal{B}(E)$ .

For  $j \in \mathbb{Z}$ ,  $\mathcal{D}_j$  stands for the collection of closed dyadic cubes of generation  $j$ , i.e. the cubes  $\lambda_{j,k} = 2^{-j}k + 2^{-j}[0, 1]^d$ , where  $k \in \mathbb{Z}^d$ . We also set  $\mathcal{D} = \bigcup_{j \in \mathbb{Z}} \mathcal{D}_j$ , and if  $\lambda = \lambda_{j,k} \in \mathcal{D}_j$  we denote  $2^{-j}k$  by  $x_\lambda$ .

For  $j \in \mathbb{Z}$ ,  $\lambda \in \mathcal{D}_j$ , and  $N \in \mathbb{N}^*$ ,  $N\lambda$  denotes the cube with same center as  $\lambda$  and radius equal to  $N \cdot 2^{-j-1}$  in  $(\mathbb{R}^d, \|\cdot\|_\infty)$ . For instance,  $3\lambda$  is the union of those  $\lambda' \in \mathcal{D}_j$  such that  $\partial\lambda \cap \partial\lambda' \neq \emptyset$ .

For  $x \in \mathbb{R}^d$ ,  $\lambda_j(x)$  stands for the closure of the unique “semi-open to the right” dyadic cube of generation  $j$  containing  $x$ .

Given  $x \in \mathbb{R}^d$  and  $r \in \mathbb{R}_+$ , the closed Euclidean ball centered at  $x$  with radius  $r$  is denoted  $B(x, r)$ . If  $E \subset \mathbb{R}^d$ ,  $|E|$  stands for the Euclidean diameter of  $E$ .

The Lebesgue measure on  $\mathbb{R}^d$  is denoted by  $\mathcal{L}^d$ , the set of Borel subsets of  $\mathbb{R}^d$  is denoted by  $\mathcal{B}(\mathbb{R}^d)$ .

The domain of a function  $g : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$  is defined as  $g^{-1}(\mathbb{R})$ , and denoted by  $\text{dom}(g)$ . If  $g$  is concave, one sets  $g'(\infty) = \lim_{t \rightarrow \infty} g'(t^+)$  and  $g'(-\infty) = \lim_{t \rightarrow -\infty} g'(t^+)$ .

The family of Hölder-Zygmund spaces is denoted  $\{\mathcal{C}^s(\mathbb{R}^d)\}_{s>0}$  (see [45, 58] for instance for thorough expositions of classical functional spaces).

**Definition 2.1.** *The set of Hölder functions on  $\mathcal{B}(\mathbb{R}^d)$  is defined as*

$$(3) \quad \mathcal{H}(\mathbb{R}^d) = \{\mu : \mathcal{B}(\mathbb{R}^d) \rightarrow \mathbb{R}_+ \cup \{\infty\} : \exists C, s > 0, \forall E \subset \mathbb{R}^d, \mu(E) \leq C|E|^s\}.$$

*Then, the set of Hölder capacities is defined as*

$$(4) \quad \mathcal{C}(\mathbb{R}^d) = \{\mu \in \mathcal{H}(\mathbb{R}^d) : \forall E, F \in \mathcal{B}(\mathbb{R}^d), E \subset F \Rightarrow \mu(E) \leq \mu(F)\}.$$

*and the set of Hölder Radon measures is defined as*

$$(5) \quad \mathcal{M}(\mathbb{R}^d) = \{\mu \in \mathcal{C}(\mathbb{R}^d) : \mu \text{ is a Radon measure}\}.$$

*The topological support  $\text{supp}(\mu)$  of  $\mu \in \mathcal{H}(\mathbb{R}^d)$  is the set of points  $x \in \mathbb{R}^d$  for which  $\mu(B(x, r)) > 0$  for every  $r > 0$ . We say that  $\mu$  is fully supported when  $\text{supp}(\mu) = \mathbb{R}^d$ .*

We also consider the sets  $\mathcal{H}([0, 1]^d)$ ,  $\mathcal{C}([0, 1]^d)$  and  $\mathcal{M}([0, 1]^d)$  consisting of set functions defined on Borel subsets of  $[0, 1]^d$ , by replacing  $\mathbb{R}^d$  by  $[0, 1]^d$  in the above definitions.

**Definition 2.2.** *For  $s > 0$ , a set function  $\mu \in \mathcal{H}(\mathbb{R}^d)$  is  $s$ -Hölder when there exists  $C > 0$  such that  $\mu(E) \leq C|E|^s$  for all  $E \in \mathcal{B}(\mathbb{R}^d)$ .*

*Then, for  $\mu \in \mathcal{H}(\mathbb{R}^d)$ ,  $s > 0$ , and  $E \in \mathbb{R}^d$ , define*

$$\mu^s(E) = \mu(E)^s \quad \text{and} \quad \mu^{(+s)}(E) = \mu(E)|E|^s$$

*and if  $\mu$  is  $s_0$ -Hölder, then for all  $s \in (0, s_0)$ , define*

$$\mu^{(-s)}(E) = \begin{cases} 0 & \text{if } |E| = 0, \\ \mu(E)|E|^{-s} & \text{if } 0 < |E| < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

Starting from  $\mu \in \mathcal{H}(\mathbb{R}^d)$ ,  $\mu^s$ ,  $\mu^{(+s)}$  and  $\mu^{(-s)}$  as defined above still belong to  $\mathcal{H}(\mathbb{R}^d)$ .

**2.2. Almost-doubling measures with prescribed multifractal behavior.** Multifractal formalisms for measures take their origin in works by physicists who proposed to characterize “strange sets” by considering, for any invariant probability measure  $\mu$  on such a set  $S$ , the partition of  $S$  into iso-Hölder sets of  $\mu$ . They further estimated the “fractal” dimensions of these sets using the Legendre transform of some free energy function, the  $L^q$ -spectrum, closely related to the generalized dimensions due to Renyi [26, 25]. Their ideas were later rigorously formalized by mathematicians (see, e.g. [13, 40, 47]).

The local behavior of elements of  $\mathcal{H}([0, 1]^d)$  will be described via their pointwise Hölder exponents, also called local dimensions in the case of measures.

**Definition 2.3.** *Let  $\mu \in \mathcal{H}([0, 1]^d)$ . For  $x \in \text{supp}(\mu)$ , we define the lower and upper pointwise Hölder exponents of  $\mu$  at  $x$  as*

$$\underline{h}_\mu(x) = \liminf_{j \rightarrow \infty} \frac{\log_2 \mu(\lambda_j(x))}{-j} \quad \text{and} \quad \bar{h}_\mu(x) = \limsup_{j \rightarrow \infty} \frac{\log_2 \mu(\lambda_j(x))}{-j}$$

*respectively. Whenever  $\underline{h}_\mu(x) = \bar{h}_\mu(x)$ , we denote this limit by  $h_\mu(x)$ .*



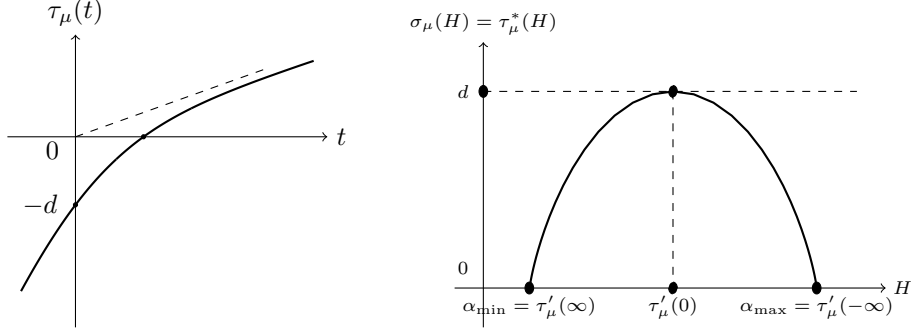


FIGURE 3. **Left:** Free energy function of  $\mu \in \mathcal{C}([0, 1]^d)$  satisfying the multifractal formalism. **Right:** The singularity spectrum of  $\mu$ .

For  $\alpha \in \mathbb{R}$ , we set

$$\begin{aligned} \underline{E}_\mu(\alpha) &= \{x \in \text{supp}(\mu) : \underline{h}_\mu(x) = \alpha\} \\ \overline{E}_\mu(\alpha) &= \{x \in \text{supp}(\mu) : \overline{h}_\mu(x) = \alpha\}, \\ E_\mu(\alpha) &= \underline{E}_\mu(\alpha) \cap \overline{E}_\mu(\alpha). \end{aligned}$$

The singularity (or multifractal) spectrum of  $\mu$  is then the mapping

$$\sigma_\mu : \alpha \in \mathbb{R} \mapsto \dim \underline{E}_\mu(\alpha).$$

**Definition 2.4.** The  $L^q$ -spectrum of  $\mu \in \mathcal{H}([0, 1]^d)$  with  $\text{supp}(\mu) \neq \emptyset$  is defined by

$$\tau_\mu : q \in \mathbb{R} \mapsto \lim_{j \rightarrow \infty} -\frac{1}{j} \log_2 \sum_{\substack{\lambda \in \mathcal{D}_j, \lambda \subset [0, 1]^d, \\ \mu(\lambda) > 0}} \mu(\lambda)^q.$$

Then, one always has (see [13, 41])

$$\sigma_\mu(\alpha) \leq \tau_\mu^*(\alpha) := \inf_{q \in \mathbb{R}} q\alpha - \tau_\mu(q).$$

In particular, if  $\mu \in \mathcal{M}([0, 1]^d)$ , since  $\tau_\mu(1) = 0$ , one has  $\sigma_\mu(\alpha) \leq \alpha$  for every  $\alpha \in \mathbb{R}$ .

**Definition 2.5.** A function  $\mu \in \mathcal{H}([0, 1]^d)$  such that  $\text{supp}(\mu) \neq \emptyset$  is said to obey the multifractal formalism over an interval  $I \subset \mathbb{R}$  when

$$(6) \quad \sigma_\mu(\alpha) = \tau_\mu^*(\alpha)$$

for all  $\alpha \in I$ . It is said to strongly obey the multifractal formalism over  $I$  when (6) still holds for all  $\alpha \in I$  after one replaced  $\underline{E}_\mu(\alpha)$  by  $E_\mu(\alpha)$  in the definition of  $\sigma_\mu$ . If  $I = \mathbb{R}$ , one simply says that the multifractal formalism holds for  $\mu$ , and that it holds strongly if one considers the sets  $E_\mu(\alpha)$ .

**Remark 2.6.** Note that one can alternatively define the lower and upper pointwise Hölder exponents at  $x \in [0, 1]^d$  in the following ways, which are equivalent:

$$\underline{h}_\mu(x) = \liminf_{r \rightarrow 0^+} \frac{\log \mu(B(x, r))}{\log(r)} \quad \text{and} \quad \overline{h}_\mu(x) = \limsup_{r \rightarrow 0^+} \frac{\log \mu(B(x, r))}{\log(r)},$$

or

$$\underline{h}_\mu(x) = \liminf_{j \rightarrow \infty} \frac{\log_2 \mu(3\lambda_j(x))}{-j} \quad \text{and} \quad \bar{h}_\mu(x) = \limsup_{j \rightarrow \infty} \frac{\log_2 \mu(3\lambda_j(x))}{-j}$$

(after defining  $\mu(A) = \mu(A \cap [0, 1]^d)$  if  $A \in \mathcal{B}(\mathbb{R}^d)$ ). In this case one naturally considers  $\mu(3\lambda)$  instead of  $\mu(\lambda)$  in the definition of the  $L^q$ -spectrum. However, in this paper we will mainly consider doubling or “almost doubling” capacities for which all the previous notions of exponents, level sets, singularity spectrum and  $L^q$ -spectrum do not depend on whether dyadic cubes or centered balls are considered.

When  $\mu \in \mathcal{M}([0, 1]^d)$ , it is known [40, 2] that  $\tau'_\mu(-\infty) < \infty$  if and only if  $\tau_\mu$  is finite in a neighborhood of  $0^-$ , and in this case  $\tau_\mu : \mathbb{R} \rightarrow \mathbb{R}$  is a non-decreasing, concave map with  $\tau_\mu(1) = 0$ . If, in addition,  $\mu$  has full support in  $[0, 1]^d$ , then  $\tau_\mu(0) = -d$ , and  $\tau_\mu^*$  reaches its maximum, equal to  $d$ , exactly over the interval  $[\tau'_\mu(0)^-, \tau'_\mu(0)^+]$ . Moreover,

$$\text{dom}(\tau_\mu^*) = [\tau'_\mu(\infty), \tau'_\mu(-\infty)] = \{\alpha \in \mathbb{R} : \tau_\mu^*(\alpha) \geq 0\}.$$

**Definition 2.7.** Let  $\mathcal{T}_{d,\mathcal{M}}$  be the set of concave increasing functions  $\tau : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\tau(1) = 0$ ,  $\tau(0) = -d$  and  $\text{dom}(\tau^*)$  is a compact subset of  $(0, \infty)$ .

Let  $\mathcal{S}_{d,\mathcal{M}}$  be the set of functions  $\sigma : \mathbb{R} \rightarrow [0, d] \cup \{-\infty\}$  such that  $\sigma$  is compactly supported with support included in  $(0, \infty)$ , concave, continuous,  $\sigma \leq \text{Id}_{\mathbb{R}}$  and there exist two exponents  $D, D' > 0$  such that  $\sigma(D) = D$  and  $\sigma(D') = d$ .

The set  $\mathcal{T}_{d,\mathcal{M}}$  is the class of admissible  $L^q$ -spectra associated with measures fully supported on  $[0, 1]^d$  that we will consider, and  $\mathcal{S}_{d,\mathcal{M}}$  is the class of admissible singularity spectra for measures strongly obeying the multifractal formalism with an  $L^q$ -spectrum in  $\mathcal{T}_{d,\mathcal{M}}$ . One easily checks that these two sets  $\mathcal{S}_{d,\mathcal{M}}$  and  $\mathcal{T}_{d,\mathcal{M}}$  are Legendre transforms of each other.

Note that  $\mathcal{S}_{d,\mathcal{M}}$  is similar to the set  $\mathcal{S}_d$  introduced in Conjecture 1.1, except that it imposes the additional conditions that  $\sigma \leq \text{Id}_{\mathbb{R}}$ , which is necessary to be the singularity spectrum of a Radon measure, and the existence of two exponents  $D, D' > 0$  such that  $\sigma(D) = D$  and  $\sigma(D') = d$ , which is necessary to be the singularity spectrum of a fully supported measure obeying the multifractal formalism (see Remark 3.5 in Section 3.1 for justifications of these facts). Observe also that,  $\mathcal{S}_d$  being defined in Conjecture 1.1,

$$\mathcal{S}_d = \{\sigma(s \cdot) : \sigma \in \mathcal{S}_{d,\mathcal{M}}, s > 0\}.$$

Given  $\sigma \in \mathcal{S}_d$ , it is natural to investigate the possibility to find a fully supported  $\mu \in \mathcal{M}([0, 1]^d)$  such that  $\mu$  obeys the multifractal formalism and satisfies  $\sigma_f = \sigma$ . We give a positive answer to this question. The measures solving the problem possess additional properties introduced now.

**Definition 2.8.** A capacity  $\mu \in \mathcal{C}(\mathbb{R}^d)$  is said to be almost doubling if there exists a non decreasing mapping  $\theta : \mathbb{N} \rightarrow \mathbb{R}_+$  with  $\lim_{j \rightarrow \infty} \frac{\theta(j)}{j} = 0$  such that

$$(7) \quad \text{for all } x \in \text{supp}(\mu) \text{ and } j \in \mathbb{N}^*, \quad \mu(3\lambda_j(x)) \leq e^{\theta(j)} \mu(\lambda_j(x)).$$

Equivalently, there is a mapping  $\theta : (0, 1] \rightarrow \mathbb{R}_+$  such that  $\lim_{r \rightarrow 0^+} \frac{\theta(r)}{\log(r)} = 0$  and for all  $x \in \text{supp}(\mu)$  and  $r \in (0, 1]$  one has

$$\mu(B(x, 2r)) \leq e^{\theta(r)} \mu(B(x, r)).$$

Also, when  $\theta$  is constant, the capacity  $\mu$  is doubling in the ‘‘classical’’ meaning.

**Definition 2.9.** *Let  $\Theta$  be the set of non decreasing functions  $\theta : \mathbb{N} \rightarrow \mathbb{R}_+$  such that  $\lim_{j \rightarrow \infty} \frac{\theta(j)}{j} = 0$ . A function  $\mu \in \mathcal{H}(\mathbb{R}^d)$  satisfies property (P) if there exist  $C, r_1, r_2 > 0$  such that:*

(P<sub>1</sub>) for all  $j \in \mathbb{N}$  and  $\lambda \in \mathcal{D}_j$ ,

$$(8) \quad C^{-1}2^{-jr_2} \leq \mu(\lambda) \leq C2^{-jr_1}.$$

(P<sub>2</sub>) There exists  $\theta \in \Theta$  such that for all  $j, j' \in \mathbb{N}$  with  $j' \geq j$ , and all  $\lambda, \tilde{\lambda} \in \mathcal{D}_j$  such that  $\partial\lambda \cap \partial\tilde{\lambda} \neq \emptyset$ , and  $\lambda' \in \mathcal{D}_{j'}$  such that  $\lambda' \subset \lambda$ :

$$(9) \quad C^{-1}2^{-\theta(j)}2^{(j'-j)r_1}\mu(\lambda') \leq \mu(\tilde{\lambda}) \leq C2^{\theta(j)}2^{(j'-j)r_2}\mu(\lambda').$$

For  $\mu \in \mathcal{P}(\mathbb{R}^d)$ , (P<sub>1</sub>) is a uniform Hölder control, from above and below, of  $\mu$ , and (P<sub>2</sub>) is a rescaled version of (P<sub>1</sub>), which implies the almost doubling property when  $\mu$  is a capacity. Our result on prescription of multifractal behavior for measures is the following.

**Theorem 2.10.** *There exists a family of measures  $\mathcal{M}_d$  in  $\mathcal{M}(\mathbb{R}^d)$  such that :*

- (1) *Every  $\mu \in \mathcal{M}_d$  is  $\mathbb{Z}^d$ -invariant, fully supported on  $\mathbb{R}^d$ , satisfies property (P), and  $\mu|_{[0,1]^d}$  strongly obeys the multifractal formalism.*
- (2)  $\mathcal{S}_{d,\mathcal{M}} = \{\sigma_{\mu|_{[0,1]^d}} : \mu \in \mathcal{M}_d\}$ .

The family  $\mathcal{M}_d \subset \mathcal{M}(\mathbb{R}^d)$  is built in Section 3, by explicitly constructing, for  $\sigma \in \mathcal{S}_{d,\mathcal{M}}$ , a fully supported Borel probability measure  $\mu$  on  $[0, 1]^d$ , which strongly obeys the multifractal formalism, and such that  $\sigma_\mu = \sigma$ . Then  $\mathcal{M}_d$  is constructed by periodisation of such measures  $\mu$ .

The claim of Theorem 2.10 regarding the multifractal properties can be equivalently stated as follows: let  $\tau \in \mathcal{S}_{d,\mathcal{M}}$ . There exists a Borel probability measure  $\mu$  with support equal to  $[0, 1]^d$ , which strongly obeys the multifractal formalism and such that  $\tau_\mu = \tau$ . This result was established in [2], but the support of the measure had to be totally disconnected. Our proof will follow quite a different method.

In order to solve the Frisch-Parisi conjecture 1.1, we will need not only  $\mathcal{M}_d$ , but also the following larger class of capacities.

**Definition 2.11.** *The set  $\mathcal{E}_d \subset \mathcal{C}(\mathbb{R}^d)$  is defined as the set of positive powers of measures  $\mu \in \mathcal{M}_d$ , i.e.*

$$(10) \quad \mathcal{E}_d = \{\mu^s : \mu \in \mathcal{M}_d, s > 0\}.$$

*An element of  $\mathcal{E}_d$  is called a multifractal environment.*

**Remark 2.12.** (1) A direct computation shows that for any  $s > 0$  and any  $\mu \in \mathcal{H}(\mathbb{R}^d)$ ,

$$\text{for every } t \in \mathbb{R}, \quad \tau_{\mu_{|[0,1]^d}^s}(t) = \tau_{\mu_{|[0,1]^d}}(st).$$

(2) It is immediate to check that as soon as  $\mu \in \mathcal{H}(\mathbb{R}^d)$  satisfies property (P), the functions  $\mu^s$ ,  $\mu^{(+s)}$ , and  $\mu^{(-s)}$  do satisfy (P) as well (when  $s$  is small enough in the case of  $\mu^{(-s)}$ ), and that  $\mu_{|[0,1]^d}^s$  has  $H \mapsto \sigma_{\mu_{|[0,1]^d}}(H/s)$  as singularity spectrum.

**2.3. Besov spaces in almost doubling environments and their wavelet characterisation.** Standard Besov spaces can be defined by using  $L^p$  moduli of smoothness, and can be characterized in terms of the behavior of the coefficients of their wavelets expansion. In order to define Besov spaces in multifractal environment considered in this paper, we begin by extending the classical definition of  $L^p$  moduli of smoothness.

**Definition 2.13.** For  $h \in \mathbb{R}^d$  and  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ , consider the finite difference operator  $\Delta_h f : x \in \mathbb{R}^d \mapsto f(x+h) - f(x)$ . Then, for  $n \geq 2$ , set  $\Delta_h^n f = \Delta_h(\Delta_h^{n-1} f)$ .

For every fully supported set function  $\mu \in \mathcal{H}(\mathbb{R}^d)$ , for every  $n \in \mathbb{N}^*$ ,  $h \in \mathbb{R}^d \setminus \{0\}$  and  $x \in \mathbb{R}^d$ , set

$$\Delta_h^{\mu,n} f(x) = \frac{\Delta_h^n f(x)}{\mu(B([x, x+nh]))},$$

where for  $x, y \in \mathbb{R}^d$ ,  $B([x, y])$  stands for the Euclidean ball of diameter  $[x, y]$ .

For  $p \in [1, \infty]$ , the  $\mu$ -adapted  $n$ -th order  $L^p$  modulus of smoothness of  $f$  is defined at any  $t > 0$  by

$$(11) \quad \omega_n^\mu(f, t, \mathbb{R}^d)_p = \sup_{t/2 \leq |h| \leq t} \|\Delta_h^{\mu,n} f\|_{L^p(\mathbb{R}^d)}.$$

Observe that when  $\mu(E) = 1$  for every set  $E$ , then  $\omega_n^\mu(f, t, \mathbb{R}^d)_p$  is a modification of the standard  $n$ -th order  $L^p$  modulus of smoothness of  $f$  defined by

$$(12) \quad \omega_n(f, t, \mathbb{R}^d)_p = \sup_{0 \leq |h| \leq t} \|\Delta_h^n f\|_{L^p(\mathbb{R}^d)}.$$

Recall that when  $s > 0$ , and  $p, q \in [1, \infty]$ , the Besov space  $B_q^{s,p}(\mathbb{R}^d)$  is the set of those functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $\|f\|_{L^p(\mathbb{R}^d)} < \infty$  and

$$(13) \quad |f|_{B_q^{s,p}(\mathbb{R}^d)} = \|(2^{js}(\omega_n(f, 2^{-j}, \mathbb{R}^d)_p)_{j \in \mathbb{N}})\|_{\ell^q(\mathbb{N})} < \infty,$$

where  $n$  is any integer larger than  $s$ . We omit on purpose the dependence in  $n$  in the notation  $|f|_{B_q^{s,p}(\mathbb{R}^d)}$ . Indeed, the norm  $\|f\|_{B_q^{s,p}(\mathbb{R}^d)} = |f|_{B_q^{s,p}(\mathbb{R}^d)} + \|f\|_{L^p(\mathbb{R}^d)}$  makes  $B_q^{s,p}(\mathbb{R}^d)$  a Banach space, and different values of  $n > s$  yield equivalent norms (see [17, Remark 3.2.2]).

**Definition 2.14** (Besov spaces in  $\mu$ -environment). Let  $\mu \in \mathcal{H}(\mathbb{R}^d)$  satisfy property (P<sub>1</sub>) of Definition 2.9 with exponents  $0 < s_1 \leq s_2$ , and consider an integer  $n \geq \lfloor s_2 + \frac{d}{p} \rfloor + 1$ .

For  $1 \leq p, q \leq \infty$ , the Besov space in  $\mu$ -environment  $B_q^{\mu,p}(\mathbb{R}^d)$  is the set of those functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $\|f\|_{L^p(\mathbb{R}^d)} < \infty$  and

$$(14) \quad |f|_{B_q^{\mu,p}(\mathbb{R}^d)} = \|2^{jd/p}(\omega_n^\mu(f, 2^{-j}, \mathbb{R}^d)_p)_{j \in \mathbb{N}}\|_{\ell^q(\mathbb{N})} < \infty.$$

We also set

$$\tilde{B}_q^{\mu,p}(\mathbb{R}^d) = \bigcap_{0 < \varepsilon < \min(s_1, 1)} B_q^{\mu^{(-\varepsilon)}, p}(\mathbb{R}^d).$$

At this stage, both  $B_q^{\mu,p}(\mathbb{R}^d)$  and  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  depend a priori on the choice of  $n$ . We are going to prove that, under the rather weak scaling like additional property (P<sub>2</sub>) of Definition 2.9, the dependence in  $n \geq \lfloor s_2 + \frac{d}{p} \rfloor + 1$  can be dropped for  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ , as well as for  $B_q^{\mu,p}(\mathbb{R}^d)$  when  $\mu$  is a doubling capacity (see Theorem 2.16 for a precise statement). Moreover, endowed with the norm  $\| \cdot \|_{L^p(\mathbb{R}^d)} + \| \cdot \|_{B_q^{\mu,p}(\mathbb{R}^d)}$ ,  $B_q^{\mu,p}(\mathbb{R}^d)$  is a Banach space, from which it follows that  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  is naturally endowed with a Frechet space structure, as the intersection of a nested family of such spaces. The Frechet spaces  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  will play a key role in the solution to the Frisch-Parisi conjecture proposed in this paper.

Recall that  $\mathcal{L}^d$  stands for the  $d$ -dimensional Lebesgue measure. Setting  $\mu = (\mathcal{L}^d)^{\frac{s}{d} - \frac{1}{p}}$ , we will see that when  $s > d/p$  the equality  $B_q^{\mu,p}(\mathbb{R}^d) = B_q^{s,p}(\mathbb{R}^d)$  holds. A multifractal element  $\mu \in \mathcal{S}(\mathbb{R}^d)$  should now be considered as defining an *heterogeneous* environment imposing local distortions in the computation of the moduli of smoothness in comparison to positive powers of  $\mathcal{L}^d$ , which are *homogeneous* in space. Like for  $B_q^{s,p}(\mathbb{R}^d)$ , in order to study the typical multifractal behavior in  $B_q^{\mu,p}(\mathbb{R}^d)$  it is essential to establish a wavelet characterization of this space. However, we obtain such a characterization only when  $\mu$  is doubling, while such a characterization is possible for  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  when  $\mu$  is almost doubling (see Theorem 2.16 again).

**Wavelet characterisations.** Let us discuss now these characterisations in detail. It is a standard result that classical Besov spaces are characterized in terms of wavelet coefficients decay. Let  $\Lambda = \bigcup_{j \in \mathbb{Z}} \Lambda_j$ , where for  $j \in \mathbb{Z}$

$$\Lambda_j = \{(i, j, k) : i \in \{1, \dots, 2^d - 1\}, k \in \mathbb{Z}^d\}.$$

Let  $\phi$  be a scaling function and  $\{\psi^{(i)}\}_{i=1, \dots, 2^d-1}$  be a family of wavelets associated with  $\phi$  so that  $(\phi, \{\psi^{(i)}\}_{i=1, \dots, 2^d-1})$  defines a multi-resolution analysis with reconstruction in  $L^2(\mathbb{R}^d)$  (see [45, Ch. 2 and 3] for a general construction).

For every  $\lambda = (i, j, k) \in \Lambda$ , denote by  $\psi_\lambda$  the function  $x \mapsto \psi^{(i)}(2^j x - k)$ . Then, the functions  $2^{dj/2} \psi_\lambda$ ,  $j \in \mathbb{Z}$ ,  $\lambda \in \Lambda_j$ , form an orthonormal basis of  $L^2(\mathbb{R}^d)$ , so that every  $f \in L^2(\mathbb{R}^d)$  can be expanded as

$$f = \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda, \quad \text{with} \quad c_\lambda = \int_{\mathbb{R}^d} 2^{dj} \psi_\lambda(x) f(x) dx$$

(pay attention to the  $L^\infty$  normalisation used to define the wavelet coefficients  $(c_\lambda)_{\lambda \in \Lambda}$ ).

**Definition 2.15.** For every  $r \in \mathbb{N}$ , we denote by  $\mathcal{F}_r$  the set of those  $\{\phi, \{\psi^{(i)}\}_{i=1, \dots, 2^d-1}\}$ , which define a multi-resolution analysis with reconstruction in  $L^2(\mathbb{R}^d)$  and such that, moreover,  $\phi$  and the  $\psi^{(i)}$  are compactly supported and  $r$  times continuously differentiable.

It is known that if  $r \in \mathbb{N}^*$  and  $\Psi \in \mathcal{F}_r$ , for each  $1 \leq i \leq 2^d - 1$  and each multi-index  $\alpha \in \mathbb{N}^d$  of length smaller than or equal to  $r$ , one has  $\int_{\mathbb{R}^d} x_1^{\alpha_1} \cdots x_d^{\alpha_d} \psi^{(i)}(x) dx = 0$  (see [45, Prop. 4, section 3.7]).

Fix  $r \in \mathbb{N}^*$  and  $\Psi \in \mathcal{F}_r$ . For any  $f \in L^p(\mathbb{R}^d)$ ,  $1 \leq p \leq \infty$ , set

$$\beta(k) = \int_{\mathbb{R}^d} f(x) \phi(x - k) dx \quad (k \in \mathbb{Z}^d).$$

Then (see [45, Ch. 6], [58], or [17, Corollary 3.6.2]), for  $r > s > d/p$ ,

$$(15) \quad f \in B_q^{s,p}(\mathbb{R}^d) \iff \begin{cases} \beta \in \ell^p(\mathbb{Z}^d), \\ (\varepsilon_j)_{j \in \mathbb{N}} \in \ell^q(\mathbb{N}), \text{ where } \varepsilon_j = \left\| \left( 2^{j(s-d/p)} c_\lambda \right)_{\lambda \in \Lambda_j} \right\|_p, \end{cases}$$

and  $f = \sum_{k \in \mathbb{Z}^d} \beta(k) \phi(\cdot - k) + \sum_{j \in \mathbb{N}} \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda$ . Moreover, the norm  $\|\beta\|_p + \|(\varepsilon_j)\|_q$  is equivalent to the norm  $\|f\|_{B_q^{s,p}(\mathbb{R}^d)}$  defined in (13). Note that the functions  $\psi^{(i)}$  then belong to  $B_q^{s,p}(\mathbb{R}^d)$ . Also,  $B_q^{s,p}(\mathbb{R}^d) \hookrightarrow B_\infty^{s-\frac{d}{p},\infty}(\mathbb{R}^d) = \mathcal{C}^{s-\frac{d}{p}}(\mathbb{R}^d)$ .

Let us now introduce the quantity

$$(16) \quad |f|_{\mu,p,q} = |f|_{\mu,p,q,\Psi} = \|(\varepsilon_j^\mu)_{j \in \mathbb{N}}\|_{\ell^q(\mathbb{N})}, \text{ where } \varepsilon_j^\mu = \left\| \left( \frac{c_\lambda}{\mu(\lambda)} \right)_{\lambda \in \Lambda_j} \right\|_p$$

and  $\mu(\lambda) = \mu(\lambda_{j,k})$  if  $\lambda = (i, j, k)$ . In (16), the wavelet coefficients are computed with the given  $\Psi \in \mathcal{F}_r$ , but we omit the dependence on  $r$  and  $\Psi$  to make the notations lighter. This is justified by the fact that in what follows,  $r$  will depend on  $\mu$  only and in the cases which are relevant to us (i.e when  $\mu$  satisfies (P)), the wavelet characterisation of the Baire topological spaces  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  will be independent of  $\Psi \in \mathcal{F}_r$ .

Our result about the wavelet characterizations of  $B_q^{\mu,p}(\mathbb{R}^d)$  and  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  is the following.

**Theorem 2.16.** *Let  $\mu \in \mathcal{C}(\mathbb{R}^d)$  be an almost doubling capacity. Let  $0 < s_1 \leq s_2$  and  $r = \lfloor s_2 + \frac{d}{p} \rfloor + 1$ . Suppose that property (P) holds for  $\mu$  with the exponents  $(r_1, r_2) = (s_1, s_2)$  and that  $B_q^{\mu,p}(\mathbb{R}^d)$  has been constructed by using the  $L^p$  moduli of smoothness of order  $n$ , for some integer  $n \geq r$ . Let  $\Psi \in \mathcal{F}_r$ .*

*For every  $\varepsilon \in (0, 1)$ , there exists a constant  $C > 1$  such that for all  $f \in L^p(\mathbb{R}^d)$ ,*

$$(17) \quad \|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu,p,q} \leq C(\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B_q^{\mu(+\varepsilon),p}(\mathbb{R}^d)}),$$

$$(18) \quad \|f\|_{L^p(\mathbb{R}^d)} + |f|_{B_q^{\mu,p}(\mathbb{R}^d)} \leq C(\|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu(+\varepsilon),p,q}).$$

*Moreover, when  $\mu$  is doubling and satisfies property (P) with  $\theta = 0$ , the norms  $\|\cdot\|_{L^p} + |\cdot|_{\mu,p,q}$  and  $\|\cdot\|_{L^p} + |\cdot|_{B_q^{\mu,p}}$  are equivalent.*

As a consequence, when  $\mu$  is doubling and satisfies (P) with  $\theta = 0$ , the space  $B_q^{\mu,p}(\mathbb{R}^d)$  possesses two equivalent definitions based either on  $L^p$  moduli of smoothness or on wavelet coefficients, and this definition is independent of the choice of  $n \geq r$  and  $\Psi \in \mathcal{F}_r$ . For  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ , when  $\mu$  satisfies property (P), combining (17) and (18) shows

that  $f \in \widetilde{B}_q^{\mu,p}(\mathbb{R}^d)$  if and only if  $\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B_q^{\mu(+\varepsilon),p}(\mathbb{R}^d)} < \infty$  for every  $\varepsilon > 0$ , hence also giving a wavelet characterization of  $\widetilde{B}_q^{\mu,p}(\mathbb{R}^d)$ .

Moreover, given  $\Psi \in \mathcal{F}_r$ , the family of Banach spaces

$$\left\{ B_\varepsilon := (B_q^{\mu(-\varepsilon),p}(\mathbb{R}^d), \| \cdot \|_{L^p(\mathbb{R}^d)} + | \cdot |_{\mu(-\varepsilon),p,q,\Psi}) \right\}_{0 < \varepsilon < \min(s_1, 1)}$$

is non decreasing, and  $B_\varepsilon \hookrightarrow B_{\varepsilon'}$  for all  $0 < \varepsilon < \varepsilon' < \min(s_1, 1)$ . This implies that the space  $\widetilde{B}_q^{\mu,p}(\mathbb{R}^d)$  can be endowed with a Frechet space structure, of which a countable basis of neighborhoods of the origin is given by

$$(19) \quad \left\{ \mathcal{N}_m = \left\{ f \in \widetilde{B}_q^{\mu,p}(\mathbb{R}^d) : \|f\|_{L^p(\mathbb{R}^d)} + |f|_{B_q^{\mu(-\frac{1}{m}),p}(\mathbb{R}^d)} < \frac{1}{m} \right\} \right\}_{\substack{m \in \mathbb{N}, \\ m > \max(1, s_1^{-1})}}.$$

**Remark 2.17.** (1) When  $(P_1)$  is satisfied, then the sequence of embeddings

$$B_q^{s_2 + \frac{d}{p}, p}(\mathbb{R}^d) \hookrightarrow B_q^{\mu,p}(\mathbb{R}^d) \hookrightarrow B_q^{s_1 + \frac{d}{p}, p}(\mathbb{R}^d) \text{ and } B_q^{\mu(+\varepsilon), p}(\mathbb{R}^d) \hookrightarrow B_q^{\mu,p}(\mathbb{R}^d) \text{ hold.}$$

(2) It is direct from the proof of Theorem 2.16 that if we make the slightly weaker assumption that property (P) holds for all  $(r_1, r_2)$  such that  $0 < r_1 < s_1 \leq s_2 < r_2$ , then the statement remains true.

By Remark 2.17(2), when  $\mu \in \mathcal{E}_d$  (see Definition 2.11), since property (P) holds with any  $(r_1, r_2)$  such that  $0 < r_1 < \tau'_\mu(\infty) \leq \tau'_\mu(-\infty) < r_2$ ,  $\widetilde{B}_q^{\mu,p}(\mathbb{R}^d)$  will always be considered as defined for an integer  $n \geq r_\mu$ , where

$$(20) \quad r_\mu = \left\lfloor \tau'_\mu(-\infty) + \frac{d}{p} \right\rfloor + 1,$$

and the wavelet characterization of  $\widetilde{B}_q^{\mu,p}(\mathbb{R}^d)$  holds with  $\Psi \in \mathcal{F}_{r_\mu}$ .

We can now present our result on the typical singularity spectrum in  $\widetilde{B}_q^{\mu,p}(\mathbb{R}^d)$ .

**2.4. Typical singularity spectrum in Besov spaces in multifractal environment.** Let us first recall Jaffard's result on the typical multifractal behavior in  $B_q^{s,p}(\mathbb{R}^d)$ .

**Theorem 2.18.** [34] *Let  $s \geq 0$  and  $(p, q) \in [1, \infty]^2$ , with  $s > d/p$ .*

$$(1) \text{ For all } f \in B_q^{s,p}(\mathbb{R}^d), \sigma_f(H) \leq \begin{cases} \min\{p(H - (s - \frac{d}{p})), d\} & \text{if } H \geq s - d/p, \\ -\infty & \text{if } H < s - d/p, \end{cases}$$

with the convention  $\infty \times 0 = d$ .

$$(2) \text{ Typical } f \in B_q^{s,p}(\mathbb{R}^d) \text{ satisfy } \sigma_f(H) = \begin{cases} p(H - (s - \frac{d}{p})) & \text{if } H \in [s - d/p, s], \\ -\infty & \text{otherwise.} \end{cases}$$

The singularity spectrum of typical functions  $f \in B_q^{s,p}(\mathbb{R}^d)$  depends only on  $s$ ,  $d$  and  $p$ , and it is affine increasing over its support whenever  $p < \infty$ . When  $p = \infty$ , the support is degenerate and the typical singularity spectrum is  $\sigma_f(H) = \mathbf{1}_{\{s\}}(H) - \infty \mathbf{1}_{\mathbb{R} \setminus \{s\}}$ .

Our result on the multifractal nature of the elements of  $\widetilde{B}_q^{\mu,p}(\mathbb{R}^d)$  when  $\mu \in \mathcal{E}_d$  (i.e. powers of measures  $\mathcal{M}_d$  defined by (10)) is the following (the validity of some multifractal formalism is dealt with in the next section).

**Theorem 2.19.** *Let  $\mu \in \mathcal{E}_d$ ,  $p, q \in [1, \infty]$ , and consider the mapping*

$$(21) \quad \zeta_{\mu,p}(t) = \begin{cases} \frac{p-t}{p} \tau_\mu \left( \frac{p}{p-t} t \right) & \text{if } t \in (-\infty, p) \\ \tau'_\mu(\infty) t & \text{if } t \in [p, \infty). \end{cases}$$

(1) For all  $f \in \widetilde{B}_q^{\mu,p}(\mathbb{R}^d)$ ,

$$(22) \quad \sigma_f(H) \leq \begin{cases} \zeta_{\mu,p}^*(H) & \text{if } H \leq \zeta'_{\mu,p}(0^+) \\ d & \text{if } H > \zeta'_{\mu,p}(0^+) \end{cases}.$$

(2) For typical functions  $f \in \widetilde{B}_q^{\mu,p}(\mathbb{R}^d)$ , one has  $\sigma_f = \zeta_{\mu,p}^*$ .

The possible features of the multifractal spectrum of typical functions in  $\widetilde{B}_q^{\mu,p}(\mathbb{R}^d)$  are investigated in detail in Section 5 (see Lemm 5.1 and Remark 5.2). There, we will see in particular that depending on the values of  $p$  and on whether  $\tau_\mu^*(\tau'_\mu(\infty))$  equals 0 or is positive, distinct phenomena may appear, see Figures 4 and 7 for a representation of the multifractal spectrum of typical functions in  $\widetilde{B}_q^{\mu,p}(\mathbb{R}^d)$ , according to whether  $\sigma_\mu(\alpha_{\min}) = 0$  or  $\sigma_\mu(\alpha_{\min}) > 0$ . Meanwhile, the three first items of the following remark provide preliminary information about this spectrum.

**Remark 2.20.** (1) *It will be proved that  $\zeta_{\mu,p}$  is always concave. Also, it is immediate that  $\zeta_{\mu,p} = \tau_\mu$  when  $p = \infty$ , so typical functions in  $\widetilde{B}_q^{\mu,\infty}(\mathbb{R}^d)$  have  $\tau_\mu^*$  as singularity spectrum.*

(2) *The support of  $\zeta_{\mu,p}^*$  is the compact subinterval  $[\zeta_{\mu,p}(\infty), \zeta'_{\mu,p}(-\infty)] \subset (0, \infty)$ . Moreover, since  $\zeta_{\mu,p}(0) = \tau_\mu(0) = -d$ , the maximum of  $\zeta_{\mu,p}^*$  is  $d$ , and it is reached at  $H$  if and only if  $H \in [\zeta'_{\mu,p}(0^+), \zeta_{\mu,p}(0^-)]$ .*

(3) *One has  $\zeta'_{\mu,p}(-\infty) \leq \tau'_\mu(-\infty) + \frac{d}{p}$  (see the comment after Proposition 5.1).*

(4) *The set of environments  $\mathcal{E}_d$  that we consider contains all the positive powers of  $\mathcal{L}^d$ . When  $s > d/p$  and  $\mu = (\mathcal{L}^d)^{s/d-1/p}$ , Theorem 2.19 coincides with Jaffard's Theorem 2.18. Indeed, in this case  $\tau_\mu(t) = (s-d/p)t - d$  so  $\tau'_\mu(-\infty) = \tau'_\mu(\infty) = s - d/p$ ,  $\tau_\mu^*(H) = d$  if  $H = s - d/p$  and  $-\infty$  otherwise. We deduce that  $\zeta_{\mu,p}(t) = st - d$  if  $t < p$  and  $\zeta_{\mu,p}(t) = (s - d/p)t$  for  $t \geq p$ , whose Legendre transform is easily seen to be the typical spectrum observed in  $B_q^{s,p}(\mathbb{R}^d)$ .*

(5) *For a doubling capacity  $\mu$  with nice scaling properties, one can expect Theorem 2.19 to be true for  $B_q^{\mu,p}(\mathbb{R}^d)$  (and not only  $\widetilde{B}_q^{\mu,p}(\mathbb{R}^d)$ ). This is the case when  $\mu$  is a positive power of a class of Gibbs measure defined in the following way: let  $\Phi$  be the set of  $\mathbb{Z}^d$ -invariant real valued Hölder continuous functions on  $\mathbb{R}^d$ . Let  $\varphi \in \Phi$ . Then, the sequence of Radon measures*

$$\nu_n(dx) = \frac{\exp(S_n \varphi(x))}{\int_{[0,1]^d} \exp(S_n \varphi(t)) \mathcal{L}^d(dt)} \mathcal{L}^d(dx), \quad \text{where } S_n \varphi(x) = \sum_{k=0}^{n-1} \varphi(2^k x),$$



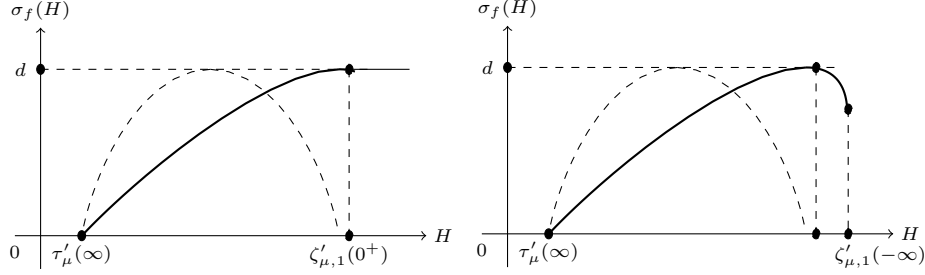


FIGURE 4. Here we take  $p = 1$ . **Left:** Upper bound for the multifractal spectrum of every  $f \in \tilde{B}_q^{\mu,1}(\mathbb{R}^d)$ . **Right:** Multifractal spectrum of a typical function in  $\tilde{B}_q^{\mu,1}(\mathbb{R}^d)$ . The dashed graph represents the (initial) multifractal spectrum of  $\mu$ . When  $p = \infty$ , the multifractal spectrum of typical  $f \in \tilde{B}_q^{\mu,\infty}(\mathbb{R}^d)$  coincides with that of  $\mu$ .

converges vaguely to a  $\mathbb{Z}^d$ -invariant Radon measure  $\nu = \nu_\varphi$  fully supported on  $\mathbb{R}^d$ , called Gibbs measure associated with  $\varphi$ . Then,  $\tau_{\nu|_{[0,1]^d}}(t) = tP(\varphi) - P(t\varphi)$ , where the topological pressure of  $\tilde{\varphi} \in \Phi$  is defined by

$$P(\tilde{\varphi}) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \int_{[0,1]^d} 2^n \exp(S_n \tilde{\varphi}(x)) \mathcal{L}^d(dx).$$

Moreover,  $\tau_{\nu|_{[0,1]^d}}$  is analytic (see [50, 52]).

It turns out that following the proofs developed in this paper when  $\mu \in \mathcal{E}_d$ , if  $\mu = \nu^s = \nu_\varphi^s$  for some  $s > 0$ , sufficient conditions for the conclusions of Theorem 2.19 to hold for typical functions in  $B_q^{\mu,p}(\mathbb{R}^d)$  and  $\tilde{B}_q^{\mu,\infty}(\mathbb{R}^d)$  are  $p = \infty$ , or  $\tau'_{\nu|_{[0,1]^d}}(\infty) = 0$ , or that the potential  $\varphi$  reaches its minimum at 0. In the general case, our result still holds but the method must be adjusted, and we will not enter into the details in this paper.

**Remark 2.21.** Let  $\mu \in \mathcal{E}_d$ . Let  $\Psi \in \mathcal{F}_{r_\mu}$  and  $\hat{B}_{q,\Psi}^{\mu,p}(\mathbb{R}^d)$  be the subspace of those  $f \in L^p(\mathbb{R}^d)$  such that  $|f|_{\mu,p,q,\Psi} < \infty$ . Endowed with the norm  $\|\cdot\|_{L^p(\mathbb{R}^d)} + \|\cdot\|_{\mu,p,q,\Psi}$ ,  $\hat{B}_{q,\Psi}^{\mu,p}(\mathbb{R}^d)$  is a Banach space, and our proof of Theorem 2.19 shows that the conclusions of this theorem do hold if one replaces  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  by  $\hat{B}_q^{\mu,p,\Psi}(\mathbb{R}^d)$ . Also, Theorem 2.16 implies that  $\hat{B}_q^{\mu,p,\Psi}(\mathbb{R}^d)$  does not depend on  $\Psi$  and equals  $B_q^{\mu,p}(\mathbb{R}^d)$  when property (P) holds with  $\theta = 0$ .

Next section presents the multifractal formalism for functions that we will use. It is based on the multifractal formalism developed by Jaffard in [35], associated with the so-called wavelet leaders, whose definition we recall below.

**2.5. Multifractal formalism for functions in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ .** Let us begin with the definition of wavelet leaders.

**Definition 2.22** (Wavelet leaders). Given  $\Psi \in \bigcup_{r \in \mathbb{N}} \mathcal{F}_r$  and  $f \in L^p_{\text{loc}}(\mathbb{R}^d)$  for some  $p \in [1, \infty]$ , denoting the wavelet coefficients of  $f$  associated with  $\Psi$  by  $(c_\lambda)_{\lambda \in \Lambda}$ , the

wavelet leader of  $f$  associated with  $\lambda \in \mathcal{D}$  (see Section 2.1 for the notations) is defined as:

$$(23) \quad L_\lambda^f = \sup\{|c_\lambda| : \lambda' = (i, j, k) \in \Lambda, \lambda'_{j,k} \subset 3\lambda\}.$$

Pointwise Hölder exponents of Hölder continuous functions are related to the wavelet leaders as follows (see [35, Corollary 1]).

**Proposition 2.23.** *Let  $r \in \mathbb{N}^*$  and  $\Psi \in \mathcal{F}_r$ . If  $f \in \mathcal{C}^\varepsilon(\mathbb{R}^d)$  for some  $\varepsilon > 0$ , then the pointwise exponent  $h_f(x_0)$  of  $f$  at  $x_0 \in \mathbb{R}^d$  (see Definition 1) satisfies  $h_f(x_0) < r$  if and only  $\liminf_{j \rightarrow \infty} \frac{\log L_{\lambda_j}^f}{\log(2^{-j})} < r$ , and in this case*

$$(24) \quad h_f(x_0) = \liminf_{j \rightarrow \infty} \frac{\log L_{\lambda_j}^f}{\log(2^{-j})}.$$

Hence, as observed by Jaffard, and rephrased in the language used in this paper, if the support of  $\sigma_f$  is bounded and sufficiently smooth wavelets  $\Psi$  are used, then the singularity spectrum  $\sigma_f$  of  $f$  coincides with the singularity spectrum of the capacity  $\nu \in \mathcal{C}(\mathbb{R}^d)$  defined by  $\nu(B) = \sup\{L_\lambda^f : \lambda \in \mathcal{D}, \lambda \subset B\}$  for all  $B \in \mathcal{B}(\mathbb{R}^d)$ .

In order to estimate from above the singularity spectrum  $\sigma_f$  of  $f \in \widetilde{B}_q^{\mu,p}(\mathbb{R}^d)$ , it is then natural to consider, exactly as it was done for the elements of  $\mathcal{H}([0, 1]^d)$ , the  $L^q$ -spectrum of  $f$  relative to  $\Psi$  defined as follows: For any  $N \in \mathbb{N}^*$ , set

$$(25) \quad \zeta_f^{N,\Psi} = \liminf_{j \rightarrow +\infty} \zeta_{f,j}^{N,\Psi}, \text{ where } \zeta_{f,j}^{N,\Psi} : t \in \mathbb{R} \mapsto -\frac{1}{j} \log_2 \sum_{\lambda \in \mathcal{D}_j, \lambda \subset N[0,1]^d, L_\lambda^f > 0} (L_\lambda^f)^t$$

Recall that with our notations (see Section 2),  $(N[0, 1]^d)_{N \in \mathbb{N}^*}$  is the increasing sequence of boxes  $[-(N-1)/2, (N+1)/2]^d$ , which cover  $\mathbb{R}^d$ . Then, observing that  $(\zeta_f^{N,\Psi})_{N \geq 1}$  is a non-increasing sequence of functions, the  $L^q$ -spectrum of  $f$  relative to  $\Psi$  is the concave function

$$(26) \quad \zeta_f^\Psi = \inf\{\zeta_f^{N,\Psi} : N \in \mathbb{N}^*\} = \lim_{N \rightarrow +\infty} \zeta_f^{N,\Psi}.$$

A remarkable fact is that  $\zeta_f^\Psi|_{\mathbb{R}_+}$  does not depend on  $\Psi$  [35, Theorem 3]. This would be the case over  $\mathbb{R}$  if the elements of  $\Psi$  belong to the Schwarz class [35, Theorem 4]. However, our wavelet characterisation of  $B_q^{\mu,p}(\mathbb{R}^d)$  makes it necessary to use compactly supported wavelets, which never belong to  $\mathcal{C}^\infty(\mathbb{R}^d)$  [19].

Also, when  $H < r$ , the Legendre transform  $(\zeta_f^\Psi)^*(H)$  of  $\zeta_f^\Psi$  at  $H$  provides an upper bound for  $\dim E_f(H)$ , i.e. one has

$$(27) \quad \sigma_f(H) \leq (\zeta_f^\Psi)^*(H).$$

We simply denote  $\zeta_f^\Psi|_{\mathbb{R}_+}$  by  $\zeta_f|_{\mathbb{R}_+}$ .

Let us now define the multifractal formalism used in this paper. It combines Jaffard's multifractal formalism associated with wavelet leaders, and a variant of it, mainly used to control the decreasing part of the singularity spectrum whenever it exists, which in

terms of scaling function corresponds to negative values of  $t$ . This variant is necessary since we will see that when  $\mu \in \mathcal{E}_d$ ,  $q < \infty$ , and the elements of  $\Psi$  are smooth enough, it is generic in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  that  $\zeta_f^\Psi|_{\mathbb{R}_+^*}$  equals  $-\infty$ , so that  $(\zeta_f^\Psi)^*(H)$  only provides the trivial upper bound  $\sigma_f(H) \leq d$  for  $H \geq (\zeta_f)^\vee(0^+)$ .

**Definition 2.24** (Multifractal formalism). *Let  $r \in \mathbb{N}^*$ . Let  $f \in \bigcup_{s>0} \mathcal{C}^s(\mathbb{R}^d)$  and suppose that  $\sigma_f$  has a compact domain included in  $(0, r)$ . Let  $I \subset \text{dom}(\sigma_f)$  be a compact interval.*

- (1) *The (wavelet leaders) multifractal formalism is said to hold for  $f$  on  $I$  when there is an integer  $\tilde{r} \geq r$  such that  $\sigma_f(H) = (\zeta_f^\Psi)^*(H)$  for all  $H \in I$ , independently of  $\Psi \in \mathcal{F}_{\tilde{r}}$ .*
- (2) *The refined (wavelet leaders) multifractal formalism is said to hold for  $f$  on  $I$  relatively to  $\Psi \in \mathcal{F}_r$  when the following property holds: there exists an increasing sequence  $(j_k)_{k \in \mathbb{N}}$  such that for all  $N \in \mathbb{N}$ ,  $\lim_{k \rightarrow \infty} \zeta_{f, j_k}^{N, \Psi} = \zeta^{(N)}$  exists, and setting  $\zeta_{f, \text{ref}}^\Psi = \lim_{N \rightarrow +\infty} \zeta^{(N)}$ , one has  $\sigma_f(H) = (\zeta_{f, \text{ref}}^\Psi)^*(H)$  for all  $H \in I$ .*

**Remark 2.25.** (1) *In the increasing part of  $\sigma_f$ , item (1) of the previous definition coincides with the multifractal formalism associated with wavelet leaders considered by Jaffard (see [35] for instance).*

*Contrarily to what happens when one considers  $\zeta_f^\Psi$  and gets (27), in general, even if there exists such a subsequence  $(j_k)_{k \in \mathbb{N}}$  making it possible to define  $\zeta_{f, \text{ref}}^\Psi$ , one cannot get the a priori inequality  $\sigma_f \leq (\zeta_{f, \text{ref}}^\Psi)^*$ . Nevertheless, the existence of  $\zeta_{f, \text{ref}}^\Psi$  emphasizes the strong property that the sequences  $(\zeta_{f, j}^{N, \Psi}(t))_{j \in \mathbb{N}}$  converge along the same subsequence for all  $N$  and  $t$ . This property will be typical in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ , and it will be valid simultaneously for countably many  $\Psi$ 's.*

(2) *If the refined multifractal formalism holds on  $I$  relatively to both  $\Psi$  and  $\tilde{\Psi}$  in  $\mathcal{F}_r$ , then  $\zeta_{f, \text{ref}}^\Psi = \sigma_f^* = \zeta_{f, \text{ref}}^{\tilde{\Psi}}$  on the interval  $\bigcup_{H \in I} \partial \sigma_{f|_I}(H)$  ( $\partial \sigma_{f|_I}$  stands for the subdifferential of the concave function  $\sigma_{f|_I}$ ).*

(3) *Inequality (27) comes from the fact that  $\dim(E_f(H) \cap N[0, 1]^d) \leq (\zeta_f^{N, \Psi})^*(H)$  for any  $H \in \mathbb{R}$  and  $N \in \mathbb{N}^*$ .*

Theorem 2.19, which states the multifractal properties of typical functions in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ , can now be completed by the following result on the validity of the multifractal formalism. Recall (21) and (20) for the definitions of  $\zeta_{\mu,p}$  and  $r_\mu$  respectively, as well as Remarks 2.17(2) and 2.20(3)).

**Theorem 2.26** (Validity of the multifractal formalism). *Let  $\mu \in \mathcal{E}_d$ .*

- (1) *For all  $f \in \tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ , one has  $\zeta_f|_{\mathbb{R}_+} \geq \zeta_{\mu,p}|_{\mathbb{R}_+}$ .*
- (2) *Typical functions  $f \in \tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  satisfy the wavelet leaders multifractal formalism on the interval  $[\zeta_{\mu,p}'(\infty), \zeta_{\mu,p}'(0^+)]$ , i.e. in the increasing part of  $\sigma_f$ , and  $\zeta_f|_{\mathbb{R}_+} = \zeta_{\mu,p}|_{\mathbb{R}_+}$ .*

- (3) (i) Let  $\Psi \in \mathcal{F}_{r_\mu}$ . Typical functions  $f \in \widetilde{B}_q^{\mu,p}(\mathbb{R}^d)$  satisfy the refined wavelet leaders multifractal formalism on  $\text{dom}(\sigma_f) = [\zeta_{\mu,p}'(\infty), \zeta_{\mu,p}'(-\infty)]$  relatively to  $\Psi$ , with  $\zeta_{f,\text{ref}}^\Psi = \sigma_f^* = \zeta_{\mu,p}$ . Moreover, if  $q < \infty$ , the property  $\zeta_f^\Psi|_{\mathbb{R}_-^*} = -\infty$  is typical as well.
- (ii) It follows that, given a countable subset  $\mathcal{F}$  of  $\mathcal{F}_{r_\mu}$ , typical functions  $f \in \widetilde{B}_q^{\mu,p}(\mathbb{R}^d)$  satisfy the refined wavelet leaders multifractal formalism on the interval  $\text{dom}(\sigma_f)$  relatively to any  $\Psi \in \mathcal{F}$ , with  $\zeta_{f,\text{ref}}^\Psi = \sigma_f^* = \zeta_{\mu,p}$ , and  $\zeta_f^\Psi|_{\mathbb{R}_-^*} = -\infty$  if  $q < \infty$ .

In other words, when  $\mu \in \mathcal{E}_d$ , for typical functions in  $\widetilde{B}_q^{\mu,p}(\mathbb{R}^d)$ , the wavelet leaders multifractal formalism holds in the increasing part of the spectrum, while its refined version holds both on the increasing and the decreasing part of the spectrum (in fact on  $\mathbb{R}$ ), but in the stronger form stated in Theorem 2.26(3)(ii), and it is not possible to substitute  $\zeta_f^\Psi$  to  $\zeta_{f,\text{ref}}$ , at least when  $q < \infty$ , since  $\zeta_f^\Psi|_{\mathbb{R}_-^*} = -\infty$ .

**Remark 2.27.** Let us come back to the case of Besov spaces. They take the form  $B_q^{\mu,p}(\mathbb{R}^d)$ , where  $\mu$  is a positive power of the Lebesgue measure. For these spaces, the wavelet characterisation also holds when the wavelets are taken in the Schwartz class. Moreover, as mentioned above,  $\zeta_f^\Psi$  does not depend on  $\Psi$  for any  $f \in B_q^{\mu,p}(\mathbb{R}^d)$ . The approach used to prove Theorem 2.26 also shows that if  $q < \infty$ , generically a function  $f$  is such that  $\zeta_f|_{\mathbb{R}_+} = \zeta_{\mu,p}|_{\mathbb{R}_+}$  and  $\zeta_f|_{\mathbb{R}_-^*} = -\infty$ .

**Remark 2.28.** Like for Besov spaces, one can let  $p$  or  $q$  take values in  $(0, 1)$  in the definition of Besov spaces in multifractal environment, and all our results are valid.

**2.6. A solution to the Frisch-Parisi conjecture. Proof of Theorem 1.2.** Combining our previous results, namely Theorems 2.10, 2.19 and 2.26, we are now able to prove Theorem 1.2.

*Proof of Theorem 1.2.* Let  $\sigma \in \mathcal{S}_d$ . Let  $\sigma_{\mathcal{M}} = \sigma(\cdot/s)$ , where  $s$  is the unique positive real number such that  $\sigma(\cdot/s) \leq \text{Id}_{\mathbb{R}}$  and there exists at least one  $H$  such that  $\sigma(H/s) = H$ . In other words,  $s$  is the unique number such that  $\sigma^*(s) = 0$ . In particular,  $\sigma_{\mathcal{M}} \in \mathcal{S}_{d,\mathcal{M}}$ . By Theorem 2.10, there exists  $\mu \in \mathcal{M}_d$  such that  $\tau_\mu = \sigma_{\mathcal{M}}^*$ .

Now, we apply Theorems 2.19 and 2.26 with the capacity  $\mu^s$ : in the Baire space  $\widetilde{B}_q^{\mu^s,\infty}(\mathbb{R}^d)$ , typical functions have  $\sigma$  as singularity spectrum, and they satisfy the wavelet leader multifractal formalism in the increasing part, and they satisfy the refined wavelet leader multifractal formalism over  $\text{supp}(\sigma)$  relatively to any  $\Psi$  in a countable family of elements of  $\mathcal{F}_{r_{\mu^s}}$ .

Hence, for any  $q \in [1, \infty]$ , the space  $\widetilde{B}_q^{\mu^s,\infty}(\mathbb{R}^d)$  provides a solution to the Conjecture 1.1 with initial data  $\sigma$ .  $\square$

**Remark 2.29** (Other solutions). Suppose  $\nu \in \mathcal{E}_d$ ,  $1 \leq p < \infty$  and  $1 \leq q \leq \infty$ . Let  $\sigma \in \mathcal{S}_d$  be the typical singularity spectrum in  $\widetilde{B}_q^{\nu,p}(\mathbb{R}^d)$  given by 2.19. Considering  $\mu \in \mathcal{M}_d$  as in the previous proof yields for all  $q' \in [1, \infty]$  the space  $\widetilde{B}_{q'}^{\mu^s,\infty}(\mathbb{R}^d)$  in which the typical multifractal structure is the same as in  $\widetilde{B}_q^{\nu,p}(\mathbb{R}^d)$ .

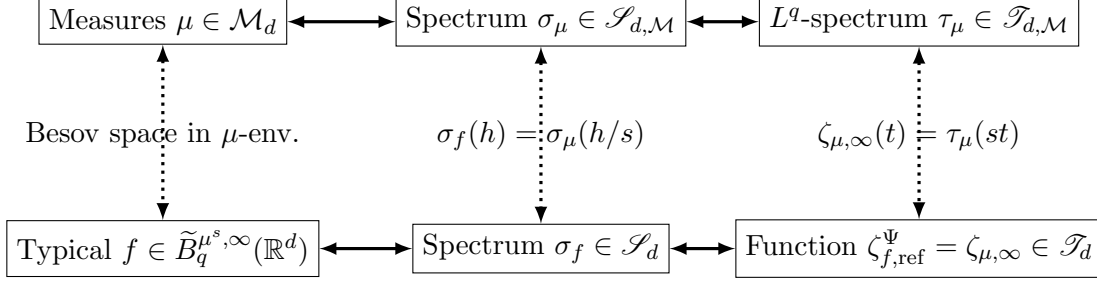


FIGURE 5. Scheme of the correspondence between the various objects and sets in the case where  $p = \infty$ , for some parameter  $s > 0$ .

It is worth recalling the approach used by Jaffard in [34] towards a solution to Conjecture 1.1. This approach can be rephrased as follows: consider an increasing continuous and concave function  $\eta$  over  $\mathbb{R}_+$  with positive slope  $\eta'(\infty)$  at  $\infty$ , such that  $\eta(0) \in [0, d]$ , and  $\eta^*$  takes values in  $[-d, 0]$  over its domain. Setting  $\zeta = \eta - d$ , one seeks for a Baire space in which the increasing part of the typical singularity spectrum is given by  $\zeta^*$ . Jaffard, who worked with the so-called homogeneous Besov spaces  $\dot{B}_q^{s,p}(\mathbb{R}^d)$ , introduced the Baire space  $V = \bigcap_{\epsilon > 0} \bigcap_{t > 0} \dot{B}_{t,\text{loc}}^{(\eta(t)-\epsilon)/t,t}(\mathbb{R}^d)$  [34] and proved that for typical functions  $f \in V$ ,  $\sigma_f = \tilde{\zeta}^*$ , where

$$\tilde{\zeta}(t) = \begin{cases} d(t/t_c - 1) & \text{if } t < t_c \\ \zeta(t) & \text{if } t \geq t_c \end{cases},$$

$t_c$  being the unique solution of  $\zeta(t_c) = 0$ . In particular,  $\sigma_f$  is necessarily increasing, with domain  $[\zeta'(\infty), d/t_c]$ , and with a linear part over the interval  $[\zeta'(t_c+), d/t_c]$ . Also,  $\sigma_f$  coincides with  $\zeta^*$  over  $[\zeta'(\infty), \zeta'(t_c+)]$ .

Moreover, in the multifractal formalism used in [34], the scaling function  $\zeta_f(t)$  is defined as  $\sup\{s \geq 0 : f \in \dot{B}_{\infty,\text{loc}}^{s/t,t}(\mathbb{R}^d)\} - d$  for  $t > 0$ , and with this definition typical functions in  $V$  satisfy  $\zeta_f = \zeta$ . Thus the associated multifractal formalism holds on  $[\zeta'(\infty), \zeta'(t_c+)]$  only. However, it can be checked that the wavelet leaders multifractal formalism does hold for  $f$  with  $\zeta_f = \tilde{\zeta}$  on  $[\zeta'(\infty), d/t_c]$ .

### Organization of the paper.

Section 3 is dedicated to the construction of the class of measures  $\mathcal{M}_d$  (Definitions 3.9 and 3.14) with prescribed multifractal behavior as described in Theorem 2.10. There, an heterogeneous mass transference principle for these measures (Proposition 3.18) is also proved. In Section 4, we establish the wavelet characterization of the space  $\tilde{B}_q^{\mu,p}$  when  $\mu$  is an almost doubling capacity satisfying property (P) (Theorem 2.16). The possible shapes of  $\zeta_{\mu,p}^*$  are investigated in Section 5, where  $\zeta_{\mu,p}^*$  is expressed in function of  $\tau_\mu^*$ . Next, in section 6, the upper bound for the singularity spectrum of all functions in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  is established (part (1) of Theorem 2.19), as a consequence of part (1) of Theorem 2.26 which is also proved there. Part (2) of Theorem 2.19 is obtained in Section 7. It consists first in building a specific function whose singularity

spectrum will turn out to be typical, and then to build a dense  $G_\delta$  set included in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  in which all functions share the same multifractal spectrum. Finally, parts (2) and (3) of Theorem 2.26 are established in Section 8.

Note that Sections 3, 4 and 5 can be read independently of the rest of the paper. The other sections lie on the results proved therein, but the arguments developed there are not used in the other proofs.

### 3. MEASURES WITH PRESCRIBED MULTIFRACTAL BEHAVIOR

We first give in Section 3.1 additional general properties associated with multifractal formalism for capacities. Section 2.2 is a preparation to the construction of the measures satisfying the requirements of Theorem 2.10. The construction is achieved when  $d = 1$  in Section 3.3. Then, in Sections 3.4 to 3.6 we check that the requirements of Theorem 2.10 are fulfilled. The construction is extended to the case  $d \geq 2$  in Section 3.7. Finally, in Section 3.8 we establish a mass transference principle associated with these measures.

#### 3.1. Additional notions related to the multifractal formalism for capacities.

Let us introduce, for  $\alpha \in \mathbb{R}$ ,

$$\begin{aligned} \underline{E}_\mu^\leq(\alpha) &= \{x \in \text{supp}(\mu) : \underline{h}_\mu(x) \leq \alpha\}, \\ \overline{E}_\mu^\geq(\alpha) &= \{x \in \text{supp}(\mu) : \overline{h}_\mu(x) \geq \alpha\}. \end{aligned}$$

The distribution of a capacity at small scales can be described through its large deviations spectrum.

**Definition 3.1.** *Let  $\mu \in \mathcal{C}([0, 1]^d)$  such that  $\text{supp}(\mu) \neq \emptyset$ . For  $I \subset \mathbb{R}$  and  $j \in \mathbb{N}^*$  define*

$$\mathcal{D}_\mu(j, I) = \left\{ \lambda \subset [0, 1]^d, \lambda \in \mathcal{D}_j : \frac{\log_2 \mu(\lambda)}{-j} \in I \right\}.$$

Then, for  $\alpha \in \mathbb{R}$  and  $\varepsilon > 0$  set

$$\mathcal{D}_\mu(j, \alpha \pm \varepsilon) = \mathcal{D}_\mu(j, [\alpha - \varepsilon, \alpha + \varepsilon]).$$

The lower and upper large deviations spectra of  $\mu$  are defined respectively as

$$\begin{aligned} \underline{\sigma}_\mu^{\text{LD}} : \alpha \in \mathbb{R} &\mapsto \lim_{\varepsilon \rightarrow 0} \liminf_{j \rightarrow \infty} \frac{\log_2 \#\mathcal{D}_\mu(j, \alpha \pm \varepsilon)}{j} \\ \text{and } \overline{\sigma}_\mu^{\text{LD}} : \alpha \in \mathbb{R} &\mapsto \lim_{\varepsilon \rightarrow 0} \limsup_{j \rightarrow +\infty} \frac{\log_2 \#\mathcal{D}_\mu(j, \alpha \pm \varepsilon)}{j}, \end{aligned}$$

with the convention  $\log(0) = -\infty$ .

Also, for  $I \subset \mathbb{R}$ , define

$$\mathcal{D}_\mu(j, I) = \left\{ \lambda \subset [0, 1]^d, \lambda \in \mathcal{D}_j : \frac{\log_2 \mu(\lambda)}{-j} \in I \right\}.$$

Next propositions complete the properties associated with multifractal analysis of capacities. Recall that the non-decreasing part of the spectrum  $\tau_\mu^*$  corresponds to the pointwise Hölder exponents  $\alpha \leq \tau_\mu'(0^-)$ , while the non-increasing part corresponds to

$\alpha \geq \tau'_\mu(0^+)$ . The following properties can be easily deduced from any of the following sources [2, 13, 47, 41, 3].

**Proposition 3.2.** *Let  $\mu \in \mathcal{C}([0, 1]^d)$  such that  $\text{supp}(\mu) \neq \emptyset$ .*

- (1) *For every  $\alpha \leq \tau'_\mu(0^-)$ ,  $\dim \underline{E}_\mu^{\leq}(\alpha) \leq \tau_\mu^*(\alpha)$ .*
- (2) *For every  $\alpha \geq \tau'_\mu(0^+)$ ,  $\dim \overline{E}_\mu^{\geq}(\alpha) \leq \tau_\mu^*(\alpha)$ .*

**Proposition 3.3.** *Let  $\mu \in \mathcal{C}([0, 1]^d)$  such that  $\text{supp}(\mu) \neq \emptyset$ . Suppose that  $\mu$  strongly obeys the multifractal formalism (Definition 2.5). One has  $\text{dom}(\tau_\mu^*) = \{\alpha \in \mathbb{R} : \tau_\mu^*(\alpha) \geq 0\}$ , and :*

- (1) *For every  $\alpha \in \mathbb{R}$ , one has*

$$\sigma_\mu(\alpha) = \dim E_\mu(\alpha) = \dim \overline{E}_\mu(\alpha) = \underline{\sigma}_\mu^{\text{LD}}(\alpha) = \overline{\sigma}_\mu^{\text{LD}}(\alpha) = \tau_\mu^*(\alpha).$$

- (2) *For every  $\alpha \leq \tau'_\mu(0^-)$ ,  $\dim \underline{E}_\mu^{\leq}(\alpha) = \tau_\mu^*(\alpha)$ .*
- (3) *For every  $\alpha \geq \tau'_\mu(0^+)$ ,  $\dim \overline{E}_\mu^{\geq}(\alpha) = \tau_\mu^*(\alpha)$ .*
- (4) *For every  $\eta > 0$  and every interval  $I \subset \text{dom}(\tau_\mu^*)$ , there exists  $\varepsilon_0 > 0$  and  $J_0 \in \mathbb{N}$  such that for every  $\varepsilon \in (0, \varepsilon_0)$  and  $j \geq J_0$ , denoting  $I + [-\varepsilon, \varepsilon]$  by  $I \pm \varepsilon$ , for  $\tilde{I} \in \{I, I \pm \varepsilon\}$  we have:*

$$\left| \frac{\log_2 \#\mathcal{D}_\mu(j, \tilde{I})}{j} - \sup_{\alpha \in I} \tau_\mu^*(\alpha) \right| \leq \varepsilon.$$

- (5) *If  $\text{dom}(\tau_\mu^*)$  is compact, then it equals  $[\tau'_\mu(\infty), \tau'_\mu(-\infty)]$  and there exists a positive decreasing sequence  $(\varepsilon_j)_{j \geq 0}$  such that for all  $j \in \mathbb{N}$  and  $\lambda \in \mathcal{D}_j$*

$$\tau'_\mu(\infty) - \varepsilon_j \leq \frac{\log_2 \mu(\lambda)}{-j} \leq \tau'_\mu(-\infty) + \varepsilon_j.$$

We will also make use of the following properties.

**Remark 3.4.** *In addition to the fact that  $\text{dom}(\tau_\mu^*) = [\tau'_\mu(\infty), \tau'_\mu(-\infty)]$ , Legendre transform properties imply that if  $t_\infty := (\tau_\mu^*)'(\tau'_\mu(\infty)) < \infty$ , then  $t_\infty = \inf\{t : \tau'_\mu(t) = \tau'_\mu(\infty)\}$ , and for all  $t \geq t_\infty$  one has  $\tau_\mu(t) = \tau'_\mu(\infty)t - \tau_\mu^*(\tau'_\mu(\infty))$ .*

*Similarly, if  $t_{-\infty} := (\tau_\mu^*)'(\tau'_\mu(-\infty)) > -\infty$ , then  $t_{-\infty} = \sup\{t : \tau'_\mu(t) = \tau'_\mu(-\infty)\}$ , and for all  $t \leq t_{-\infty}$  one has  $\tau_\mu(t) = \tau'_\mu(-\infty)t - \tau_\mu^*(\tau'_\mu(-\infty))$ .*

**Remark 3.5.** *When  $\mu$  is a positive measure, one has  $\tau_\mu^*(\alpha) = \alpha$  if and only if  $\alpha \in [\tau'_\mu(1^+), \tau'_\mu(1^-)]$  [46]. This justifies that if  $\mu$  obeys the multifractal formalism there must exist  $D$  such that  $\sigma_\mu(D) = D$ . Moreover, it is also clear that if  $\mu$  obeys the multifractal formalism and  $\mu$  is fully supported, any  $D' \in [\tau'_\mu(0^+), \tau'_\mu(0^-)]$  is such that  $\sigma_\mu(D') = \tau_\mu^*(D') = -\tau_\mu(0) = d$ .*

We now prove Theorem 2.10 in the case  $d = 1$ . The  $d$ -dimensional case follows immediately (see Section 3.7),

**3.2. A family of probability vectors associated with  $\sigma \in \mathcal{S}_{1,\mathcal{M}}$ .** Fix  $\sigma \in \mathcal{S}_{1,\mathcal{M}}$  (recall Definition 2.7). Our objective is to build an almost doubling measure  $\mu$  supported on the interval  $[0, 1]$  satisfying both (P) and the multifractal formalism strongly, with  $\tau_\mu^* = \sigma$ . Write  $\text{dom}(\sigma) = [\alpha_{\min}, \alpha_{\max}]$ . We suppose that  $\alpha_{\min} < \alpha_{\max}$ , for otherwise  $\alpha_{\min} = 1 = \alpha_{\max}$  and taking for  $\mu$  the Lebesgue measure on  $[0, 1]$  yields a solution to the inverse problem studied in this Section 3.

Let us start by introducing two parameters  $D, D'$  defined as follows:

- if  $\sigma(1) = 1$ , set  $D = D' = 1$ .
- if  $\sigma(1) \neq 1$ , let  $0 < D < 1 < D'$  be such that  $\sigma(D) = D$  and  $\sigma(D') = 1$ .

Then, fix an integer  $N_0$  large enough so that for all  $N \geq N_0$ , setting  $\varepsilon_N = 2 \log_2(N)/N$ , there exists a subset  $A_N = \{\alpha_{N,i} : i = 1, \dots, 2m_N\}$  of  $[\alpha_{\min}, \alpha_{\max}]$  satisfying:

- $m_N \leq 2N(\alpha_{\max} - \alpha_{\min})$ ;
- $D, D' \in A_N$ ;
- for every  $i \in \{1, \dots, m_N - 1\}$ ,  $(4N)^{-1} < \alpha_{N,i+1} - \alpha_{N,i} < N^{-1}$ ;
- the following inclusions hold:

$$(28) \quad A_N \subset \sigma^{-1}\left(\left[\frac{1}{N} + \varepsilon_N, 1\right]\right) \subset \bigcup_{i=1}^{m_N} \left[\alpha_{N,i} - \frac{1}{N}, \alpha_{N,i} + \frac{1}{N}\right];$$

- for every  $i \in \{m_N + 1, \dots, 2m_N\}$ ,  $\alpha_{N,i} = \alpha_{N,2m_N-i+1}$ ;
- if  $\sigma(\alpha_{\min}) > 0$ , then  $\alpha_{N,1} = \alpha_{\min}$ .

The continuity of  $\sigma$  is used to get (28), and when  $D \neq D'$  the above conditions impose that  $|D - D'| \geq (4N)^{-1}$ .

Heuristically, the intervals  $\left[\alpha_{N,i} - \frac{1}{N}, \alpha_{N,i} + \frac{1}{N}\right]$  form a covering of  $\text{dom}(\sigma)$  (apart from the extreme points of  $\text{dom}(\sigma)$  when  $\sigma$  vanishes there) by small intervals that do not overlap too much.

We denote by  $i_N$  (resp.  $i'_N$ ) the index in  $[1, m_N]$  such that  $D = \alpha_{N,i_N}$  (resp.  $D' = \alpha_{N,i'_N}$ ). Note that  $i_N = i'_N$  if and only if  $D = D' = 1$ , and that it may happen that  $\alpha_{\min} = D$ .

For each  $1 \leq i \leq m_N$ , such that  $i \notin \{i_N, i'_N\}$ , set

$$(29) \quad R_{N,i} = \left\lfloor 2^{N(\sigma(\alpha_{N,i}) - \varepsilon_N) - 1} \right\rfloor.$$

Note that for every  $i$ ,  $1 \leq R_{N,i} \leq 2^N N^{-2}$  due to (28).

When  $D = D'$ ,  $i_N = i'_N$  and we set

$$R_{N,i_N} = 2^{N-1} - \sum_{i=1, i \neq i_N}^{m_n} R_{N,i}.$$

When  $D < D'$ ,  $i_N < i'_N$  and we set

$$(30) \quad R_{N,i_N} = \left\lfloor 2^{N\sigma(\alpha_{N,i_N}) - 1} \right\rfloor = \left\lfloor 2^{ND-1} \right\rfloor \quad \text{and} \quad R_{N,i'_N} = 2^{N-1} - \sum_{i=1, i \neq i'_N}^{m_n} R_{N,i}.$$



In all cases, by construction

$$\sum_{i=1, i \neq i'_N}^{m_N} R_{N,i} \leq m_N 2^{N-1} N^{-2} + \mathbf{1}_{\{D \neq D'\}} 2^{ND-1} = o(2^{N-1}) \quad \text{as } N \rightarrow \infty$$

since the term  $\mathbf{1}_{\{D \neq D'\}} 2^{ND-1}$  appears if and only if  $D < 1$ . This also implies that

$$(31) \quad R_{N,i'_N} \geq 2^{N-1} - \sum_{i=1, i \neq i'_N}^{m_N} R_{N,i} = 2^{N-1}(1 + o(1)).$$

Without restriction, we choose  $N_0$  large enough so that for all  $N \geq N_0$ ,

$$(32) \quad \sum_{i=1, i \neq i'_N}^{m_N} R_{N,i} \leq 2^{N-2}.$$

Finally, for  $N \geq N_0$  and  $m_N < i \leq 2m_N$ , set  $R_{N,i} = R_{N,2m_N-i+1}$ , so that

$$\sum_{i=1}^{2m_N} R_{N,i} = 2^N.$$

We now introduce a collection of exponents  $(\beta_{N,i})_{0 \leq i \leq 2^N-1}$  by setting, for all  $1 \leq j \leq 2m_N$ ,

$$(33) \quad \beta_{N,i} = \alpha_{N,j} \quad \text{if } \sum_{k=1}^{j-1} R_{N,k} \leq i < \sum_{k=1}^j R_{N,k}.$$

In other words,  $(\beta_{N,i})_{0 \leq i \leq 2^N-1}$  is obtained by repeating  $R_{N,1}$  times the value  $\alpha_{N,1}$ ,  $R_{N,2}$  times the value  $\alpha_{N,2}$ , and so on.

**Lemma 3.6.** *Let  $p_N = (p_{N,i})_{0 \leq i \leq 2^N-1}$  be the probability vector defined by*

$$p_{N,i} = \frac{2^{-N\beta_{N,i}}}{\sum_{j=0}^{2^N-1} 2^{-N\beta_{N,j}}}.$$

For  $N$  large enough,

$$(34) \quad p_{N,i} 2^{N\beta_{N,i}} = 1 + \varepsilon_{N,i},$$

where  $\varepsilon_{N,i} = O(N^{-1})$  uniformly in  $0 \leq i < 2^N$ . Moreover, if  $|i - i'| \leq 1$ , then

$$(35) \quad \frac{p_{N,i}}{p_{N,i'}} \in [2^{-1}, 2].$$

Also,  $p_{N,0} = p_{N,2^N-1}$ .

*Proof.* By definition we have

$$p_{N,i} 2^{N\beta_{N,i}} = \frac{1}{2 \sum_{j=1}^{m_N} 2^{-N\alpha_{N,j}} R_{N,j}}.$$

In order to estimate  $p_{N,i}2^{N\beta_{N,i}}$  uniformly in  $i$ , recall that  $\sigma \leq \text{Id}_{\mathbb{R}}$ , so that using the definition of  $R_{N,i}$  and  $\varepsilon_N$ , one gets

$$\begin{aligned} \lfloor 2^{ND-1} \rfloor 2^{-ND} &= R_{N,i_N} 2^{-N\alpha_{N,i_N}} \\ &\leq \sum_{i=1}^{m_N} 2^{-N\alpha_{N,i}} R_{N,i} \\ &\leq \sum_{1 \leq i \neq i_N, i'_N \leq m_N} 2^{N(\sigma(\alpha_{N,i}) - \alpha_{N,i} - \varepsilon_N)} + R_{N,i_N} 2^{-ND} + \mathbf{1}_{D \neq D'} R_{N,i'_N} 2^{-ND'} \\ &\leq m_N N^{-2} + 2^{-ND} \lfloor 2^{ND-1} \rfloor + \mathbf{1}_{D \neq D'} 2^{N(1-D')}. \end{aligned}$$

Also, recall that when  $D \neq D'$ ,  $D < 1$  and  $D' > 1$ . Consequently, since  $\lfloor 2^{ND-1} \rfloor 2^{-ND} = 1/2 + o(1)$ , we deduce that (34) holds.

The fact that (35) holds if  $0 \leq i, i' \leq 2^N - 1$  and  $|i - i'| \leq 1$  follows from the choice  $\alpha_{N,i+1} - \alpha_{N,i} \leq N^{-1}$ .

Finally,  $p_{N,0} = p_{N,2^N-1}$  by definition of these parameters.  $\square$

Next we construct the desired measure.

**3.3. Construction of the measure  $\mu_\sigma$  associated with  $\sigma \in \mathcal{S}_{1,\mathcal{M}}$ .** We construct a Moran measure  $\mu_\sigma$  by using concatenation of pieces of Bernoulli product measures associated with the probability vectors  $(p_N)_{N \geq N_0}$ . The good property of  $(p_N)_{N \geq N_0}$  is that when  $N$  goes to infinity, the singularity spectrum of the Bernoulli product measure associated with  $p_N$  converges pointwise to  $\sigma$ . This comes from the fact that each  $p_N$  is built so that, heuristically, there are  $2^{N\sigma(\alpha_{N,i})}$  weights of order  $2^{-N\alpha_{N,i}}$  and the  $\alpha_{N,i}$  tend to be more or less uniformly distributed in the domain of  $\sigma$ .

We introduce further ingredients:

- For  $N \geq N_0$ , we fix an integer  $\ell_N \geq N^2$ ;
- we consider the product space

$$\Sigma = \prod_{N=N_0}^{\infty} \{0, \dots, 2^N - 1\}^{\ell_N};$$

- for  $N \geq N_0$ , if  $g = \ell + \sum_{n=N_0}^{N-1} \ell_n$  with  $1 \leq \ell \leq \ell_N$ , and

$$(J_{N_0}, J_{N_0+1}, \dots, J_{N-1}, J_N) \in \left( \prod_{n=N_0}^{N-1} \{0, \dots, 2^n - 1\}^{\ell_n} \right) \times \{0, \dots, 2^N - 1\}^\ell,$$

then  $[J_{N_0} \cdot J_{N_0+1} \cdots J_N]$  is the cylinder consisting of those elements in  $\Sigma$  with common prefix  $J_{N_0} \cdot J_{N_0+1} \cdots J_N$ ;

- $\Sigma_g$  and  $\mathcal{C}_g$  stand for the set of words and the set of cylinders in  $\Sigma$  of length (or generation)  $g$  respectively;
- the space  $\Sigma$  is endowed with the  $\sigma$ -field  $\mathcal{B}$  generated by the cylinders.

**Definition 3.7.** *The measure  $\nu_\sigma$  on  $(\Sigma, \mathcal{B})$  is defined as follows. For all  $N \geq N_0$ , for all  $1 \leq \ell \leq \ell_N$ , if  $g = \ell + \sum_{n=N_0}^{N-1} \ell_n$  and  $[J_{N_0} \cdot J_{N_0+1} \cdots J_N] \in \mathcal{C}_g$ , we set*

$$(36) \quad \nu_\sigma([J_{N_0} \cdot J_{N_0+1} \cdots J_N]) = \left( \prod_{n=N_0}^{N-1} \prod_{k=1}^{\ell_n} p_{n,j_{n,k}} \right) \prod_{k=1}^{\ell} p_{N,k},$$

where :

- for  $N_0 \leq n \leq N-1$ ,  $J_n = j_{n,1} \cdots j_{n,\ell_n} \in \{0, \dots, 2^n - 1\}^{\ell_n}$
- $J_N = j_{N,1} \cdots j_{N,\ell} \in \{0, \dots, 2^N - 1\}^\ell$ .

**Remark 3.8.** *Formula (36) could be written*

$$\nu_\sigma([J_{N_0} \cdot J_{N_0+1} \cdots J_N]) = \prod_{n=N_0}^N \mu_n(J_n),$$

where  $\mu_n$  is the Bernoulli measure associated with the parameters  $p_n = (p_{n,i})_{i=0, \dots, 2^n - 1}$ .

It is immediate to check that (36) is consistent, in the sense that for every integers  $g' > g \geq 1$ , for every cylinder  $J \in \mathcal{C}_g$ ,  $\nu_\sigma(J) = \sum_{J' \in \mathcal{C}_{g'}, J' \subset J} \nu_\sigma(J')$ , and  $\nu_\sigma(\Sigma) = 1$ .

By construction, using (34) we can deduce that there exists  $C > 0$  such that for each  $N \geq N_0$  and  $(J_n)_{N_0 \leq n \leq N} \in \prod_{n=N_0}^N \{0, \dots, 2^n - 1\}^{\ell_n}$ ,

$$\nu_\sigma([J_{N_0} \cdot J_{N_0+1} \cdots J_N]) \leq \prod_{n=N_0}^N ((1 + C/n) 2^{-n\alpha_{\min}})^{\ell_n},$$

hence  $\nu_\sigma$  is atomless since the right hand side tends to 0 as  $N$  tends to infinity.

Every  $g \in \mathbb{N}^*$  writes in a unique way under the form  $g = \ell + \sum_{n=N_0}^{N-1} \ell_n$  with  $N \geq N_0$  and  $1 \leq \ell \leq \ell_N$ . We set

$$(37) \quad \gamma(g) = N\ell + \sum_{n=N_0}^{N-1} n\ell_n.$$

The space  $\Sigma$  provides a natural coding of  $[0, 1]$ . Indeed, considering the coding map

$$(38) \quad \pi : x = \left( (x_{N,k})_{k=1}^{\ell_N} \right)_{N \geq N_0} \in \Sigma \mapsto \sum_{N=N_0}^{\infty} 2^{-\sum_{n=N_0}^{N-1} n\ell_n} \sum_{k=1}^{\ell_N} x_{N,k} 2^{-kN} \in [0, 1],$$

for each  $g \in \mathbb{N}^*$ ,  $\pi$  maps bijectively the elements of  $\mathcal{C}_g$  onto the set of closed dyadic subintervals of generation  $\gamma(g)$  of  $[0, 1]$ .

**Definition 3.9.** *For every  $\sigma \in \mathcal{S}_{1, \mathcal{M}}$ , consider the Borel probability measure on  $[0, 1]$*

$$\tilde{\mu}_\sigma = \nu_\sigma \circ \pi^{-1},$$

where  $\nu_\sigma$  is the measure constructed above (36). Then,  $\mu_\sigma$  is defined as the natural periodized version of  $\tilde{\mu}_\sigma$ , i.e. the  $\mathbb{Z}$ -invariant measure

$$\mu_\sigma : B \in \mathcal{B}(\mathbb{R}) \mapsto \sum_{k \in \mathbb{Z}} \mu((B \cap [k, k+1)) - k).$$

We set

$$\mathcal{M}_1 = \{\mu_\sigma : \sigma \in \mathcal{S}_{1,\mathcal{M}}\} \subset \mathcal{M}(\mathbb{R}).$$

We say that  $\mu_\sigma$  and  $\tilde{\mu}_\sigma$  are associated with  $\sigma \in \mathcal{S}_{1,\mathcal{M}}$ .

**Proposition 3.10.** *Every  $\mu \in \mathcal{M}_1$  satisfies the property (P) in Definition 2.9.*

*Moreover, if  $\mu$  is associated with  $\sigma \in \mathcal{S}_{1,\mathcal{M}}$ , then  $\mu|_{[0,1]}$  has  $\sigma$  as multifractal spectrum, and it strongly obeys the multifractal formalism on  $\mathbb{R}_+$ .*

Observe that since  $\nu_\sigma$  is atomless and  $\pi$  is 1-to-1 outside a countable set of points of  $\Sigma$ , for any closed dyadic subinterval  $\lambda$  of  $[0, 1]$  of generation  $n \in \gamma(\mathbb{N}^*)$ , we have  $\mu_\sigma(\lambda) = \nu_\sigma([w])$ , where  $[w]$  is the unique cylinder of generation  $\gamma^{-1}(n)$  such that  $\pi([w]) = \lambda$ .

Next sections are devoted to the proofs of the various properties concerning  $\mu_\sigma$ , which, in particular, yield Proposition 3.10.

For the rest of this section,  $\sigma \in \mathcal{S}_{1,\mathcal{M}}$  is fixed, and we simply call  $\nu$  and  $\mu$  the measures  $\mu \in \mathcal{M}_1$  associated with  $\sigma$ .

### 3.4. The measure $\mu$ satisfies property (P).

**Lemma 3.11.** *The measure  $\mu$  is almost doubling.*

*Proof.* Let  $g \in \mathbb{N}^*$  and write it under the form  $g = \ell + \sum_{n=N_0}^{N-1} \ell_n \in \mathbb{N}$  with  $N \geq N_0$  and  $1 \leq \ell \leq \ell_N$ .

First, note that if  $g$ , hence  $N$ , is large enough, the term  $1 + \varepsilon_{N,i}$  in (34) is greater than  $1/2$  and smaller than  $3/2$ . Hence, for any  $1 \leq i \leq 2m_N$ ,

$$(39) \quad 2^{-N(\alpha_{\max} + \tilde{\varepsilon}_N)} \leq p_{N,i} \leq 2^{-N(\alpha_{\min} - \tilde{\varepsilon}_N)},$$

where  $(\tilde{\varepsilon}_n)_{n \geq 1}$  is a non-increasing sequence (independent of  $i$ ), which converges to 0 as  $n$  tends to infinity.

Now fix  $\varepsilon \in (0, \alpha_{\min})$ . We start by dealing with the dyadic intervals of generation  $\gamma(g)$ .

Consider two closed dyadic subintervals  $\lambda$  and  $\hat{\lambda}$  of  $[0, 1]$  of generation  $\gamma(g)$  such that  $\lambda$  is the left neighbor of  $\hat{\lambda}$ . By construction,  $\lambda$  and  $\hat{\lambda}$  are the images of two cylinders  $[J]$  and  $[\tilde{J}]$  in  $\mathcal{C}_g$  such that, denoting by  $u$  the longest common prefix of the words  $J$  and  $\tilde{J}$ , there exist  $N \geq N_0$  and  $0 \leq j < 2^N - 2$  such that  $J = u \cdot j \cdot v$  and  $\tilde{J} = u \cdot (j+1) \cdot \tilde{v}$ , where either  $v$  and  $\tilde{v}$  are empty words, or all letters of  $v$  equals  $2^N - 1$  and all letters of  $\tilde{v}$  are 0. From (35) and the fact that  $p_{N,0} = p_{N,2^N-1}$ , one deduces that

$$(40) \quad 2^{-1} \leq \frac{\mu(\lambda)}{\mu(\hat{\lambda})} \leq 2.$$

Consider now two neighboring intervals  $\lambda$  and  $\hat{\lambda}$  of generation  $j$ , where  $\gamma(g) < j < \gamma(g) + N$ . Let  $\lambda'$  and  $\hat{\lambda}'$  be the elements of  $\mathcal{D}_{\gamma(g)}$  which contain  $\lambda$  and  $\hat{\lambda}$  respectively. These intervals are either equal or neighbors. By construction, if  $N$  is large enough, one has

$$(41) \quad 2^{-N(\alpha_{\max} + \varepsilon)} \leq \frac{\mu(\lambda)}{\mu(\lambda')} \leq 2^{-N(\alpha_{\min} - \varepsilon)}$$

(the same being true for  $\widehat{\lambda}$  and  $\widehat{\lambda}'$ ), hence

$$(42) \quad \frac{\mu(\lambda')}{\mu(\widehat{\lambda}')} 2^{-(\alpha_{\max} - \alpha_{\min} + 2\varepsilon)N} \leq \frac{\mu(\lambda)}{\mu(\widehat{\lambda})} = \frac{\mu(\lambda)}{\mu(\lambda')} \frac{\mu(\lambda')}{\mu(\widehat{\lambda}')} \frac{\mu(\widehat{\lambda}')}{\mu(\widehat{\lambda})} \leq \frac{\mu(\lambda')}{\mu(\widehat{\lambda}')} 2^{(\alpha_{\max} - \alpha_{\min} + 2\varepsilon)N}.$$

Let

$$(43) \quad \theta(j) = \begin{cases} 0 & \text{if } 0 \leq j < N_0 \ell_{N_0} \\ (1 + (\alpha_{\max} - \alpha_{\min} + 2\varepsilon))N, & \text{if } \gamma(g) \leq j < \gamma(g) + N. \end{cases}$$

Applying (40) to  $\lambda'$  and  $\widehat{\lambda}'$ , we deduce from (42) that for  $j$  large enough

$$(44) \quad 2^{-\theta(j)} \leq \frac{\mu(\lambda)}{\mu(\widehat{\lambda})} \leq 2^{\theta(j)},$$

Also,  $\theta(j)/j \leq (1 + (\alpha_{\max} - \alpha_{\min} + 2\varepsilon))N/\gamma(g)$  which tends to 0 as  $j \rightarrow \infty$ . This follows from the fact that  $\gamma(g) \geq \sum_{n=N_0}^{N-1} n \ell_n \gg N^2$  as  $N \rightarrow \infty$  since  $\ell_n \geq n^2$  for all  $n \geq N_0$ .

Upon adding a constant to  $\theta$ , we conclude that  $\mu|_{[0,1]}$  is almost doubling in the sense of Definition 2.8. To get that  $\mu$  is almost doubling on  $\mathbb{R}$ , it is enough to observe that by construction for any  $g \in \mathbb{N}$  we have  $\mu|_{[1-2^{-\gamma(g)},1]}(\cdot + 1 - 2^{-\gamma(g)}) = \mu|_{[0,2^{-\gamma(g)]}$ .  $\square$

**Lemma 3.12.** *The measure  $\mu$  satisfies (P).*

*Proof.* Due to the property pointed out above, i.e.  $\mu|_{[1-2^{-\gamma(g)},1]}(\cdot + 1 - 2^{-\gamma(g)}) = \mu|_{[0,2^{-\gamma(g)]}$  for all  $g \in \mathbb{N}^*$ , it is enough to consider subintervals of the interval  $[0, 1]$ .

Let  $\varepsilon > 0$ . For  $N \geq N_0$  and  $g = \ell + \sum_{n=N_0}^{N-1} \ell_n$  with  $1 \leq \ell \leq \ell_N$ , any dyadic interval  $\lambda \in \mathcal{D}_j$  with  $\gamma(g) \leq j < \gamma(g) + N$  satisfies, if  $N$  is large enough

$$2^{-(\gamma(g)+N)(\alpha_{\max} + \varepsilon/2)} \leq \mu(\lambda) \leq 2^{-\gamma(g)(\alpha_{\min} - \varepsilon/2)}.$$

(use (39) for instance). By our choice for  $\ell_N$ , for  $\gamma(g) \leq j < \gamma(g) + N$ ,  $\gamma(g)/j$  converges to 1 as  $j \rightarrow \infty$ . Hence, for  $j$  large enough

$$(45) \quad 2^{-j(\alpha_{\max} + \varepsilon)} \leq \mu(\lambda) \leq 2^{-j(\alpha_{\min} - \varepsilon)}.$$

So, (8) is satisfied with  $r_2 = \alpha_{\max} + \varepsilon$  and  $r_1 = \alpha_{\min} - \varepsilon$ , and some constant  $C > 0$ . This yields property (P<sub>1</sub>).

Let us now prove (P<sub>2</sub>). Let  $j, j' \in \mathbb{N}^*$  with  $j' > j$ , and consider two neighbouring dyadic intervals  $\lambda, \widehat{\lambda} \in \mathcal{D}_j$ , and an interval  $\lambda' \in \mathcal{D}_{j'}$  such that  $\lambda' \subset \lambda$ .

Let  $g, g' \in \mathbb{N}^*$  and  $N' \geq N \geq N_0$  such that:

- $\gamma(g) \leq j < \gamma(g) + N$ , where  $g = \ell + \sum_{n=N_0}^{N-1} \ell_n$  and  $1 \leq \ell \leq \ell_N$
- $\gamma(g') \leq j' < \gamma(g') + N'$ , where  $g' = \ell' + \sum_{n=N_0}^{N'-1} \ell_n$  and  $1 \leq \ell' \leq \ell_{N'}$ .

Due to the doubling property of  $\mu$  applied to  $\lambda$  and  $\widehat{\lambda}$ , we have

$$(46) \quad 2^{-\theta(j)} \frac{\mu(\lambda)}{\mu(\lambda')} \leq \frac{\mu(\widehat{\lambda})}{\mu(\lambda')} = \frac{\mu(\widehat{\lambda})}{\mu(\lambda)} \frac{\mu(\lambda)}{\mu(\lambda')} \leq 2^{\theta(j)} \frac{\mu(\lambda)}{\mu(\lambda')}.$$

For  $J \leq j$ , denote by  $\lambda_{|J}$  the unique element of  $\mathcal{D}_J$  which contains  $\lambda$ , and for  $j < J \leq j'$  denote by  $\lambda'_{|J}$  the unique element  $\tilde{\lambda}$  of  $\mathcal{D}_J$  such that  $\lambda' \subset \tilde{\lambda} \subset \lambda$ . We have

$$\frac{\mu(\lambda_{|\gamma(g)+N})}{\mu(\lambda'_{|\gamma(g')})} \leq \frac{\mu(\lambda)}{\mu(\lambda')} \leq \frac{\mu(\lambda_{|\gamma(g)})}{\mu(\lambda'_{|\gamma(g')+N'})}.$$

It is easily seen that  $N + N' = o(j) + o(j' - j)$  as  $j, j' \rightarrow \infty$ . Consequently, using the multiplicative structure of  $\mu$  and (39) yields a function  $\tilde{\theta} \in \Theta$ , as well as a constant  $C \geq 1$ , depending on  $\mu$  only, such that

$$(47) \quad C^{-1} 2^{-j\tilde{\theta}(j)} 2^{(j'-j)(\alpha_{\min}-\varepsilon)} \leq \frac{\mu(\lambda)}{\mu(\lambda')} \leq C 2^{j\tilde{\theta}(j)} 2^{(j'-j)(\alpha_{\max}+\varepsilon)}.$$

Incorporating (47) in (46) shows that (P<sub>2</sub>) holds with the same exponents  $r_1$  and  $r_2$  as (P<sub>1</sub>), and replacing  $\theta$  by  $\theta + \tilde{\theta}$  in (P<sub>1</sub>) yields the validity of (P<sub>1</sub>) and (P<sub>2</sub>) with the same element of  $\Theta$ .  $\square$

**Remark 3.13.** *We deduce from the previous estimates that for every  $\varepsilon > 0$ , there exists  $j_\varepsilon \in \mathbb{N}$  such that for all  $j' \geq j \geq j_\varepsilon$ , for all  $\lambda, \tilde{\lambda} \in \mathcal{D}_j$  such that  $\partial\lambda \cap \partial\tilde{\lambda} \neq \emptyset$ , and all  $\lambda' \in \mathcal{D}_{j'}$  such that  $\lambda' \subset \lambda$ ,*

$$(48) \quad \mu(\lambda') \leq \mu(\tilde{\lambda}) 2^{j\varepsilon} 2^{-(j'-j)(\alpha-\varepsilon)}.$$

*This inequality will be useful in finding an upper bound for the typical singularity spectrum in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ .*

**3.5. The  $L^q$ -spectrum of  $\mu_{|[0,1]}$  equals  $\sigma^*$ .** Let  $\tau = \sigma^*$ . Since  $\sigma \in \mathcal{S}_{1,\mathcal{M}}$ , we have  $\tau \in \mathcal{S}_{1,\mathcal{M}}$ .

We simply denote  $\mu_{|[0,1]}$  by  $\mu$ . For all  $j \in \mathbb{N}$ , let

$$\mathcal{D}_j^0 = \{\lambda \in \mathcal{D}_j : \lambda \subset [0, 1]^d\}.$$

Fix  $t \in \mathbb{R}$  and  $g = \ell + \sum_{n=N_0}^{N-1} \ell_n$  with  $N \geq N_0$  and  $1 \leq \ell \leq \ell_N$ .

The multiplicative structure defining  $\nu$  and  $\mu$  using concatenation of pieces of Bernoulli product measures yields

$$(49) \quad \sum_{\lambda \in \mathcal{D}_{\gamma(g)}^0} \mu(\lambda)^t = \left( \prod_{n=N_0}^{N-1} \left( \sum_{j=0}^{2^n-1} p_{n,j}^t \right)^{\ell_n} \right) \cdot \left( \sum_{j=0}^{2^N-1} p_{N,j}^t \right)^\ell.$$

For each  $n \geq N_0$ , using (34), one has  $C_{n,t}^{-1} 2^{-nt\beta_{n,i}} \leq p_{n,i}^t \leq 2^{-nt\beta_{n,i}} C_{n,t}$  where  $C_{n,t}$  tends to 1 when  $n \rightarrow \infty$  (and does not depend on  $i \in \{0, \dots, 2^n - 1\}$ ). Hence, using (33), the definition of the  $R_{n,i}$  and the inequality  $2R_{n,i} \leq 2^{n\sigma(\alpha_{n,i})}$  which follows from (29), we get

$$\begin{aligned} \sum_{j=0}^{2^n-1} p_{n,j}^t &\leq C_{n,t} \sum_{j=0}^{2^n-1} 2^{-tn\beta_{n,j}} \leq C_{n,t} \sum_{i=1}^{m_n} 2^{n(\sigma(\alpha_{n,i})-t\alpha_{n,i})} \\ &\leq C_{n,t} m_n 2^{-n \inf\{t\alpha - \sigma(\alpha) : \alpha \in \text{dom}(\sigma)\}} = C_{n,t} m_n 2^{-n\tau(t)}. \end{aligned}$$

Consequently,

$$\sum_{\lambda \in \mathcal{D}_{\gamma(g)}^0} \mu(\lambda)^t \leq 2^{-\gamma(g)\tau(t)} \cdot \left( \prod_{n=N_0}^{N-1} (C_{n,t} m_n)^{\ell_n} \right) \cdot (C_{N,t} m_N)^\ell.$$

Since  $\log(m_n) = o(n)$ ,  $\ell \log(m_N) + \sum_{n=N_0}^{N-1} \ell_n \log m_n = o(\gamma(g))$ . Combining this with the fact that  $(C_{n,t})_{n \geq N_0}$  converges to 1, one deduces that  $\left( \prod_{n=N_0}^{N-1} (C_{n,t} m_n)^{\ell_n} \right) \cdot (C_{N,t} m_N)^\ell = 2^{o(\gamma(g))}$  and

$$\tau_\mu(t) \geq \liminf_{g \rightarrow \infty} \frac{-1}{\gamma(g)} \log_2 \sum_{\lambda \in \mathcal{D}_{\gamma(g)}^0} \mu(\lambda)^t \geq \tau(t).$$

Let us move to the upper bound for  $\tau_\mu(t)$ .

Suppose first that  $\sigma(\tau'(t^+)) > 0$ . For  $n$  large enough, say  $N \geq N'_0 \geq N_0$ , choose an integer  $1 \leq i_{n,t} \leq m_n$ , distinct from  $i_n$  and  $i'_n$ , such that  $|\alpha_{n,i_{n,t}} - \tau'(t^+)| \leq 1/n$ . The fact that  $\sigma = \tau^*$  implies that  $t\tau'(t^+) - \tau(t) = \sigma(\tau'(t^+))$ . Moreover, the continuity of  $\sigma$  implies that  $\lim_{n \rightarrow \infty} \eta_n = 0$ , where  $\eta_n = \sigma(\alpha_{n,i_{n,t}}) - t\alpha_{n,i_{n,t}} + \tau(t)$ . Bounding from below the sums in (49) by the sum only over those integers  $j$  such that  $\beta_{n,j} = \alpha_{n,i_{n,t}}$  (see (33)), and using (29) again to bound  $2R_{N,i_{n,t}}$  from below, one gets

$$\begin{aligned} \sum_{\lambda \in \mathcal{D}_{\gamma(g)}^0} \mu(\lambda)^t &\geq \left( \prod_{n=N_0}^{N'_0-1} \left( \sum_{j=0}^{2^n-1} p_{n,j}^t \right)^{\ell_n} \right) \cdot \left( \prod_{n=N'_0}^{N-1} \left( C_{n,t}^{-1} \lfloor 2^{n(\sigma(\alpha_{n,i_{n,t}}) - \varepsilon_n)} \rfloor 2^{-tn\alpha_{n,i_{n,t}}} \right)^{\ell_n} \right) \\ &\quad \cdot \left( C_{N,t}^{-1} \lfloor 2^{N(\sigma(\alpha_{N,i_{N,t}}) - \varepsilon_N)} \rfloor 2^{-tN\alpha_{N,i_{N,t}}} \right)^\ell. \end{aligned}$$

Recalling that  $\varepsilon_n = \frac{2 \log_2 n}{n}$ , and setting  $C_t = \prod_{n=N_0}^{N'_0-1} \left( \sum_{j=0}^{2^n-1} p_{n,j}^t \right)^{\ell_n}$ , we get

$$\begin{aligned} \sum_{\lambda \in \mathcal{D}_{\gamma(g)}^0} \mu(\lambda)^t &\geq C_t \left( \prod_{n=N'_0}^{N-1} \left( C_{n,t}^{-1} \frac{2^{n(\sigma(\alpha_{n,i_{n,t}}) - t\alpha_{n,i_{n,t}})}}{4n^2} \right)^{\ell_n} \right) \cdot \left( C_{N,t}^{-1} \frac{2^{N(\sigma(\alpha_{N,i_{N,t}}) - t\alpha_{N,i_{N,t}})}}{4N^2} \right)^\ell \\ &= C_t 2^{-\gamma(g)\tau(t)} \left( \prod_{n=N'_0}^{N-1} \left( C_{n,t}^{-1} \frac{2^{n\eta_n}}{4n^2} \right)^{\ell_n} \right) \cdot \left( C_{N,t}^{-1} \frac{2^{N\eta_N}}{4N^2} \right)^\ell \\ &= 2^{-\gamma(g)(\tau(t)+o(1))} \end{aligned}$$

as  $g \rightarrow +\infty$ , where we used that  $\log(C_{n,t}) + n\eta_n + \log(4n^2) = o(n)$ . The last lines imply that

$$(50) \quad \lim_{g \rightarrow \infty} -\frac{1}{n} \log_2 \sum_{\lambda \in \mathcal{D}_{\gamma(g)}^0} \mu(\lambda)^t = \tau(t).$$

For the integers  $n$  such that  $\gamma(g) \leq n < \gamma(g) + N$ , one remarks that if  $\lambda \in \mathcal{D}_{\gamma(g)}^0$  by (41) one has

$$2^{(n-\gamma(g))(1-|t|(\alpha_{\max}+\varepsilon))} \leq \sum_{\lambda' \in \mathcal{D}_n, \lambda' \subset \lambda} \frac{\mu(\lambda')^t}{\mu(\lambda)^t} \leq 2^{(n-\gamma(g))((1+|t|)(\alpha_{\max}+\varepsilon))}.$$

Since  $n - \gamma(g) = o(\gamma(g))$  as  $g \rightarrow \infty$ , one deduces that

$$\sum_{\lambda \in \mathcal{D}_n^0} \mu(\lambda)^t = 2^{o(\gamma(g))} \sum_{I \in \mathcal{D}_{\gamma(g)}^0} \mu(I)^t.$$

This, combined with (50), yields

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log_2 \sum_{\lambda \in \mathcal{D}_n^0} \mu(\lambda)^t = \tau(t).$$

Hence  $\tau_\mu(t) = \tau(t) = \sigma^*(t)$  on the interval of those  $t \in \mathbb{R}$  such that  $\sigma(\tau'(t^+)) > 0$ .

It remains us to consider the extremal case  $\sigma(\tau'(t^+)) = 0$ , which may happen only if  $\tau'(t^+) \in \{\alpha_{\min}, \alpha_{\max}\}$ .

Suppose that  $\tau'(t^+) = \alpha_{\min}$  and  $\sigma(\alpha_{\min}) = 0$ . One has  $0 = \sigma(\alpha_{\min}) = \tau^*(\alpha_{\min}) = t^+ \tau'(t^+) - \tau(t)$ , so  $\tau(t) = t\alpha_{\min}$ , and  $t_0 = \min\{t \in \mathbb{R} : \tau(t) = \alpha_{\min}t\} < \infty$ . In addition,  $t_0 > 0$  since  $\tau(0) < 0$ . Also, for  $t \in [0, t_0)$ ,  $\sigma(\tau'(t^+)) \in (0, 1]$  and we know from the first part of this proof that  $\tau_\mu(t) = \tau(t)$  on this interval  $[0, t_0)$ . To conclude, let us show that this last equality holds over the whole interval  $[t_0, \infty)$  as well.

At first, for all  $t \geq t_0$ ,  $\varepsilon \in (0, t_0)$  and  $n \in \mathbb{N}$ , by subadditivity of  $x \geq 0 \mapsto x^{t/(t_0-\varepsilon)}$ ,

$$\sum_{\lambda \in \mathcal{D}_n^0} \mu(\lambda)^t \leq \left( \sum_{\lambda \in \mathcal{D}_n^0} \mu(\lambda)^{t_0-\varepsilon} \right)^{t/(t_0-\varepsilon)},$$

so

$$(51) \quad \tau(t) = \liminf_{n \rightarrow \infty} -\frac{1}{n} \log_2 \sum_{\lambda \in \mathcal{D}_n^0} \mu(\lambda)^t \geq \frac{t}{t_0 - \varepsilon} \tau(t_0 - \varepsilon).$$

On the other hand, consider the interval  $[0, 2^{-\gamma(g)}]$  in  $\mathcal{D}_{\gamma(g)}$ . Its  $\mu$ -mass is by construction  $2^{-\gamma(g)\alpha_{\min}}$ , so

$$\limsup_{n \rightarrow \infty} -\frac{1}{n} \log_2 \sum_{\lambda \in \mathcal{D}_n^0} \mu(\lambda)^t \leq \limsup_{g \rightarrow \infty} -\frac{1}{\gamma(g)} \log_2 2^{-t\gamma(g)\alpha_{\min}} = \alpha_{\min}t.$$

Letting  $\varepsilon \rightarrow 0$  in (51) and since  $\alpha_{\min} = \tau(t_0)/t_0$ , one gets that  $\tau_\mu(t) = \alpha_{\min}t = \tau(t)$ .

The case  $\tau'(t^+) = \alpha_{\max}$  and  $\sigma(\alpha_{\max}) = 0$  works similarly by considering  $t_0 = \max\{t \in \mathbb{R} : \tau(t) = \alpha_{\max}t\} \in (-\infty, 0)$ , and the element of  $\mathcal{D}_{\gamma(g)}$  whose  $\mu$ -mass is minimal.



**3.6. The multifractal formalism holds strongly for  $\mu$ , with  $\sigma_\mu = \sigma$ .** First, the fact that  $E_\mu(\alpha) = \emptyset$  for all  $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$  and that  $\dim E_\mu(\alpha) \leq \sigma(\alpha)$  for  $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ , follows from Proposition 3.2 and the previous section where we proved that  $\tau_\mu^* = \sigma$  (so  $\tau_\mu^*(\alpha) = -\infty$  if  $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$ ).

Further, it follows from the construction and the choice of the weights  $p_{n,i}$  that there exist points  $x$  at which  $h_\mu(x) = \alpha_{\min}$  and points  $x$  at which  $h_\mu(x) = \alpha_{\max}$ . Hence,  $\sigma_\mu(\alpha_{\min}) \geq 0$  and  $\sigma_\mu(\alpha_{\max}) \geq 0$ .

In particular, if  $\sigma(\alpha_{\min}) = 0$  (*resp.*  $\sigma(\alpha_{\max}) = 0$ ), then  $\sigma_\mu(\alpha_{\min}) = 0$  (*resp.*  $\sigma_\mu(\alpha_{\max}) = 0$ ) and the multifractal formalism holds strongly at  $\alpha_{\min}$  (*resp.*  $\alpha_{\max}$ ).

Now, fix  $\alpha \in [\alpha_{\min}, \alpha_{\max}]$  such that  $\sigma(\alpha) > 0$ . For each  $N \geq N_0$ , let

$$(52) \quad \mathcal{J}_{N,\alpha} = \{j \in \{0, \dots, 2^N - 1\} : j \text{ is odd and } |\beta_{N,j} - \alpha| \leq N^{-1}\}.$$

Let  $\varepsilon > 0$ . Recalling the definitions of Section 3.2 we first observe that the exponents  $\beta_{N,j}$  considered in the definition of  $\mathcal{J}_{N,\alpha}$  correspond to at most four distinct exponents  $\alpha_{N,i}$ . This observation, together with the continuity of  $\sigma$  and the definition of the numbers  $R_{N,i}$  imply that for  $N$  large enough we have

$$(53) \quad 2^{N(\sigma(\alpha) - \varepsilon)} \leq \#\mathcal{J}_{N,\alpha} \leq 2^{N(\sigma(\alpha) + \varepsilon)}.$$

Consider the measure  $\nu_\alpha$  supported on

$$\Sigma_\alpha = \prod_{n=N_0}^{\infty} \mathcal{J}_{n,\alpha}^{\ell_n} \subset \Sigma$$

defined by setting, for each  $N \geq N_0$ ,  $0 \leq \ell < \ell_N$  and for every word  $J_{N_0} \cdot J_{N_0+1} \cdots J_N \in \left( \prod_{n=N_0}^{N-1} \{0, \dots, 2^n - 1\}^{\ell_n} \right) \times \{0, \dots, 2^N - 1\}^{\ell}$ :

$$\nu_\alpha([J_{N_0} \cdots J_N]) = \begin{cases} (\#\mathcal{J}_{N,\alpha})^{-\ell} \prod_{n=N_0}^{N-1} (\#\mathcal{J}_{n,\alpha})^{-\ell_n} & \text{if } [J_{N_0} \cdots J_N] \cap \Sigma_\alpha \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

One checks that this last formula is consistent, and the measure  $\nu_\alpha$  is well-defined and atomless. Consider now  $\mu_\alpha = \nu_\alpha \circ \pi^{-1}$ , the push-forward measure of  $\nu_\alpha$  on the interval  $[0, 1]$ , see (38). Heuristically, due to (52) it should be expected that  $\mu_\alpha$  is concentrated on points of  $[0, 1]$  where the measure  $\mu$  has a local dimension equal to  $\alpha$ . This is indeed the case.

For all  $\omega \in \Sigma_\alpha$ , denote by  $[\omega]_g$  the cylinder of generation  $g \in \mathbb{N}$  which contains  $\omega$ . From the definition of  $\mathcal{J}_{N,\alpha}$ ,

$$\alpha - \varepsilon \leq \liminf_{g \rightarrow +\infty} -\frac{1}{\gamma(g)} \log (\mu(\pi([\omega]_g))) \leq \limsup_{g \rightarrow \infty} -\frac{1}{\gamma(g)} \log (\mu(\pi([\omega]_g))) \leq \alpha + \varepsilon.$$

Since this holds for every choice of  $\varepsilon > 0$ , one has

$$\lim_{g \rightarrow \infty} -\frac{1}{\gamma(g)} \log (\mu(\pi([\omega]_g))) = \alpha.$$

Moreover,  $\lim_{g \rightarrow \infty} \frac{\gamma(g+1)}{\gamma(g)} = 1$  and  $\mu$  is almost doubling, so  $\pi(\Sigma_\alpha) \subset E_\mu(\alpha)$ .

On the other hand, from (53) one deduces that

$$\sigma(\alpha) - \varepsilon \leq \liminf_{g \rightarrow \infty} \frac{-1}{\gamma(g)} \log (\mu_\alpha(\pi([\omega]_g))) \leq \limsup_{g \rightarrow \infty} \frac{-1}{\gamma(g)} \log (\mu_\alpha(\pi([\omega]_g))) \leq \sigma(\alpha) + \varepsilon.$$

Again, this holds for every choice of  $\varepsilon > 0$ , hence

$$\lim_{g \rightarrow \infty} -\frac{1}{\gamma(g)} \log (\mu_\alpha(\pi([\omega]_g))) = \sigma(\alpha).$$

Since  $\lim_{g \rightarrow \infty} \frac{\gamma(g+1)}{\gamma(g)} = 1$ , the measure  $\mu_\alpha$ , which is supported by  $\pi(\Sigma_\alpha)$ , is exact dimensional with dimension  $\sigma(\alpha)$ , so  $\dim(\Sigma_\alpha) \geq \sigma(\alpha)$ .

The combination of the last two facts imply that  $\sigma_\mu(\alpha) = \dim E_\mu(\alpha) \geq \sigma(\alpha)$ . Since the converse inequality holds true by the multifractal formalism, the proof is complete.

**3.7. The case  $d \geq 2$ .** If  $\sigma \in \mathcal{S}_{d,\mathcal{M}}$ , then the map  $\tilde{\sigma} : \alpha \in \mathbb{R} \mapsto d^{-1}\sigma(d \cdot \alpha)$  belongs to  $\mathcal{S}_{1,\mathcal{M}}$ . Let  $\tilde{\mu}_{\tilde{\sigma}}$  be the measure associated with  $\tilde{\sigma}$  as built in the previous sections in dimension 1. Then, it is easily checked that the tensor product measure  $\mu = \tilde{\mu}_{\tilde{\sigma}}^{\otimes d}$  possesses all the required properties.

In addition, for all  $\alpha \in \text{dom}(\sigma)$ , if  $\tilde{\nu}_{d^{-1}\alpha}$  is the measure built in Section 3.6 associated with the exponent  $d^{-1}\alpha$ , then the measure  $\nu_\alpha := (\tilde{\nu}_{d^{-1}\alpha})^{\otimes d}$  satisfies the same properties as the one in Section 3.6.

**Definition 3.14.** Set  $\mathcal{M}_d = \{\mu^{\otimes d} : \mu \in \mathcal{M}_1\}$ .

By construction, for an outer measure  $\mu \in \mathcal{M}_d$  and its associated auxiliary measures  $\nu_\alpha$ , the inequalities (45), (48) and (53) and all those of Section 3.6 still hold true.

We end this Section 3 with a property which will play a key role in determining the singularity spectrum of typical elements in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  when  $p < \infty$  and  $\sigma_\mu(\alpha_{\min}) > 0$ .

**3.8. A conditioned ubiquity property associated with the elements of  $\mathcal{M}_d$ .**

Let  $\mu \in \mathcal{M}_d$ . In this section, we deal with those points  $x \in \mathbb{R}^d$  which are infinitely often close to dyadic vectors  $2^{-j}k \in \mathbb{R}^d$  such that the order of magnitude of  $\mu(\lambda_{j,k})$  is  $2^{-j\alpha_{\min}}$ .

**Definition 3.15.** A dyadic vector  $2^{-j}k$ ,  $k \in \mathbb{Z}^d$ ,  $j \in \mathbb{N}$  is irreducible when  $k \in \mathbb{Z}^d \setminus (2\mathbb{Z})^d$ .

The irreducible representation of a dyadic vector  $2^{-j}k$  with  $k \in \mathbb{Z}^d$ ,  $j \in \mathbb{N}$ , is the unique irreducible dyadic number  $\bar{k}2^{-\bar{j}}$  such that  $k2^{-j} = \bar{k}2^{-\bar{j}}$ .

If  $\lambda = 2^{-j}(k + [0, 1]^d) \in \mathcal{D}_j$ , then its associated irreducible cube is  $\bar{\lambda} := 2^{-\bar{j}}(\bar{k} + [0, 1]^d) \in \mathcal{D}_{\bar{j}}$ , where  $2^{-\bar{j}}\bar{k}$  is the irreducible representation of  $2^{-j}k$ .

Observe that  $\lambda$  is the dyadic cube of generation  $j$  located at the ‘‘bottom-left’’ corner of  $\bar{\lambda}$ . We write  $\lambda = \bar{\lambda} \cdot [0, 2^{-(j-\bar{j})}]^d$ , the concatenation meaning that  $\lambda$  equals the image of  $[0, 2^{-(j-\bar{j})}]^d$  by the canonical isometry which maps  $[0, 1]^d$  onto  $\bar{\lambda}$ .

Also, by construction of  $\mu$ , for all integers  $j, j' \geq 0$  and  $\lambda \in \mathcal{D}_j$ , one has

$$(54) \quad \mu(\lambda \cdot [0, 2^{-j'}]^d) = \mu(\lambda)2^{-j\varepsilon_\lambda}2^{-j'(\alpha_{\min} + \varepsilon_{\lambda,j'})},$$

where  $\lim_{j \rightarrow \infty} \sup\{|\varepsilon_\lambda| : \lambda \in \mathcal{D}_j\} = 0$  and  $\lim_{j' \rightarrow \infty} \sup\{|\varepsilon_{\lambda, j'}| : \lambda \in \bigcup_{j \in \mathbb{N}} \mathcal{D}_j\} = 0$ . This property will be applied several times to dyadic cubes associated to irreducible dyadic vectors.

**Definition 3.16.** For  $\delta > 1$ ,  $\eta > 0$ , and  $j \geq 1$ , let  $(j)_\delta$  be the largest integer in  $\gamma(\mathbb{N}) \cap [0, j/\delta]$  (recall the definition (37) of the mapping  $\gamma$ ).

For any positive sequence  $\eta = (\eta_j)_{j \geq 1}$ , let us define the set

$$X_j(\delta, \eta) = \left\{ k2^{-(j)_\delta} \in [0, 1]^d : \begin{cases} k \in \mathbb{Z}^d \setminus 2\mathbb{Z}^d, \\ \mu(2^{-(j)_\delta}(k + [0, 1]^d)) \geq 2^{-(j)_\delta(\alpha_{\min} + \eta_{(j)_\delta})} \\ \mu(2^{-(j)_\delta}k + 2^{-j}[0, 1]^d) \geq 2^{-j(\alpha_{\min} + \eta_j)} \end{cases} \right\}.$$

Recall that by construction and (45),  $\mu(2^{-(j)_\delta}(k + [0, 1]^d)) \leq 2^{-(j)_\delta(\alpha_{\min} - \varepsilon)}$  and  $\mu(2^{-(j)_\delta}k + 2^{-j}[0, 1]^d) \leq 2^{-j(\alpha_{\min} - \varepsilon)}$ . Hence, this set  $X_j(\delta, \eta)$  contains irreducible dyadic points of generation  $(j)_\delta$  whose  $\mu$ -mass is controlled both at generation  $(j)_\delta$  and at generation  $j$  by the exponent  $\alpha_{\min}$  (note that  $(j)_\delta \sim j/\delta$ ).

**Definition 3.17.** For any increasing sequence of integers  $(j_n)_{n \geq 1}$ , and any positive sequence  $\eta = (\eta_j)_{j \geq 1}$ , set

$$S(\delta, \eta, (j_n)_{n \geq 1}) = \bigcap_{N \geq 1} \bigcup_{n \geq N} \bigcup_{x \in X_{j_n}(\delta, \eta)} (x + 2^{-j_n}[0, 1]^d).$$

Heuristically, this set contains points which are approximated at rate  $\delta$  by irreducible dyadic points  $k2^{-j_n(\delta)}$  whose local dimension for  $\mu$  is locally controlled by  $\alpha_{\min}$  at generations  $j_n$  and  $j_n(\delta)$ . The terminology ‘‘approximation rate  $\delta$ ’’ comes from the fact that  $j_n(\delta) \sim j_n/\delta$ .

Recall that the lower Hausdorff dimension of a Borel probability measure  $\nu$  on  $\mathbb{R}^d$  is the infimum of the Hausdorff dimension of the Borel sets of positive  $\nu$ -measure.

**Proposition 3.18.** Suppose that  $\sigma_\mu(\alpha_{\min}) > 0$ .

There is a positive sequence  $\eta = (\eta_j)_{j \geq 1}$  converging to 0 when  $j$  tends to  $\infty$  such that for any  $\delta > 1$ , for any increasing sequence of integers  $(j_n)_{n \geq 1}$ , there exists a Borel probability measure  $\nu$  on  $\mathbb{R}^d$  of lower Hausdorff dimension larger than or equal to  $\sigma_\mu(\alpha_{\min})/\delta$ , and such that  $\nu(S(\delta, \eta, (j_n)_{n \geq 1})) = 1$ .

In particular,  $\dim S(\delta, \eta, (j_n)_{n \geq 1}) \geq \sigma_\mu(\alpha_{\min})/\delta$ .

**Remark 3.19.** The previous result is proved in [34] in the case that  $\mu$  is the Lebesgue measure.

*Proof.* We first deal with the case  $d = 1$ . We simply denote  $\sigma_\mu$  by  $\sigma$ .

Fix  $\delta > 1$  and an increasing sequence of integers  $(j_n)_{n \geq 1}$ . We are going to construct a Cantor subset  $\mathcal{K}$  of  $S(\delta, \eta, (j_n)_{n \geq 1})$  and a Borel probability measure  $\nu$  supported on  $\mathcal{K}$  such that for all closed dyadic subcube  $\lambda$  of  $[0, 1]^d$  of generation  $j \geq 0$ , one has  $\nu(\lambda) \leq 2^{-j(\delta^{-1}\sigma(\alpha_{\min}) - \psi(j))}$ , where the function  $\psi : \mathbb{N} \rightarrow (0, \infty)$  tends to 0 as  $n \rightarrow \infty$ . The mass distribution principle allows then to conclude that  $\dim S(\delta, \eta, (j_n)_{n \geq 1}) \geq \sigma(\alpha_{\min})/\delta$ .

**Preliminary observation.** Recall the construction of the measure  $\mu$  and the notations of Section 3.2. We start with a definition.

**Definition 3.20.** We say that a point  $x \in [0, 1]$  satisfies property  $P(\alpha_{\min}, g)$ ,  $g \in \mathbb{N}^*$ , if there exists a word  $w \in \Sigma_g$  such that  $x \in \pi([w])$  and after writing  $g = \ell + \sum_{n=N_0}^{N-1} \ell_n$ , with  $N \geq N_0$  and  $1 \leq \ell \leq \ell_N$ , one has  $\beta_{n,j} = \alpha_{\min}$  for all  $N_0 \leq n \leq N-1$  and  $1 \leq j \leq \ell_n$ , as well as  $\beta_{N,j} = \alpha_{\min}$  for all  $1 \leq j \leq \ell$ .

It is direct to see that there exists a sequence  $(\eta_j)_{j \geq 1}$  such that for all  $x \in [0, 1]$ , for all  $g \geq 1$ , if  $x$  satisfies property  $P(\alpha_{\min}, g)$ , then for all  $1 \leq j \leq \gamma(g)$ , one has

$$\mu(\lambda_j(x)) \geq 2^{-j(\alpha_{\min} + \eta_j)}.$$

We fix such a sequence  $\eta = (\eta_j)_{j \geq 1}$ .

We now proceed in three steps. Notations and definitions of Section 3.6 will be.

**Step 1:** Construction of a family of measures  $(\nu^\lambda)_{\lambda \in \mathcal{D}}$ .

Below we construct a family of auxiliary measures indexed by the closed dyadic subintervals of  $[0, 1]$ , in a very similar way to that used to get the measure  $\mu_{\alpha_{\min}}$  built in Section 3.6.

Let us introduce a notation: for  $j \in \mathbb{N}^*$ , set

$$N(j) = \begin{cases} N_0 & \text{if } 1 \leq j \leq \ell_{N_0} N_0, \\ N & \text{if } j > \ell_{N_0} N_0 \text{ and } \gamma(\sum_{n=N_0}^{N-1} \ell_n) < j \leq \gamma(\sum_{n=N_0}^N \ell_n). \end{cases}$$

Let  $N \geq N_0 + 1$ ,  $1 \leq \ell \leq \ell_N$ , and  $g = \ell + \sum_{n=N_0}^{N-1} \ell_n$ . Let  $J$  be an integer such that  $\gamma(g-1) < J \leq \gamma(g)$ . Note that  $J \geq j_0 := \ell_{N_0} N_0 + 1$ .

We fix  $\lambda \in \mathcal{D}_J$ , and construct a measure  $\nu^\lambda$  supported on  $\lambda$  as follows:

For each  $n \geq N = N(J)$ , consider

$$(55) \quad \mathcal{J}_{n, \alpha_{\min}} = \{j \in \{0, \dots, 2^n - 1\} : j \text{ is odd and } \beta_{n,j} = \alpha_{\min}\}.$$

Using (29) and (30), we can get that

$$(56) \quad \#\mathcal{J}_{n, \alpha_{\min}} \geq 2^{n(\sigma(\alpha_{\min}) - 2\varepsilon_n)}.$$

Writing  $\lambda = K2^{-J} + 2^{-J}[0, 1]$ , denote by  $\lambda_g \subset \lambda$  the dyadic subinterval  $K2^{-J} + 2^{-\gamma(g)}[0, 1]$  and  $[w_{\lambda_g}]$  the unique cylinder such that  $\pi([w_{\lambda_g}]) = \lambda_g$ . Observe that  $[w_{\lambda_g}] \in \mathcal{C}_g$ , the set of cylinders of generation  $g$  in  $\Sigma$ . Then, consider the set

$$\Sigma^\lambda = \{w_{\lambda_g}\} \times (\mathcal{J}_{N, \alpha_{\min}})^{\ell_N - \ell} \times \prod_{n=N+1}^{\infty} (\mathcal{J}_{n, \alpha_{\min}})^{\ell_n} \subset \Sigma,$$

and for each  $n \geq N$  and  $w \in \Sigma_g \times \{0, \dots, 2^N - 1\}^{\ell_N - \ell} \times \prod_{k=N+1}^n \{0, \dots, 2^k - 1\}^{\ell_k}$  one sets

$$\rho^\lambda([w]) = \begin{cases} (\#\mathcal{J}_{N, \alpha_{\min}})^{-\ell_N + \ell} \prod_{k=N+1}^n (\#\mathcal{J}_{k, \alpha_{\min}})^{-\ell_k} & \text{if } [w] \cap \Sigma^\lambda \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

This yields an atomless measure  $\rho$  whose support is  $\Sigma^\lambda$ , and we define  $\nu^\lambda = \rho^\lambda \circ \pi^{-1}$ .

By construction of  $\nu^\lambda$ , using (56), for  $g' \geq g$  and  $\lambda' \in \mathcal{D}_{\gamma(g')}$ , one has either  $\nu^\lambda(\lambda') = 0$  or  $\lambda' \cap \pi(\Sigma^\lambda) \neq \emptyset$  and

$$\nu^\lambda(\lambda') \leq 2^{-(\gamma(g')-\gamma(g))(\sigma(\alpha_{\min})-2\varepsilon_{N(J)})} \leq 2^{-(\gamma(g')-J)(\sigma(\alpha_{\min})-2\varepsilon_{N(J)})} 2^{N(J)\sigma(\alpha_{\min})}.$$

Consequently, for every  $g' \geq g$  and  $\gamma(g') < j \leq \gamma(g' + 1)$ , for  $\lambda' \in \mathcal{D}_j$  one has

$$(57) \quad \nu^\lambda(\lambda') \leq 2^{-(j-J)(\sigma(\alpha_{\min})-2\varepsilon_{N(J)})} 2^{2N(j)\sigma(\alpha_{\min})}.$$

This inequality extends easily to all integers  $j$  such that  $J \leq j \leq \gamma(g)$  and  $\lambda' \in \mathcal{D}_j$ .

**Remark 3.21.** *By construction, since only odd integers  $j$  are considered in the definition of the sets  $\mathcal{I}_{n,\alpha_{\min}}$ , if  $\widehat{\lambda} \not\subseteq \lambda$  and  $\nu^\lambda(\widehat{\lambda}) > 0$ , then  $\widehat{\lambda} = \lambda_{\widehat{j},\widehat{k}}$  with  $\widehat{k}2^{-\widehat{j}}$  irreducible. Moreover, writing  $\gamma(\widehat{g}) < \widehat{j} \leq \gamma(\widehat{g} + 1)$ , if property  $P(\alpha_{\min}, g)$  (see Definition 3.20) holds for all  $x \in \lambda$ , then  $P(\alpha_{\min}, \widehat{g})$  holds for all  $x \in \widehat{\lambda}$ .*

We finally set  $\nu^\lambda = \nu^{[0,2^{-j_0}]}$  if  $\lambda \in \bigcup_{j=1}^{j_0-1} \mathcal{D}_j$  and  $\lambda \subset [0, 1]$ .

**Step 2:** Construction of a Cantor set  $\mathcal{K} \subset S(\delta, (\eta_j)_{j \geq 1}, (j_n)_{n \geq 1})$  and a Borel probability measure  $\nu$  supported on  $K$ .

Recall that  $j_0 = N_0 \ell_{N_0} + 1$ . Set  $n_1 = 0$ ,  $\mathcal{G}_1 = \{[0, 2^{-j_0}]\}$  and define a function  $\nu$  on  $\mathcal{G}_1$  by  $\nu([0, 2^{-j_0}]) = 1$ . Note that  $\gamma(\ell_0) < j_0 \leq \gamma(\ell_0 + 1)$ , and that for all  $x \in [0, 2^{-j_0}]$ , property  $P(\alpha_{\min}, \ell_0)$  holds.

Let  $p$  be a positive integer. Suppose that we have constructed  $p$  families  $\mathcal{G}_1, \dots, \mathcal{G}_p$  of closed dyadic intervals, as well as  $p$  integers  $0 = n_1 < n_2 < \dots < n_p$  such that:

- (a) for every  $k \in \{1, \dots, p\}$ , writing  $\gamma(g) < j_{n_k} \leq \gamma(g + 1)$ , every  $x \in \lambda$  satisfies property  $P(\alpha_{\min}, g)$ ;
- (b) for every  $k \in \{2, \dots, p\}$ ,  $(j_{n_k})_\delta \geq j_0$ ;
- (c) for every  $k \in \{2, \dots, p\}$ ,  $\mathcal{G}_k \subset \{x + 2^{-j_{n_k}}[0, 1]^d : x \in X_{j_{n_k}}(\delta, \eta)\} \subset \mathcal{D}_{j_{n_k}}$ ;
- (d) for every  $k \in \{2, \dots, p\}$ , the irreducible intervals  $\bar{\lambda}, \lambda \in \mathcal{G}_k$ , are pairwise disjoint;
- (e) for every  $k \in \{2, \dots, p\}$  and every element of  $\lambda \in \mathcal{G}_k$ , there is a unique  $\lambda^\dagger \in \mathcal{G}_{k-1}$  such that  $\lambda \subset \bar{\lambda} \subset \lambda^\dagger$ ;
- (f) the measure  $\nu$  is defined on the  $\sigma$ -algebra generated by the elements of  $\bigcup_{k=1}^p \mathcal{G}_k$  by the following formula: for all  $2 \leq k \leq p$  and  $\lambda \in \mathcal{G}_k$ ,

$$\nu(\lambda) := \nu(\lambda^\dagger) \nu^{\lambda^\dagger}(\bar{\lambda});$$

- (g) for all  $2 \leq k \leq p$  and  $\lambda \in \mathcal{G}_k$ ,

$$(58) \quad \nu(\lambda) \leq 2^{-j_{n_k}(\delta^{-1}\sigma(\alpha_{\min})-3\varepsilon_{N(j_{n_{k-1}})})}.$$

Let us explain how to build  $n_{p+1}$  and  $\mathcal{G}_{p+1}$ .

Write  $\gamma(g) < j_{n_p} \leq \gamma(g + 1)$ , where  $g = \ell + \sum_{n=N_0}^{N-1} \ell_n \in \mathbb{N}$  with  $N \geq N_0$  and  $1 \leq \ell \leq \ell_N$ .

Fix  $n_{p+1}$  so that  $\gamma(g + 1) \leq j := (j_{n_{p+1}})_\delta$  (other constraints on  $n_{p+1}$  will be given a few lines below).

Consider  $\lambda^\dagger \in \mathcal{G}_p$ . For every  $\widehat{\lambda} \in \mathcal{D}_j$  with  $\widehat{\lambda} \subset \lambda^\dagger$  and  $\nu^{\lambda^\dagger}(\widehat{\lambda}) > 0$ , due to (57) one has

$$\nu(\lambda^\dagger)\nu^{\lambda^\dagger}(\widehat{\lambda}) \leq \nu(\lambda^\dagger)2^{-(j-j_{n_p})(\sigma(\alpha_{\min})-2\varepsilon_{N(j_{n_p})})}2^{2N(j)\sigma(\alpha_{\min})}.$$

Observe that  $N(j)/j$  tends to 0 as  $j$  tends to  $\infty$ . Hence, choosing  $n_{p+1}$  large enough, by (58) applied to  $\nu(\lambda^\dagger)$ , one gets  $\nu(\lambda^\dagger)\nu^{\lambda^\dagger}(\widehat{\lambda}) \leq 2^{-j_{n_{p+1}}(\delta^{-1}\sigma(\alpha_{\min})-3\varepsilon_{N(j_{n_p})})}$ .

Now one sets

$$(59) \quad \mathcal{G}_{p+1} = \bigcup_{\lambda^\dagger \in \mathcal{G}_p} \left\{ k2^{-(j_{n_{p+1}})\delta} + 2^{-j_{n_{p+1}}}[0, 1] : \left\{ \widehat{\lambda} = k2^{-(j_{n_{p+1}})\delta} + 2^{-(j_{n_{p+1}})\delta}[0, 1] \subset \lambda^\dagger \right\} \right\}.$$

By construction,  $\mathcal{G}_{p+1} \subset \mathcal{D}_{j_{n_{p+1}}}$ , and each interval  $\lambda \in \mathcal{G}_{p+1}$  is the left-most interval inside the corresponding interval  $\widehat{\lambda} \in \mathcal{D}_{(j_{n_{p+1}})\delta}$ . It follows from this, (a) and remark 3.21 that property (a) holds at generation  $p+1$  as well.

Next, for every  $\lambda \in \mathcal{G}_{p+1}$  associated with  $\widehat{\lambda} \in \mathcal{D}_{(j_{n_{p+1}})\delta}$  and  $\lambda^\dagger \in \mathcal{G}_p$ , one finally sets  $\nu(\lambda) = \nu(\lambda^\dagger)\nu^{\lambda^\dagger}(\widehat{\lambda})$ .

The previous construction and the above remarks show that all the items (a)-(g) above hold with  $p+1$  as well.

Finally, we define

$$\mathcal{K} = \bigcap_{p \geq 1} \bigcup_{\lambda \in \mathcal{G}_p} \lambda,$$

and the function  $\nu$  defined on the elements of  $\bigcup_{p \geq 1} \mathcal{G}_p$  extends to a Borel probability measure on  $[0, 1]$ , whose topological support is  $\mathcal{K}$ . It is direct to check that  $\nu$  is atomless, and that due to property (d) and the preliminary observation,  $\mathcal{K} \subset S(\delta, \eta, (j_n)_{n \geq 1})$ .

**Step 3:** Let us estimate the scaling properties of  $\nu$  to get a lower bound for its lower Hausdorff dimension.

Fix  $\lambda$  a closed dyadic subinterval of  $[0, 1]$  of generation  $j \geq j_{n_2}$  such that the interior of  $\lambda$  intersects  $\mathcal{K}$ . Let  $p \geq 2$  be the smallest integer such that the interior of  $\lambda$  intersects at least two elements of  $\mathcal{G}_p$ . We have  $j \leq j_{n_p}$ .

Let  $\lambda^\dagger$  the unique element of  $\mathcal{G}_{p-1}$  such that the interior of  $\lambda$  intersects  $\lambda^\dagger$ . Since  $\nu$  is atomless,  $\nu(\lambda) \leq \nu(\lambda^\dagger)$ . In addition,  $\nu(\lambda) = \nu(\lambda^\dagger)\nu^{\lambda^\dagger}(\widehat{\lambda})$  where  $\widehat{\lambda}$  is associated with  $\lambda$  as in (59).

Consequently, denoting  $\varepsilon_{N(j_{n_p})}$  simply by  $\tilde{\varepsilon}_p$ , if  $j \leq j_{n_{p-1}}$  then

$$\nu(\lambda) \leq \nu(\lambda^\dagger) \leq 2^{-j_{n_{p-1}}(\delta^{-1}\sigma(\alpha_{\min})-3\tilde{\varepsilon}_{p-2})} \leq 2^{-j(\delta^{-1}\sigma(\alpha_{\min})-3\tilde{\varepsilon}_{p-2})},$$

and if  $j > j_{n_{p-1}}$ , then by (57) and (58), one has

$$\begin{aligned}\nu(\lambda) &= \nu(\lambda^\dagger) \nu^{\lambda^\dagger}(\widehat{\lambda}) \\ &\leq 2^{-j_{n_{p-1}}(\delta^{-1}\sigma(\alpha_{\min}) - 3\tilde{\epsilon}_{p-2})} 2^{-(j-j_{n_{p-1}})(\sigma(\alpha_{\min}) - 2\tilde{\epsilon}_{p-1})} 2^{2N(j)\sigma(\alpha_{\min})} \\ &= 2^{-j(\delta^{-1}\sigma(\alpha_{\min}) - \varphi(\lambda))},\end{aligned}$$

where

$$\varphi(\lambda) = 3\tilde{\epsilon}_{p-2} + \frac{(j - j_{n_{p-1}})(\sigma(\alpha_{\min})(\delta^{-1} - 1) + 3\tilde{\epsilon}_{p-2} - 2\tilde{\epsilon}_{p-1}) + 2N(j)\sigma(\alpha_{\min})}{j}.$$

Observe that  $\varphi(\lambda) \leq 6\tilde{\epsilon}_{p-2} + \frac{2N(j)\sigma(\alpha_{\min})}{j}$ , and that when  $j$  tends to infinity,

$\min\{p \geq 2 : \text{for some } \lambda \in \mathcal{D}_j, \text{ the interior of } \lambda \text{ intersects at least 2 elements of } \mathcal{G}_p\}$

tends to  $\infty$ . Consequently,  $\tilde{\epsilon}_{p-2}$  converges uniformly to 0 over  $\{\lambda \in \mathcal{D}_j, \text{Int}(\lambda) \cap \mathcal{K} \neq \emptyset\}$  as  $j \rightarrow \infty$ . Moreover,  $\frac{N(j)}{j}$  tend to zero when  $j \rightarrow \infty$ . Consequently, there exists a function  $\psi : \mathbb{N} \rightarrow (0, \infty)$  such that  $\lim_{j \rightarrow \infty} \psi(j) = 0$  and for every closed dyadic subinterval  $\lambda$  of generation  $j \geq 0$ ,

$$\nu(\lambda) \leq 2^{-j(\delta^{-1}\sigma(\alpha_{\min}) - \psi(j))}.$$

In particular the lower Hausdorff dimension of  $\nu$  is not less than  $\sigma(\alpha_{\min})/\delta$ . Since  $\mathcal{K} \subset S(\delta, \eta, (j_n)_{n \geq 1})$ ,  $\nu(\mathcal{K}) = 1$ , we get  $\dim S(\delta, \eta, (j_n)_{n \geq 1}) \geq \delta^{-1}\sigma(\alpha_{\min})$ , and the conclusions of Proposition 3.18 holds in dimension 1.

For the case  $d \geq 1$ , we know by Section 3.7 that a measure  $\mu \in \mathcal{M}_d$  is equal to  $\mu_1^{\otimes d}$  for some  $\mu \in \mathcal{M}_1$ . Hence, with the definitions and notations introduced earlier in this section, the tensor product measure  $\nu^{\otimes d}$  of the measure  $\nu$  associated above with the measure  $\mu_1$  satisfies the conclusions of Proposition 3.18 in any dimension  $d$ .  $\square$

#### 4. WAVELET CHARACTERIZATION OF $B_q^{\mu,p}(\mathbb{R}^d)$ AND $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$

In this section we prove Theorem 2.16. We start with some definitions, and two basic lemmas in Section 4.1. Then, we prove Theorem 2.16 when  $\in [1, \infty)$  in Section 4.2. The much simpler case  $p = \infty$  is left to the reader who will easily adapt the lines used to treat the case  $p < \infty$ .

**4.1. Preliminary definitions and observations.** We start by extending the definition of the moduli of smoothness (11) and (12) to all sets  $\Omega \subset \mathbb{R}^d$ .

**Definition 4.1.** *Let  $\Omega \subset \mathbb{R}^d$ . For  $h \in \mathbb{R}^d$ , let  $\Omega_{h,n} = \{x \in \Omega : x + kh \in \Omega, k = 1, \dots, n\}$ . Then, for  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $\mu \in \mathcal{H}(\mathbb{R}^d)$ ,  $t > 0$  and  $n \geq 1$  set*

$$(60) \quad \omega_n^\mu(f, t, \Omega)_p = \sup_{t/2 \leq |h| \leq t} \|\Delta_h^{\mu,n} f\|_{L^p(\Omega_{h,n})}$$

and

$$(61) \quad \omega_n(f, t, \Omega)_p = \sup_{0 \leq |h| \leq t} \|\Delta_h^n f\|_{L^p(\Omega_{h,n})}$$

Let  $\mu \in \mathcal{C}(\mathbb{R}^d)$  be an almost doubling capacity such that property (P) holds with exponents  $0 < s_1 \leq s_2$ . Let  $n \geq r = \lfloor s_2 + \frac{d}{p} \rfloor + 1$  and  $\Psi = (\phi, \{\psi^{(i)}\}_{i=1, \dots, 2^d-1}) \in \mathcal{F}_r$  (see Definition ??).

Also, recall that for  $\lambda = (i, j, k) \in \Lambda_j$ ,  $\psi_\lambda(x) = \psi^{(i)}(2^j x - k)$ . It follows from the construction of  $\Psi$  (see [45, Section 3.8]) that there exists an integer  $N_\Psi \in \mathbb{N}^*$  such that  $\text{supp}(\phi)$  and  $\text{supp}(\psi^{(i)})$  are included in  $N_\Psi[0, 1]^d$ . Our proofs will use some estimates established in [17]. These estimates require to associate to each  $\lambda = (i, j, k) \in \Lambda_j$ , a larger cube  $\tilde{\lambda}$  described in the following definition.

**Definition 4.2.** For each  $\lambda = (i, j, k) \in \Lambda_j$ , set

$$\tilde{\lambda} = \lambda_{j,k} + 2^{-j}(\text{supp}(\phi) - \text{supp}(\phi)).$$

Note that  $\lambda_{j,k} \subset \text{supp}(\psi_\lambda) \subset \tilde{\lambda} \subset 3N_\Psi \lambda_{j,k}$ , the second inclusion coming from the construction of compactly supported wavelets (see [45, Section 3.8]).

For every  $j \in \mathbb{N}$ , the cubes  $(\tilde{\lambda})_{\lambda \in \Lambda_j}$  do not overlap too much, in the sense that

$$(62) \quad K_\Psi := \sup_{j \in \mathbb{N}} \sup_{\lambda \in \Lambda_j} \#\{\lambda' \in \Lambda_j : \tilde{\lambda} \cap \tilde{\lambda}' \neq \emptyset\} < \infty.$$

**Lemma 4.3.** Let  $p \in [1, \infty)$  and  $n \in \mathbb{N}^*$ . There exists a constant  $C_{d,n,p}$  (depending on  $p$ ,  $n$ , and  $d$  only) such that for all  $f \in L^p_{\text{loc}}(\mathbb{R}^d)$ ,  $t > 0$  and  $\lambda \in \Lambda$ , the following inequality holds:

$$\omega_n(f, t, \tilde{\lambda})_p^p \leq C_{d,n,p} t^{-d} \int_{t \leq |y| \leq 4nt} \int_{\tilde{\lambda} + B(0, 2nt)} |\Delta_y^n f(x)|^p dx dy.$$

*Proof.* The approach follows the lines of the proof of [17, inequality (3.3.17)], where a similar inequality is proved, the first integration being done over the cube  $\|y\|_\infty \leq t$ , and the second one over  $\tilde{\lambda}_{y,n}$  instead of  $\tilde{\lambda} + B(0, nt)$ .

Fix  $f$ ,  $t$  and  $\lambda$  as in the statement. For any  $h, y \in \mathbb{R}^d$ , recall the following equality (see (3.3.19) in [17]):

$$\Delta_h^n f(x) = \sum_{k=1}^n (-1)^k \binom{n}{k} [\Delta_{ky}^n f(x + kh) - \Delta_{h+ky}^n f(x)].$$

Integrating  $|\Delta_h^n f|^p$  over  $\tilde{\lambda}_{h,n}$ , one deduces that for some constant  $C_{n,p} > 0$ , when  $|h| \leq t$ ,

$$\begin{aligned} \|\Delta_h^n f\|_{L^p(\tilde{\lambda}_{h,n})}^p &\leq C_{n,p} \sum_{k=1}^n \|\Delta_{ky}^n f(\cdot + kh)\|_{L^p(\tilde{\lambda}_{h,n})}^p + \|\Delta_{h+ky}^n f\|_{L^p(\tilde{\lambda}_{h,n})}^p \\ &\leq C_{n,p} \sum_{k=1}^n \|\Delta_{ky}^n f\|_{L^p(\tilde{\lambda} + B(0, 2nt))}^p + \|\Delta_{h+ky}^n f\|_{L^p(\tilde{\lambda} + B(0, 2nt))}^p. \end{aligned}$$

Then, integrating with respect to  $y$  over  $B(0, 3t) \setminus B(0, 2t)$  yields:

$$C_d t^d \|\Delta_h^n f\|_{L^p(\tilde{\lambda}_{h,n})}^p \leq C_{n,p} \sum_{k=1}^n \int_{2t \leq |y| \leq 3t} \int_{\tilde{\lambda} + B(0, 2nt)} |\Delta_{ky}^n f(x)|^p + |\Delta_{h+ky}^n f(x)|^p dx dy,$$



where  $C_d = \mathcal{L}^d(B(0, 3) \setminus B(0, 2))$ . Then, operating the change of variable  $y' = ky$  in each term of the sum, one obtains

$$\begin{aligned} t^d \|\Delta_h^n f\|_{L^p(\tilde{\lambda}_{h,n})}^p &\leq C_d^{-1} C_{n,p} \sum_{k=1}^n \int_{2kt \leq |y| \leq 3kt} \int_{\tilde{\lambda} + B(0, 2nt)} |\Delta_y^n f(x)|^p + |\Delta_{h+y}^n f(x)|^p \, dx dy \\ &\leq 2n C_d^{-1} C_{n,p} \int_{t \leq |y| \leq 4nt} \int_{\tilde{\lambda} + B(0, 2nt)} |\Delta_y^n f(x)|^p \, dx dy. \end{aligned}$$

where one used that  $t \leq |h+y| \leq 4nt$  when  $|h| \leq t$  and  $|y| \geq 2t$ . The previous upper bound being independent of  $h \in B(0, t)$ , one concludes that

$$\omega_n(f, t, \tilde{\lambda})_p^p = \sup_{0 \leq |h| \leq t} \|\Delta_h^n f\|_{L^p(\tilde{\lambda}_{h,n})}^p \leq \frac{2n C_d^{-1} C_{n,p}}{t^d} \int_{t \leq |y| \leq 4nt} \int_{\tilde{\lambda} + B(0, 2nt)} |\Delta_y^n f(x)|^p \, dx dy,$$

hence the conclusion.  $\square$

**Lemma 4.4.** *Let  $\mu \in \mathcal{C}(\mathbb{R}^d)$  and suppose that  $\mu$  satisfies the almost doubling property. Fix  $\varepsilon > 0$ . There exists a constant  $C = C_\varepsilon \geq 1$  depending on  $n$  and  $\mu$  only such that for every  $j \in \mathbb{N}$  and  $\lambda \in \Lambda_j$ , for every  $x \in \tilde{\lambda} + B(0, 2n2^{-j})$  and  $y \in \mathbb{R}^d$  such that  $2^{-j} \leq |y| \leq 4n2^{-j}$ , for every  $f : \tilde{\lambda} \rightarrow \mathbb{R}$ , the following properties hold:*

$$(63) \quad \frac{\mu(B([x, x + ny]))}{\mu(\lambda)} \leq C(n|y|)^{-\varepsilon},$$

$$(64) \quad \frac{|\Delta_y^n f(x)|}{\mu(\lambda)} \leq C \frac{|\Delta_y^n f(x)|}{\mu^{(+\varepsilon)}(B([x, x + ny]))}.$$

*Proof.* Inequality (63) follows easily from the definition of the almost doubling property (7), and inequality (64) directly from (63) and the definition of  $\mu^{(+\varepsilon)}$ .  $\square$

**4.2. Proof of Theorem 2.16 when  $1 \leq p < \infty$ .** Let us now explain our approach to get Theorem 2.16 when  $p \in [1, \infty)$ . Recall that  $B_q^{\mu,p}(\mathbb{R}^d)$  is supposed to be defined via  $L^p$  moduli of smoothness of order  $n \geq r = \lfloor s_2 + d/p + 1 \rfloor$ , and that  $\Psi$  belongs to  $\mathcal{F}_r$ .

We first prove in Section 4.2.1 that,  $n \geq r$  being fixed, (17) holds for any  $\varepsilon > 0$  when  $B_q^{\mu^{(+\varepsilon)},p}(\mathbb{R}^d)$  is defined via the  $L^p$  modulus of smoothness of order  $n$  and any  $\Psi \in \mathcal{F}_n$ . This is not exactly the statement, since one wants to obtain (17) for any  $\Psi \in \mathcal{F}_r$ .

Then we prove in Section 4.2.2 that (18) holds for any  $\varepsilon \in (0, 1)$  and any  $\Psi \in \mathcal{F}_r$  when  $B_q^{\mu,p}(\mathbb{R}^d)$  is defined via the  $L^p$  modulus of smoothness of order exactly equal to  $r$  (this is exactly the statement of Theorem 2.16). Since  $\mathcal{F}_n \subset \mathcal{F}_r$ , the statement also holds for  $\Psi \in \mathcal{F}_n$ .

Finally, we conclude that (17) holds for any  $\varepsilon \in (0, 1)$  and any  $\Psi \in \mathcal{F}_r$ , by applying first (17) with the environment  $\mu$ , the  $n$ -th order difference operator,  $\varepsilon/3$  and any wavelet  $\tilde{\Psi} \in \mathcal{F}_n$ , then (18) with the environment  $\mu^{(+\varepsilon/3)}$ , the  $r$ -th order difference operator,  $\varepsilon/3$  and the same  $\tilde{\Psi} \in \mathcal{F}_n$ , and finally (17) with the environment  $\mu^{(+2\varepsilon/3)}$ , the  $r$ -th order difference operator,  $\varepsilon/3$  and  $\Psi \in \mathcal{F}_r$ .

4.2.1. *Proof of inequality (17) in Theorem 2.16.* Assume that  $\Psi \in \mathcal{F}_n \subset \mathcal{F}_r$ . Fix  $\varepsilon > 0$ ,  $f \in L^p(\mathbb{R}^d)$  and  $j \in \mathbb{N}$ .

Due to the local approximation of  $f$  by polynomials (equation (3.3.13) in [17]), there exists a polynomial  $P_\lambda$  of degree  $\leq n$  such that

$$(65) \quad \|f - P_\lambda\|_{L^p(\tilde{\lambda})} \leq C\omega_n(f, 2^{-j}, \tilde{\lambda})_p,$$

where  $C$  depends on  $n$  and  $p$  only. This implies, using the fact that  $\psi_\lambda$  is orthogonal to any polynomial of degree less than or equal to  $n$ , that

$$\frac{|c_\lambda|}{\mu(\lambda)} = \frac{\left| 2^{jd} \int_{\mathbb{R}^d} (f(x) - P_\lambda(x)) \psi_\lambda(x) dx \right|}{\mu(\lambda)}.$$

Then, Hölder's inequality (where  $p'$  stands for  $\frac{p}{p-1}$ ) yields (recall that  $\text{supp}(\psi_\lambda) \subset \tilde{\lambda}$ )

$$(66) \quad \begin{aligned} \frac{|c_\lambda|}{\mu(\lambda)} &\leq 2^{jd} \frac{\|\psi_\lambda\|_{L^{p'}(\mathbb{R}^d)} \|f - P_\lambda\|_{L^p(\tilde{\lambda})}}{\mu(\lambda)} \\ &\leq \tilde{C} 2^{jd/p} \frac{\omega_n(f, 2^{-j}, \tilde{\lambda})_p}{\mu(\lambda)}, \end{aligned}$$

where  $\tilde{C} = C \sup \left\{ \|\psi^{(i)}\|_{L^{p'}(\mathbb{R}^d)} : 1 \leq i \leq 2^d - 1 \right\}$ .

Then, Lemma 4.3 gives

$$\left( \frac{|c_\lambda|}{\mu(\lambda)} \right)^p \leq C_{d,n,p} \tilde{C}^p 2^{2dj} \int_{2^{-j} \leq |y| \leq 4n2^{-j}} \int_{\tilde{\lambda} + B(0, 2n2^{-j})} \frac{|\Delta_y^n f(x)|^p}{\mu(\lambda)^p} dx dy,$$

and due to Lemma 4.4, there exists  $C_\varepsilon$  depending on  $(\varepsilon, n, p, \Psi)$  such that:

$$\begin{aligned} \left( \frac{|c_\lambda|}{\mu(\lambda)} \right)^p &\leq C_{d,n,p} C_\varepsilon^p 2^{2dj} \int_{2^{-j} \leq |y| \leq 4n2^{-j}} \int_{\tilde{\lambda} + B(0, 2n2^{-j})} |\Delta_y^{\mu^{(+\varepsilon)}, n} f(x)|^p dx dy \\ &\leq C_{d,n,p} C_\varepsilon^p \sum_{k=0}^{\lfloor \log_2(4n) \rfloor} 2^{2dj} \int_{2^{-j+k} \leq |y| \leq 2^{-j+k+1}} \int_{\tilde{\lambda} + B(0, 2n2^{-j})} |\Delta_y^{\mu^{(+\varepsilon)}, n} f(x)|^p dx dy. \end{aligned}$$

Set  $k_n = \lfloor \log_2(4n) \rfloor$ . By (62), there exists a constant  $K_{\Psi, n} > 0$  depending on  $(\Psi, n)$  only such that any  $\lambda \in \Lambda_j$  is covered by at most  $K_{\Psi, n}$  sets of the form  $\tilde{\lambda}' + B(0, 2n2^{-j})$  with  $\lambda' \in \Lambda_j$ . It follows that

$$\sum_{\lambda \in \Lambda_j} \left( \frac{|c_\lambda|}{\mu(\lambda)} \right)^p \leq K_{\Psi, n} C_{d,n,p} C_\varepsilon^p \sum_{k=0}^{k_n} 2^{2dj} \int_{2^{-j+k} \leq |y| \leq 2^{-j+k+1}} \int_{\mathbb{R}^d} |\Delta_y^{\mu^{(+\varepsilon)}, n} f(x)|^p dx dy.$$

Recalling the definition (11) of  $\omega_n^{\mu^{(+\varepsilon)}}(f, t, \mathbb{R}^d)$ , each double integral above is bounded by  $2^{d(-j+k+1)} \omega_n^{\mu^{(+\varepsilon)}}(f, 2^{-j+k+1}, \mathbb{R}^d)_p^p$ . Since  $2^{d(k+1)} \leq 2^{d(k_n+1)} \leq (8n)^d$ , one has

$$\sum_{\lambda \in \Lambda_j} \left( \frac{|c_\lambda|}{\mu(\lambda)} \right)^p \leq C_1^p \sum_{k=0}^{k_n} 2^{dj} \omega_n^{\mu^{(+\varepsilon)}}(f, 2^{-j+k+1}, \mathbb{R}^d)_p^p,$$

where  $C_1 = ((8n)^d K_{\Psi,n} C_{d,n,p})^{1/p} C_\varepsilon$ .

Suppose now that  $q \in [1, \infty)$  (the case  $q = \infty$  is obvious). The previous estimates together with the subadditivity of  $t \geq 0 \mapsto t^{1/p}$  and the convexity of  $t \geq 0 \mapsto t^q$  yield

$$\left\| \left( \frac{c_\lambda}{\mu(\lambda)} \right)_{\lambda \in \Lambda_j} \right\|_{\ell^p(\Lambda_j)}^q \leq C_1^q (k_n + 1)^{q-1} \sum_{k=0}^{k_n} (2^{dj/p} \omega_n^{\mu^{(+\varepsilon)}}(f, 2^{-j+k+1}, \mathbb{R}^d)_p)^q.$$

Observe that there is  $C_2 \geq 1$  such that for  $0 \leq j \leq k_n$ ,  $2^{dj/p} \omega_n^{\mu^{(+\varepsilon)}}(f, 2^j, \mathbb{R}^d)_p \leq C_2 \|f\|_{L^p(\mathbb{R}^d)}$ . Consequently,

$$\sum_{j \geq 0} \left\| \left( \frac{c_\lambda}{\mu(\lambda)} \right)_{\lambda \in \Lambda_j} \right\|_{\ell^p(\Lambda_j)}^q \leq C_1^q C_2^q (k_n + 1)^{q+1} \left( \|f\|_{L^p(\mathbb{R}^d)}^q + \sum_{j \geq 0} (2^{dj/p} \omega_n^{\mu^{(+\varepsilon)}}(f, 2^{-j}, \mathbb{R}^d)_p)^q \right),$$

which implies that  $\|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu,p,q} \leq C(\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B_q^{\mu^{(+\varepsilon)},p}(\mathbb{R}^d)})$  for some constant  $C > 0$  independent of  $f$ . Hence, (17) holds when  $\Psi \in \mathcal{F}_n$ .

4.2.2. *Proof of inequality (18) in Theorem 2.16.* Fix  $\varepsilon > 0$  and  $f \in L^p(\mathbb{R}^d)$ . We will need the following lemma.

**Lemma 4.5.** *Let  $s \in \left(s_2 + \frac{d}{p}, s_2 + \frac{d}{p} + 1\right)$ . There exist a constant  $C$  and a sequence  $(\tilde{\varepsilon}_m)_{m \in \mathbb{N}} \in \ell^q(\mathbb{N})$ , independent of  $f$ , such that for all  $j, J \geq 0$ , defining  $f_j = \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda$ , one has*

$$(67) \quad \omega_n^\mu(f_j, 2^{-J}, \mathbb{R}^d)_p \leq C 2^{-jd/p} \min(1, 2^{(j-J)(s-s_2)} \tilde{\varepsilon}_{J-j}) \left( \sum_{\lambda \in \Lambda_j} \left( \frac{|c_\lambda|}{\mu^{(+\varepsilon)}(\lambda)} \right)^p \right)^{1/p},$$

with the convention that  $\tilde{\varepsilon}_m = 1$  when  $m < 0$ .

*Proof.* Inspired by the proof of [17, Theorem 3.4.3], we distinguish two cases:

**Case 1:**  $J < j$ . In order to prove (67) let us begin by simply writing

$$(68) \quad \omega_n^\mu(f_j, 2^{-J}, \mathbb{R}^d)_p^p = \sup_{2^{-J-1} \leq |h| \leq 2^{-J}} \sum_{\lambda' \in \mathcal{D}_J} \int_{\lambda'} \frac{\left| \sum_{\lambda \in \Lambda_j} c_\lambda \Delta_h^n \psi_\lambda(x) \right|^p}{\mu(B(x, x + nh))^p} dx.$$

Note now that if  $x \in \lambda' \in \mathcal{D}_J$ ,  $\lambda \in \Lambda_j$  and  $h \in \mathbb{R}^d$  is such that  $2^{-J-1} \leq |h| \leq 2^{-J}$ , then  $\Delta_h^n \psi_\lambda(x) = 0$  if  $x \notin \bigcup_{k=0}^n \text{supp}(\psi_\lambda) - kh$ . Also, there exists an integer  $N$  depending on  $(n, \Psi)$  only such that  $x \in \lambda' \in \mathcal{D}_J$  and  $x \in \bigcup_{k=0}^n \text{supp}(\psi_\lambda) - kh$  implies  $\lambda \subset N\lambda'$ . Moreover, using the almost doubling property of  $\mu$ , there exists a constant  $C$  depending on  $(\mu, n, \Psi, \varepsilon)$  only such that for all integers  $j \geq 1$  and  $0 \leq J < j$ ,  $h \in \mathbb{R}^d$  such that  $2^{-J-1} \leq |h| \leq 2^{-J}$  and  $x \in \bigcup_{k=0}^n \text{supp}(\psi_\lambda) + kh$ , one has

$$(69) \quad \mu^{(+\varepsilon)}(\lambda) = 2^{-j\varepsilon} \mu(\lambda) \leq 2^{-j\varepsilon} \mu(N\lambda') \leq C \mu(B(x, x + nh)).$$

Consequently, (68) and the second inequality of (69), together with the equality  $\Delta_h^n \psi_\lambda = \sum_{k=0}^n (-1)^k \binom{n}{k} \psi_\lambda(\cdot + (n-k)h)$ , the bound  $\binom{n}{k} \leq 2^n$ , and the convexity

inequality  $(\sum_{k=0}^n |z_k|)^p \leq (n+1)^{p-1} \sum_{k=0}^n |z_k|^p$  yield

$$\begin{aligned}
 & \omega_n^\mu(f_j, 2^{-J}, \mathbb{R}^d)_p^p \\
 & \leq C^p \sup_{2^{-J-1} \leq |h| \leq 2^{-J}} \sum_{\lambda' \in \mathcal{D}_J} \frac{2^{j\varepsilon p}}{\mu(N\lambda')^p} \int_{\lambda'} \left| \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda'} c_\lambda \Delta_h^n \psi_\lambda(x) \right|^p dx \\
 & \leq C^p \sup_{2^{-J-1} \leq |h| \leq 2^{-J}} \sum_{\lambda' \in \mathcal{D}_J} \frac{2^{j\varepsilon p}}{\mu(N\lambda')^p} \int_{\mathbb{R}^d} \left| \sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda'} c_\lambda \psi_\lambda(x + (n-k)h) \right|^p dx \\
 & \leq C^p (n+1)^{p-1} 2^{np} \sum_{\lambda' \in \mathcal{D}_J} \frac{2^{j\varepsilon p}}{\mu(N\lambda')^p} \int_{\mathbb{R}^d} \left| \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda'} c_\lambda \psi_\lambda(x) \right|^p dx.
 \end{aligned}$$

Moreover, according to [45, Ch. 6, Prop. 7], there exists  $C' > 0$  depending on  $\Psi$  only such that

$$\int_{\mathbb{R}^d} \left| \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda'} c_\lambda \psi_\lambda(x) \right|^p dx \leq C'^p 2^{-jd} \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda'} |c_\lambda|^p.$$

Consequently, using the first inequality of (69), we get

$$\omega_n^\mu(f_j, 2^{-J}, \mathbb{R}^d)_p^p \leq (CC')^p (n+1)^{p-1} 2^{np} \sum_{\lambda' \in \mathcal{D}_J} 2^{-jd} \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda'} \left( \frac{|c_\lambda|}{\mu^{(+\varepsilon)}(\lambda)} \right)^p.$$

Finally, (67) comes from the fact that there exists an integer  $\tilde{N}$  independent of  $J$  and  $j$  such that for each  $\lambda \in \Lambda_j$ , there are less than  $\tilde{N}$  cubes  $\lambda' \in \mathcal{D}_J$  such that  $\lambda \subset N\lambda'$ .

**Case 2:**  $J \geq j$ . Let us start with a few observations. First, by assumption,  $\psi^{(i)} \in B_q^{s,p}(\mathbb{R}^d)$ , hence

$$\omega_n(\psi^{(i)}, 2^{j-J}, \mathbb{R}^d)_p \leq 2^{(j-J)s} \tilde{\varepsilon}_{J-j}^{(i)},$$

where  $(\tilde{\varepsilon}_m^{(i)})_{m \geq 1} \in \ell^q(\mathbb{N}^*)$  and  $\|\tilde{\varepsilon}^{(i)}\|_{\ell^q(\mathbb{N}^*)} \leq \|\psi^{(i)}\|_{B_q^{s,p}}$ . Consequently, for all  $\lambda \in \Lambda_j$  one has

$$(70) \quad \omega_n(\psi_\lambda, 2^{-J}, \mathbb{R}^d)_p \leq 2^{(j-J)s} 2^{-jd/p} \tilde{\varepsilon}_{J-j},$$

where  $\tilde{\varepsilon}_{J-j} = \sup_i \tilde{\varepsilon}_{J-j}^{(i)}$ .

Next, there exists an integer  $N$  independent of  $j$  and  $J$  such that for all  $x \in \mathbb{R}^d$  and  $h \in \mathbb{R}^d$  such that  $2^{-J-1} \leq |h| \leq 2^{-J}$ , one has  $B([x, x + nh]) \subset N\lambda_j(x)$ , and  $\Delta_h^n \psi_\lambda(x)$  may not vanish only if  $\lambda = (i, j, k)$  is such that  $\lambda_{j,k} \subset N\lambda_j(x)$ . Moreover, there exists a dyadic cube  $\lambda' \subset B(x, x + nh)$  of generation  $J+3$  as well as a dyadic cube  $\lambda'' \subset N\lambda_j(x)$  of generation  $j$ , included in  $N\lambda_j(x)$  such that  $\lambda' \subset \lambda''$ . By construction, for all  $\mathcal{D}_j \ni \lambda_{j,k} \subset N\lambda_j(x)$ , one has

$$\mu(B([x, x + nh]))^{-1} \leq \mu(\lambda')^{-1} = \frac{\mu(\lambda'')}{\mu(\lambda')} \frac{\mu(\lambda_{j,k})}{\mu(\lambda'')} \mu(\lambda_{j,k})^{-1},$$

and using property (P<sub>2</sub>) to control from above  $\frac{\mu(\lambda_{j,k})}{\mu(\lambda'')}$  by  $O(2^{N^d \theta(j)})$  and  $\frac{\mu(\lambda'')}{\mu(\lambda')}$  by  $O(2^{\theta(j)} 2^{(J-j)s_2})$ , as well as (P<sub>1</sub>) to control  $2^{\theta(j)}$  from above by  $|\lambda|^{-\varepsilon}$ , there exists a

constant  $C$  depending on  $(\mu, n, \varepsilon)$  only such that

$$\mu(B([x, x + nh]))^{-1} \leq C2^{(J-j)s_2}(\mu^{(+\varepsilon)}(\lambda_{j,k}))^{-1}.$$

The previous observations yield (recall that if  $(i, j, k) \in \Lambda_j$  we also denote  $\mu(\lambda_{j,k})$  by  $\mu(\lambda)$ , and  $\lambda \subset N\lambda_j(x)$  means  $\lambda_{j,k} \subset N\lambda_j(x)$ )

$$\omega_n^\mu(f_j, 2^{-J}, \mathbb{R}^d)_p^p \leq C^p 2^{(J-j)s_2} \sup_{2^{-J-1} \leq |h| \leq 2^{-J}} \int_{\mathbb{R}^d} \left( \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda_j(x)} \frac{|c_\lambda|}{\mu^{(+\varepsilon)}(\lambda)} |\Delta_h^n \psi_\lambda(x)| \right)^p dx.$$

Without loss of generality, suppose that  $N$  is odd. Then  $\#\{\lambda \in \Lambda_j : \lambda \subset N\lambda_j(x)\} = N^d$ , and for each  $x \in \mathbb{R}^d$ , we have

$$\left( \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda_j(x)} \frac{|c_\lambda|}{\mu^{(+\varepsilon)}(\lambda)} |\Delta_h^n \psi_\lambda(x)| \right)^p \leq N^{d(p-1)} \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda_j(x)} \left( \frac{|c_\lambda|}{\mu^{(+\varepsilon)}(\lambda)} \right)^p |\Delta_h^n \psi_\lambda(x)|^p.$$

Since each element of  $\mathcal{D}_j$  belongs to  $N^d$  cube of the form  $N\lambda'$  with  $\lambda' \in \mathcal{D}_j$ , we get

$$\omega_n^\mu(f_j, 2^{-J}, \mathbb{R}^d)_p^p \leq C^p N^{dp} 2^{(J-j)s_2} \sum_{\lambda \in \Lambda_j} \left( \frac{|c_\lambda|}{\mu^{(+\varepsilon)}(\lambda)} \right)^p \omega_n(\psi_\lambda, 2^{-J}, \mathbb{R}^d)_p^p,$$

hence the conclusion due to (70).  $\square$

We can now prove (18). Fix  $\varepsilon \in (0, 1)$ . Setting  $\tilde{f} = f - \sum_{j=0}^\infty f_j$ , the triangle inequality yields

$$(71) \quad \omega_n^\mu(f, 2^{-J}, \mathbb{R}^d)_p \leq \omega_n^\mu(\tilde{f}, 2^{-J}, \mathbb{R}^d)_p + \sum_{j=0}^\infty \omega_n^\mu(f_j, 2^{-J}, \mathbb{R}^d)_p.$$

The  $\ell^q$  norm of the first term on the right hand side of the above inequality (corresponding to low frequencies) can be controlled as follows. Using property (P),

$$(72) \quad 2^{Jd/p} \omega_n^\mu(\tilde{f}, 2^{-J}, \mathbb{R}^d)_p \leq 2^{J(s_2 + \varepsilon + d/p)} \omega_n(\tilde{f}, 2^{-J}, \mathbb{R}^d)_p.$$

Observe that, since  $\tilde{f}$  is obtained by removing from  $f$  the terms of low frequencies, we have  $\tilde{f} \in B_q^{s', p}(\mathbb{R}^d)$  for all  $s' \in (d/p, r)$  and  $q \in [1, \infty]$ , as can be checked using (15). In particular, choosing  $s' = s_2 + \varepsilon + d/p$ , we have  $|\tilde{f}|_{(\mathcal{L}^d)^{\frac{s'+\varepsilon+d/p}{d}-\frac{1}{p}, p, q}} = 0 =$

$|\tilde{f}|_{(\mathcal{L}^d)^{\frac{s_1+\varepsilon+d/p}{d}-\frac{1}{p}, p, q}}$ . Then, using the equivalence of norms recalled after (15), there is

a constant  $\tilde{C}$  depending on  $(d, \varepsilon, \mu, p, q)$  such that

$$\begin{aligned} \|(2^{Jd/p} \omega_n^\mu(\tilde{f}, 2^{-J}, \mathbb{R}^d)_p)_{J \in \mathbb{N}}\|_{\ell^q(\mathbb{N})} &\leq \|\tilde{f}\|_{B_q^{s_2+\varepsilon+d/p, p}(\mathbb{R}^d)} \\ &\leq \tilde{C} (\|f\|_{L^p(\mathbb{R}^d)} + |\tilde{f}|_{(\mathcal{L}^d)^{\frac{s_2+\varepsilon+d/p}{d}-\frac{1}{p}, p, q}}) \\ &= \tilde{C} (\|f\|_{L^p(\mathbb{R}^d)} + |\tilde{f}|_{(\mathcal{L}^d)^{\frac{s_1+\varepsilon+d/p}{d}-\frac{1}{p}, p, q}}) \\ &\leq \tilde{C} (\|f\|_{L^p(\mathbb{R}^d)} + |f|_{(\mathcal{L}^d)^{\frac{s_1+\varepsilon+d/p}{d}-\frac{1}{p}, p, q}}) \\ &\leq \tilde{C}' (\|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu^{(+\varepsilon), p, q}}), \end{aligned}$$

where the last inequality is a consequence of property (P<sub>1</sub>) (which implies that  $\mu(\lambda) \leq C2^{-js_1} = C\mathcal{L}^d(\lambda)^{\frac{s_1+d/p}{d}-1/p}$  for all  $j \in \mathbb{N}$  and  $\lambda \in \mathcal{D}_j$ ), and  $\tilde{C}'$  depends on  $C$  and  $\tilde{C}$ .

Next we control the  $\ell^q$  norm of the second term of the right hand side in (71).

Set  $A_j = \left\| \left( \frac{|\mathcal{C}_\lambda|}{\mu^{(+\varepsilon)}(\lambda)} \right)_{\lambda \in \Lambda_j} \right\|_{\ell^p(\Lambda_j)}$ . By Lemma 4.5, modifying the value of the constant  $C$  there if necessary, one may assume that the sequence  $\tilde{\varepsilon}$  is bounded by 1 from above. Hence, when  $j \leq J$  one has  $\omega_n^\mu(f_j, 2^{-J}, \mathbb{R}^d)_p \leq C2^{-jd/p}2^{(j-J)(s-s_2)}A_j$ , while when  $j > J$ , one has  $\omega_n^\mu(f_j, 2^{-J}, \mathbb{R}^d)_p \leq C2^{-jd/p}A_j$ . Consequently,

$$2^{Jd/p}\omega_n^\mu\left(\sum_{j \geq 0} f_j, 2^{-J}, \mathbb{R}^d\right)_p \leq 2^{Jd/p} \sum_{j=0}^J 2^{-jd/p+(j-J)(s-s_2)}A_j + 2^{Jd/p} \sum_{j=J+1}^{\infty} 2^{-jd/p}A_j,$$

which implies

$$\|2^{Jd/p}\omega_n^\mu(f, 2^{-J}, \mathbb{R}^d)_p\|_{\ell^q(\mathbb{N})} \leq C(\|(\alpha_J)_{J \geq 1}\|_{\ell^q(\mathbb{N})} + \|(\beta_J)_{J \geq 1}\|_{\ell^q(\mathbb{N})})$$

where

$$\alpha_J := 2^{Jd/p} \sum_{j=0}^J 2^{-jd/p+(j-J)(s-s_2)}A_j \quad \text{and} \quad \beta_J := 2^{Jd/p} \sum_{j=J+1}^{\infty} 2^{-jd/p}A_j.$$

Recall now the two following Hardy's inequalities (see, e.g. (3.5.27) and (3.5.36) in [17]): let  $q \in [1, \infty]$  as well as  $0 < \gamma < \delta$ . There exists a constant  $K > 0$  such that :

- if  $(a_j)_{j \in \mathbb{N}}$  is a non negative sequence and for  $J \in \mathbb{N}$  one sets  $b_J = 2^{-\delta J} \sum_{j=0}^J 2^{j\delta} a_j$ , then  $\|(2^{\gamma J} b_J)_{J \geq 1}\|_{\ell^q(\mathbb{N})} \leq K \|(2^{\gamma J} a_J)_{J \geq 0}\|_{\ell^q(\mathbb{N})}$ .
- if  $(a_j)_{j \in \mathbb{N}}$  is a non negative sequence and for  $J \in \mathbb{N}$  one sets  $b_J = \sum_{j \geq J} a_j$ , then  $\|(2^{\gamma J} b_J)_{J \in \mathbb{N}}\|_{\ell^q(\mathbb{N})} \leq K \|(2^{\gamma J} a_J)_{J \geq 0}\|_{\ell^q(\mathbb{N})}$ .

Let  $\delta = s - s_2$  and  $\gamma = d/p$ . Applying the first Hardy's inequality with  $a_j = 2^{-jd/p}A_j$  yields

$$\|(\alpha_j)_{j \in \mathbb{N}}\|_{\ell^q(\mathbb{N})} \leq K \|(A_J)_{J \in \mathbb{N}}\|_{\ell^q(\mathbb{N})},$$

while applying the second one with  $a_j = 2^{-jd/p}A_j$  and  $\gamma = d/p$ , one obtains

$$\|(\beta_J)_{J \in \mathbb{N}}\|_{\ell^q(\mathbb{N})} \leq K \|(A_J)_{J \in \mathbb{N}}\|_{\ell^q(\mathbb{N})}.$$

Finally,

$$\|2^{Jd/p}\omega_n^\mu(f, 2^{-J}, \mathbb{R}^d)_p\|_{\ell^q(\mathbb{N})} \leq (C' + 2CK)(\|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu^{(+\varepsilon), p, q}}),$$

which implies (18).

Although we do not elaborate on this in this paper, it is certainly worth investigating the relationship between the Besov spaces in multifractal environment and the following analog of Sobolev space in multifractal environment.

**Definition 4.6.** Let  $\mu$  be a probability measure on  $\mathbb{R}^d$ ,  $s > 0$ ,  $p \geq 1$ . A function  $f$  belongs to  $W_p^{\mu, s}(\mathbb{R}^d)$  if and only if  $\|f\|_{W_p^{\mu, s}(\mathbb{R}^d)} < \infty$ , where  $\|f\|_{W_p^{\mu, s}(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)} +$

$|f|_{W_p^{\mu,s}(\mathbb{R}^d)}$  and

$$|f|_{W_p^{\mu,s}(\mathbb{R}^d)} := \iint_{([0,1]^d)^2} \frac{|f(x) - f(y)|^p}{\mu(B([x,y]))^{sp} |x-y|^{2d}} dx dy < \infty.$$

## 5. THE SHAPE OF THE TYPICAL SINGULARITY SPECTRUM IN $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ WHEN $\mu \in \mathcal{E}_d$

Given  $\mu \in \mathcal{E}_d$ , Theorem 2.19(2) claims that the singularity spectrum of typical functions in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  is given by the Legendre transform  $\zeta_{\mu,p}^*$  of the function  $\zeta_{\mu,p}$ , which is explicitly given in terms of  $\tau_\mu$ . In this section, we give an explicit formula for  $\zeta_{\mu,p}^*$  in terms of  $\tau_\mu^*$ , i.e.  $\sigma_\mu$ , and we discuss the possible shapes and features of  $\zeta_{\mu,p}^*$ , as well as those of  $\zeta_{\mu,p}$ .

To express  $\zeta_{\mu,p}^*$  in terms of  $\tau_\mu^*$  we need to introduce the following mapping:

$$(73) \quad \theta_p : \alpha \in [\tau'_\mu(\infty), \tau'_\mu(-\infty)] \mapsto \alpha + \frac{\tau_\mu^*(\alpha)}{p},$$

see Figure 6. Notice that  $\theta_\infty$  is just the identity map.

If  $\tau'_\mu(\infty) \neq \tau'_\mu(-\infty)$ , the map  $\theta_p$  is increasing on  $\partial(\tau_\mu^*) \cap [-p, \infty)$ , where  $\partial(\tau_\mu^*)$  stands for the sub-differential of the concave map  $\tau_\mu^*$ .

If  $-p \notin \partial(\tau_\mu^*)$ , we set  $\alpha_p = \tau'_\mu(-\infty)$ ; otherwise, let  $\alpha_p$  be the unique  $\alpha$  such that  $-p \in \partial(\tau_\mu^*)(\alpha)$ . Note that necessarily  $\alpha_p \geq \tau'_\mu(0^+)$  (since  $\tau_\mu^*$  is increasing over the interval  $[\tau'_\mu(\infty), \tau'_\mu(0^+))$ ), and that  $\theta_p$  is increasing on  $[\tau'_\mu(\infty), \alpha_p]$  and decreasing on  $[\alpha_p, \tau'_\mu(-\infty)]$ . Moreover,  $\tau_\mu(-p) = (\tau_\mu^*)^*(-p) = -\alpha_p p - \tau_\mu^*(\alpha_p) = -p\theta_p(\alpha_p)$ . Consequently,

$$\theta_p(\alpha_p) = \begin{cases} \frac{\tau_\mu(-p)}{-p} & \text{if } -p \in \partial(\tau_\mu^*) \\ \tau'_\mu(-\infty) + \frac{\tau_\mu^*(\tau'_\mu(-\infty))}{p} & \text{otherwise;} \end{cases}$$

In particular, according to Remark 3.4, if  $-p \notin \partial(\tau_\mu^*)$ , then  $(\tau_\mu^*)'(\tau'_\mu(-\infty)) > -\infty$  so that  $\tau_\mu$  is linear near  $-\infty$ , and so is  $\zeta_{\mu,p}$ , with the formula  $\zeta_{\mu,p}(t) = (\tau'_\mu(-\infty) + \frac{\tau_\mu^*(\tau'_\mu(-\infty))}{p})t - \tau_\mu^*(\tau'_\mu(-\infty))$ .

In any case,  $\theta_p$  reaches its maximum at  $\alpha_p$ . Let  $\theta_p^{-1}$  be the inverse branch of  $\theta_p$  over  $[\theta_p(\tau'_\mu(\infty)), \theta_p(\alpha_p)]$ . We can rewrite the Legendre transform of  $\zeta_{\mu,p}$  as follows:

**Proposition 5.1.** *Let  $\mu \in \mathcal{E}_d$ . One has*

$$(74) \quad \zeta_{\mu,p}^*(H) = \begin{cases} p(H - \tau'_\mu(\infty)) & \text{if } H \in [\tau'_\mu(\infty), \theta_p(\tau'_\mu(\infty))] \\ \tau_\mu^*(\theta_p^{-1}(H)) & \text{if } H \in [\theta_p(\tau'_\mu(\infty)), \theta_p(\alpha_p)] \\ -\infty & \text{if } H \notin [\tau'_\mu(\infty), \theta_p(\alpha_p)]. \end{cases}$$

Next remark gathers various facts regarding  $\zeta_{\mu,p}$  and  $\zeta_{\mu,p}^*$ , which directly follow from the proposition and its proof, or from the definition of  $\zeta_{\mu,p}$ .

**Remark 5.2.** (1) *As an immediate consequence of the proposition we get that  $\tau'_\mu(\infty) = \zeta'_{\mu,p}(\infty)$  and  $\theta_p(\alpha_p) = \zeta'_{\mu,p}(-\infty)$ , though these equalities can be directly checked. Also, by definition of  $\theta_p$ ,  $\zeta'_{\mu,p}(-\infty) \leq \tau'_\mu(-\infty) + \frac{d}{p}$ .*

(2) When  $p = \infty$ ,  $\zeta_{\mu, \infty} = \tau_\mu$ .

(3) When  $\tau_\mu^*(\tau'_\mu(\infty)) = 0$  (i.e. when  $\theta_p(\tau'_\mu(\infty)) = \tau'_\mu(\infty)$ ), the function  $\zeta_{\mu, p}^*$  reduces to the map  $H \mapsto \tau_\mu^*(\theta_p^{-1}(H))$  on the interval  $[\theta_p(\tau'_\mu(\infty)), \theta_p(\alpha_p)]$ .

(4) When  $\tau_\mu^*(\tau'_\mu(\infty)) > 0$  and  $p \in [1, \infty)$ , (equivalently, when  $\theta_p(\tau'_\mu(\infty)) > \tau'_\mu(\infty)$ ),  $\zeta_{\mu, p}^*$  is linear over  $[\tau'_\mu(\infty), \theta_p(\tau'_\mu(\infty))]$ . This is the signature of the fact that  $\zeta_{\mu, p}$  is not differentiable at  $p$ , where one has  $\zeta'_{\mu, p}(p^+) = \tau'_\mu(\infty)$  and  $\zeta'_{\mu, p}(p^-) = \theta_p(\tau'_\mu(\infty))$ .

Note that this affine part of the singularity spectrum  $\zeta_{\mu, p}^*$  of typical functions  $f \in \tilde{B}_q^{\mu, p}(\mathbb{R}^d)$  is obtained as a consequence of the heterogeneous ubiquity property stated in Proposition 3.18.

Also, if  $[\theta_p(\tau'_\mu(\infty)), \theta_p(\alpha_p)]$  is non trivial,  $\zeta_{\mu, p}^*$  is concave on this interval.

Moreover, using the notations of Remark 3.4,  $\zeta_{\mu, p}^*$  is differentiable at  $\theta_p(\tau'_\mu(\infty))$  if and only if  $t_\infty = (\tau_\mu^*)'(\tau'_\mu(\infty)) = \infty$ . Otherwise, one has  $(\zeta_{\mu, p}^*)'(\theta_p(\tau'_\mu(\infty))^+) = \frac{t_\infty}{t_\infty + p} p < p = (\zeta_{\mu, p}^*)'(\theta_p(\tau'_\mu(\infty))^-)$ . This implies that  $\zeta_{\mu, p}$  is affine over the interval  $[\frac{t_\infty}{t_\infty + p} p, p]$ , with slope  $\theta_p(\tau'_\mu(\infty))$ .

See Figures 4 and 7 for some examples of the shape of the spectrum of typical functions  $f \in \tilde{B}_q^{\mu, p}(\mathbb{R}^d)$ .

(5) When  $-p \notin \partial(\tau_\mu^*)$ , one has  $t_{-\infty} > -\infty$ , so both  $\tau_\mu$  and  $\zeta_{\mu, p}$  are affine near  $-\infty$ .

*Proof of Proposition 5.1.* The case  $p = \infty$  is trivial, so we assume  $p \in [1, \infty)$ .

Recall that we defined  $\alpha_{\min} = \tau'_\mu(\infty)$ . Let  $\chi$  be the mapping defined on the right hand side of (74). We are going to prove that  $\chi^* = \zeta_{\mu, p}$  as defined by (21). Next, we will check that the function  $\chi$ , which is continuous over its domain, is concave. This and the Legendre duality will get  $\zeta_{\mu, p}^* = \chi$ .

It is convenient to write  $\chi^* = \min(\chi_1^*, \chi_2^*)$  where, for  $t \in \mathbb{R}$ ,

$$\begin{aligned} \chi_1^*(t) &= \inf\{tH - p(H - \alpha_{\min}) : H \in [\alpha_{\min}, \theta_p(\alpha_{\min})]\} \\ \chi_2^*(t) &= \inf\{tH - \tau_\mu^*(\theta_p^{-1}(H)) : H \in [\theta_p(\alpha_{\min}), \theta_p(\alpha_p)]\}. \end{aligned}$$

If  $t \neq p$ , we set

$$t_p = \frac{pt}{p - t}.$$

Then whenever it exists, let  $\tilde{\alpha}_t$  be the unique real number such that

$$t_p \in [(\tau_\mu^*)'(\tilde{\alpha}_t^+), (\tau_\mu^*)'(\tilde{\alpha}_t^-)].$$

Otherwise, set  $\tilde{\alpha}_t = \alpha_{\min}$ .

First fix  $t > p$ . The mapping  $H \mapsto tH - p(H - \alpha_{\min})$  is increasing, hence  $\chi_1^*(t) = t\alpha_{\min}$ . Setting  $\alpha = \theta_p^{-1}(H)$  for  $H \in [\theta_p(\alpha_{\min}), \theta_p(\alpha_p)]$ , one has

$$\chi_2^*(t) = \inf_{\alpha \in [\alpha_{\min}, \alpha_p]} \tilde{\chi}(\alpha)$$

where

$$\tilde{\chi}(\alpha) = t\theta_p(\alpha) - \tau_\mu^*(\alpha).$$

Then, differentiating (formally)  $\tilde{\chi}$  gives  $\tilde{\chi}'(\alpha) = t + \frac{t-p}{p}(\tau_\mu^*)'(\alpha)$ . Since  $\tau_\mu^*$  is non decreasing over  $[\alpha_{\min}, \tau'_\mu(0^+)]$ , for  $\tilde{\chi}$  not to be non decreasing,  $\tilde{\alpha}_t$  must exist and be



distinct from  $\alpha_{\min}$ , except if  $\tau_\mu$  is linear over  $[0, \infty)$ , in which case  $\alpha_{\min} = \tau'_\mu(0^+)$ . Suppose  $\tilde{\alpha}_t > \alpha_{\min}$ . Since  $t > p$ , we have  $\frac{t}{t-p} \in [1, \infty)$ , hence  $t_p \in (-\infty, -p]$ , so that  $\tilde{\alpha}_t \in (\alpha_p, \infty)$ . Consequently,  $\tilde{\chi}$  is non decreasing on  $[\alpha_{\min}, \alpha_p]$  and the infimum defining  $\chi_2^*$  is reached at  $\alpha_{\min}$ , where it equals  $t\alpha_{\min} + \frac{t-p}{p}\tau_\mu^*(\alpha_{\min}) \geq t\alpha_{\min}$ . One concludes that  $\chi^*(t) = t\alpha_{\min}$  as desired, and it is easily checked that the same holds if  $\tilde{\alpha}_t = \alpha_{\min}$ .

The case  $t = p$  follows by continuity.

Fix now  $t < p$ . The mapping  $H \mapsto tH - p(H - \alpha_{\min})$  is decreasing, so  $\chi_1^*(t) = (t-p)\theta_p(\alpha_{\min}) + p\alpha_{\min} = t\alpha_{\min} + \frac{t-p}{p}\tau_\mu^*(\alpha_{\min})$ . To determine  $\chi_2^*(t)$ , we distinguish the cases  $t_p \leq (\tau_\mu^*)'(\alpha_{\min})$  and  $t_p > (\tau_\mu^*)'(\alpha_{\min})$ .

Suppose first that  $t_p \leq (\tau_\mu^*)'(\alpha_{\min})$ . Noting that  $t_p > -p$ , we deduce that the function  $\tilde{\chi}$  now reaches its minimum at  $\tilde{\alpha}_t$ , which necessarily belongs to  $[\alpha_{\min}, \alpha_p]$ . Consequently,

$$\chi_2^*(t) = t\theta_p(\tilde{\alpha}_t) - \tau_\mu^*(\tilde{\alpha}_t).$$

By definition, whenever  $\tilde{\alpha}_t \neq \alpha_{\min}$ ,  $\tau_\mu^*(\tilde{\alpha}_t) = t_p\tilde{\alpha}_t - \tau_\mu(t_p)$ , so after simplification one gets in this case

$$\chi_2^*(t) = t \left( \tilde{\alpha}_t + \frac{\tau_\mu^*(\tilde{\alpha}_t)}{p} \right) - \tau_\mu^*(\tilde{\alpha}_t) = \frac{p-t}{p}\tau_\mu(t_p).$$

Noting that  $\chi_2^*(\alpha_{\min}) = \chi_1^*(\alpha_{\min})$ , we can conclude that (21) holds when  $\tilde{\alpha}_t = \alpha_{\min}$ , or when  $\tilde{\alpha}_t \neq \alpha_{\min}$  and if, moreover,  $t\alpha_{\min} + \frac{t-p}{p}\tau_\mu^*(\alpha_{\min}) \geq \frac{p-t}{p}\tau_\mu(t_p)$ . If  $\tilde{\alpha}_t \neq \alpha_{\min}$ , using that  $\tau_\mu^*(\tilde{\alpha}_t) = t_p\tilde{\alpha}_t - \tau_\mu(t_p)$ , the previous inequality is equivalent to  $t\alpha_{\min} + \frac{t-p}{p}\tau_\mu^*(\alpha_{\min}) \geq t\tilde{\alpha}_t - \frac{p-t}{p}\tau_\mu^*(\tilde{\alpha}_t)$ , i.e.  $t(\tilde{\alpha}_t - \alpha_{\min}) \leq \frac{p-t}{p}(\tau_\mu^*(\tilde{\alpha}_t) - \tau_\mu^*(\alpha_{\min}))$ , that is

$$\frac{\tau_\mu^*(\tilde{\alpha}_t) - \tau_\mu^*(\alpha_{\min})}{\tilde{\alpha}_t - \alpha_{\min}} \geq t_p \in \partial(\tau_\mu^*)(\tilde{\alpha}_t).$$

But this inequality does hold due to the concavity of  $\tau_\mu^*$ .

It remains the case where  $t < p$  and  $t_p > (\tau_\mu^*)'(\alpha_{\min})$ . In this case,  $t_\infty = (\tau_\mu^*)'(\alpha_{\min}) < \infty$ , which implies that  $\tau_\mu$  is linear over  $[t_\infty, \infty)$ : specifically,  $\tau_\mu(t) = \alpha_{\min}t - \tau_\mu^*(\alpha_{\min}) +$  for all  $t \geq t_\infty$ . Also, since  $t_p > (\tau_\mu^*)'(\alpha_{\min}) > 0$ ,  $\tilde{\alpha}_t = \alpha_{\min}$  and  $\partial\tilde{\chi} \subset (0, \infty)$ . In particular  $\tilde{\chi}$  reaches its minimum at  $\alpha_{\min}$ . Consequently,  $\chi_2^*(t) = \chi_1^*(t) = t\alpha_{\min} + \frac{t-p}{p}\tau_\mu^*(\alpha_{\min})$ . Since  $t_p \geq t_\infty$  and  $\tau_\mu$  is affine on  $[t_\infty, \infty)$ , it follows that  $\chi^*(t) = \frac{p-t}{p}\tau_\mu(t_p)$ , as announced.

Note that the previous case corresponds to  $\frac{t_\infty}{t_\infty+p}p < t < p$ . In regard to the form taken by  $\zeta_{\mu,p}^*$ , it is convenient to rewrite  $\zeta_{\mu,p}(t) = \theta_p(\alpha_{\min})t - \tau_\mu^*(\alpha_{\min})$ .

Now we prove that  $\chi$  is concave. We assume that the domain of  $\chi$  is not reduced to  $[\alpha_{\min}, \theta_p(\alpha_{\min})]$ , for otherwise the conclusion is trivial.

Let us start by explaining why  $\chi$  is concave over  $[\theta_p(\alpha_{\min}), \theta_p(\alpha_p)]$ . To do so, assume first that  $\tau_\mu^*$  is differentiable over  $(\alpha_{\min}, \theta_p^{-1}(\alpha_p))$ . Then this is also the case for  $\theta_p^{-1}$  over  $(\theta_p(\alpha_{\min}), \theta_p(\alpha_p))$ . For  $H \in (\theta_p(\alpha_{\min}), \theta_p(\alpha_p))$ , denoting  $\theta_p^{-1}(H)$  by  $\alpha$  and  $(\tau_\mu^*)'(\alpha)$  by  $t$ , one gets  $\chi'(H) = \frac{t}{1+t/p}$ , which is increasing as a function of  $t$ . Since  $H = \theta_p(\alpha)$  is an increasing function of  $\alpha$  and  $\alpha$  is a decreasing function of  $t$  ( $\tau_\mu$  is concave), it follows

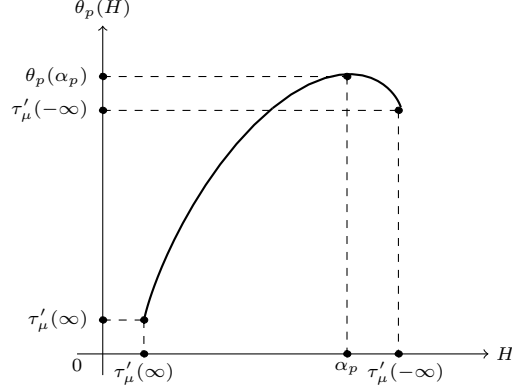


FIGURE 6. The mapping  $\theta_p$  when  $\sigma_\mu(\tau'_\mu(\infty)) = \sigma_\mu(\tau'_\mu(-\infty)) = 0$ .

that  $\chi'$  is increasing over  $(\theta_p(\alpha_{\min}), \theta_p(\alpha_p))$ . Hence  $\chi$  is concave over  $[\theta_p(\alpha_{\min}), \theta_p(\alpha_p)]$ . If  $\tau_\mu^*$  has point of non differentiability over  $(\alpha_{\min}, \theta_p^{-1}(\alpha_p))$ , we get the conclusion by approximating it by the differentiable  $L^q$ -spectra associated with the Bernoulli product generated by the probability vectors used to construct  $\mu$ .

Finally, if  $\theta_p(\alpha_{\min}) > \alpha_{\min}$ , i.e.  $\tau_\mu^*(\alpha_{\min}) > 0$ , to get that  $\chi$  is concave, it is enough to check that  $\chi'(\theta_p(\alpha_{\min}^+)) \leq p = \chi'(\theta_p(\alpha_{\min}^-))$ . With the notations used above, we have to distinguish the cases  $(\tau_\mu^*)'(\alpha_{\min}) = t_\infty < \infty$  and  $t_\infty = \infty$ . A direct computation then yields  $\chi'(\theta_p(\alpha_{\min}^+)) = p$  if  $t_\infty = \infty$  and  $\chi'(\theta_p(\alpha_{\min}^+)) = \frac{t_\infty}{t_\infty + p}p$  otherwise.  $\square$

## 6. LOWER BOUND FOR THE $L^q$ -SPECTRUM, AND UPPER BOUND FOR THE SINGULARITY SPECTRUM IN $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ , WHEN $\mu \in \mathcal{E}_d$

This section uses the notions of wavelet leaders and  $L^q$ -spectrum of a function introduced in Section 2.5. We are going to prove item (1) of Theorem 2.19 by establishing a non trivial general lower bound for the  $L^q$ -spectrum of any element of  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  when  $\mu \in \mathcal{E}_d$  (Theorem 2.26(1)).

The main result of this section is the following. Recall the definition (20) of  $r_\mu$ .

**Theorem 6.1.** *Let  $\mu \in \mathcal{E}_d$  and  $p, q \in [1, \infty]$ . Let  $\Psi \in \mathcal{F}_{r_\mu}$ . For all  $f \in L^p(\mathbb{R}^d)$  such that  $|f_{\mu,p,q}| < \infty$ , one has  $\zeta_f|_{\mathbb{R}_+} \geq \zeta_{\mu,p}|_{\mathbb{R}_+}$ .*

It is implicit in Theorem 6.1 that the semi-norm  $|f_{\mu,p,q}|$  defined in (16) is computed using the wavelet  $\Psi \in \mathcal{F}_{r_\mu}$  which is fixed by the statement.

It yields the following corollary.

**Corollary 6.2.** *Let  $\mu \in \mathcal{E}_d$  and  $p, q \in [1, \infty]$ . For all  $f \in \tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ , one has:*

- (1)  $\zeta_f|_{\mathbb{R}_+} \geq \zeta_{\mu,p}|_{\mathbb{R}_+}$ , i.e. the claim of Theorem 2.26(1) holds true.
- (2) For all  $H \in \mathbb{R}$ ,

$$\sigma_f(H) \leq \begin{cases} \zeta_{\mu,p}^*(H) & \text{if } H \leq \zeta'_{\mu,p}(0^+) \\ d & \text{if } H > \zeta'_{\mu,p}(0^+) \end{cases},$$

i.e. part (1) of Theorem 2.19 holds true.

*Proof.* Part (1) follows from the definition of  $\widetilde{B}_q^{\mu,p}(\mathbb{R}^d)$  and the continuity of  $\zeta_{\mu^{(-\varepsilon)},p|\mathbb{R}_+}$  as a function of  $\varepsilon$ . Part (2) is then a consequence of (27).  $\square$

To obtain Theorem 6.1, we estimate, for any  $f \in L^p(\mathbb{R}^d)$  such that  $|f_{\mu,p,q}| < \infty$  and any  $N \in \mathbb{N}$ , the upper large deviations spectrum of the wavelet leaders  $(L_\lambda^f)_{\lambda \in N[0,1]^d}$  associated with some given wavelets  $\Psi \in \mathcal{F}_{r_\mu}$ , which we fix for the rest of this section (see the definition of  $N\lambda$  at the beginning of Section 2).

**Definition 6.3.** Let  $f \in L_{\text{loc}}^1(\mathbb{R}^d)$  and  $N \in \mathbb{N}^*$ . For  $H \in \mathbb{R}$  and  $\varepsilon > 0$ , set

$$H \pm \varepsilon = [H - \varepsilon, H + \varepsilon].$$

For any compact subinterval  $I$  of  $\mathbb{R}$  set

$$\mathcal{D}_f^N(j, I) = \left\{ \lambda \in \mathcal{D}_j : \lambda \subset N[0, 1]^d, \frac{\log_2 |L_\lambda^f|}{-j} \in I \right\},$$

where the wavelet coefficients are computed with  $\Psi$ .

Then the upper wavelet leaders large deviation spectrum of  $f$  associated with  $\Psi$  and  $N[0, 1]^d$  is defined as

$$\bar{\sigma}_f^{\text{LD},N}(H) = \lim_{\varepsilon \rightarrow 0} \limsup_{j \rightarrow +\infty} \frac{\log_2 \#\mathcal{D}_f^N(j, H \pm \varepsilon)}{j}.$$

**Proposition 6.4.** Let  $\mu \in \mathcal{E}_d$  and  $p, q \in [1, \infty]$ . For all  $f \in L^p(\mathbb{R}^d)$  such that  $|f_{\mu,p,q}| < \infty$ , and all  $N \in \mathbb{N}$ , one has

$$(75) \quad \bar{\sigma}_f^{\text{LD},N}(H) \leq \begin{cases} \zeta_{\mu,p}^*(H) & \text{if } H \leq \zeta'_{\mu,p}(0^+) \\ d & \text{if } H > \zeta'_{\mu,p}(0^+) \end{cases}.$$

Let us assume this proposition has been proven and explain how Theorem 6.1 follows.

*Proof of Theorem 6.1.* Note that by large deviations theory, the function  $\zeta_f^{N,\Psi}$  defined in (25) is the Legendre transform of the concave hull of  $\bar{\sigma}_f^{\text{LD},N}$  [35]. By Proposition 6.4, this concave hull is dominated by the concave function  $\max(\zeta_{\mu,p}^*, d)$ , whose Legendre transform is easily seen to be equal to  $\zeta_{\mu,p|\mathbb{R}_+}$  over  $\mathbb{R}_+$  and equal to  $-\infty$  over  $\mathbb{R}_+^*$ . Consequently,  $\zeta_f^{N,\Psi}|_{\mathbb{R}_+} \geq \zeta_{\mu,p|\mathbb{R}_+}$ , which is enough to conclude since  $\zeta_f^\Psi|_{\mathbb{R}_+}$  does not depend on  $\Psi$ .  $\square$

Proving Proposition 6.4 requires some large deviations estimates on the distribution of the wavelet coefficients of  $f$  under the constraint imposed by the condition  $|f_{\mu,p,\infty}| < \infty$ , which holds automatically if  $|f_{\mu,p,q}| < \infty$ .

**6.1. Preliminary large deviations estimates on the distribution of wavelet coefficients when  $|f_{\mu,p,\infty}| < \infty$ .** If  $\mu \in \mathcal{C}(\mathbb{R}^d)$ ,  $I_H$  and  $I_\alpha$  are two compact subintervals of  $\mathbb{R}$ , and  $f \in L^1_{\text{loc}}(\mathbb{R}^d)$  has  $(c_\lambda)_{\lambda \in \Lambda}$  as wavelet coefficients, let

$$\Lambda_{f,\mu}(j, I_H, I_\alpha) = \left\{ \lambda = (i, j, k) \in \Lambda : \lambda_{j,k} \subset 3[0, 1]^d, \begin{cases} \frac{\log_2 |c_\lambda|}{-j} \in I_H \\ \frac{\log_2 \mu(\lambda_{j,k})}{-j} \in I_\alpha \end{cases} \right\}.$$

Heuristically,  $\Lambda_{f,\mu}(j, I_H, I_\alpha)$  contains cubes of generation  $j$  whose  $\mu$ -capacity is of order of magnitude  $2^{-j\alpha}$  with  $\alpha \in I_\alpha$  and whose associated wavelet coefficient is of order of magnitude  $2^{-jh}$  with  $h \in I_H$ . We consider  $3[0, 1]^d$  rather than  $[0, 1]^d$  because the computation of wavelet leaders on  $[0, 1]^d$  requires some knowledge of  $\mu$  and  $f$  in this neighbourhood of  $[0, 1]^d$ .

We are interested in estimating the cardinality of  $\Lambda_{f,\mu}(j, I_H, I_\alpha)$  in order to get a control of the wavelets leaders large deviations spectrum under the assumptions of Proposition 6.4.

In the next lemma we adopt the convention  $\infty \times x = \infty$  for  $x \geq 0$ . Recall that for any interval  $I$  and  $\varepsilon > 0$ ,  $I \pm \varepsilon$  stands for  $I + [-\varepsilon, \varepsilon]$ .

**Lemma 6.5.** *Let  $\mu \in \mathcal{E}_d$  and  $p \in [1, \infty]$ . Let  $\alpha_{\min} = \tau'_\mu(\infty)$  and  $\alpha_{\max} = \tau'_\mu(-\infty)$ . Let  $f \in L^p(\mathbb{R}^d)$  be such that  $|f|_{\mu,p,\infty} < \infty$  and let  $I_H, I_\alpha$  be two compact subintervals of  $\mathbb{R}$ .*

- (1) *If  $\max I_H < \min I_\alpha$ , then  $\Lambda_{f,\mu}(j, I_H, I_\alpha) = \emptyset$  for  $j$  large enough.*
- (2) *If  $I_\alpha \subset [\alpha_{\min}, \alpha_{\max}]$  and  $\min I_\alpha \leq \min I_H$ , then for every  $\eta > 0$ , there exists  $\varepsilon_0 > 0$  and  $J_0 \in \mathbb{N}$  such that for every  $\varepsilon \in [0, \varepsilon_0]$  and  $j \geq J_0$ :*

$$(76) \quad \frac{\log_2 \#\Lambda_{f,\mu}(j, I_H \pm \varepsilon, I_\alpha \pm \varepsilon)}{j} \leq \max_{\beta \in I_\alpha \cap [0, \max I_H]} \min(p(\max I_H - \beta), \tau_\mu^*(\beta)) + \eta.$$

*Proof.* We treat the  $p < \infty$  and leave the simpler case  $p = \infty$  to the reader.

- (1) Recall that by definition  $\sup_{j \in \mathbb{N}} \left\| \left( \frac{c_\lambda}{\mu(\lambda)} \right)_{\lambda \in \Lambda_j} \right\|_p < \infty$ . There is  $C_f \geq 1$  such that

$$(77) \quad \sup_{j \in \mathbb{N}} \left\| \left( \frac{c_\lambda}{\mu(\lambda)} \right)_{\lambda \in \Lambda_j} \right\|_p \leq C_f.$$

It follows that item (1) holds true, for otherwise (77) would be contradicted.

- (2) Fix  $\eta, \varepsilon > 0$  and set  $\tilde{H} = \max(I_H)$ . Since  $I_\alpha$  is compact and  $\tau_\mu^*$  is continuous over its domain, there are finitely many numbers  $\alpha_0 < \dots < \alpha_m$  such that  $I_\alpha = \bigcup_{\ell=0}^{m-1} [\alpha_\ell, \alpha_{\ell+1}]$  and for every  $\ell$ ,  $\alpha_{\ell+1} - \alpha_\ell \leq \eta/p$  and  $|\tau_\mu^*(\beta) - \tau_\mu^*(\beta')| \leq \eta$  for all  $\beta, \beta' \in [\alpha_\ell, \alpha_{\ell+1}]$ .

Let  $j \in \mathbb{N}$ . Consider the subset  $\Lambda_{f,\mu}(j, I_H, [\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon)$  of  $\Lambda_{f,\mu}(j, I_H \pm \varepsilon, I_\alpha \pm \varepsilon)$ . With each cube  $\lambda \in \Lambda_{f,\mu}(j, I_H \pm \varepsilon, [\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon)$  is associated a wavelet coefficient  $c_\lambda$  whose absolute value is at least equal to  $2^{-j(\tilde{H}+\varepsilon)}$ . Thus, for each  $\ell \in \{0, \dots, m-1\}$ ,

$$(78) \quad C_f^p \geq \sum_{\lambda \in \Lambda_j} \left( \frac{|c_\lambda|}{\mu(\lambda)} \right)^p \geq \sum_{\lambda \in \Lambda_{f,\mu}(j, I_H \pm \varepsilon, [\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon)} \left( \frac{2^{-j(\tilde{H}+\varepsilon)}}{2^{-j(\alpha_\ell - \varepsilon)}} \right)^p.$$

**Remark 6.6.** We recall again that when  $\lambda = (i, j, k) \in \Lambda_j$ , sometimes we make a slight abuse of notation by considering when necessary  $\lambda$  as the dyadic cube  $\lambda_{j,k} \in \mathcal{D}_j$ . So we write  $\mu(\lambda)$  for  $\mu(\lambda_{j,k})$ , and when  $I$  is a subset of  $\mathbb{R}^d$  we may write  $\lambda \subset I$  meaning that  $\lambda_{j,k} \subset I$ .

It follows from (78) that

$$\#\Lambda_{f,\mu}(j, I_H, [\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon) \leq C_f^p 2^{jp(\tilde{H} - \alpha_\ell + 2\varepsilon)}.$$

On the other hand, observe that for each  $j \geq 0$ , one has

$$\Lambda_{f,\mu}(j, I_H \pm \varepsilon, [\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon) \subset \left\{ \lambda = (i, j, k) \in \Lambda : \lambda \subset 3[0, 1]^d, \frac{\log_2 \mu(\lambda)}{-j} \in I \right\},$$

where  $I = [\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon \cap [0, \tilde{H} + \varepsilon]$ . Applying Proposition 3.3(4) to each interval  $[\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon \cap [0, \tilde{H} + \varepsilon]$ , one finds  $\varepsilon_0 > 0$  and  $J_0 \in \mathbb{N}$  such that for all  $\varepsilon \in (0, \varepsilon_0]$ ,  $0 \leq \ell \leq m-1$  and  $j \geq J_0$ ,

$$\#\mathcal{D}_\mu(j, [\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon \cap [0, \tilde{H} + \varepsilon]) \leq \#\mathcal{D}_\mu(j, ([\alpha_\ell, \alpha_{\ell+1}] \cap [0, \tilde{H}]) \pm 2\varepsilon) \leq 2^{j(\gamma_\ell + \eta)},$$

where  $\gamma_\ell = \max\{\tau_\mu^*(\beta) : \beta \in [\alpha_\ell, \alpha_{\ell+1}] \cap [0, \tilde{H}]\}$ . Then, taking into account the fact that  $\mu$  is  $\mathbb{Z}^d$ -invariant, as well as the fact that with each dyadic cube  $\lambda_{j,k}$  are associated  $2^d - 1$  wavelet coefficients, one obtains

$$\#\Lambda_{f,\mu}(j, I_H \pm \varepsilon, [\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon) \leq 3^d (2^d - 1) 2^{j(\gamma_\ell + \eta)}.$$

Combining the previous estimates yields, one gets for  $\varepsilon \in (0, \varepsilon_0]$  and  $j \geq J_0$

$$\begin{aligned} \#\Lambda_{f,\mu}(j, I_H, I_\alpha \pm \varepsilon) &\leq \sum_{\ell=0}^{m-1} \#\Lambda_{f,\mu}(j, I_H, [\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon) \\ &\leq \sum_{\ell=0}^{m-1} \min(C_f^p 2^{jp(\tilde{H} - \alpha_\ell + 2\varepsilon)}, 3^d (2^d - 1) \cdot 2^{j(\gamma_\ell + \eta)}) \\ &\leq 3^d (2^d - 1) C_f^p m \max\{2^{j \min(p(\tilde{H} - \alpha_\ell + 2\varepsilon), \gamma_\ell + \eta)} : \ell = 0, 1, \dots, m-1\}. \end{aligned}$$

Also, the constraints imposed to the exponents  $\alpha_\ell$  imply that

$$\begin{aligned} &\max\{ \min(p(\tilde{H} - \alpha_\ell + 2\varepsilon), \gamma_\ell + \eta) : \ell = 0, 1, \dots, m-1 \} \\ &\leq \max\{ \min(p(\tilde{H} - \beta), \tau_\mu^*(\beta)) : \beta \in I_\alpha \cap [0, \tilde{H}] \} + 2\varepsilon + 3\eta. \end{aligned}$$

Taking  $\varepsilon_0 \leq \eta/p$  and  $J_0$  so large that  $2^{J_0\eta} \geq 3^d (2^d - 1) C_f^p m$ , we finally get the desired upper bound (76) (with  $6\eta$  instead of  $\eta$ ).  $\square$

We are now ready to get an upper bound for the wavelet leaders upper large deviations spectrum of  $f$ .

**6.2. Proof of Proposition 6.4.** First notice that since  $\mu$  is  $\mathbb{Z}^d$ -invariant, due to the definition of  $|\cdot|_{\mu,p,q}$ , any general upper bound for  $\bar{\sigma}_{f_{[0,1]^d}}^{\text{LD},1}$  holds for  $\bar{\sigma}_f^{\text{LD},N}$ . Thus, without loss of generality we prove that  $\bar{\sigma}_f^{\text{LD},1}$  is upper bounded by the right hand side of (75).

This proof is rather involved because we must treat with care all the possible interactions between the values  $\mu(\lambda)$  and the wavelet coefficients  $c_\lambda$  which contribute to determine the wavelet leaders with a given order of magnitude.

Note that the inequality  $\bar{\sigma}_f^{\text{LD},1} \leq d$  obviously holds. So it is enough to deal with the case  $H \leq \zeta'_{\mu,p}(0^+)$ .

We fix  $H \leq \zeta'_{\mu,p}(0^+)$  and for  $\varepsilon > 0$  small enough estimate  $\#\mathcal{D}_f^1(j, H \pm \varepsilon)$  from above (recall Definition 6.3). Specifically, we establish (75) under the equivalent form: there exist  $C, c > 0$  such that for any  $\eta > 0$ , if  $\varepsilon_0 \in (0, \eta]$  is chosen small enough, then for  $j$  large enough, for all  $\varepsilon \in (0, \varepsilon_0)$ ,

$$(79) \quad \#\mathcal{D}_f^1(j, H \pm \varepsilon) \leq Cj2^{j(\zeta_{\mu,p}^*(H)+c\eta)}.$$

Since  $|f_{\mu,p,\infty}| < \infty$ , there exists  $C > 0$  such that  $|c_\lambda| \leq C\mu(\lambda)$  for every  $\lambda \in \bigcup_{j \geq 0} \Lambda_j$  (recall Remark 6.6).

Without loss of generality, we suppose that the above constant is equal to 1 and

$$|c_\lambda| \leq \mu(\lambda) \text{ for every } \lambda \in \bigcup_{j \geq 0} \Lambda_j.$$

Recall the definition (23) of wavelet leaders:  $L_\lambda^f = \sup\{|c_{\lambda'}| : \lambda' = (i, j, k) \in \Lambda, \lambda' \subset 3\lambda\}$ .

We start with the following key observations. A dyadic cube  $\lambda$  belongs to  $\mathcal{D}_f^1(j, H \pm \varepsilon)$  if and only if:

- $\lambda \subset [0, 1]^d$ ;
- there exists a dyadic cube  $\lambda' \subset 3\lambda$  of generation  $j' \geq j$  as well as  $i \in \{1, \dots, 2^d - 1\}$  and  $k' \in \mathbb{Z}^d$  such that  $\lambda' = \lambda_{j',k'}$ , and  $|c_{(i,j',k')}| = 2^{-j'H'}$  with  $H' \in \frac{j}{j'}[H - \varepsilon, H + \varepsilon]$ ;
- moreover, if  $j$ , hence  $j'$ , is large enough, Proposition 6.5(2) implies that  $|c_{(i,j',k')}| \leq 2^{-j'\alpha_{\min}/2}$ . So  $H' \geq \alpha_{\min}/2$ . It follows that  $j' \leq 2j/\alpha_{\min}$ .

We will use these properties (mainly the second one) repeatedly in what follows.

Now we distinguishing three cases.

**Case 1 :**  $H < \alpha_{\min}$ .

Note that  $\zeta_{\mu,p}^*(H) = -\infty$ . Suppose that  $\varepsilon > 0$  is so small that  $\alpha_{\min} - \varepsilon > H + \varepsilon$ . Due to Proposition 3.3(5), and the observation made just above, for  $j$  large enough we have

$$\#\mathcal{D}_f(j, H \pm \varepsilon) \leq \sum_{j \leq j' \leq 2j/\alpha_{\min}} \Lambda_{f,\mu}(j', [0, H + \varepsilon], I_\alpha),$$

with  $I_\alpha = [\alpha_{\min} - \varepsilon, \alpha_{\max} + \varepsilon]$ . However,  $H + \varepsilon < \alpha_{\min} - \varepsilon$ , so by Lemma 6.5,  $\mathcal{D}_f(j', H \pm \varepsilon) = \emptyset$ . This implies (75), i.e.  $\bar{\sigma}_{f_{[0,1]^d}}^{\text{LD},1}(H) = -\infty$ .

To deal with the other cases, we discretise the interval  $[\alpha_{\min}, H]$ .

Fix  $\eta > 0$ ,  $\varepsilon_0 \in (0, \min(1/2, \alpha_{\min}/2), \eta)$ , and split the interval  $[\alpha_{\min}, H]$  into finitely many contiguous closed intervals  $I_1, \dots, I_m$  ( $m = m(\varepsilon_0)$ ) such that

- $|I_\ell| \leq \varepsilon_0$  for every  $\ell \in \{1, \dots, m\}$ ,
- Writing  $I_\ell = [h_\ell, h_{\ell+1}]$ , one has  $1 \leq h_{\ell+1}/h_\ell \leq 1 + \varepsilon_0$  for every  $1 \leq \ell \leq m$ .

In particular,  $H/h_\ell \geq 1$  for every  $\ell$ .

Now, applying Proposition 6.5(2) to each pair  $\{I_\ell, I_{\ell'}\}$ , one can find  $\varepsilon \in (0, \varepsilon_0)$  and  $J_0 \in \mathbb{N}$  such that for all  $j \geq J_0$ , for all  $1 \leq \ell' \leq \ell \leq m$ ,

$$(80) \quad \frac{\log_2 \#\Lambda_{f,\mu}(j, I_\ell \pm \varepsilon, I_{\ell'} \pm \varepsilon)}{j} \leq d(\ell, \ell') + \eta$$

where

$$(81) \quad d(\ell, \ell') = \max \{ \min(p(h_{\ell+1} - \beta), \tau_\mu^*(\beta)) : \beta \in I_{\ell'} \}.$$

As observed above, if  $j \geq J_0$  and  $\lambda \in \mathcal{D}_f(j, H \pm \varepsilon)$ , there exists  $j' \geq j$  and  $\lambda' = (i, j', k') \in \Lambda_{j'}$  such that  $\lambda' \subset 3\lambda$  and  $|c_{\lambda'}| = 2^{-j'H'}$  with  $H' \in \frac{j}{j'}[H \pm \varepsilon]$ . Moreover there exists  $1 \leq \ell' \leq \ell \leq m$  such that  $\lambda' \in \mathcal{D}_{f,\mu}(j, I_\ell \pm \varepsilon, I_{\ell'} \pm \varepsilon)$ .

In addition,  $H' \in I_\ell \pm \varepsilon \subset I_\ell \pm \varepsilon_0$ ,  $j' \in \frac{j}{H'}[H \pm \varepsilon] \subset \left[ j \frac{H-\varepsilon_0}{h_{\ell+1}+\varepsilon_0}, j \frac{H+\varepsilon_0}{h_\ell-\varepsilon_0} \right]$ , and  $h_{\ell+1} \leq H$ . Consequently,

$$(82) \quad \mathcal{D}_f(j, H \pm \varepsilon) \subset \bigcup_{1 \leq \ell' \leq \ell \leq m} \mathcal{D}_f^{\ell, \ell'}(j, H \pm \varepsilon),$$

where (recall Remark 6.6)

$$\mathcal{D}_f^{\ell, \ell'}(j, H \pm \varepsilon) = \bigcup_{\substack{j' \in \left[ \frac{H-\varepsilon_0}{h_{\ell+1}+\varepsilon_0}, \frac{H+\varepsilon_0}{h_\ell-\varepsilon_0} \right]}} \left\{ \lambda \in \mathcal{D}_j \cap [0, 1]^d : \left\{ \begin{array}{l} \exists \lambda' \in \Lambda_{f,\mu}(j', I_\ell \pm \varepsilon, I_{\ell'} \pm \varepsilon) \\ \text{such that } \lambda' \subset 3\lambda \end{array} \right. \right\}.$$

**Case 2:**  $\alpha_{\min} \leq H < \theta_p(\alpha_{\min}) = \alpha_{\min} + \frac{\tau_\mu^*(\alpha_{\min})}{p}$ . This case is non empty only when  $\tau_\mu^*(\alpha_{\min}) > 0$ .

Let  $j \geq J_0$ , and  $1 \leq \ell' \leq \ell \leq m$ . Since  $h_{\ell+1} \leq H$ , one necessarily has  $d(\ell, \ell') \leq p(h_{\ell+1} - h_{\ell'}) \leq p(h_\ell - \alpha_{\min})$ . Thus, if  $j' \in \left[ j \frac{H-\varepsilon_0}{h_{\ell+1}+\varepsilon_0}, j \frac{H+\varepsilon_0}{h_\ell-\varepsilon_0} \right]$ , then  $j'd(\ell, \ell') \leq j(pH \frac{h_\ell - \alpha_{\min}}{h_\ell} + O(\varepsilon_0) + \eta)$ , where  $O(\varepsilon_0)$  is independent of  $\ell$ . Taking the supremum over  $\ell$  yields

$$j'(d(\ell, \ell') + \eta) \leq j(p(H - \alpha_{\min}) + O(\varepsilon_0) + \eta) = j(\zeta_{\mu,p}^*(H) + O(\varepsilon_0) + \eta).$$

Consequently, since (82) implies

$$\#\mathcal{D}_f(j, H \pm \varepsilon) \leq \sum_{1 \leq \ell' \leq \ell \leq m} \sum_{j' \in \left[ j \frac{H-\varepsilon_0}{h_{\ell+1}+\varepsilon_0}, j \frac{H+\varepsilon_0}{h_\ell-\varepsilon_0} \right]} \#\Lambda_{f,\mu}(j', I_\ell \pm \varepsilon, I_{\ell'} \pm \varepsilon),$$

inequality (80) combined with the previous remarks yields

$$\#\mathcal{D}_f(j, H \pm \varepsilon) \leq m^2 j \frac{H + \varepsilon_0}{\alpha_{\min} - \varepsilon_0} 2^{j(\zeta_{\mu,p}^*(H) + O(\varepsilon_0) + \eta)}.$$

**Case 3:**  $\theta_p(\alpha_{\min}) \leq H \leq \zeta'_{\mu,p}(0^+) = \theta_p(\tau'_\mu(0^+))$ .

This case will be subdivided into four subcases in order to estimate  $\#\mathcal{D}_f^{\ell,\ell'}(j, H \pm \varepsilon)$ .

Recall the definition (81) of  $d(\ell, \ell')$ . This quantity can easily be expressed in terms of the mappings  $\theta_p$  defined in (73) and  $\tau_\mu^*$ . The mapping  $\theta_p$  is an increasing map over  $[\alpha_{\min}, \alpha_p]$  and  $\alpha_p \geq \tau'_\mu(0^+)$ , so using that  $h_\ell \leq H$ , one deduces that

$$(83) \quad d(\ell, \ell') = \begin{cases} \tau_\mu^*(h_{\ell'+1}) & \text{if } h_{\ell'+1} \leq \theta_p^{-1}(h_{\ell+1}), \\ p(h_{\ell+1} - h_{\ell'}) & \text{if } h_{\ell'} \geq \theta_p^{-1}(h_{\ell+1}), \\ \tau_\mu^*(\theta_p^{-1}(h_{\ell+1})) = \zeta_{\mu,p}^*(h_{\ell+1}) & \text{otherwise.} \end{cases}$$

Moreover, the maximum of the three possible values is always  $\zeta_{\mu,p}^*(h_{\ell+1})$ .

**Subcase (a):**  $\frac{H}{h_{\ell+1}}d(\ell, \ell') \leq \zeta_{\mu,p}^*(H)$ . Using the definition of  $\mathcal{D}_f^{\ell,\ell'}(j, H \pm \varepsilon)$ , inequality (80), the fact that  $\frac{H+\varepsilon_0}{h_{\ell+1}-\varepsilon_0} = \frac{H}{h_{\ell+1}} + O(\varepsilon_0)$  and  $d(\ell, \ell') \leq \frac{H}{h_{\ell+1}}d(\ell, \ell') \leq \zeta_{\mu,p}^*(H)$ , for  $j \geq J_0$  we get

$$\begin{aligned} \#\mathcal{D}_f^{\ell,\ell'}(j, H \pm \varepsilon) &\leq \sum_{j' \in \left[ j \frac{H-\varepsilon_0}{h_{\ell+1}+\varepsilon_0}, j \frac{H+\varepsilon_0}{h_{\ell+1}-\varepsilon_0} \right]} \#\Lambda_{f,\mu}(j', I_\ell \pm \varepsilon, I_{\ell'} \pm \varepsilon) \\ &\leq j \frac{H + \varepsilon_0}{\alpha_{\min} - \varepsilon_0} 2^{j(\zeta_{\mu,p}^*(H) + O(\varepsilon_0) + \eta)}. \end{aligned}$$

**Subcase (b):**  $\frac{H}{h_{\ell+1}}d(\ell, \ell') > \zeta_{\mu,p}^*(H)$  and  $h_{\ell'+1} \leq \theta_p^{-1}(h_{\ell+1})$ .

A technical lemma is needed.

**Lemma 6.7.** *For every  $j$  large enough,*

$$\mathcal{D}_f^{\ell,\ell'}(j, H \pm \varepsilon) \subset \mathcal{D}_\mu \left( j, \left[ \alpha_{\min}, \alpha_{\min} + \frac{H}{h_{\ell+1}}(h_{\ell'+1} - \alpha_{\min}) \right] \pm O(\varepsilon_0) \right),$$

where  $O(\varepsilon_0)$  is independent of  $(\ell, \ell')$ .

*Proof.* Take  $\lambda \in \mathcal{D}_f^{\ell,\ell'}(j, H \pm \varepsilon)$  and  $j' \in \left[ j \frac{H-\varepsilon_0}{h_{\ell+1}+\varepsilon_0}, j \frac{H+\varepsilon_0}{h_{\ell+1}-\varepsilon_0} \right]$  such that there exists  $\lambda' = (i, j', k') \in \Lambda_{f,\mu}(j', I_\ell \pm \varepsilon, I_{\ell'} \pm \varepsilon)$  for which  $\lambda' \subset 3\lambda$ .

Denote by  $\widehat{\lambda}$  the unique dyadic interval of  $\mathcal{D}_j$  that containing  $\lambda'$ . One has  $\mu(\widehat{\lambda}) = \mu(\lambda') \frac{\mu(\widehat{\lambda})}{\mu(\lambda')}$ , and by construction of  $\mu$ ,  $\frac{\mu(\widehat{\lambda})}{\mu(\lambda')} \geq 2^{(j'-j)(\alpha_{\min}-\varepsilon_0)}$  (recall (41)).

Moreover, the property (P) satisfied by  $\mu$  gives that  $\mu(\lambda) \geq 2^{-j\varepsilon_0} \mu(\widehat{\lambda})$ . Consequently, since  $\mu(\lambda') \geq 2^{-j'(h_{\ell'+1}+\varepsilon_0)}$ , one concludes that

$$\frac{\log \mu(\lambda)}{-j \log(2)} \leq \alpha_{\min} + \frac{j'}{j}(h_{\ell'+1} + \varepsilon_0) - (\alpha_{\min} - \varepsilon_0) = \alpha_{\min} + \frac{H}{h_{\ell+1}}(h_{\ell'+1} - \alpha_{\min}) + O(\varepsilon_0),$$

where  $O(\varepsilon_0)$  is independent of  $(\ell, \ell')$ . This yields the result.  $\square$



Let us now bound  $\alpha_{\min} + \frac{H}{h_{\ell+1}}(h_{\ell'+1} - \alpha_{\min})$  from above. Thanks to (83),  $h_{\ell'+1} \leq \theta_p^{-1}(h_{\ell+1})$  implies  $d(\ell, \ell') = \tau_\mu^*(h_{\ell'+1})$ . Since  $\theta_p^{-1}(h_{\ell+1}) \leq \theta_p^{-1}(H) \leq \tau_\mu'(0^+)$  and  $\tau_\mu^*$  is non decreasing over  $[\alpha_{\min}, \tau_\mu'(0^+)]$ , one has

$$\frac{H}{h_{\ell+1}}\tau_\mu^*(\theta_p^{-1}(h_{\ell+1})) \geq \frac{H}{h_{\ell+1}}\tau_\mu^*(h_{\ell'+1}) > \zeta_{\mu,p}^*(H) = \tau_\mu^*(\theta_p^{-1}(H)),$$

from which one deduces that

$$(84) \quad \frac{\tau_\mu^*(\theta_p^{-1}(h_{\ell+1}))}{h_{\ell+1}} > \frac{\tau_\mu^*(\theta_p^{-1}(H))}{H}.$$

Observe that the definition of  $\theta_p$  implies that

$$(85) \quad \theta_p^{-1}(\beta) + p^{-1}\tau_\mu^*(\theta_p^{-1}(\beta)) = \beta$$

for all  $\beta \in [\alpha_{\min}, \zeta_{\mu,p}'(0^+)]$ . Applying (85) to both sides of (84) yields

$$(86) \quad \frac{\theta_p^{-1}(h_{\ell+1})}{h_{\ell+1}} < \frac{\theta_p^{-1}(H)}{H},$$

and since  $\frac{H}{h_{\ell+1}} > 1$ , the following series of inequalities hold:

$$(87) \quad \alpha_{\min} + \frac{H}{h_{\ell+1}}(h_{\ell'+1} - \alpha_{\min}) \leq \frac{H}{h_{\ell+1}}h_{\ell'+1} \leq \frac{H}{h_{\ell+1}}\theta_p^{-1}(h_{\ell+1}) \leq \theta_p^{-1}(H).$$

Consequently, Lemma 6.7 yields

$$(88) \quad \mathcal{D}_f^{\ell,\ell'}(j, H \pm \varepsilon) \subset \mathcal{D}_\mu(j, [\alpha_{\min}, \theta_p^{-1}(H)] \pm O(\varepsilon_0)).$$

Recall  $\tau_\mu^*$  is continuous and non-decreasing over  $[\alpha_{\min}, \theta_p^{-1}(H)]$  by Proposition 3.3(4). Hence, choosing initially  $\varepsilon_0$  small enough yields for  $j$  large enough that

$$(89) \quad \#\mathcal{D}_f^{\ell,\ell'}(j, H \pm \varepsilon) \leq 2^{j(\tau_\mu^*(\theta_p^{-1}(H)) + \eta)} = 2^{j(\zeta_{\mu,p}^*(H) + \eta)}.$$

**Subcase (c):**  $\frac{H}{h_{\ell+1}}d(\ell, \ell') > \zeta_{\mu,p}^*(H)$  and  $h_{\ell'} \geq \theta_p^{-1}(h_{\ell+1})$ .

By (83),  $d(\ell, \ell') = p(h_{\ell+1} - h_{\ell'})$ . Consequently,  $h_{\ell'} = h_{\ell'} - \frac{d(\ell, \ell')}{p} < h_{\ell+1} - \frac{h_{\ell+1}}{H}\zeta_{\mu,p}^*(H)/p$ , and

$$\alpha_{\min} + \frac{H}{h_{\ell+1}}(h_{\ell'+1} - \alpha_{\min}) \leq \frac{H}{h_{\ell+1}}h_{\ell'+1} < \frac{H}{h_{\ell+1}}\left(h_{\ell+1} - \frac{h_{\ell+1}}{H}\frac{\zeta_{\mu,p}^*(H)}{p}\right) + H\frac{h_{\ell+1} - h_{\ell'}}{h_{\ell+1}}.$$

Thus,

$$(90) \quad \alpha_{\min} + \frac{H}{h_{\ell+1}}(h_{\ell'+1} - \alpha_{\min}) \leq H - \frac{\zeta_{\mu,p}^*(H)}{p} + O(\varepsilon_0) = \theta_p^{-1}(H) + O(\varepsilon_0),$$

where again  $O(\varepsilon_0)$  is independent of  $(\ell, \ell')$ . Hence, one concludes by Lemma 6.7 that (89) holds in this subcase as well.

**Subcase (d):**  $\frac{H}{h_{\ell+1}}d(\ell, \ell') > \zeta_{\mu,p}^*(H)$  and  $h_{\ell'} < \theta_p^{-1}(h_{\ell+1}) < h_{\ell'+1}$ .

Here one has  $h_{\ell'+1} \leq (1 + \varepsilon_0)h_{\ell'} \leq (1 + \varepsilon_0)\theta_p^{-1}(h_{\ell+1})$ , so

$$(91) \quad \begin{aligned} \alpha_{\min} + \frac{H}{h_{\ell+1}}(h_{\ell'+1} - \alpha_{\min}) &\leq (1 + \varepsilon_0)\frac{H}{h_{\ell+1}}\theta_p^{-1}(h_{\ell+1}) + \alpha_{\min} \left(1 - \frac{H}{h_{\ell'+1}}\right) \\ &\leq (1 + \varepsilon_0)\frac{H}{h_{\ell+1}}\theta_p^{-1}(h_{\ell+1}). \end{aligned}$$

Also, (83) gives  $d(\ell, \ell') = \tau_\mu^*(\theta_p^{-1}(h_{\ell+1}))$ , so  $\frac{H}{h_{\ell+1}}d(\ell, \ell') > \zeta_{\mu,p}^*(H)$  is equivalent to (84), and it implies (86). Finally, arguing as in the subcase (b) and using (91), one sees that (90) holds, from which one deduces that (89) holds once again.

Collecting the estimates obtained along the cases considered above, (79) is proved, and so is Proposition 6.4.

## 7. TYPICAL SINGULARITY SPECTRUM IN $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$

In this section we compute the singularity spectrum of typical functions in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  when  $\mu \in \mathcal{E}_d$ , proving item (2) of Theorem 2.19.

The general strategy is similar to that used to derive the generic multifractal behavior in classical Besov spaces. First, a *saturation* function is built, whose multifractal structure is precisely the one hoped to be generic in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ . Then, this function is used to perturb a countable family of dense sets in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ , in order to obtain a countable family of dense open sets on the intersection of which the desired multifractal behavior holds. However, the construction of the saturation function as well as the multifractal analysis of typical functions are much more delicate than in Besov spaces, i.e. when  $\mu$  is a power of  $\mathcal{L}^d$ .

The environment  $\mu \in \mathcal{E}_d$  is fixed for the rest of this section, as well as  $(p, q) \in [1, \infty]^2$  and  $\Psi \in \mathcal{F}_{r_\mu}$ .

**7.1. A saturation function.** In this section, a saturation function  $g^{\mu,p,q} \in \tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  is built via its wavelet coefficients, which are *as large as possible* in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ , and its wavelet leaders are estimated.

The definition of  $g^{\mu,p,q}$  demands some preparation.

For every  $N \in \mathbb{N}^*$ , if  $[\alpha_{\min}, \alpha_{\max}] = [\tau'_\mu(\infty), \tau'_\mu(-\infty)]$  is not a singleton, it is possible to find an integer  $M_N$  such that the interval  $[\alpha_{\min}, \alpha_{\max}] = [\tau'_\mu(\infty), \tau'_\mu(-\infty)]$  can be split into  $M_N$  non-trivial contiguous closed intervals  $I_1^N, I_2^N, \dots, I_{M_N}^N$  satisfying for every  $i \in \{1, \dots, M_N\}$ ,

$$(92) \quad |I_i^N| \leq 1/N \quad \text{and} \quad \max\{|\tau_\mu^*(\alpha) - \tau_\mu^*(\alpha')| : \alpha, \alpha' \in I_i^N\} \leq 1/N$$

We also take the sequence  $(M_N)_{N \geq 1}$  increasing.

If  $\alpha_{\min} = \alpha_{\max}$ , we fix an increasing and positive sequence of integers  $(M_N)_{N \in \mathbb{N}^*}$ , and set  $I_i^N = \{\alpha_{\min}\}$  for all  $1 \leq i \leq M_N$ .

In any case, item (4) of Proposition 3.3 yields a decreasing sequence  $(\eta_N)_{N \in \mathbb{N}^*}$  converging to 0 as  $N \rightarrow \infty$ , and for all  $N \in \mathbb{N}^*$ ,  $M_N$  integers  $J_{N,1}, J_{N,2}, \dots, J_{N,M_N}$ ,

such that for every  $i \in \{1, \dots, M_N\}$ , for every  $j \geq J_{N,i}$ ,

$$(93) \quad \left| \frac{\log_2 \#\mathcal{D}_\mu(j, I_i^N \pm 1/N)}{j} - \max_{\alpha \in I_i^N} \tau_\mu^*(\alpha) \right| \leq \eta_N.$$

Without loss of generality, we assume that  $\eta_N \geq 1/N$ .

Then, define inductively an increasing sequence of integers  $(J_N)_{N \in \mathbb{N}^*}$  such that:

$$(94) \quad \begin{cases} \forall N \geq 1, J_N \geq \max\{J_{N,i} : i \in \{1, \dots, M_N\}\} \\ \forall N \geq 2, M_N \leq 2^{J_N \eta_{N-1}}, \\ \forall N \geq 3, J_{N-1} \eta_{N-2} < J_N \eta_{N-1}. \end{cases}$$

Moreover, we can require that for every  $j \geq J_N$  and  $\lambda \in \mathcal{D}_j$ :

$$2^{-j(\alpha_{\max} + 1/N)} \leq \mu(\lambda) \leq 2^{-j(\alpha_{\min} - 1/N)}.$$

This is possible due (45).

When  $J_N \leq j < J_{N+1}$ , set  $N_j = N$ . Since we required that  $(J_N \eta_{N-1})_{N \geq 2}$  is an increasing sequence, the sequence  $(j \eta_{N_j-1})_{j \geq J_2}$  is increasing as well.

Finally, let us introduce some coefficients depending on the elements  $\lambda \in \Lambda_j$ :

- If  $L \in \mathbb{Z}^d$ ,  $j \geq J_2$  and  $\lambda \in \Lambda_j^L = \{\lambda = (i, j, k) \in \Lambda_j : \lambda_{j,k} \subset L + [0, 1]^d\}$ , set

$$(95) \quad w_\lambda = \begin{cases} \frac{2^{-\frac{3j\eta_{N_j-1}}{p}}}{j^{\frac{1}{p} + \frac{2}{q}} (1 + \|L\|)^{\frac{d+1}{p}}} & \text{if } p < \infty \\ j^{-\frac{2}{q}} & \text{if } p = \infty, \end{cases}$$

with the convention  $\frac{2}{\infty} = 0$ .

- If  $j \geq J_2$  and  $\lambda = (i, j, k) \in \Lambda_j$ , set  $\alpha_{j,k} = \frac{\log_2 \mu(\lambda_{j,k})}{-j}$  and

$$\alpha_\lambda = \begin{cases} \alpha_{j,k} & \text{if } \alpha_{j,k} \in [\alpha_{\min}, \alpha_{\max}], \\ \alpha_{\min} & \text{if } \alpha_{j,k} < \alpha_{\min}, \\ \alpha_{\max} & \text{if } \alpha_{j,k} > \alpha_{\max}. \end{cases}$$

**Remark 7.1.** Note that  $\tilde{\varepsilon}_\lambda = \frac{\log_2 \mu(\lambda)}{-j} - \alpha_\lambda$  tends to 0 uniformly in  $\lambda \in \mathcal{D}_j$  as  $j \rightarrow \infty$ .

Before defining the saturation function, we recall Definition 3.15, and to  $\lambda = (i, j, k) \in \Lambda_j$ , we associate the irreducible dyadic cube  $\bar{\lambda} := \lambda_{\bar{j}, \bar{k}}$  such that  $2^{-\bar{j}\bar{k}} = 2^{-j}k$  with  $0 \leq \bar{j} \leq j$  and  $\bar{k} \in \mathbb{Z}^d \setminus (2\mathbb{Z})^d$ .

**Definition 7.2.** The saturation function  $g^{\mu,p,q} : \mathbb{R}^d \rightarrow \mathbb{R}$  is defined by its wavelet coefficients in the wavelet basis associated with  $\Psi$ , denoted by  $(c_\lambda^{\mu,p,q})_{\lambda \in \Lambda}$ , as follows:

- $c_\lambda^{\mu,p,q} = 0$  if  $\lambda \in \bigcup_{j < J_2} \Lambda_j$ .

- If  $j \geq J_2$  and  $\lambda = (i, j, k) \in \Lambda_j$ , we set

$$(96) \quad c_\lambda^{\mu,p,q} = \begin{cases} w_\lambda \cdot \mu(\lambda_{j,k}) 2^{-\bar{j} \frac{\tau_\mu^*(\alpha_{\bar{\lambda}})}{p}} & \text{if } p < \infty \\ w_\lambda \cdot \mu(\lambda_{j,k}) & \text{if } p = \infty. \end{cases}$$

**Remark 7.3.** (1) Note that  $c_\lambda^{\mu,p,q}$  does not depend on  $i$  if  $\lambda = (i, j, k)$ . Consequently,  $c_\lambda^{\mu,p,q}$  is defined without ambiguity by the same formula for  $\lambda \in \mathcal{D}_j$ .

(2) The choice of  $\bar{j}$  and  $\bar{\lambda}$  in the exponent  $2^{-\bar{j} \frac{\tau_\mu^*(\alpha_{\bar{\lambda}})}{p}}$  in (96) implies that at a given generation  $j$ , the wavelet coefficients of  $g^{\mu,p,q}$  display several order of magnitudes. One can also guess from this choice that approximation by dyadic numbers will play an important role in our analysis, since the local behavior of  $g^{\mu,p,q}$  around a point  $x$  will depend on how close  $x$  is to the dyadic numbers.

(3) When  $p < \infty$  and  $\tau_\mu^*(\alpha_{\min}) = 0$ , in (96)  $\mu(\lambda) 2^{-\bar{j} \frac{\tau_\mu^*(\alpha_{\bar{\lambda}})}{p}}$  can be replaced by the simpler term  $\mu(\lambda)$  and still get a relevant saturation function  $g^{\mu,p,q}$ . When  $\tau_\mu^*(\alpha_{\min}) > 0$ , the situation is more subtle, the ubiquity properties pointed out in Proposition 3.18 come into play.

**Lemma 7.4.** The function  $g^{\mu,p,q} = \sum_{j \geq J_2} \sum_{\lambda \in \Lambda_j} c_\lambda^{\mu,p,q} \psi_\lambda$  belongs to  $\widetilde{B}_q^{\mu,p}(\mathbb{R}^d)$ .

*Proof.* Suppose that  $p < \infty$ .

For  $j \in \mathbb{N}$  and  $L \in \mathbb{Z}^d$ , set  $\mathcal{D}_j^L = \{\lambda \in \mathcal{D}_j : \lambda \subset L + [0, 1]^d\}$  and  $\Lambda_j^L = \{(i, j, k) \in \Lambda_j : \lambda_{j,k} \in \mathcal{D}_j^L\}$ .

Recall that for  $\lambda = (i, j, k)$ ,  $\mu(\lambda)$  stands for  $\mu(\lambda_{j,k})$ .

Let us define, for  $j \geq J_2$  and  $L \in \mathbb{Z}^d$ , the sum  $A_{j,L} = \sum_{\lambda \in \Lambda_j^L} \left( \frac{|c_\lambda^{\mu,p,q}|}{\mu(\lambda)} \right)^p$ . To prove that  $g^{\mu,p,q} \in B_q^{\mu,p}(\mathbb{R}^d)$ , it is enough to show that  $A_j := \left( \sum_{L \in \mathbb{Z}^d} A_{j,L} \right)^{1/p} \in \ell^q(\mathbb{N})$ .

For  $j \in [J_N, J_{N+1})$ , one has

$$(97) \quad \begin{aligned} A_{j,L} &= \sum_{\lambda \in \Lambda_j^L} \left( \frac{2^{-3j\eta_{N_j-1}/p} \mu(\lambda) 2^{-\bar{j} \frac{\tau_\mu^*(\alpha_{\bar{\lambda}})}{p}}}{j^{\frac{1}{p} + \frac{2}{q}} (1 + \|L\|)^{(d+1)/p} \mu(\lambda)} \right)^p \\ &= (2^d - 1) \frac{2^{-3j\eta_{N_j-1}}}{j^{1 + \frac{2p}{q}} (1 + \|L\|)^{(d+1)}} \sum_{\lambda \in \mathcal{D}_j^0} 2^{-\bar{j} \tau_\mu^*(\alpha_{\bar{\lambda}})} \end{aligned}$$

where the factor  $2^{d-1}$  comes from the fact that  $c_\lambda^{\mu,p,q}$ ,  $\lambda = (i, j, k)$ , is independent of  $i \in \{1, \dots, 2^{d-1}\}$ .

Observe that if  $\lambda \in \mathcal{D}_j$  and  $\bar{\lambda}$  is the cube associated with its irreducible representation, then one can write  $\lambda = \bar{\lambda} \cdot [0, 2^{-(j-\bar{j})}]^d$ , the concatenation meaning that  $\lambda$  equals the image of  $[0, 2^{-(j-\bar{j})}]^d$  by the canonical isometry that sends  $[0, 1]^d$  to  $\bar{\lambda}$ .

Then, after regrouping in (97) the terms according to the generation of their irreducible representation, one has

$$\begin{aligned}
A_{j,L} &= (2^d - 1) \frac{2^{-3j\eta_{N_j-1}}}{j^{1+\frac{2p}{q}} (1 + \|L\|)^{(d+1)}} \left( 1 + \sum_{J=1}^j \sum_{\lambda \in \mathcal{D}_J^0 \setminus (\mathcal{D}_{J-1}^0 \cdot [0,2^{-1}]^d)} 2^{-J\tau_\mu^*(\alpha_\lambda)} \right) \\
&\leq 2^d \frac{2^{-3j\eta_{N_j-1}}}{j^{1+\frac{2p}{q}} (1 + \|L\|)^{(d+1)}} \left( 1 + \sum_{J=1}^j \sum_{\lambda \in \mathcal{D}_J^0} 2^{-J\tau_\mu^*(\alpha_\lambda)} \right) \\
(98) \quad &= 2^d \frac{2^{-3j\eta_{N_j-1}}}{j^{1+\frac{2p}{q}} (1 + \|L\|)^{(d+1)}} \left( \sum_{J=0}^{J_1-1} + \left( \sum_{N=1}^{N_j-1} \sum_{J=J_N}^{J_{N+1}-1} \right) + \sum_{J=J_{N_j}}^j \right) \sum_{\lambda \in \mathcal{D}_J^0} 2^{-J\tau_\mu^*(\alpha_\lambda)}.
\end{aligned}$$

For each  $J_N \leq J < J_{N+1}$ , using (92) and then (93), we obtain

$$\begin{aligned}
\sum_{\lambda \in \mathcal{D}_J^0} 2^{-J\tau_\mu^*(\alpha_\lambda)} &\leq \sum_{i=1}^{M_{N_j}} \sum_{\lambda \in \mathcal{D}_\mu(j, I_i^{N_j} \pm 1/N)} 2^{-J(\max\{\tau_\mu^*(\alpha) : \alpha \in I_i^{N_j}\} - 1/N_j)} \\
&\leq \sum_{i=1}^{M_{N_j}} 2^{J(\max\{\tau_\mu^*(\alpha) : \alpha \in I_i^{N_j}\} + \eta_{N_j})} 2^{-J(\max\{\tau_\mu^*(\alpha) : \alpha \in I_i^{N_j}\} - 1/N_j)} \\
&= M_{N_j} 2^{J(\eta_{N_j} + 1/N_j)} \leq M_{N_j} 2^{2J\eta_{N_j}}.
\end{aligned}$$

Consequently, by (94),

$$\begin{aligned}
&\left( \left( \sum_{N=1}^{N_j-1} \sum_{J=J_N}^{J_{N+1}-1} \right) + \sum_{J=J_{N_j}}^j \right) \sum_{\lambda \in \mathcal{D}_J^0} 2^{-J\tau_\mu^*(\alpha_\lambda)} \\
&\leq \sum_{N=1}^{N_j-1} \sum_{J=J_N}^{J_{N+1}-1} M_N 2^{2J\eta_N} + \sum_{J=J_{N_j}}^j M_{N_j} 2^{2J\eta_{N_j}} \\
&\leq \sum_{N=1}^{N_j-1} (J_{N+1} - J_N) M_N 2^{2J_{N+1}\eta_N} + (j - J_{N_j} + 1) M_{N_j} 2^{2j\eta_{N_j}} \\
&\leq j M_{N_j} 2^{2j\eta_{N_j-1}},
\end{aligned}$$

since all terms  $M_N 2^{2J_{N+1}\eta_N}$  are less than  $M_{N_j} 2^{2j\eta_{N_j-1}}$ .

Setting  $C_\mu = \sum_{J=0}^{J_1-1} \sum_{\lambda \in \mathcal{D}_J^0} 2^{-J\tau_\mu^*(\alpha_\lambda)}$ , it follows still from (94) that

$$A_{j,L} \leq 2^d \frac{M_{N_j} 2^{-j\eta_{N_j-1}}}{j^{\frac{2p}{q}} (1 + \|L\|)^{(d+1)}} (C_\mu + 1) \leq \frac{2^d (C_\mu + 1)}{j^{\frac{2p}{q}} (1 + \|L\|)^{(d+1)}}.$$

Finally,

$$\left( \sum_{L \in \mathbb{Z}^d} A_{j,L} \right)^{1/p} = \left\| \left( \frac{c_\lambda^{\mu,p,q}}{\mu(\lambda)} \right)_{\lambda \in \Lambda_j} \right\|_p = O(j^{-2/q}),$$

hence  $\left(\left\|\left(\frac{c_{\lambda}^{\mu,p,q}}{\mu(\lambda)}\right)_{\lambda \in \Lambda_j}\right\|_p\right)_{j \in \mathbb{N}}$  is in  $\ell^q(\mathbb{N})$ . This implies that  $g^{\mu,p,q} \in B_q^{\mu,p}(\mathbb{R}^d)$ .

If  $p = \infty$ , the estimate is much simpler and left to the reader.  $\square$

Recall Remark 7.3(1). Next lemma shows that the wavelet leader (recall (23))  $L_{\lambda}^{g^{\mu,p,q}}$  of  $g^{\mu,p,q}$  at  $\lambda \in \mathcal{D}_j$  is essentially comparable to the wavelet coefficients  $c_{\lambda'}^{\mu,p,q}$  indexed by the cubes  $\lambda$  of generation  $j$  which neighbour  $\lambda$ . This property will be useful to estimate the  $L^q$ -spectrum of  $g^{\mu,p,q}$  relative to  $\Psi$ .

**Lemma 7.5.** *Fix  $L \in \mathbb{Z}^d$ . For every  $\varepsilon > 0$ , there exists  $J^\varepsilon \in \mathbb{N}$  such that if  $j \geq J^\varepsilon$ , for every  $\lambda \in \mathcal{D}_j^L$ ,*

$$\tilde{c}_{\lambda}^{\mu,p,q} \leq L_{\lambda}^{g^{\mu,p,q}} \leq 2^{j\varepsilon} \tilde{c}_{\lambda}^{\mu,p,q},$$

where  $\tilde{c}_{\lambda}^{\mu,p,q} = \max\{c_{\lambda}^{\mu,p,q} : \lambda \in \mathcal{D}_j, \tilde{\lambda} \subset 3\lambda\}$ .

*Proof.* It is enough to prove the result for  $L = 0$ . Let  $\varepsilon, \varepsilon' \in (0, 1)$ . Let  $j \geq 1$  and  $\lambda \in \mathcal{D}_j^0$ . Let us begin with some remarks.

First, in (96), the term  $w_{\lambda}$  depends only on  $j$ , and is decreasing with  $j$ .

Second, if  $\lambda' \subset \lambda$ ,  $\mu(\lambda') \leq \mu(\lambda)$  since  $\mu \in \mathcal{C}(\mathbb{R}^d)$ .

Next, observe that if  $\lambda' \subset \lambda$ , the irreducible cubes  $\bar{\lambda}' \in \mathcal{D}_{\bar{j}'}$  and  $\bar{\lambda} \in \mathcal{D}_{\bar{j}}$  associated with  $\lambda'$  and  $\lambda$ , respectively, are such that  $\bar{j} \leq \bar{j}'$ . Moreover, if  $\bar{j} \leq j p \varepsilon / (2d)$ , then  $c_{\lambda'}^{\mu,p,q} 2^{j\varepsilon} \geq w_{\lambda} \mu(\lambda)$  so  $c_{\lambda'}^{\mu,p,q} \leq c_{\lambda}^{\mu,p,q} 2^{j\varepsilon}$ . If  $\bar{j} > j p \varepsilon / (2d)$  and  $j$  is large enough, by construction of  $\mu$ , either  $\bar{j}' - \bar{j} \leq \varepsilon' \bar{j}'$  or one has

$$(99) \quad \bar{j}' \alpha_{\bar{\lambda}'} = \bar{j} \alpha_{\bar{\lambda}} + (\bar{j}' - \bar{j}) \alpha$$

for some  $\alpha \in [\alpha_{\min} - \varepsilon, \alpha_{\max} + \varepsilon]$ . We can choose  $\varepsilon'$  small enough so that if  $\bar{j}' - \bar{j} \leq \varepsilon' \bar{j}'$ , then  $|\alpha_{\bar{\lambda}'} - \alpha_{\bar{\lambda}}|$  is so small that  $|\bar{j}' \tau_{\mu}^*(\alpha_{\bar{\lambda}'} - \bar{j} \tau_{\mu}^*(\alpha_{\bar{\lambda}}))| \leq \bar{j} p \varepsilon$ , and once again  $c_{\lambda'}^{\mu,p,q} \leq c_{\lambda}^{\mu,p,q} 2^{j\varepsilon}$ . Such an  $\varepsilon'$  being fixed, if  $\bar{j}' - \bar{j} > \varepsilon' \bar{j}'$ , (99) and the concavity of  $\tau_{\mu}^*$  then implies that for some  $\tilde{\varepsilon}$  independent of  $j$  and  $j'$ ,

$$\bar{j}' \tau_{\mu}^*(\alpha_{\bar{\lambda}'} - \bar{j} \tau_{\mu}^*(\alpha_{\bar{\lambda}})) \geq \bar{j} \tau_{\mu}^*(\alpha_{\bar{\lambda}}) + (\bar{j}' - \bar{j}) (\tau_{\mu}^*(\alpha^*) - \tilde{\varepsilon}), \quad \text{where } \alpha^* = \begin{cases} \alpha & \text{when } \alpha \in [\alpha_{\min}, \alpha_{\max}] \\ \alpha_{\max} & \text{when } \alpha \geq \alpha_{\max} \\ \alpha_{\min} & \text{when } \alpha \leq \alpha_{\min} \end{cases}$$

(we must be careful because  $\alpha$  may not belong to  $[\alpha_{\min}, \alpha_{\max}]$  and this case  $\tau_{\mu}^*(\alpha) = -\infty$ ). In particular,  $\bar{j}' \tau_{\mu}^*(\alpha_{\bar{\lambda}'} - \bar{j} \tau_{\mu}^*(\alpha_{\bar{\lambda}})) \geq \bar{j} \tau_{\mu}^*(\alpha_{\bar{\lambda}}) - (\bar{j}' - \bar{j}) \tilde{\varepsilon}$ , hence

$$2^{-\bar{j}' \tau_{\mu}^*(\alpha_{\bar{\lambda}'})/p} \leq 2^{-\bar{j} \tau_{\mu}^*(\alpha_{\bar{\lambda}})/p} 2^{(\bar{j}' - \bar{j}) \tilde{\varepsilon}/p} \leq 2^{-\bar{j} \tau_{\mu}^*(\alpha_{\bar{\lambda}})/p} 2^{\bar{j}' \tilde{\varepsilon}/p}.$$

Note that :

- Since  $\mu$  satisfies (P), there exists  $M \in \mathbb{N}^*$  such that for every  $\hat{\lambda} \in \mathcal{D}_{Mj}$  one has  $\mu(\hat{\lambda}) \leq 2^{-j(d/p + 2\alpha_{\max} + 1)}$ . Due to the definition of the wavelet coefficients of  $g^{\mu,p,q}$ , this implies that if  $j' \geq Mj$ , then  $c_{\lambda'}^{\mu,p,q} \leq c_{\lambda}^{\mu,p,q}$ . In other words, the relevant resolutions  $j'$ , i.e. the  $j'$  such that there may exist  $\lambda' \in \Lambda_{j'}$  with  $c_{\lambda'}^{\mu,p,q}$  greater than  $c_{\lambda}^{\mu,p,q}$ , verify necessarily  $j' \leq Mj$ .
- $\tilde{\varepsilon}$  can be chosen as small as necessary when  $j$  tends to infinity, in particular less than  $p\varepsilon / (M(d/p + 2\alpha_{\max} + 1))$ .

These two observations imply that for  $j' \geq j$  large enough, independently on  $\lambda \in \mathcal{D}_j^0$  and  $\lambda' \in \mathcal{D}_{j'}$  such that  $\lambda' \subset \lambda$  and  $\bar{j}' - \bar{j} > \varepsilon' \bar{j}'$ , one has

$$2^{-\bar{j}'} \tau_\mu^*(\alpha_{\bar{\lambda}'})/p \leq 2^{-\bar{j}} \tau_\mu^*(\alpha_{\bar{\lambda}})/p 2^{j\varepsilon}.$$

Putting together all the previous information yields that for  $j' \geq j$  large enough, for all  $\lambda \in \mathcal{D}_j^0$  and all  $\lambda' \in \mathcal{D}_{j'}$  such that  $\lambda' \subset \lambda$ , one has  $c_{\lambda'}^{\mu,p,q} \leq c_\lambda^{\mu,p,q} 2^{j\varepsilon}$ . However, the same property holds true for all  $j', j$  large enough, for all  $\tilde{\lambda} \in \mathcal{D}_j$  such that  $\tilde{\lambda} \subset 3[0, 1]^d$  and  $\lambda' \in \mathcal{D}_{j'}$  such that  $\lambda' \subset \tilde{\lambda}$ . This yields the desired property.  $\square$

**7.2. The singularity spectrum of the saturation function  $g^{\mu,p,q}$  and some of its perturbations.** We now determine the singularity spectrum of  $g^{\mu,p,q}$ , and more generally of any function whose wavelet coefficients are “comparable” to those of  $g^{\mu,p,q}$  over infinitely many generations.

**Proposition 7.6.** *Let  $f \in \tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  such that for any  $L \in \mathbb{Z}^d$ , there exists an increasing sequence of integers  $(j_n)_{n \in \mathbb{N}}$ , and a positive sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  converging to 0 such that for all  $n \geq 1$  and  $\lambda = (i, j_n, k) \in \Lambda_{j_n}$  such that  $\lambda_{j_n,k} \subset L + 3[0, 1]^d$  the inequality  $2^{-j_n \varepsilon_n} c_\lambda^{\mu,p,q} \leq |c_\lambda^f|$  holds. Then  $\sigma_f = \sigma_{g^{\mu,p,q}} = \zeta_{\mu,p}^*$ .*

*Proof.* We suppose that  $p < \infty$ , the other case being simpler and easily deduced from arguments similar to those developed below. Fix  $(j_n)_{n \in \mathbb{N}}$  and  $(\varepsilon_n)_{n \in \mathbb{N}}$  as in the statement.

It is enough to prove that  $\dim E_f(H) \cap (L + [0, 1]^d) = \zeta_{\mu,p}^*(H)$  for all  $H \in \mathbb{R}$  and  $L \in \mathbb{Z}^d$ . Without loss of generality we work with  $L = 0$  and show that  $\dim E_f(H) \cap [0, 1]^d = \zeta_{\mu,p}^*(H)$  for all  $H \in \mathbb{R}$ .

Note that the characterization (24) and the assumptions on  $(j_n)_{n \in \mathbb{N}}$  imply that for all  $x \in [0, 1]^d$ ,

$$(100) \quad \liminf_{n \rightarrow +\infty} \frac{\log c_{\lambda_{j_n}(x)}^{\mu,p,q}}{\log 2^{-j_n}} \geq \liminf_{n \rightarrow +\infty} \frac{\log |c_{\lambda_{j_n}(x)}^f|}{\log 2^{-j_n}} \geq \liminf_{j \rightarrow +\infty} \frac{\log L_j^f(x)}{\log 2^{-j}} \geq h_f(x).$$

• **The upper bound**  $\sigma_f \leq \zeta_{\mu,p}^*$ . Theorem 2.19(1) gives  $\sigma_f(H) \leq \zeta_{\mu,p}^*(H)$  for all  $H \leq \zeta_{\mu,p}'(0^+)$ . Note also that  $\zeta_{\mu,p}^*(H) = d$  for all  $H \in [\zeta_{\mu,p}'(0^+), \zeta_{\mu,p}'(0^-)]$ . Hence it remains us to treat the case  $H > \zeta_{\mu,p}'(0^-)$ , which corresponds to the decreasing part of the spectrum of  $g$  (and  $f$ ).

Fix  $H > \zeta_{\mu,p}'(0^-)$  and  $x \in [0, 1]^d$  such that  $h_f(x) \geq H$ .

By (100),

$$(101) \quad \liminf_{n \rightarrow +\infty} \frac{\log c_{\lambda_{j_n}(x)}^{\mu,p,q}}{\log 2^{-j_n}} \geq H.$$

For the cube  $\lambda_{j_n}(x)$ , denote by  $\overline{\lambda_{j_n}(x)} \in \mathcal{D}_{\bar{j}_n}$  its irreducible representation, and write  $\lambda_{j_n}(x) = \overline{\lambda_{j_n}(x)} \cdot [0, 2^{-(j_n - \bar{j}_n)}]^d$ , using the concatenation of cubes introduced just after

Definition 3.15. One has

$$\begin{aligned}
 \frac{\log c_{\lambda_{j_n}(x)}^{\mu,p,q}}{\log 2^{-j_n}} &= \frac{\log w_{\lambda_{j_n}(x)} \mu(\lambda_{j_n}(x)) 2^{-\frac{\tau_\mu^*(\alpha_{\overline{\lambda_{j_n}(x)}})}{p}}}{\log 2^{-j_n}} \\
 (102) \qquad &= \frac{\log_2 w_{\lambda_{j_n}(x)}}{j_n} + \frac{\log_2 \mu(\lambda_{j_n}(x))}{-j_n} + \frac{\overline{j_n} \tau_\mu^*(\alpha_{\overline{\lambda_{j_n}(x)}})}{j_n p}.
 \end{aligned}$$

Due to (54) and Remark 7.1, dropping the dependence in  $x$  in  $\lambda_{j_n}(x)$  and  $\overline{\lambda_{j_n}(x)}$  in the formulas, we get

$$\begin{aligned}
 \frac{\log_2 \mu(\lambda_{j_n})}{-j_n} &= \frac{\overline{j_n} \log_2 \mu(\lambda_{\overline{j_n}})}{j_n - \overline{j_n}} + \left(1 - \frac{\overline{j_n}}{j_n}\right) (\alpha_{\min} + \varepsilon_{\overline{\lambda_{j_n}}, j_n - \overline{j_n}}) \\
 &= \frac{\overline{j_n}}{j_n} \alpha_{\overline{\lambda_{j_n}}} + \frac{\overline{j_n}}{j_n} \tilde{\varepsilon}_{\overline{\lambda_{j_n}}} + \frac{\overline{j_n}}{j_n} \varepsilon_{\overline{\lambda_{j_n}}} + \left(1 - \frac{\overline{j_n}}{j_n}\right) (\alpha_{\min} + \varepsilon_{\overline{\lambda_{j_n}}, j_n - \overline{j_n}}),
 \end{aligned}$$

which combined with (102) yields

$$(103) \qquad \frac{\log c_{\lambda_{j_n}(x)}^{\mu,p,q}}{\log 2^{-j_n}} = \frac{\overline{j_n}}{j_n} \theta_p(\alpha_{\overline{\lambda_{j_n}(x)}}) + \left(1 - \frac{\overline{j_n}}{j_n}\right) \alpha_{\min} + r_n(x),$$

where

$$r_n(x) = \frac{\overline{j_n}}{j_n} \tilde{\varepsilon}_{\overline{\lambda_{j_n}(x)}} + \frac{\overline{j_n}}{j_n} \varepsilon_{\overline{\lambda_{j_n}}} + \left(1 - \frac{\overline{j_n}}{j_n}\right) \varepsilon_{\overline{\lambda_{j_n}(x)}, j_n - \overline{j_n}} + \frac{\log_2 w_{\lambda_{j_n}(x)}}{j_n}.$$

One has  $\lim_{n \rightarrow \infty} r_n(x) = 0$ . Indeed, using the properties of the family  $\{|\tilde{\varepsilon}_\lambda|\}_{\lambda \in \bigcup_{j \in \mathbb{N}} \mathcal{D}_j}$  (see Remark 7.1), denoting by  $C$  its supremum, we can get that for all  $\eta \in (0, 1)$ , for  $n$  large enough,  $\frac{\overline{j_n}}{j_n} > \eta$  implies  $|\tilde{\varepsilon}_{\overline{\lambda_{j_n}(x)}}| \leq \eta$  since  $\overline{j_n}$  is then large, while  $\left|\frac{\overline{j_n}}{j_n} \tilde{\varepsilon}_{\overline{\lambda_{j_n}(x)}}\right| \leq C\eta$  if  $\frac{\overline{j_n}}{j_n} \leq \eta$ . In any case, for  $n$  large enough one must have  $\left|\frac{\overline{j_n}}{j_n} \tilde{\varepsilon}_{\overline{\lambda_{j_n}(x)}}\right| \leq (C+1)\eta$ . The same treatment can be done with  $\frac{\overline{j_n}}{j_n} \varepsilon_{\overline{\lambda_{j_n}(x)}}$ . Also, denoting by  $C'$  the supremum of the family  $\{|\varepsilon_{\lambda, j'}|\}_{\lambda \in \bigcup_{j \in \mathbb{N}} \mathcal{D}_j, j' \in \mathbb{N}}$  (again, see (54)), for all  $\eta \in (0, 1)$ , either  $\frac{\overline{j_n}}{j_n} > 1 - \eta$  or  $j_n - \overline{j_n} \geq \eta j_n$ , hence for  $n$  large enough, one necessarily has  $\left|\left(1 - \frac{\overline{j_n}}{j_n}\right) \varepsilon_{\overline{\lambda_{j_n}(x)}, j_n - \overline{j_n}}\right| \leq (C'+1)\eta$ . Since it is clear that  $\frac{\log_2 w_{\lambda_{j_n}(x)}}{j_n}$  converges to 0 as  $n$  tends to  $\infty$ , we get the desired conclusion.

Note now that  $\theta_p(\alpha) \geq \alpha_{\min}$  for all  $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ . Since  $\alpha_{\min} \leq \zeta'_{\mu,p}(0^-) < H$ , (101) and (103) together imply that necessarily, for every  $\varepsilon > 0$ ,  $\theta_p(\alpha_{\overline{\lambda_{j_n}(x)}}) \geq H - \varepsilon$  for infinitely many integers  $n$ . Hence,  $\overline{h}_\mu(x) \geq \theta_p^{-1}(H)$ , and  $x \in \overline{E}_\mu^{\geq}(\theta_p^{-1}(H))$ .

As a conclusion,  $E_f(H) \subset \overline{E}_\mu^{\geq}(\theta_p^{-1}(H))$ . Since  $\theta_p^{-1}(H) \geq \tau'_\mu(0^-)$  lies in the decreasing part of the singularity spectrum of  $\mu$ , Proposition 3.3(3) yields that  $\dim E_f(H) \leq \tau_\mu^*(\theta_p^{-1}(H))$ . This is the desired upper bound.

- **The lower bound**  $\sigma_f \geq \zeta_{\mu,p}^*$  **over the range**  $[\alpha_{\min}, \theta_p(\alpha_p)] = [\zeta'_{\mu,p}(\infty), \zeta'_{\mu,p}(-\infty)]$ . Two cases must be distinguished.



**Case 1:**  $H \in [\theta_p(\alpha_{\min}), \theta_p(\alpha_p)]$ .

Let  $\alpha \in [\alpha_{\min}, \alpha_p]$  such that  $H = \theta_p(\alpha)$ . Our goal is to show that  $\sigma_f(H) = \dim E_f(H) \geq \zeta_{\mu,p}^*(H) = \tau_\mu^*(\alpha)$ . To achieve this, we prove that  $\mu_\alpha(E_f(H)) > 0$ , where  $\mu_\alpha$  is the measure built in Section 3.7. Since  $\mu_\alpha$  is exact dimensional with exponent  $\tau_\mu^*(\alpha)$ , this yields the claim.

For any  $H' \geq 0$  set

$$E_f^{\leq}(H') := \{y \in [0, 1]^d : h_f(y) \leq H'\}.$$

We need two lemmas.

**Lemma 7.7.** *For every  $\eta > 0$ ,  $\mu_\alpha(E_\mu(\alpha) \cap E_f^{\leq}(H - \eta)) = 0$ .*

*Proof.* Fix  $J_0 \in \mathbb{N}$  and  $\varepsilon \in (0, \eta/8)$ , and set

$$E_{\mu,\varepsilon,J_0}(\alpha) = \left\{x \in [0, 1]^d : \forall J \geq J_0, \forall \lambda \subset 3\lambda_J(x), \lambda \in \mathcal{D}_J, 2^{-J(\alpha+\varepsilon)} \leq \mu(\lambda) \leq 2^{-J(\alpha-\varepsilon)}\right\}.$$

Let  $x \in E_{\mu,\varepsilon,J_0}(\alpha) \cap E_f^{\leq}(H - \eta)$ . By (24), there are infinitely many integers  $J \geq J_0$  for which  $L_J^f(x) \geq 2^{-J(H-2\eta)}$ . For such a generation  $J$ , there is  $j \geq J$  and  $\lambda = (i, j, k) \in \Lambda_j$  with  $\lambda_{j,k} \subset 3\lambda_J(x)$  such that  $|c_\lambda^f| \geq 2^{-J(H-\eta/2)}$ . Using that we can also assume that  $\mu(\lambda_{j,k}) \leq \mu(\lambda_J(x))2^{J\varepsilon}2^{-(j-J)\alpha_{\min}/2}$  (due to (48)), the definition of  $E_{\mu,\varepsilon,J_0}(\alpha)$  and the fact that  $\theta_p(\alpha) = \alpha + \tau_\mu^*(\alpha)/p = H$ , one gets that

$$(104) \quad \frac{|c_\lambda^f|}{\mu(\lambda_{j,k})} \geq 2^{(j-J)\frac{\alpha_{\min}}{2}} 2^{-J(H-\alpha-\frac{\eta}{2}+2\varepsilon)} \geq 2^{(j-J)\frac{\alpha_{\min}}{2}} 2^{-J\left(\frac{\tau_\mu^*(\alpha)}{p}-\frac{\eta}{4}\right)}.$$

Now for  $j \geq J \geq J_0$  define the set

$$(105) \quad \mathcal{D}_{\varepsilon,J,j} = \left\{ \lambda \in \mathcal{D}_J : \left\{ \begin{array}{l} \lambda \cap E_{\mu,\varepsilon,J_0}(\alpha) \neq \emptyset \text{ and} \\ \exists \lambda' = (i, j, k) \in \Lambda_j, \lambda' \subset 3\lambda, |c_{\lambda'}^f| \geq 2^{-J(H-\frac{\eta}{2})} \end{array} \right. \right\}.$$

Since  $f \in \tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ , we have  $f \in B_q^{\mu^{(-\frac{\eta}{8})},p}(\mathbb{R}^d)$ , so  $\sum_{\lambda \in \Lambda_j} \left(2^{-j\frac{\eta}{8}} \frac{|c_\lambda^f|}{\mu(\lambda)}\right)^p = C < \infty$  and both (105) and (104) imply

$$C \geq \sum_{\lambda \in \mathcal{D}_{\varepsilon,J,j}} 2^{-jp\frac{\eta}{8}} \left(2^{(j-J)\frac{\alpha_{\min}}{2}} 2^{-J\left(\frac{\tau_\mu^*(\alpha)}{p}-\frac{\eta}{4}\right)}\right)^p.$$

This yields

$$\#\mathcal{D}_{\varepsilon,J,j} \leq C 2^{-(j-J)p\frac{\alpha_{\min}}{2}} 2^{J(\tau_\mu^*(\alpha)-p\frac{\eta}{8})}.$$

In particular,  $\mathcal{D}_{\varepsilon,J,j} = \emptyset$  for  $j \geq J(p\frac{\alpha_{\min}}{2} + \tau_\mu^*(\alpha_{\min}))$ . Note that

$$E_{\mu,\varepsilon,J_0}(\alpha) \cap E_f^{\leq}(H - \eta) \subset \bigcap_{J \geq J_0} \bigcup_{j \geq J} \bigcup_{\lambda \in \mathcal{D}_{\varepsilon,J,j}} \lambda.$$

For any  $\delta > 0$ , denote by  $\mathcal{H}_\delta^s$  the pre- $s$ -Hausdorff measure on  $\mathbb{R}^d$  associated with coverings by sets of diameter less than or equal to  $\delta$ . Using  $\bigcup_{j \geq J} \bigcup_{\lambda \in \mathcal{D}_{\varepsilon,J,j}} \lambda$  as covering

of  $E_{\mu,\varepsilon,J_0}(\alpha) \cap E_f^{\leq}(H - \eta)$ , one deduces that for every  $J \geq J_0$ ,

$$\begin{aligned} \mathcal{H}_{\sqrt{d} \cdot 2^{-J}}^s(E_{\mu,\varepsilon,J_0}(\alpha) \cap E_f^{\leq}(H - \eta)) &\leq \sum_{J \leq j \leq J(p \frac{\alpha_{\min}}{2} + \tau_{\mu}^*(\alpha_{\min}))} (\#\mathcal{D}_{\varepsilon,J,j}) (\sqrt{d} \cdot 2^{-j})^s \\ &\leq (\sqrt{d})^s C \left( \sum_{m \geq 0} 2^{-mp \frac{\alpha_{\min}}{2}} \right) 2^{J(\tau_{\mu}^*(\alpha) - p \frac{\eta}{8} - s)}, \end{aligned}$$

which tends to zero as soon as  $s > \tau_{\mu}^*(\alpha) - p \frac{\eta}{8}$ . It follows that

$$\dim(E_{\mu,\varepsilon,J_0}(\alpha) \cap E_f^{\leq}(H - \eta)) \leq \tau_{\mu}^*(\alpha) - p \frac{\eta}{8},$$

and thus  $\mu_{\alpha}(E_{\mu,\varepsilon,J_0}(\alpha) \cap E_f^{\leq}(H - \eta)) = 0$ , because  $\mu_{\alpha}$  may give a positive mass to a set  $E$  only if  $\dim E \geq \tau_{\mu}^*(\alpha)$ .

To conclude, observe that the almost doubling property of  $\mu$  yields

$$E_{\mu}(\alpha) = \bigcap_{m \geq 1} \bigcup_{J_0 \in \mathbb{N}} E_{\mu, \frac{1}{m}, J_0}(\alpha).$$

This equality combined with the previous estimate on  $\mu_{\alpha}$  gives  $\mu_{\alpha}(E_{\mu}(\alpha) \cap E_f^{\leq}(H - \eta)) = 0$ .  $\square$

The second lemma states that  $\mu_{\alpha}$  may give a mass only to points which are not well approximated by dyadic vectors.

**Lemma 7.8.** *For every  $x$ , call  $\overline{\lambda_j(x)} \in \mathcal{D}_{\frac{j(x)}{j}}$  the irreducible representation of  $\lambda_j(x)$ .*

*For  $\mu_{\alpha}$ -almost every  $x$ , one has  $\lim_{n \rightarrow +\infty} \frac{j_n(x)}{j_n} = 1$ .*

*Proof.* Fix  $\delta > 1$ . For  $j \in \mathbb{N}^*$ , let  $E_{\mu}(\alpha, \delta, j) = \{x \in E_{\mu}(\alpha) : \frac{j(x)}{j} \leq \delta^{-1}\}$  and

$$E_{\mu}(\alpha, \delta) := \left\{ x \in E_{\mu}(\alpha) : \liminf_{j \rightarrow +\infty} \frac{j(x)}{j} \leq \delta^{-1} \right\} = \limsup_{j \rightarrow +\infty} E_{\mu}(\alpha, \delta, j).$$

For  $\varepsilon > 0$ , let

$$F_{\mu}(\alpha, j, \varepsilon) = \{x \in [0, 1]^d : \forall j' \geq j, 2^{-j'(\alpha+\varepsilon)} \leq \mu(\lambda_{j'}(x)) \leq 2^{-j'(\alpha-\varepsilon)}\}.$$

Setting  $j_{\delta} = \lfloor j/\delta \rfloor$ , the following inclusion holds :

$$E_{\mu}(\alpha, \delta) \subset \bigcap_{\varepsilon > 0} \bigcap_{J \geq 1} \bigcup_{j \geq J} \bigcup_{\substack{\lambda_{j_{\delta},k} \in \mathcal{D}_{j_{\delta}}: \\ \lambda_{j_{\delta},k} \cap F_{\mu}(\alpha, j_{\delta}, \varepsilon) \neq \emptyset}} B(k2^{-j_{\delta}}, 2^{-j}).$$

Using Proposition 3.3(1) or (4), for every fixed  $\varepsilon > 0$ , one sees that the cardinality of  $\{\lambda_{j_{\delta},k} \in \mathcal{D}_{j_{\delta}} : \lambda_{j_{\delta},k} \cap F_{\mu}(\alpha, j_{\delta}, \varepsilon) \neq \emptyset\}$  is less than  $2^{j_{\delta}(\tau_{\mu}^*(\alpha) + \varepsilon)}$  when  $j$  is large.

Combining this with the previous inclusion, one can construct coverings of  $E_{\mu}(\alpha, \delta)$  by sets of the form  $\bigcup_{j \geq J} \bigcup_{\substack{\lambda_{j_{\delta},k} \in \mathcal{D}_{j_{\delta}}: \\ \lambda_{j_{\delta},k} \cap F_{\mu}(\alpha, j_{\delta}, \varepsilon) \neq \emptyset}} B(2^{-j_{\delta}} k, 2^{-j})$ , and it is easily seen that

$\dim E_{\mu}(\alpha, \delta) \leq \tau_{\mu}^*(\alpha)/\delta$ , hence  $\mu_{\alpha}(E_{\mu}(\alpha, \delta)) = 0$ , again because  $\mu_{\alpha}$  may give a positive mass to a set  $E$  only if  $\dim E \geq \tau_{\mu}^*(\alpha)$ .

Since this holds for all  $\delta > 1$ , one concludes that  $\liminf_{j \rightarrow \infty} \frac{\overline{j(x)}}{j} = 1$  for  $\mu_\alpha$ -almost every  $x$ , and in particular  $\lim_{n \rightarrow \infty} \frac{\overline{j_n(x)}}{j_n} = 1$ .  $\square$

Recall that for  $\mu_\alpha$ -almost every  $x$ ,  $\lim_{j \rightarrow \infty} \alpha_{\lambda_j(x)} = \alpha$ . From this, (103) and the last lemma, one deduces that  $h_f(x) \leq \theta_p(\alpha) = H$  for  $\mu_\alpha$ -almost every  $x$ , i.e.  $\mu_\alpha(E_f^{\leq}(H)) = 1$  (the equality  $h_f(x) = H$  does not hold in general, since (103) is true only for a subsequence of integers  $(j_n)_{n \geq 1}$ ).

However, combining all the above results, one concludes that

$$\begin{aligned} \mu_\alpha(E_f(H)) &= \mu_\alpha(E_\mu(\alpha) \cap E_f(H)) \\ &\geq \mu_\alpha(E_\mu(\alpha) \cap E_f^{\leq}(H)) - \sum_{m \geq 1} \mu_\alpha(E_\mu(\alpha) \cap E_f^{\leq}(H - 1/m)) = 1. \end{aligned}$$

This proves that necessarily  $\dim E_f(H) \geq \tau_\mu^*(\alpha)$ , as expected.

**Case 2:**  $H \in [\alpha_{\min}, \theta_p(\alpha_{\min}))$ : this corresponds to the affine part of the spectrum, which occurs only when  $\sigma_\mu(\alpha_{\min}) = \tau_\mu^*(\alpha_{\min}) > 0$ , see Figure 7.

If  $H \in [\alpha_{\min}, \theta_p(\alpha_{\min}))$ , write  $H = \alpha_{\min} + \frac{\tau_\mu^*(\alpha_{\min})}{\delta p}$ , where  $\delta > 1$ . We can apply Proposition 3.18 (which is established when  $\mu \in \mathcal{M}_d$  but immediately extends to the case where  $\mu$  is a power of an element of  $\mathcal{M}_d$ ), to the sequence  $(j_n)_{n \in \mathbb{N}}$  given by the statement of Proposition 7.6: the set  $S(\delta, (\eta_j)_{j \in \mathbb{N}^*}, (j_n)_{n \in \mathbb{N}})$  (where  $(\eta_j)_{j \in \mathbb{N}^*}$  depends only on  $\mu$ ) supports a Borel probability measure  $\nu$  of lower Hausdorff dimension at least  $\tau_\mu^*(\alpha_{\min})/\delta = p(H - \alpha_{\min}) = \zeta_{\mu,p}^*(H)$ .

For  $x \in S(\delta, (\eta_j)_{j \in \mathbb{N}^*}, (j_n)_{n \in \mathbb{N}})$ , one checks that

$$h_f(x) \leq \liminf_{n \rightarrow \infty} \frac{\log c_{j_n}^{\mu,p,q}(x)}{\log 2^{-j_n}} \leq \alpha_{\min} + \frac{\tau_\mu^*(\alpha_{\min})}{\delta p} = H.$$

In addition,  $\{y \in [0, 1]^d : h_f(y) < H\} = \bigcup_{m \geq 1} E_f^{\leq}(H - 1/m)$ , and each set  $E_f^{\leq}(H - 1/m)$  has a  $\nu$ -measure equal to 0, since due to Proposition 3.3(2) applied to the capacity provided by the leaders of  $f$ ,  $\dim E_f^{\leq}(H - 1/m) \leq (\zeta_f^\Psi)^*(H - 1/m) < \zeta_{\mu,p}^*(H)$ . Consequently,  $\nu(E_f(H)) = 1$  and  $\dim E_f(H) \geq \zeta_{\mu,p}^*(H)$ .

Finally, if  $H = \alpha_{\min}$ , the set  $F = \bigcap_{p \in \mathbb{N}} S(p, (\eta_j)_{j \geq 1}, (j_n)_{n \in \mathbb{N}})$  is easily seen to be non empty (by taking  $\delta = p$  at step  $p$  of the construction in the proof of proposition 3.18) and to be included in  $E_f^{\leq}(\alpha_{\min})$ , by using the previous estimates. However we know that  $E_f^{\leq}(h) = \emptyset$  for all  $h < \alpha_{\min}$  by Theorem 2.19. Consequently,  $E_f^{\leq}(\alpha_{\min}) = E_f^{\leq}(\alpha_{\min}) \neq \emptyset$ , so  $\sigma_f(\alpha_{\min}) = \dim E_f(\alpha_{\min}) \geq 0$ .  $\square$

**7.3. Typical multifractal behavior in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ .** We finally prove item (2) of Theorem 2.19, hence obtaining the multifractal behavior of typical functions in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ .

Recall the definition (19) of the basis  $\{\mathcal{N}_m\}_{\substack{m \in \mathbb{N} \\ m > \max(1, s_1^{-1})}}$  of neighborhoods of the origin in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ .

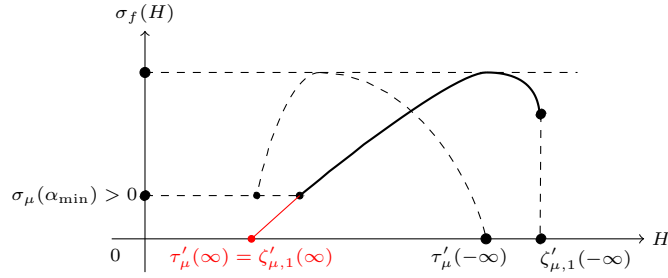


FIGURE 7. Case where  $\sigma_\mu(\alpha_{\min}) > 0$  and  $p = 1$ : the dashed graph represents the spectrum of  $\mu$ , the plain graph represents the multifractal spectrum  $\sigma_f$  of typical functions  $f \in \tilde{B}_q^{\mu,1}(\mathbb{R}^d)$ . An affine segment (in red) with slope  $p = 1$  appears in the spectrum  $\sigma_f$ .

For every integer  $m > m_0 = \lfloor \max(1, s_1^{-1}) \rfloor + 1$ , set

$$V_m = \left\{ f \in \tilde{B}_q^{\mu,p}(\mathbb{R}^d) : \forall j \geq J_2, \forall \lambda \in \Lambda_j, \frac{|c_\lambda^f|}{c_\lambda^{\mu,p,q}} \in m^{-1}\{1, \dots, m^2\} \right\}.$$

Then let

$$(106) \quad \mathcal{G} = \limsup_{m \rightarrow \infty} (V_m + \mathcal{V}_m)$$

where  $\mathcal{V}_m = N_{2^{\lceil m \log(m) \rceil}}$ . Each  $\bigcup_{\ell \geq m} V_\ell$ ,  $m \geq m_0$ , is dense in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ , so  $\mathcal{G}$  is a dense  $G_\delta$  set.

When  $f \in \mathcal{G}$ , there exists an increasing sequence  $(j_n)_{n \geq 0}$  such that  $f \in V_{j_n} + \mathcal{V}_{j_n}$  for all  $n \geq 0$ .

Fix  $L \in \mathbb{Z}^d$ . Looking at the particular generation  $j_n$ , for all  $\lambda \in \Lambda_{j_n}$  such that  $\lambda \subset L + 3[0, 1]^d$ , by definition of  $V_{j_n}$  and  $\tilde{N}_{2^{\lceil j_n \log(j_n) \rceil}}$ , the lower bound  $|c_\lambda^f| \geq j_n^{-1} c_\lambda^{\mu,p,q} - 2^{-\lceil j_n \log(j_n) \rceil} \mu(\lambda) 2^{j_n 2^{-j_n \log(j_n)}}$  holds. By construction of the coefficients  $c_\lambda^{\mu,p,q}$ , this implies that for  $n$  large enough one has  $|c_\lambda^f| \geq j_n^{-1} c_\lambda^{\mu,p,q} / 2$ , hence there exists a positive sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  converging to 0 such that  $|c_\lambda^f| \geq 2^{-j_n \varepsilon_n} |c_\lambda^{\mu,p,q}|$  for all  $\lambda \in \Lambda_{j_n}$  such that  $\lambda \subset L + 3[0, 1]^d$ . Consequently, Proposition 7.6 yields  $\sigma_f = \sigma_{g^{\mu,p,q}} = \zeta_{\mu,p}^*$ .

**Remark 7.9.** In fact, the definition of  $V_{j_n}$  and  $\tilde{N}_{2^{\lceil j_n \log(j_n) \rceil}}$ , as well as that of  $c_\lambda^{\mu,p,q}$  show that if  $(j_n)_{n \geq 1}$  is an increasing sequence of integers and  $f \in \bigcap_{n \geq 1} V_{j_n} + \mathcal{V}_{j_n}$ , then for all  $N, K \in \mathbb{N}^*$ , for all  $n \geq 1$  large enough and  $\lambda \in \bigcup_{j=j_n}^{K j_n} \Lambda_{j,n}$  such that  $\lambda \subset N[0, 1]^d$ , one has

$$\frac{1}{2j_n} c_\lambda^{\mu,p,q} \leq |c_\lambda^f| \leq 2j_n c_\lambda^{\mu,p,q}.$$

These bounds will be useful to estimate the  $L^q$ -spectrum of  $f$ .

## 8. VALIDITY OF THE MULTIFRACTAL FORMALISM.

Recall that the multifractal formalism used in this paper, was defined in Section 2.5. In this last section, we first discuss the validity of the multifractal formalism for the saturation function  $g^{\mu,p,q}$ . This will be useful to establish part (3) of Theorem 2.26 in Section 8.3, while Section 8.2 provides the proof of part (2) of Theorem 2.26.

8.1. Validity of the multifractal formalism for the saturation function  $g^{\mu,p,q}$ .

**Proposition 8.1.** *For the function  $g^{\mu,p,q}$ , the wavelet leaders multifractal formalism holds on the interval  $[\zeta'_{\mu,p}(\infty), \zeta'_{\mu,p}(0^+)]$ , and its refined version holds on the interval  $[\zeta'^{N,\Psi}_{\mu,p}(\infty), \zeta'^{N,\Psi}_{\mu,p}(-\infty)]$ .*

Moreover, for all  $N \in \mathbb{N}^*$ , one has  $\lim_{j \rightarrow \infty} \zeta_{g^{\mu,p,q},j}^{N,\Psi} = \zeta_{\mu,p}$  (see (25) for the definition of  $\zeta_{g^{\mu,p,q},j}^{N,\Psi}$ ).

The second part of the statement shows that for  $g^{\mu,p,q}$  we have a stronger property than the convergence of the sequence  $(\zeta_{g^{\mu,p,q},j}^{N,\Psi})_{j \geq 1}$  along a subsequence, which is required for the refined wavelet leaders formalism to hold.

*Proof.* Suppose that we have proved that for all  $N \in \mathbb{N}^*$ , one has  $\lim_{j \rightarrow \infty} \zeta_{g^{\mu,p,q},j}^{N,\Psi} = \zeta_{\mu,p}$ . In particular  $\zeta_{g^{\mu,p,q}}^{N,\Psi} = \zeta_{\mu,p}$  for all  $N \in \mathbb{N}^*$ , so  $\zeta_{g^{\mu,p,q}}^{N,\Psi} = \zeta_{\mu,p}$ . It was proved in the previous section that  $\sigma_{g^{\mu,p,q}} = \zeta_{\mu,p}^*$ . This is enough to get the desired conclusion about the validity of the multifractal formalism for  $g^{\mu,p,q}$ .

Now, fix  $N \in \mathbb{N}^*$ . Let us prove that  $\lim_{j \rightarrow \infty} \zeta_{g^{\mu,p,q},j}^{N,\Psi} = \zeta_{\mu,p}$ .

The  $\mathbb{Z}^d$ -invariance of  $\mu$  and the definition of  $g^{\mu,p,q}$  show that it is enough to prove that  $\lim_{j \rightarrow \infty} j^{-1} \log \sum_{\lambda \in \mathcal{D}_j^0} (L_\lambda^{g^{\mu,p,q}})^t = \zeta_{\mu,p}(t)$ .

Fix  $t \in \mathbb{R}$ . Recall Remark 7.3(1) and Lemma 7.5. We leave the reader check that due to these two facts,

$$\lim_{j \rightarrow \infty} j^{-1} \log \frac{\sum_{\lambda \in \mathcal{D}_j^0} (L_\lambda^{g^{\mu,p,q}})^t}{\sum_{\lambda \in \mathcal{D}_j^0} (c_\lambda^{g^{\mu,p,q}})^t} = 0.$$

Moreover, by definition of the coefficients  $c_\lambda^{g^{\mu,p,q}}$ , we also have

$$\lim_{j \rightarrow \infty} j^{-1} \log \frac{\sum_{\lambda \in \mathcal{D}_j^0} (c_\lambda^{g^{\mu,p,q}})^t}{B_j} = 0, \text{ where } B_j = \sum_{\lambda \in \mathcal{D}_j^0} \left( \mu(\lambda) 2^{-j \frac{\tau_\mu^*(\alpha_\lambda)}{p}} \right)^t.$$

Thus, we must prove that  $\lim_{j \rightarrow \infty} j^{-1} \log_2(B_j) = \zeta_{\mu,p}(t)$ . If  $p = \infty$ , this was proved when  $\mu$  is an element of  $\mathcal{M}_d$  in Section 3.5, but in the general case where  $\mu$  is a positive power of such a measure the result holds as well by a direct calculation.

Assume now that  $p < \infty$ . Fix  $t \in \mathbb{R}^*$ , the case  $t = 0$  being obvious. Denote by  $s(t)$  the sign of  $t$ .

Fix  $\varepsilon > 0$ . Using the same decomposition as that used in the proof of Lemma 7.4, we can write

$$B_j = \sum_{J=0}^j \sum_{\lambda \in \mathcal{D}_j^0 \setminus (\mathcal{D}_{j-1}^0 \cdot [0, 2^{-1}]^d)} \mu(\lambda \cdot [0, 2^{-(j-J)}]^d) t 2^{-\frac{t}{p} J \tau_\mu^*(\alpha_\lambda)}.$$

Then, from (54) we deduce that there exists a positive sequence  $(C_j)_{j \geq 1}$  depending on  $t$  and  $\mu$  such that  $\lim_{j \rightarrow \infty} \frac{\log(C_j)}{j} = 0$  and for all  $j \geq 1$

$$C_j^{-1} B(j, \alpha_{\min} + s(t)\varepsilon) \leq B_j \leq C_j B(j, \alpha_{\min} - s(t)\varepsilon),$$

where

$$(107) \quad B(j, \beta) = 2^{-j t \beta} + \sum_{J=1}^j 2^{-(j-J) t \beta} \sum_{\lambda \in \mathcal{D}_j^0 \setminus (\mathcal{D}_{j-1}^0 \cdot [0, 2^{-1}]^d)} \mu(\lambda) t 2^{-\frac{t}{p} J \tau_\mu^*(\alpha_\lambda)}.$$

Next, using that  $\mu$  is almost doubling, we deduce from (107) the existence of another positive sequence  $(\tilde{C}_j)_{j \geq 1}$  depending on  $t$  and  $\mu$  such that  $\lim_{j \rightarrow \infty} \frac{\log(\tilde{C}_j)}{j} = 0$  and

$$(108) \quad \tilde{C}_j^{-1} \tilde{B}(j, \alpha_{\min} + s(t)\varepsilon) \leq B_j \leq \tilde{C}_j \tilde{B}(j, \alpha_{\min} - s(t)\varepsilon),$$

where

$$(109) \quad \tilde{B}(j, \beta) = \sum_{J=0}^j 2^{-(j-J) t \beta} \sum_{\lambda \in \mathcal{D}_j^0} \mu(\lambda) t 2^{-\frac{t}{p} J \tau_\mu^*(\alpha_\lambda)}.$$

We now estimate  $\sum_{\lambda \in \mathcal{D}_j^0} \mu(\lambda) t 2^{-\frac{t}{p} J \tau_\mu^*(\alpha_\lambda)}$ . Using Proposition 3.3(4), we split the interval  $[\alpha_{\min}, \alpha_{\max}]$  into  $M$  contiguous intervals  $I_i = [\alpha_i, \alpha_{i+1}]$ ,  $i = 1, \dots, M$  of length less than  $\varepsilon$  such that for every  $i \in \{1, \dots, M\}$ ,

$$\left| \sup_{\alpha \in I_i} \tau_\mu^*(\alpha) - \frac{\log_2 \# \mathcal{D}_\mu(j, I_i)}{j} \right| \leq \varepsilon \quad \text{and} \quad \sup_{\alpha, \alpha' \in I_i} |\tau_\mu^*(\alpha) - \tau_\mu^*(\alpha')| \leq \varepsilon.$$

Also, by Remark 7.1, there exists  $C \geq 1$  such that for all  $\ell \in \mathbb{N}$  and  $\lambda \in \mathcal{D}_\ell^0$ , one has  $C^{-1} 2^{-\ell(\alpha_\lambda + \varepsilon)} \leq \mu(\lambda) \leq C 2^{-\ell(\alpha_\lambda - \varepsilon)}$ .

It follows from the previous information that

$$\sum_{\lambda \in \mathcal{D}_j^0} \mu(\lambda) t 2^{-\frac{t}{p} J \tau_\mu^*(\alpha_\lambda)} \begin{cases} \leq C^{|t|} \sum_{i=1}^M 2^{J(\tau_\mu^*(\alpha_i) + \varepsilon)} 2^{-J t (\alpha_i - 2s(t)\varepsilon)} 2^{-\frac{t}{p} J (\tau_\mu^*(\alpha_i) - s(t)\varepsilon)} \\ \geq C^{-|t|} \sum_{i=1}^M 2^{J(\tau_\mu^*(\alpha_i) - \varepsilon)} 2^{-J t (\alpha_i + 2s(t)\varepsilon)} 2^{-\frac{t}{p} J (\tau_\mu^*(\alpha_i) + s(t)\varepsilon)}, \end{cases}$$

which implies that

$$(110) \quad \sum_{\lambda \in \mathcal{D}_j^0} \mu(\lambda) t 2^{-\frac{t}{p} J \tau_\mu^*(\alpha_\lambda)} = m_J(t, \varepsilon) \sum_{i=1}^M 2^{-J(t\theta_p(\alpha_i) - \tau_\mu^*(\alpha_i))}$$

where  $|\log(m_J(t, \varepsilon))| \leq |t| \log(C) + (1 + 2|t| + \frac{|t|}{p}) J \varepsilon$ .

We can assume without loss of generality that there exists  $1 \leq i \leq M$  such that  $t\theta_p(\alpha_i) - \tau_\mu^*(\alpha_i) = \min\{t\theta_p(\alpha) - \tau_\mu^*(\alpha) : \alpha \in [\alpha_{\min}, \alpha_{\max}]\} =: \tilde{\zeta}(t)$ . Then, incorporating (110) in (109) yields

$$(111) \quad \tilde{B}(j, \beta) = \sum_{J=0}^j 2^{-(j-J)t\beta} \tilde{m}_J(t, \varepsilon) 2^{-J\tilde{\zeta}(t)},$$

where  $|\log(\tilde{m}_J(t, \varepsilon))| \leq \log(M) + |t| \log(C) + (1 + 2|t| + \frac{|t|}{p})J\varepsilon$ . Incorporating (111) in (108) then implies

$$(112) \quad B_j = \hat{m}_j(t, \varepsilon) 2^{-jt\alpha_{\min}} \sum_{J=0}^j \bar{m}_J(t, \varepsilon) \tilde{m}_J(t, \varepsilon) 2^{-J(\tilde{\zeta}(t) - t\alpha_{\min})},$$

where  $\max(|\log(\bar{m}_j(t, \varepsilon)), |\log(\hat{m}_j(t, \varepsilon))|) \leq j|t|\varepsilon + \log(\tilde{C}_j)$ .

It follows from (112) and the fact that  $\varepsilon$  is arbitrary that  $\tilde{\zeta}(t) - t\alpha_{\min} \geq 0$  implies  $\lim_{j \rightarrow \infty} \frac{\log_2(B_j)}{-j} = t\alpha_{\min}$ , while  $\tilde{\zeta}(t) - t\alpha_{\min} \leq 0$  implies  $\lim_{j \rightarrow \infty} \frac{\log_2(B_j)}{-j} = \tilde{\zeta}(t)$ .

Finally, let us determine  $\tilde{\zeta}(t)$  and then the sign of  $\tilde{\zeta}(t) - t\alpha_{\min}$ . According to the previous observation, this will give the desired conclusion.

We distinguish two cases.

Suppose first that  $[\alpha_{\min}, \alpha_{\max}]$  is trivial. Then,  $\tau_\mu(s) = \alpha_{\min}s - d$  for all  $s \in \mathbb{R}$ , and  $\zeta_{\mu,p}(s) = (\alpha_{\min} + \frac{d}{p})s - d$  for  $s < p$  and  $\zeta_{\mu,p}(s) = \alpha_{\min}s$  for  $s \geq p$ . Also, we directly have  $\tilde{\zeta}(t) = t\alpha_{\min} + \left(\frac{t}{p} - 1\right)d$ . Thus  $\tilde{\zeta}(t) = \zeta_{\mu,p}(t)$  when  $t < p$ . Moreover,  $\tilde{\zeta}(t) - t\alpha_{\min} = \left(\frac{t}{p} - 1\right)\tau_\mu^*(\alpha_{\min})$ , which is non negative if and only if  $t \geq p$ . In addition, when  $p \geq t$  one has  $\zeta_{\mu,p}(t) = t\alpha_{\min}$ .

Assume next that  $[\alpha_{\min}, \alpha_{\max}]$  is non trivial. Suppose that  $t \geq p$ . The mapping  $g : \alpha \in [\alpha_{\min}, \alpha_{\max}] \mapsto t\theta_p(\alpha) - \tau_\mu^*(\alpha) = t\alpha + \left(\frac{t}{p} - 1\right)\tau_\mu^*(\alpha)$  is concave, so it attains its minimum  $\tilde{\zeta}_p(t)$  at either  $\alpha_{\min}$  or  $\alpha_{\max}$ . In any case,  $\tilde{\zeta}_p(t) - t\alpha_{\min} \geq 0$ . Moreover,  $\zeta_{\mu,p}(t) = t\alpha_{\min}$ .

Suppose now that  $t < p$ . Using the notations and arguments of the proof of Proposition 5.1, we have that either  $t_p = \frac{pt}{p-t} \leq t_\infty = (\tau_\mu^*)'(\alpha_{\min})$ , and the convex function  $g$  attains its minimum  $\frac{p-t}{p}\tau_\mu\left(\frac{p}{p-t}t\right) = \zeta_{\mu,p}(t)$  at  $\tilde{\alpha}_t$ , i.e.  $\tilde{\zeta}(t) = \zeta_{\mu,p}(t)$ , or  $t_p > t_\infty$ . In this later case  $g$  is increasing and attains its minimum  $t\alpha_{\min} + \left(\frac{t}{p} - 1\right)\tau_\mu^*(\alpha_{\min}) = \zeta_{\mu,p}(t)$  at  $\alpha_{\min}$ , i.e.  $\tilde{\zeta}(t) = \zeta_{\mu,p}(t)$  as well. In both cases,  $\tilde{\zeta}(t) - t\alpha_{\min} \leq \tilde{\zeta}(t) - g(\alpha_{\min}) \leq 0$ .  $\square$

**8.2. Proof of Theorem 2.26(2).** As recalled in the introduction, it is known [35] that for any smooth function  $\sigma_f \leq \zeta_f^*$ . Since it was proved in Section 7.3 that  $\sigma_f = (\zeta_{\mu,p}^\Psi)^*$  for typical functions in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ , for such functions one necessarily has  $\zeta_f \leq \zeta_{\mu,p}$  by inverse Legendre transform. Simultaneously, Theorem 6.1 states that  $\zeta_f|_{\mathbb{R}_+} = \zeta_f^\Psi|_{\mathbb{R}_+} \geq \zeta_{\mu,p}|_{\mathbb{R}_+}$ , which yields the desired result.

**8.3. Proof of Theorem 2.26(3).** It is enough to prove part (i). Then part (ii) follows from the fact that the class or residual sets is stable by countable intersection.

Let  $f \in \mathcal{G}$ , where  $\mathcal{G}$  is the  $G_\delta$  set defined by (106), and consider a sequence  $(j_n)$  such that  $f \in V_{j_n} + \mathcal{V}_{j_n}$  for all  $n \geq 1$ . Fix  $N \in \mathbb{N}^*$ . We prove that  $\zeta_{f,j_n}^{\Psi,N}$  converges pointwise to  $\zeta_{\mu,p}$  as  $n \rightarrow +\infty$ , which is enough to show that the refined wavelet leaders multifractal formalism holds relatively to  $\Psi$  over  $[\zeta_{\mu,p}'(\infty), \zeta_{\mu,p}'(-\infty)]$ , since it was established that  $\sigma_f = \zeta_{\mu,p}^*$ .

Since a function  $f \in \mathcal{G}$  belongs to  $\mathcal{C}^{\alpha_{\min}-\varepsilon}(\mathbb{R}^d)$  (for every  $\varepsilon > 0$ ), one has  $|c_\lambda^f| \leq 2^{-j(\alpha_{\min}-\varepsilon)}$  for every large  $j$  and  $\lambda \in \Lambda_j$  such that  $\lambda \subset (N+1)[0, 1]^d$ . Take  $\varepsilon = \alpha_{\min}/2$ . Also, by construction,  $c_\lambda^{\mu,p,q} \geq 2^{-2j\alpha_{\max}}$ . We deduce from the previous fact and Remark 7.9 applied with  $K = \lfloor 4\alpha_{\max}/\alpha_{\min} \rfloor + 1$  that when  $n$  is large, for all  $j \geq j_n$  and  $\lambda \in \Lambda_j$  such that  $\lambda \subset (N+1)[0, 1]^d$ , either  $j \in \{j_n, \dots, Kj_n\}$  the wavelet coefficient  $c_\lambda^f$  of  $f$  satisfies  $\frac{1}{2j_n} c_\lambda^{\mu,p,q} \leq |c_\lambda^f| \leq 2j_n c_\lambda^{\mu,p,q}$ , or  $j > Kj_n$  and  $|c_\lambda^f| \leq c_\lambda^{\mu,p,q}$ . This implies that for all  $\lambda \in \mathcal{D}_{j_n}$  such that  $\lambda \subset N[0, 1]^d$ , the wavelets leaders leader  $L_\lambda^f$  of  $f$  satisfies

$$\frac{1}{2j_n} L_\lambda^{g^{\mu,p,q}} \leq L_\lambda^f \leq 2j_n L_\lambda^{g^{\mu,p,q}}.$$

Consequently,  $\lim_{n \rightarrow \infty} j_n^{-1} \log_2 \left( \frac{\zeta_{f,j_n}^{\Psi,N}}{\zeta_{g,j_n}^{\mu,p,q}} \right) = 0$ , and due Proposition 8.1, we get the desired convergence of  $\zeta_{f,j_n}^{\Psi,N}$  to  $\zeta_{\mu,p}$  as  $n \rightarrow \infty$ .

Finally, when  $q < \infty$ , to establish that for a typical  $f \in \tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  one has  $\zeta_f^\Psi|_{\mathbb{R}_-^*} = -\infty$ , consider for all  $m \in \mathbb{N}^*$  the set

$$\tilde{V}_m = \left\{ f \in \tilde{B}_q^{\mu,p}(\mathbb{R}^d) : \forall m \leq j \leq m \log(m), \forall \lambda \in \Lambda_j, c_\lambda^f = 0 \right\}.$$

The set  $\limsup_{m \rightarrow \infty} \tilde{V}_m$  is dense in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  and

$$\tilde{\mathcal{G}} = \mathcal{G} \cap \limsup_{m \rightarrow \infty} (\tilde{V}_m + \mathcal{V}_m).$$

is a dense  $G_\delta$ -set. When  $f \in \tilde{\mathcal{G}}$ , there exists an increasing sequence of integers  $(m_n)_{n \in \mathbb{N}}$  such that  $f \in \tilde{V}_{m_n} + \mathcal{V}_{m_n}$  for all  $n \in \mathbb{N}$ . It is easily checked that for any  $A > 0$  and  $N \in \mathbb{N}$ , for  $n$  large enough, if  $\lambda \in \mathcal{D}_{m_n}$  and  $\lambda \subset N[0, 1]^d$ , one has  $L_\lambda^f \leq 2^{-Am_n}$ . This implies that for  $t < 0$ ,

$$\sum_{\lambda \in \mathcal{D}_{m_n}, \lambda \subset N[0,1]^d} \mathbf{1}_{L_\lambda^f > 0} (L_\lambda^f)^t \geq \#\{\lambda \in \mathcal{D}_{m_n}, \lambda \subset N[0, 1]^d : L_\lambda^f > 0\} \cdot 2^{-Atm_n},$$

hence  $\zeta_f^{\Psi, \lambda^{(N)}}(t) \leq At$ . Consequently,  $A$  being arbitrary and  $t < 0$ , the desired conclusion holds.

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