# About the unification type of K + $[][]\bot$

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## 1 Introduction

The unification problem in a propositional logic is to determine, given a formula $\varphi$, whether there exists a substitution $\sigma$ such that $\sigma(\varphi)$ is in that logic [1]. In that case, $\sigma$ is a unifier of $\varphi$. When a unifiable formula has minimal complete sets of unifiers, it is either infinitary, finitary, or unitary, depending on the cardinality of its minimal complete sets of unifiers. Otherwise, it is nullary. Within the context of elementary unification, it is known that $\mathbf{Alt}_1$ is nullary [8], $\mathbf{S}5$ and $\mathbf{S}4.3$ are unitary [10, 11, 12], transitive modal logics like $\mathbf{K}4$ and $\mathbf{S}4$ are finitary [13, 15], $\mathbf{KD}45$, $\mathbf{K}45$ and $\mathbf{K}4.2^+$ are unitary [14, 16], $\mathbf{K}$ is nullary [17] and $\mathbf{K}4\mathbf{D}1$ is unitary [18]. The unification types of the description logics $\mathcal{E}L$ and $\mathcal{F}L_0$ are known too: both of them are nullary [2, 3]. In this paper, we prove that in modal logic $\mathbf{K} + \Box\Box\bot$ — the least normal modal logic containing the formula $\Box\Box\bot$ — unifiable formulas are either unitary, or finitary[1].

## 2 Preliminaries

Let $S$ be a finite set. We will write $\|S\|$ for the cardinality of $S$. If $S$ is non-empty then for all equivalence relations $\sim$ on $S$ and for all $T\subseteq S$, $T/\sim$ will denote the quotient set of $T$ modulo $\sim$.

**Proposition 1.** *Let $T$ be a finite set. If $S$ is non-empty then for all equivalence relations $\sim$ on $S$, $\|S/\sim\|\leq\|T\|\leq\|S\|$ iff there exists a surjective function $f$ from $S$ to $T$ such that for all $\alpha,\beta\in S$, if $f(\alpha)=f(\beta)$ then $\alpha\sim\beta$.*

Proposition 1 will be used twice in the proof of Proposition 11.

## 3 Syntax

Let $\mathbf{VAR}$ be a countably infinite set of *variables* (with typical members denoted $x$, $y$, etc). Let $(x_1, x_2, \ldots)$ be an enumeration of $\mathbf{VAR}$ without repetitions. Let $n\geq1$. The set $\mathbf{FOR}_n$ of all $n$-*formulas* (with typical members denoted $\varphi$, $\psi$, etc) is inductively defined by:

- $\varphi,\psi ::= x_i \mid \bot \mid \neg\varphi \mid (\varphi\vee\psi) \mid \Box\varphi$ where $i\in\{1,\ldots,n\}$.

We adopt the standard rules for omission of the parentheses. The connectives $\top$, $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined by the usual abbreviations. We have also a connective $\Diamond$ which is defined by $\Diamond\varphi ::= \neg\Box\neg\varphi$. For all $\varphi\in\mathbf{FOR}_n$, we respectively write "$\varphi^0$" and "$\varphi^1$" to mean "$\neg\varphi$" and "$\varphi$". From now on,


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$\boxed{\textbf{we write “}\mathbf{L}_2\textbf{” to mean “}\mathbf{K}+\Box\Box\bot\textbf{”.}}$

Let $\equiv_n$ be the equivalence relation on $\mathbf{FOR}_n$ defined by:

- $\varphi\equiv_n\psi$ iff $\varphi\leftrightarrow\psi\in\mathbf{L}_2$.

**Proposition 2.** $\equiv_n$ *possesses finitely many equivalence classes.*

An *$n$-substitution* is a couple $(k,\sigma)$ where $k\geq 1$ and $\sigma$ is a homomorphism from $\mathbf{FOR}_n$ to $\mathbf{FOR}_k$. Let $\mathbf{SUB}_n$ be the set of all $n$-substitutions. The equivalence relation $\simeq_n$ on $\mathbf{SUB}_n$ is defined by:

- $(k,\sigma)\simeq_n(l,\tau)$ iff for all $i\in\{1,\ldots,n\}$, $\sigma(x_i)\leftrightarrow\tau(x_i)\in\mathbf{L}_2$.

The preorder $\preccurlyeq_n$ on $\mathbf{SUB}_n$ is defined by:

- $(k,\sigma)\preccurlyeq_n(l,\tau)$ iff there exists a $k$-substitution $(m,\upsilon)$ such that for all $i\in\{1,\ldots,n\}$, $\upsilon(\sigma(x_i))\leftrightarrow\tau(x_i)\in\mathbf{L}_2$.

# 4 Semantics

Let $n\geq 1$. An *$n$-tuple of bits* (denoted $\alpha$, $\beta$, etc) is a function from $\{1,\ldots,n\}$ to $\{0,1\}$. Such function should be understood as a propositional valuation of the variables $x_1,\ldots,x_n$: for all $i\in\{1,\ldots,n\}$, if $\alpha_i=0$ then it is interpreted to mean "$x_i$ is false" else it is interpreted to mean "$x_i$ is true". Let $\mathbf{BIT}_n$ be the set of all $n$-tuples of bits. An *$n$-model* is a structure of the form $(\alpha,S)$ where $\alpha\in\mathbf{BIT}_n$ and $S\subseteq\mathbf{BIT}_n$. Such structure should be understood as a tree-like Kripke model of depth at most 1: $\alpha$ is the valuation of its root node and $S$ is the set of the valuations of its non-root nodes. Let $\mathbf{MOD}_n$ be the set of all $n$-models. We shall say that an $n$-model $(\alpha,S)$ is *degenerated* if $S=\emptyset$. Let $\mathbf{MOD}_n^{\mathsf{deg}}$ be the set of all degenerated $n$-models. Notice that $\|\mathbf{MOD}_n^{\mathsf{deg}}\|=2^n$. Notice also that for all sets $S$ of $n$-tuples of bits, $S\times\{\emptyset\}$ is a set of degenerated $n$-models. The binary relation $\models_n$ of *$n$-satisfiability* between $\mathbf{MOD}_n$ and $\mathbf{FOR}_n$ is defined as expected. In particular,

- $(\alpha,S)\models_n x_i$ iff $\alpha_i=1$ where $i\in\{1,\ldots,n\}$,
- $(\alpha,S)\models_n\Box\varphi$ iff for all $\beta\in S$, $(\beta,\emptyset)\models_n\varphi$.

As a result, $(\alpha,S)\models_n\Diamond\varphi$ iff there exists $\beta\in S$ such that $(\beta,\emptyset)\models_n\varphi$.

**Proposition 3.** *For all $\varphi\in\mathbf{FOR}_n$, $\varphi\in\mathbf{L}_2$ iff for all $(\alpha,S)\in\mathbf{MOD}_n$, $(\alpha,S)\models_n\varphi$.*

For all $\alpha\in\mathbf{BIT}_n$, the $n$-formula

- $\bar{x}^\alpha=\bigwedge\{x_i^{\alpha_i}\ :\ i\in\{1,\ldots,n\}\}$

exactly characterizes the propositional valuation represented by $\alpha$. For all $(\alpha,S)\in\mathbf{MOD}_n$, the $n$-formula

- $\mathbf{for}_n(\alpha,S)=\bar{x}^\alpha\wedge\Box\bigvee\{\bar{x}^\gamma\ :\ \gamma\in S\}\wedge\bigwedge\{\Diamond\bar{x}^\gamma\ :\ \gamma\in S\}$

exactly characterizes the tree-like Kripke model of depth at most 1 represented by $(\alpha,S)$.

**Proposition 4.** *Let $(\alpha,S),(\beta,T)\in\mathbf{MOD}_n$. The following conditions are equivalent: **(i)** $(\alpha,S)=(\beta,T)$; **(ii)** $(\alpha,S)\models_n\mathbf{for}_n(\beta,T)$.*

**Proposition 5.** *Let $(k,\sigma)\in\mathbf{SUB}_n$. For all $(\alpha,S)\in\mathbf{MOD}_k$, there exists $(\beta,T)\in\mathbf{MOD}_n$ such that $(\alpha,S)\models_k\sigma(\mathbf{for}_n(\beta,T))$.*

**Proposition 6.** *Let $(k,\sigma)\in\mathbf{SUB}_n$. Let $(\alpha,S)\in\mathbf{MOD}_k$. For all $(\beta,T),(\gamma,U)\in\mathbf{MOD}_n$, if $(\alpha,S)\models_k \sigma(\mathbf{for}_n(\beta,T))$ and $(\alpha,S)\models_k\sigma(\mathbf{for}_n(\gamma,U))$ then $(\beta,T)=(\gamma,U)$.*

For all $k\geq 1$, a $(k,n)$-*morphism* is a function $f$ from $\mathbf{MOD}_k$ to $\mathbf{MOD}_n$ such that for all $(\alpha,S)\in\mathbf{MOD}_k$ and for all $(\beta,T)\in\mathbf{MOD}_n$, if $f(\alpha,S)=(\beta,T)$ then[2]

**forward condition:** for all $\gamma\in S$, there exists $\delta\in T$ such that $f(\gamma,\emptyset)=(\delta,\emptyset)$,

**backward condition:** for all $\delta\in T$, there exists $\gamma\in S$ such that $f(\gamma,\emptyset)=(\delta,\emptyset)$.

**Proposition 7.** *Let $k\geq 1$. Let $f$ be a $(k,n)$-morphism. Let $(\beta,T)\in\mathbf{MOD}_k$ and $(\gamma,U)\in\mathbf{MOD}_n$. If $f(\beta,T)=(\gamma,U)$ then the following conditions hold: **(i)** the image by $f$ of $T\times\{\emptyset\}$ is equal to $U\times\{\emptyset\}$; **(ii)** $T=\emptyset$ iff $U=\emptyset$.*

**Proposition 8.** *Let $k\geq 1$. Let $f$ be a $(k,n)$-morphism. Let $(\beta,T)\in\mathbf{MOD}_k$ and $(\gamma,U)\in\mathbf{MOD}_n$. If the following conditions hold then $f(\beta,T)=(\gamma,U)$: **(i)** $f(\beta,T)\models_n\bar{x}^\gamma$; **(ii)** for all $\delta\in T$, there exists $\epsilon\in U$ such that $f(\delta,\emptyset)=(\epsilon,\emptyset)$; **(iii)** for all $\epsilon\in U$, there exists $\delta\in T$ such that $f(\delta,\emptyset)=(\epsilon,\emptyset)$.*

# 5 Unification

Let $n\geq 1$. An $n$-*unifier* of $\varphi\in\mathbf{FOR}_n$ is an $n$-substitution $(k,\sigma)$ such that $\sigma(\varphi)\in\mathbf{L}_2$. We shall say that $\varphi\in\mathbf{FOR}_n$ is $n$-*unifiable* if there exists an $n$-unifier of $\varphi$. We shall say that a set $\Sigma$ of $n$-unifiers of an $n$-unifiable $\varphi\in\mathbf{FOR}_n$ is $n$-*complete* if for all $n$-unifiers $(k,\sigma)$ of $\varphi$, there exists $(l,\tau)\in\Sigma$ such that $(l,\tau)\preccurlyeq_n(k,\sigma)$. As is well-known, for all $\varphi\in\mathbf{FOR}_n$, if $\varphi$ is $n$-unifiable then for all minimal $n$-complete sets $\Sigma,\Delta$ of $n$-unifiers of $\varphi$, $\Sigma$ and $\Delta$ have the same cardinality. Then, an important question is the following: when $\varphi\in\mathbf{FOR}_n$ is $n$-unifiable, is there a minimal $n$-complete set of $n$-unifiers of $\varphi$? When the answer is "yes", how large is this set? For all $n$-unifiable $\varphi\in\mathbf{FOR}_n$, we shall say that:

- $\varphi$ is $n$-*nullary* if there exists no minimal complete set of unifiers of $\varphi$,
- $\varphi$ is $n$-*infinitary* if there exists a minimal complete set of unifiers of $\varphi$ with infinite cardinality,
- $\varphi$ is $n$-*finitary* if there exists a minimal complete set of unifiers of $\varphi$ with finite cardinality $\geq 2$,
- $\varphi$ is $n$-*unitary* if there exists a minimal complete set of unifiers of $\varphi$ with cardinality $1$.

**Proposition 9.** *The $n$-unifiable $n$-formula $\Diamond x_1\rightarrow\Box x_1$ is $n$-finitary.*

For all $n$-unifiable $\varphi\in\mathbf{FOR}_n$ and for all $\pi\geq 1$, we shall say that $\varphi$ is $n$-$\pi$-*reasonable* if for all $n$-unifiers $(k,\sigma)$ of $\varphi$, if $k\geq\pi$ then there exists an $n$-unifier $(l,\tau)$ of $\varphi$ such that $(l,\tau)\preccurlyeq_n(k,\sigma)$ and $l\leq\pi$.

**Proposition 10.** *Let $\varphi\in\mathbf{FOR}_n$ be $n$-unifiable and $\pi\geq 1$. If $\varphi$ is $n$-$\pi$-reasonable then $\varphi$ is either $n$-finitary, or $n$-unitary.*

# 6 Main results

Let $n\geq 1$.

**Proposition 11.** *Let $k\geq n$. For all $(k,n)$-morphisms $g$, there exists a surjective $(k,n)$-morphism $f$ such that for all $(\alpha,S),(\beta,T)\in\mathbf{MOD}_k$, if $f(\alpha,S)=f(\beta,T)$ then $g(\alpha,S)=g(\beta,T)$.*

[2] The morphisms described here should not be mistaken for the bounded morphisms usually considered in modal logic [9, Definition 2.10]. In particular, in the above definition, there is no condition related to the propositional valuation of the variables.

*Proof.* Let $g$ be a $(k,n)$-morphism. Let $\sim_k$ be the equivalence relation on $\mathbf{MOD}_k$ defined by:

- $(\alpha,S)\sim_k(\beta,T)$ iff $g(\alpha,S)=g(\beta,T)$.

**Lemma 1.** *1.* $\|\mathbf{MOD}_k^{\mathsf{deg}}/\sim_k\|\leq\|\mathbf{MOD}_n^{\mathsf{deg}}\|$,

*2.* $\|\mathbf{MOD}_n^{\mathsf{deg}}\|\leq\|\mathbf{MOD}_k^{\mathsf{deg}}\|$.

Hence, by Proposition 1, there exists a surjective function $f^{\mathsf{deg}}$ from $\mathbf{MOD}_k^{\mathsf{deg}}$ to $\mathbf{MOD}_n^{\mathsf{deg}}$ such that for all $(\alpha,\emptyset),(\beta,\emptyset)\in\mathbf{MOD}_k^{\mathsf{deg}}$, if $f^{\mathsf{deg}}(\alpha,\emptyset)=f^{\mathsf{deg}}(\beta,\emptyset)$ then $(\alpha,\emptyset)\sim_k(\beta,\emptyset)$.

**Lemma 2.** *For all non-empty sets $S,T$ of $k$-tuples of bits, if the images by $f^{\mathsf{deg}}$ of $S\times\{\emptyset\}$ and $T\times\{\emptyset\}$ are equal then the images by $g$ of $S\times\{\emptyset\}$ and $T\times\{\emptyset\}$ are equal.*

For all non-empty sets $E$ of $n$-tuples of bits, let

- $f^\circ(E)$ be the set of all $(\alpha,S)\in\mathbf{MOD}_k\setminus\mathbf{MOD}_k^{\mathsf{deg}}$ such that the image by $f^{\mathsf{deg}}$ of $S\times\{\emptyset\}$ is equal to $E\times\{\emptyset\}$,
- $f^\bullet(E)$ be the set of all $(\alpha,S)\in\mathbf{MOD}_n\setminus\mathbf{MOD}_n^{\mathsf{deg}}$ such that $S=E$.

Notice that since $f^{\mathsf{deg}}$ is surjective, therefore $\|f^\circ(E)\|\geq2^k$. Notice also that $\|f^\bullet(E)\|=2^n$.

**Lemma 3.** *For all non-empty sets $E$ of $n$-tuples of bits,*

*1.* $\|f^\circ(E)/\sim_k\|\leq\|f^\bullet(E)\|$,

*2.* $\|f^\bullet(E)\|\leq\|f^\circ(E)\|$.

Thus, for all non-empty sets $E$ of $n$-tuples of bits, by Proposition 1, there exists a surjective function $f^E$ from $f^\circ(E)$ to $f^\bullet(E)$ such that for all $(\alpha,S),(\beta,T)\in f^\circ(E)$, if $f^E(\alpha,S)=f^E(\beta,T)$ then $(\alpha,S)\sim_k(\beta,T)$. Let $f$ be the function from $\mathbf{MOD}_k$ to $\mathbf{MOD}_n$ such that for all $(\alpha,\emptyset)\in\mathbf{MOD}_k^{\mathsf{deg}}$,

- $f(\alpha,\emptyset)=f^{\mathsf{deg}}(\alpha,\emptyset)$

and for all $(\alpha,S)\in\mathbf{MOD}_k\setminus\mathbf{MOD}_k^{\mathsf{deg}}$, $E$ being the non-empty set of $n$-tuples of bits such that the image by $f^{\mathsf{deg}}$ of $S\times\{\emptyset\}$ is equal to $E\times\{\emptyset\}$,

- $f(\alpha,S)=f^E(\alpha,S)$.

**Lemma 4.** *$f$ is surjective.*

**Lemma 5.** *$f$ is a $(k,n)$-morphism.*

**Lemma 6.** *For all $(\alpha,S),(\beta,T)\in\mathbf{MOD}_k$, if $f(\alpha,S)=f(\beta,T)$ then $g(\alpha,S)=g(\beta,T)$.*

This finishes the proof of Proposition 11. □

**Proposition 12.** *For all $\varphi\in\mathbf{FOR}_n$, if $\varphi$ is $n$-unifiable then $\varphi$ is $n$-$n$-reasonable.*

*Proof.* Let $\varphi\in\mathbf{FOR}_n$. Suppose $\varphi$ is $n$-unifiable. Let $(k,\sigma)$ be an $n$-unifier of $\varphi$ such that $k\geq n$. Hence, $\sigma(\varphi)\in\mathbf{L}_2$. Let $g$ be the function from $\mathbf{MOD}_k$ to $\mathbf{MOD}_n$ such that for all $(\alpha,S)\in\mathbf{MOD}_k$,

- $g(\alpha,S)$ is the $(\beta,T)\in\mathbf{MOD}_n$ such that $(\alpha,S)\models_k\sigma(\mathbf{for}_n(\beta,T))$.

Notice that by Propositions 5 and 6, $g$ is well-defined.

**Lemma 7.** *$g$ is a $(k,n)$-morphism.*

**Lemma 8.** *For all $(\alpha,S),(\beta,T)\in\mathbf{MOD}_k$, if $g(\alpha,S)=g(\beta,T)$ then for all $i\in\{1,\ldots,n\}$, $(\alpha,S)\models_k\sigma(x_i)$ iff $(\beta,T)\models_k\sigma(x_i)$.*

Since $k\geq n$ therefore by Proposition 11 and Lemma 7, let $f$ be a surjective $(k,n)$-morphism such that for all $(\alpha,S),(\beta,T)\in\mathbf{MOD}_k$, if $f(\alpha,S)=f(\beta,T)$ then $g(\alpha,S)=g(\beta,T)$. Let $(n,\tau),(k,\nu)$ be the $n$-substitutions defined by:

- $\tau(x_i)=\bigvee\{\mathbf{for}_n(f(\alpha,S)):\ (\alpha,S)\in\mathbf{MOD}_k$ is such that $(\alpha,S)\models_k\sigma(x_i)\}$ where $i\in\{1,\ldots,n\}$,
- $\nu(x_i)=\bigvee\{\mathbf{for}_k(\alpha,S):\ (\alpha,S)\in\mathbf{MOD}_k$ is such that $f(\alpha,S)\models_n x_i\}$ where $i\in\{1,\ldots,n\}$.

**Lemma 9.** *Let $\psi\in\mathbf{FOR}_n$. For all $(\beta,T)\in\mathbf{MOD}_n$, the following conditions are equivalent: **(i)** there exists $(\alpha,S)\in\mathbf{MOD}_k$ such that $f(\alpha,S)=(\beta,T)$ and $(\alpha,S)\models_k\sigma(\psi)$; **(ii)** for all $(\alpha,S)\in\mathbf{MOD}_k$, if $f(\alpha,S)=(\beta,T)$ then $(\alpha,S)\models_k\sigma(\psi)$; **(iii)** $(\beta,T)\models_n\tau(\psi)$.*

**Lemma 10.** *For all $(\beta,T)\in\mathbf{MOD}_k$ and for all $i\in\{1,\ldots,n\}$, $(\beta,T)\models_k\nu(x_i)$ iff $f(\beta,T)\models_n x_i$.*

**Lemma 11.** *Let $(\beta,T)\in\mathbf{MOD}_k$ and $(\gamma,U)\in\mathbf{MOD}_n$. The following conditions are equivalent: **(i)** $f(\beta,T)=(\gamma,U)$; **(ii)** $(\beta,T)\models_k\nu(\mathbf{for}_n(\gamma,U))$.*

**Lemma 12.** *For all $(\beta,T)\in\mathbf{MOD}_k$ and for all $i\in\{1,\ldots,n\}$, $(\beta,T)\models_k\nu(\tau(x_i))$ iff $(\beta,T)\models_k\sigma(x_i)$.*

Since $\sigma(\varphi)\in\mathbf{L}_2$, therefore by Proposition 3, for all $(\alpha,S)\in\mathbf{MOD}_k$, $(\alpha,S)\models_k\sigma(\varphi)$. Thus, by Lemma 9, for all $(\beta,T)\in\mathbf{MOD}_n$, $(\beta,T)\models_n\tau(\varphi)$. Consequently, by Proposition 3, $\tau(\varphi)\in\mathbf{L}_2$. Hence, $(n,\tau)$ is an $n$-unifier of $\varphi$. Since by Lemma 12, $(n,\tau)\preccurlyeq_n(k,\sigma)$, therefore $\varphi$ is $n$-$n$-reasonable. $\Box$

**Theorem 1.** *For all $\varphi\in\mathbf{FOR}_n$, if $\varphi$ is $n$-unifiable then $\varphi$ is either $n$-finitary, or $n$-unitary.*

# 7 Conclusion

In this paper, within the context of elementary unification, we have proved Theorem 1 asserting that in $\mathbf{K}+\Box\Box\bot$, unifiable formulas are either finitary, or unitary. We believe that in our line of reasoning, the main properties of $\mathbf{K}+\Box\Box\bot$ are the ones given in Propositions 2, 5 and 6. Proposition 2 says that $\mathbf{K}+\Box\Box\bot$ is locally tabular[3] — it is used in the proof of Proposition 10. Propositions 5 and 6 give us the possibility to define the function $g$ — they are used in the proof of Proposition 12. Notice that Theorem 1 is an immediate consequence of Propositions 10 and 12. Here are open questions:

1. determine the unification type of the locally tabular modal logic $\mathbf{K}+\Box^d\bot$ for each $d\geq 3$,
2. determine the unification types of other locally tabular modal logics like the ones studied in [19, 20, 21],
3. determine the unification types of the modal logics $\mathbf{KB}$, $\mathbf{KD}$ and $\mathbf{KT}$.

We conjecture that within the context of elementary unification, the modal logics mentioned in Items 1 and 2 are either finitary, or unitary. As for the modal logics considered in Item 3, it is only known that $\mathbf{KD}$ and $\mathbf{KT}$ are not unitary within the context of elementary unification and $\mathbf{KB}$, $\mathbf{KD}$ and $\mathbf{KT}$ are nullary within the context of unification with parameters [4, 5, 6].

[3]A modal logic $\mathbf{L}$ is *locally tabular* if for all $n\geq 1$, the equivalence relation $\equiv_n$ on $\mathbf{FOR}_n$ defined by

- $\varphi\equiv_n\psi$ iff $\varphi\leftrightarrow\psi\in\mathbf{L}$

possesses finitely many equivalence classes. The most popular of all locally tabular modal logics is probably $\mathbf{S}5$. See [19, 20, 21] for other examples of locally tabular modal logics.

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# Appendix

**Proof of Proposition 1:** See [7]. Notice that Proposition 1 is used exactly twice in the proof of Proposition 11: once immediately after Lemma 1 and once immediately after Lemma 3.

**Proof of Proposition 2:** This result follows from [9, Proposition 2.29] and the following fact: for all $\varphi \in \mathbf{FOR}_n$, there exists $\psi \in \mathbf{FOR}_n$ such that the degree of $\psi$ is less than 2 and $\varphi \leftrightarrow \psi \in \mathbf{L}_2$.

**Proof of Proposition 3:** See [9, Chapter 4] for proofs of similar completeness results.

**Proof of Proposition 4:** This result follows from the definition of $\mathbf{for}_n$.

**Proof of Proposition 5:** Let $(\alpha,S)\in\mathbf{MOD}_k$. Let $\beta$ be the $n$-tuple of bits such that for all $i\in\{1,\ldots,n\}$, if $(\alpha,S)\not\models_k\sigma(x_i)$ then $\beta_i=0$ else $\beta_i=1$. Let $T$ be the set of $n$-tuples of bits such that for all $\delta\in\mathbf{BIT}_n$, $\delta\in T$ iff there exists $\gamma\in S$ such that for all $i\in\{1,\ldots,n\}$, if $(\gamma,\emptyset)\not\models_k\sigma(x_i)$ then $\delta_i=0$ else $\delta_i=1$. Obviously, $(\alpha,S)\models_k\sigma(\mathbf{for}_n(\beta,T))$.

**Proof of Proposition 6:** This result follows from the following facts: ***(i)*** for all $(\beta,T),(\gamma,U)\in\mathbf{MOD}_n$, if $(\alpha,S)\models_k\sigma(\bar{x}^\beta)$ and $(\alpha,S)\models_k\sigma(\bar{x}^\gamma)$ then $\beta=\gamma$; ***(ii)*** for all $(\beta,T),(\gamma,U)\in\mathbf{MOD}_n$, if $(\alpha,S)\models_k\sigma(\Box\bigvee\{\bar{x}^\delta:\ \delta\in T\}\wedge\bigwedge\{\Diamond\bar{x}^\delta:\ \delta\in T\})$ and $(\alpha,S)\models_k\sigma(\Box\bigvee\{\bar{x}^\epsilon:\ \epsilon\in U\}\wedge\bigwedge\{\Diamond\bar{x}^\epsilon:\ \epsilon\in U\})$ then $T=U$.

**Proof of Proposition 7:** This result follows from the definition of $(k,n)$-morphisms.

**Proof of Proposition 8:** Suppose the following conditions holds: ***(i)*** $f(\beta,T)\models_n\bar{x}^\gamma$; ***(ii)*** for all $\delta\in T$, there exists $\epsilon\in U$ such that $f(\delta,\emptyset)=(\epsilon,\emptyset)$; ***(iii)*** for all $\epsilon\in U$, there exists $\delta\in T$ such that $f(\delta,\emptyset)=(\epsilon,\emptyset)$. Let $(\gamma',U')\in\mathbf{MOD}_n$ be such that $f(\beta,T)=(\gamma',U')$. Since $f(\beta,T)\models_n\bar{x}^\gamma$, therefore $(\gamma',U')\models_n\bar{x}^\gamma$. Hence, obviously, $\gamma'=\gamma$. Since $f$ is a $(k,n)$-morphism, therefore the following conditions hold: ***(iv)*** for all $\delta'\in T$, there exists $\epsilon'\in U'$ such that $f(\delta',\emptyset)=(\epsilon',\emptyset)$; ***(v)*** for all $\epsilon'\in U'$, there exists $\delta'\in T$ such that $f(\delta',\emptyset)=(\epsilon',\emptyset)$. Since for all $\delta\in T$, there exists $\epsilon\in U$ such that $f(\delta,\emptyset)=(\epsilon,\emptyset)$, therefore $U'\subseteq U$. Moreover, since for all $\epsilon\in U$, there exists $\delta\in T$ such that $f(\delta,\emptyset)=(\epsilon,\emptyset)$, therefore $U'\supseteq U$. Thus, $U'=U$. Since $f(\beta,T)=(\gamma',U')$ and $\gamma'=\gamma$, therefore $f(\beta,T)=(\gamma,U)$.

**Proof of Proposition 9:** Let $(n,\sigma)$ and $(n,\tau)$ be the $n$-substitutions defined by:

- $\sigma(x_1)=\Box\bot\vee x_1$,
- $\tau(x_1)=\Diamond\top\wedge x_1$,
- $\sigma(x_i)=x_i$ for each $i\in\{2,\ldots,n\}$,
- $\tau(x_i)=x_i$ for each $i\in\{2,\ldots,n\}$.

Obviously, $\Diamond\sigma(x_1)\to\Box\sigma(x_1)\in\mathbf{L}_2$ and $\Diamond\tau(x_1)\to\Box\tau(x_1)\in\mathbf{L}_2$. Hence, $(n,\sigma)$ and $(n,\tau)$ are $n$-unifiers of $\Diamond x_1\to\Box x_1$. Thus, $\Diamond x_1\to\Box x_1$ is $n$-unifiable. In order to prove that $\Diamond x_1\to\Box x_1$ is $n$-finitary, it suffices to prove that $\{(n,\sigma),(n,\tau)\}$ is a minimal $n$-complete set of $n$-unifiers of $\Diamond x_1\to\Box x_1$.

**$n$-completeness of $\{(n,\sigma),(n,\tau)\}$:** Let $(k,\upsilon)$ be an arbitrary $n$-unifier of $\Diamond x_1\to\Box x_1$. Consequently, $\Diamond\upsilon(x_1)\to\Box\upsilon(x_1)\in\mathbf{L}_2$. By standard reasoning in modal logic, it follows that either $\Box\bot\to\upsilon(x_1)\in\mathbf{L}_2$, or $\upsilon(x_1)\to\Diamond\top\in\mathbf{L}_2$. In the former case[4], it follows immediately that $\upsilon(\sigma(x_1))\leftrightarrow\upsilon(x_1)\in\mathbf{L}_2$. Hence, $(n,\sigma)\preccurlyeq_n(k,\upsilon)$.

**Minimality of $\{(n,\sigma),(n,\tau)\}$:** For the sake of the contradiction, suppose $\{(n,\sigma),(n,\tau)\}$ is not minimal. Consequently, either $(n,\sigma)\preccurlyeq_n(n,\tau)$, or $(n,\tau)\preccurlyeq_n(n,\sigma)$. In the former case[5], there exists an $n$-substitution $(k,\upsilon)$ such that $\upsilon(\sigma(x_1))\leftrightarrow\tau(x_1)\in\mathbf{L}_2$. Hence, $\Box\bot\vee\upsilon(x_1)\leftrightarrow\Diamond\top\wedge x_1\in\mathbf{L}_2$. Thus, $\Box\bot\to\Diamond\top\in\mathbf{L}_2$: a contradiction.

[4]In the latter case, the proof can be similarly done.
[5]In the latter case, the proof can be similarly done.

**Proof of Proposition 10:** Suppose $\varphi$ is $n$-$\pi$-reasonable. Let $\Sigma$ be the set of all $n$-unifiers of $\varphi$. Notice that $\Sigma$ is $n$-complete. Let $\Sigma'$ be the set of $n$-substitutions obtained from $\Sigma$ by keeping only the $n$-substitutions $(k,\sigma)$ such that $k\leq\pi$. Since $\Sigma$ is $n$-complete and $\varphi$ is $n$-$\pi$-reasonable, therefore $\Sigma'$ is $n$-complete. Let $\Sigma''$ be the set of $n$-substitutions obtained from $\Sigma'$ by keeping only one representative of each equivalence class modulo $\simeq_n$. Since $\Sigma'$ is $n$-complete, therefore $\Sigma''$ is $n$-complete. Moreover, since $\equiv_n$ possesses finitely many equivalence classes, therefore $\Sigma''$ is finite. Hence, $\varphi$ is either $n$-finitary, or $n$-unitary.

**Proof of Lemma 1:** (1) For all $(\alpha,\emptyset)\in\mathbf{MOD}_k^{\mathsf{deg}}$, $[(\alpha,\emptyset)]$ will denote the equivalence class modulo $\sim_k$ with $(\alpha,\emptyset)$ as its representative. Let $h$ be the function from $\mathbf{MOD}_k^{\mathsf{deg}}/{\sim_k}$ to $\mathbf{MOD}_n^{\mathsf{deg}}$ such that for all $(\alpha,\emptyset)\in\mathbf{MOD}_k^{\mathsf{deg}}$ $h([(\alpha,\emptyset)])=g(\alpha,\emptyset)$. Since obviously, $h$ is injective, therefore $\|\mathbf{MOD}_k^{\mathsf{deg}}/{\sim_k}\|\leq \|\mathbf{MOD}_n^{\mathsf{deg}}\|$.

(2) Since $k\geq n$, therefore $2^n\leq 2^k$. Hence, $\|\mathbf{MOD}_n^{\mathsf{deg}}\|\leq\|\mathbf{MOD}_k^{\mathsf{deg}}\|$.

**Proof of Lemma 2:** Let $S,T$ be non-empty sets of $k$-tuples of bits. Suppose the images by $f^{\mathsf{deg}}$ of $S\times\{\emptyset\}$ and $T\times\{\emptyset\}$ are equal. For the sake of the contradiction, suppose the images by $g$ of $S\times\{\emptyset\}$ and $T\times\{\emptyset\}$ are not equal. Hence, there exists $(\gamma,U)\in\mathbf{MOD}_n$ such that either $(\gamma,U)\in g(S\times\{\emptyset\})$ and $(\gamma,U)\not\in g(T\times\{\emptyset\})$, or $(\gamma,U)\in g(T\times\{\emptyset\})$ and $(\gamma,U)\not\in g(S\times\{\emptyset\})$. In the former case[6], let $\alpha\in S$ be such that $g(\alpha,\emptyset)=(\gamma,U)$. Since the images by $f^{\mathsf{deg}}$ of $S\times\{\emptyset\}$ and $T\times\{\emptyset\}$ are equal, therefore there exists $\beta\in T$ such that $f^{\mathsf{deg}}(\alpha,\emptyset)=f^{\mathsf{deg}}(\beta,\emptyset)$. Thus, $(\alpha,\emptyset)\sim_k(\beta,\emptyset)$. Consequently, $g(\alpha,\emptyset)=g(\beta,\emptyset)$. Since $g(\alpha,\emptyset)=(\gamma,U)$, therefore $g(\beta,\emptyset)=(\gamma,U)$. Since $\beta\in T$, therefore $(\gamma,U)\in g(T\times\{\emptyset\})$: a contradiction.

**Proof of Lemma 3:** Let $E$ be a non-empty set of $n$-tuples of bits.

(1) For the sake of the contradiction, suppose $\|f^\circ(E)/{\sim_k}\|>\|f^\bullet(E)\|$. Hence, there exists $p>2^n$ and there exists $(\alpha_1,S_1),\ldots,(\alpha_p,S_p)\in f^\circ(E)$ such that for all $i,j\in\{1,\ldots,p\}$, if $i\neq j$ then $(\alpha_i,S_i)\not\sim_k(\alpha_j,S_j)$. Thus, for all $i,j\in\{1,\ldots,p\}$, if $i\neq j$ then $g(\alpha_i,S_i)\neq g(\alpha_j,S_j)$. Moreover, $f^{\mathsf{deg}}(S_1\times\{\emptyset\})=E\times\{\emptyset\}$, $\ldots$, $f^{\mathsf{deg}}(S_p\times\{\emptyset\})=E\times\{\emptyset\}$. Consequently, for all $i,j\in\{1,\ldots,p\}$, if $i\neq j$ then $f^{\mathsf{deg}}(S_i\times\{\emptyset\})=f^{\mathsf{deg}}(S_j\times\{\emptyset\})$. Hence, by Lemma 2, for all $i,j\in\{1,\ldots,p\}$, if $i\neq j$ then $g(S_i\times\{\emptyset\})=g(S_j\times\{\emptyset\})$. Let $(\beta_1,T_1),\ldots,(\beta_p,T_p)\in\mathbf{MOD}_n$ be such that $g(\alpha_1,S_1)=(\beta_1,T_1)$, $\ldots$, $g(\alpha_p,S_p)=(\beta_p,T_p)$. Since for all $i,j\in\{1,\ldots,p\}$, if $i\neq j$ then $g(\alpha_i,S_i)\neq g(\alpha_j,S_j)$, therefore for all $i,j\in\{1,\ldots,p\}$, if $i\neq j$ then $(\beta_i,T_i)\neq(\beta_j,T_j)$. Moreover, by Proposition 7, $g(S_1\times\{\emptyset\})=T_1\times\{\emptyset\}$, $\ldots$, $g(S_p\times\{\emptyset\})=T_p\times\{\emptyset\}$. Since for all $i,j\in\{1,\ldots,p\}$, if $i\neq j$ then $g(S_i\times\{\emptyset\})=g(S_j\times\{\emptyset\})$, therefore for all $i,j\in\{1,\ldots,p\}$, if $i\neq j$ then $T_i=T_j$. Since for all $i,j\in\{1,\ldots,p\}$, if $i\neq j$ then $(\beta_i,T_i)\neq(\beta_j,T_j)$, therefore for all $i,j\in\{1,\ldots,p\}$, if $i\neq j$ then $\beta_i\neq\beta_j$. Thus, $p\leq 2^n$: a contradiction.

(2) Since $k\geq n$, therefore $2^n\leq 2^k$. Consequently, $\|f^\bullet(E)\|\leq\|f^\circ(E)\|$.

**Proof of Lemma 4:** Let $(\beta,T)\in\mathbf{MOD}_n$. We consider the following 2 cases.

**Case $T=\emptyset$:** Hence, $(\beta,T)\in\mathbf{MOD}_n^{deg}$. Since $f^{deg}$ is surjective, therefore there exists $(\alpha,\emptyset)\in\mathbf{MOD}_k^{deg}$ such that $f^{deg}(\alpha,\emptyset)=(\beta,T)$. Thus, $f(\alpha,\emptyset)=(\beta,T)$.

**Case $T$ is a non-empty set of $n$-tuples of bits:** Consequently, $(\beta,T)\in\mathbf{MOD}_n\setminus\mathbf{MOD}_n^{deg}$. Obviously, $(\beta,T)\in f^\bullet(T)$. Since $f^T$ is surjective, therefore there exists $(\alpha,S)\in f^\circ(T)$ such that $f^T(\alpha,S)=(\beta,T)$. Hence, the image by $f^{deg}$ of $S\times\{\emptyset\}$ is equal to $T\times\{\emptyset\}$. Thus, $f(\alpha,S)=f^T(\alpha,S)$. Since

[6] In the latter case, the proof can be similarly done.

$f^T(\alpha,S)=(\beta,T)$, therefore $f(\alpha,S)=(\beta,T)$.

**Proof of Lemma 5:** Let $(\alpha,S)\in\mathbf{MOD}_k$ and $(\beta,T)\in\mathbf{MOD}_n$ be such that $f(\alpha,S)=(\beta,T)$.

**Forward condition:** Let $\gamma\in S$. Hence, $S$ is a non-empty set of $k$-tuples of bits and $T$ is a non-empty set of $n$-tuples of bits. Thus, $(\alpha,S)\in\mathbf{MOD}_k\setminus\mathbf{MOD}_k^{deg}$. Consequently, $E$ being the non-empty set of $n$-tuples of bits such that the image by $f^{\mathtt{deg}}$ of $S\times\{\emptyset\}$ is equal to $E\times\{\emptyset\}$, $f(\alpha,S)=f^E(\alpha,S)$. Since $f(\alpha,S)=(\beta,T)$, therefore $f^E(\alpha,S)=(\beta,T)$. Hence, $E=T$. Since $E$ is the non-empty set of $n$-tuples of bits such that the image by $f^{\mathtt{deg}}$ of $S\times\{\emptyset\}$ is equal to $E\times\{\emptyset\}$, therefore the image by $f^{\mathtt{deg}}$ of $S\times\{\emptyset\}$ is equal to $T\times\{\emptyset\}$. Since $\gamma\in S$, therefore there exists $\delta\in T$ such that $f^{\mathtt{deg}}(\gamma,\emptyset)=(\delta,\emptyset)$. Thus, $f(\gamma,\emptyset)=(\delta,\emptyset)$.

**Backward condition:** Let $\delta\in T$. Consequently, $T$ is a non-empty set of $n$-tuples of bits and $S$ is a non-empty set of $k$-tuples of bits. Hence, $E$ being the non-empty set of $n$-tuples of bits such that the image by $f^{\mathtt{deg}}$ of $S\times\{\emptyset\}$ is equal to $E\times\{\emptyset\}$, $f(\alpha,S)=f^E(\alpha,S)$. Since $f(\alpha,S)=(\beta,T)$, therefore $f^E(\alpha,S)=(\beta,T)$. Thus, $E=T$. Since $E$ is the non-empty set of $n$-tuples of bits such that the image by $f^{\mathtt{deg}}$ of $S\times\{\emptyset\}$ is equal to $E\times\{\emptyset\}$, therefore the image by $f^{\mathtt{deg}}$ of $S\times\{\emptyset\}$ is equal to $T\times\{\emptyset\}$. Since $\delta\in T$, therefore there exists $\gamma\in S$ such that $f^{\mathtt{deg}}(\gamma,\emptyset)=(\delta,\emptyset)$. Consequently, $f(\gamma,\emptyset)=(\delta,\emptyset)$.

**Proof of Lemma 6:** Let $(\alpha,S),(\beta,T)\in\mathbf{MOD}_k$. Suppose $f(\alpha,S)=f(\beta,T)$. We consider the following 2 cases.

**Case $S=\emptyset$.** Hence, $f(\alpha,S)=f^{\mathtt{deg}}(\alpha,\emptyset)$. Since $f(\alpha,S)=f(\beta,T)$, therefore $f^{\mathtt{deg}}(\alpha,\emptyset)=f(\beta,T)$. Thus, $T=\emptyset$. Consequently, $f(\beta,T)=f^{\mathtt{deg}}(\beta,\emptyset)$. Since $f^{\mathtt{deg}}(\alpha,\emptyset)=f(\beta,T)$, therefore $f^{\mathtt{deg}}(\alpha,\emptyset)=f^{\mathtt{deg}}(\beta,\emptyset)$. Hence, $(\alpha,\emptyset)\sim_k(\beta,\emptyset)$. Thus, $g(\alpha,\emptyset)=g(\beta,\emptyset)$. Since $S=\emptyset$ and $T=\emptyset$, therefore $g(\alpha,S)=g(\beta,T)$.

**Case $S$ is a non-empty set of $k$-tuples of bits.** Consequently, $E$ being the non-empty set of $n$-tuples of bits such that the image by $f^{\mathtt{deg}}$ of $S\times\{\emptyset\}$ is equal to $E\times\{\emptyset\}$, $f(\alpha,S)=f^E(\alpha,S)$. Since $f(\alpha,S)=f(\beta,T)$, therefore $f^E(\alpha,S)=f(\beta,T)$. Hence, $E$ is a non-empty set of $n$-tuples of bits such that the image by $f^{\mathtt{deg}}$ of $T\times\{\emptyset\}$ is equal to $E\times\{\emptyset\}$. Thus, $f(\beta,T)=f^E(\beta,T)$. Since $f^E(\alpha,S)=f(\beta,T)$, therefore $f^E(\alpha,S)=f^E(\beta,T)$. Consequently, $(\alpha,S)\sim_k(\beta,T)$. Hence, $g(\alpha,S)=g(\beta,T)$.

**Proof of Lemma 7:** This result follows from the definition of $g$.

**Proof of Lemma 8:** This result follows from the definition of $g$.

**Proof of Lemma 9:** By induction on $\psi$.

**Proof of Lemma 10:** By Proposition 4.

**Proof of Lemma 11:** By Lemma 10 and Proposition 8.

**Proof of Lemma 12:** By Lemma 11.

**Proof of Theorem 1:** By Propositions 10 and 12.