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Fourier series analysis of the (pseudodifferential) Dirichlet to Neumann operator for a layer of dielectric material

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1. Introduction

One considers a general cylinder $\Omega = \Theta \times (-\infty, +\infty)$, where Θ^c is characterized by $\{n > 0, M(s) + n\vec{n}(s)\}$, $M(s)$ describes, for $s \in [0, L]$, a closed convex curve, characterized by its curvilinear absciss s and the unit outgoing normal vector $\vec{n}(s)$. The body Θ is constituted of a perfectly conducting core supplemented by a layer of dielectric material $\{-l < n < 0\}$.

The problem of replacing the Helmholtz equation with coefficients ϵ, μ in this layer ($(\Delta + \epsilon\mu\omega^2)u = 0$) and with the permittivity and permeability coefficients of the vacuum in $\Theta^c \times (-\infty, +\infty)$ by the Helmholtz equation $(\Delta + \epsilon_0\mu_0\omega^2)u = 0$ in $\Theta^c \times (-\infty, +\infty)$ supplemented by a boundary condition on $\partial\Theta^c \times (-\infty, +\infty)$ is a very classical and useful idea. One is left to determining this boundary condition. This condition is generally called the Dirichlet to Neumann operator on the boundary, and is frequently deduced through a factorization of the Helmholtz operator.

Under the assumption $\Im\epsilon\mu < 0$ and a rather sharp (and somewhat unphysical) assumption (H0) $l\omega \to c_0, 0 < c_0 < +\infty$ when $\omega \to +\infty$ (hence assuming that the size of the layer depends on the wavelength of the wave), we derived the impedance boundary condition in [15], using microlocal analysis. Microlocal analysis led to the exact behavior, in term of the size of the layer, in the high frequency regime, using a local approximation of the surface, which enabled the author to obtain the diffracted wave. Theorem 1 p 1042 of [15] gives the leading order term of the impedance operator, which corresponds to the 'tangent plane approximation'.

In the present paper, we deal with the two following cases (ϵ, μ do not depend on ω):

$$\Im\epsilon\mu < 0, \tag{1}$$

where the convention of sign comes from the time convention of a solution with $e^{i\omega t}$,

$$\epsilon\mu \in \mathbb{R}_+^*. \tag{2}$$

We need, in each case, to adress a specific problem:
i) in the case $\epsilon\mu \in \mathbb{R}_+^*$, we have to avoid resonances, id est cases where the problem

$$\begin{cases} (\Delta + \epsilon\mu\omega^2)u = 0, C \\ u|_{n=-l} = 0 \\ u|_{n=0} = u_0 \end{cases}$$

has no solution. Note that, as $z \in (-\infty, +\infty)$, there is always continuous spectrum, hence one shall deal with either a fixed arbitrary of k_z, Fourier variable in z or consider a finite periodic cylinder where $z \in (-\infty, +\infty)$ is replaced with $z \in [0, L]$ with periodic boundary conditions in z

ii) in the case $\epsilon\mu \notin \mathbb{R}_+$ there is no, in general, nontrivial solutions in $\mathcal{S}'(\mathbb{R}^3)$ solutions of $(\Delta + \epsilon\mu\omega^2)u = 0$, hence the use of Fourier transform should not be possible. However, a variational result yields for the Dirichlet problem a unique solution in $H^1(\Omega)$, hence Fourier analysis of this unique solution is possible.

Recall that, in [15], we introduced the metric $g(x, \eta)$ induced by the Euclidean metric on the boundary $\partial\Omega$, and obtained that there exists an operator (the Dirichlet to Neumann operator) such that $\partial_n u|_{n=0} = Op(\mathcal{D})(u|_{n=0})$. Denote by

$$\nu(x, \eta) = \sqrt{\epsilon\mu - g_{11}(x)\eta_2^1 - 2g_{12}(x)\eta_1\eta_2 - g_{22}(x)\eta_2^2}, \Im\nu < 0. \tag{3}$$

The principal symbol of $\mathcal{D}$ is

$$\omega \frac{\nu(x, \eta)}{\tan l\omega\nu(x, \eta)}.$$

Notice that the limit, when $\omega l \to +\infty$, of this principal symbol, under the assumption (1), is $i\omega\sqrt{\epsilon\mu - g_{11}(x)\eta_2^1 - 2g_{12}(x$

thanks to the relation $\tan l\omega\sqrt{\epsilon\mu - g_{11}(x)\eta_2^1 - 2g_{12}(x)\eta_1\eta_2 - g_{22}(x)\eta_2^2} \to -i$, exponentially.

We are interested in this paper on the behavior of the lower order term of the impedance operator (in the scalar case) for measuring the effect of the curvature of the body in the high frequency regime, that is the leading order term of r, which is of order 0 in ω.

We restrict ourselves to an infinite cylinder or ellipse, for which one does not have, in the case $\epsilon\mu \in \mathbb{R}_+^*$, any solution of the problem (see Lemma 1) in this Introduction for an explanation), for reasons that are explained later. We can also consider a cylinder of finite length $\Theta \times [0, L]$ with periodic boundary conditions in z for which $k_z \in \frac{2\pi}{L}\mathbb{Z}$.

Using the ideas developed in the thesis of P. Payen, along with classical formulae stated by B. Stupfel [28] (formulae that go back to Etienne [10] (1961) or Bates [2] (1975)), where, for a cylinder, the solution is expressed in terms of Bessel functions (hence taking into account the polar coordinates system), we derive here, using Fourier modes in the θ variable (with θ denoting the Euler angle, see [3] for example), a more complete version of the Calderòn operator and in particular prove estimates on the modes and provide an asymptotic analysis of the Dirichlet to Neumann operator (treating most of the time the scalar Helmholtz case).

The main feature of this paper is to consider, in the case of cylindrical coordinates, k_z and $\frac{p}{R}$ as **high frequency variables**, and to consider as well the case where the waves are evanescent, meaning that we consider the division in elliptic, hyperbolic, and glancing regions for the high frequency Helmholtz operator.

In this Introduction, we recall the classical case of a plane layer $B = \mathbb{R}^2 \times [-l, 0]$ to outline the main features of the cases we shall look at. We will construct uniquely the solution of a boundary problem in this layer, and use this solution to derive a relation between u and $\partial_{x_3} u$ on $x_3 = 0$. The first notion is the notion of **resonances**.

Lemma 1. 1. *The resonances of $-\Delta$ in B are the values of ω such that there exists $(k_1, k_2, n) \in \mathbb{R}^2 \times \mathbb{N}^*$ such that $\omega^2\epsilon\mu = k_1^2 + k_2^2 + \frac{n^2\pi^2}{l^2}$. There are no resonances if and only if $\epsilon\mu \notin \mathbb{R}_+^*$.*

2. *If one replaces B by its finite counterpart $[0, L_1] \times [0, L_2] \times [-l, 0]$, and if one assumes $\epsilon\mu \in \mathbb{R}_+^*$, then the resonances are the values of (n_1, n_2) such that $\omega^2\epsilon\mu = n_1^2\frac{4\pi^2}{L_1^2} + n_2^2\frac{4\pi^2}{L_2^2} + \frac{n^2\pi^2}{l^2}$.*

Note that there is no resonance when $\epsilon\mu \notin \mathbb{R}$. Note also that there is no trivial solution in $L^2(\mathbb{R}^3)$ of $(\Delta + \epsilon\mu\omega^2)u = 0$, as well as in $\mathcal{S}'(\mathbb{R}^3)$ in this case. Indeed, if u were a solution in $\mathcal{S}'(\mathbb{R}^3)$, one would have

$$(\epsilon\mu\omega^2 - k_1^2 - k_2^2 - \frac{\pi^2 n^2}{l^2})\hat{u} = 0$$

hence $\hat{u} = 0$ thanks to $(\epsilon\mu\omega^2 - k_1^2 - k_2^2 - \frac{\pi^2 n^2}{l^2})^{-1} \in C^\infty(\mathbb{R}^2)$.

It is also classical that 0 is the unique solution of

$$(\Delta + \omega^2\epsilon\mu)u = 0, C, u|_{\Gamma_-} = 0, u|_{\Gamma_+},$$

thanks to the variational formulation

$$\{\forall v \in H^1(C), a(u, v) = 0\} \Rightarrow u = 0, \tag{4}$$

the sesquilinear form considered being

$$a(u, v) = i\Im\epsilon\mu \int_C u\bar{v}dx + \int_C (\Re\epsilon\mu u\bar{v} - \nabla u\nabla\bar{v})dx$$

and one applies

Lemma 2. *The sesquilinear form a is coercive on $H^1(C)$.*

The proof of this Lemma can be found in Sebelin et al [24] for example, it is nevertheless classical, using the inequality

$$|a(u, u)| \geq \sqrt{(\Re(\epsilon\mu\omega^2))^2 + \frac{1}{2}(\Im(\epsilon\mu\omega^2))^2 - \sqrt{(\Re(\epsilon\mu\omega^2))^4 + \frac{1}{4}(\Im(\epsilon\mu\omega^2))^4}}\|u\|^2_{H^1(C)}.$$

Introduce some notations now. For all $(k_1, k_2) \in \mathbb{R}^2$, denote by

$$k_\perp = \sqrt{\omega^2\epsilon\mu - k_1^2 - k_2^2}, \Im k_\perp < 0. \tag{5}$$

One checks, in the case $\Im\epsilon\mu <$ independent on ω, that there exists $\alpha_0 > 0$ such that

$$\forall (k_1, k_2) \in \mathbb{R}^2, \Im k_\perp \leq -\omega\alpha_0. \tag{6}$$

One notices as well that the case $k_1^2 + k_2^2 > \Re\epsilon\mu\omega^2$ is covered in this expression. Note that in this case, $k_\perp = a_*(k_1, k_2) + ib_*(k_1, k_2)$, where $2a_*b_* = \Im\epsilon\mu\omega^2$, and $0 < a_*(k_1, k_2) = (\frac{1}{2}(|\epsilon\mu\omega^2 - k_1^2 - k_2^2|^2 + \Re\epsilon\mu\omega^2 - k_1^2 - k_2^2))^{\frac{1}{2}}$, a_* goes to 0 when $k_1^2 + k_2^2 \to +\infty$ and $b_* = O(\sqrt{k_1^2 + k_2^2})$.

In the case $\epsilon\mu \in \mathbb{R}_+^*$, we adopt the same notation for the equivalent following quantity:

$$k_\perp = \sqrt{\omega^2\epsilon\mu - k_1^2 - k_2^2}, \omega^2\epsilon\mu - k_1^2 - k_2^2 \geq 0, i\sqrt{k_1^2 + k_2^2 - \omega^2\epsilon\mu}, k_1^2 + k_2^2 - \omega^2\epsilon\mu > 0,$$

but, evidently, (6) is not true anymore.

Lemma 3. 1. *Assume $\epsilon\mu \in \mathbb{R}_+^*$. Provided that $\omega^2\epsilon\mu$ is not a resonance of $-\Delta$ on B, for all $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^2)$, there exists a unique solution in B of*

$$\begin{cases} (\Delta + \omega^2\epsilon\mu)u = 0 \\ u(., ., -l) = 0 \\ u(., ., 0) = u_0. \end{cases} \tag{7}$$

- *The quantity $\omega^2\epsilon\mu$ is always an element of the spectrum of $-\Delta$ (continuous spectrum).*
- *However, fix (k_1, k_2) and consider now the ODE on $[-l, 0]$. Assume $\epsilon\mu\omega^2 - k_1^2 - k_2^2 > 0$. Denote by $s_0 = min_{n\in\mathbb{Z}}|\frac{\pi}{l}\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2} - n|$,*

$$|\hat{u}(k_1, k_2, x_3)| \leq \frac{1}{\sin \pi s_0}.$$

This case (and similar cases) will be called throughout the paper the hyperbolic case. It is the case where a wave can propagate inside the layer without attenuation.

- *Similarly, if we fix (k_1, k_2) and if $\epsilon\mu\omega^2 - k_1^2 - k_2^2 < 0$, $|\hat{u}(k_1, k_2, x_3)| \leq 1$. This case can be called the elliptic case (there is no $k_3 \in \mathbb{R}$ such that $\epsilon\mu\omega^2 = k_1^2 + k_2^2 + k_3^2 = 0$.)*

2. *For $\epsilon\mu \notin \mathbb{R}_+^*$, even though a general solution of $(\Delta + \epsilon\mu\omega^2)u = 0$ belonging to $C^2([-l, 0], \mathcal{S}'(\mathbb{R}^2))$ is necessarily 0, there exists a 'suitable' solution in $H^1(C)$, which has a partial Fourier transform $\hat{u}$ (it is a consequence of Lemma 2). In addition, the Fourier transform satisfies the pointwise estimate:*

$$|\hat{u}(k_1, k_2, x_3)| \leq \frac{2}{1 - e^{-l\omega\alpha_0}}.$$

3. *In both cases, and apart from resonances for $\epsilon\mu \in \mathbb{R}_+^*$, there exists a linear operator $\mathcal{DTN}$, called the Dirichlet to Neumann operator, such that*

$$\partial_{x_3} u(., ., 0) = C(u(., ., 0)).$$

It is given by

$$\partial_{x_3}\hat{u}(k_1, k_2, 0) = ik_\perp \frac{1 + e^{-2ik_\perp l}}{1 - e^{-2ik_\perp l}}\hat{u}(k_1, k_2, 0) = ik_\perp \frac{1 + e^{-2ia_* l + 2b_* l}}{1 - e^{-2ia_* l + 2b_* l}}.$$

The Dirichlet to Neumann operator is characterized by the Fourier multiplier

$$\frac{k_\perp}{\tan(k_\perp l)} = l\frac{\cos k_\perp l}{\mathrm{sinc}_\perp\, l} = \frac{ik_\perp}{\tanh(b_* - ia_*)l} = \frac{ib_* - a_*}{\tanh(b_* - ia_*)l} = (ib_* - a_*)\frac{\cosh(b_* - ia_*)l}{\sinh(b_* - ia_*)l}, \tag{8}$$

the latter equality being valid also for $k_\perp = 0$. Note that, when $\omega \to +\infty$, the Fourier multiplier satisfies

$$ik_\perp \frac{1 + e^{-2ik_\perp l}}{1 - e^{-2ik_\perp l}} = ik_\perp(1 + O(e^{2l\Im k_\perp})),$$

which remainder term is exponentially decaying in ω.
For $u_0 \in \mathcal{S}(\mathbb{R}^2)$,

$$\partial_{x_3} u(x_1, x_2, 0) = \frac{1}{(2\pi)^2}\int\int il\frac{\cos k_\perp l}{\mathrm{sinc}\, k_\perp l}\hat{u}_0(k_1, k_2)dk_1 dk_2 = \frac{1}{(2\pi)^2}\int\int ik_\perp \frac{\cos k_\perp l}{\sin k_\perp l}\hat{u}_0(k_1, k_2)dk_1 dk_2.$$

4. *If one considers the case of $B_{L_1,L_2} = \{(0 \leq x \leq L_1, 0 \leq y \leq L_2, -l \leq z \leq 0\}$, $(k_1, k_2) \in \frac{2\pi}{L_1}\mathbb{Z} \times \frac{2\pi}{L_2}\mathbb{Z}$ and the previous results are true in the discrete Fourier operators set-up.*

This result (proven in Annex 1), which is straightforward to obtain, gives the aim of the present papier.

It has been proven [5] that, outside resonances, the problem of inhomogeneous boundary conditions has a unique solution. This defines the Calderòn operator. Our aim in this paper is to obtain, for $\epsilon\mu \notin \mathbb{R}_+^*$, an

estimate on the exact solutions in a layer which prove that these solutions are in $\mathcal{S}'(R^2) \times C^2([-l, 0]$ and deduce explicit versions of the Calderòn operator for more complicated geometries than the plane layer. Section 3 solves explicitly the Dirichlet problem in the cylindrical and in the elliptical geometry. The study of the solution and of the Dirichlet to Neumann operator for the cylindrical geometry, where the formal solutions after separation of variables are known is also well known, is the aim of Sections 4 and 5 of this paper. Section 6 gives the expression and the asymptotics of the Dirichlet to Neumann, in the case of two elliptical cylinders: in one case, the Calderòn is a Fourier multiplier operator, diagonal on each Fourier mode, while in the other case, it is a convolution Fourier operator. In Section 7, we study the asymptotics of the Calderòn operator for one layer of dielectric material for the Maxwell equations.

Remark that, if $\Omega - C$ is not bounded and if $\epsilon\mu \in \mathbb{R}_+^*$, the continuous spectrum of the operator prevents the existence of the Calderòn operator. It is interesting to define a weaker form of the Calderòn analysis in this case, which is done by considering the expressions for a Fourier mode k_z.

The Helmholtz operator inside the layer[1] is

$$\frac{1}{1+\kappa(s)n}\frac{\partial}{\partial s}[\frac{1}{1+\kappa(s)n}\frac{\partial u}{\partial s}] + \frac{1}{1+\kappa(s)n}\frac{\partial}{\partial n}[(1+\kappa(s)n)\frac{\partial u}{\partial n}] + (\omega^2\epsilon\mu - k_z^2)u = 0. \tag{9}$$

This paper is organised as follows. Section 2 states general results already known on the Dirichlet to Neumann operator, and states the main result, that is asymptotics of the Dirichlet to Neumann operator for a circular ring and an elliptic ring, both in the case $\epsilon\mu \in \mathbb{R}_+^*$, where one has to deal with resonances, and in the case $\Im\epsilon\mu < 0$, independent on ω.

Section 3 prepares the calculations by identifying Fourier modes (which will be defined precisely for the elliptic ring and which are clear for the circular ring) and derive equations for the 'radial' variable, which is deduced from the definition of modes.

Section 4 shows that the exact solution of the Dirichlet problem for each mode belongs to L^∞, hence proving that the Fourier transform approach makes sense (because, as usual, one always assumes that the solution has a Fourier transform in z and has (bounded) Fourier coefficients for the angular variable), indeed we prove that the assumption of having the right to consider a Fourier transform and series indeed lead to a function belonging to S'. One also introduces in this Section the relevant asymptotics of the Bessel functions for p, k_z, ω large, which are precious tools detailed in Subsection 4.3.

Section 5 expresses and gives the asymptotics of the Dirichlet to Neumann operator for the case of a cylindrical ring, and Section 6 does the same for the case of elliptic rings (with two different cases).

Finally, the asymptotics of the Calderon operator (which yields, in the case of Maxwell equations in the cylindrical ring, the expression of $\vec{n} \wedge \vec{H}$ in terms of $\vec{E}_t$ on the outer boundary for $\vec{E}_t$ satisfying the null Dirichlet condition on the inner boundary, are derived in Section **??**.

2. General results

Consider C a 2d bounded domain with a smooth boundary, and assume $\partial C = \Gamma_- \cup \Gamma_+$, such that there exists an ellipse $E \subset \mathbb{R}^2$ such that $\Gamma_- \subset E$ and $\Gamma_+ \subset E^c$. An annulus, or a layer on a perfectly conducting body of cylinder shape is a model for this. The following Theorem is classical:

Theorem 1. 1. *There exists a sequence λ_n of eigenvalues of $-\Delta$ in $H_0^1(C)$ (this operator being denoted by $-\Delta_D$), each eigenvalue is of multiplicity 1, they are strictly positive and normalized eigenfunctions form an orthonormal basis of $L^2(C)$.*

[1] which is the operator acting on the components of the electromagnetic field for the harmonic Maxwell's equations

2. *If k^2 is not an eigenvalue of $-\Delta_D$, the problem*

$$\begin{cases} (\Delta + k^2)u = 0 \text{ in } C \\ u = u_0 \text{ on } \partial C \end{cases} \tag{10}$$

has a unique solution through Fredholm alternative.

3. *When $\epsilon\mu \notin \mathbb{R}$, and for $\omega \in \mathbb{R}$, the problem*

$$\begin{cases} (\Delta + \omega^2\epsilon\mu)u = 0 \text{ in } C \\ u = 0 \text{ on } \Gamma_- \\ u = f \text{ on } \Gamma_+ \end{cases}$$

has a unique solution in $H^1(C)$ and the operator $f \to \partial_n u|_{\Gamma_+}$ is called the Dirichlet to Neumann operator.

The proof of this Theorem can be, in particular, found in Cessenat [5]. We do not reproduce this proof here. Note only that it is a consequence of Lemma 2.

Another result we recall here is the following (proven in [15])

Theorem 2. *Assume (H0). Let (η_1, η_2) be the cotangent coordinates near a point x_0 of the boundary.*

Under the quite restrictive condition 'ωl finite when $\omega \to +\infty$', the principal symbol of the Dirichlet to Neumann operator is

$$\omega \frac{\nu(x,\eta)}{\tan l\omega\nu(x,\eta)}.$$

where ν is given by (3).

One interprets in the case of a cylinder the previous result as

Remark 1. *The principal part of the Dirichlet to Neumann operator reduces to $i\sqrt{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}}$ in the case of the cylinder in the hyperbolic regime. Its approximation by an operator of order 2 is $\partial_n u = [i\omega\sqrt{\epsilon\mu} + \frac{i}{2\omega\sqrt{\epsilon\mu}}(\partial^2_{z^2} + \frac{1}{R^2}\partial^2_{\theta^2})]u$, or the condition*

$$\partial_t\partial_n u = \sqrt{\epsilon\mu}\partial^2_{t^2}u - \frac{1}{2\sqrt{\epsilon\mu}}(\partial^2_{z^2} + \frac{1}{R^2}\partial^2_{\theta^2})u.$$

Our aim in this paper is to improve the general result obtained in [15] under the assumption ωl finite to two particular geometries: the cylindrical layer and the elliptic layer, in order to obtain the two first terms in ω of the expansion of the Dirichlet to Neumann symbol, in the Fourier-discrete Fourier space. Assume that the dielectric constants satisfy $\Im\epsilon\mu < 0$, independent on ω.

In the infinite cylindrical layer geometry ($r_0 \le r \le R$), denote by k_z the wave number in z, p the Fourier mode. Assume in addition that the wave numbers $\frac{k_z}{\omega}$, $\frac{p}{\omega R}$ are bounded when $\omega \to +\infty$ (high frequency regime), and in addition $\Re\epsilon\mu - \frac{k_z^2}{\omega^2} - \frac{p^2}{R^2\omega^2} > 0$ (hyperbolic regime). Define

$$k_3 = \sqrt{\epsilon\mu\omega^2 - k_z^2} := a_0 + ib_0, a_0 > 0, b_0 < 0, \forall k_z \in \mathbf{R}, \tag{11}$$

$$k_\perp = \sqrt{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}}, \Im k_\perp < 0, \forall p, \forall k_z \in \mathbf{R}. \tag{12}$$

Denoting by $a_n(k_3\rho)$ and $b_n(k_3\rho)$ the Floquet numbers of the Mathieu equation.

We have the Theorem (a more precise statement is given below)

Theorem 3. 1. *In the case of the cylindrical layer, and under the conditions*

- *the dissipation condition* $\Im\epsilon\mu < 0$*, independent of* ω
- *the hyperbolic hypothesis* $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2} > 0$*,*

the leading order term in ω *of the Dirichlet to Neumann operator is* $i\omega\sqrt{\epsilon\mu - \frac{k_z^2}{\omega^2} - \frac{p^2}{R^2\omega^2}} = ik_\perp$*. The symbol of the Dirichlet to Neumann operator is, up to lower order terms,* $ik_\perp - \frac{1}{2R}\frac{k_3^2}{k_\perp^2}$

2. *This result is also valid in the elliptic case* $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2} < 0$*, where one notes that*

$$ik_\perp = \sqrt{k_z^2 + \frac{p^2}{R^2} - \epsilon\mu\omega^2},$$

the choice of the square root is uniform in $\mathbb{C} - (-\infty, 0)$*.*

and in the case of a elliptic layer, in the hyperbolic regime, $\Im\epsilon\mu < 0$, independent on ω:

Theorem 4. *In the infinite elliptical layer geometry (characterized by* $\mathcal{E} = \{(\rho\cosh u\cos v, \sinh u\sin v, z), u_0 \leq u \leq u_1, z \in \mathbb{R}\}$*), let* $a_n(\sqrt{\epsilon\mu\omega^2 - k_z^2}\rho)$ *be the* $n-th$ *Floquet eigenvalue for the Mathieu equation (edge of the* $n-th$ *band of the spectrum).*

Assume that there exists $\delta_0 > 0$ *such that* $\delta_0 \leq |\frac{a_n(\sqrt{\epsilon\mu\omega^2 - k_z^2}\rho)}{\omega^2}| \leq \frac{1}{\delta_0}$ *bounded independently of* ω*,* $\Re\epsilon\mu - \frac{k_z^2}{\omega^2} - \frac{a_n(\sqrt{\epsilon\mu\omega^2 - k_z^2}\rho)}{\rho^2\omega^2} > 0$ *(which means n of order* ω*).*

Denote by g_n *the normalized eigenvector associated with the Floquet eigenvalue* $a_n(\sqrt{\epsilon\mu\omega^2 - k_z^2}\rho)$*.*

1. *The leading order term in* ω *of the Dirichlet to Neumann operator on the basis* $\{g_n\}$ *is*

$$C_0(v, k_z, n) = i\omega\frac{\sqrt{\frac{1}{2}(\epsilon\mu\omega^2 - k_z^2)\rho^2\cosh^2 u_1 - \frac{a_n(\sqrt{\epsilon\mu\omega^2 - k_z^2}\rho)}{\omega^2}}}{\sqrt{\cosh^2 u_1\cos^2 v + \sinh^2 u_2\sin^2 v}}$$

and the term of order 0 is 0 (which means that the leading order term contains the contribution of the radius of curvature at each point).

2. *This Fourier multiplier is not independent on v, that is*

$$U(v) = \sum_n U_n g_n(v) \Rightarrow C_0(U) = \sum_n C_0(v, k_z, n)U_n g_n(v).$$

It is a pseudodifferential operator in (v, n) *and a Fourier multiplier in* k_z*.*

3. Exact resolution of the Calderòn problem for a ring and an elliptic ring using special functions

3.1. The case of the infinite cylindrical ring

The program of this section is to perform the following analysis:

After Fourier transform[2] in z, θ, we obtain a formal solution $u(r, \theta, z) = \sum_p(\alpha_p(k_z)J_p(k_3 r) + \beta_p(k_z)Y_p(k_3 r))e^{ip\theta}$, J_p, Y_p being the classical Bessel functions, k_3 is the square root of $\epsilon\mu\omega^2 - k_z^2$. This sum has a meaning when it is finite in p, but nothing is known about the behavior in k_z.

[2]Note that it is a Fourier series in θ and a Fourier transform in z

After using the Dirichlet boundary condition at $r = r_0$, we obtain, still formally, all solutions of the Helmholtz equation in the annulus which satisfy the Dirichlet boundary condition at $r = r_0$. Its formal expression is $\sum_p a_p(k_z)u_p(r,k_z)e^{ip\theta}$, $u_p(R,k_z) = 1$.

Introduce the following Definition

Definition 1. *If ν is real and $\lambda > 0$, the roots of $J_\nu(z)Y_\nu(\lambda z) - J_\nu(z)Y_\nu(\lambda z) = 0$ are real and simple. They are denoted by the increasing sequence $z_n(\nu,\lambda)$. It is stated in (9.5.27) of [1].*

Similar to the result obtained in the Introduction, we have (the formulae here are classical, see Stupfel [28] for example), but not the conditions on ω.

Lemma 4. *Let $\epsilon\mu \in \mathbb{R}_*^+$. The resonances of the operator $-\Delta$ are the values of ω such that there exists at least a n, and a couple $(p,k_z) \in \mathbb{Z}\times\mathbb{R}$ such that $r_0^2(\epsilon\mu\omega^2 - k_z^2) = z_n(p,\frac{R}{r_0})$, where $z_n(\lambda)$ is defined in Definition 1.*

Every value of ω is a resonance in this case when $k_z \in \mathbb{R}$.

If k_z is fixed, we have an infinite sequence of resonances of the operator $-\Delta + k_z^2$. If $\epsilon\mu \notin \mathbb{R}$, there are no resonances.

If one restricts to the 'periodic cylinder' $\{(r,\theta,z), 0 \le r_0, \theta \in [0,2\pi], z \in [0,L]\}$ with periodicity conditions in z, we have $k_z \in \frac{2\pi}{L}\mathbb{Z}$, hence a dispersion relation, in the case $\epsilon\mu \in \mathbb{R}_+$, $\epsilon\mu\omega^2 = \frac{z_n}{r_0^2} + (\frac{2\pi}{L})^2q^2$, $(n,q) \in \mathbb{N}\times\mathbb{Z}$.

add a proof and comments One has the following

Proposition 1. *Provided that $\omega^2\epsilon\mu$ is not a resonance of $-\Delta + k_z^2$ on C, for all $u_0 \in H^{\frac{1}{2}}(S_R) \simeq H^{\frac{1}{2}}([0,2\pi])$ there exists a unique solution in $H^1(C)$ of*

$$\begin{cases} (\Delta + \omega^2\epsilon\mu)u = 0 \\ u(r_0,.,.) = 0 \\ u(R,.,.) = u_0. \end{cases} \tag{13}$$

It has a partial Fourier transform in z and Fourier coefficient in θ. This partial Fourier transform in z and Fourier series in θ is

$$\hat{u}(r,\theta,k_z) = \sum_p a^p(k_z)\frac{J_p(k_3r)Y_p(k_3r_0) - Y_p(k_3r)J_p(k_3r_0)}{J_p(k_3R)Y_p(k_3r_0) - Y_p(k_3R)J_p(k_3r_0)}e^{ip\theta}, \tag{14}$$

where $a^p(k_z)$ is the $p-th$ Fourier coefficient of $\hat{u}_0(k_z,\theta)$. As $\frac{J_p(k_3r)Y_p(k_3r_0)-Y_p(k_3r)J_p(k_3r_0)}{J_p(k_3R)Y_p(k_3r_0)-Y_p(k_3R)J_p(k_3r_0)}$ satisfies the inequality of Proposition 5, we check that (14) belongs to $L^\infty \times L^2$ hence its inverse Fourier transform exists.

Analysis: After partial Fourier transform in x_3, the equation reads

$$(\Delta + k_3^2)\hat{u} = 0,$$

where we kept the notation Δ for the Laplace operator in $\mathbb{R}^2$. The expansion $\hat{u}(r,\theta) = \sum a_p(r)e^{ip\theta}$, where $\sum |a_p|^2 < +\infty$ (for example) yields the classical Bessel equation on a_p:

$$\frac{1}{r}\frac{d}{dr}(r\frac{da_p}{dr}) + (\omega^2\epsilon\mu - k_z^2 - \frac{p^2}{r^2})a_p = \frac{1}{r}\frac{d}{dr}(r\frac{da_p}{dr}) + (k_3^2 - \frac{p^2}{r^2})a_p = 0. \tag{15}$$

This Bessel equation implies that there exists (α_p, β_p) depending on k_z such that $a_p(r) = \alpha_p J_p(k_3 r) + \beta_p H_p^{(2)}(k_3 r)$. Note, as in the Introduction, that it is a necessary (but not necessary and sufficient condition) as a general solution of the ODE.

Define the functions of $r \in [r_0, R]$:

$$d_p(k_3 r) := J_p(k_3 r_0) Y_p(k_3 r) - Y_p(k_3 r_0) J_p(k_3 r), dd_p(k_3 r) := J'_p(k_3 r_0) Y_p(k_3 r) - Y'_p(k_3 r_0) J_p(k_3 r). \quad (16)$$

The first one is a solution of the Bessel equation which vanishes at $r = r_0$ and the second quantity is a solution of the Bessel equation which derivative vanishes at $r = r_0$. It is a pair of fundamental solutions of the Bessel equation, under the condition:

$$J'_p(k_3 r_0) Y_p(k_3 r_0) - Y'_p(k_3 r_0) J_p(k_3 r_0) \neq 0. \quad (17)$$

Remark that, if one picks any pair of independent solutions of the Bessel equations $\{f_p, g_p\}$, there exists a constant $C(f_p, g_p)$ such that $d_p(k_3 r) = C(f_p, g_p)[f_p(k_3 r_0) g_p(k_3 r) - g_p(k_3 r_0) f_p(k_3 r)]$ and $dd_p(k_3 r) = C(f_p, g_p)[f'_p(k_3 r_0) g_p(k_3 r) - g'_p(k_3 r_0) f_p(k_3 r)]$. Hence we can choose any pair of independent solutions instead of J_p and of Y_p for the representation of d_p and dd_p.

3.2. The exact representation of solutions of the Helmholtz equation for elliptic coordinates

In this Section, we rely on the classical approach using Mathieu and modified Mathieu functions. Mathieu functions are periodic functions which form a basis of $L^2([0, 2\pi])$ and modified Mathieu functions are the analogous of the Bessel and Hankel functions. B. Stupfel [28] already used such an approach to compute a good estimate

The (transversal) change of variable in which the Laplacian operator is diagonal is

$$(x, y) = \rho(\cosh u \cos v, \sinh u \sin v).$$

If one wants to use this on the ellipse $\Omega = \{(x, y), \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\}$ of constants $a, b, a > b$, one defines $\rho = \sqrt{a^2 - b^2}$ and u_0 such that $\tanh u_0 = \frac{b}{a}$. The open set Ω is characterized by $\{(u, v), (\frac{\cosh u}{\cosh u_0})^2 \cos^2 v + (\frac{\sinh u}{\sinh u_0})^2 \sin^2 v \leq 1, \}$, its boundary being $\partial\Omega = \{u = u_0, v \in (0, 2\pi]\}$.

All solutions of $(\Delta + \omega^2 \epsilon\mu) u = 0$ are linear combinations of solutions of the form

$$u(x, y, z) = F(u) G(v) \phi(z),$$

where ϕ satisfies $\phi''(z) + c\phi(z) = 0$ and F and G are solutions of

$$F''(u) + (\frac{\omega^2 \epsilon\mu - c}{2} \rho^2 \cosh 2u - a) F = 0, \quad (18)$$

$$G''(v) + (a - \frac{\omega^2 \epsilon\mu - c}{2} \rho^2 \cos 2v) G = 0, \quad (19)$$

thanks to the transformation of the Helmholtz equation into

$$[\frac{\partial^2}{\partial z^2} + \frac{2}{\rho^2(\cosh 2u - \cos 2v)} (\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}) + \epsilon\mu\omega^2] u = 0.$$

The natural choice in our analysis is to consider $c = k_z^2$. Let $U(x,y) = F(u)G(v)$. For all $(a, k_3\rho)$, there exists (through Floquet theory, as mentionned in [1], 20.3.1) $\nu(a, k_3\rho)$ such that $g_\pm(v) = e^{\pm i\nu(a,k_3\rho)}P_{a,k_3\rho}(\pm v)$ is a pair of independent solutions of (19), where $P_{a,k_3\rho}$ are periodic functions.

Let $a_n(k_3\rho)$ and $b_n(k_3\rho)$ be the unique solutions, respectively, of $\nu(a, k_3\rho) = n$ and $\nu(a, k_3\rho) = -n$.

Lemma 5. *The eigenvalues of the Mathieu operator $-\frac{d^2}{dv^2} + \frac{k_3^2\rho^2}{2}\cos 2v$ on $L^2([0, 2\pi])$, with Bloch boundary conditions, are $a_n(k_3\rho), b_n(k_3\rho)$, and the associated eigenfunctions, with a suitable normalization are denoted by $ce_n(v, k_3\rho)$ and $se_n(v, k_3\rho)$.*

The notation recalling that each of which is closely related to the cosine and sine (namely for $k_3\rho = 0$, $a_n(0) = b_n(0) = n^2$ and $ce_n(v) = \cos nv$, $se_n(v) = \sin nv$).

Let $c_n(k_3\rho) = a_n(k_3\rho)$ for $n \geq 1$, $c_n(k_3\rho) = b_{-n}(k_3\rho)$ for $n \leq -1$.

Lemma 6. *The equation (18)*

$$F'' - (c_n(k_3\rho) - \frac{k_3^2\rho^2}{2}\cosh 2u)F = 0$$

has a pair of canonical solutions, even and odd respectively, denoted by $C_{|n|}^{k_3\rho}, S_{|n|}^{k_3\rho}$. where n positive stands for $c_n(k_3\rho) = a_n(k_3\rho)$ and $n \geq 1$, $c_n(k_3\rho) = b_n(k_3\rho)$ and $n \leq -1$.

Lemma 7. *The periodic in v solutions of the Helmholtz equations $(\Delta + k_3^2)U = 0$ in $\mathbb{C}^2(\mathcal{E}(A, B))$ are*

$$U_n(x,y) = (AC_{|n|}^{k_3\rho} + BS_{|n|}^{k_3\rho})(u)g_n(v), n \in \mathbb{Z}.$$

All the items of these Lemma are well known through [1] or through [8].

3.3. Estimates of the quantities $a_n(k_3\rho)$ and $b_n(k_3\rho)$ for n large in the high frequency regime.

Let us begin with the case $\epsilon\mu \in \mathbb{R}$.

Proposition 2. *For $\Im\epsilon\mu = 0$, impose $\delta_0 > 0$ small enough.*

1. *For n such that the n−th Floquet band of the Mathieu equation is contained in $[-\frac{\epsilon\mu-\eta^2}{2}\rho^2, -\frac{\epsilon\mu-\eta^2}{2}\rho + \delta_0]$,*
$$a_n(k_3\rho) \simeq (2n+1)\sqrt{\epsilon\mu\omega^2 - k_z^2}.$$
2. *For n such that the n−th Floquet band of the Mathieu equation is in $[-\frac{\epsilon\mu-\eta^2}{2}\rho^2 + \delta_0, \frac{\epsilon\mu-\eta^2}{2}\rho - \delta_0]$ (that is not in the bottom of the well but still in the image of the potential), there exists b(n) solution of (21) such that $a_n(k_3\rho), b_n(k_3\rho)$ satisfy both:*
$a_n(k_3\rho) \simeq h^{-1}b(n)$.
3. *For n such that the n−th Floquet band of the Mathieu equation is contained in $[\frac{\epsilon\mu-\eta^2}{2}\rho^2 + \delta_0, +\infty)$, there exists $N(n) \geq 1$ such that*
$$a_n(k_3\rho) \simeq (N(n))^2.$$

Proof. Note first that $h^{-2}a_n(k_3\rho)$ and $h^{-2}b_n(k_3\rho)$ are the edges of each band, and that $a_n(k_3\rho) - b_n(k_3\rho)$ is exponentially small for h small or n large.

It is straightforward in this case to write (19), for $c = k_z^2 = \omega^2\eta^2$ and $h = \omega^{-1}$ as

$$-h^2G''(v) + \frac{\epsilon\mu - \eta^2}{2}\rho^2\cos 2vG = h^2aG.$$

The potential $V(v) = \frac{\epsilon\mu-\eta^2}{2}\rho^2 \cos 2v$ is a periodic potential, of minimum $-\frac{\epsilon\mu-\eta^2}{2}\rho^2$, of maximum $\frac{\epsilon\mu-\eta^2}{2}\rho^2$.

The Floquet theory predicts that three regimes are available for the bands

1. E in the neighborhood of the minimum $-\frac{\epsilon\mu-\eta^2}{2}\rho^2$ (where results of Harrel [13] are of use, and mostly the results of Keller and Weinstein [14]),
2. E is in the neighborhood of $\frac{\epsilon\mu-\eta^2}{2}\rho^2$ (where results of Marz [18] study the precise behavior of the bands)
3. $E > \frac{\epsilon\mu-\eta^2}{2}\rho^2$, where one can use the methods and WKB expansions (see Grigis and Sjostrand [11] for details)

We concentrate first on the elliptic case $E > \frac{\epsilon\mu-\eta^2}{2}\rho^2$.

In what we call here the hyperbolic case, we may apply the results of Exercises 12.1 and 12.2 of Chapter 12 of [11], where all values of E greater than the maximum of the potential, called E_N, are given by the solution of the **Bohr-Sommerfeld quantization condition**, for $N \geq 1$:

$$\int_0^\pi \sqrt{E - \frac{\epsilon\mu-\eta^2}{2}\rho^2 \cos 2v}dv + h^2 d(E,h) = \pi h N \tag{20}$$

where $d(E,h)$ is constructed through the formal WKB solution $a(v,h)e^{ih^{-1}\phi(v)}$, $\phi(v) = \int_0^\pi \sqrt{E - \frac{\epsilon\mu-\eta^2}{2}\rho^2 \cos 2v'}dv'$, $a(v+\pi,h) = e^{ihd(E,h)}a(v,h)$. Details are given in Section 3.4. The label N does not correspond to the label n: we have $N = n - \sharp\{n, [E^n_{min}, E^n_{max}] \subset [-\frac{\epsilon\mu-\eta^2}{2}\rho^2, \frac{\epsilon\mu-\eta^2}{2}\rho^2]\}$, assuming that no band contains the maximum of the potential (hence $n \geq \sharp\{n, [E^n_{min}, E^n_{max}] \subset [-\frac{\epsilon\mu-\eta^2}{2}\rho^2, \frac{\epsilon\mu-\eta^2}{2}\rho^2]\} + 1$.

It is then reasonable to assert that, for n large, $\pi\sqrt{E_{N(n)}}$ is well approximated by $\pi h N(n)$, leading to $E_N \simeq h^2(N(n))^2$, hence $a_N(k_3\rho) \simeq (N(n))^2$ as in the case of the Fourier equation $-h^2u'' = Eu$. It is then meaningful to consider the high frequency regime to be the regime when $N(n)h$ is bounded. Note that, in this approximation, $\rho\sqrt{\epsilon\mu-\eta^2}$ does not appear anymore at the first order.

The extension to $\Im\epsilon\mu \neq 0$ is straightforward; the definition of the quantity $d(E,h)$ does not change (it belongs to $\mathbb{C}$). The Bohr-Sommerfeld quantization condition is (20) with $E \in \mathbb{C}$.

In the case $E + \frac{\epsilon\mu-\eta^2}{2}\rho^2 \in [0, \delta_0]$, the results of Harrel [13] yield a behavior for $a_n(k_3\rho)$ of the form $\omega\sqrt{\epsilon\mu-\eta^2}\rho(n+\frac{1}{2})\pi$ (the value of the second derivative of the potential at its minimum point being $2(\epsilon\mu - \eta^2)\rho^2$), and the choice $n = h^{-1}\delta\rho$ yields as well the same estimate for h small enough. More precisely, for all $\delta_0 < \frac{\epsilon\mu-\eta^2}{2}$, the inequality $E < \frac{\epsilon\mu-\eta^2}{2}\rho^2$ implies

$$\epsilon\mu - \eta^2 - 2E\rho^2 > 0,$$

and this is called the hyperbolic region, while the case $E \simeq \frac{\epsilon\mu-\eta^2}{2}\rho^2$ is the case described by Marz, and can be called the glancing region.

The precise description is made in Keller and Weinstein's work [14].

Let n be the label of a band included in $[-\frac{\epsilon\mu-\eta^2}{2}, \frac{\epsilon\mu-\eta^2}{2}]$. For all K and for all δ small enough, there exists h_* such that, for $h < h_*$, if

$$Kh^{\delta-1} \leq n \leq \frac{1}{\pi h}\int_0^\pi \sqrt{\frac{\epsilon\mu-\eta^2}{2}(1-\cos 2v)}dv = \frac{1}{\pi h}\int_0^\pi \sqrt{\frac{\epsilon\mu-\eta^2}{2}(1-\cos 2v)}dv = \frac{2}{\pi h}\rho\sqrt{\epsilon\mu-\eta^2},$$

then there exists $b(n)$, given by (21)

$$E^+_n, E^-_{n+1} \simeq b(n)h^{-1} \mp 2(-1)^n\delta_n$$

where δ_n is exponential in $e^{-h^{-1}}$, estimated by a turning point analysis, the turning points $0 < v_0^n < v_1^n < \pi$ being the solutions of

$$\rho^2 \frac{\epsilon\mu - \eta^2}{2}(\cos 2v_j^n + 1) = b(n), \int_{v_1^n}^{v_0^n+\pi} \sqrt{b(n) - \frac{\epsilon\mu - \eta^2}{2}\rho^2(\cos 2v + 1)}dv = (n + \frac{1}{2})h\pi. \tag{21}$$

while $b(n)$ is exponentially close to E_n^+. □

An important remark is that we have the **Weyl's law**. It reads

Proposition 3. *The number of eigenvalues of $H = -h^2\frac{d^2}{dv^2} + V(v)$, with boundary conditions $\psi(-\frac{\pi}{2}) = \psi(\frac{\pi}{2})$, $\psi'(-\frac{\pi}{2}) = \psi'(\frac{\pi}{2})$, which are smaller than E is given by the estimate*

$$N(E) \simeq \frac{1}{2\pi h}\int\int_{\xi^2+V(v)<E} dvd\xi.$$

In particular, the number of eigenvalues which are smaller than the maximum of the potential is

$$N(E) \simeq \frac{1}{2\pi h}\int\int_{\xi^2<V_{max}-V(v)} dvd\xi.$$

For $V(v) = \frac{\epsilon\mu-\eta^2}{2}\rho^2\cos 2v$, it is thus

$$N(E) \simeq \frac{2}{\pi h}\rho\sqrt{\epsilon\mu - \eta^2}.$$

Let us consider now the case $\epsilon\mu \notin \mathbb{R}$. Recall that we assumed $k_z^2 < \Re\epsilon\mu\omega^2$ and ω large.

In a first part of this Section, we derive results for $\Im\epsilon\mu = 0$, which is a more familiar case for the Floquet theory and the band structure of the spectrum of a periodic operator, where the bands are more conveniently defined for $E \in \mathbf{R}$. Indeed, it is known from Reed and Simon [23] that the spectrum of $-h^2D^2 + V$ is a band spectrum $\cup_i[E_{min}^i, E_{max}^i]$ included in $(\min V, +\infty)$. The edges of the bands are the solutions of the Floquet problems (see a description in [4] of the classical results)

$$\begin{array}{l}(-h^2D^2 + V - E)\psi = 0 \text{ on } (-\frac{\pi}{2}, \frac{\pi}{2}), \psi(-\frac{\pi}{2}) = \psi(\frac{\pi}{2}), \psi'(-\frac{\pi}{2}) = \psi'(\frac{\pi}{2}) \\ (-h^2D^2 + V - E)\psi = 0 \text{ on } (-\frac{\pi}{2}, \frac{\pi}{2}), \psi(-\frac{\pi}{2}) = -\psi(\frac{\pi}{2}), \psi'(-\frac{\pi}{2}) = -\psi'(\frac{\pi}{2}).\end{array} \tag{22}$$

Definition 2. *Consider the ellipse characterized by (ρ, u_0) such that $a = \rho\cosh u_0, b = \rho\sinh u_0$.*

We say that the n−th Floquet mode of the associated Mathieu equation is in the elliptic regime when $\Re a_n(k_3\rho) > \frac{\Re\epsilon\mu\omega^2-k_z^2}{2}\rho^2\cosh 2u_0$.

We say that the n−th Floquet mode of the associated Mathieu equation is in the hyperbolic regime when $\Re a_n(k_3\rho) < \frac{\Re\epsilon\mu\omega^2-k_z^2}{2}\rho^2\cosh 2u_0$.

In the case of complex coefficients, recall first that the problem $(\Delta + \omega^2\epsilon\mu)u = 0$ has no solution in $\mathcal{S}'(\mathbb{R}^d)$, hence we cannot get estimates which ensure that a generic solution has a Fourier transform. One can observe this in Section 4.3, where the modulus of one of the solutions is exponentially growing in p or in k_z. Hence we could have difficulties with a symbol and Fourier integral analysis, which requires using Fourier modes. Two options are at our disposal:

- use analytic symbols, where a Gaussian growth is, for example, allowed (see Sjostrand [27])

- use the limiting absorption principe, id est we shall say that the point considered in the symplectic coordinates on the boundary $u = u_0$: (v_0, z_0, n, k_z) is in the elliptic (respectively hyperbolic) regime if the limit of the symbol $a_n(k_3\rho) - \frac{\epsilon\mu\omega^2 - k_z^2}{2}\rho^2 \cosh 2u_0$ when $\Im\epsilon\mu \to 0$ is in the (real) elliptic (respectively hyperbolic) regime.

When $\Im\epsilon\mu \neq 0$, the Floquet problems (22) still have solutions E, which are complex now (as well as the equation (20) is a complex equation), which defines as well $a_n(k_3\rho), b_n(k_3\rho) \in \mathbb{C}$. We shall thus say that

$$\Re a_n(k_3\rho) > \rho^2 \frac{\Re\epsilon\mu\omega^2 - k_z^2}{2} \cosh 2u_0 \tag{23}$$

defines the elliptic region and, in addition, nh is bounded below in this case for $h \to 0_+$, and for all n satisfying the inequality (23), $a_n(k_3\rho)h^2$ is of order $n^2\pi^2\rho^2\frac{\epsilon\mu - \eta^2}{2}$.

In a similar way, we shall say that

$$\Re a_n(k_3\rho) < \rho^2 \frac{\Re\epsilon\mu\omega^2 - k_z^2}{2} \cosh 2u_0 \tag{24}$$

defines the hyperbolic region (by inspection of the modified Mathieu equation), along also nh bounded when $h \to 0_+$. We add the assumption nh bounded below to avoid the case of a fixed number of modes.

In this case, it is observed that, though we cannot define the minimum value of $\frac{\epsilon\mu\omega^2 - k_z^2}{2}\rho^2 \cos 2v$ (it is a complex function), the minimum value of $\cos 2v$ is -1 for $v = \frac{\pi}{2}$, one has the estimate, for v close to $\frac{\pi}{2}$

$$\frac{\epsilon\mu\omega^2 - k_z^2}{2}\rho^2 \cos 2v = -\frac{\epsilon\mu\omega^2 - k_z^2}{2}\rho^2 + (\epsilon\mu\omega^2 - k_z^2)\rho^2(v - \frac{\pi}{2})^2 + O((v - \frac{\pi}{2})^4),$$

hence eigenvalues close to

$$\tilde{E}_n = -\frac{\epsilon\mu\omega^2 - k_z^2}{2}\rho^2 + \sqrt{\epsilon\mu\omega^2 - k_z^2}\rho^2(2n + 1).$$

3.4. Eigenvalues of the Mathieu equation (Grigis-Sjostrand)

One consider the Mathieu equation

$$-G''(v) + (\frac{1}{2}(\epsilon\mu\omega^2 - k_z^2)\cos 2v - a)G = 0,$$

where a must be identified so that we get a periodic solution (*of period π or 2π?*). When $\omega^2\rho^2$ is small, $a \simeq n^2$. But this is not the case here. When a is close to the minimum of this function one uses Harrell. In the general case, assume $h\omega = 1$. The equation writes

$$P_h G := -h^2 G'' + \frac{1}{2}(\epsilon\mu - \eta^2)\cos 2vG = ah^2 C.$$

We use the procedure described by Grigis and Sjostrand ([11], chapter 15). Assume that $E > \frac{1}{2}|\epsilon\mu\omega^2 - k_z^2|$. Consider the eikonal equation

$$(\phi_v')^2 + \frac{1}{2}(\epsilon\mu - \eta^2)\cos 2v - E = 0,$$

that is $\phi(v) = \int_0^v \sqrt{E - \frac{1}{2}(\epsilon\mu - \eta^2)\cos 2s}ds$, with the canonical choice of the determination of the root. It is then classical to deduce a sequence $a_j(v, E)$ such that a WKB solution is

$$(\sum a_j(v, E)h^j)e^{i\frac{\phi(v)}{h}}, a_0(v, E) = (\sqrt{E - \frac{1}{2}(\epsilon\mu - \eta^2)\cos 2v})^{-\frac{1}{4}}.$$

One observes that there exists $c(E, h)$ and thus $d(E, h)$ such that

$$a(v + \pi, E, h) = c(E, h)a(v, E, h), c(E, h) = e^{ihd(E,h)}$$

where c and d have expansions in h, $c(E, 0) = 1$. If one denotes, for h small enough, the unique solution $E_p(h)$ of

$$\int_0^\pi \sqrt{E - \frac{1}{2}(\epsilon\mu - \eta^2)\cos 2s}ds + h^2 d(E, h) = \pi ph, \tag{25}$$

then for all N there exists h_N such that for $0 < h < h_N$, the interval $(E_p(h) - h^N, E_p(h) + h^N)$ contains at least two eigenvalues of the operator P_h, denoted by $ea_p(h), eb_p(h)$.

It is then a consequence of this result that $a_p(k_3\rho) = \omega^2 ea_p(h), b_p(k_3\rho) = eb_p(h)\omega^2$, which is to be used in the Modified Mathieu equation.

3.5. Precise approximations of the eigenvalues

This Section is a remark, allowing us to have a very precise estimate of eigenvalues, using the special functions, namely the Elliptic integrals (as in [8], [1]). It is based on the observation that

$$\begin{aligned}\int_0^\pi \sqrt{E - \tfrac{1}{2}(\epsilon\mu - \eta^2)\rho^2\cos 2s}ds &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{E + \tfrac{1}{2}(\epsilon\mu - \eta^2)\rho^2 - (\epsilon\mu - \eta^2)\rho^2\sin^2 s}ds \\ &= 2\sqrt{E + \tfrac{1}{2}(\epsilon\mu - \eta^2)\rho^2}\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2 s}ds,\end{aligned}$$

with $k^2 = \rho^2\frac{\epsilon\mu - \eta^2}{E + \frac{1}{2}(\epsilon\mu - \eta^2)}$ hence

$$\int_0^\pi \sqrt{E - \frac{1}{2}(\epsilon\mu - \eta^2)\rho^2\cos 2s}ds = 2\sqrt{E + \frac{1}{2}(\epsilon\mu - \eta^2)\rho^2}E(\frac{\pi}{2}, k) \tag{26}$$

as well as, for v_0, v_1 solving $b - \frac{\epsilon\mu - \eta^2}{2}\rho^2(\cos 2v - 1) = 0$ that is $0 < v_0 < v_1 = \pi - v_0$ such that

$$\int_{v_1}^{v_0+\pi} \sqrt{b - \frac{\epsilon\mu - \eta^2}{2}\rho^2(\cos 2v - 1)}dv = \int_{v_1}^{v_0+\pi} \sqrt{b - (\epsilon\mu - \eta^2)\rho^2\cos^2 v}dv = 2\sqrt{b}\int_0^{v_+} \sqrt{1 - \frac{\sin^2 v}{\sin^2 v_+}}dv,$$

that is

$$\int_{v_1}^{v_0+\pi} \sqrt{b - \frac{\epsilon\mu - \eta^2}{2}\rho^2(\cos 2v - 1)}dv = 2\sqrt{b}E(v_+, \frac{1}{\sin v_+}), \sin v_+ = \frac{b}{\rho^2(\epsilon\mu - \eta^2)}. \tag{27}$$

$$2\sqrt{E + \frac{1}{2}(\epsilon\mu - \eta^2)\rho^2}E(\frac{\pi}{2}, k) = \pi Nh, 2\sqrt{b}E(v_+, \frac{1}{\sin v_+}) = (n + \frac{1}{2})\pi h$$

Note that such relations extends to the complex plane, hence defining the eigenvalues even when $\epsilon\mu \notin \mathbb{R}$. The equations we have to solve are

4. Analysis of the asymptotic regimes for cylindrical coordinates and proof of an estimate on the solution

In this Section, we derive estimates, both in the case $\Im\epsilon\mu < 0$ (where existence and uniqueness result of a solution in H^1 of the Dirichlet problem in the layer is proven using a variational formulation) and in the case $\epsilon\mu \in \mathbb{R}_+$, where one needs to avoid the resonances. The first subsection is devoted to the case $\epsilon\mu \in \mathbb{R}_+$, where we can rely on a classical analysis of second order ODEs.

4.1. Rigorous estimates of the solution for $\epsilon\mu \in \mathbb{R}_+$

In this Section, we treat the two following cases:

$$q(r) := -k_z^2 - \frac{p^2 - \frac{1}{4}}{r^2} + \epsilon\mu\omega^2 > 0, r \in [r_0, R] \tag{28}$$

$$k(r) := k_z^2 + \frac{p^2 - \frac{1}{4}}{r^2} - \epsilon\mu\omega^2 > 0, r \in [r_0, R], \tag{29}$$

The function $r^{\frac{1}{2}} J_p(k_3 r)$ satisfies the Whittaker equation

$$w'' = (k_z^2 + \frac{p^2 - \frac{1}{4}}{r^2} - \epsilon\mu\omega^2)w. \tag{30}$$

Let w be the unique solution of (30) satisfying $w(r_0) = 0$ and $w'(r_0) = 1$. Define $\theta(r)$ such that (ρ, θ) is the unique solution of (32) with $\rho(r_0) = 1, \theta(r_0) = 0$. We prove

Proposition 4. *For (p, k_z) satisfying (29), $0 \leq \frac{d_p(k_3 r)}{d_p(k_3 R)} \leq 1$ for all $r \in [r_0, R]$.*

For (p, k_z) satisfying (28),

$$|\frac{d_p(k_3 r)}{d_p(k_3 R)}| \leq \frac{C}{|\sin\theta(R)|}.$$

We consider in all what follows a solution w of (30) satisfying $w(r_0) = 0$. In both cases (k_z, p satisfying (29) or (28)) the function $\frac{d_p(k_3 r)}{d_p(k_3 R)}$ is equal, when defined, to $\sqrt{\frac{R}{r}}\frac{w(r)}{w(R)}$, even when $k_3 = i|k_3|$.

When k_z, p satisfies (29), one has

Lemma 8. *Let w be the unique solution of (30) in the case (29) satisfying $w(r_0) = 0, w'(r_0) = 1$.*

For all $r \in [r_0, R]$

$$w(r) \leq \frac{\sinh\sqrt{max_{[r_0,R]}k}(r - r_0)}{\sqrt{max_{[r_0,R]}k}}, w'(r) \leq \cosh\sqrt{max_{[r_0,R]}k}(r - r_0).$$

In addition, for all $r \in [r_0, R]$, the unique solution of (30) in the case (29) satisfying $w(r_0) = 0, w(R) = 1$ is strictly increasing, hence $w(r) \leq 1$ for all $r \in [r_0, R]$.

Proof. We first check that, in the case (29), there exists a neighborhood of r_0 such that $w(r) > 0$ for $r > r_0$, hence $w''(r) > 0$ when w' is increasing, hence for $r \geq r_0$, $w'(r) \geq 1$ hence $w(r) \geq r - r_0$.

Assume now that there exists $r_* > r_0$ such that $w(r_*) = 0$, and denote by the same symbol the smallest value of this r_*. There exists at least one point r_1 in $[r_0, r_*]$ where $w'(r_1) = 0$. If r_1 is the smallest greater than r_0, w' is positive on $[r_0, R_1]$ hence w is increasing hence w'' is positive hence w' is increasing: contradiction.

Finally, there is no point greater that r_0 for which w is zero. This proves that w is strictly positive. From $w'w" = k(r)ww'$, one deduces after integration $\frac{1}{2}((w'(r))^2 - 1) \leq \frac{1}{2}\max(k(r))(w(r))^2$.

Introducing $w(r) = \alpha \sinh \phi(r)$, one checks that, after the choice $\alpha^2 \max_{[r_0,R]}(k(r)) = 1$, that $(\phi')^2 \leq \max(k(r))_{[r_0,R]}$, $\phi(r_0) = 0$ hence the estimate on $\phi(r)$ and the inequalities of Lemma 8. □

Lemma 9. *Let w be a solution of (30) in the case (28) satisfying $w(r_0) = 0$. One has the inequality:*

$$(w'(r))^2 + q(r)(w(r))^2 \leq \frac{q(r)}{q(R)}((w'(R))^2 + q(R)(w(R))^2).$$

Proof. One checks that

$$\frac{d}{dr}[(w'(r))^2 + q(r)(w(r))^2] = 2w'(r)(w''(r) + q(r)w(r)) + q'(r)(w(r))^2 = q'(r)(w(r))^2 = \frac{q'(r)}{q(r)}(q(r)(w(r))^2),$$

hence, thanks to $\frac{q'}{q} > 0$,

$$\frac{d}{dr}[(w'(r))^2 + q(r)(w(r))^2] \leq \frac{q'(r)}{q(r)}[(w'(r))^2 + q(r)(w(r))^2],$$

from which one deduces $\frac{d}{dr}(\frac{(w'(r))^2+q(r)(w(r))^2}{q(r)}) \leq 0$ hence the inequality of Lemma 9.

□

In addition, if one introduces $\rho(r)\sin\theta(r) = w(r), \rho(r)\cos\theta(r) = \frac{w'(r)}{\sqrt{\epsilon\mu\omega^2 - k_z^2}}$, $\rho(r) > 0$, one shows that $\theta' > 0$, and one deduces

Lemma 10.
1. *The resonances are the solutions of* $\sin(\theta(R)) = 0$. *If ω is large, given, the number of points of resonances in $[r_0, R]$ is $O(\omega)$.*
2. *If ω is not a resonance, $\rho(R) = \frac{w(R)}{\sin(\theta(R))}$ is well defined.*

Proof. The second item is a La Pallice's statement, to acknowledge that the problem has a unique solution. This holds as well as the first part of the first item, because there is no solution of (30) satisfying $w(r_0) = 0$ and $w(R) = 1$ when $\sin(\theta(R)) = 0$.

The second part of the first item comes from the fact that there exists $\delta_0 > 0$ such that $\theta'(r) \geq \delta_0\omega$, hence $\theta(r) \geq \delta_0\omega(r - r_0)$. Hence, as there exists $O(\omega)$ points such that $\delta_0\omega(r - r_0) = n\pi$, the same result holds for θ (see Coddington-Levinson [7] Th. 1.1 Chap. 8). □

Of course we have the exact (but useless) form of the Dirichlet to Neumann operator in the hyperbolic regime for the mode p:

$$\frac{d}{dr}(\frac{d'_p(k_3 r)}{d_p(k_3 R)})|_{r=R} = k_3 \frac{d'_p(k_3 R)}{d_p(k_3 R)} = \omega\sqrt{\epsilon\mu - \eta^2}\frac{\cos\theta(R)}{\sin\theta(R)}. \tag{31}$$

One deduces that

Lemma 11. *If ω is not a resonance and if w solution of (30) satisfies $w(r_0) = 0$, $w(R) = 1$*

$$\forall r \in [r_0, R], (w(r))^2 + \frac{(w'(r))^2}{q(r)} \leq (w(R))^2 + \frac{(w'(R))^2}{q(R)} \leq 1 + \frac{\epsilon\mu\omega^2 - k_z^2}{q(R)}\frac{\cos(\theta(R))}{\sin(\theta(R))}$$

hence w is uniformly bounded on $[r_0, R]$.

This is a consequence of the system that ρ, θ satisfies (recall $k_z = \omega\eta$):

$$\begin{cases} \frac{\rho'}{\rho}(r) = \frac{\sin\theta\cos\theta}{\omega\sqrt{\epsilon\mu-\eta^2}}(\frac{p^2-\frac{1}{4}}{r^2}) \\ \theta'(r) = \omega\sqrt{\epsilon\mu-\eta^2}[\cos^2\theta + \frac{q(r)}{\omega^2\epsilon\mu-k_z^2}\sin^2\theta]. \end{cases} \tag{32}$$

Observe that we can deduce a name for each of the regimes described in (29) or (28) thanks to the estimates on the solution. The regime (28) leads to an oscillating solution $\rho(r)\sin(\theta(r))$, hence one calls this regime the **hyperbolic regime**. The regime (29) yields exponentially growing and decaying solutions, which renders the name **elliptic regime** soundful. This will be detailed in Subsection 4.4.

From these two items, one gets Proposition 4. It is a consequence of the two following items:

- In the elliptic case, there exists a unique solution of (30) such that $w(r_0) = 0$ and $w(R) = 1$. It is uniformly bounded by 1 on $[r_0, R]$.
- In the hyperbolic case, consider the unique solution of (32) such that $\rho(r_0) = 1$ and $\theta(r_0) = 0$. The resonances are the values of ω such that there exists a (r, n) such that $\theta(r) = n\pi$. If ω is not a resonance, then there exists a unique solution of (30) such that $w(r_0) = 0$, $w(R) = 1$. This solution satisfies thus

$$|w(r)| \leq \frac{C}{|\sin\theta(R)|}.$$

Proposition 4 implies that is in $C^1([r_0, R], L^\infty(\mathbf{R}\times\mathbb{N}))$ hence $w \in C^1([r_0, R], \mathcal{S}'\times l^\infty)$ hence it is the Fourier transform in θ, z of an element of $w \in C^1([r_0, R], \mathcal{S}'(\mathbf{R}\times[0, 2\pi])$.

When (p, k_z) neither satisfies (29) or (28), that is

$$-k_z^2 - \frac{p^2-\frac{1}{4}}{R^2} + \epsilon\mu\omega^2 > 0 > -k_z^2 - \frac{p^2-\frac{1}{4}}{r_0^2} + \epsilon\mu\omega^2,$$

denote by r_e the unique point such that $k(r_e) = 0$. Let w^* such that $w^*(r_e) > 0, (w^*)'(r_e) \geq 0$, w^* is strictly increasing on $[r_e, R]$, and the estimate reads $w^*(r_e) \leq w^*(r) \leq w^*(R)$ for all $r \in [r_e, R]$. One then checks that one is in the case (28) on $[r_0, r_e]$. Let w^* satisfying $w^*(r_e) = 0$, then $\frac{w^*(r)}{(w^*)'(r_e)}$ is also strictly increasing on $[r_e, R]$. Altogether, for w satisfying $w(r_0) = 0, w'(r_0) = 0$ we are in one of the two cases, and one can solve readily $w(R) = 1$.

Lemma 12. *There is no resonance for this point, and, in addition, we have an uniform bound of the unique solution w of the Whittaker equation (30) on* $[r_0, R]$.

Proof. The proof is identical to the proof of Lemma 20. □

The next section uses the asymptotic expansions of the Bessel functions to obtain estimates of the quantities introduced in the present section and deduce the Dirichlet to Neumann operator, as well as the conditions for resonances.

The second subsection deals with the case $\epsilon\mu$ complex, where the analysis of the complex system of ODEs is not straightforward.

4.2. Estimates on $\frac{d_p(k_3 r)}{d_p(k_3 R)}$ using asymptotic results on the Bessel functions.

The aim of this subsection is to prove the fundamental result

Proposition 5. *Assume $\Im \epsilon\mu < 0$, independent on ω. There exists a constant K such that*

$$\forall r \in [r_0, R], \forall (k_z, p) \in \mathbb{R} \times \mathbb{N}, |\frac{d_p(k_3 r)}{d_p(k_3 R)}| \leq K.$$

The proof of this Proposition is the aim of subsection 4.5 and one uses the known behavior of the Bessel functions to derive an result identical to (3), thanks to precise estimates on Bessel functions. The precise estimates are listed in Section 4.3, in five cases.

4.3. Bessel functions toolbox

In this subsection, we describe the representations of the Bessel functions that will be used in this paper. We concentrate mainly on $J_p + iY_p = H_p^{(1)}$ and $J_p - iY_p = H_p^{(2)}$. The variables z, β, a are defined in (44). We have the representations:

1. In the case of p fixed and of the argument of the Bessel functions large, let $\chi_p(Z) = Z - (\frac{1}{2}p + \frac{1}{4}\pi)$. There exists four functions P^*, Q^*, R^*, S^*, where one gets estimates of remainder terms in the asymptotic expansions, such that

$$H_p^{(1)}(Z) = \frac{2}{\pi Z}[P^*(p,Z) + iQ^*(p,Z)]e^{i\chi_p(Z)}, (H_p^{(1)})'(Z) = \frac{2}{\pi Z}[iR^*(p,Z) - S^*(p,Z)]e^{i\chi_p(Z)}, \quad (33)$$

and the conjugate solution which is $H_p^{(2)}$. We have in addition

$$P^*(p,Z) = 1+O(Z^{-2}), 8ZQ^*(p,Z) = 4p^2-1+O(Z^{-2}), R^*(p,Z) = 1+O(Z^{-2}), 8ZS*(p,Z) = 4p^2+3+O(Z^{-2}).$$

This one was used already in [4] and one notices that all the constants in $O(z^{-2})$ depend on p. More precisely, the expansion of P in inverse powers of Z is $\sum_{j\geq 0} Z^{-2j} a_j(p)$, where $a_j(p)$ is a polynomial of degree $2j$.

2. In the hyperbolic case $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r^2} > 0$, let $\Psi(p,\beta) = p(\tan\beta - \beta)$. From [1], (9.3.15) to (9.3.19), there exists four functions L, M, N, O depending on p and β such that

$$J_p(\frac{p}{\cos\beta})+iY_p(\frac{p}{\cos\beta}) = \sqrt{\frac{2}{\pi p \tan\beta}}(L-iM)e^{i\Psi}, J_p'(\frac{p}{\cos\beta})+iY_p'(\frac{p}{\cos\beta}) = i\sqrt{\frac{\sin 2\beta}{\pi p}}(N+iO)e^{i\Psi}, \quad (34)$$

where L, N, pM, pO have an asymptotic expansion in inverse powers of p^2 when p large, and $L(p,\beta) = 1 + O(p^{-2})$, $N(p,\beta) = 1 + O(p^{-2})$, $pM(p,\beta) = a(\beta) + O(p^{-1}), pO(p,\beta) = b(\beta) + O(p^{-1})$, where $a(\beta) = \frac{1}{8\tan\beta} + \frac{5}{24\tan^3\beta}$, $b(\beta) = \frac{3}{8\tan\beta} + \frac{7}{24\tan^3\beta}$.

3. In the elliptic case, namely $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r^2} < 0$, we use 10.20 in [8]. Note that z and ζ introduced in (44) can be extended to $z \in \mathbb{C}$. There exists A, B, C, D, continuous functions on ζ, depending on p, such that

$$\begin{cases} H_p^{(1)}(pz) \simeq 2e^{-\frac{i\pi}{3}}(\frac{\zeta}{1-z^2})^{\frac{1}{4}}[p^{-\frac{1}{3}}Ai(e^{\frac{2i\pi}{3}}p^{\frac{2}{3}}\zeta)A(\zeta,p) - p^{-\frac{5}{3}}e^{\frac{2i\pi}{3}}Ai'(e^{\frac{2i\pi}{3}}p^{\frac{2}{3}}\zeta)\zeta^{-\frac{1}{2}}B(\zeta,p)] \\ (H_p^{(1)})'(pz) \simeq \frac{4e^{-\frac{2i\pi}{3}}}{z}(\frac{1-z^2}{\zeta})^{\frac{1}{4}}[-p^{-\frac{4}{3}}e^{-\frac{2i\pi}{3}}Ai(e^{\frac{2i\pi}{3}}p^{\frac{2}{3}}\zeta)\zeta^{\frac{1}{2}}C(\zeta,p) + p^{-\frac{2}{3}}Ai'(e^{\frac{2i\pi}{3}}p^{\frac{2}{3}}\zeta)D(\zeta,p)], \end{cases} \quad (35)$$

where

$$A(\zeta,p)=\sum_{k=0}^{+\infty}\frac{A_k(\zeta)}{p^{2k}},$$

and similar uniform expansions for B, C, D (the notations for B and C are slightly different to the ones appearing in10.20 in [8], $A_0(\zeta)=D_0(\zeta)=v_0, B_0(\zeta)=C_0(\zeta)=u_0$).

First notice that $a(p,\zeta)=\frac{Ai(p^{\frac{2}{3}}\zeta)}{Bi(p^{\frac{2}{3}}\zeta)}$ is exponentially small as $\omega\to+\infty$ when $\Re\zeta^{\frac{3}{2}}>0$ and that $b(p,\zeta)=\frac{Bi(p^{\frac{2}{3}}\zeta)}{Ai(p^{\frac{2}{3}}\zeta)}$ is exponentially small as $\omega\to+\infty$ when $\Re\zeta^{\frac{3}{2}}<0$.

Introduce

$$k_{\pm}(p,\zeta)=-\zeta^{-\frac{1}{2}}p^{-\frac{4}{3}}e^{\pm 2i\frac{\pi}{3}}\frac{Ai'}{Ai}(e^{\pm\frac{2}{3}i\pi}p^{\frac{2}{3}}\zeta)\frac{B(\zeta,p)}{A(\zeta,p)},$$

$$\tau_{\pm}(p,\zeta)=-\zeta^{\frac{1}{2}}p^{-\frac{2}{3}}e^{\mp 2i\frac{\pi}{3}}\frac{Ai}{Ai'}(e^{\pm\frac{2}{3}i\pi}p^{\frac{2}{3}}\zeta)\frac{C(\zeta,p)}{D(\zeta,p)},$$

which both satisfy uniformly $k_{\pm}=O(p^{-1}), \tau_{\pm}=O(p^{-1})$. More precisely one has

$$k_{\pm}(p,\zeta)=-p^{-1}\frac{u_0}{v_0}(1+O(p^{-1})), \tau_{\pm}(\zeta,p)=-p^{-1}\frac{u_0}{v_0}(1+O(p^{-1})).$$

Estimates (35) yield

$$H_p^{(1)}(pz)=2e^{-i\frac{\pi}{3}}p^{-\frac{1}{3}}(\frac{\zeta}{1-z^2})^{\frac{1}{4}}Ai(e^{\frac{2i\pi}{3}}p^{\frac{2}{3}}\zeta)A(p,\zeta)[1+k_+(p,\zeta)],$$

and a similar estimate on $H_p^{(2)}(pz)$.

We also observe that

$$Ai(e^{\frac{2i\pi}{3}}p^{\frac{2}{3}}\zeta)=\frac{1}{2}e^{i\frac{\pi}{3}}Bi(p^{\frac{2}{3}}\zeta)(-i+a(\zeta))=\frac{1}{2}e^{i\frac{\pi}{3}}Ai(p^{\frac{2}{3}}\zeta)(1-ib(\zeta)), \tag{36}$$

from which one deduces the behavior of $H_p^{(j)}(pz)$ according to the sign of $\Re\zeta^{\frac{3}{2}}$:

$$\begin{aligned}H_p^{(1)}(pz) &= (\tfrac{\zeta}{1-z^2})^{\frac{1}{4}}p^{-\frac{1}{3}}A(\zeta,p)(1+k_+(p,\zeta)]Bi(p^{\frac{2}{3}}\zeta)(-i+a(\zeta))\\ &= (\tfrac{\zeta}{1-z^2})^{\frac{1}{4}}p^{-\frac{1}{3}}A(\zeta,p)(1+k_+(p,\zeta))Ai(p^{\frac{2}{3}}\zeta)(1-ib(\zeta)),\end{aligned} \tag{37}$$

not writing the similar estimate on $H_p^{(2)}(pz)$ (one needs to replace k_+ by k_-, and all the conjugate coefficients), stressing the role of the sign of $\Re\zeta^{\frac{3}{2}}$: the equivalent of $H_p^{(1)}(pz)$ for $\Re\zeta^{\frac{3}{2}}>0$ is $-i(\frac{\zeta}{1-z^2})^{\frac{1}{4}}p^{-\frac{1}{3}}A(\zeta,p)Bi(p^{\frac{2}{3}}\zeta)$ and is $(\frac{\zeta}{1-z^2})^{\frac{1}{4}}p^{-\frac{1}{3}}A(\zeta,p)Ai(p^{\frac{2}{3}}\zeta)$ for $\Re\zeta^{\frac{3}{2}}<0$.

Introducing also $\tilde{a}(\zeta)=\frac{Ai'(p^{\frac{2}{3}}\zeta)}{Bi'(p^{\frac{2}{3}}\zeta)}$, $\tilde{b}(\zeta)=\frac{Bi'(p^{\frac{2}{3}}\zeta)}{Ai'(p^{\frac{2}{3}}\zeta)}$, one gets

$$\begin{aligned}(H_p^{(1)})'(pz) &= \tfrac{2e^{-\frac{2i\pi}{3}}}{z}(\tfrac{1-z^2}{\zeta})^{\frac{1}{4}}p^{-\frac{2}{3}}D(\zeta,p)(1+\tau_+(p,\zeta)]Bi'(p^{\frac{2}{3}}\zeta)(-i+\tilde{a}(\zeta))\\ &= \tfrac{2e^{-\frac{2i\pi}{3}}}{z}(\tfrac{1-z^2}{\zeta})^{\frac{1}{4}}p^{-\frac{2}{3}}D(\zeta,p)(1+\tau_+(p,\zeta)]Ai'(p^{\frac{2}{3}}\zeta)(1-i\tilde{b}(\zeta)),\end{aligned} \tag{38}$$

It will be useful as well to obtain the following equalities on $Y_p(pz)$ and $J_p(pz)$, using (10.20.4) and (10.20.5) of [8] and the expressions (different from the definitions mentioned) $A(\zeta,p)=v_0+O(p^{-2})$, $B(\zeta,p)=u_0+O(p^{-2})$, $C(\zeta,p)=v_0+O(p^{-2})$, $D(\zeta,p)=u_0+O(p^{-2})$, $u_0=v_0=1$:

$$\left\{\begin{array}{ll} J_p(pz) & = (\frac{4\zeta}{1-z^2})^{\frac{1}{4}} p^{-\frac{1}{3}} [A(\zeta,p)Ai(p^{\frac{2}{3}}\zeta) - p^{-\frac{4}{3}}\zeta^{-\frac{1}{2}} B(\zeta,p)Ai'(p^{\frac{2}{3}}\zeta)] \\ & = (\frac{4\zeta}{1-z^2})^{\frac{1}{4}} p^{-\frac{1}{3}} A(\zeta,p)(1+k(p,\zeta))Ai(p^{\frac{2}{3}}\zeta) \\ Y_p(pz) & = -(\frac{4\zeta}{1-z^2})^{\frac{1}{4}} p^{-\frac{1}{3}} [A(\zeta,p)Bi(p^{\frac{2}{3}}\zeta) - p^{-\frac{4}{3}}\zeta^{-\frac{1}{2}} B(\zeta,p)Bi'(p^{\frac{2}{3}}\zeta)] \\ & = -(\frac{4\zeta}{1-z^2})^{\frac{1}{4}} p^{-\frac{1}{3}} A(\zeta,p)(1+\tilde{k}(p,\zeta))Bi(p^{\frac{2}{3}}\zeta) \\ J_p'(pz) & = -\frac{2}{z}(\frac{1-z^2}{4\zeta})^{\frac{1}{4}} p^{-\frac{2}{3}} [D(\zeta,p)Ai'(p^{\frac{2}{3}}\zeta) - p^{-\frac{2}{3}}\zeta^{\frac{1}{2}} C(\zeta,p)Ai(p^{\frac{2}{3}}\zeta)] \\ & = -\frac{2}{z}(\frac{1-z^2}{4\zeta})^{\frac{1}{4}} p^{-\frac{2}{3}} D(\zeta,p)(1+\tau(p,\zeta))Ai'(p^{\frac{2}{3}}\zeta) \\ Y_p'(pz) & = \frac{2}{z}(\frac{1-z^2}{4\zeta})^{\frac{1}{4}} p^{-\frac{2}{3}} [D(\zeta,p)Bi'(p^{\frac{2}{3}}\zeta) - p^{-\frac{2}{3}}\zeta^{\frac{1}{2}} C(\zeta,p)Bi(p^{\frac{2}{3}}\zeta)] \\ & = \frac{2}{z}(\frac{1-z^2}{4\zeta})^{\frac{1}{4}} p^{-\frac{2}{3}} D(\zeta,p)(1+\tilde{\tau}(p,\zeta))Bi'(p^{\frac{2}{3}}\zeta), \end{array}\right. \tag{39}$$

where this equality defines $k, \tilde{k}, \tau, \tilde{\tau}$ satisfy $k(\zeta,p) = p^{-1} + O(p^{-2})$, $\tilde{k}(\zeta,p) = -p^{-1} + O(p^{-2})$, $\tau(p,\zeta)) = p^{-1} + O(p^{-2})$, $\tilde{\tau}(p,\zeta)) = -p^{-1} + O(p^{-2})$, in order to deal with the totally elliptic case. We have

Lemma 13. *Let $\delta_* > 0$ be arbitrary. For all ζ, $|\zeta| \geq 1$ and $|arg(\zeta)| \leq \pi - \delta_*$*

- *The function $Ai(p^{\frac{2}{3}}\zeta)(p^{\frac{2}{3}}\zeta)^{\frac{1}{4}} e^{\frac{2}{3}p\zeta^{\frac{3}{2}}}$ is bounded uniformly,*
- *The function $Bi(p^{\frac{2}{3}}\zeta)(p^{\frac{2}{3}}\zeta)^{\frac{1}{4}} e^{-\frac{2}{3}p\zeta^{\frac{3}{2}}}$ is bounded uniformly,*
- *The function $(p^{\frac{2}{3}}\zeta)^{\frac{1}{2}} \frac{Ai}{Ai'}(p^{\frac{2}{3}}\zeta)$ is bounded uniformly,*
- *The function $(p^{\frac{2}{3}}\zeta)^{\frac{1}{2}} \frac{Bi}{Bi'}(p^{\frac{2}{3}}\zeta)$ is bounded uniformly.*

All these quantities have an uniform expansion in powers in $(p^{\frac{2}{3}}\zeta)^{-\frac{3}{2}}$.
All these terms have a finite limit $p \to +\infty$.

4. In the glancing case (namely $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r^2} \simeq 0$), there exists four functions P, Q, R, S depending on a, p, defined in 10.19.9 of [8] such that

$$P(a,p) = \sum_{k=0}^{+\infty} \frac{P_k(a)}{p^{\frac{2k}{3}}}$$

and similar equalities for Q, R, S such that one has the following equalities (and the conjugate ones), recalling $Ai(-X) - iBi(-X) = \frac{1}{2}e^{-i\frac{\pi}{3}}Ai(e^{-i\frac{\pi}{3}}X)$ and $Ai'(-X) - iBi'(-X) = \frac{1}{2}e^{i\frac{\pi}{3}}Ai'(e^{-i\frac{\pi}{3}}X)$:

$$\left\{\begin{array}{ll} H_p^{(1)}(p + ap^{\frac{1}{3}}) & = \frac{2^{\frac{1}{3}}}{p^{\frac{1}{3}}}(Ai - iBi)(-2^{\frac{1}{3}}a)P + \frac{2^{\frac{2}{3}}}{p}(Ai' - iBi')(-2^{\frac{1}{3}}a)Q \\ & = \frac{2^{\frac{4}{3}}}{p^{\frac{1}{3}}}e^{-i\frac{\pi}{3}}Ai(e^{-i\frac{\pi}{3}}2^{\frac{1}{3}}a)P + \frac{2^{\frac{5}{3}}}{p}e^{i\frac{\pi}{3}}Ai'(e^{-i\frac{\pi}{3}}2^{\frac{1}{3}}a)Q \\ (H_p^{(1)})'(p + ap^{\frac{1}{3}}) & = -\frac{2^{\frac{1}{3}}}{p^{\frac{2}{3}}}(Ai'(-2^{\frac{1}{3}}a) - iBi'(-2^{\frac{1}{3}}a))R + \frac{2^{\frac{1}{3}}}{p^{\frac{4}{3}}}(Ai(-2^{\frac{1}{3}}a) - iBi(-2^{\frac{1}{3}}a))S \\ & = -\frac{2^{\frac{5}{3}}}{p^{\frac{2}{3}}}e^{i\frac{\pi}{3}}Ai'(e^{-i\frac{\pi}{3}}2^{\frac{1}{3}}a)R + \frac{2^{\frac{1}{3}}}{p^{\frac{4}{3}}}e^{-i\frac{\pi}{3}}Ai(e^{-i\frac{\pi}{3}}2^{\frac{1}{3}}a)S. \end{array}\right. \tag{40}$$

One has the following asymptotic estimates on P, Q, R, S:

$$P = 1 - \frac{a}{5p^{\frac{2}{3}}} + O(p^{-\frac{4}{3}}), Q = \frac{3}{10}a^2 + O(p^{-\frac{2}{3}}), R = 1 - \frac{4a}{5p^{\frac{2}{3}}} + O(p^{-\frac{4}{3}}), S = \frac{3}{5}a^2 - \frac{1}{5} + O(p^{-\frac{2}{3}}).$$

The first case, which is the most classical one, will nevertheless be of less use than the others in this analysis. It is the case of a fixed p and of $k_3 r$ large. It relies on the equalities (33), which yield:

$$\frac{d_p(k_3 r)}{d_p(k_3 R)} = \frac{R}{r} e^{-ik_3(R-r)} \frac{e^{-2ik_3(r-r_0)}(P_* - iQ_*)(k_3 r_0)(P_* + iQ_*)(k_3 r) - (P_* + iQ_*)(k_3 r_0)(P_* - iQ_*)(k_3 r)}{e^{-2ik_3(R-r_0)}(P_* - iQ_*)(k_3 r_0)(P_* + iQ_*)(k_3 R) - (P_* + iQ_*)(k_3 r_0)(P_* - iQ_*)(k_3 R)}.$$

As $\Im k_3 < 0$, one has $|e^{-2ik_3(R-r_0)}| = e^{2\Im k_3(R-r_0)} \leq 1$ and $|e^{-2ik_3(r-r_0)}| \leq 1$, along with $|e^{-ik_3(R-r)}| \leq 1$. Hence there exists $C(p)$ such that, for $\omega \geq 1$

$$|\frac{d_p(k_3 r)}{d_p(k_3 R)}| \leq C(p), \forall k_z, \forall r \in [r_0, R].$$

This bound is not uniform in p, hence this result shall only be able to control $\sum_{p=0}^{P} a_p(k_z) e^{ip\theta}$ in k_z in order to perform an inverse Fourier transform, but cannot be used to obtain a behavior in $(p; k_z)$ for the regularity when $(p, k_z) \to +\infty$.

We need, for dealing with (p, k_z), to consider p AND k_z large. In classical situations, this corresponds to what can be called the **high frequency regime**, where both k_z and $\frac{p}{r}$ are of order of magnitude ω. It is however necessary to have estimates for all $(k_z, p) \in \mathbb{R} \times \mathbb{N}$,

We rely on the following definition, based on $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2} > \Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r_0^2}$ for all (p, k_z), under the condition

$$\Re\epsilon\mu\omega^2 > k_z^2 : \tag{41}$$

Definition 3. *One has the following classification on (p, k_z) (it is a classification on the wave vector or dual vector)*

1. *One is in the totally elliptic region when $0 > \Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}$.*
2. *One is in the mixed-elliptic-hyperbolic region when $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2} > 0 > \Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r_0^2}$.*
3. *One is in the totally hyperbolic region when $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r_0^2} > 0$.*

In cases 2 and 3, one has

$$k_z^2 + \frac{p^2}{R^2} \leq C\omega^2. \tag{42}$$

One denotes by $k_\perp$, $\Im k_\perp < 0$ (as in the plane layer)

$$k_\perp = \sqrt{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}} \tag{43}$$

An explanation of the names of the zones is provided below in Section 4.4. Note for the moment that we adopted a classification similar to what is deduced from the behavior outlined in Proposition 4 using inequalities (29) or (28).

Let us prove (42).

In the case 'mixed elliptic hyperbolic', it is enough to choose $C = \Re\epsilon\mu$.

In the case 'totally hyperbolic', one has

$$k_z^2+\frac{p^2}{r_0^2}<k_z^2+\frac{\Re\epsilon\mu\omega^2-k_z^2}{r_0^2}R^2\leq\Re\epsilon\mu\frac{R^2}{r_0^2}\omega^2,$$

hence $C=\Re\epsilon\mu\frac{R^2}{r_0^2}$ is possible. We will thus introduce the following variables, which replace the variable r in the asymptotic analysis in (ω,k_z,p):

$$\begin{array}{ll}k_z^2+\frac{p^2}{r^2}>\Re\epsilon\mu\omega^2: & z=\frac{k_3r}{p},\zeta=(\frac{3}{2}\int_z^1\frac{\sqrt{1-t^2}}{t}dt)^{\frac{2}{3}},\int_z^1\frac{\sqrt{1-t^2}}{t}dt=\ln\frac{1+\sqrt{1-z^2}}{z}-\sqrt{1-z^2}\\ k_z^2+\frac{p^2}{r^2}<\Re\epsilon\mu\omega^2: & \frac{p}{\cos\beta}=k_3r,\sin\beta=\frac{\sqrt{\epsilon\mu\omega^2-k_z^2-\frac{p^2}{r^2}}}{k_3},\Re\sin\beta>0\\ k_z^2+\frac{p^2}{r^2}\simeq\Re\epsilon\mu\omega^2: & k_3r=p+ap^{\frac{1}{3}}.\end{array}\tag{44}$$

Remark 2. *In the hyperbolic, mixed, or elliptic regimes, β,z,ζ, when introduced, have a finite limit when $\omega\to+\infty$. In the glancing regime and $\Im\epsilon\mu$ independent on ω, $a=O(\omega^{\frac{2}{3}})$ and we are not in the range of application of this regime. If $\Im\epsilon\mu=\frac{\sigma}{i\omega}$, we are in the range $a=O(\omega^{\frac{1}{6}})$ for $\Re\epsilon\mu-\frac{k_z^2}{\omega^2}-\frac{p^2}{r^2\omega^2}=0$.*

Remark 3. *For $\Re\epsilon\mu\omega^2-k_z^2-\frac{p^2}{r^2}\geq0$, introduce a_0 and b_0 such that $k_3=a_0+ib_0$, $\alpha_0(r)$ and $\beta_0(r)$ such that $k_3\sin\beta=\alpha_0(r)+i\beta_0(r)$, and $a(r)$ and $b(r)$ such that $\sin\beta=a(r)+ib(r)$.*

For $\Re\epsilon\mu\omega^2-k_z^2-\frac{p^2}{r^2}\leq0$, introduce $p(r),q(r)$ such that $p\zeta^{\frac{1}{2}}\frac{d\zeta}{dr}=-(p(r)+iq(r))$.

They are uniquely determined through $b_0<0$, $\alpha_0(r)>0$ (condition crucial for the analysis, coming from the assumption $\sin\beta>0$ when $\epsilon\mu\in\mathbf{R}$ (see [8])) and $p(r)>0$ (choice of the determination of the square root as indicated in [8]), and one has

$$\beta_0(r)=\frac{p^2\Im\epsilon\mu\omega^2}{r^2|k_3|^4r\alpha_0(r)}<0,q(r)=-\frac{\Im\epsilon\mu\omega^2}{2p(r)}>0.$$

Proof. Let $\Re\epsilon\mu\omega^2-k_z^2-\frac{p^2}{r^2}>0$. One has $k_3^2\sin^2\beta=k_3^2-k_3^2\cos^2\beta=k_3^2-\frac{p^2}{r^2}=k_\perp^2$. Moreover $\sin^2\beta=1-\frac{p^2}{r^2k_3^2}=1-\frac{p^2\bar{k}_3^2}{r^2|k_3|^4}$, from which one deduces (imaginary part)

$$2a(r)b(r)=\frac{2a_0b_0p^2}{r^2|k_3|^4}.$$

As $2a_0b_0=\Im\epsilon\mu\omega^2<0$, one has $a(r)b(r)<0$ hence $b(r)<0$. Observe then that $\Im k_3\sin\beta=b_0a(r)+a_0b(r)$, which yields $\beta_0=\Im k_3\sin\beta<0$.

Let $\Re\epsilon\mu\omega^2-k_z^2-\frac{p^2}{r^2}<0$. From $pz=k_3r$, and $\frac{2}{3}\zeta^{\frac{3}{2}}=\int_z^1\frac{\sqrt{1-t^2}}{t}dt$, one deduces $\zeta^{\frac{1}{2}}\frac{d\zeta}{dr}=-\frac{\sqrt{1-z^2}}{z}\frac{d\zeta}{dr}=-\frac{\sqrt{1-z^2}}{z}\frac{k_3}{p}$.

One deduces (observe that one uses $\lambda\sqrt{e}=\sqrt{\lambda^2e}$ for all $\lambda>0$)

$$p\zeta^{\frac{1}{2}}\frac{d\zeta}{dr}=-p\frac{\sqrt{1-\frac{k_3^2r^2}{p^2}}}{\frac{k_3r}{p}}\frac{k_3}{p}=-p\frac{\sqrt{1-\frac{k_3^2r^2}{p^2}}}{r}=-\sqrt{\frac{p^2}{r^2}-k_3^2}=-\sqrt{\frac{p^2}{r^2}+k_z^2-\epsilon\mu\omega^2}.$$

Hence $2p(s)q(s)=-\Im\epsilon\mu\omega^2$, which gives the second result.

□

Let $r_e > 0$ solve

$$\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r^2} = 0, \tag{45}$$

id est $r_e = \frac{\sqrt{\Re\epsilon\mu\omega^2 - k_z^2}}{p} R$. Fixing $\eta = \frac{k_z}{\omega}$ and $q = \frac{p}{\omega R}$ (dimensionless numbers), we observe

$$r_e = \frac{\sqrt{\Re\epsilon\mu - \eta^2}}{q} R.$$

Lemma 14. 1. *The identity holds true*

$$i\Psi(r_e) = -\frac{2}{3} p\zeta^{\frac{3}{2}}(r_e).$$

2. *We have* $\Re\frac{2}{3} p\zeta^{\frac{3}{2}}(r_e) < 0$.
3. $\Re i\Psi(\beta(r)) > 0$ *for all* $r > r_e$, *and is an increasing function on* $r > r_e$,
4. *There exists* $r_- < r_e$ *such that* $\Re\frac{2}{3} p\zeta_e^{\frac{3}{2}} = \int_{r_-}^{r_e} p(s)ds$ *for* $r < r_-$, *and* $\Re\frac{2}{3}\zeta^{\frac{3}{2}} > 0$ *for* $r > r_-$, *and it is a decreasing function of* r.

Remark that the point $r = r_e$ is the only point where the limit of Φ and the limit of ζ can be computed simultaneously according to our definitions.

Proof. Let $\psi(r) = \Psi(\beta(r))$. One has $\psi'(r) = \frac{d\beta}{dr}\Psi'(\beta(r)) = \frac{d\beta}{dr} p\tan^2\beta(r)$. Using $-\sin\beta(r)\frac{d\beta}{dr} = -\frac{k_3}{pr^2}$, one gets

$$\psi'(r) = k_3 \sin\beta(r) = \alpha(r) + i\beta(r).$$

Let $g(r) = \frac{2}{3} p\zeta^{\frac{3}{2}}(z(r))$. One already obtained

$$g'(r) = -(p(r) + iq(r)).$$

• For all r such that $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r^2} \geq 0$, $\alpha^2(r) - \beta^2(r) = \Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r^2}$, $2\alpha\beta = \Im\epsilon\mu\omega^2$, $\alpha > 0$, $\beta < 0$.

• For all r such that $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r^2} \leq 0$, $p^2(r) - q^2(r) = \frac{p^2}{r^2} + k_z^2 - \Re\epsilon\mu\omega^2$, $2pq = -\Im\epsilon\mu\omega^2$, $p > 0, q > 0$. One deduces $\alpha^2 - \beta^2 = q^2 - p^2$, $\alpha\beta = -pq$.

One checks that $\alpha^2(r_e) = q^2(r_e)$, $\beta^2(r_e) = q^2(r_e)$, $2\alpha(r_e)\beta(r_e) = -2p(r_e)q(r_e)$, hence, using the sign conditions, $\alpha(r_e) = q(r_e), \beta(r_e) = -p(r_e)$ and $i(\alpha(r_e) + i\beta(r_e)) = p(r_e) + iq(r_e)$.

The function Ψ satisfies $\Psi(0) = 0$, hence the complex continuation of ψ satisfies $\psi(r_*) = 0$ for $r_* \in \mathbb{C}$ such that $\beta(r_*) = \alpha(r_*) = 0$, that is r_* given by $k_3 r_* = p$. One deduces $i\psi(r_e) = \int_{r_*}^{r_e}(-\beta(s) + i\alpha(s))ds$. Similarly, the function ζ satisfies $\frac{2}{3} p\zeta^{\frac{3}{2}}(1) = 0$, hence $g(r) := \frac{2}{3} p\zeta^{\frac{3}{2}}(z(r))$ satisfies $g(r_*) = 0$ (point such that $z = 1$ *id est* $p = k_3 r_*$ as well). The equality obtained above shows $g(r_e) = -\int_{r_*}^{r_e}(p(s) + iq(s))ds$. Using the identity $p(s) = -\beta(s)$, $q(s) = \alpha(s)$, one obtains

$$g(r_e) = -i\psi(r_e).$$

This summarizes as the first item of Lemma 14.

For the second item, introduce $\alpha = \frac{-\Im\epsilon\mu\omega^2}{\sqrt{\Re\epsilon\mu\omega^2 - k_z^2}} > 0$. From $\frac{2}{3}\zeta_e^{\frac{3}{2}} = \int_{z_e}^1 \frac{\sqrt{1-t^2}}{t} dt = \int_{\sqrt{1-i\alpha}}^1 \frac{\sqrt{1-t^2}}{t} dt$, and denoting by $iu = 1 - t^2$, *id est* $dt = -\frac{idu}{2t}$, one obtains

$$\frac{2}{3}\zeta_e^{\frac{3}{2}} = i^{\frac{3}{2}} \int_0^\alpha \frac{\sqrt{u}}{2(1-iu)} du = i^{\frac{3}{2}} \int_0^\alpha \frac{\sqrt{u}(1+iu)}{2(1+u^2)} du = i^{\frac{3}{2}} \int_0^\alpha \frac{\sqrt{u}}{2(1+u^2)} du + i^{\frac{5}{2}} \int_0^\alpha \frac{u\sqrt{u}}{2(1+u^2)} du$$

where $i^{\frac{3}{2}}$ and $i^{\frac{5}{2}}$ are both of negative real part, hence the inequality.

For the expressions of $\alpha(r)$ and $\beta(r)$, one has

$$\alpha^2(r) - \beta^2(r) = \frac{p^2}{r_e^2} - \frac{p^2}{r^2}, \alpha^2(r) + \beta^2(r) = |i\Im\epsilon\mu\omega^2 + \frac{p^2}{r_e^2} - \frac{p^2}{r^2}|$$

which extends the expression of $\alpha(r), \beta(r)$ for $r < r_e$, and one observes that, extending also the same expression for $p(r)$ and $q(r)$, $i(\alpha(r) + i\beta(r)) = p(r) + iq(r)$. We may thus notice that one could formally deduce the identity $i\Psi(\beta(r)) = -\frac{2}{3}p\zeta^{\frac{3}{2}}(z(r))$ for all r, by extending the equalities defining p, q, α, β.

The third item uses $\Re i\Psi(r) = \Re i\Psi(r_e) - \int_{r_e}^{r} \beta(s)ds$, with $\beta < 0$, $r > r_e$ and $\Re i\Psi(r_e) > 0$, which implies that, for all $r > r_e$, $\Re i\Psi(r) \geq \Re i\Psi(r_e) > 0$.

For the last item, use $\Re\frac{2}{3}p\zeta^{\frac{3}{2}}(z(r)) = \Re\frac{2}{3}p\zeta_e^{\frac{3}{2}} - \int_{r_e}^{r} p(s)ds = \Re\frac{2}{3}p\zeta_e^{\frac{3}{2}} + \int_{r}^{r_e} p(s)ds$. As there exists $\delta_0 > 0$ such that $\delta_0 \leq sp(s) \leq 2\delta_0$ for s large enough, $\int_r^{r_e} p(s)ds \to +\infty$ when $r \to 0_+$, there is a unique solution to $\Re\frac{2}{3}p\zeta_e^{\frac{3}{2}} + \int_r^{r_e} p(s)ds = 0$, that one calls r_-. As $\Re\frac{2}{3}\Re\zeta^{\frac{3}{2}}$ is strictly decreasing, $\Re\frac{2}{3}\Re\zeta^{\frac{3}{2}} > 0$ for $r < r_-$ and $\Re\frac{2}{3}\Re\zeta^{\frac{3}{2}} < 0$ for $r > r_-$.

Remark 4. *We could also have deduced the equalities above, using the fact that $\zeta(z)$ is solution of the ODE*

$$\zeta(\frac{d\zeta}{dz})^2 = \frac{1 - z^2}{z^2},$$

which transforms, with $z = \frac{k_3 r}{p}$, into

$$p\zeta(\frac{d\zeta}{dr})^2 = \frac{p^2}{r^2} + k_z^2 - \epsilon\mu\omega^2 = (p(r) + iq(r))^2.$$

□

It is useful to introduce and recall the following notations:

$$k_3 = \sqrt{\epsilon\mu\omega^2 - k_z^2}, k_\perp = \sqrt{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}}, k_l = \frac{p}{R}, \cos\beta_* = \frac{k_l}{k_3}, \sin\beta_* = \frac{k_\perp}{k_3}, \tan\beta_* = \frac{k_\perp}{k_l}. \quad (46)$$

4.4. Eikonal equations.

We rely on the previous asymptotics of the Bessel and Hankel functions in the case where there exists $K > 0$ such that $k_z^2 + \frac{p^2}{r^2} \leq K\omega^2$. It is always the case, as it was noticed, when one is in the hyperbolic, top glancing, bottom glancing, and mixed hyperbolic elliptic regimes on $[r_0, R]$, and we enforce it additionally when $\frac{p^2}{R^2} + k_z^2 > \Re\epsilon\mu\omega^2$. This is called the asymptotic regime[3]. The PDO that one considers is

$$r^{-1}\partial_r(r\partial_r) + \partial_{z^2}^2 + r^{-2}\partial_{\theta^2}^2 + \omega^2\epsilon\mu.$$

After a change of unknown of the form $v = r^{\frac{1}{2}}u$ and a change of notation $r = R + n$, it writes, in the high-frequency regime, as the action of the operator $\partial_{n^2}^2 + R(n, i^{-1}\omega^{-1}\partial_z, i^{-1}\partial_\theta)$ (k_z as the dual variable of z and p as the (discrete) dual variable of θ):

[3]We avoid the case where k_z, p grow more than ω

$$R(n, k_z, p) = \epsilon\mu\omega^2 - k_z^2 - \frac{p^2 - \frac{1}{4}}{(n+R)^2}.$$

The eikonal equation associated with this PDE is thus

$$(\partial_r \Phi)^2 + (R+n)^{-2}(\partial_\theta \Phi)^2 + \epsilon\mu\omega^2 - k_z^2 = 0. \tag{47}$$

In the classical analysis of operators of the form $\partial_{n^2}^2 + R(n, x, i^{-1}\partial_x)$, in the case $\epsilon\mu \in \mathbb{R}_+^*$ for which the principal symbol is real, the elliptic region is characterized as the set of points $(x_0, \xi_0) \in T^*\partial\Omega$ for which $R(0, x_0, \xi_0) < 0$. In the case of the Helmholtz equation in cylindrical coordinates, we observe that, considering p as the discrete Fourier variable in θ (equivalent to $\frac{p}{R}$ Fourier variable for the curvilinear absciss $s = R\theta$), this corresponds, when $\epsilon\mu \in \mathbb{R}_+^*$, to $\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2} < 0$. In a similar fashion, the hyperbolic region is characterized by $\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2} > 0$, and the glancing region is characterized by $\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2} = 0$. In the latter case, as $\partial_n R(0, k_z, p) = \frac{2p^2}{R^3} > 0$, we fall in the category of **strictly diffractive points** (which is not suprising, because this corresponds to the model problem of the strictly diffractive analysis).

Even though the classical decomposition of the space of (k_z, p) into elliptic, hyperbolic, and glancing region respectively for $R(0, k_z, p) < 0$, $R(0, k_z, p) > 0$ and $R(0, k_z, p) = 0$, where $n = 0$ is equivalent to $r = R$ is valid for a real principal symbol, we shall use the same terminology[4], **by convention**, in the case $\epsilon\mu \notin \mathbb{R}$ is replaced by $\Re(\epsilon\mu)$:

- Elliptic regime: $k_z^2 + \frac{p^2}{R^2} > \Re\epsilon\mu\omega^2$
- Hyperbolic regime: $k_z^2 + \frac{p^2}{R^2} \leq \Re\epsilon\mu\omega^2$

The glancing region is not of use here because the distance of $k_z^2 + \frac{p^2}{R^2}$ to $\epsilon\mu\omega^2$ is always of order ω^2 at least. We have:

• Elliptic region: $\frac{\Re k_3 R}{p} < 1$. One has thus to study asymptotics of $H_p^{(j)}(pz)$, where z is defined in (44). We use the estimates (37) of Section 4.3. The asymptotic solutions use the Airy functions $Ai(e^{\pm\frac{2i\pi}{3}} p^{\frac{2}{3}}\zeta)$, associated with the phase

$$G(\theta, r) = p\theta \pm i\frac{2}{3} p\zeta^{\frac{3}{2}},$$

where ζ is defined in (44).

• Hyperbolic region: we consider β defined in (44).

We deduce from the asymptotic expansion of the Hankel functions an equivalent for the Dirichlet to Neumann operator is a consequence of the analysis of $H_p^{(1)}(k_3 r)$ when p and k_3 are of the same order. Following Abramovitz and Stegun [1], one has the following phase for the Bessel functions (estimates (34) of Section 4.3)

$$\Psi(p, \beta) = p(\tan\beta - \beta).$$

and the total phase is

$$\Phi(\theta, r) = \Psi(p, \beta) + p\theta.$$

[4] we can use a limiting absorption principle with $\Im\epsilon\mu \to 0-$

Lemma 15. *For each (k_z, p), define r_e such that $\Re\epsilon\mu\omega^2 - k_z^2 = \frac{p^2}{r_e^2}$.*

1. *For $r > r_e$, use (34). The phase $\Phi : (\theta, r) \to \Psi(p, \beta) + p\theta$ is a solution of the eikonal equation*

$$(\partial_r \Phi(r,\theta))^2 + (\frac{\partial_\theta \Phi(r,\theta)}{r})^2 = k_3^2 = \omega^2\epsilon\mu - k_z^2.$$

2. *For $r < r_e$, use (37). The phase $G : (\theta, r) \to p\theta - -\frac{2}{3}\zeta^{\frac{3}{2}}$ is solution of the eikonal equation as well:*

$$(\partial_r G(r,\theta))^2 + (\frac{\partial_\theta G(r,\theta)}{r})^2 = k_3^2 = \omega^2\epsilon\mu - k_z^2.$$

3. *For $r \simeq r_e$, use (40). The phase $h : (\theta, r) \to p\theta \pm \frac{2}{3}(2^{\frac{1}{3}}a)^{\frac{3}{2}}$, with $k_3 r = p + ap^{\frac{1}{3}}$ solves as well*

$$(\partial_r h)^2 + \frac{1}{r^2}(\partial_\theta h)^2 = \omega^2\epsilon\mu - k_z^2.$$

Proof. Consider

$$\Phi(r,\theta) = p(\theta + \tan\beta - \beta), \cos\beta = \frac{p}{k_3 r}.$$

One has

$$\partial_r \Phi = p\partial_r\beta \tan^2\beta, \partial_\theta \Phi = p.$$

Using $k_3 r = \frac{p}{\cos\beta}$, $\partial_r f = \frac{1}{r\tan\beta}\partial_\beta f$, hence

$$(\partial_r \Phi(r,\theta))^2 + (\frac{\partial_\theta \Phi(r,\theta)}{r})^2 = k_3^2.$$

This is the eikonal equation associated with the PDE. In the case of $r = R + n$ (where the description corresponds to a circle of radius R and n is the algebraic distance to the circle), one finds

$$(\partial_n \phi(n,\theta))^2 = k_3^2 - \frac{p^2}{(n+R)^2}.$$

The first eikonal equation is obtained.

Consider now G. One notices

$$\partial_r G = \pm ip\partial_r(\frac{2}{3}\zeta^{\frac{3}{2}}), \partial_\theta G = p$$

Using $p\partial_r = k_3\partial_z$ and the expression of $\frac{2}{3}\zeta^{\frac{3}{2}}$, one obtains

$$\partial_r G = \pm ik_3\frac{\sqrt{1-z^2}}{z}.$$

One then obtains

$$(\partial_r G)^2 + r^{-2}(\partial_\theta G)^2 = -k_3^2\frac{1-z^2}{z^2} + \frac{p^2}{r^2} = k_3^2 + \frac{p^2}{r^2} - k_3^2\frac{p^2}{k_3^2 r^2} = k_3^2.$$

This phase also solves the eikonal equation. □

We will continue this section by using the classification of Definition 3. One needs to observe the following thing

Remark 5. *The Dirichlet to Neumann operator uses the asymptotics of the Hankel functions at $r = r_0$ and at $r = R$ and of the derivatives of the Hankel functions at $r = R$. This means that it uses the same phase function at the numerator and at the denominator, denoted by $l(r_0, R, k_z, p, \omega)$, $\Re l(r_0, R, k_z, p, \omega) > 0$ and the exact DTN for the mode p is of the form*

$$\frac{n_+ e^l - n_- e^{-l}}{d_+ e^l - d_- e^{-l}} = \frac{n_+ - n_- e^{-2l}}{d_+ - d_- e^{-2l}}.$$

we will thus only take care of the coefficients $n_\pm, d_\pm$ in the analysis.

4.5. Estimates on the solution

We have the estimate

Proposition 6. *Assume that there exists C constant such that $\frac{p^2}{r_0^2} + k_z^2 \leq C\omega^2$. For all M large enough, there exists a constant D such that, for all $\omega \geq M$, for all p, k_z, for all $r \in [r_0, R]$*

$$|\frac{d_p(k_3 r)}{d_p(k_3 R)}| \leq D.$$

Our aim is to obtain the estimate of Proposition 6. We will use the following equalities of the toolbox:

1. In the hyperbolic case, we use (34) of Section 4.3,
2. In the elliptic case, we use (37) of Section 4.3,
3. In the mixed-elliptic-hyperbolic case, for all $r \in [r_0, R]$, we use (37) for the behavior at r_0, (34) for the behavior at R.

This proposition 6 is detailed above, so that we can see the different cases that are involved here. We state two detailed results.

Proposition 7.

- *For all (p, k_z) satisfying $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r_0^2} > 0$ we are in the totally hyperbolic region. There exists a constant $D_0 > 0$ such that*

$$|\frac{d_p(k_3 r)}{d_p(k_3 R)}| \leq D_0 e^{\Im(\Psi_* - \Psi(\beta(r)))} \leq D_0. \tag{48}$$

- *For all (p, k_z) such that*

$$\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2} < 0 < \Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r_0^2}$$

assume that *r_e does not depend on ω. One is in the mixed elliptic-hyperbolic region.*

$\star$ For $r_+ < r_0 < r < r_e$, there exists a constant K such that

$$|\frac{J_p(k_3 r)Y_p(k_3 r_0) - Y_p(k_3 r)J_p(k_3 r_0)}{J_p(k_3 R)Y_p(k_3 r_0) - Y_p(k_3 R)J_p(k_3 r_0)}| \leq= K e^{\frac{4}{3}p\Re\zeta_0^{\frac{3}{2}} - \Re(\frac{2}{3}p\zeta^{\frac{3}{2}} + i\Psi_*)} \leq K. \tag{49}$$

$\star$ For $r_0 < r < r_+$, there exists a constant K_0 such that

$$|\frac{J_p(k_3 r)Y_p(k_3 r_0) - Y_p(k_3 r)J_p(k_3 r_0)}{J_p(k_3 R)Y_p(k_3 r_0) - Y_p(k_3 R)J_p(k_3 r_0)}| \leq G_0 e^{\frac{2}{3}p\Re(\zeta^{\frac{3}{2}} - \zeta_0^{\frac{3}{2}})} \leq K_0 e^{-\Re(\frac{2}{3}p\zeta^{\frac{3}{2}} + i\Psi_*)} \leq K_0. \tag{50}$$

One notes that, as $|k_3^2 r^2 - p^2| \geq |\Im\epsilon\mu|\omega^2$, one is never in the glancing case.

Proof. We consider the totally hyperbolic regime. Recall that $\Re i(\Psi(\beta(r)) - \Psi(\beta_0)) > 0$ because $\Re i\Psi$ is increasing. For notational simplicity, we write Ψ_0 instead of $\Psi(p,\beta(r_0)) = \Psi(p,\beta_0)$, L_0, M_0, L, M, L_*, M_* instead of, respectively, $L(p,\beta_0), M(p,\beta_0)$ and so on.

$$\frac{J_p(k_3r)Y_p(k_3r_0) - Y_p(k_3r)J_p(k_3r_0)}{J_p(k_3R)Y_p(k_3r_0) - Y_p(k_3R)J_p(k_3r_0)} = e^{i(\Psi(\beta(r))-\Psi_*)}\frac{(L_0 - iM_0)(L + iM) - (L_0 + iM_0)(L - iM)e^{2i(\Psi_0-\Psi(\beta(r)))}}{(L_0 - iM_0)(L_* + iM_*) - (L_0 + iM_0)(L_* - iM_*)e^{2i(\Psi_0-\Psi_*)}},$$

As $|e^{2i(\Psi_0-\Psi_*)}| \leq 1$, $|e^{2i(\Psi_0-\Psi(\beta(r)))}| \leq 1$, using precise uniform bounds on L and M, one checks that there exists a constant $D_0 > 0$, **independent on** $p; k_z$, such that

$$|\frac{J_p(k_3r)Y_p(k_3r_0) - Y_p(k_3r)J_p(k_3r_0)}{J_p(k_3R)Y_p(k_3r_0) - Y_p(k_3R)J_p(k_3r_0)}| \leq D_0 e^{\Im(\Psi_*-\Psi(\beta))} \leq D_0.$$

As k_z and $\frac{p}{r}$ are controlled by ω, we get an uniform control of β, β_0, β_* hence of L, M and other quantities above.

Let us look now at the elliptic-hyperbolic case.

For $r_0 < r < r_e$, it is natural to use the previous result (because we use the elliptic representation at r_0 and at r). Assume that $r_0 > r_-$ first. Note that $\Re(i\Psi_* + \frac{2}{3}p\zeta^{\frac{3}{2}}) > 0$ and recall that $\Re\zeta_0^{\frac{3}{2}} < 0$.

$$\frac{J_p(pz_0)Y_p(pz) - J_p(pz)Y_p(pz_0)}{J_p(pz_0)Y_p(\frac{p}{\cos\beta_*}) - J_p(\frac{p}{\cos\beta_*})Y_p(pz_0)} = \frac{e^{\frac{2}{3}p(\zeta_0^{\frac{3}{2}}-\zeta^{\frac{3}{2}})}}{e^{i\Psi_*-\frac{2}{3}p\zeta_0^{\frac{3}{2}}}}\frac{J_p(pz_0)e^{\frac{2}{3}p\zeta_0^{\frac{3}{2}}}Y_p(pz)e^{-\frac{2}{3}p\zeta^{\frac{3}{2}}}e^{-\frac{4}{3}p\Re(\zeta_0^{\frac{3}{2}}-\zeta^{\frac{3}{2}})} - J_p(pz)e^{\frac{2}{3}p\zeta^{\frac{3}{2}}}Y_p(pz_0)e^{-\frac{2}{3}p\zeta_0^{\frac{3}{2}}}}{J_p(pz_0)e^{\frac{2}{3}p\zeta_0^{\frac{3}{2}}}Y_p(\frac{p}{\cos\beta_*})e^{-i\Psi_*} - J_p(\frac{p}{\cos\beta_*})e^{-i\Psi_*}[Y_p(pz_0)e^{-\frac{2}{3}p\zeta_0^{\frac{3}{2}}}]e^{\frac{4}{3}p\zeta_0^{\frac{3}{2}}}}$$

Using $J_p(pz_0)e^{\frac{2}{3}p\zeta_0^{\frac{3}{2}}}$, $Y_p(pz)e^{-\frac{2}{3}p\zeta^{\frac{3}{2}}}$, $J_p(pz)e^{\frac{2}{3}p\zeta^{\frac{3}{2}}}$, $Y_p(pz_0)e^{-\frac{2}{3}p\zeta_0^{\frac{3}{2}}}$ bounded, $|e^{\frac{4}{3}p\zeta_0^{\frac{3}{2}}}| < 1$, $|e^{-\frac{4}{3}p\Re(\zeta_0^{\frac{3}{2}}-\zeta^{\frac{3}{2}})}| < 1$, one gets, with the equality

$$\sqrt{\frac{\pi p\tan\beta_*}{2}}J_p(\frac{p}{\cos\beta_*}) = \frac{1}{2}[(L-iM)e^{i\Psi_*}+(L+iM)e^{-i\Psi_*}],\ \sqrt{\frac{\pi p\tan\beta_*}{2}}Y_p(\frac{p}{\cos\beta_*}) = \frac{1}{2i}[(L-iM)e^{i\Psi_*}-(L+iM)e^{-i\Psi_*}], \tag{51}$$

$$|\frac{d_p(k_3r)}{d_p(k_3R)}| \leq K|e^{\frac{2}{3}p(\zeta_0^{\frac{3}{2}}-\zeta^{\frac{3}{2}})-i\Psi_*+\frac{2}{3}\zeta_0^{\frac{3}{2}}}| = Ke^{\frac{4}{3}p\Re\zeta_0^{\frac{3}{2}}-\Re(\frac{2}{3}p\zeta^{\frac{3}{2}}+i\Psi_*)} \leq K.$$

Assume now $r_0 < r < r_-$,

$$\frac{J_p(pz_0)Y_p(pz) - J_p(pz)Y_p(pz_0)}{J_p(pz_0)Y_p(\frac{p}{\cos\beta_*}) - J_p(\frac{p}{\cos\beta_*})Y_p(pz_0)} = \frac{e^{\frac{2}{3}p(\zeta_0^{\frac{3}{2}}-\zeta^{\frac{3}{2}})}}{e^{i\Psi_*+\frac{2}{3}p\zeta_0^{\frac{3}{2}}}}\frac{J_p(pz_0)e^{\frac{2}{3}p\zeta_0^{\frac{3}{2}}}Y_p(pz)e^{-\frac{2}{3}p\zeta^{\frac{3}{2}}}e^{-\frac{4}{3}p\Re(\zeta_0^{\frac{3}{2}}-\zeta^{\frac{3}{2}})} - J_p(pz)e^{\frac{2}{3}p\zeta^{\frac{3}{2}}}Y_p(pz_0)e^{-\frac{2}{3}p\zeta_0^{\frac{3}{2}}}}{J_p(pz_0)e^{\frac{2}{3}p\zeta_0^{\frac{3}{2}}-\frac{4}{3}p\zeta_0^{\frac{3}{2}}}Y_p(\frac{p}{\cos\beta_*})e^{-i\Psi_*} - J_p(\frac{p}{\cos\beta_*})e^{-i\Psi_*}Y_p(pz_0)e^{-\frac{2}{3}p\zeta_0^{\frac{3}{2}}}}.$$

Use again the estimate (51), the inequality $\Re\frac{2}{3}\zeta_0^{\frac{3}{2}} > 0$ (because $r_0 < r_+$), $\Re(\zeta_0^{\frac{3}{2}} - \zeta^{\frac{3}{2}}) > 0$, we obtain

$$|\frac{d_p(k_3r)}{d_p(k_3R)}| \leq K_0|e^{-\frac{2}{3}p\zeta^{\frac{3}{2}}-i\Psi_*}| = Ke^{-\Re(\frac{2}{3}p\zeta^{\frac{3}{2}}+i\Psi_*)} \leq K.$$

□

Proposition 8 deals with the totally elliptic case (that is $k_z^2 + \frac{p^2}{R^2} > \Re\epsilon\mu\omega^2$). It enables to treat the case where $\frac{k_z^2+\frac{p^2}{R^2}}{\omega^2} \to +\infty$ when $\omega \to +\infty$.

Proposition 8. *Let $\Re\epsilon\mu\omega^2 - k_z^2 > 0$. For all (p, k_z) satisfying $k_z^2 + \frac{p^2}{r_0^2} \leq C\omega^2$ and $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2} < 0$. We say we are in the totally elliptic region.*

Let $r_0 < r \leq R$. There exists a constant K_0 such that one has, for ω large enough

$$|\frac{J_p(k_3r)Y_p(k_3r_0) - Y_p(k_3r)J_p(k_3r_0)}{J_p(k_3R)Y_p(k_3r_0) - Y_p(k_3R)J_p(k_3r_0)}| \leq e^{-\frac{2}{3}p\Re(\zeta^{\frac{3}{2}} - \zeta_*^{\frac{3}{2}})}K_0 \leq K_0. \tag{52}$$

More precisely, for any $\delta > 0$ small enough, there exists ω_δ such that there exists K_δ such that (52) is uniformly true for all $r \in [r_0 + \delta, R]$.

Proof. In the totally elliptic regime, introduce the notations z, z_0, z_* and ζ, ζ_0, ζ_* corresponding to $pz = k_3r$, $pz_0 = k_3r_0$, $pz_* = k_3R$.

One notices that the functions $H_p^{(j)}(pz)$ contains $e^{\frac{2}{3}p\zeta^{\frac{3}{2}}}$ as well as $e^{-\frac{2}{3}p\zeta^{\frac{3}{2}}}$, because it contains $Ai(p^{\frac{2}{3}}\zeta)$ as well as $Bi(p^{\frac{2}{3}}\zeta)$. It is then more relevant to distinguish the role of Ai and Bi, and use the Bessel functions as in (39)

$$\frac{d_p(k_3r)}{d_p(k_3R)} = \frac{J_p(pz)Y_p(pz_0) - J_p(pz_0)Y_p(pz)}{J_p(pz_*)Y_p(pz_0) - J_p(pz_0)Y_p(pz_*)} = \frac{J_p(pz)}{J_p(pz_*)}\frac{1-\tau}{1-\tau_*},$$

where

$$\tau = \frac{Y_p(pz)J_p(pz_0)}{J_p(pz)Y_p(pz_0)}, \tau_* = \frac{Y_p(pz_*)J_p(pz_0)}{J_p(pz_*)Y_p(pz_0)}.$$

We use (39) to deduce estimates for τ and τ_*. We then have to study estimates for $\frac{Bi(p^{\frac{2}{3}}\zeta)Ai(p^{\frac{2}{3}}\zeta_0)}{Ai(p^{\frac{2}{3}}\zeta)Bi(p^{\frac{2}{3}}\zeta_0)}$. As the classical estimates for the Airy functions ([8] 9.7(ii)) are valid uniformly in the sector $|argz| \leq \pi - \delta_*$, one concludes that

$$\frac{Bi(p^{\frac{2}{3}}\zeta)e^{-\frac{2}{3}p\zeta^{\frac{3}{2}}}Ai(p^{\frac{2}{3}}\zeta_0)e^{\frac{2}{3}p\zeta_0^{\frac{3}{2}}}}{Ai(p^{\frac{2}{3}}\zeta)e^{\frac{2}{3}p\zeta^{\frac{3}{2}}}Bi(p^{\frac{2}{3}}\zeta_0)e^{-\frac{2}{3}p\zeta_0^{\frac{3}{2}}}}$$

is bounded, hence there exists M such that

$$|\frac{Bi(p^{\frac{2}{3}}\zeta)e^{-\frac{2}{3}p\zeta^{\frac{3}{2}}}Ai(p^{\frac{2}{3}}\zeta_0)e^{\frac{2}{3}p\zeta_0^{\frac{3}{2}}}}{Ai(p^{\frac{2}{3}}\zeta)e^{\frac{2}{3}p\zeta^{\frac{3}{2}}}Bi(p^{\frac{2}{3}}\zeta_0)e^{-\frac{2}{3}p\zeta_0^{\frac{3}{2}}}}| = |\frac{Bi(p^{\frac{2}{3}}\zeta)Ai(p^{\frac{2}{3}}\zeta_0)}{Ai(p^{\frac{2}{3}}\zeta)Bi(p^{\frac{2}{3}}\zeta_0)}||e^{-\frac{4}{3}p(\zeta^{\frac{3}{2}} - \zeta_0^{\frac{3}{2}})}| \leq M$$

This shows that

$$|\tau| \leq Me^{-\frac{4}{3}\Re(\zeta_0^{\frac{3}{2}} - \zeta^{\frac{3}{2}})}, |\tau_*| \leq Me^{-\frac{4}{3}\Re(\zeta_0^{\frac{3}{2}} - \zeta_*^{\frac{3}{2}})}$$

and one uses $Me^{-\frac{4}{3}\Re(\zeta_0^{\frac{3}{2}} - \zeta_*^{\frac{3}{2}})} < \frac{1}{2}$ and $Me^{-\frac{4}{3}\Re(\zeta_0^{\frac{3}{2}} - \zeta^{\frac{3}{2}})} < \frac{1}{2}$ for $r > r_0$ for large ω to conclude that there exists K such that $|\frac{1-\tau}{1-\tau_*}| \leq K$ for all r, $r_0 < r \leq R$ fixed, which imply Proposition 8. The precise statement at the end of this proposition comes from the fact that for all (p, k_z) satisfying $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2} < 0$, $p \geq C\omega$, $\tau \to 0$ exponentially in ω.

□

Finally, we adress the case (which implies that k_z^2 grows when $\omega \to +\infty$):

$$k_z^2 \geq \Re \epsilon\mu\omega^2. \tag{53}$$

Proposition 9. *There exists a constant E such that, for all $r \in [r_0, R]$, for all p, for all k_z satisfying (53)*

$$|\frac{d_p(k_3 r)}{d_p(k_3 R)}| \leq E.$$

The proof of this proposition comes from the uniform representation of the Bessel functions for large p, even for z such that the phase of z is in $(-\pi + \delta, \pi - \delta)$. Indeed, if $pz = k_3 r$ and if $k_3^2 = \epsilon\mu\omega^2 - k_z^2$, $z = \frac{k_3 r}{p}$ is of argument belonging to $(-\frac{\pi}{2}, 0)$ because k_3^2 has a negative real part and a negative imaginary part, hence the argument of k_3^2 is in $(-\pi, -\frac{\pi}{2})$. One deduces that, on the line $k_3 \frac{r}{p}$ which is included in $(-\frac{\pi}{2}, 0)$, $\zeta^{\frac{3}{2}}$ is increasing.

Hence one has

$$\frac{d_p(k_3 r)}{d_p(k_3 R)} = e^{-\frac{2}{3}p(\zeta^{\frac{3}{2}} - \zeta_*^{\frac{3}{2}})} \frac{e^{\frac{4}{3}p(\zeta_0^{\frac{3}{2}} - \zeta^{\frac{3}{2}})} J_p(k_3 r) Y_p(k_3 r_0) e^{\frac{2}{3}p(\zeta^{\frac{3}{2}} - \zeta_0^{\frac{3}{2}})} - J_p(k_3 r_0) Y_p(k_3 r) e^{\frac{2}{3}p(\zeta_0^{\frac{3}{2}} - \zeta^{\frac{3}{2}})}}{e^{\frac{4}{3}p(\zeta_0^{\frac{3}{2}} - \zeta_*^{\frac{3}{2}})} J_p(k_3 R) Y_p(k_3 r_0) e^{\frac{2}{3}p(\zeta_*^{\frac{3}{2}} - \zeta_0^{\frac{3}{2}})} - J_p(k_3 r_0) Y_p(k_3 R) e^{\frac{2}{3}p(\zeta_0^{\frac{3}{2}} - \zeta_*^{\frac{3}{2}})}}.$$

One observes that $pJ_p(k_3 r)Y_p(k_3 r_0)e^{\frac{2}{3}p(\zeta^{\frac{3}{2}} - \zeta_0^{\frac{3}{2}})}$, $pJ_p(k_3 r_0)Y_p(k_3 r)e^{\frac{2}{3}p(\zeta_0^{\frac{3}{2}} - \zeta^{\frac{3}{2}})}$, $pJ_p(k_3 R)Y_p(k_3 r_0)e^{\frac{2}{3}p(\zeta_*^{\frac{3}{2}} - \zeta_0^{\frac{3}{2}})}$, $pJ_p(k_3 r_0)Y_p(k_3 R)$ are all bounded when p, k_z is large (thanks to the coefficient $p^{-\frac{1}{3}}$ as coefficient in the asymptotic expansion of $J_p(pz)$ and the coefficient $(p^{\frac{2}{3}})^{-\frac{1}{4}}$ through the asymptotics of the Airy function). Using these estimates one deduces that $\frac{d_p(k_3 r)}{d_p(k_3 R)}$ is bounded, uniformly in ζ, ζ_0, ζ_*, when $p \to +\infty$ and $\omega \to +\infty$. This result still holds when p is finite and ω tending to $+\infty$ which imply $k_z \to +\infty$. Proposition 9 is proven.

Proposition 6 is a consequence of Proposition 7 and Proposition 8. Proposition 5 is then a consequence of Proposition 6 and of Proposition 9.

5. The Dirichlet to Neumann operator for the cylindrical ring

The first subsection is devoted to the result stated for the plane layer on the form and the regularity of the Dirichlet to Neumann operator (which is an operator acting on functions on $\{x_3 = 0\}$).

5.1. Proof of the result of the Introduction for the plane boundary

Lemma 16. *The Dirichlet to Neumann multiplier is a Fourier multiplier from $H^s(\mathbb{R}^2)$ to $H^{s-1}(\mathbb{R}^2)$, in particular for $s = \frac{1}{2}$ whenever $\epsilon\mu \notin \mathbb{R}_+^*$. It is not defined when $\epsilon\mu \in \mathbb{R}_+^*$. However, if $\epsilon\mu \in \mathbb{R}_+^*$, for each (k_1, k_2) given, there exists a sequence $(\omega_n(\sqrt{k_1^2 + k_2^2}))_n$ of values of ω such that the Fourier multiplier is not defined for ω belonging to this sequence. When ω does not belong to this sequence, the Fourier multiplier is well defined.*

Proof. Begin with the last item. In the case $\epsilon\mu \in \mathbb{R}_+^*$, there exists at least $(k_1, k_2) \in \mathbb{R}^2$ such that $\epsilon\mu\omega^2 = k_1^2 + k_2^2$, hence there exists at least a resonance, hence no uniqueness of the solution. The Dirichlet to Neumann multiplier is never defined as an operator on $\mathcal{S}'(\mathbb{R}^2)$. This is a consequence of the fact that the domain is unbounded, hence continuous spectrum exist and, in particular, for all (k_1, k_2) there exists at

least an associated point of the continuous spectrum. However, for (k_1, k_2) given, define $\omega_n(\sqrt{k_1^2+k_2^2}) = \sqrt{\frac{k_1^2+k_2^2+\frac{\pi^2 n^2}{l^2}}{\epsilon\mu}}$. These are values for which $\sin k_\perp l = 0$, hence non uniqueness of the solution. For a fixed value of ω which is not in this list, there exists α_1 such that $|\sin k_\perp l| \geq \alpha_1$, which ensures the existence and uniqueness of the solution of the ODE on $[-l, 0]$.

Consider now $\epsilon\mu \notin \mathbb{R}$ and assume $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^2)$ (that is $(1+k_1^2+k_2^2)^{\frac{1}{4}}\hat{u}_0(k_1,k_2) \in L^2(\mathbb{R}^2)$), introduce $U(k_1,k_2,x_3) = \frac{\sin k_\perp(x+l)}{\sin k_\perp l}\hat{u}_0(k_1,k_2)$. Estimate (76) shows that $U \in L^2(\mathbb{R}^2 \times [-l,0])$, hence

$$u(x_1,x_2,x_3) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{\sin k_\perp(x_3+l)}{\sin k_\perp l}\hat{u}_0(k_1,k_2)e^{ik_1x_1+ik_2x_2}dk_1dk_2$$

is a function of $L^2(\mathbb{R}^2 \times [-l,0])$. Its Fourier transform solves (75). Hence U is a L^2 solution of

$$\begin{cases} (\Delta + \omega^2\epsilon\mu)U = 0 \\ U(-l) = 0 \\ U(0) = u_0. \end{cases}$$

It is straightforward to check that $U \in H^1(\mathbb{R}^2 \times [-l,0])$.

The trace of the normal derivative is thus given through

$$\partial_{x_3}\hat{u}(0,k_1,k_2) = k_\perp \frac{\cos(k_\perp l)}{\sin k_\perp l}\hat{u}_0(k_1,k_2)$$

from which one deduces that $\partial_{x_3}\hat{u}(0,k_1,k_2) = \hat{C}(k_1,k_2)\hat{u}_0(k_1,k_2)$ and that $|\hat{C}| \leq M\sqrt{k_1^2+k_2^2}$. One has thus $(1+k_1^2+k_2^2)^{-\frac{1}{4}}\partial_{x_3}\hat{u}(0,k_1,k_2) \leq M(1+k_1^2+k_2^2)^{\frac{1}{4}}\hat{u}_0(k_1,k_2)$, hence $\hat{C}$ is a Fourier multiplier, and $\partial_{x_3}u(.,.,0) = Cu(.,.,0)$ where C goes from $H^{\frac{1}{2}}(\mathbb{R}^2)$ to $H^{-\frac{1}{2}}(\mathbb{R}^2)$.

□

5.2. The Dirichlet to Neumann operator for the cylindrical layer

Proposition 10. *The Dirichlet to Neumann operator C is a Fourier multiplier, which expression is, for each (p,k_z) such that $k_3 \neq 0$ and k_3 is not a resonance of the problem*

$$C(p,k_z) = k_3 \frac{H_p^{(2)}(k_3r_0)(H_p^{(1)})'(k_3R) - H_p^{(1)}(k_3r_0)(H_p^{(2)})'(k_3R)}{H_p^{(2)}(k_3r_0)H_p^{(1)}(k_3R) - H_p^{(1)}(k_3r_0)H_p^{(2)}(k_3R)}.$$

In the case $k_3 = 0$, one obtains, for $\tau = \frac{R}{r_0}$

$$C(p,k_z) = \frac{p}{R}\frac{\tau^p + \tau^{-p}}{\tau^p - \tau^{-p}}.$$

Proof. Consider $a^p(k_z) \in L^2(\mathbf{R})$. Thanks to Proposition 6, the formal solution (14) written in Proposition 1 satisfies

$$(k_z,r) \to a^p(k_z)\frac{H_p^{(2)}(k_3r_0)H_p^{(1)}(k_3r) - H_p^{(1)}(k_3r_0)H_p^{(2)}(k_3r)}{H_p^{(2)}(k_3r_0)H_p^{(1)}(k_3R) - H_p^{(1)}(k_3r_0)H_p^{(2)}(k_3R)} \in L^2([r_0,R]\times\mathbf{R})$$

and if $a^p(k_z) \in H^1(\mathbf{R}_{k_z}) \times h^1([0,2\pi])$,

$$(k_z, r, \theta) \to a^p(k_z) \frac{H_p^{(2)}(k_3 r_0) H_p^{(1)}(k_3 r) - H_p^{(1)}(k_3 r_0) H_p^{(2)}(k_3 r)}{H_p^{(2)}(k_3 r_0) H_p^{(1)}(k_3 R) - H_p^{(1)}(k_3 r_0) H_p^{(2)}(k_3 R)} e^{ip\theta} \in H^1([r_0, R] \times [0, 2\pi] \times \mathbf{R}_{k_z}).$$

The derivative in r is indeed in L^2 thanks to $|k_3| \leq |k_z|$, and one uses $k_z \to k_3 a^p(k_z)$. Similarily, the derivative in θ is in L^2 thanks to $pa^p(k_z) \in l^2$. One can consider $\partial_r u(R, ., .)$, which is a Fourier multiplier of the corresponding mode of $u(R, ., .)$, which defines the Dirichlet to Neumann operator C from $H^1(S_R)$ to $L^2(S_R)$.

This lemma complements the results of (13), (14) of [28] by characterizing the values where (16) is not fulfilled as resonances of the problem, and by asserting estimates on the solution. Of course, one has also

$$C(p, k_z) = k_3 \frac{Y_p(k_3 r_0) J_p'(k_3 R) - J_p(k_3 r_0) Y_p'(k_3 R)}{Y_p(k_3 r_0) J_p(k_3 R) - J_p(k_3 r_0) Y_p(k_3 R)}. \tag{54}$$

Let us prove finally the result for $k_3 = 0$ (that is $\omega^2 \epsilon\mu - k_z^2 = 0$). The Helmholtz equation reads, for $U(r, \theta, z) = e^{ik_z z + ip\theta} u(r)$

$$\frac{1}{r} \partial_r (r \partial_r u) - \frac{p^2}{r^2} u = 0.$$

Solutions of this ODE are $u(r) = Ar^p + Br^{-p}$, hence the Dirichlet boundary condition at $r = r_0$ yields $u(r) = a(r^p r_0^{-p} - r_0^p r^{-p})$, hence at $r = R$ one deduces $u(R) = a_p(R^p r_0^{-p} - r_0^p R^{-p})$ and $\partial_r u(R) = pa_p(R^{p-1} r_0^{-p} + r_0^p R^{-p-1})$, from which one deduces the Dirichlet to Neumann operator as a Fourier multiplier

$$C(p, k_z) = p \frac{R^{p-1} r_0^{-p} + r_0^p R^{-p-1}}{R^p r_0^{-p} - r_0^p R^{-p}},$$

which rewrites, denoting by $\tau = \frac{r_1}{r_0}$

$$C(p, k_z) = \frac{p}{R} \frac{\tau^p + \tau^{-p}}{\tau^p - \tau^{-p}},$$

and the Dirichlet to Neumann operator is the operator such that

$$\sum_p a_p e^{ip\theta + i\sqrt{\epsilon\mu}\omega z} \to R^{-1} \sum_p p a_p \frac{\tau^p + \tau^{-p}}{\tau^p - \tau^{-p}} e^{ip\theta + i\sqrt{\epsilon\mu}\omega z}.$$

It is worth noticing that $\sum |p| a_p^2 < +\infty$ implies $\sum_{p \neq 0} |p|^{-1} (pa_p)^2 (\frac{\tau^p + \tau^{-p}}{\tau^p - \tau^{-p}})^2 < +\infty$ hence one checks this expression sends $h^{\frac{1}{2}}([0, 2\pi])$ onto $h^{-\frac{1}{2}}([0, 2\pi])$.

□

5.3. Asymptotic estimates of the Dirichlet to Neumann operator for k_z, p of order of magnitude ω for $\epsilon\mu \in \mathbb{R}_+^*$ away from resonances

In this case, the use of the exact solutions is necessary, because the situation is different from the case $\Im \epsilon\mu < 0$ where, in the hyperbolic regime, it has been easy to choose the Fourier integral operator which is exponentially growing in l in the region $[0, R - r_0]$ and the one which is exponentially decaying in l, and all the calculations done in the previous Sections amount to choosing the leading order term in the layer. On

the contrary, in the case $\epsilon\mu \in \mathbb{R}$, the two solutions which could be characterized as incoming and outgoing are not easy to identify, the solution of the homogeneous Dirichlet problem is a linear combination of both.

We then use the asymptotic expansions in p, k_z.

Let ϕ_0, ϕ_* the quantities $\tan\beta_0 - \beta_0$, (resp. $\tan\beta_* - \beta_*$) depending only on η, θ and not on ω. Introduce the solutions $\omega_n(\theta, \eta, r_0, R)$ of

$$\tan(\theta R(\phi_* - \phi_0)\omega) = -\frac{1}{\theta R\omega}\frac{p(L_0M_* - L_*M_0)}{L_0L_* + M_0M_*}. \tag{55}$$

We prove

Proposition 11. *There exists a constant c' and a constant D such that, for all n, for all ω such that*

$$|\omega - \omega_n(\theta, \eta, r_0, R)| \geq \frac{2c'}{\omega_n(\theta, \eta, r_0, R)},$$

one has

$$|C(p, k_z) + \omega\frac{\sqrt{\epsilon\mu - \eta^2}}{\tan(\Psi_* - \Psi_0)} + \sqrt{\epsilon\mu - \eta^2}\frac{(b_0 + d_*)\sin^2(\Psi_* - \Psi_0) - (c_* - d_0)\cos^2(\Psi_* - \Psi_0)}{\sin^2(\Psi_* - \Psi_0)}| \leq \frac{D}{\omega|\sin^3(\Psi_* - \Psi_0)|}.$$

Proof. The solutions are, thanks to Proposition 4, which shows that $\frac{d_p(k_3r)}{d_p(k_3R)}$ is in $C^0([r_0, R], \mathcal{S}')$, hence is the Fourier transform of an element of $C^0([r_0, R], \mathcal{S}')$, which yields to the representation:

$$\sum a_p e^{ip\theta}\frac{J_p(k_3r)Y_p(k_3r_0) - Y_p(k_3r)J_p(k_3r_0)}{J_p(k_3R)Y_p(k_3r_0) - Y_p(k_3R)J_p(k_3r_0)}.$$

The existence and uniqueness of the solution of for a given p is given by the equality equivalent to the non resonance condition of Lemma 11:

$$J_p(k_3R)Y_p(k_3r_0) - Y_p(k_3R)J_p(k_3r_0) = 0. \tag{56}$$

Lemma 17. *Let $\frac{p}{\omega R} = \theta$, $\eta = \frac{k_z}{\omega}$ independent on ω and $\epsilon\mu - \theta^2 - \eta^2 > 0$. There exists a sequence $\omega_n(\theta, \eta, r_0, R)$ such that (56) is equivalent to 'There exists a n such that $\omega = \omega_n(\theta, \eta, r_0, R)$'.*

Proof. Let us use, in the hyperbolic regime $\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r_0^2} > 0$, the asymptotic representation of the Bessel functions. The conditions for resonances is

$$(L_0 - iM_0)(L_* + iM_*)e^{i(\Psi_* - \Psi_0)} - (L_0 + iM_0)(L_* - iM_*)e^{-i(\Psi_* - \Psi_0)} = 0,$$

which is equivalent to

$$[L_0L_* + M_0M_*]\sin(\Psi_* - \Psi_0) + (L_0M_* - L_*M_0]\cos(\Psi_* - \Psi_0) = 0,$$

that is

$$\tan(\Psi_* - \Psi_0) = -\frac{L_0M_* - L_*M_0}{L_0L_* + M_0M_*}. \tag{57}$$

One considers the case where $\frac{p}{\omega R} = \theta$ is independent on ω. Equation (57) rewrites

$$\tan(\theta R(\phi_* - \phi_0)\omega) = -\frac{1}{\theta R\omega}\frac{p(L_0M_* - L_*M_0)}{L_0L_* + M_0M_*}. \tag{58}$$

It is equivalent to (55). By inspection of the behavior of the function $\frac{p(L_0M_*-L_*M_0)}{L_0L_*+M_0M_*}$, which is uniformly bounded in ω thanks to the uniform expansions of L_*, M_*, L_0, M_0, one obtains existence and uniqueness of each term of the sequence $\omega_n(\theta,\eta,r_0,R) \in (\frac{\frac{\pi}{2}+(n-1)\pi}{\theta R(\phi_*-\phi_0)}, \frac{n\pi}{\theta R(\phi_*-\phi_0)})$ of solutions of (56). These points are close to the intersections of the graph of the tangent function and the graph of $\frac{C}{\omega}$. □

We are now ready to express, for $\omega \notin \omega_n(\theta,\eta,r_0,R)$, the Dirichlet to Neumann operator and its leading order term.

Proof. We use

$$\begin{aligned} C(p,k_z) &= -ik_3 \frac{(L_0-iM_0)(N_*-iO_*)e^{i(\Psi_*-\Psi_0)}+(L_0+iM_0)(N_*+iO_*)e^{-i(\Psi_*-\Psi_0)}}{2i(L_0L_*+M_0M_*)\sin(\Psi_*-\Psi_0)+2i(L_0M_*-L_*M_0)\cos(\Psi_*-\Psi_0)} \\ &= -k_3 \frac{(L_0N_*-M_0O_*)\cos(\Psi_*-\Psi_0)+(M_0N_*+O_*L_0)\sin(\Psi_*-\Psi_0)}{(L_0L_*+M_0M_*)\sin(\Psi_*-\Psi_0)+(L_0M_*-L_*M_0)\cos(\Psi_*-\Psi_0)} \\ &= -k_3 \frac{L_0N_*-M_0O_*}{L_0L_*+M_0M_*} \frac{\cos(\Psi_*-\Psi_0)+\frac{M_0N_*+O_*L_0}{L_0N_*-M_0O_*}\sin(\Psi_*-\Psi_0)}{\sin(\Psi_*-\Psi_0)+\frac{L_0M_*-L_*M_0}{L_0L_*+M_0M_*}\cos(\Psi_*-\Psi_0)}. \end{aligned}$$

If one wants to have a leading order term in ω of this symbol, it is necessary at the same time to avoid $\sin(\Psi_*-\Psi_0)=0$ and $\sin(\Psi_*-\Psi_0)+\frac{L_0M_*-L_*M_0}{L_0L_*+M_0M_*}\cos(\Psi_*-\Psi_0)$. The latter condition writes $\omega \neq \omega_n(\theta,\eta,r_0,R)$, hence we have thus to ensure that ω satisfies an inequality of the type $|\omega-\omega_n(\theta,\eta,r_0,R)| \geq c_*$ (c_* to be determined) in order to avoid $\sin(\Psi_*-\Psi_0)+\frac{L_0M_*-L_*M_0}{L_0L_*+M_0M_*}\cos(\Psi_*-\Psi_0)$ AND $\sin(\Psi_*-\Psi_0)$ different from 0. For this purpose, denote by ϕ_* such that

$$\tan\phi_* = \frac{L_0M_*-L_*M_0}{L_0L_*+M_0M_*}.$$

One observes that $|\tan\phi_*| \leq \frac{C}{\theta\omega R}$ with C independent on ω. Hence there exists a constant c', and a unique ϕ_* such that $|\phi_*| \leq \frac{c'}{\omega}$. If one assumes $|\omega-\omega_n(\theta,\eta,r_0,R)| \leq \frac{2c'}{\omega_n(\theta,\eta,r_0,R)}$ for all $n \geq 1$, then $|\sin(\Psi_*-\Psi_0)| \geq \frac{c'}{\omega_n(\theta,\eta,r_0,R)}$, and one can expand the expression. One obtains, observing that $\frac{L_0N_*-M_0O_*}{L_0L_*+M_0M_*} = 1+O(p^{-2})$, $(\frac{L_0M_*-L_*M_0}{L_0L_*+M_0M_*})^2 = O(p^{-2})$, $p(L_0N_*-M_0O_*) = c_*-b_0+O(p^{-1})$, $p(M_0N_*+O_*L_0) = d_*+b_0+O(p^{-1})$,

$$|C(p,k_z)+\omega\frac{\sqrt{\epsilon\mu-\eta^2}}{\tan(\Psi_*-\Psi_0)}+\sqrt{\epsilon\mu-\eta^2}\frac{(b_0+d_*)\sin^2(\Psi_*-\Psi_0)-(c_*-d_0)\cos^2(\Psi_*-\Psi_0)}{\sin^2(\Psi_*-\Psi_0)}| \leq \frac{D}{\omega|\sin^3(\Psi_*-\Psi_0)|}.$$

□

□

One can link this result with the result stated in Theorem 2, through

Remark 6. *Assume $\Im\epsilon\mu < 0$, independent on ω, and assume $l\omega$ is a constant, independent on ω. Then*

$$C(p,k_z) = \frac{\sqrt{\epsilon\mu\omega^2-k_z^2-k_l^2}}{\tan lk_\perp} + \frac{1}{R}\frac{(a_*-a_0)\cos^2 lk_\perp-(a_0+b_*)\sin^2 lk_\perp}{\sin^2 lk_\perp}\frac{k_\perp}{k_l}+O(\omega^{-1})$$

Proof. We obtained $\frac{d\Psi}{dr} = k_3\sin\beta(r)$, from which, for $r=R$, one deduces $\Psi'(R)=k_\perp$. Hence, for l small, $\Psi_*-\Psi_0 = lk_\perp = l\omega\frac{k_\perp}{\omega}$. One checks, in addition, that $lk_\perp = \sqrt{(\Re\epsilon\mu\omega^2-k_z^2-k_l^2)l+i\Im\epsilon\mu l\omega}$, hence $lk_\perp$ is a complex number which distance from the real axis is strictly positive, hence $\tan lk_\perp$ is not zero.

One has

$$C(p,k_z)=k_\perp\frac{\cos lk_\perp-\frac{a_0+b}{p}\sin lk_\perp}{\sin lk_\perp-\frac{a_*-a_0}{p}\cos lk_\perp}=\frac{k_\perp}{\tan lk_\perp}+\frac{1}{R}\frac{(a_*-a_0)\cos^2 lk_\perp-(a_0+b_*)\sin^2 lk_\perp}{\sin^2 lk_\perp}\frac{k_\perp}{k_l}+O(\omega^{-1}).$$

□

5.4. High frequency expansion of the DTN Fourier multiplier for a fixed mode

In this section, for p fixed, we describe a high frequency equivalent ($\omega\to+\infty$, or $|k_3|\to+\infty$) of each mode of the Dirichlet to Neumann operator.

As it is easier to manipulate the oscillating phases $e^{\pm i\Psi}$ hence we shall use $(H_p^{(1)},H_p^{(2)})$ as the suitable pair of independent solutions for this analysis and use the equality (33) of Section 4.3. One gets

Proposition 12. *Let $C>0$ and $|p|\le M$ is fixed. Assume $\frac{|k_z|}{\omega}<\Re\epsilon\mu$.*

- *The leading order term, for $\omega\to+\infty$ of the Fourier multiplier for the Dirichlet to Neumann operator is $C(p,k_z)\simeq k_3\frac{\cos k_3l}{\sin k_3l}$. Note that the influence of the Fourier mode has just disappeared in this equivalent; it gives rise to a lower order term.*
- *One has the estimate*

$$C(p,k_z)=k_3\frac{\cos k_3l+\frac{1}{8k_3}(\frac{4p^2-1}{r_0}-\frac{4p^2+3}{R})\sin k_3l}{\sin k_3l-\frac{1}{8k_3}(\frac{4p^2-1}{r_0}-\frac{4p^2-1}{R})\cos k_3l}(1+O(\omega^{-2})).$$

- *In the case $\Im\epsilon\mu\neq0$, ($\Im k_3\to-\infty$ when $\omega\to+\infty$),*

$$C(p,k_z)=(ik_3-\frac{1}{2R}+O(\omega^{-1}))(1+O(e^{2\Im k_3l})).$$

Remark 7. *Note that the Item 2 of Proposition 12 shows the dependency of the Fourier multiplier in terms of the radius of curvature. This dependency is not uniform in p because the remainder term is not.*

Proof. Let us prove first the last item. Using the equivalence $\cos k_3l\simeq\frac{1}{2}e^{-ik_3l}$, $\sin k_3l\simeq\frac{i}{2}e^{-ik_3l}$, thanks to $\Re k_3>0$, $\Im k_3<0$, one gets

$$C(p,k_z)\simeq k_3\frac{1+\frac{i}{8k_3}(\frac{4p^2-1}{r_0}+\frac{4p^2+3}{R})}{-i+\frac{1}{8k_3}(\frac{4p^2-1}{r_0}-\frac{4p^2-1}{R})}\simeq ik_3(1+\frac{i}{8k_3}(\frac{4p^2-1}{r_0}+\frac{4p^2+3}{R})-\frac{i}{8k_3}(\frac{4p^2-1}{r_0}-\frac{4p^2-1}{R})),$$

hence the result.

For proving the first item, it is enough to replace P by 1 and Q by 0. In this case $H_p^{(1)}(z)\simeq\frac{2}{\pi z}e^{i\chi_p(z)}$, $(H_p^{(1)})'(z)\simeq\frac{2}{\pi z}ie^{i\chi_p(z)}$, and one has thus

$$C(p,k_z)\simeq k_3\frac{e^{i\chi_p(k_3r_0)}(-ie^{-i\chi_p(k_3R)})-e^{-i\chi_p(k_3r_0)}ie^{i\chi_p(k_3R)}}{e^{i\chi_p(k_3r_0)}ie^{-i\chi_p(k_3R)}-e^{-i\chi_p(k_3r_0)}e^{i\chi_p(k_3R)}},$$

that is

$$C(p,k_z)\simeq k_3\frac{\cos k_3(R-r_0)}{\sin k_3(R-r_0)}=ik_3\frac{\cos k_3l}{\sin k_3l},l=R-r_0.$$

The expansion at a higher order yields the following term of the expansion in k_3, it depends on p. One uses

$$\begin{aligned} H_p^{(1)}(kr_0)(H_p^{(2)})'(kR) &= \frac{2}{\pi\sqrt{r_0R}}e^{i\chi_p(r_0)-i\chi_p(R)}(1+i\frac{4p^2-1}{8k_3r_0})(-i-\frac{4p^2+3}{8k_3R})(1+O(k_3^{-2})) \\ &= \frac{2}{\pi\sqrt{r_0R}}e^{i\chi_p(r_0)-i\chi_p(R)}(-i+\frac{1}{8k_3}(\frac{4p^2-1}{r_0}-\frac{4p^2+3}{R})+O(k_3^{-2})), \end{aligned}$$

as well as the conjugate term $H_p^{(2)}(kr_0)(H_p^{(1)})'(kR)$, and

$$\begin{aligned} H_p^{(1)}(kr_0)H_p^{(2)}(kR) &= \frac{2}{\pi\sqrt{r_0R}}e^{i\chi_p(r_0)-i\chi_p(R)}(1+i\frac{4p^2-1}{8k_3r_0})(1-i\frac{4p^2-1}{8k_3R})(1+O(k_3^{-2})) \\ &= \frac{2}{\pi\sqrt{r_0R}}e^{i\chi_p(r_0)-i\chi_p(R)}(1+i\frac{(4p^2-1)}{8k_3}(\frac{1}{r_0}-\frac{1}{R})+O(k_3^{-2})), \end{aligned}$$

and its conjugate expression $H_p^{(2)}(kr_0)H_p^{(1)}(kR)$.

Collecting, one deduces

$$\begin{aligned} C(p,k_z) &= k_3\frac{e^{i\chi_p(r_0)-i\chi_p(R)}(-i+\frac{1}{8k_3}(\frac{4p^2-1}{r_0}-\frac{4p^2+3}{R}))-e^{-i\chi_p(r_0)+i\chi_p(R)}(i+\frac{1}{8k_3}(\frac{4p^2-1}{r_0}-\frac{4p^2+3}{R}))+O(k_3^{-2})}{e^{i\chi_p(r_0)-i\chi_p(R)}(1+\frac{i(4p^2-1)}{8k_3}(\frac{1}{r_0}-\frac{1}{R}))-e^{-i\chi_p(r_0)+i\chi_p(R)}(1-\frac{i(4p^2-1)}{8k_3}(\frac{1}{r_0}-\frac{1}{R}))+O(k_3^{-2})} \\ &= k_3\frac{-2i\cos k_3(r_0-R)+\frac{1}{8k_3}(\frac{4p^2-1}{r_0}-\frac{4p^2+3}{R})(2i\sin k_3(r_0-R))+O(k_3^{-2})}{2i\sin k_3(r_0-R)+2i\cos k_3(r_0-R)\frac{(4p^2-1)}{8k_3}(\frac{1}{r_0}-\frac{1}{R})+O(k_3^{-2})} \\ &= k_3\frac{\cos k_3l+\frac{1}{8k_3}(\frac{4p^2-1}{r_0}-\frac{4p^2+3}{R})\sin k_3l+O(k_3^{-2})}{\sin k_3l-\frac{(4p^2-1)}{8k_3}(\frac{1}{r_0}-\frac{1}{R})\cos k_3l+O(k_3^{-2})}. \end{aligned}$$

This proves the second Item of Proposition 12 when p is fixed and ω goes to $+\infty$ (including the case $k_z=\eta\omega$, $\eta\in\mathbb{R}$).

Finally $\frac{\cos k_3l}{\sin k_3l}=i\frac{e^{ik_3l}+e^{-ik_3l}}{e^{ik_3l}-e^{-ik_3l}}=i\frac{e^{-2ik_3l}+1}{1-e^{-2ik_3l}}\to i, \omega\to+\infty$. □

We shall now study a complete asymptotic regime (p, k_z, ω going to ∞).

5.5. High frequency analysis for the exact solution in cylindrical coordinates in the high frequency regime in (k_z, p): the hyperbolic region.

The case studied in this section is

$$\Re\epsilon\mu\omega^2-k_z^2-\frac{p^2}{R^2}>\Re\epsilon\mu\omega^2-k_z^2-\frac{p^2}{r_0^2}>0.$$

In the totally high frequency regime we are able to find the analogous of Proposition 12.

Proposition 13. *Assume $\frac{k_z}{\omega}=\eta$, $|\eta|<\sqrt{\Re\epsilon\mu}$ and $\omega^2\Re\epsilon\mu-k_z^2-\frac{p^2}{r_0^2}>0$. Let β_*,β_0 be given by $\cos\beta_*=\frac{k_3}{\sqrt{\epsilon\mu\omega^2-k_z^2-\frac{p^2}{R^2}}}=\frac{\sqrt{\epsilon\mu\omega^2-k_z^2}}{\sqrt{\epsilon\mu\omega^2-k_z^2-\frac{p^2}{R^2}}}$ (and a similar definition for β_0). Denote by $\Psi_*=p(\tan\beta_*-\beta_*)$, $\Psi_0=p(\tan\beta_0-\beta_0)$. When $J_p(k_3r_0)Y_p(k_3R)-Y_P(k_3r_0)J_p(k_3R)\neq0$, the Dirichlet to Neumann operator is equal to*

$$C(p,k_z)=-ik_3\sin\beta_*\frac{(L_0-iM_0)(N_*-iO_*)+(L_0+iM_0)(N_*+iO_*)e^{-2i(\Psi_*-\Psi_0)}}{(L_0-iM_0)(L_*+iM_*)-(L_0+iM_0)(L_*-iM_*)e^{-2i(\Psi_*-\Psi_0)}}.$$

Let $\theta=\frac{p}{\omega R}$. Define in addition $k_\perp=\sqrt{\epsilon\mu\omega^2-k_z^2-\omega^2\theta^2}$ as the (complex) normal transmitted wave number, the tangent waves vectors are $(\omega\theta, k_z)$, and β_* is an angle of incidence.

Proposition 14. *Assume $\Im \epsilon\mu < 0$ independent on ω. The following asymptotic expansion holds:*

$$\begin{aligned} C(p,k_z) &= [i\sqrt{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}} - \frac{1}{R}\frac{\sqrt{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}}}{\frac{p}{R}}(a_* + b_*) + O(\omega^{-1})](1 + O(e^{-2\Im(\Psi_* - \Psi_0)})) \\ &= [ik_\perp - \frac{1}{2R}\frac{k_3^2}{k_\perp^2} + O(\omega^{-1})](1 + O(e^{-2\omega\delta_*})) \end{aligned}$$

Proof. Using Bessel functions toolbox, one obtains

$$C(p,k_z) = k_3 \frac{(H_p^{(2)})'(\frac{p}{\cos\beta_*})H_p^{(1)}(\frac{p}{\cos\beta_0}) - (H_p^{(1)})'(\frac{p}{\cos\beta_*})H_p^{(2)}(\frac{p}{\cos\beta_0})}{H_p^{(2)}(\frac{p}{\cos\beta_*})H_p^{(1)}(\frac{p}{\cos\beta_0}) - H_p^{(1)}(\frac{p}{\cos\beta_*})H_p^{(2)}(\frac{p}{\cos\beta_0})},$$

$$C(p,k_z) = -ik_3 \frac{\sqrt{\frac{\sin 2\beta_*}{\pi p}}\sqrt{\frac{2}{\pi p\tan\beta_0}}(e^{i\Psi_0 - i\Psi_*}(L_0 - iM_0)(N_* - iO_*) + (N_* + iO_*)(L_0 + iM_0)e^{i\Psi_* - i\Psi_0})}{\sqrt{\frac{2}{\pi p\tan\beta_0}}\sqrt{\frac{2}{\pi p\tan\beta_*}}(e^{i\Psi_0 - i\Psi_*}(L_0 - iM_0)(L_* + iM_*) - (L_0 + iM_0)(L_* - iM_*)e^{i\Psi_* - i\Psi_0})}.$$

The expansion in inverse powers of ω is given through

$$\begin{aligned} C(p,k_z) &= -ik_3\sin\beta_* \frac{e^{i\Psi_0 - i\Psi_*}(1 - i\frac{a_0}{p})(1 - i\frac{b_*}{p}) + (1 + i\frac{b_*}{p})(1 + i\frac{a_0}{p})e^{i\Psi_* - i\Psi_0}}{e^{i\Psi_0 - i\Psi}(1 - i\frac{a_0}{p})(1 + i\frac{a_*}{p}) - (1 + i\frac{a_0}{p})(1 - i\frac{a_*}{p})e^{i\Psi_* - i\Psi_0}} \\ &= k_3\sin\beta_* \frac{\cos(\Psi_* - \Psi_0) - \frac{a_0 + b_*}{p}\sin(\Psi_* - \Psi_0)}{\sin(\Psi_* - \Psi_0) - \frac{a_* - a_0}{p}\cos(\Psi_* - \Psi_0)}(1 + O(p^{-2})). \end{aligned}$$

Lemma 14 shows that $\Re i(\Psi(\beta_*) - \Psi(\beta_0)) > 0$, hence there exists δ_0 such that $e^{-2i(\Psi(\beta_*) - \Psi(\beta_0))} = O(e^{-2\delta_0\omega})$. Using

$$C(p,k_z) = -ik_3\sin\beta_* \frac{e^{i\Psi_0 - i\Psi_*}(L_0 - iM_0)(N_* - iO_*) + (N_* + iO_*)(L_0 + iM_0)e^{i\Psi_* - i\Psi_0}}{e^{i\Psi_0 - i\Psi_*}(L_0 - iM_0)(L_* + iM_*) - (L_0 + iM_0)(L_* - iM_*)e^{i\Psi_* - i\Psi_0}},$$

one obtains

$$C(p,k_z) = ik_3\sin\beta_* \frac{N_* + iO_*}{L_* - iM_*}(1 + O(e^{-2\Im(\Psi_* - \Psi_0)})),$$

from which one deduces

$$C(p,k_z) = (ik_3\sin\beta_* - k_3\sin\beta_* \frac{a(\beta_*) + b(\beta_*)}{p} + O(\omega^{-1}))(1 + O(e^{-2\Im(\Psi_* - \Psi_0)})).$$

As

$$k_3\sin\beta_* \frac{a(\beta_*) + b(\beta_*)}{p} = pR\frac{1}{2p\tan\beta_*}(1 + \frac{1}{\tan^2\beta_*}) = \frac{1}{2R\sin^2\beta_*},$$

we get that the leading order term of the Dirichlet to Neumann operator is

$$i\omega\sqrt{\epsilon\mu - \frac{k_z^2}{\omega^2} - \frac{p^2}{\omega^2 R^2}} = ik_\perp$$

and that the lower order term is

$$-\frac{1}{2R\sin^2\beta_*} = -\frac{1}{2R}\frac{\epsilon\mu\omega^2 - k_z^2}{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}} = -\frac{1}{2R}\frac{k_3^2}{k_\perp^2}.$$

Note that this result can also be obtained as a consequence of the limit

$$\frac{\cos(\Psi_* - \Psi_0)}{\sin(\Psi_* - \Psi_0)} = -i\frac{1 + e^{2i(\Psi_* - \Psi_0)}}{1 + e^{2i(\Psi_* - \Psi_0)}} \to -i.$$

□

5.6. Dirichlet to Neumann operator for the mixed- hyperbolic-elliptic case

We are in the case

$$\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2} > 0 > \Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r_0^2}.$$

Note that, contrary to the hyperbolic regime, an exponentially growing term appears both in $H_p^{(1)}(k_3 r_0)$ and in $H_p^{(2)}(k_3 r_0)$. This corresponds to the branchs identified in [17] for the decomposition near a turning point of the solution of a second order ODE.

Proposition 15. *Assume $\Im\epsilon\mu < 0$, independent on ω and $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2} > 0 > \Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r_0^2}$.*

The Dirichlet to Neumann operator is given by

$$\begin{aligned} C(p,k_z) &= ik_3 \sin\beta_* \tfrac{N_*+iO_*}{L_*-iM_*}(1+O(e^{-2\Re(i\Psi_*)}))(1+O(e^{\frac{4}{3}p\Re\zeta_0^{\frac{3}{2}}})) \\ &= (ik_\perp - \tfrac{k_3^2}{2Rk_\perp^2} + O(p^{-2}))(1+O(e^{\frac{4}{3}p\Re\zeta_0^{\frac{3}{2}}}))(1+O(e^{-2\Re(i\Psi_*)})). \end{aligned}$$

Proof. Use

$$C(p,k_z) = k_3 \frac{(H_p^{(1)})'(k_3R)H_p^{(2)}(k_3r_0) - (H_p^{(2)})'(k_3R)H_p^{(1)}(k_3r_0)}{(H_p^{(1)})'(k_3R)H_p^{(2)}(k_3r_0) - (H_p^{(2)})'(k_3R)H_p^{(1)}(k_3r_0)} = k_3 \frac{(H_p^{(1)})'(k_3R)\frac{H_p^{(2)}(pz_0)}{H_p^{(1)}(pz_0)} - (H_p^{(2)})'(k_3R)}{(H_p^{(1)})'(k_3R)\frac{H_p^{(2)}(pz_0)}{H_p^{(1)}(pz_0)} - (H_p^{(2)})'(k_3R)}.$$

We use the expressions (39) which are more convenient. We assume that .

Assume first that $\Re\zeta_0^{\frac{3}{2}} < 0$. In this case, $Ai(p^{\frac{2}{3}}\zeta)$ is exponentially growing and $Bi(p^{\frac{2}{3}}\zeta)$ is exponentially decaying. Through

$$\frac{H_p^{(2)}(pz_0)}{H_p^{(1)}(pz_0)} = \frac{J_p(pz_0) - iY_p(pz_0)}{J_p(pz_0) + iY_p(pz_0)}$$

one observes that this term is of order of magnitude 1. Indeed, one observes first that

$$A(p,\zeta_0)Ai(p^{\frac{2}{3}}\zeta_0) - \zeta_0^{-\frac{1}{2}}p^{-\frac{4}{3}}B(p,\zeta_0)Ai'(p^{\frac{2}{3}}\zeta_0) = A(p,\zeta_0)Ai(p^{\frac{2}{3}}\zeta_0)[1 - \frac{Ai'}{Ai}(p^{\frac{2}{3}}\zeta_0)p^{-\frac{4}{3}}\zeta^{-\frac{1}{2}}\frac{B}{A}(p,\zeta_0)],$$

and one has $\frac{B}{A}(p,\zeta_0) = 1 + O(p^{-2})$ and $\frac{Ai'}{Ai}(p^{\frac{2}{3}}\zeta_0)p^{-\frac{4}{3}}\zeta^{-\frac{1}{2}} = -p^{-1}(1+O(p^{-1}))$ (the last estimate coming from the expansion of the Airy function with terms of order $X^{-\frac{3n}{2}}$).

Hence

$$A(p,\zeta_0)Ai(p^{\frac{2}{3}}\zeta_0) - \zeta_0^{-\frac{1}{2}}p^{-\frac{4}{3}}B(p,\zeta_0)Ai'(p^{\frac{2}{3}}\zeta_0) = A(p,\zeta_0)Ai(p^{\frac{2}{3}}\zeta_0)[1 + \frac{1}{p} + O(p^{-2})]. \tag{59}$$

In a similar fashion, thanks to $\frac{Bi'}{Bi}(p^{\frac{2}{3}}\zeta_0)p^{-\frac{4}{3}}\zeta^{-\frac{1}{2}} = p^{-1}(1+O(p^{-1}))$,

$$A(p,\zeta_0)Bi(p^{\frac{2}{3}}\zeta_0) - \zeta_0^{-\frac{1}{2}}p^{-\frac{4}{3}}B(p,\zeta_0)Bi'(p^{\frac{2}{3}}\zeta_0) = A(p,\zeta_0)Bi(p^{\frac{2}{3}}\zeta_0)[1 - \frac{1}{p} + O(p^{-2})]. \tag{60}$$

One deduces, thanks to $\frac{Bi}{Ai}(p^{\frac{2}{3}}\zeta_0) = O(e^{\frac{4}{3}p\Re\zeta_0^{\frac{3}{2}}})$,

$$\frac{H_p^{(2)}(pz_0)}{H_p^{(1)}(pz_0)} = \frac{J_p(pz_0) - iY_p(pz_0)}{J_p(pz_0) + iY_p(pz_0)} = \frac{1 + \frac{1}{p} + O(p^{-2}) - iO(e^{\frac{4}{3}p\Re\zeta_0^{\frac{3}{2}}})}{1 + \frac{1}{p} + O(p^{-2}) + iO(e^{\frac{4}{3}p\Re\zeta_0^{\frac{3}{2}}})} = 1 + O(e^{\frac{4}{3}p\Re\zeta_0^{\frac{3}{2}}}).$$

As

$$C(p,k_z) = -ik_3\sin\beta_* \frac{(N_* - iO_*)e^{-i\Psi_*} + (N_* + iO_*)e^{i\Psi_*}\frac{H_p^{(2)}(pz_0)}{H_p^{(1)}(pz_0)}}{(L_* + iM_*)e^{-i\Psi_*} - (L_* - iM_*)e^{-i\Psi_*}\frac{H_p^{(2)}(pz_0)}{H_p^{(1)}(pz_0)}},$$

one obtains the same estimate as before in the case $\Im\epsilon\mu < 0$ independent on ω, which is the estimate based on $\Re(i\Psi_*) > 0$:

$$ik_3\sin\beta_*\frac{N_* + iO_*}{L_* - iM_*}(1+O(e^{-2\Re(i\Psi_*)}))(1+O(e^{\frac{4}{3}p\Re\zeta_0^{\frac{3}{2}}})) = ik_3\sin\beta_*(1+i\frac{a(\beta_*)+b(\beta_*)}{p}+O(p^{-2}))(1+O(e^{-2\Re(i\Psi_*)}))(1+O(e^{\frac{4}{3}p\Re\zeta_0^{\frac{3}{2}}}))$$

This ends the proof of Proposition 15. □

5.7. Dirichlet to Neumann operator in the elliptic region

The case studied in this Section is

$$\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2} < \Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r_0^2} < 0.$$

One introduces, according to the Bessel functions toolbox:

$$z_0 = \frac{k_3 r_0}{p}, z_* = \frac{k_3 R}{p}, \zeta_*, \zeta_0.$$

Denote by A_0, B_0 the quantities $A(\zeta_0, p), B(\zeta_0, p)$, and A_*, B_*, C_*, D_* the quantities $A(\zeta_*, p), B(\zeta_*, p), C(\zeta_*, p), D(\zeta_*, p)$.

Proposition 16. 1. *The Dirichlet to Neumann operator is*

$$C(p,k_z) = k_3\frac{2}{z_*}\sqrt{\frac{1-z_*^2}{4\zeta_*}}p^{-\frac{1}{3}}\frac{A_0D_*(1+\tau_*)(1+\tilde{k}_0)Ai'(p^{\frac{2}{3}}\zeta_*)Bi(p^{\frac{2}{3}}\zeta_0) - A_0D_*(1+\tilde{\tau}_*)(1+k_0)Bi'(p^{\frac{2}{3}}\zeta_*)Ai(p^{\frac{2}{3}}\zeta_0)}{A_0A_*(1+k_*)(1+\tilde{k}_0)Ai(p^{\frac{2}{3}}\zeta_*)Bi(p^{\frac{2}{3}}\zeta_0) - A_0D_*(1+\tilde{k}_*)(1+k_0)Bi(p^{\frac{2}{3}}\zeta_*)Ai(p^{\frac{2}{3}}\zeta_0)}.$$

2. *Assume $\Im\epsilon\mu < 0$, independent on ω and assume $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r_0^2} < 0$. One has*

$$C(p,k_z) = [\sqrt{k_z^2 + \frac{p^2}{R^2} - \epsilon\mu\omega^2} - \frac{1}{R}\frac{\sqrt{k_z^2 + \frac{p^2}{R^2} - \epsilon\mu\omega^2}}{\frac{p}{R}}\zeta_*^{-\frac{3}{2}}(d_1 - c_1) + O(p^{-1})](1 + O(e^{-\frac{4}{3}p\Re(\zeta_0^{\frac{3}{2}} - \zeta_*^{\frac{3}{2}})})).$$

Proof. Instead of using the Hankel functions in this case, we use the expressions (39). Indeed, the behavior in the elliptic case is rather an exponentially growing or decaying solution instead of oscillatory representations. One has

$$C(p,k_z)=k_3\frac{J_p'(pz_*)Y_p(pz_0)-Y_p'(pz_*)J_p(pz_0)}{J_p(pz_*)Y_p(pz_0)-Y_p(pz_*)J_p(pz_0)}.$$

Assume $\Im\epsilon\mu<0$ independent on ω. We use Lemma 13.

One has $\frac{2}{3}p\Re(\zeta_0^{\frac{3}{2}}-\zeta_*^{\frac{3}{2}})>0$ hence $Ai(p^{\frac{2}{3}}\zeta_*)Bi(p^{\frac{2}{3}}\zeta_0)$ is the leading term in these equalities, hence $J_p(pz_*)Y_p(pz_0)$ and $J_p'(pz_*)Y_p(pz_0)$ are the two dominant terms, hence

$$C(p,k_z)=k_3\frac{J_p'(pz_*)Y_p(pz_0)}{J_p(pz_*)Y_p(pz_0)}\frac{1-\frac{Y_p'(pz_*)J_p(pz_0)}{J_p'(pz_*)Y_p(pz_0)}}{1-\frac{Y_p(pz_*)J_p(pz_0)}{J_p(pz_*)Y_p(pz_0)}}=k_3\frac{J_p'(pz_*)}{J_p(pz_*)}(1+O(e^{-\frac{4}{3}p\Re(\zeta_0^{\frac{3}{2}}-\zeta_*^{\frac{3}{2}})})).$$

One has

$$\frac{J_p'(pz_*)}{J_p(pz_*)}=-\frac{2}{z_*}(\frac{1-z_*^2}{4\zeta_*})^{\frac{1}{2}}p^{-\frac{1}{3}}\frac{D(\zeta_*,p)}{A(\zeta_*,p)}\frac{Ai'(p^{\frac{2}{3}}\zeta_*)}{Ai(p^{\frac{2}{3}}\zeta_*)}\frac{1-p^{-\frac{2}{3}}\zeta_*^{\frac{1}{2}}\frac{C}{D}\frac{Ai}{Ai'}(p^{\frac{2}{3}}\zeta_*)}{1-p^{-\frac{4}{3}}\zeta_*^{-\frac{1}{2}}\frac{B}{A}\frac{Ai'}{Ai}(p^{\frac{2}{3}}\zeta_*)}.$$

We observe that

$$-p^{-\frac{2}{3}}\zeta_*^{\frac{1}{2}}\frac{C}{D}\frac{Ai}{Ai'}(p^{\frac{2}{3}}\zeta_*)=\frac{C}{D}p^{-1}(-(p^{\frac{2}{3}}\zeta_*)^{\frac{1}{2}}\frac{Ai}{Ai'}(p^{\frac{2}{3}}\zeta_*))\simeq p^{-1},$$
$$-p^{-\frac{4}{3}}\zeta_*^{-\frac{1}{2}}\frac{B}{A}\frac{Ai'}{Ai}(p^{\frac{2}{3}}\zeta_*)=\frac{B}{A}p^{-1}(-(p^{\frac{2}{3}}\zeta_*)^{-\frac{1}{2}}\frac{Ai'}{Ai}(p^{\frac{2}{3}}\zeta_*))\simeq p^{-1},$$

hence

$$\frac{1-p^{-\frac{2}{3}}\zeta_*^{\frac{1}{2}}\frac{C}{D}\frac{Ai}{Ai'}(p^{\frac{2}{3}}\zeta_*)}{1-p^{-\frac{4}{3}}\zeta_*^{-\frac{1}{2}}\frac{B}{A}\frac{Ai'}{Ai}(p^{\frac{2}{3}}\zeta_*)}=1+O(p^{-2}).$$

One has also $\frac{D}{A}(\zeta_*,p)=1+\frac{D_1(\zeta_*)-A_1(\zeta_*)}{p^2}+O(p^{-4})=1+O(p^{-2})$. One is thus left with the estimate of $\frac{Ai}{Ai'}(p^{\frac{2}{3}}\zeta_*)$, where we use 10.4.59 of [1], which yields

$$-(p^{\frac{2}{3}}\zeta_*)^{-\frac{1}{2}}\frac{Ai'}{Ai}(p^{\frac{2}{3}}\zeta_*)=1-\frac{d_1-c_1}{(p^{\frac{2}{3}}\zeta_*)^{\frac{3}{2}}}+O(p^{-2})=1-\frac{d_1-c_1}{p}\zeta_*^{-\frac{3}{2}}+O(p^{-2}),$$

hence

$$\frac{J_p'(pz_*)}{J_p(pz_*)}=(\frac{1-z_*^2}{z_*})^{\frac{1}{2}}(1-\frac{d_1-c_1}{p}\zeta_*^{-\frac{3}{2}})(1+O(p^{-2})).$$

One has $k_3\frac{\sqrt{1-z_*^2}}{z_*}=\sqrt{k_z^2+\frac{p^2}{R^2}-\epsilon\mu\omega^2}$, hence the result of the proposition.

□

Remark 8. *Surprisingly, the expression of the lower order term in $\frac{1}{R}$ is not the same for the hyperbolic and the elliptic regime.*

6. Dirichlet to Neumann operator for elliptic layers

6.1. The Dirichlet to Neumann operator

Thanks to the representation with Mathieu and modified Mathieu functions, one can solve (rather explicitely) the Dirichlet problem with homogeneous Dirichlet condition on $\{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\}:=\partial\Omega$. Using the notation ρ, u_0 as above, one gets

Lemma 18. *Let C be a bounded open set of $\mathbb{R}^2$. Assume that $\{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\}\subset C$, with $d(\partial\Omega,\partial C)>0$. Define $\rho=\sqrt{a^2-b^2}$ and u_0 such that $\tanh u_0=\frac{b}{a}$. Define u_1 such that $C\subset\{(x,y),\frac{x^2}{\cosh^2 u_1}+\frac{y^2}{\sinh^2 u_1}\leq 1\}$. The function U is a $C^2(C)$ solution of the Helmholtz equation which is equal to 0 on the ellipse if and only if there exists $\alpha_n, n\in\mathbb{Z}$ such that*

$$U(x,y)=\sum_{n\in\mathbb{Z}}\alpha_n(C_{|n|}(u)S_{|n|}(u_0)-C_{|n|}(u_0)S_{|n|}(u))g_n(v),$$

provided that $\gamma_n: U\to\frac{1}{2\pi}\int_0^{2\pi}U(x,y)g_n(v)dv$ is continuous for each n. Note that $e^{ik_z z}u$ is NOT the Fourier transform of a solution of $(\Delta+\omega^2\epsilon\mu)U=0$ because one cannot assert, for the moment, that each term in in $\mathcal{S}'$ as estimates on α_n are not provided.

The first item is a consequence of the continuity (note that the space where U lives is not given here, and such equalities are only formal if the sum is infinite) of the application γ_n for calculation the Dirichlet boundary condition.

Once all these results are proven, we are ready to study two cases for evaluating the Dirichlet to Neumann multiplier, the first subsection below deals with the case of the cofocal ellipses (where the parameter ρ is the same for both ellipses) and the second subsection deals with the case of homothetic ellipses.

6.2. Dirichlet to Neumann operator for an elliptic-type layer with same focal points for the two boundaries

B. Stupfel [28] already used such an approach to study the following problem: Ω is as usual a perfectly conducting body, supplemented with a layer which is also with an elliptic boundary $\Gamma_1=\{(x,y),\frac{x^2}{a_1^1}+\frac{y^2}{b_1^2}=1\}$, under the assumption that $a_1^2-b_1^2=a^2-b^2$, $a_1\geq a$, $b_1\geq b$. This means that the two elliptic boundaries $\partial\Omega$ and $\Gamma-1$ are ellipses sharing the same focal points. In this set-up, if one denotes by u_1 such that $\tanh u_1=\frac{b_1}{a_1}$, then $\{(x,y),\frac{x^2}{a_1^1}+\frac{y^2}{b_1^2}\leq 1\}-\Omega=\{u_0\leq u\leq u_1, v\in[0,2\pi]\}$. In this case, the result reads exactly as in the case of the cylinder. It is stated in Section 6 of [28], using expressions (67), (70) and all expressions that follow. The coefficients of the Calderòn operator are given by the results of [28], but we rephrase it in the language of operators.

Lemma 19. *The Dirichlet to Neumann multiplier acts on the orthonormal Hilbert base of $L^2([0,2\pi])$ $\{g_m\}$ as:*

$$\begin{aligned}C(k_z)(U) &= \textstyle\sum_n \frac{C'_{|n|}(u_1)S_{|n|}(u_0)-C_{|n|}(u_0)S'_{|n|}(u_1)}{C_{|n|}(u_1)S_{|n|}(u_0)-C_{|n|}(u_0)S_{|n|}(u_1)}\frac{g_n(v)}{\rho\sqrt{\cosh^2 u_1\sin^2 v+\sinh^2 u_1\cos^2 v}}U_n\\ &= \textstyle\sum_{n,m}\frac{C'_{|n|}(u_1)S_{|n|}(u_0)-C_{|n|}(u_0)S'_{|n|}(u_1)}{C_{|n|}(u_1)S_{|n|}(u_0)-C_{|n|}(u_0)S_{|n|}(u_1)}\rho^{-1}K_{nm}U_n g_m(v)\end{aligned}$$

for $U=\sum_n U_n g_n(v)$, or

$$\int_0^{2\pi}C(k_z)(U)(w)g_m(w)dw=\sum_{n,m}\frac{C'_{|n|}(u_1)S_{|n|}(u_0)-C_{|n|}(u_0)S'_{|n|}(u_1)}{C_{|n|}(u_1)S_{|n|}(u_0)-C_{|n|}(u_0)S_{|n|}(u_1)}K_{nm}\int_0^{2\pi}U(v')g_n(v')dv'.$$

It is a discrete pseudo-differential operator.[5]

Remark 9. *This result is easier to obtain, when one considers the Dirichlet boundary condition on the inner boundary, when one is able to decompose the solution on periodic modes.*

Proof. Let us start by identifying the normal derivative, which is the crucial point of this first result. A tangent vector to the boundary is $\vec{t} = (-\rho \cosh u \sin v, \rho \sinh u \cos v)$, hence

$$\vec{n} = \frac{1}{\sqrt{\cosh^2 u \sin^2 v + \sinh^2 u \cos^2 v}}(\sinh u \cos v, \cosh u \sin v),$$

hence

$$\partial_n f = \frac{1}{\rho\sqrt{\cosh^2 u_1 \sin^2 v + \sinh^2 u_1 \cos^2 v}}\partial_u f.$$

We thus deduce

$$\begin{aligned}\partial_n U &= \frac{1}{\rho\sqrt{\cosh^2 u_1 \sin^2 v+\sinh^2 u_1 \cos^2 v}} \sum_{n\in\mathbb{Z}} \alpha_n (C'_{|n|}(u_1)S_{|n|}(u_0) - C_{|n|}(u_0)S'_{|n|}(u_1))g_n(v) \\ &= \rho^{-1} \sum_{n\in\mathbb{Z}} \alpha_n (C'_{|n|}(u_1)S_{|n|}(u_0) - C_{|n|}(u_0)S'_{|n|}(u_1)) \frac{g_n(v)}{\sqrt{\cosh^2 u_1 \sin^2 v+\sinh^2 u_1 \cos^2 v}}\end{aligned}$$

We then define the matrix

$$K_{mn} = \frac{1}{2\pi}\int_0^{2\pi} \frac{g_n(v)g_m(v)}{\sqrt{\cosh^2 u_1 \sin^2 v + \sinh^2 u_1 \cos^2 v}} dv$$

and use

$$\frac{g_n(v)}{\sqrt{\cosh^2 u_1 \sin^2 v + \sinh^2 u_1 \cos^2 v}} = \sum_m K_{nm} g_m(v)$$

to obtain the result of Lemma 19.

□

As in the case of the Bessel functions, we have a classification of the regions to be studied. The following classification applies:

1. totally elliptic region if $\Re a_n(k_3\rho) - \frac{\Re\epsilon\mu\omega^2 - k_z^2}{2}\cosh 2u_0 > \Re a_n(k_3\rho) - \frac{\Re\epsilon\mu\omega^2 - k_z^2}{2}\cosh 2u_1 > 0$,
2. mixed elliptic-hyperbolic region if $\Re a_n(k_3\rho) - \frac{\Re\epsilon\mu\omega^2 - k_z^2}{2}\cosh 2u_0 > 0 > \Re a_n(k_3\rho) - \frac{\Re\epsilon\mu\omega^2 - k_z^2}{2}\cosh 2u_1$,
3. hyperbolic region if $0 > \Re a_n(k_3\rho) - \frac{\Re\epsilon\mu\omega^2 - k_z^2}{2}\cosh 2u_0 > \Re a_n(k_3\rho) - \frac{\Re\epsilon\mu\omega^2 - k_z^2}{2}\cosh 2u_1$,
4. top glancing region if $\Re a_n(k_3\rho) - \frac{\Re\epsilon\mu\omega^2 - k_z^2}{2}\cosh 2u_0 \simeq 0$.

Proposition 17. *The asymptotics of the coefficient* $M_n = \frac{C'_{|n|}(u_1)S_{|n|}(u_0) - C_{|n|}(u_0)S'_{|n|}(u_1)}{C_{|n|}(u_1)S_{|n|}(u_0) - C_{|n|}(u_0)S_{|n|}(u_1)}$ *is, in the hyperbolic regime (and* $\omega\eta = k_z < \sqrt{\Re\epsilon\mu}\omega$*)*

$$M_n = \pm i\sqrt{a_n(k_3\rho)}\sqrt{\frac{k_3^2\rho^2}{2a_n(k_3\rho)}\cosh 2u_1 - 1}(1 + O(\omega^{-2})) = \pm\omega\rho\sqrt{\frac{\epsilon\mu - \eta^2}{2}\cosh 2u_1 - \frac{a_n(k_3\rho)}{\rho^2\omega^2}}(1 + O(\omega^{-2})).$$

[5]The action of a discrete pseudo-differential operator expresses as $Op(a(v,n))(U)(v) = \sum_n a(v,n)g_n(v)U_n$, by comparison with $Op(a(v,\xi))f(x) = \frac{1}{2\pi}\int e^{ix.\xi}a(v,\xi)\hat{f}(\xi)d\xi$.

Remark 10. *We notice that the Helmholtz equation (after Fourier transform in z) writes*

$$[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + \frac{1}{2}(\epsilon\mu\omega^2 - k_z^2)\rho^2(\cosh 2u - \cos 2v)]u = 0,$$

and the mode analysis of this equation yields

$$[\frac{\partial^2}{\partial u^2} + \frac{1}{2}(\epsilon\mu\omega^2 - k_z^2)\rho^2 \cosh 2u - a_n(k_3\rho)]u = 0,$$

which principal symbol leads exactly to $\pm i\omega\rho\sqrt{\frac{1}{2}(\epsilon\mu - \eta^2)\cosh 2u - \frac{a_n(k_3\rho)}{\rho^2\omega^2}}$.

Remark 11. *The coefficient obtained is thus*

$$\pm i\frac{\sqrt{\frac{\epsilon\mu-\eta^2}{2}\cosh 2u_1 - \frac{a_n(k_3\rho)}{\rho^2\omega^2}}}{\sqrt{\cosh^2 u_1 \sin^2 v + \sinh^2 u_1 \cos^2 v}} = \pm i\frac{\sqrt{\frac{\epsilon\mu-\eta^2}{2}\cosh^2 u_1 + (\frac{\epsilon\mu-\eta^2}{2} - \frac{a_n(k_3\rho)}{\rho^2\omega^2})\sinh^2 u_1}}{\sqrt{\cosh^2 u_1 \sin^2 v + \sinh^2 u_1 \cos^2 v}},$$

which shows the influence of the radius of curvature at each point of the boundary (characterized by v).

Proof. We use the asymptotic expansions of the modified Mathieu functions described in Sharples [26], namely the following UNIFORM asymptotic expansion of

$$W_j(u, \theta, x), \frac{dW_j}{dx}(u, \theta, x)$$

where u, θ, x are defined in [26], and will be recalled here. They are obtained through the modified Mathieu functions $M_p^{(j)}((z, h)$ described in [8]. Of use will be also the seminal paper of Olver [20], where uniform asymptotic expressions for solution of ordinary differential equations with a large parameter are discussed in the elliptic, hyperbolic or glancing zones.

Collecting the notations of Sharple, the equation (4) is $y'' = (\lambda - 2h^2\cosh 2z)y$, which yields $u = z$, $\lambda = a_n(k_3\rho)$, $h^2 = \frac{k_3^2\rho^2}{4}$, that is $h = \frac{k_3\rho}{2}$.

6.3. Behavior of the solutions of the modified Mathieu equation

The modified Mathieu equation is thus $y''(z) = [a_n(k_3\rho)(1 - \frac{2h^2}{a_n(k_3\rho)}\cosh 2z)]y$. Recall that $k_3^2 = \epsilon\mu\omega^2 - k_z^2$, which means that

$$\frac{2h^2}{a_n(k_3\rho)} = \frac{k_3^2\rho^2}{a_n(k_3\rho)} = \frac{\epsilon\mu\omega^2 - k_z^2}{\frac{a_n(k_3\rho)}{\rho^2}}.$$

The case $\epsilon\mu \in \mathbb{R}$ is easy to treat, Proposition 4 allows to treat the case where $q(u) = \frac{1}{2}\rho^2(\epsilon\mu\omega^2 - k_z^2)\cosh 2u - a_n(k_3\rho)$ when $a_n(k_3\rho) < \frac{1}{2}\rho^2(\epsilon\mu\omega^2 - k_z^2)\cosh 2u_0$ (where the result is identical to (28)), and the case where $a_n(k_3\rho) > \frac{1}{2}\rho^2(\epsilon\mu\omega^2 - k_z^2)\cosh 2u_1$ for $k_z^2 < \epsilon\mu\omega^2$ and also, regardless of the value of $a_n(k_3\rho)$ when $k_z^2 \geq \epsilon\mu\omega^2$, where the result is identical to (29). All the Lemmas of Subsection 4.1 stay true and one states the results in Subsection 6.4.

Let us treat now the case $\epsilon\mu \notin \mathbb{R}_+$.

Sharples treats the case $h \notin \mathbb{R}$ as well, hence the case $h = \frac{k_3\rho}{\sqrt{a_n(k_3\rho)}}$ is covered. The angle θ is defined by $h = |h|e^{i\theta}$, hence $k_z^2 < \Re\epsilon\mu\omega^2$ allows to follow the branch associated with $\theta \in]-\frac{\pi}{4}, 0]$. One notes that $\frac{k_3\rho}{n}$ corresponds exactly to $\frac{1}{\sin\beta}$ introduced for the hyperbolic region for the cylinder.

In the elliptic case, we define the variable ξ such that

$$(\frac{d\xi}{du})^2 = 1 - \frac{2h^2}{a_n(k_3\rho)}\cosh 2u, \tag{61}$$

for an asymptotic representation with exponentials $e^{\pm\sqrt{a_n(k_3\rho)}\xi}$, where $\sqrt{a_n(k_3\rho)}\xi$ represents the phase solution of the reduced eikonal equation.

In the hyperbolic case, we define the variable Ξ such that

$$(\frac{d\Xi}{du})^2 = \frac{2h^2}{a_n(k_3\rho)}\cosh 2u - 1, \tag{62}$$

In the glancing case, we define ζ such that

$$\zeta(\frac{d\zeta}{du})^2 = 1 - \frac{2h^2}{a_n(k_3\rho)}\cosh 2u, \tag{63}$$

for an asymptotic representation with Airy functions, where $(a_n(k_3\rho))^{\frac{1}{3}}\zeta$ is the argument of the Airy functions considered.

We denote, for simplification, by $N = \sqrt{a_n(k_3\rho)}$. It is NOT an approximation of n, and, in the high frequency regime, $N \simeq \omega\sqrt{\frac{(2n+1)\rho}{\omega}}\sqrt{\epsilon\mu - \eta^2}$ (dimension of the inverse of a velocity?or dimensioness?).

In the hyperbolic case, we check that the modified Mathieu equation becomes (18)

$$y'' = (iN)^2(\frac{d\Xi}{dz})^2 y,$$

which corresponds in Sharples'analysis to the case $h = iN$, $|h| = |N|$, and $h = |h|e^{i\frac{\pi}{2}+\theta}$, which yields the change of variable $u = xe^{-i\frac{\pi}{2}-\theta}$, leading to $hu = iNu = ie^{i\theta}|h|u = |h|x$.

In the case of the Airy function analysis, $W = (\frac{d\zeta}{dz})^{-\frac{1}{2}}y$ solves the equation $\frac{d^2W}{d\zeta^2} = (N^2\zeta + f(\zeta))W$, which yields solutions as

$$W_j(N,\theta,u) = P_j(N^{\frac{2}{3}}\zeta))P_j(N,u) + P'_j(N^{\frac{2}{3}}\zeta))Q_j(N,u)$$

$$\frac{d}{d\zeta}W_j(N,\theta,u) = N^{\frac{2}{3}}P'_j(n^{\frac{2}{3}}\zeta))R_j(N,u) + P_j(N^{\frac{2}{3}}\zeta))S_j(N,u)$$

where $P_1(X) = Ai(X)$, $P_2(X) = Ai(e^{-\frac{2i\pi}{3}}X, P_3(X) = Ai(e^{\frac{2i\pi}{3}}X)$, $P_4(X) = Bi(X)$.

The analysis of the conditions on which one can apply the method of Olver [20] and the results of Sharples [26], relies on the inequality, for the ODE $\frac{d^2F}{dz^2} = (h^2 + f(\theta,z,u))F$ is essentially $|f(\theta,z,u)| \leq \frac{C}{1+|z|^{1+\sigma}}$ and additional conditions under the hypothesis that f has an expansion in u^{-1}. This is the case when one introduces k_z, n of order ω, and we transform the modified Mathieu ODE (18) by using Ξ as new variable. Indeed it writes

$$\frac{d\Xi}{du}\frac{d}{d\Xi}(\frac{d\Xi}{du}\frac{dy}{d\Xi}) = (iN)^2(\frac{d\Xi}{dz})^2 y,$$

hence

$$\frac{d}{d\Xi}(\frac{d\Xi}{du})\frac{dy}{d\Xi} + \frac{d\Xi}{du}\frac{d^2y}{d\Xi^2} = (iN)^2\frac{d\Xi}{dz}y.$$

Introducing $W = (\frac{d\Xi}{du})^{\frac{1}{2}}y$, one deduces

$$\frac{d^2W}{d\Xi^2} = [(iN)^2 + (\frac{d\Xi}{du})^{-\frac{1}{2}}\frac{d^2}{d\Xi^2}((\frac{d\Xi}{du})^{\frac{1}{2}})]W,$$

where the bound on $(\frac{d\Xi}{du})^{-\frac{1}{2}}\frac{d^2}{d\Xi^2}((\frac{d\Xi}{du})^{\frac{1}{2}})$ is fulfilled (we are on compact sets).

Application of Theorem A of [20] leads to

$$W_{\pm}(\Xi,n)=e^{\pm iN\Xi}(\sum_s\frac{(\pm)^sA_s^{\pm}(\xi)}{N^s}),\ \frac{d}{d\Xi}W_{\pm}(\Xi,N)=\pm iNe^{\pm in\Xi}(\sum_s\frac{(\pm)^sB_s^{\pm}(\xi)}{N^s}),$$

and we deduce in this case that, with $y_{\pm}(u,N)=(\frac{d\Xi}{du})^{-\frac{1}{2}}W_{\pm}(\Xi(u),N)$,

$$\frac{d}{du}(y_{\pm})(u,N)=(\frac{d\Xi}{du})^{-\frac{1}{2}}[\frac{d\Xi}{du}\frac{dW_{\pm}}{d\Xi}+(\frac{d\Xi}{du})^{\frac{1}{2}}\frac{d}{du}((\frac{d\Xi}{du})^{-\frac{1}{2}})W_{\pm}],$$

which becomes

$$\frac{d}{du}(y_{\pm})(u,N)=(\frac{d\Xi}{du})^{-\frac{1}{2}}e^{iN\Xi}[\pm\frac{d\Xi}{du}iNB^{\pm}(\Xi,N)+(\frac{d\Xi}{du})^{\frac{1}{2}}\frac{d}{du}((\frac{d\Xi}{du})^{-\frac{1}{2}})A^{\pm}(\Xi,N)],$$

where $A^{\pm}$ and $B^{\pm}$ are the coefficients whose expansion in inverse powers of n is uniform for n large enough. Note that $A^{\pm}=1\pm\frac{A_1}{iN}+O(N^{-2})$, $B^{\pm}=1\pm\frac{B_1}{iN}+O(N^{-2})$, A_1 and B_1 are given by the expressions of [25].

The evaluation of the Dirichlet to Nemann passes through the computation

$$\frac{y_+'(u_1)y_-(u_0)-y_-'(u_1)y_+(u_0)}{y_+(u_1)y_-(u_0)-y_-(u_1)y_+(u_0)}$$

which amounts to, after all the needed simplifications

$$iN\frac{d\Xi}{du}(u_1)\frac{e^{iN(\Xi(u_1)-\Xi(u_0))}B^+(u_1)A^-(u_0)+e^{-iN(\Xi(u_1)-\Xi(u_0))}B^-(u_1)A^+(u_0)}{e^{iN(\Xi(u_1)-\Xi(u_0))}A^+(u_1)A^-(u_0)-e^{-iN(\Xi(u_1)-\Xi(u_0))}A^-(u_1)A^+(u_0)}.$$

In the case $\Im(\Xi(u_1)-\Xi(u_0))>0$, one has the estimate

$$\frac{y_+'(u_1)y_-(u_0)-y_-'(u_1)y_+(u_0)}{y_+(u_1)y_-(u_0)-y_-(u_1)y_+(u_0)}=-iN\frac{d\Xi}{du}(u_1)(1+\frac{A_1-B_1}{iN}+O(N^{-2}))(1+e^{-2N\Im(\Xi(u_1)-\Xi(u_0))}),$$

and in the case $\Im(\Xi(u_1)-\Xi(u_0))<0$

$$\frac{y_+'(u_1)y_-(u_0)-y_-'(u_1)y_+(u_0)}{y_+(u_1)y_-(u_0)-y_-(u_1)y_+(u_0)}=iN\frac{d\Xi}{du}(u_1)(1-\frac{A_1-B_1}{iN}+O(N^{-2}))(1+e^{-2N\Im(\Xi(u_1)-\Xi(u_0))}).$$

Using $A_1=B_1$, one gets the estimate.

□

The second case of interest is the case called above mixed-eliiptic-hyperbolic. In this case, the previous analysis is used at $u=u_1$, while new uniform expansions are used at $u=u_0$.

Proposition 18. *The asymptotics of the coefficient* $M_n=\frac{C'_{|n|}(u_1)S_{|n|}(u_0)-C_{|n|}(u_0)S'_{|n|}(u_1)}{C_{|n|}(u_1)S_{|n|}(u_0)-C_{|n|}(u_0)S_{|n|}(u_1)}$ *is, in the mixed elliptic-hyperbolic regime (and* $\omega\eta=k_z<\sqrt{\Re\epsilon\mu}\omega$*)*

$$M_n=\pm iN\sqrt{\frac{k_3^2\rho^2}{2a_n(k_3\rho)}\cosh 2u_1-1}(1+O(N^{-2}))=\pm\omega\rho\sqrt{\frac{\epsilon\mu-\eta^2}{2}\cosh 2u_1-\frac{N^2}{\rho^2\omega^2}}(1+O(N^{-2})).$$

Proof. One begins with the analysis of the solutions near $u = u_0$.

In this case, one recalls that one introduced ξ above, and there exists two solutions, respectively denoted by $E_\pm(u)$, such that

$$E_\pm(\xi) = (\frac{d\xi}{du})^{\frac{1}{2}} y_\pm(u)$$

solution of

$$\frac{d^2E_\pm}{d\xi^2} = (N^2 + \tau(\xi, N))E_\pm,$$

such that

$$E_\pm(\xi) = e^{\pm N\xi}(1 + \sum_{j\geq 1} \frac{A_j(\xi)}{N}),$$

as well as $\frac{dE_\pm}{d\xi} = \pm Ne^{\pm N\xi}(1 + \sum_{j\geq 1} \frac{B_j(\xi)}{N})$.

However, the relation between the solutions denoted by $W_\pm$ in the proof of the previous proposition and the solutions described above has to be carefully deduced from the branch cuts analysis of the solutions. The situation was different in the case of the Bessel functions, where the phase analysis either for the Hankel solutions and for the Bessel solution has been studied before. □

6.4. The case $\epsilon\mu \in \mathbb{R}_*^+$: bounds of the solution and DTN

Consider $\epsilon\mu \in \mathbb{R}_*^+$.

The equation on G, in the case $k_z^2 > \epsilon\mu\omega^2$, is

$$-\omega^{-2}G''(v) - (\frac{a}{\omega^2} + \frac{\eta^2 - \epsilon\mu}{2}\rho^2 \cos 2v)G = 0$$

that is

$$-\omega^{-2}G''(v) + (\frac{\eta^2 - \epsilon\mu}{2}\rho^2(1 - \cos 2v))G = (\frac{a}{\omega^2} + \frac{\eta^2 - \epsilon\mu}{2}\rho^2)G.$$

The minimum of the potential is 0, which means that $\frac{a}{\omega^2} + \frac{\eta^2-\epsilon\mu}{2}\rho^2 > 0$.

The equation on F is thus

$$F''(u) = (a + \frac{k_z^2 - \epsilon\mu\omega^2}{2} \cosh 2u)F.$$

The inequality $\frac{a}{\omega^2} + \frac{\eta^2-\epsilon\mu}{2}\rho^2 > 0$ yields

$$k(u) := a + \frac{k_z^2 - \epsilon\mu\omega^2}{2} \cosh 2u > \frac{k_z^2 - \epsilon\mu\omega^2}{2}(\cosh 2u - 1),$$

hence $u \to k(u)$ has the same properties as k introduced in (29). This implies that there exists a unique solution of the equation (18) satisfying $F(u_0) = 0, F(u_1) = 0$, which indeeds satisfies $0 \leq F(u) \leq 1$ for all $u \in [u_0, u_1]$.

Lemma 20. *Let a be a Floquet mode for Equation (19). When $\frac{\epsilon\mu\omega^2-k_z^2}{2}\rho^2 \cosh 2u_1 - a < 0$, there exists a unique solution w of (18) such that $w(u_0) = 0$ and $w(u_1) = 1$. The function w satisfies*

$$|w(u)| \leq M, u \in [u_0, u_1].$$

Proof. In the case $k_z^2 < \epsilon\mu\omega^2$, two cases are discussed:

• the case where the Floquet mode a satisfies

$$a > \frac{\omega^2\epsilon\mu - k_z^2}{2}\rho^2 \cosh 2u_1$$

where one is in the elliptic case and $u \to k(u) := a - \frac{\omega^2\epsilon\mu - k_z^2}{2}\rho^2 \cosh 2u$ is strictly positive on $[u_0, u_1]$ hence one has again (29) and the property $0 = F(u_0) \leq F(u) \leq F(u_1) = 1$ for all $u \in [u_0, u_1]$.

• the case where the Floquet mode a satisfies

$$a < \frac{\omega^2\epsilon\mu - k_z^2}{2}\rho^2 \cosh 2u_0$$

where one is in the hyperbolic case and we introduce $q(u) := \frac{\omega^2\epsilon\mu - k_z^2}{2}\rho^2 \cosh 2u - a$, which is positive, it is q introduced in (28). The same result holds (with a representation of w solution of $w''(u) + q(u)w(u) = 0$ by $\sqrt{\epsilon\mu}\omega w(u) = \rho(u)\sin\theta(u), w'(u) = \rho(u)\cos\theta(u)$), that is $\sin(\theta(u))$ has $O(\omega)$ roots on $[u_0, u_1]$ as well.

Finally consider the last case

$$\frac{\omega^2\epsilon\mu - k_z^2}{2}\rho^2 \cosh 2u_0 < a < \frac{\omega^2\epsilon\mu - k_z^2}{2}\rho^2 \cosh 2u_1.$$

There exists u_e such that $\frac{\omega^2\epsilon\mu - k_z^2}{2}\rho^2 \cosh 2u_e = a$. Three cases are available:

$w(r_e) = 0$ and (without loss of generality),$w'(r_e) > 0$,

$w(r_e) > 0$ and $w'(r_e) \geq 0$,

$w(r_e) > 0$ and $w'(r_e) < 0$.

In the two first cases, $w > 0$ on (r_e, R) hence $\frac{1}{w(R)}w$ satisfies $\frac{1}{w(R)}w(r_0) = 0$, $\frac{1}{w(R)}w(R) = 1$ and $\frac{1}{w(R)}w(r) \in [0, 1]$ fpr $r \in [r_e, R]$. Similarly, this holds also in the third case and $w > 0$ on $[r_e, R]$.

In the third case and there is a root of w larger than r_e, denote by r_e^* this root. If $w'(r_e^*) < 0$, one deduces $0 \leq \frac{1}{w(R)}w \leq 1$ for $r \in [r_e^*, R]$.

□

6.5. Elliptical cylinder with homothetic boundaries

We concentrate in this Section on a slightly different problem, where the two ellipses $\partial\Omega$ and Γ_1, instead of having the same focal points, are homothetic: one considers Ω as a perfectly conducting body, $\gamma = \partial\Omega$ and $\Gamma := \{(x, y), \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1\}$, where $\frac{B}{A} = \frac{b}{a}$. We are able to prove a more explicit result than the general theorem of this Section, but however less explicit than the results stated by Stupfel [28]:

Proposition 19. *Assume that* $\{(x, y, z), 1 \leq \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq \frac{A^2}{a^2}, z \in \mathbb{R}\}$ *is a dielectric material, of dielectric constants* ϵ *and* μ*. Let* k_z *be a wave number, with the assumption* $\Re(\epsilon\mu\omega^2) - k_z^2 > 0$*. Denote by* $k_3 = \sqrt{\epsilon\mu\omega^2 - k_z^2}$*. We call* g_m *the family of periodic solutions of the Mathieu equation, and* (C_m, S_m) *the associated solutions of the modified Mathieu equation.*

If $\epsilon\mu \notin \mathbb{R}$*,* k_3 *is not a resonance of (10), the Dirichlet to Neumann multiplier is well defined as an operator on* $l^2(\{g_m, m \in \mathbb{Z}\})$ *where this space is the set of limits of* $\sum_{m\in K}\alpha_m g_m$*,* $\sum_{m\in K}|\alpha_m|^2 < C$*,* C *independent on* K*. Its matrix (infinite) is given using (67) below by*

$$C_{ml} = \sum_{n\in\mathbb{Z}} r_{nl}[D_{nm}S_n(u_0) - E_{nm}C_n(u_0)].$$

where $(r_{nl})_{n,l}(I_{nm}S_n(u_0) - J_{nm}C_n(u_0))_{n,m} = \delta_{nm}$.

Proof. Assume in this proof that $\rho = \sqrt{a^2 - b^2}$, which fixes the system of variables (u, v). The same formal analysis as before yields $U = \sum_n \alpha_n (C_{|n|}(u) S_{|n|}(u_0) - S_{|n|}(u) C_{|n|}(u_0)) g_n(v)$, where the Mathieu and modified Mathieu functions are associated with $k_3\rho$. In order to obtain the Dirichlet to Neumann operator, it is needed to find $(\alpha_n)_n$ in terms of $U|_\Gamma$, more precisely its expansion on $(g_n)_n$. Let us call K the application which yields $(\frac{1}{2\pi}\int_0^{2\pi} U|_\Gamma(v) g_n(v) dv)_n$ in terms of α_n. One needs to invert K. As we are in an Hilbert separable space of infinite dimension, it is sufficient to be able to write K (or any operator deduced from K as $Id - T$, where T satisfies $\|T\| < 1$. This can be achieved if one can prove $\sup \frac{|T(\alpha)|_{l^2}}{|\alpha|_{l^2}} < 1$. Hence the choice of coefficients $(\alpha_n)_n$ is not the best choice.

Note that, for $(x, y) \in \Gamma$, $\frac{\rho^2 \cosh^2 u \cos^2 v}{A^2} + \frac{\rho^2 \sinh^2 u \sin^2 v}{B^2} = 1$, which is equivalent to $\sinh^2 u = \frac{1 - \frac{\rho^2}{A^2}\cos^2 v}{\frac{\rho^2}{A^2}\cos^2 v + \frac{\rho^2}{B^2}\sin^2 v}$, or

$$\tanh^2 u = \frac{1 - \frac{\rho^2}{A^2}\cos^2 v}{1 + \frac{\rho^2}{B^2}\sin^2 v}.$$

It is easy to see that

$$\tanh^2 u_0 < \frac{1}{1 + \frac{\rho^2}{B^2}} \leq \tanh^2 u \leq 1 - \frac{\rho^2}{A^2}, \forall v \in [0, 2\pi].$$

We denote by u_1 and u_2 such that $\tanh^2 u_1 = \frac{1}{1+\frac{\rho^2}{B^2}}$, $\tanh^2 u_2 = 1 - \frac{\rho^2}{A^2}$, such that $u_0 < u_1 < u_2$. Use then the estimate (**??**) and choose $u_3 > u_2$. Denote by

$$\beta_n = \alpha_n \kappa_n, \kappa_n = (C_n'(u_0) S_n(u_0) - S_n'(u_0) C_n(u_0)) \frac{\sinh(k_n(u_0)(u_3 - u_0))}{k_n(u_0)}. \tag{64}$$

One has

$$U(\rho \cosh u \cos v, \rho \sinh u \sin v) = \sum_n \beta_n \frac{C_n(u) S_n(u_0) - S_n(u) C_n(u_0)}{\kappa_n} g_n(v).$$

Define

$$T((\beta_n)_n) = (\frac{1}{2\pi}\int_0^{2\pi} \sum_n \beta_n (1 - \frac{C_n(u) S_n(u_0) - S_n(u) C_n(u_0)}{\kappa_n}) g_n(v) g_m(v) dv)_m.$$

One checks that

$$U|_\Gamma = \sum_n \beta_n \frac{C_n(\psi(v)) S_n(u_0) - S_n(\psi(v)) C_n(u_0)}{\kappa_n} g_n(v),$$

and that, denoting by $\mathcal{T}$ the operator

$$\mathcal{T}(\sum_n \beta_n g_n) = \sum_n \beta_n g_n - U|_\Gamma,$$

the coefficients of $\mathcal{T}(\sum_n \beta_n g_n)$ on the Mathieu complete orthonormal family (g_n) of $L^2([0, 2\pi])$ is $T((\beta_n)_n)$. In addition, estimate (**??**) proves that

$$\int_0^{2\pi} (\sum_n \beta_n g_n - U|_\Gamma)^2 dv \leq |1 - \frac{C_n(u) S_n(u_0) - S_n(u) C_n(u_0)}{\kappa_n}|_\infty \sum \beta_n^2.$$

As one has $1 - \frac{C_n(u) S_n(u_0) - S_n(u) C_n(u_0)}{\kappa_n} \leq 1 - \frac{\sinh k_n(u_0)(u_2 - u_0)}{\sinh k_n(u_0)(u_3 - u_0)}$, one deduces that $\mathcal{T}$ satisfies $\|\mathcal{T}\| < 1$, hence its inverse is $\sum_{p \geq 0} (\mathcal{T})^p$. This proof relies only on modes $|n| \geq n_0$, hence on a subspace of $L^2([0, 2\pi])$ of finite codimension.

We thus proved that $(\beta_n)_n$ is known when $U|_\Gamma$ is known. This yields

$$\beta_n = \sum_m G_{nm}\mu_m, U|_\Gamma = \sum_n \mu_n g_n. \tag{65}$$

Note that one can compute G_{nm} for each n by considering the problem $(\Delta + k_3^2)U = 0, U|_{\partial\Omega} = 0, U|_\Gamma = g_n$, which has a unique solution in $H^1(C - \Omega)$.

□

Assume for now on that k_3 does not lead to a resonance of the problem (10). Assume that, on the surface Γ (concentric ellipse), we impose $U_\Gamma \in L^2([0, 2\pi])$, as a function of ϕ such that $x = A\cos\phi, y = B\sin\phi$. This surface has not a simple expression in terms of u, v. The equation of Γ is $u = \Psi(v)$ such that

$$\sinh u = \sqrt{\frac{1 - \frac{\rho^2}{A^2}\cos^2 v}{\frac{\rho^2}{B^2} - (\frac{1}{B^2} - \frac{1}{A^2})\rho^2\cos^2 v}} \Leftrightarrow u = \psi(v). \tag{66}$$

The trace of u on Γ is

$$U|_\Gamma = U_\Gamma = \sum_{n\in\mathbb{Z}} \alpha_n(C_{|n|}(\psi(v))S_{|n|}(u_0) - C_{|n|}(u_0)S_{|n|}(\psi(v)))g_n(v),$$

where the variable v describes $[0, 2\pi]$. Two equivalent possibilities are thus available:
• assume that the expansion of $u|_\Gamma$ uses Fourier series expansions,
• assume that the expansion of $u|_\Gamma$ follows Mathieu series expansion.
The normal derivative on Γ is $\vec{n}.\nabla$, and as the equation of Γ writes $\sinh^2 u = f(\cos^2 v)$, where $f(X) = \frac{A^2-\rho^2X}{e^2\rho^2-\rho^2(e^2-1)X}$, $e = \frac{b}{a} = \frac{B}{A}$, a normal vector is given through $\sinh 2udu = -\sin 2vf'(\cos^2 v)dv$, that is $N = (\sinh 2u, \sin 2vf'(\cos^2 v))$, hence the normal outgoing unit vector is $n = (\sin^2 2u+(f'(\cos^2 v)\sin^2 2v)^{-\frac{1}{2}}N|_{u=\psi(v)}$.

One then uses the following coefficients, from which one deduces a linear transform in $l^2([0, 2\pi])$, using the orthogonality of the Mathieu functions:

$$\begin{aligned}
I_{nm} &= \tfrac{1}{2\pi}\int_0^{2\pi} C_{|n|}(\psi(v))g_n(v)g_m(v)dv,\\
J_{nm} &= \tfrac{1}{2\pi}\int_0^{2\pi} S_{|n|}(\psi(v))g_n(v)g_m(v)dv,\\
D_{nm} &= \tfrac{1}{2\pi}\int_0^{2\pi} \vec{n}(v).[C'_{|n|}(\psi(v))g_n(v), C_{|n|}(\psi(v))g'_n(v)]g_m(v)dv,\\
E_{nm} &= \tfrac{1}{2\pi}\int_0^{2\pi} \vec{n}(v).[S'_{|n|}(\psi(v))g_n(v), S_{|n|}(\psi(v))g'_n(v)]g_m(v)dv.
\end{aligned} \tag{67}$$

These quantities are just diagonal matrices when we have the Bessel functions and the usual Fourier modes because in this case $\psi(v)$ does not depend on v (for the cylinder).

Let U be the unique solution of the problem (apart for k_3 in the set of resonances of the Dirichlet problem)

$$\begin{cases}
(\Delta + k_3^2)U = 0, 1 < \frac{x^2}{a^2} + \frac{y^2}{b^2} < \frac{A^2}{a^2},\\
U|_{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} = 0,\\
U|_{\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{A}{(}a} = F \in L^2([0, 2\pi]).(x, y) = \frac{A}{a}(c\cos\theta, b\sin\theta).
\end{cases}$$

One has (denoting $C_{-n} = C_n$ and $S_{-n} = S_n$ for all $n \in \mathbb{N}$),

$$U = \sum_{n\in\mathbb{Z}} \alpha_n(C_n(u)S_n(u_0) - S_n(u)C_n(u_0))g_n(v),$$

which imply that

$$\frac{1}{2\pi}\int_0^{2\pi} U|_\Gamma g_{-m}(v)dv = \sum_{n\in\mathbb{Z}} \alpha_n (I_{nm}S_n(u_0) - J_{nm}C_n(u_0)).$$

Let $F \in L^2([0,2\pi])$. As the family g_n is a complete family of $L^2(\Gamma)$, there exists a sequence F_n such that $F = \sum_{n\in\mathbb{Z}} F_m g_m$. As this system has a unique solution there exists $(r_{nm})_{n,m}$ such that

$$\alpha_n = \sum_{m\in\mathbb{Z}} r_{nm}F_m.$$

On the other side,

$$\partial_n U = \vec{n}.\nabla U|_\Gamma = \sum_{n\in\mathbb{Z}} \alpha_n \vec{n}.[(C_n'(u)S_n(u_0) - S_n'(u)C_n(u_0))g_n(v), C_n(u)S_n(u_0) - S_n(u)C_n(u_0))g_n'(v)]|_{u=\psi(v)}$$

which imply

$$\frac{1}{2\pi}\int_0^{2\pi} \partial_n U g_{-m}(v)dv = \sum_{n\in\mathbb{Z}} \alpha_n [D_{nm}S_n(u_0) - E_{nm}C_n(u_0)].$$

Finally, one has

$$\frac{1}{2\pi}\int_0^{2\pi} \partial_n U g_{-m}(v)dv = \sum_{n\in\mathbb{Z},l\in\mathbb{Z}} r_{nl}F_l [D_{nm}S_n(u_0) - E_{nm}C_n(u_0)],$$

which imply that, in the basis (g_n), the Dirichlet to Neumann multiplier is characterized by the matrix

$$C_{ml} = \sum_{n\in\mathbb{Z}} r_{nl}[D_{nm}S_n(u_0) - E_{nm}C_n(u_0)].$$

7. The Calderòn operator for a cylindrical layer in 3D for the Maxwell equations

This section is a follow-up of one of the chapters of the thesis of Pierre Payen [22]. Indeed, in Stupfel [28] the case of a 2D cylindrical layer is thouroughly described, with exact solutions. But, as this was a 2D problem, the case of oblique incidence (*id est* a wave vector with a non zero z component) was not accounted for. In Payen [22] a procedure for constructing the Calderòn operator is described, but an explicit formula for this operator has to be deduced from the expressions therein. This is the aim of this Section. We use there the same set-up as the set-up used in [16]. Denote by

$$K(r,n,k_z) = \begin{pmatrix} \frac{k_z n}{r} & \epsilon\mu\omega^2 - k_z^2 \\ -(\epsilon\mu\omega^2 - k_z^2) & -\frac{k_z n}{r} \end{pmatrix}.$$

Lemma 21. *The system of Maxwell equations is equivalent to*

$$\begin{cases} \begin{cases} \frac{in}{r}E_z - ik_z E_\theta = i\omega\mu H_r \\ \frac{in}{r}H_z - ik_z H_\theta = -i\omega\epsilon E_r \end{cases} \\ \begin{pmatrix} \frac{dH_z}{dr} \\ \frac{1}{r}\frac{d}{dr}(rH_\theta) \end{pmatrix} = \frac{i}{\mu\omega}K\begin{pmatrix} E_z \\ E_\theta \end{pmatrix}, \begin{pmatrix} \frac{dE_z}{dr} \\ \frac{1}{r}\frac{d}{dr}(rE_\theta) \end{pmatrix} = -\frac{i}{\epsilon\omega}K\begin{pmatrix} H_z \\ H_\theta \end{pmatrix}. \end{cases}$$

The functions E_z and H_z are solution of the Bessel equation (15).

Proof. The system of Maxwell equations writes

$$\left\{\begin{array}{l} \frac{in}{r}E_z - ik_zE_\theta = i\omega\mu H_r \\ \frac{in}{r}H_z - ik_zH_\theta = -i\omega\epsilon E_r \\ \\ \\ ik_zE_r - \frac{dE_z}{dr} = i\omega\mu H_\theta \\ \frac{1}{r}[\frac{d}{dr}(rE_\theta) - inE_r] = i\omega\mu H_z \\ \\ ik_zH_r - \frac{dH_z}{dr} = -i\omega\epsilon E_\theta \\ \frac{1}{r}[\frac{d}{dr}(rH_\theta) - inH_r] = -i\omega\epsilon E_z \end{array}\right. \tag{68}$$

The two first equations yield H_r in terms of (E_θ, E_z) and E_r in terms of (H_θ, H_z), hence the first result of Lemma 21. Note that this result decouples the ODE system, allowing it to be written with a matrix $\begin{pmatrix} 0 & \frac{i}{\mu\omega}K \\ -\frac{i}{\epsilon\omega}K & 0 \end{pmatrix}$. Note also that, if one introduces $\eta = \frac{k}{\omega}$ and $\delta(r) = \frac{n}{r\omega}$, one has

$$\frac{1}{\omega^2}K = \begin{pmatrix} \eta\delta(r) & \epsilon\mu - \eta^2 \\ -(\epsilon\mu - \eta^2) & -\eta\delta(r) \end{pmatrix}.$$

As for the Bessel equations on (E_z, H_z), it comes from the vectorial Helmholtz equation that E and H solve, and we observe that, while the equations on (E_θ, E_r) on one side and on (H_θ, H_r) are coupled, the equation on E_z or on H_z is a scalar equation.

□

The Calderòn operator given by what follows ($C(r_0, R) \times R$ denoting the infinite cylindrical annulus between $r = r_0$ and $r = R$, $S(r)$ denoting the surface of the infinite cylinder of radius r, e_r is the normal unit outgoing vector to a point of the boundary $S(r)$ or $S(R)$):

Theorem 5. *If there is no resonance, the problem of Maxwell equations in $C(r_0, R) \times \mathbb{R}$, supplemented with $E|_{S(r_0)} = 0$ on $S(r_0)$ and with $-(E \wedge e_r) \wedge e_r|_{S(R)} = \vec{E_0} \in L^2(S(R), T(S(R))$ has a unique solution in $C(r_0, R) \times \mathbb{R}$, denoted by $\vec{E}(\vec{E_0})$. One has thus $e_r \wedge \vec{H}_{S(R)} = C(E_0)$, where C is the interior Calderòn operator.*

This Theorem is a theorem of [5].

Lemma 22. 1. *If the domain is $C(r_0, R) \times R$ and if $\varepsilon\mu \notin \mathbb{R}_+^*$, there is no resonance.*

2. *If the domain is $C(r_0, R) \times [0, L]$, and $d_n(k_3R)(dd_n)'(k_3R) \neq 0$, there is no resonance.*

Introduce now, in the case $d_n(k_3R)dd_n(k_3R) \neq 0$, the functions

$$\begin{array}{rcl} S_n(k_3r) & = & \frac{J_n'(k_3r_0)Y_n(k_3r) - Y_n'(k_3r_0)J_n(k_3r)}{J_n'(k_3r_0)Y_n'(k_3R) - Y_n'(k_3r_0)J_n'(k_3R)}, \\ \\ T_n(k_3r) & = & \frac{J_n(k_3r_0)Y_n'(k_3r) - Y_n(k_3r_0)J_n'(k_3r)}{J_n(k_3r_0)Y_n(k_3R) - Y_n(k_3r_0)J_n(k_3R)}, \\ \\ Q_n(k_3r) & = & \frac{J_n(k_3r_0)Y_n(k_3r) - Y_n(k_3r_0)J_n(k_3r)}{J_n(k_3r_0)Y_n(k_3R) - Y_n(k_3r_0)J_n(k_3R)}, \\ \\ R_n(k_3r) & = & \frac{J_n'(k_3r_0)Y_n'(k_3r) - Y_n'(k_3r_0)J_n'(k_3r)}{J_n'(k_3r_0)Y_n'(k_3R) - Y_n'(k_3r_0)J_n'(k_3R)}. \end{array} \tag{69}$$

We notice that these functions do not depend on the choice of the pair of fundamental solutions of the Bessel equation considered.

Lemma 23. *The Calderòn operator for the cylindrical annulus is given by*

$$\begin{pmatrix} -H_\theta \\ H_z \end{pmatrix} = \begin{pmatrix} -\frac{i\omega\epsilon}{k_3}[T_n(k_3R) - \frac{ik_z^2n^2}{k_3^2R^2\omega^2\epsilon\mu}S_n(k_3R)] & \frac{ik_zn}{k_3R\omega\mu}S_n(k_3R) \\ \frac{ik_zn}{k_3R\omega\mu}S_n(k_3R) & \frac{ik_3}{\omega\mu}S_n(k_3R) \end{pmatrix}\begin{pmatrix} E_z \\ E_\theta \end{pmatrix}.$$

Remark 12. *One hoped that the leading order term of this operator is $\sqrt{\frac{\epsilon}{\mu}}$, at the first order, which is the case because the approximation of $S_n(k_3R)$ is $-\frac{i}{\sin\beta_*}$ while the approximation of $T_n(k_3R)$ is $i\sin\beta_*$ as it is stated in Lemma 24.*

Remark 13. *The impedance operator, which expresses $n\wedge E$ in terms of $-n\wedge n\wedge H$, is*

$$\begin{pmatrix} -E_\theta \\ E_z \end{pmatrix} = \begin{pmatrix} \frac{i\omega\mu}{k_3}[(S_n(k_3R))^{-1} - \frac{ik_z^2n^2}{k_3^2R^2\omega^2\epsilon\mu}(T_n(k_3R))^{-1}] & -\frac{ik_zn}{k_3R\omega\epsilon}(T_n(k_3R))^{-1} \\ -\frac{ik_zn}{k_3R\omega\epsilon}(T_n(k_3R))^{-1} & -\frac{ik_3}{\omega\epsilon}(T_n(k_3R))^{-1} \end{pmatrix}\begin{pmatrix} H_z \\ H_\theta \end{pmatrix}.$$

At normal incidence it is diagonal and equal to

$$\sqrt{\frac{\mu}{\epsilon}}\begin{pmatrix} i(S_n(\sqrt{\epsilon\mu}\omega R))^{-1} & 0 \\ 0 & -i(T_n(\sqrt{\epsilon\mu}\omega R))^{-1} \end{pmatrix}$$

For $n=0$ it is also diagonal and equal to

$$\begin{pmatrix} \frac{i}{\sqrt{\frac{\mu}{\epsilon}-\frac{k_z^2}{\omega^2}}}(S_0(k_3R))^{-1} & 0 \\ 0 & -i\sqrt{\frac{\mu}{\epsilon}-\frac{k_z^2}{\omega^2}}(T_0(k_3R))^{-1} \end{pmatrix}.$$

One recognizes a classical form for the impedance matrix.

Proof. Consider for the moment formal solutions of the Helmholtz equations obtained on E_z, H_z. After Fourier transform in z and expanding in Fourier series in θ, one has, for $k_3\neq 0$

$$\begin{cases} E_z = aJ_n(k_3r) + bY_n(k_3r) \\ H_z = cJ_n(k_3r) + dY_n(k_3r). \end{cases}$$

We now deduce H_θ and E_θ through the equalities 3 and 5 of (68), after replacing E_r, H_r using the equalities 1 and 2 of (68) for $k_3\neq 0$:

$$\begin{cases} H_\theta = \frac{i\omega\epsilon}{k_3^2}[\frac{dE_z}{dr} + \frac{ik_zn}{\omega\epsilon r}H_z] = \frac{i\omega\epsilon}{k_3^2}\frac{dE_z}{dr} - \frac{k_zn}{k_3^2r}H_z \\ E_\theta = -\frac{i\omega\mu}{k_3^2}[\frac{dH_z}{dr} - \frac{ink_z}{\omega\mu r}E_z] = -\frac{i\omega\mu}{k_3^2}\frac{dH_z}{dr} - \frac{nk_z}{k_3^2r}E_z \end{cases}$$

that is

$$\begin{cases} H_\theta = \frac{i\omega\epsilon}{k_3}(aJ_n' + bY_n') - \frac{k_zn}{k_3^2r}(cJ_n + dY_n) \\ E_\theta = -\frac{i\omega\mu}{k_3}(cJ_n' + dY_n') - \frac{k_zn}{k_3^2r}(aJ_n + bY_n). \end{cases} \tag{70}$$

Any solution of the Cauchy problem on $(E_z, E_\theta, H_z, H_\theta)$ stated in Lemma 21, in the case $k_3\neq 0$ can be written

$$\begin{pmatrix} E_z \\ E_\theta \\ H_z = \frac{ik_3}{\omega\mu}(E_\theta^0 + \frac{k_zn}{k_3^2R}E_z^0)S_n(R)\frac{d_n(r)}{d_n(R)} \\ H_\theta \end{pmatrix} = a\begin{pmatrix} J_n(k_3r) \\ -\frac{nk_z}{k_3^2r}J_n(k_3r) \\ 0 \\ \frac{i\omega\epsilon}{k_3}J_n'(k_3r) \end{pmatrix} + b\begin{pmatrix} Y_n(k_3r) \\ -\frac{nk_z}{k_3^2r}Y_n(k_3r) \\ 0 \\ \frac{i\omega\epsilon}{k_3}Y_n'(k_3r) \end{pmatrix} + c\begin{pmatrix} 0 \\ -\frac{i\omega\mu}{k_3}J_n'(k_3r) \\ J_n(k_3r) \\ -\frac{k_zn}{k_3^2r}J_n(k_3r) \end{pmatrix} + d\begin{pmatrix} 0 \\ -\frac{i\omega\mu}{k_3}Y_n'(k_3r) \\ Y_n(k_3r) \\ -\frac{k_zn}{k_3^2r}Y_n(k_3r) \end{pmatrix},$$

which defines a fundamental basis of the Cauchy problem.

For one annulus, the boundary conditions $E_\theta(r_0) = 0$, $E_z(r_0) = 0$ yield

$$\begin{cases} aJ_n(k_3 r_0) + bY_n(k_3 r_0) = 0 \\ cJ_n'(k_3 r_0) + dY_n'(k_3 r_0) = 0. \end{cases}$$

The boundary conditions at $r = R$ yield

$$\begin{cases} aJ_n(k_3 R) + bY_n(k_3 R) = E_z^0 \\ cJ_n'(k_3 R) + dY_n'(k_3 R) = \frac{ik_z n}{k_3 R\omega\mu} E_z^0 + \frac{ik_3}{\omega\mu} E_\theta^0 = \frac{ik_3}{\omega\mu}[E_\theta^0 + \frac{k_z n}{k_3^2 R} E_z^0]. \end{cases}$$

Collecting, we have

$$\begin{cases} aJ_n(k_3 r_0) + bY_n(k_3 r_0) = 0 \\ aJ_n(k_3 R) + bY_n(k_3 R) = E_z^0 \\ cJ_n'(k_3 r_0) + dY_n'(k_3 r_0) = 0 \\ cJ_n'(k_3 R) + dY_n'(k_3 R) = \frac{ik_3}{\omega\mu}[E_\theta^0 + \frac{k_z n}{k_3^2 R} E_z^0], \end{cases}$$

which gives two decoupled systems. Hence, under $d_n(k_3 R)(dd_n)'(k_3 R) \neq 0$ there is existence and uniqueness of a solution of this system. Note that the condition $d_n(k_3 R)(dd_n)'(k_3 R) \neq 0$ do not depend on the pair of independent solutions of the Bessel equation chosen because there exists, for any pair $\{f, g\}$ of independent solutions of Equation (15), a constant $C(f, g)$ such that $f(r_0)g(R) - f(R)g(r_0) = C(f, g)d_n$. The condition $d_n(R)(dd_n)'(R) = 0$ was obtained in Pierre Payen's thesis, as well as the Fourier multiplier (with an equivalent method).

The solution of the system on (a, b, c, d) is:

$$\begin{cases} a = -\frac{Y_n(k_3 r_0)}{d_n(k_3 R)} E_z^0 \\ b = \frac{J_n(k_3 r_0)}{d_n(k_3 R)} E_z^0 \\ c = -\frac{Y_n'(k_3 r_0)}{D_n(k_3 R)} \frac{ik_3}{\omega\mu}[E_\theta^0 + \frac{k_z n}{k_3^2 R} E_z^0] \\ d = \frac{J_n'(k_3 r_0)}{D_n(k_3 R)} \frac{ik_3}{\omega\mu}[E_\theta^0 + \frac{k_z n}{k_3^2 R} E_z^0]. \end{cases} \tag{71}$$

One deduces

$$E_z(r) = E_z^0 S_n(k_3 r), H_z(r) = \frac{ik_3}{\omega\mu}[\frac{k_z n}{k_3^2 R} E_z^0 + E_\theta^0] S_n(k_3 r).$$

We use then (70) to deduce the two other components. In particular

$$\frac{dE_z}{dr}(r) = k_3 E_z^0 T_n(k_3 r), \frac{dH_z}{dr}(r) = \frac{ik_3^2}{\omega\mu}[E_\theta^0 + \frac{ik_z n}{k_3^2 R} E_z^0] R_n(k_3 r),$$

which yields

$$\begin{cases} H_\theta = \frac{i\omega\epsilon}{k_3} E_z^0 T_n(k_3 r) - \frac{ik_z n}{k_3 r\omega\mu}[E_\theta^0 + \frac{ik_z n}{k_3^2 R} E_z^0] S_n(k_3 r), \\ E_\theta = [E_\theta^0 + \frac{ik_z n}{k_3^2 R} E_z^0] R_n(k_3 r) - \frac{k_z n}{k_3^2 r} E_z^0 Q_n(k_3 r). \end{cases}$$

Collecting, the electromagnetic fields that were obtained here are

$$\begin{cases} E_z(r) = Q_n(k_3 r) E_z^0, \\ E_\theta(r) = [E_\theta^0 + \frac{k_z n}{k_3^2 R} E_z^0] R_n(k_3 r) - \frac{nk_z}{k_3^2 r} Q_n(k_3 r) E_z^0 \\ H_z(r) = \frac{ik_3}{\omega\mu}[E_\theta^0 + \frac{k_z n}{k_3^2 R} E_z^0] S_n(k_3 r) \\ H_\theta(r) = -\frac{ik_z n}{k_3 \omega\mu r}[E_\theta^0 + \frac{k_z n}{k_3^2 R} E_z^0] S_n(k_3 r) + \frac{i\omega\epsilon}{k_3} E_z^0 T_n(k_3 r) \end{cases} \tag{72}$$

As before, these expressions do not depend on the pair of independent solutions of the Bessel equation chosen.

Finally,

$$\begin{cases} H_z(R) = \frac{ik_3}{\omega\mu}[E^0_\theta + \frac{k_z n}{k_3^2 R}E^0_z]S_n(R) \\ H_\theta(R) = -\frac{ik_z n}{k_3\omega\mu R}[E^0_\theta + \frac{k_z n}{k_3^2 R}E^0_z]S_n(R) + \frac{i\omega\epsilon}{k_3}E^0_z T_n(R) = \frac{i\omega\epsilon}{k_3}E^0_z[T_n(R) - \frac{k_z^2 n^2}{k_3^2 R^2\omega^2\epsilon\mu}S_n(R)] - \frac{ik_z n}{k_3\omega\mu R}E^0_\theta, \end{cases}$$

which yields the Calderòn operator and ends the proof of Lemma 23. □

Lemma 24. *Let*

$$F = \Psi - \Psi_0 = p(\tan\beta_* - \beta_* - \tan\beta_0 + \beta_0).$$

The following asymptotic expansions hold, under the hypotheses described in Proposition 14 and the notation $F = \Psi - \Psi_0$

$$T_n(k_3R) = -i\sin\beta_* \frac{(L_0 - iM_0)(N - iO)e^{-iF} + (L_0 + iM_0)(N + iO)e^{iF}}{(L_0 - iM_0)(L + iM)e^{-iF} - (L_0 + iM_0)(L - iM)e^{iF}},$$

or

$$T_n(k_3R) = \sin\beta_* \frac{\cos F - \frac{a_0+b}{p}\sin F}{\sin F - \frac{a-a_0}{p}\cos F}(1 + O(p^{-2}).$$

Similarly,

$$S_n(k_3R) = \frac{i}{\sin\beta_*} \frac{(N_0 + iO_0)(L_* + iM_*)e^{-iF} + (N_0 - iO_0)(L_* - iM_*)e^{-iF}}{(N_0 + iO_0)(N_* - iO_*)e^{-iF} - (N_0 - iO_0)(N_* + iO_*)e^{iF}},$$

which yields

$$S_n(k_3R) = -\frac{1}{\sin\beta_*} \frac{\cos F + \frac{b_0+a_*}{p}\sin F}{\sin F + \frac{b_*-b_0}{p}\cos F}(1 + O(p^{-2}).$$

and its counterpart when $\Im\epsilon\mu$ does not depend on ω:

Lemma 25. *Assume that* $\Im\epsilon\mu < 0$ *is a constant independent on* ω*. Depending on the sign of* $\Im F$*, one obtains, up to terms which are* $O(e^{-2|\Im F|})$*:*

$$S_n(k_3R) = \pm\frac{i}{\sin\beta_*} - (b + a)\frac{k_3}{p\sin\beta_*} + O(\omega^{-1}),$$

$$T_n(k_3R) = \pm i\sin\beta_* + (b + a)\frac{k_3\sin\beta_*}{p} + O(\omega^{-1}).$$

Remark 14. *The conditions for existence and uniqueness of a solution for a multi-layer annulus is not the intersection of the conditions for each layer. It is, for example, for a double layer* $(detM_E)(detM_H) \neq 0$*, where* M_E*,* M_H *are given by (73), (74). This condition does not depend, again, on the choice of the pair of independent solutions of the Bessel equation chosen in each layer.*

For example, for two layers of dielectric materials, of dielectric constants $(\epsilon, \mu), r_0 < r < r_1$ and $(\epsilon_1, \mu_1), r_1 < r < R$, introduce $k_3^1 = \sqrt{\epsilon_1\mu_1\omega^2 - k_z^2}$. The condition $\det M_E \neq 0$ rewrites

$$(k_3^1)^{-1}(J_n(k_3r_0)Y_n(k_3r_1) - J_n(k_3r_1)Y_n(k_3r_0))[J_n(k_3^1R)Y_n'(k_3^1r_1) - Y_n(k_3^1R)J_n'(k_3^1r_1)]$$
$$\neq$$
$$(k_3)^{-1}(J_n(k_3r_0)Y_n'(k_3r_1) - J_n'(k_3r_1)Y_n(k_3r_0))[J_n(k_3^1R)Y_n(k_3^1r_1) - Y_n'(k_3^1R)(k_3^1r_1)].$$

The proof of Remark 14 comes from the expression of the continuity at the interface $r = r_1$ of the fields (E_z, H_θ), that is the continuity of $(E_z, \frac{dE_z}{dr})$, and the Dirichlet boundary conditions on E_z at $r = r_0$ and $r = R$, leading to the determinant of the matrix M:

$$M_E := \begin{pmatrix} J_n(k_3r_0) & Y_n(k_3r_0) & 0 & 0 \\ J_n(k_3r_1) & Y_n(k_3r_1) & -J_n(k_3^1r_1) & -Y_n(k_3^1r_1) \\ (k_3)^{-1}J_n'(k_3r_1) & (k_3)^{-1}Y_n'(k_3r_1) & -(k_3^1)^{-1}J_n'(k_3^1r_1) & (k_3^1)^{-1}Y_n'(k_3r_1) \\ 0 & 0 & J_n(k_3^1R) & Y_n(k_3^1R) \end{pmatrix}, \tag{73}$$

the system to be solved being

$$M_E \begin{pmatrix} a \\ b \\ a_1 \\ b_1 \end{pmatrix} = E_z^0 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

In a similar fashion, the system on (c, d, c_1, d_1) reads

$$M_H \begin{pmatrix} c \\ d \\ c_1 \\ d_1 \end{pmatrix} = (\frac{(k_3^1)^2}{i\omega\mu_1}E_\theta^0 + \frac{ikn}{i\omega\mu_1 R}E_z^0) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \frac{k_z n}{r_1}[\frac{1}{(k_3^1)^2}(a_1J_n'(k_3^1r_1) + b_1Y_n'(k_3^1r_1)) - \frac{1}{k_3^2}(aJ_n'(k_3r_1) + bY_n'(k_3r_1)) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

where

$$M_H := \begin{pmatrix} J_n'(k_3r_0) & Y_n'(k_3r_0) & 0 & 0 \\ J_n(k_3r_1) & Y_n(k_3r_1) & -J_n(k_3^1r_1) & -Y_n(k_3^1r_1) \\ \frac{i\omega\mu}{k_3^2}J_n'(k_3r_1) & \frac{i\omega\mu}{k_3^2}Y_n'(k_3r_1) & \frac{i\omega\mu_1}{(k_3^1)^2}J_n'(k_3^1r_1) & \frac{i\omega\mu_1}{(k_3^1)^2}Y_n'(k_3r_1) \\ 0 & 0 & J_n'(k_3^1R) & Y_n'(k_3^1R) \end{pmatrix}. \tag{74}$$

No attempt to solve these systems is to be done here. See [22] for a procedure leading in most cases to the values of the traces on $r = R$, even if it does not rely on the global system.

8. Annex

8.1. Pseudodifferential discrete operators

We define in this Section what is a pseudodifferential operator acting on $\mathbb{R} \times [0, 2\pi]$. Consider the usual algebra of symbols on $T^*(\mathbb{R}^2)$, and assume this symbol is periodic in θ, of period 2π, satisfying the classical estimates in $S^m_{1,0}$. The action of such a symbol $a(z, \theta, \xi, \eta)$ is, as usual

$$Op(a)f(z, \theta) = \frac{1}{(2\pi)^2}\int\int a(z, \theta, \xi, \eta)\hat{f}(\xi, \eta)e^{ix\xi + i\theta\eta}d\eta d\xi$$

If F is a function in $L^2(\mathbb{R}\times[0,2\pi])$, periodic in θ, consider the associated function on $\mathbb{R}^2$ where $\tilde{F}(z,\theta) := F(z,\theta-2\pi[\frac{\theta}{2\pi}])$. Using the Fourier series of F:

$$F(z,\theta)=\sum_n F_n(z)e^{in\theta}$$

one defines directly $\tilde{F}(z,\theta)=\sum_n F_n(z)e^{in\theta}$, and $\tilde{F}\in\mathcal{S}'(\mathbb{R}^2)$, and

$$\mathcal{F}(\tilde{F})(\xi,\eta)=2\pi\sum_n \hat{F}_n(\xi)\delta_n.$$

One has thus

$$\begin{array}{rl} Op(a)\tilde{F}(z,\theta)= & \frac{1}{2\pi}<\sum_n \hat{F}_n(\xi)\delta_n, a(z,\theta,\xi,\eta)e^{iz\xi+i\theta\eta}> \\ & =\sum_n \frac{1}{2\pi}<\hat{F}_n(\xi), a(z,\theta,\xi,n)e^{iz\xi}e^{in\theta}> \\ & =\sum_n e^{in\theta}(\frac{1}{2\pi}\int a(z,\theta,\xi,n)e^{iz\xi}\hat{F}_n(\xi)d\xi). \end{array}$$

If, in addition, the symbol does not depend on z (that is the operator does not depend on z, as it will be the case here)

$$Op(a)\tilde{F}(z,\theta)=\sum_n e^{in\theta}(\frac{1}{2\pi}\int a(\theta,\xi,n)e^{iz\xi}\hat{F}_n(\xi)d\xi),$$

and

$$\begin{array}{rl} \frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta}Op(a)\tilde{F}(z,\theta)d\theta & =\sum_n \frac{1}{2\pi}\int(\frac{1}{2\pi}\int_0^{2\pi} a(\theta,\xi,n)e^{i(n-m)\theta}d\theta)e^{iz\xi}\hat{F}_n(\xi)d\xi \\ & =\sum_n \frac{1}{2\pi}\int a_{m-n}(\xi,n)e^{iz\xi}\hat{F}_n(\xi)d\xi \end{array}$$

where $a_p(\xi,n)$ is the $p-$th Fourier coefficient of $a(\theta,\xi,n)$. As this definition comes from the classical definition of pseudodifferential calculus, the pseudodifferential calculus extends to this case.

8.2. Proof of Lemma 3

Proof. Begin by a necessary condition. For this purpose, consider the partial Fourier transform in (x_1,x_2) of $u\in\mathcal{S}'(\mathbb{R}^2,C^2([-l,0]))$. One has the following ODE problem on $\hat{u}\in\mathcal{S}'(\mathbb{R}^2,C^2([-l,0]))$

$$\left\{\begin{array}{l} (\frac{d^2}{dx_3^2}+\omega^2\epsilon\mu-k_1^2-k_2^2)\hat{u}=0 \\ \hat{u}(-l)=0 \\ \hat{u}(0)=\hat{u}_0(k_1,k_2). \end{array}\right. \tag{75}$$

All solutions in $C^\infty(\mathbb{R})$ of this problem read

$$\hat{u}(x_3)=ae^{ik_\perp x_3}+be^{-ik_\perp x_3}.$$

It is important to check that, for all $K>0$, one has the implication

$$\forall(k_1,k_2)\in\mathbb{R}^2, |a|\leq K|e^{ilk_\perp}|, |b|\leq K\Rightarrow|\hat{u}(x_3)|\leq 2K, -l\leq x_3\leq 0$$

ensuring that the conditions on a and b are sufficient conditions for u being in $L^\infty(\mathbb{R}^2\times[-l,0])\subset\mathcal{S}'(\mathbb{R}^2,L^\infty([-l,0]))$, which is a sufficient condition for showing that $\hat{u}$ is the Fourier transform of a tempered distribution.

Under such conditions, one gets $a+b=\hat{u}_0, ae^{-ik_\perp l}+be^{ik_\perp l}=0$, from which one gets

$$a=\frac{\hat{u}_0}{1-e^{-2ik_\perp l}}, b=-\frac{\hat{u}_0e^{-2ik_\perp l}}{1-e^{-2ik_\perp l}}.$$

The solution, if it belongs to $\mathcal{S}'(\mathbb{R}^2, C^2([-l,0]))$ is then

$$\hat{u}(x_3) = \hat{u}_0[\frac{e^{ik_\perp x_3}}{1-e^{-2ik_\perp l}} - \frac{e^{-ik_\perp x_3 - 2ik_\perp l}}{1-e^{-2ik_\perp l}}] = \hat{u}_0 \frac{\sin(k_\perp(x_3+l))}{\sin(k_\perp l)}.$$

As $\Im k_\perp \leq -\alpha_0\omega$, we have

$$\frac{\sin(k_\perp(x_3+l))}{\sin(k_\perp l)} = e^{-\Im k_\perp x_3} \frac{e^{ik_\perp^0(x+l)} - e^{-ik_\perp^0(x_3+l)+2k_\perp^1(x_3+l)}}{e^{ik_\perp^0 v} - e^{-ik_\perp^0 l + 2k_\perp^1 l}}$$

hence, as $-l \leq x_3 \leq 0$, this expression satisfies the estimate

$$|\frac{\sin(k_\perp(x_3+l))}{\sin(k_\perp l)}| \leq \frac{2}{1-e^{-2\Im k_\perp l}} \leq \frac{2}{1-e^{-2\alpha_0\omega l}}, \tag{76}$$

and similar estimates hold for $\hat{u}'$ and $\hat{u}''$, hence $\hat{u}(x_3)$ belongs to $\mathcal{S}'(\mathbb{R}^2, C^2([-l_0,0]))$ for all $a^0 \in \mathcal{S}(\mathbb{R}^2)$.

□

Assume now that $\epsilon\mu \in \mathbf{R}$.

Proof. Two cases occur only: $k_\perp \in \mathbb{R}^*$ (when $\epsilon\mu\omega^2 - k_1^2 - k_2^2 > 0$) or $\Re k_\perp = 0$ (otherwise).

In the first case, that is for $k_1^2 + k_2^2 < \omega^2\epsilon\mu$, the system on (a,b) is $a+b = \hat{u}_0$, $ae^{-ik_\perp l} + be^{ik_\perp l} = 0$, which yields $a(1-e^{-2ik_\perp l}) = \hat{u}_0$. The condition $(1-e^{-2ik_\perp l}) \neq 0$ grants uniqueness of the solution. If there exists n such that $\omega^2\epsilon\mu = k_1^2 + k_2^2 + \frac{n^2\pi^2}{l^2}$, the function $A\sin\frac{n\pi}{l}x_3$ is a non zero solution of the system (75) for $\hat{u}_0 = 0$. This yields the resonances of the problem. However, this only ensures that the distribution $A\sin\frac{n\pi}{l}x_3$ in $\mathcal{S}'(R^2, C^2([-l,0]))$ where $\hat{A}$ is supported on $k_1^2 + k_2^2 = \omega^2\epsilon\mu - \frac{n^2\pi^2}{l^2}$, is a nonzero solution of (75) with $u_0 = 0$.

If (k_1, k_2) is such that $\omega^2\epsilon\mu - k_1^2 - k_2^2 \neq \frac{n^2\pi^2}{l^2}$, the notation s_0 imply that

$$n_0 \in \{\frac{l}{\pi}\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2}, \frac{l}{\pi}\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2} + 1\}.$$

In this case, for n_0 even, $l\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2} - n_0\pi \geq s_0\pi$ or $(n_0+1)\pi - l\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2} \geq s_0\pi$, hence $\sin(l\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2}) \geq \sin\pi s_0$ (and similar inequalites for n_0 odd). This means that

$$|\frac{\sin(l+x_3)\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2}}{\sin l\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2}}| \leq \frac{1}{\sin\pi s_0}.$$

In the case $k_\perp = 0$, the equation reads $(\hat{u})'' = 0$, hence $\hat{u}(x_3) = a(k_1,k_2)x_3 + b(k_1,k_2)$, the Dirichlet boundary conditions yields $\hat{u}(x_3) = a(k_1,k_2)(x_3+l)$, and $\partial_{x_3}\hat{u}(0) = a(k_1,k_2)$, hence $\hat{u}(x_3) = \hat{u}_0\frac{x_3+l}{l}$.

In the second case, that is for $k_1^2 + k_2^2 > \omega^2\epsilon\mu$, $\Im k_\perp = \sqrt{-\epsilon\mu\omega^2 + k_1^2 + k_2^2}$ and the formal[6] solution is

$$\hat{u}(k_1, k_2, x_3) = a(k)[e^{\Im k_\perp(x_3+l)} - e^{-\Im k_\perp(x_3+l)}],$$

[6]because this is not an element of $\mathcal{S}'(\mathbb{R}^2)$ because of the exponential growth in $\sqrt{k_1^2+k_2^2}$ for at least one term

and if one computes the value at $x_3 = 0$, one finds $a(k)(2\sinh l\Im k_\perp) = \hat{u}(k_1, k_2, 0)$. One then deduces

$$\hat{u}(k_1, k_2, x_3) = \hat{u}(k_1, k_2, 0)\frac{\sinh(x_3 + l)\Im k_\perp}{\sinh l\Im k_\perp},$$

and $\frac{\sinh(x_3+l)\Im k_\perp}{\sinh l\Im k_\perp}$ is increasing, hence the estimate $|\hat{u}(k_1, k_2, x_3)| \leq |\hat{u}(k_1, k_2, 0)|$, $\forall (k_1, k_2)$, $k_1^2 + k_2^2 > \omega^2\epsilon\mu$. This shows that this formal solution belongs to $C^2(\mathcal{S}'(\mathbb{R}^2), [-l, 0])$ hence it was possible to use the Fourier transform.

This proves the first item of Lemma 3 and explains the term resonance for values of (k_1, k_2) associated with n through Lemma 1. □

8.3. Radius of curvature of an ellipse

The normal derivative and the radius of curvature for the ellipse are expressed as follows

Lemma 26. *The normal unit vector to the ellipse at the point* $\rho(\cosh u_1 \cos v, \sinh u_1 \sin v)$ *is*

$$\vec{n} = \frac{1}{\sqrt{\sinh^2 u_1 \cos^2 v + \cosh^2 u_1 \sin^2 v}}(\sinh u_1 \cos v, \cosh u_1 \sin v)$$

and the radius of curvature of the ellipse at this point is

$$\rho(s) = \rho\frac{(\cosh^2 u_1 \cos^2 v + \sinh^2 u_1 \sin^2 v)^{\frac{3}{2}}}{\cosh u_1 \sinh u_1}.$$

Proof. A tangent vector to the boundary $u = u_1$ is $(\cosh u_1 \sin v, -\sinh u_1 \cos v)$. The curvilinear abscissa is given through $\rho\|\frac{d}{ds}((\cosh u_1 \cos v, \sinh u_1 \sin v))\| = 1$, that is $\frac{dv}{ds}\sqrt{\cosh^2 u_1 \cos^2 v + \sinh^2 u_1 \sin^2 v} = \rho^{-1}$. The normal unit vector is

$$\frac{1}{\sqrt{\sinh^2 u_1 \cos^2 v + \cosh^2 u_1 \sin^2 v}}(\sinh u_1 \cos v, \cosh u_1 \sin v).$$

One knows that, for a unit tangent vector $\frac{1}{\sqrt{a^2+b^2}}(a(s), b(s))$, its derivative being $\frac{a'b-ab'}{(a^2+b^2)^{\frac{3}{2}}}(b, -a)$, the radius of curvature is $\frac{a^2+b^2}{a'b-ab'}$. With $\frac{dv}{ds} = (\rho\sqrt{\cosh^2 u_1 \cos^2 v + \sinh^2 u_1 \sin^2 v})^{-1}$, along with

$$a'b - ab' = \frac{dv}{ds}(\frac{da}{dv}b - \frac{db}{dv}a) = \frac{dv}{ds}(-\cosh u_1 \sinh u_1 \cos^2 v - \sinh u_1 \cosh u_1 \sin^2 v) = -\frac{dv}{ds}\cosh u_1 \sinh u_1,$$

one gets

$$\rho(s) = \frac{\rho}{\cosh u_1 \sinh u_1}(\cosh^2 u_1 \cos^2 v + \sinh^2 u_1 \sin^2 v)^{\frac{3}{2}}$$

which ends the proof.

□

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