



# Exponential Runge-Kutta methods for Vlasov-Poisson equation

Josselin Massot, Nicolas Crouseilles, Anaïs Crestetto

## ► To cite this version:

Josselin Massot, Nicolas Crouseilles, Anaïs Crestetto. Exponential Runge-Kutta methods for Vlasov-Poisson equation. Numerical Methods for Multiscale Models arising in Physics and Biology, Jun 2019, Nantes, France. hal-03196935

**HAL Id: hal-03196935**

**<https://hal.science/hal-03196935>**

Submitted on 13 Apr 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



# Exponential Runge-Kutta methods for Vlasov-Poisson equation

Josselin MASSOT<sup>1,2</sup> & Nicolas CROUSEILLES<sup>2,1</sup> & Anaïs CRESTETTO<sup>3</sup>

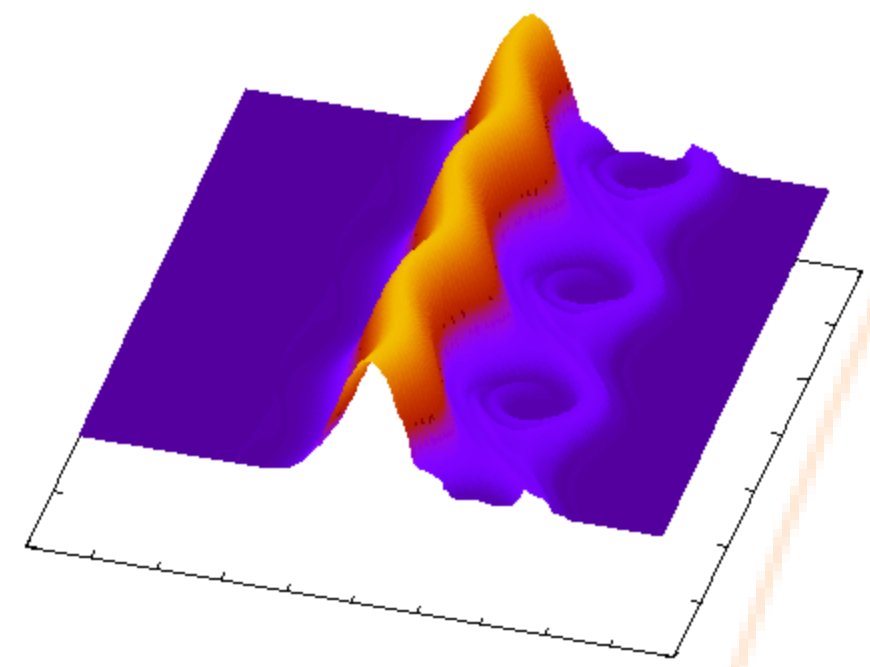
1: IRMAR – Université de Rennes, 2: Inria Rennes – Bretagne Atlantique, 3: LMJL – Université de Nantes

## 1 ▶ Vlasov-Poisson equation

Vlasov equation plays an important role in the modeling of plasma physics, it is a non-linear transport in  $(x, v)$  of a density distribution  $f = f(t, x, v)$ ; coupled with Poisson equation for estimate electric field  $E = E(t, x)$ :

$$\begin{cases} \partial_t f + v \partial_x f + E \partial_v f = 0 \\ \partial_x E = \int_{\mathbb{R}} f dv - 1 \end{cases} \quad (\text{VP})$$

$$x \in \Omega \subset \mathbb{R}, v \in \mathbb{R}$$



Example of filamentation process in a solution

High order methods are needed in phase space  $(x, v)$  to study filamentation.

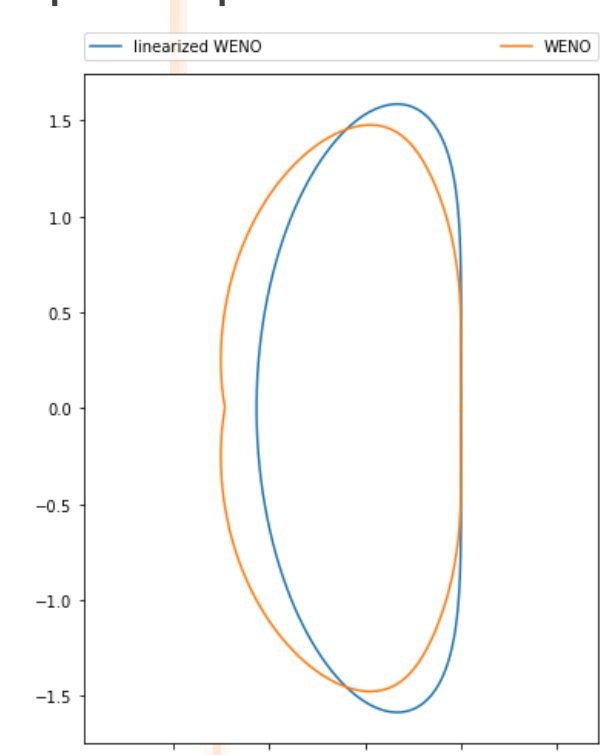
## 2 ▶ Simplified model

We study the stability problem on a simplify model:

$$\partial_t u + a \partial_x u = 0. \quad (*)$$

We discretize in space with **WENO** method [1], and we test various Runge-Kutta methods for the time integrator. We apply **von Neumann analysis** on space discretization to study amplification factor as a curve in complex plane  $\mathbb{C}$ .

- Von Neumann analysis usually used on linear method: first studies on stability of WENO method [2], and CFL estimate [3] have been done on *linearized WENO*.
- We can try to apply von Neumann analysis to the full non-linearized WENO method. WENO method is weakly non-linear then we can compute amplification factor.



Amplification factor of WENO method (linearized or not)

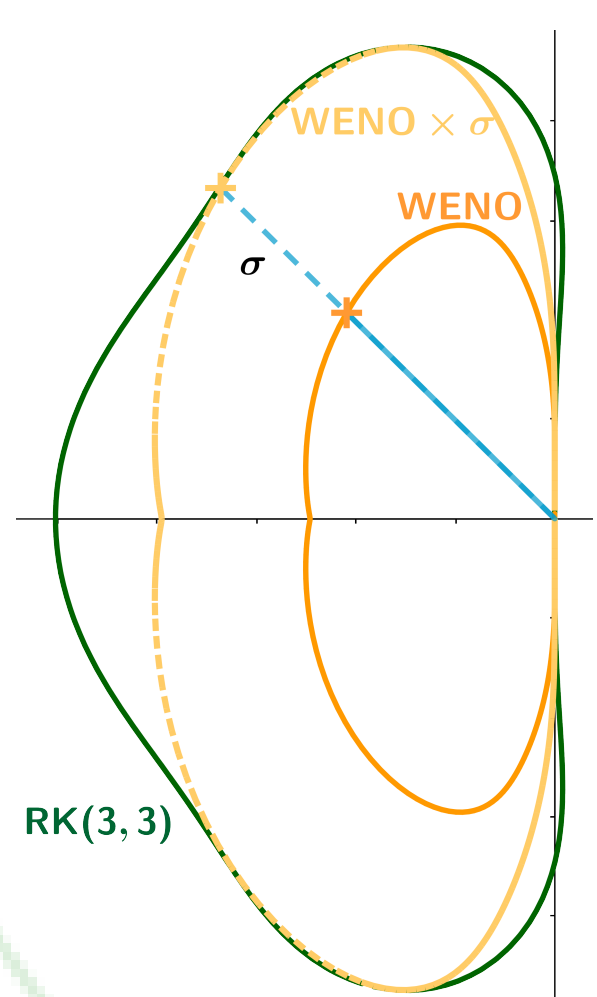
We define stability domain of an explicit Runge-Kutta method  $\text{RK}(s, n)$  as:

$$\mathcal{D}_{(s, n)} = \{z \in \mathbb{C} : |p_{(s, n)}(z)| \leq 1\}$$

with  $s$  the number of stages,  $n$  the order and  $p_{(s, n)}$  the stability function of  $\text{RK}(s, n)$ .

## 3 ▶ Automatic CFL estimate

It is possible to interpret CFL number between time integrator and space method as the biggest homothety ratio that wedges all the amplification factor curve into the stability domain of considered Runge-Kutta methods. We estimate **numerically** the CFL  $\sigma$  (respectively  $\sigma^\ell$ ) for various couples  $\text{RK}(s, n)$  with WENO (respectively linearized WENO). *All CFL have been approved with a long time simulation of (\*)*.



Geometric CFL interpretation

$$\Delta t \leq \frac{\sigma}{a} \Delta x$$

Time integrator	$\sigma^\ell$	$\sigma$
RK(3,3)	1.43	1.60
RK(4,4)	1.73	1.68
RK(6,5) (DP5)	1.78	2.06
RK(8,6)	2.53	2.56

## 4 ▶ Exponential integrator on Vlasov-Poisson equation

We apply a Fourier transform in  $x$  direction of (VP):

$$\partial_t \hat{f} + i\kappa v \hat{f} + \widehat{(E \partial_v f)} = 0 \quad \text{of the form} \quad \partial_t u + Lu + N(u) = 0.$$

Next we use an exponential integrator (Duhamel formula):

$$\partial_t (e^{i\kappa v t} \hat{f}) + e^{i\kappa v t} \widehat{(E \partial_v f)} = 0 \quad \text{of the form} \quad \partial_t (e^{Lt} u) + e^{Lt} N(u) = 0.$$

With this exponential form we can use a classical Runge-Kutta method, in this context we call them Lawson methods or Integrating factor Runge-Kutta methods [4].

Stability function of Lawson schemes can be expressed in terms of the underlying Runge-Kutta method:

$$p_{\text{Lawson}(s, n)}(z) = p_{(s, n)}(z) e^{L \Delta t}$$

with  $L = i\kappa v \in i\mathbb{R}$  so stability domain is the same. Our numerical CFL study, on simplified model, still works on Duhamel formulation of Vlasov-Poisson.

## 5 ▶ The chosen one

We are interested in the numerical cost  $\frac{\Delta t}{s}$  of  $\text{RK}(s, n)$ . To compare each time integrator, we compute total energy in Vlasov-Poisson system:

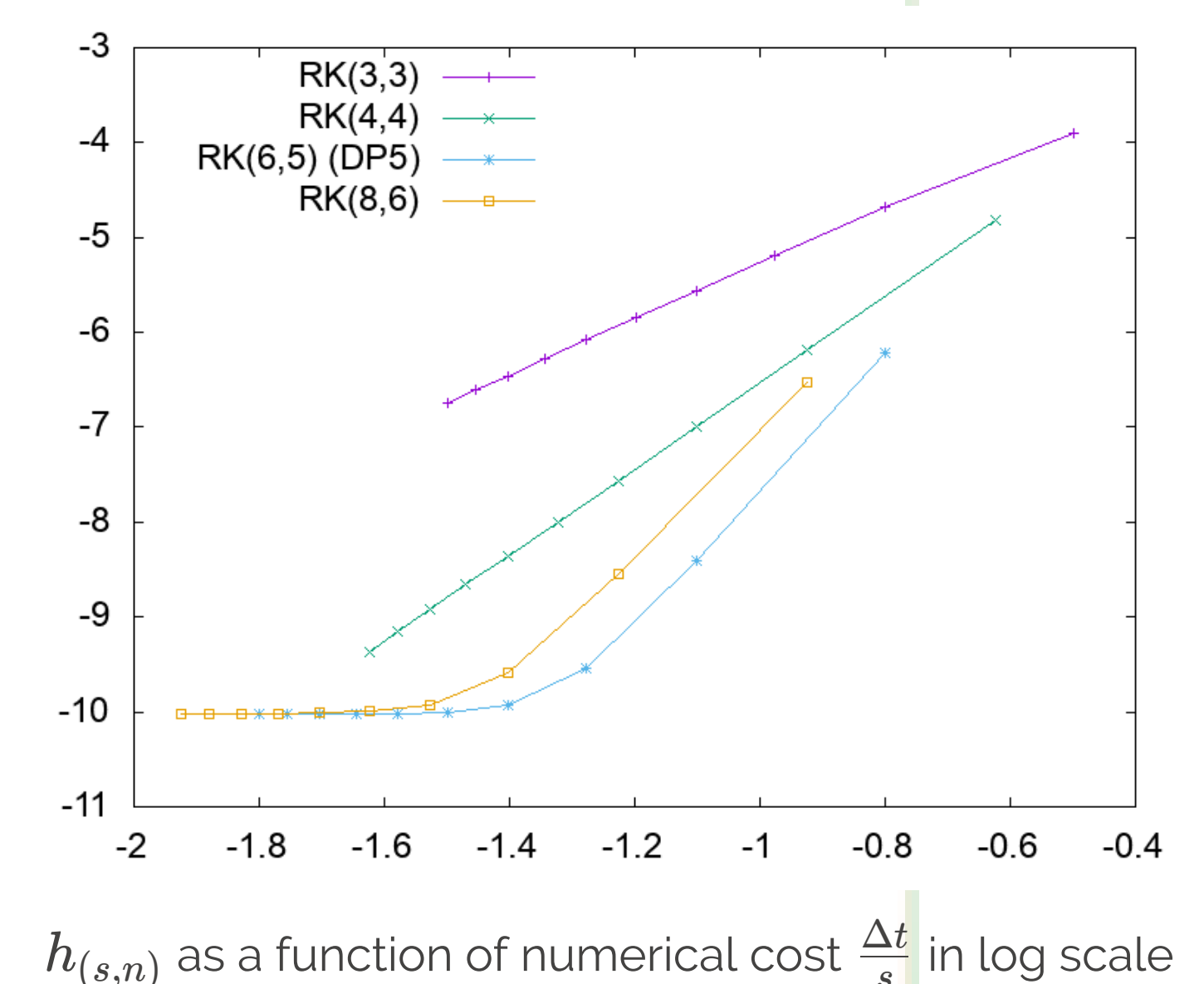
$$H(t) = \int_{\Omega} \int_{\mathbb{R}} v^2 f dv dx + \int_{\Omega} E^2 dx$$

which is preserved in time. We propose to select the best method by considering:

$$h_{(s, n)} : \frac{\Delta t}{s} \mapsto \left\| \frac{H(t) - H(0)}{H(0)} \right\|_{\infty}$$

Tested strategy:

- Spectral scheme in  $x$ ,
- WENO method in  $v$ ,
- Lawson method of the underlying  $\text{RK}(s, n)$  in time.

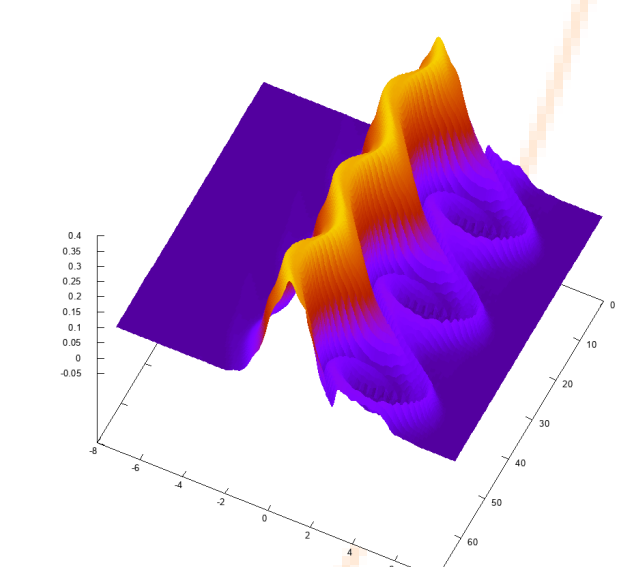
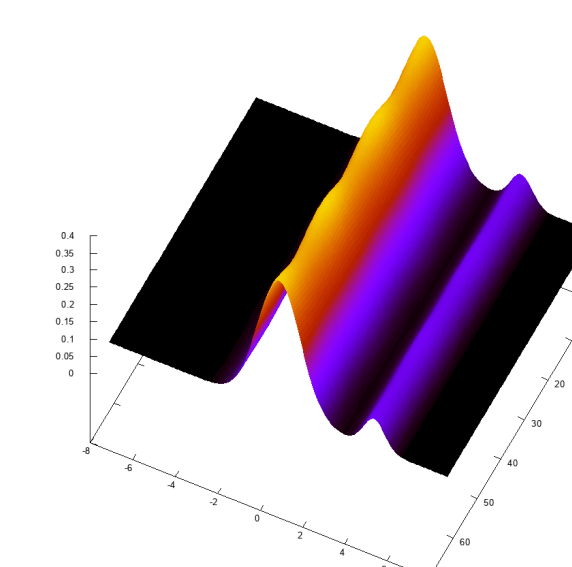


## 6 ▶ Conclusion

- Promising approach for Vlasov-Poisson simulation, *less stages than splitting strategy*.
- Selected Runge-Kutta method is Dormand-Prince RK(6,5), this is an embedded Runge-Kutta method (adaptative stepsize, relaxation of the CFL).
- Extension to multi-dimensional Vlasov-Poisson is easier than splitting strategy.

Simulation of the *bump on tail* test case :

$$f(t=0, x, v) = \left( \frac{0.9}{\sqrt{2} \cdot \pi} e^{-\frac{|v|^2}{2}} + \frac{0.2}{\sqrt{2} \cdot \pi} e^{-2|v-4.5|^2} \right) (1 + 0.04 \cdot \cos(0.3 \cdot x))$$



Bump on tail simulation ( $t=0$  on the left,  $t=20$  on the right)

## References

- [1] Efficient Implementation of Weighted ENO Schemes, G.-S. JIANG and C.-W. SHU *Journal of Computational Physics* (1996)
- [2] Linear instability of the fifth-order WENO method, R. WANG and R. J. SPITERI *Journal on Numerical Analysis* (2007)

- [3] On the Linear Stability of the Fifth-Order WENO Discretization, M. MOTAMED and C. B. MACDONALD and S. J. RUUTH *Journal of Scientific Computing* (2010)
- [4] Strong Stability Preserving Integrating Factor Runge-Kutta Methods, L. ISHERWOOD and Z. J. GRANT and S. GOTTLIEB (2018) (2018)