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A comparative study of polynomial-type chaos expansions for indicator functions

F. Bourgey[*], E. Gobet[†], C. Rey[‡]

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Abstract

We propose a thorough comparison of polynomial chaos expansion (PCE) for indicator functions of the form $\mathbf{1}_{c\leq X}$ for some threshold parameter $c \in \mathbb{R}$ and a random variable X associated with classical orthogonal polynomials. We provide tight global and localized L^2 estimates for the resulting truncation of the PCE and numerical experiments support the tightness of the error estimates. We also compare the theoretical and numerical accuracy of PCE when extra quantile/probability transforms are applied, revealing different optimal choices according to the value of c in the center and the tails of the distribution of X.

KEY WORDS: Metamodeling, orthogonal polynomials, polynomial chaos expansion.

MSC2010: 33C45, 40A30, 41A10, 62L20.

1 Introduction

We study a refined polynomial chaos expansion (PCE) for indicator functions of the form $\mathbf{1}_{c\leq X}$ for a general scalar random variable X and some parameter $c \in \mathbb{R}$. The random variable X is distributed according to $\nu(\mathrm{d}x) = w(x)\mathrm{d}x$, a probability measure absolutely continuous with respect to the Lebesgue measure defined in a finite or infinite interval $I_w = (a, b)$ of $\mathbb{R}$. To perform our PCE, we work with orthogonal polynomial sets (OPS), that is, sets of polynomial functions $(p_n)_{n\in\mathbb{N}}$ of degree n associated with the probability measure ν, and satisfying an orthogonal property of the form:

$$\langle p_n, p_m\rangle_{L^2(\nu)} := \int_{I_w} p_n(x)p_m(x)\nu(\mathrm{d}x) = h_n\delta_{nm}, \quad \text{for all } n, m \in \mathbb{N}, \tag{1.1}$$

where $\delta_{nm} = 1$ if $n = m$ or 0 and

$$h_n := \|p_n\|^2_{L^2(\nu)} = \int_a^b p_n(x)^2\nu(\mathrm{d}x). \tag{1.2}$$

We assume the existence of $\kappa > 0$ such that

$$\int_{\mathbb{R}} e^{\kappa|x|}\nu(\mathrm{d}x) < +\infty, \tag{1.3}$$

which ensures the existence of an OPS associated with ν, that the squared L^2 norm is finite, i.e., $h_n < +\infty$, and that the OPS associated with ν is dense in $L^2(\mathbb{R}, \nu)$ (see, e.g., [14, Theorem

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3.4] for a review of sufficient conditions ensuring the density of an OPS and connections with the moment problem).

From the polynomial denseness property available under Assumption (1.3), we have that for every $f \in L^2(\nu)$, the following PCE holds (see, e.g., Kakutani [22], [17] or [33]),

$$f \overset{L^2(\nu)}{=} \sum_{n=0}^{+\infty} \gamma_n(f) p_n, \qquad \gamma_n(f) = \frac{\langle f, p_n \rangle_{L^2(\nu)}}{\|p_n\|^2_{L^2(\nu)}}. \tag{1.4}$$

"Homogeneous chaos" (or "Wiener chaos expansion") was first introduced by [37] using Hermite polynomials, i.e., an OPS when ν is a Gaussian measure (see Table 1), to model stochastic processes with Gaussian random variables. It was then extensively used in [17] for Hermite-chaos when dealing with the normal distribution (see also [30] for Charlier-chaos expansion for the Poisson distribution). A general and unified extension of the Wiener chaos expansion (referred to as PCE) to an arbitrary probability measure associated with any orthogonal polynomials belonging to the so-called Askey-scheme [3] was proposed in [38]. PCE is now standard and extensively used in the uncertainty quantification community, and is, with Gaussian processes, one of the two main classes of surrogate models for sensitivity analysis (see, e.g., [26] and references therein).

In our specific approach, we focus on the discontinuous case $f : x \mapsto \mathbf{1}_{\{c \leq x\}}$ for some $c \in \mathbb{R}$. In practice, one has no choice but to truncate the PCE (infinite sum) at some order $N \in \mathbb{N}^*$, and approximate the indicator function with

$$\mathbf{1}_{c \leq X} \approx \sum_{n=0}^{N} \gamma_n(c) p_n(X). \tag{1.5}$$

From the orthogonality condition (1.1), the related L^2 truncation error writes as

$$\mathcal{E}_N(c) := \mathbb{E}\left[\left|\mathbf{1}_{c \leq X} - \sum_{n=0}^{N} \gamma_n(c) p_n(X)\right|^2\right]^{\frac{1}{2}} = \left|\sum_{n=N+1}^{+\infty} h_n \gamma_n(c)^2\right|^{\frac{1}{2}}. \tag{1.6}$$

Objectives. In this paper, we focus on the case where $(p_n)_{n \in \mathbb{N}}$ is a classical OPS (COPS), i.e., ν is either a gamma, beta, or normal distribution. The respective orthogonal polynomials are then called Laguerre $\mathrm{L}_n^{(\alpha)}$ with $\alpha > -1$, $I_w = (0, \infty)$, $w(x) \propto x^\alpha e^{-x}$, or Jacobi $\mathrm{P}_n^{(\alpha,\beta)}$ with $\alpha, \beta > -1$, $I_w = (-1, 1)$, $w(x) \propto (1-x)^\alpha (1+x)^\beta$, or Hermite polynomials with $I_w = \mathbb{R}$, $w(x) \propto e^{-\frac{x^2}{2}}$ (see Table 1 for some detailed properties and notations regarding the COPS). Note that condition (1.3) can easily be checked for the COPS and ensures the existence and denseness in $L^2(\mathbb{R}, \nu)$. Restricting to COPS will be crucial for deriving recurrence relations regarding the truncation parameters (see Proposition 2.2) thanks to Rodrigues' formula (see Theorem 2.1).

Our first objective is to provide (tight) global and local L^2 truncation error estimates for $\mathcal{E}_N(c)$ (see Theorems 2.2–2.4). Our main motivation comes from [8, Theorem 2.6] in which the authors provide uniform estimates in c for the L^2 error (when dealing with Hermite polynomials) of order $N^{-\frac{1}{4}}$. Surprisingly, though the order is relatively low, the PCE seems to work exceptionally well on their practical applications only with a few N for large values of c. Such a behavior probably indicates that the uniform estimate hides an extra term (most likely exponential in c), which ensures the approximation to be effective for appropriate ranges of value for c. We thus aim at exhibiting the dependence in c for $\mathcal{E}_N(c)$ and seeking whether such a behavior is specific to the Hermite case or holds also for Jacobi and Laguerre polynomials.

Our second objective concerns the choice of COPS one should consider when performing the expansion w.r.t. X. Indeed, for two probability measures $\nu, \tilde{\nu}$ (beta, gamma, or normal)

with respective c.d.f. F_ν, $F_{\tilde{\nu}}$, and respective OPS $(p_n)_{n\in\mathbb{N}}$, $(\tilde{p}_n)_{n\in\mathbb{N}}$, the following probabilistic transformation always holds (see section 3.2),

$$\mathbf{1}_{c\leq X} = \mathbf{1}_{F_\nu(c)\leq F_\nu(X)} \overset{d}{=} \mathbf{1}_{\tilde{c}\leq \tilde{X}}, \tag{1.7}$$

where $\tilde{c} = T(c)$ and $\tilde{X} = T(X) \overset{d}{=} \tilde{\nu}$ for $T = F_{\tilde{\nu}}^{-1} \circ F_\nu$ defined as the composition of the quantile and probability transforms. A natural question arising is whether is it optimal in L^2 sense to approximate $\mathbf{1}_{c\leq X}$ with the PCE $\sum_{n=0}^{N} \gamma_n(c)p_n(X)$ or with the chaos expansion $\sum_{n=0}^{N} \gamma_n(\tilde{c})\tilde{p}_n(\tilde{X})$, which is not a polynomial in the variable X anymore. For the examples of first-order ordinary differential equation [38] and of the coupled Navier–Stokes structure equations [39], the authors numerically illustrate that it is optimal to perform the PCE w.r.t. X rather than any $\tilde{X}$. However, regarding our problem of approximating $\mathbf{1}_{c\leq X}$, we will demonstrate that the optimal convergence is not so simple and depends on whether c is in the bulk or in the tails of the distribution.

Motivation and applications. We provide two instances motivating the current study. In [8], the authors investigate a metamodel for approximating the distribution of

$$\mathcal{L}_K := \sum_{k=1}^{K} l_k \mathbf{1}_{c_k\leq X},$$

which reads as the credit portfolio loss in a Gaussian copula model ($X \overset{d}{=} \mathcal{N}(0,1)$), where $X, c_1, \dots, c_K$ are independent Gaussian random variables and $(l_k)_{k=1,\dots,K}$ are deterministic coefficients. Expanding each indicator function using a PCE and truncating at order N gives

$$\mathcal{L}_K \approx \mathcal{L}_{K,N} := \sum_{k=1}^{K} l_k \sum_{n=0}^{N} \gamma(c_k)p_n(X) = \sum_{n=0}^{N} \underbrace{\Big(\sum_{k=1}^{K} l_k\gamma(c_k)\Big)}_{:=\varepsilon_{K,n}} p_n(X).$$

The advantage of such a decomposition is the possibility of approximating the vector $(\varepsilon_{K,n})_{n\in\mathbb{N}}$ by a Gaussian vector (using central limit theorem arguments). When sampling $\mathcal{L}_K$, this approach is particularly efficient for large K. The global error analysis relies much on bounds on $\mathcal{E}_N(c)$ for many c. Indeed, following the proof of [8, Theorem 2.6], one has

$$\mathbb{E}\left[(\mathcal{L}_K - \mathcal{L}_{K,N})^2\right]^{\frac{1}{2}} \leq \sum_{k=1}^{K} |l_k| \, \mathbb{E}\left[\mathcal{E}_N(c_k)\right],$$

showing the importance of pointwise control of $\mathcal{E}_N(c)$ w.r.t. N and c.

A second example motivating our study is related to random graphs generated as follows: the adjacency value between two vertices i and j is defined by $Y_{i,j} = \mathbf{1}_{s_{i,j}\leq X_{i,j}}$ for some scalar random variable $X_{i,j}$ and a parameter $s_{i,j} \in \mathbb{R}$. The random graphs topic is a well-developed research area, see [5, 36]. See also [24, Chapters 5 and 6] for a reference on stochastic block models. Any model will specify the dependence between the random variables $X_{i,j}$. For instance, in [18] the $X_{i,j}$'s form a Gaussian vector correlated to a single Gaussian factor X. Computing the weighted degree of a vertex i boils down to defining a quantity of the form $\mathcal{L}_K$ where K is the number of other vertices of the graph. Expanding this quantity via a PCE allows to get a metamodel for the weighted degree, under a form that can be simulated more efficiently, especially for large graphs. Beyond convergence rate issues with respect to the chaos truncation, choosing the type of chaos expansion (e.g., the quantile/probability transform T in (1.7)) is relevant and will be addressed in the current work. For a recent study about asymptotics of the degree $\mathcal{L}_K$ of random graphs, see [12].

Background results. Approximating a function f with a truncated series of orthogonal polynomials is somehow standard. The extensive study of orthogonal polynomials is available in the influential works of [34], or more recently [32], where various approximation issues are also studied. The interest in quantifying convergence rates has increased with the expansion of numerical analysis, and for which one has to approximate a function with a polynomial like in spectral methods, see, e.g., [9, 4]. In these references, the function is assumed to be smooth: convergence rates in L^2 norm for Jacobi polynomials are given, for instance, in [9, Chapter 9] and write as N^{-k}, where k represents the smoothness of f and N is the polynomial degree. For unbounded support (Laguerre, Hermite), the rate becomes $N^{-\frac{k}{2}}$, see [15, section 6.7]. To the best of our knowledge, the rate of convergence for non-smooth functions (like the indicator function) has not been considered in the literature before. In the (infinity-dimensional) stochastic analysis community, in which PCE with Gaussian noise is considered, people relate the chaos truncation error to the smoothness in the Malliavin sense. More precisely, from [29, Proposition 1.2.2], if F is a square integrable random variable with the Wiener chaos expansion $F = \sum_{n=0}^{\infty} J_n F$ (J_n being the projection on the nth chaos), then $F \in \mathbb{D}^{k,2}$ if and only if $\sum_{n=1}^{+\infty} n^k \|J_n F\|_2^2 < +\infty$ where $\mathbb{D}^{k,2}$ refers to the usual Sobolev space, see, e.g., [29, Equation 1.32] for a definition. Consequently, as soon as $F \in \mathbb{D}^{k,2}$, then

$$\|F - \sum_{n=0}^{N} J_n F\|_2 = \Big(\sum_{n=N+1}^{+\infty} \|J_n F\|_2^2 \Big)^{\frac{1}{2}} \leq (1+N)^{-\frac{k}{2}} \Big(\sum_{n=N+1}^{+\infty} n^k \|J_n F\|_2^2 \Big)^{\frac{1}{2}} = \mathcal{O}(N^{-\frac{k}{2}}),$$

which corresponds to the same rate obtained for the Hermite polynomials. The case $F = \mathbf{1}_{c \leq W_1}$, where $(W_t)_{t \geq 0}$ is a standard Brownian motion, corresponds to a fractional smoothness case (see [16]) with regularity $k = \frac{1}{2}$ leading to an L^2 truncation error of $\mathcal{O}(N^{-\frac{1}{4}})$ uniform in c (see also [8, Theorem 2.6]). In Theorem 2.2, we establish a finer result obtaining the same error in N but keeping the dependence on c for Hermite polynomials, and extending the analysis to the Laguerre (Theorem 2.3) and Jacobi (Theorem 2.4) polynomials.

Our contributions. When ν is associated with a COPS, we prove a stronger result than the $L^2(\nu)$ equality in (1.4), and establish the pointwise convergence $\sum_{n=0}^{N} \gamma_n(c) p_n(X) \to \mathbf{1}_{c \leq X}$ for any $X \in I_w \setminus \{c\}$ as $N \to +\infty$. For a fixed not-too-extreme c, we establish (see section 2.3) that it is optimal (in the L^2 sense) to perform a PCE w.r.t. the Jacobi polynomials as it exhibits an L^2 error of order $\mathcal{O}(N^{-\frac{1}{2}})$ as opposed to $\mathcal{O}(N^{-\frac{1}{4}})$ for the Laguerre and Hermite polynomials. Despite the poor convergence rate of order $\mathcal{O}(N^{-\frac{1}{4}})$, we exhibit an exponential factor of the form $e^{-\frac{c^2}{4}}$ (resp. $e^{-\frac{c}{2}}$) for the Hermite (resp. Laguerre) polynomials, ensuring a faster decrease for large values of $|c|$ (see Remark 3 for a precise discussion).

Regarding the optimal probabilistic transformation (see (1.7)), we consider the problem of approximating the random $\mathbf{1}_{c \leq X}$ where $X \overset{d}{=} \mathcal{U}([0,1])$ and $c \in (0,1)$, and investigate the case where $\tilde{p}_n(\cdot)$ can be any COPS, see section 3.2.1. For a fixed N, we show both theoretically (see Lemma 3.1) and numerically (see Figure 5(a)) that when $c \to 0$ (extremely low values) the smallest L^2 error is either achieved by the Jacobi polynomials provided that its associated parameters satisfy the constraints $\alpha < \beta - \frac{1}{2}$, or the Hermite polynomials otherwise. Similarly when $c \to 1$ (extremely large values), we demonstrate that either the Jacobi polynomials attain the smallest error when $\beta < \alpha - \frac{1}{2}$, or the Hermite and Laguerre polynomials which share the same error order (see section 3.2.1). For no extreme values of c, we demonstrate that the optimal transformation is attained with Jacobi polynomials with small parameters (see section 3.2).

Organization of the paper. In section 2, we start by recalling the main properties and characterizations of OPS and COPS (see Theorem 2.1). We then derive an explicit PCE for the COPS (see Proposition 2.1) when dealing with indicators functions of the form $\mathbf{1}_{c \leq X}$. We also

provide three-term recurrence relations for each chaos expansion coefficient $\gamma_n(c)$ (see Proposition 2.2) which will be shown to be crucial for numerical applications. In section 2.3, local and global estimates of the L^2 truncation error (1.6) are provided when dealing with the COPS (see Theorem 2.2–2.4) and illustrated in section 3.1. To assess the performance of one classical orthogonal polynomial compared to another depending on the threshold parameter c, a precise theoretical and numerical study is conducted in section 3, and we conclude in section 4. Proofs are postponed to section 5.

Special functions ([31, Chapter 5]).

- C.d.f. of the standard normal distribution:

$$\Phi(x) := \int_{-\infty}^{x} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} \mathrm{d}t, \quad x \in \mathbb{R}.$$

- Upper incomplete gamma and regularized lower incomplete gamma functions:

$$\Gamma_\alpha(z) := \int_z^{+\infty} t^{\alpha-1} e^{-t} \mathrm{d}t, \qquad \mathscr{G}_\alpha(z) := \frac{1}{\Gamma_\alpha} \int_0^z t^{\alpha-1} e^{-t} \mathrm{d}t, \quad \alpha > 0,\, z \geq 0, \tag{1.8}$$

  and we write $\Gamma_\alpha := \Gamma_\alpha(0)$ for the usual gamma function.

- Lower incomplete beta function and its regularized version:

$$\mathrm{B}_{\alpha,\beta}(x) := \int_0^x t^{\alpha-1} (1-t)^{\beta-1} \, \mathrm{d}t, \qquad \mathscr{B}_{\alpha,\beta}(x) := \frac{\mathrm{B}_{\alpha,\beta}(x)}{\mathrm{B}_{\alpha,\beta}}, \quad \alpha, \beta > 0,\, x \in [0,1], \tag{1.9}$$

  and we write $\mathrm{B}_{\alpha,\beta} := \mathrm{B}_{\alpha,\beta}(1)$ for the usual beta function. It is well known that the beta function can be expressed in terms of the gamma function (see, e.g., [31, 5.12.1]):

$$\mathrm{B}_{\alpha,\beta} = \frac{\Gamma_\alpha \Gamma_\beta}{\Gamma_{\alpha+\beta}}. \tag{1.10}$$

Probability distributions (see, e.g., [20, Chapters 17, 18] [21, Chapter 25]).

- The random variable X has a beta distribution with parameters $\alpha, \beta > 0$, written $X \overset{d}{=} \mathrm{Beta}(\alpha, \beta)$, if its c.d.f. is given by $\mathbb{P}(X \leq x) = \mathscr{B}_{\alpha,\beta}(x)$.

- The random variable X has a gamma distribution with parameters $\alpha, \beta > 0$, written $X \overset{d}{=} \mathrm{Gamma}(\alpha, \beta)$, if its c.d.f. is given by $\mathbb{P}(X \leq x) = \mathscr{G}_\alpha(\beta x)$. For any $\lambda > 0$, we have $\lambda X \overset{d}{=} \mathrm{Gamma}\left(\alpha, \frac{\beta}{\lambda}\right)$.

2 Orthogonal polynomials and polynomial chaos expansion

2.1 Reminders on orthogonal polynomials

We start by recalling results on orthogonal polynomials. General references are [34], [10], and for extensive reviews of their characterizations and main properties see, e.g., [1] and [31, Chapter 18].

Askey-scheme. In this work, we only focus on COPS, that is, continuous orthogonal polynomials. Discrete orthogonal polynomials also exist when X is a discrete random variable (see, e.g., [28]). The continuous and discrete orthogonal polynomials can be listed according to the so-called Askey-scheme [3] of hypergeometric orthogonal polynomials. All of such polynomials can be expressed in terms of hypergeometric functions ${}_pF_q(\cdot)$ and are listed according to their free real parameters (see [31, Figure 18.21.1] and [23]). For the continuous (resp. discrete) orthogonal polynomials, the most general polynomials are the Wilson (resp. Racah) polynomials which depend on four parameters. Then, the Continuous dual Hahn and Continuous Hahn (resp. Hahn and Dual Hahn) with three parameters, the Meixner–Pollaczek and Jacobi polynomials (resp. Meixner and Krawtchouk) with two parameters, the Laguerre polynomials (resp. Charlier) with one parameter, and finally the Hermite polynomials.

Recurrence relation. A remarkable property of orthogonal polynomials is that they necessarily solve a three-term recurrence relation of the form ([34, Theorem 3.2.1]):

$$p_{n+1}(x) = (A_n x + B_n)\, p_n(x) - C_n p_{n-1}(x), \quad n \in \mathbb{N}^*, \tag{2.1}$$

where A_n, B_n, C_n are constants with $A_n > 0, C_n > 0$. Note that the converse is also true and known as Favard's Theorem.

Classical orthogonal polynomials set (COPS). In this work, we will consider a subclass of continuous orthogonal polynomials commonly referred to as the classical orthogonal polynomials set or COPS. These polynomials are the only ones to satisfy Rodrigues' formula (2.2) and the limit conditions (2.3), which will be essential to derive closed-form expressions and recurrence relations for the chaos coefficients $\gamma_n(\cdot)$ (see Propositions 2.1 and 2.2). This motivates our choice to focus on the COPS which can be completely characterized as follows.

Theorem 2.1. *(see [1, section 5], [2, Theorem 1.2]) The set of orthogonal polynomials $(p_n(x) : n \in \mathbb{N})$ associated with some measure $\nu(\mathrm{d}x) = w(x)\mathrm{d}x$ is said to be a COPS if and only if it satisfies one of the following equivalent assertions:*

1. *For any $n \in \mathbb{N}$, $p_n(\cdot)$ satisfies a second order linear differential equation of the Sturm–Liouville type:*
$$F(x)\frac{\mathrm{d}^2 p_n}{\mathrm{d}x^2}(x) + G(x)\frac{\mathrm{d}p_n}{\mathrm{d}x}(x) + \lambda_n p_n(x) = 0,$$
where F and G are two polynomials with $\deg(F) \leq 2$ and $\deg(G) = 1$, and are both independent of n, and λ_n is a constant independent of x.

2. *The family of polynomials $\left(\frac{\mathrm{d}p_n}{\mathrm{d}x}(x), n \in \mathbb{N}\right)$ forms also a COPS.*

3. *They all satisfy a Rodrigues' type formula of the form:*
$$p_n(x) = \frac{1}{\kappa_n w(x)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[F(x)^n w(x)\right], \tag{2.2}$$
for some coefficients κ_n.

4. *The weight function $w(\cdot)$ satisfies the following Pearson's type differential equation*
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[F(x)w(x)\right] = G(x)w(x). \tag{2.3}$$

The only classical orthogonal polynomials are the Hermite, Laguerre (with parameter $\alpha > -1$), and the Jacobi (with parameters $\alpha, \beta > -1$) orthogonal polynomials, for which some of

the properties are gathered in Table 1 (see, e.g., [31]). One easily checks that the following limit conditions hold for the COPS (with $I_w = (a, b)$ defined in Table 1),

$$\lim_{x \to a} w(x)F(x) = \lim_{x \to b} w(x)F(x) = 0.$$

The Legendre polynomials are particular Jacobi polynomials with $\alpha = \beta = 0$.

Name	Hermite	Laguerre	Jacobi	Legendre
Parameters		$\alpha > -1$	$\alpha, \beta > -1$	$\alpha = \beta = 0$
$p_n(x)$	$\mathrm{He}_n(x)$	$\mathrm{L}_n^{(\alpha)}(x)$	$\mathrm{P}_n^{(\alpha,\beta)}(x)$	$\mathrm{P}_n(x)$
$I_w = (a, b)$	$(-\infty, +\infty)$	$(0, +\infty)$	$(-1, 1)$	$(-1, 1)$
$w(x)$	$\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$	$\frac{x^\alpha e^{-x}}{\Gamma_{\alpha+1}}$	$\frac{(1-x)^\alpha(1+x)^\beta}{2^{\alpha+\beta+1}\mathrm{B}_{\alpha+1,\beta+1}}$	$\frac{1}{2}$
ν	$\mathcal{N}(0, 1)$	$\mathrm{Gamma}(\alpha + 1, 1)$	$1 - 2\mathrm{Beta}(\alpha + 1, \beta + 1)$	$\mathcal{U}(-1, 1)$
$p_0(x)$	1	1	1	1
$p_1(x)$	x	$1 + \alpha - x$	$\alpha + 1 + \frac{(\alpha+\beta+2)(x-1)}{2}$	x
h_n	$n!$	$\frac{1}{n!}\frac{\Gamma_{n+\alpha+1}}{\Gamma_{\alpha+1}}$	$\frac{\Gamma_{n+\alpha+1}\Gamma_{n+\beta+1}}{(2n+\alpha+\beta+1)\Gamma_{n+\alpha+\beta+1}n!\mathrm{B}_{\alpha+1,\beta+1}}$	$\frac{2}{2n+1}$
A_n	1	$-\frac{1}{n+1}$	$\frac{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}{2(n+1)(\alpha+\beta+n+1)}$	$\frac{2n+1}{n+1}$
B_n	0	$\frac{2n+\alpha+1}{n+1}$	$\frac{(\alpha^2-\beta^2)(\alpha+\beta+2n+1)}{2(n+1)(\alpha+\beta+n+1)(\alpha+\beta+2n)}$	0
C_n	n	$\frac{n+\alpha}{n+1}$	$\frac{(\alpha+n)(\beta+n)(\alpha+\beta+2n+2)}{(n+1)(\alpha+\beta+n+1)(\alpha+\beta+2n)}$	$\frac{n}{n+1}$
$F(x)$	1	x	$1 - x^2$	$1 - x^2$
$G(x)$	$-x$	$\alpha + 1 - x$	$\beta - \alpha - (\alpha + \beta + 2)x$	$-2x$
κ_n	$(-1)^n$	$n!$	$(-2)^n n!$	$(-2)^n n!$

Table 1: Some properties for the classical orthogonal polynomials.

2.2 Chaos decomposition of the indicator function for the COPS

For the COPS, we show that the convergence (1.4)-(1.5) holds not only in $L^2(\nu)$ but also pointwise except on c. Furthermore, all coefficients $\gamma_n(\cdot)$ admit a closed-form representation.

Proposition 2.1. *Let $(p_n)_{n\in\mathbb{N}}$ be a COPS w.r.t. $\nu(\mathrm{d}x) = w(x)\mathrm{d}x$, and let $c \in I_w$. Then,*

$$\forall x \in I_w \setminus \{c\}, \quad \mathbf{1}_{c \le x} = \sum_{n=0}^{+\infty} \gamma_n(c) p_n(x),$$

where the coefficients $\gamma_n(\cdot)$ have been defined in (1.4). Furthermore, for every $n \ge 2$,

- *If $\nu = \mathcal{N}(0, 1)$,*

$$\gamma_0(c) = \Phi(-c), \quad \gamma_1(c) = \frac{e^{-\frac{c^2}{2}}}{\sqrt{2\pi}}, \quad \gamma_n(c) = \frac{e^{-\frac{c^2}{2}}\mathrm{He}_{n-1}(c)}{n!\sqrt{2\pi}}. \tag{2.4}$$

- *If $\nu = \mathrm{Gamma}(\alpha + 1, 1)$,*

$$\gamma_0(c) = \frac{\Gamma_{\alpha+1}(c)}{\Gamma_{\alpha+1}}, \quad \gamma_1(c) = -\frac{c^{1+\alpha}e^{-c}}{\Gamma_{\alpha+2}}, \quad \gamma_n(c) = -\frac{(n-1)!}{\Gamma_{n+\alpha+1}}c^{\alpha+1}e^{-c}\mathrm{L}_{n-1}^{(\alpha+1)}(c). \tag{2.5}$$

- *If $\nu = 1 - 2\mathrm{Beta}(\alpha + 1, \beta + 1)$,*

$$\gamma_0(c) = \mathscr{B}_{\alpha+1,\beta+1}\Big(\frac{1-c}{2}\Big),$$

$$\gamma_1(c) = \frac{\alpha+\beta+3}{(\alpha+1)(\beta+1)} \frac{(\alpha+1)\mathrm{B}_{\alpha+1,\beta+1}\left(\frac{1-c}{2}\right) - (\alpha+\beta+2)\,\mathrm{B}_{\alpha+2,\beta+1}\left(\frac{1-c}{2}\right)}{\mathrm{B}_{\alpha+1,\beta+1}}, \tag{2.6}$$
$$\gamma_n(c) = \frac{(2n+\alpha+\beta+1)\,\Gamma_{n+\alpha+\beta+1}\,(n-1)!}{2^{\alpha+\beta+2}\Gamma_{n+\alpha+1}\Gamma_{n+\beta+1}}(1-c)^{\alpha+1}\,(1+c)^{\beta+1}\,\mathrm{P}_{n-1}^{(\alpha+1,\beta+1)}(c).$$

For the reader's convenience, we have specified the first two values for $\gamma_n(\cdot)$. They will serve for the following proposition that provides a three-term recurrence relation for the chaos coefficients $\gamma_n(\cdot)$. The main interest relies on the fact that the implementation of the metamodel (see Proposition 2.1) may be computationally intensive.

Proposition 2.2 (Recurrence relation for the $\gamma_n(\cdot)$)**.** *For every $n \in \mathbb{N}$ and for any COPS, the following three-term recurrence relation holds for the parameters $\gamma_n(c)$,*

$$\gamma_{n+2}(c) = (B_{n+1} + A_{n+1}c)\frac{h_{n+1}}{h_{n+2}}\gamma_{n+1}(c) - C_{n+1}\frac{h_n}{h_{n+2}}\gamma_n(c) + \frac{h_{n+1}}{h_{n+2}}A_{n+1}\int_c^b \gamma_{n+1}(x)\mathrm{d}x. \tag{2.7}$$

In particular, depending on the COPS, if $\nu = \mathcal{N}(0,1)$,

$$\gamma_{n+2}(c) = \frac{c}{(n+2)}\gamma_{n+1}(c) - \frac{n}{(n+1)(n+2)}\gamma_n(c), \tag{2.8}$$

or if $\nu = \mathrm{Gamma}(\alpha+1, 1)$,

$$\gamma_{n+2}(c) = \frac{2n+\alpha+2-c}{n+\alpha+2}\gamma_{n+1}(c) - \frac{n}{n+\alpha+2}\gamma_n(c), \tag{2.9}$$

or if $\nu = 1 - 2\mathrm{Beta}(\alpha+1, \beta+1)$,

$$\gamma_{n+2}(c) = D(n,\alpha,\beta,c)\,\gamma_{n+1}(c) + E(n,\alpha,\beta)\,\gamma_n(c), \tag{2.10}$$

where

$$D(n,\alpha,\beta,c) = \frac{(\alpha+\beta+n+2)(\alpha+\beta+2n+5)\left[(\alpha-\beta)(\alpha+\beta+2) + c(\alpha+\beta+2n+2)(\alpha+\beta+2n+4)\right]}{2(\alpha+n+2)(\beta+n+2)(\alpha+\beta+n+3)(\alpha+\beta+2n+2)},$$
$$E(n,\alpha,\beta) = -\frac{n(\alpha+\beta+n+1)(\alpha+\beta+n+2)(\alpha+\beta+2n+4)(\alpha+\beta+2n+5)}{(\alpha+n+2)(\beta+n+2)(\alpha+\beta+n+3)(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}.$$

Remark 1. For the Legendre polynomials, setting $\alpha = \beta = 0$ in (2.6) and (2.10), we have the simplifications

$$\gamma_{n+2}(c) = \frac{(2n+5)c}{n+3}\gamma_{n+1}(c) - \frac{(2n+5)\,n}{(n+3)\,(2n+1)}\gamma_n(c). \tag{2.11}$$

$$\gamma_0(c) = \frac{1}{2}(1-c), \quad \gamma_1(c) = \frac{3}{4}\left(1-c^2\right), \quad \gamma_n(c) = \frac{2n+1}{4n}\left(1-c^2\right)\mathrm{P}_{n-1}^{(1,1)}(c). \tag{2.12}$$

Remark 2. Suppose that, instead of $\mathbf{1}_{c\le X}$, we are interested in deriving a PCE for the more general random variable $\ell(X)\mathbf{1}_{c\le X}$, for a given function ℓ. Then, the chaos coefficients $\Gamma_{\ell,n}(c)$ for $\ell(X)\mathbf{1}_{c\le X}$ can be expressed as linear combinations of $\gamma_n(c)$ and admit a semi-closed form (see [7, Chapter 4, section 4.2.1] for a detailed discussion).

2.3 L^2 error

In this section, we provide estimates for the L^2 error between $\mathbf{1}_{c\le X}$ and $\sum_{n=0}^{N}\gamma_n(c)p_n(X)$ which is given by (see (1.6)),

$$\mathcal{E}_N(c) = \left|\sum_{n=N+1}^{+\infty} h_n\gamma_n(c)^2\right|^{\frac{1}{2}}.$$

We will explicit the dependence of the basis for the L^2 error, i.e., we write $\mathcal{E}_N^{\mathrm{He}}(\cdot)$ for the Hermite polynomials, $\mathcal{E}_N^{\mathrm{Le}}(\cdot)$ for the Legendre polynomials, $\mathcal{E}_N^{\mathrm{J}}(\cdot)$ for the Jacobi polynomials and $\mathcal{E}_N^{\mathrm{La}}(\cdot)$ for the Laguerre polynomials.

Theorem 2.2 (Estimates for the Hermite polynomials)**.** *There exists a universal constant $A > 0$ such that for every $N \in \mathbb{N}^*$,*

$$\mathcal{E}_N^{\mathrm{He}}(c) \leq A e^{-\frac{c^2}{4}} \begin{cases} N^{-\frac{1}{4}}, & |c| \in [0, (4N)^{\frac{1}{2}}], \quad (2.13) \\ \left(\frac{\sqrt{N}}{|c|}\right)^{\frac{1}{4}} N^{-\frac{1}{12}}, & |c| \in [(2N)^{\frac{1}{2}}, N^{\frac{39}{14}}], \quad (2.14) \\ \left(\frac{|c|}{\sqrt{N}}\right)^{\frac{1}{3}} N^{-\frac{17}{12}}, & |c| \in [N^{\frac{39}{14}}, +\infty). \quad (2.15) \end{cases}$$

Theorem 2.3 (Estimates for the Laguerre polynomials)**.** *Let $0 < \eta < 4$. Then, there exists a constant $A > 0$ (depending on η and α) such that for every $N \in \mathbb{N}^*$,*

$$\mathcal{E}_N^{\mathrm{La}}(c) \leq A e^{-\frac{c}{2}} \begin{cases} \left(\frac{c}{N}\right)^{\frac{\alpha}{2}+\frac{1}{4}} N^{\frac{\alpha}{2}}, & c \in (0, (4-\eta)N], \quad (2.16) \\ \left(\frac{c}{N}\right)^{\frac{\alpha}{2}+\frac{5}{12}} N^{\frac{\alpha}{2}+\frac{1}{6}}, & c \in [2N, +\infty). \quad (2.17) \end{cases}$$

Theorem 2.4 (Estimates for the Jacobi polynomials)**.** *For any $\alpha, \beta > -1$, there exists a constant $A > 0$ (independent of α and β) such that for every $N \in \mathbb{N}^*$,*

$$\mathcal{E}_N^{\mathrm{J}}(c) \leq A \left(\frac{2 + \sqrt{(\alpha+1)^2 + (\beta+1)^2}}{2^{\alpha+\beta} \mathrm{B}_{\alpha+1,\beta+1}}\right)^{\frac{1}{2}} (1-c)^{\frac{\alpha}{2}+\frac{1}{4}} (1+c)^{\frac{\beta}{2}+\frac{1}{4}} N^{-\frac{1}{2}}, \quad c \in [-1, 1]. \quad (2.18)$$

Remark 3. Let us comment on each of the estimates.

First, for a given value of c (see estimates (2.13), (2.16), (2.18)), we retrieve the expected convergence rate w.r.t. N aforementioned in the introduction (paragraph on *background results*). For a probability measure ν with infinite support $I_w = \mathbb{R}$ or $I_w = \mathbb{R}^+$ (Hermite or Laguerre polynomials), the L^2 error converges as $N^{-\frac{1}{4}}$, while for the case with finite support (Jacobi polynomials), it becomes $N^{-\frac{1}{2}}$.

Second, for a given value of N, we get an accurate estimate on the dependence of the error w.r.t. c.

- For Hermite or Laguerre polynomials, we obtain an exponential term in c which decays fastly as c goes in the tails of the distribution ν. This explains well the accurate results in the numerical experiments of [8]: these approximations were used for stochastic but large values of c, and despite the low values of N, the exponential factor helps much in getting a small global error.

- For Jacobi polynomials, we obtain a polynomial factor which goes to 0 as $c = \pm 1$ as soon as $\alpha, \beta > -\frac{1}{2}$: this is the analog of the exponential term of the Hermite/Laguerre cases. The factor may explode in the case α or $\beta < -\frac{1}{2}$: this is not surprising since in such a case, the measure ν is concentrated at one or/and the other ending points of the interval $I_w = (-1, 1)$.

- In all cases, the L^2 truncation error goes to 0 for extreme values of c, that is when $\mathbf{1}_{c \leq X} = 1$ or 0.

The interplay between c and N is detailed. This is potentially useful when c is taken large, as a function of N.

- For Jacobi polynomials, the dependence on c and N is separable, see (2.18), it is presumably due to the property of compact support of the underlying measure ν.

- For Hermite polynomials, Theorem 2.2 exhibits different regimes for $\mathcal{E}_N^{\mathrm{He}}(c)e^{\frac{c^2}{4}}$ depending on the relative value of c and N. More precisely, when $|c| \leq \sqrt{4N}$, $\mathcal{E}_N^{\mathrm{He}}(c)e^{\frac{c^2}{4}}$ decreases as $N^{-\frac{1}{4}}$. When $|c|$ is larger than $\sqrt{2N}$, different regimes occur: in any case the exponential term shows that $\mathcal{E}_N^{\mathrm{He}}(c)$ is exponentially small but our estimates gives a further refinement (beyond the exponential scale with a polynomial scale in c and N). The jump in the polynomial scale comes from the abrupt change of behavior of Hermite polynomial $\mathrm{He}_N(c)$ around its maximum value which is attained around $c \approx \sqrt{4N}$ (see the estimates of Lemma 5.2).

- For Laguerre polynomials, we obtain also a refined estimate beyond the exponential scale, i.e., in the polynomial scale. Here again, the jump in the polynomial term is due to the rapid change of behavior of Laguerre polynomial $\mathrm{L}_N^{(\alpha)}(c)$ around its maximum achieved at $c \approx 2N$.

As opposed to the estimates in Theorem 2.3 for the Laguerre polynomials, note that the constant in front of the estimates for the Jacobi polynomials in Theorem 2.4 does not depend on the Jacobi parameters α and β.

For the Legendre polynomials, setting $\alpha = \beta = 0$ in (2.18), there exists a universal constant $B > 0$ such that $\mathcal{E}_N^{\mathrm{Le}}(c) \leq B\left(1-c^2\right)^{\frac{1}{4}} N^{-\frac{1}{2}}, \quad c \in [-1,1]$.

For a fixed $c \in [-1,1]$ and $N \geq 1$, let us comment on the behavior of estimate (2.18) for large values of α, β. Set $F(\alpha,\beta) := \left(\frac{2+\sqrt{(\alpha+1)^2+(\beta+1)^2}}{2^{\alpha+\beta}\mathrm{B}_{\alpha+1,\beta+1}}\right)^{\frac{1}{2}}$ for $\alpha, \beta > -1$ and recall that $\Gamma_x \underset{x\to+\infty}{\sim} e^{-x}x^x(\frac{2\pi}{x})^{\frac{1}{2}}$ (see [31, 5.11.3]). For a fixed $\alpha > -1$, we readily obtain $F(\alpha,\beta) \underset{\beta\to+\infty}{\sim} \frac{1}{2^{\frac{\alpha}{2}}\Gamma_{\alpha+1}^{\frac{1}{2}}} \frac{\beta^{\frac{\alpha}{2}+1}}{2^{\frac{\beta}{2}}} \to 0$, where we have used the identity (1.10). As F is symmetric, the same limit holds for a fixed $\beta > -1$ and large α. However, when $\alpha = \beta$ a similar study shows $F(\alpha,\alpha) \underset{\alpha\to+\infty}{\sim} \frac{(2\alpha)^{\frac{3}{4}}}{\pi^{\frac{1}{4}}} \to +\infty$. Though the L^2 error decreases as $N^{-\frac{1}{2}}$, the multiplicative constant can be large when α, β are large. In fact, we can obtain another estimate for $\mathcal{E}_N^{\mathrm{J}}(c)$ with a multiplicative constant which is uniform in α, β but with a convergence rate $N^{-\frac{1}{4}}$ instead of $N^{-\frac{1}{2}}$. Evoking [19, Theorem 1.1] instead of [27, Theorem 1] in the proof of Theorem 2.4, one may prove that there exists a universal constant $A > 0$ such that

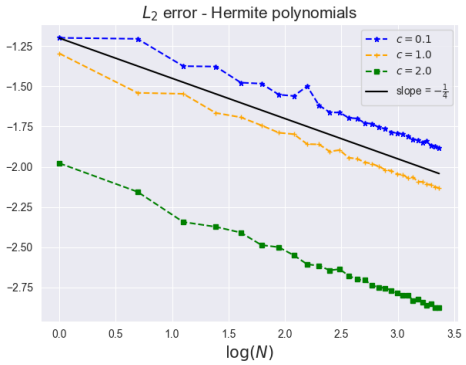
$$\mathcal{E}_N^{\mathrm{J}}(c) \leq A(1-c)^{\frac{\alpha}{2}+\frac{1}{4}}(1+c)^{\frac{\beta}{2}+\frac{1}{4}} N^{-\frac{1}{4}}, c \in [-1,1] \tag{2.19}$$

where the order $N^{-\frac{1}{4}}$ is known to be sharp when $\alpha, \beta \to +\infty$.

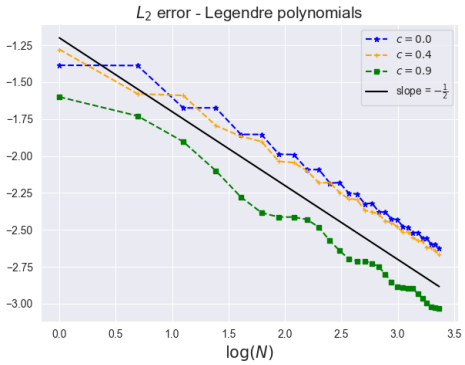
3 Numerical tests

3.1 L^2 errors comparison w.r.t. N for fixed c

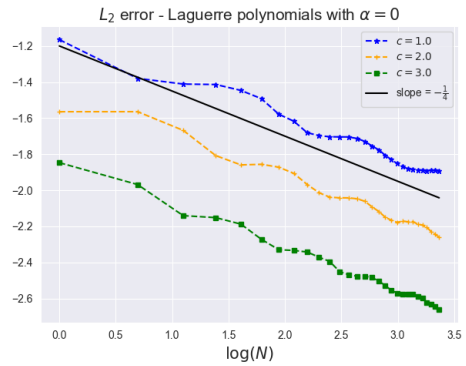
Let us fix c. To illustrate the estimates found in Theorems 2.2–2.4 for the L^2 error, we wish to retrieve the order $\mathcal{O}\left(N^{-\frac{1}{2}}\right)$ for the Jacobi polynomials and $\mathcal{O}\left(N^{-\frac{1}{4}}\right)$ for the other orthogonal polynomials. In Figure 1, we then plot $\log\left(\mathcal{E}_N(c)\right)$ w.r.t. $\log(N)$ for different fixed values of c and truncation parameters $N \in [1,30]$. Note that even if we have explicit formulas for all L^2 errors associated to the COPS, we do not use them as they entail numerical instability when evaluating polynomials at very high degree n and large values for c. To overcome this issue, we proceed with a Monte Carlo simulation and directly estimate the expectation $\mathbb{E}\left[\left|\mathbf{1}_{c\leq X} - \sum_{n=0}^N \gamma_n(c)p_n(X)\right|^2\right]^{\frac{1}{2}}$ in (1.6) with $M = 5\times 10^5$ samples. We do not report the confidence interval as the error is negligible (less than 0.1% in relative error).



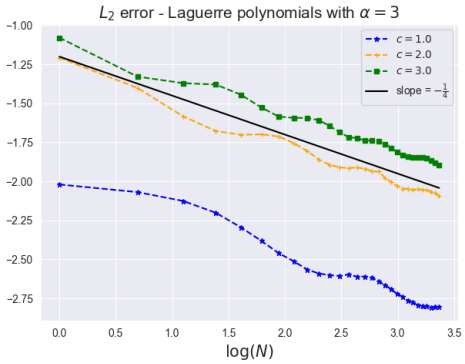
(a)



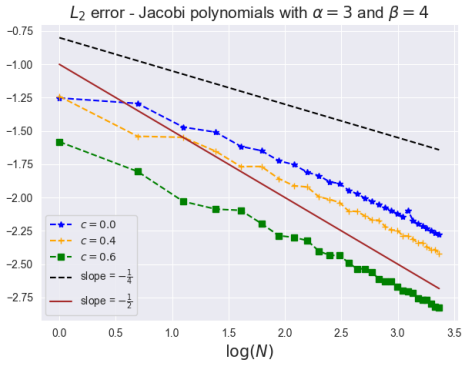
(b)



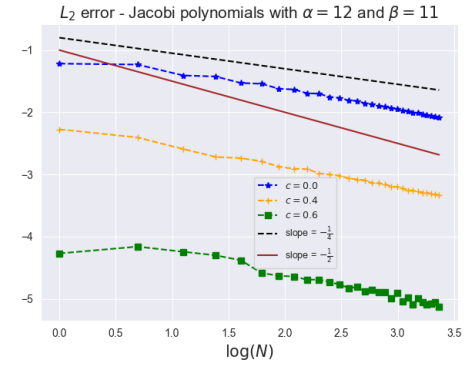
(c)



(d)



(e)



(f)

Figure 1: $\log \mathcal{E}_N(c)$ w.r.t. $\log N$ for the COPS for different parameters c.

For the Hermite (Figure 1(a)) and Laguerre (Figures 1(c) and 1(d)) polynomials, we retrieve the error of order $N^{-\frac{1}{4}}$ exhibited in (2.13) and (2.16). This seems to support the idea that the order of our estimates is tight, i.e., the speed in N for the error upper bound is optimal. To prove that the error is precisely asymptotically equivalent to Cst $\times N^{-\frac{1}{4}}$ seems to be hopeless for Laguerre polynomials since the error oscillates as N increases: our error upper bound seems to be the best result we can achieve. Similarly, for Jacobi (Figure 1(e)) with no-extreme values of α, β, and Legendre (1(b)) polynomials, the numerical results confirm that the error behaves as $N^{-\frac{1}{2}}$. For large α and β, Figure 1(f) seems to indicate an error of order $N^{-\frac{1}{4}}$ and illustrates the competition between large values of N and α, β (see Remark 3 and (2.19)).

3.2 Best selection of COPS, L^2 errors comparison w.r.t. the quantile q

In this section, we are concerned with the choice of COPS one should consider when performing the PCE of $\mathbf{1}_{c\leq X}$ w.r.t. X, while playing with the degree of freedom of a continuous increasing transformation T, see the introduction. Referring to (1.7), we have $\mathbf{1}_{c\leq X} = \mathbf{1}_{T(c)\leq T(X)} =$ and

$$\mathbf{1}_{c\leq X} = \begin{cases} \sum_{n\geq 0} \gamma_n^{\mathrm{He}}(T^{\mathrm{He}}(c))\mathrm{He}_n(T^{\mathrm{He}}(X)) & \text{if } T^{\mathrm{He}}(X) \overset{d}{=} \mathcal{N}(0,1), \\ \sum_{n\geq 0} \gamma_n^{\mathrm{La}}(T^{\mathrm{La}}(c))\mathrm{L}_n^{(\alpha)}(T^{\mathrm{La}}(X)) & \text{if } T^{\mathrm{La}}(X) \overset{d}{=} \mathrm{Gamma}\,(\alpha+1,1), \\ \sum_{n\geq 0} \gamma_n^{\mathrm{J}}(T^{\mathrm{J}}(c))\mathrm{P}_n^{(\alpha,\beta)}(T^{\mathrm{J}}(X)) & \text{if } T^{\mathrm{J}}(X) \overset{d}{=} 1-2\mathrm{Beta}\,(\alpha+1,\beta+1), \\ \sum_{n\geq 0} \gamma_n^{\mathrm{Le}}(T^{\mathrm{Le}}(c))\mathrm{P}_n(T^{\mathrm{Le}}(X)) & \text{if } T^{\mathrm{Le}}(X) \overset{d}{=} \mathcal{U}\,(-1,1). \end{cases}$$

The L^2 error between the left hand side and the truncated sum (at order N) of the right hand side writes respectively as $\mathcal{E}_N^{\mathrm{He}}\left(T^{\mathrm{He}}(c)\right), \mathcal{E}_N^{\mathrm{La}}\left(T^{\mathrm{La}}(c)\right), \mathcal{E}_N^{\mathrm{J}}\left(T^{\mathrm{J}}(q)\right), \mathcal{E}_N^{\mathrm{Le}}\left(T^{\mathrm{Le}}(c)\right)$. To get easily interpretable results, we choose $X \overset{d}{=} \mathcal{U}\,(0,1)$ so that $c =: q \in (0,1)$ reads as a quantile. The quantities $T(c)$ become respectively

$$c_{\mathrm{He}}(q) := \Phi^{-1}(q) \in (-\infty, +\infty), \qquad c_{\mathrm{La}}(q) := \mathscr{G}_{\alpha+1}^{-1}(q) \in (0, +\infty),$$
$$c_{\mathrm{Le}}(q) := 2q-1 \in (-1,1), \qquad c_{\mathrm{J}}(q) := 2\mathscr{B}_{\alpha+1,\beta+1}^{-1}(q) - 1 \in (-1,1).$$

We seek to compare $\mathcal{E}_N^{\mathrm{He}}\left(c_{\mathrm{He}}(q)\right), \mathcal{E}_N^{\mathrm{La}}\left(c_{\mathrm{La}}(q)\right), \mathcal{E}_N^{\mathrm{J}}\left(c_{\mathrm{J}}(q)\right), \mathcal{E}_N^{\mathrm{Le}}\left(c_{\mathrm{Le}}(q)\right)$ for all $q \in (0,1)$, in order to assess the best choice of COPS for each level of quantile q. We will see that depending on the value q, it might be more adequate (in terms of L^2 error) to decompose on one basis compared to another. In all our numerical experiments, we choose $M = 5 \times 10^5$ Monte Carlo samples for evaluating the L^2 error, plot the associated confidence interval, and take 20 evenly-spaced quantiles $q \in [0.05, 0.95]$. Note that, due to the high number of Monte Carlo samples, the confidence intervals are almost indistinguishable in all of our experiments.

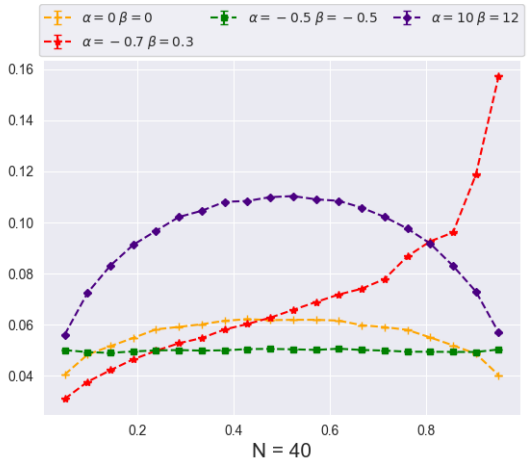


Figure 2: L^2 error for the Jacobi polynomials w.r.t. $q \in (0,1)$ for $N = 40$.

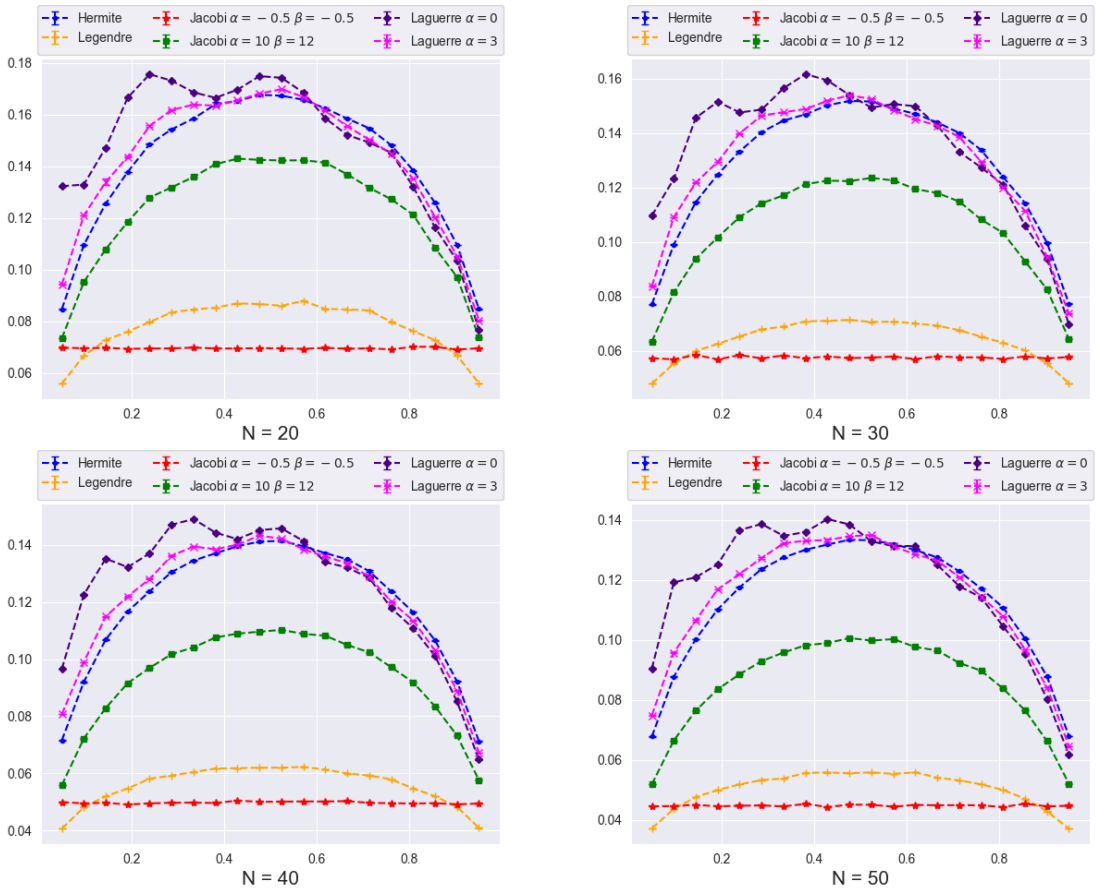


Figure 3: L^2 error for the COPS w.r.t. $q \in (0,1)$ and for $N \in \{20, 30, 40, 50\}$.

From (2.18), recall that $\mathcal{E}_N^{\mathrm{J}}(c_{\mathrm{J}}(q)) \leq A(\alpha,\beta)(1-c_{\mathrm{J}}(q))^{\frac{\alpha}{2}+\frac{1}{4}}(\alpha,\beta)(1+c_{\mathrm{J}}(q))^{\frac{\beta}{2}+\frac{1}{4}}N^{-\frac{1}{2}}$, for every $q \in [0,1]$ and where $(\alpha,\beta) \mapsto A(\alpha,\beta)$ is a function (independent of q and N) exploding when $\alpha, \beta \to +\infty$. Consequently, for a fixed N and assuming that our estimate (2.18) is tight, we expect $q \mapsto \mathcal{E}_N^{\mathrm{J}}(c_{\mathrm{J}}(q))$ to be constant when $\alpha = \beta = -\frac{1}{2}$ (Chebychev polynomials of the first kind, see, e.g., [31, Table 18.3.1]), to increase when α, β increase simultaneously, to explode when $c_{\mathrm{J}}(q) \to 1 \iff q \to 1$ for $\alpha < -\frac{1}{2}$ and/or when $c_{\mathrm{J}}(q) \to -1 \iff q \to 0$ for $\beta < -\frac{1}{2}$. Such behaviors are numerically confirmed by Figure 2 in which we plot the L^2 errors for the Jacobi polynomials for different values of α, β. More precisely, the L^2 error is constant for all values of q when $\alpha = \beta = -\frac{1}{2}$ (green line with the squares), explodes when $\alpha = -0.7, \beta = 0.3$ when q approaches 1 (red line with the stars), and increases when α, β increase simultaneously (compare the Legendre polynomials in yellow with the crosses to the Jacobi polynomials with $\alpha = 10, \beta = 12$ in indigo with the diamonds). As a conclusion, note that the Legendre polynomials are not necessarily the best orthogonal polynomials, that the Jacobi polynomials for $\alpha \to -1$ (resp. $\beta \to -1$) perform well for low (resp. large) quantiles, and that taking large values of α, β is never the optimal choice.

In Figure 3, we plot the L^2 error for the Jacobi (with $\alpha = \beta = 0$ and $\alpha = \beta = -0.5$, and $\alpha = 10, \beta = 12$), Laguerre ($\alpha = 0$ and $\alpha = 3$), and Hermite polynomials for $N \in \{20, 30, 40, 50\}$. From Theorems 2.2–2.4 and their respective errors order in N, we expect the Jacobi polynomials to provide the smallest L^2 error compared to the Hermite and Laguerre polynomials for no-extreme values of q, i.e., $0 \ll q \ll 1$. This is precisely the behavior observed in Figure 3. Notice also that the L^2 errors for the Hermite and Laguerre polynomials are very close. Consequently, in view of Figure 3, using Jacobi polynomials with small α, β is undoubtedly the best choice.

To assess the tightness in q and N of the estimates found in Theorems 2.2–2.4, we also plot the ratio between the L^2 error and the corresponding upper bound for every orthogonal

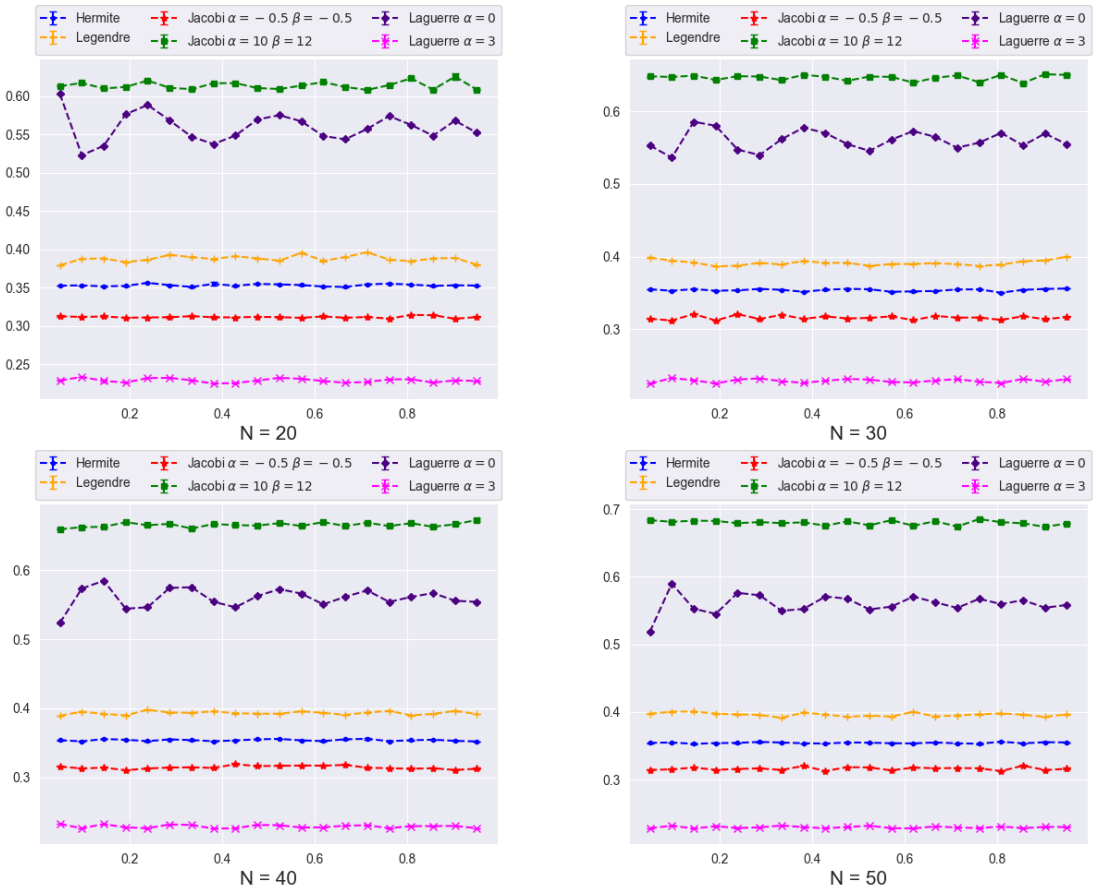


Figure 4: Ratio between the L^2 error and the error estimates of Theorems 2.2–2.4 w.r.t. the quantile $q \in (0,1)$ for different truncation parameters $N \in \{20, 30, 40, 50\}$.

polynomial, i.e., we plot the functions

$$
\begin{aligned}
&q \mapsto e^{\frac{c_{\mathrm{He}}(q)^2}{4}} N^{\frac{1}{4}} \mathcal{E}_N^{\mathrm{He}}\left(c_{\mathrm{He}}(q)\right), \\
&q \mapsto c_{\mathrm{La}}(q)^{-\frac{\alpha}{2}-\frac{1}{4}} e^{\frac{c_{\mathrm{La}}(q)}{2}} N^{\frac{1}{4}} \mathcal{E}_N^{\mathrm{La}}\left(c_{\mathrm{La}}(q)\right), \\
&q \mapsto \left(1-c_{\mathrm{J}}(q)\right)^{-\frac{\alpha}{2}-\frac{1}{4}} \left(1+c_{\mathrm{J}}(q)\right)^{-\frac{\beta}{2}-\frac{1}{4}} N^{\frac{1}{2}} \mathcal{E}_N^{\mathrm{J}}\left(c_{\mathrm{J}}(q)\right), \\
&q \mapsto \left(1-c_{\mathrm{Le}}(q)^2\right)^{-\frac{1}{4}} N^{\frac{1}{2}} \mathcal{E}_N^{\mathrm{Le}}\left(c_{\mathrm{Le}}(q)\right).
\end{aligned}
$$

We expect these ratios to be quite constant for all quantiles $q \in (0,1)$ and all N. Note that for the Hermite (resp. Laguerre) polynomials, we use estimate (2.13) (resp. (2.16)) being valid for $|c| \in \left[0, (4N)^{\frac{1}{2}}\right]$ (resp. $c \in [0, 3N]$). Actually, if we set a small truncation parameter $N = 5$, the above thresholds in N correspond to a normal quantile $\Phi(20^{\frac{1}{2}}) \approx 0.999996$ and a gamma quantile $\mathscr{G}_{\alpha+1}(15) \approx 0.999995$ where we have chosen $\alpha = 1$. This means that for all usual quantiles, estimates (2.13) and (2.16) are to be used. From Figure 4, we clearly observe that all ratios remain quite constant w.r.t. q and N, confirming that all our theoretical estimates of Theorems 2.2–2.4 are tight in both q and N.

3.2.1 Extreme quantiles

The Chebychev polynomials of the first kind (Jacobi with $\alpha = \beta = -\frac{1}{2}$) are among the most accurate orthogonal polynomials for not-too-extreme values of q. As displayed in Figure 3, this seems different when dealing with extremely small ($q \to 0^+$) or large ($q \to 1^-$) quantiles.

Assuming that all estimates exposed in Theorems 2.2–2.4 are tight, the next lemma provides a precise statement of the performance of every polynomial for extreme quantiles, i.e., we give the exact asymptotics of the upper-bounds depending on q of the L^2 errors for extreme quantiles. When $q \to 1^-$, we have to use (2.15) valid when $c_{\mathrm{He}}(q) \to \pm\infty$ and (2.17) valid when $c_{\mathrm{La}}(q) \to +\infty$.

Lemma 3.1 (Asymptotic expansions for the upper-bounds of the L^2 errors for extreme quantiles)**.** *For the Hermite polynomials, we have*

$$|c_{\mathrm{He}}(q)|^{\frac{1}{3}}\, e^{-\frac{c_{\mathrm{He}}(q)^2}{4}} \sim \begin{cases} 2^{\frac{2}{3}}\pi^{\frac{1}{4}}\, |\ln q|^{\frac{5}{12}}\, q^{\frac{1}{2}}, & (q \to 0^+), \quad (3.1) \\ 2^{\frac{2}{3}}\pi^{\frac{1}{4}}\, |\ln(1-q)|^{\frac{5}{12}}\, (1-q)^{\frac{1}{2}}, & (q \to 1^-). \quad (3.2) \end{cases}$$

For the Laguerre polynomials, we have

$$c_{\mathrm{La}}(q)^{\frac{\alpha}{2}+\frac{1}{4}} e^{-\frac{c_{\mathrm{La}}(q)}{2}} \underset{q\to 0^+}{\sim} \Gamma_{\alpha+2}^{\frac{2\alpha+1}{4(\alpha+1)}}\, q^{\frac{2\alpha+1}{4(\alpha+1)}}, \quad (3.3)$$

$$c_{\mathrm{La}}(q)^{\frac{\alpha}{2}+\frac{5}{12}} e^{-\frac{c_{\mathrm{La}}(q)}{2}} \underset{q\to 1^-}{\sim} \Gamma_{\alpha+1}^{\frac{1}{2}} (1-q)^{\frac{1}{2}}\, |\ln(1-q)|^{\frac{5}{12}}\,. \quad (3.4)$$

For the Jacobi polynomials, we have

$$(1-c_{\mathrm{J}}(q))^{\frac{\alpha}{2}+\frac{1}{4}} (1+c_{\mathrm{J}}(q))^{\frac{\beta}{2}+\frac{1}{4}} \sim 2^{\frac{\alpha+\beta+1}{2}} \begin{cases} ((\alpha+1)\mathrm{B}_{\alpha+1,\beta+1} q)^{\frac{2\beta+1}{4(\alpha+1)}}, & (q \to 0^+), \quad (3.5) \\ ((\beta+1)\mathrm{B}_{\beta+1,\alpha+1}(1-q))^{\frac{2\alpha+1}{4(\beta+1)}}, & (q \to 1^-). \quad (3.6) \end{cases}$$

Remark 4. To avoid any confusion, denote α_{J}, β_{J} (resp. α_{La}) the parameters associated with the Jacobi (resp. Laguerre) polynomials.

From (3.5) and (3.6), note that the asymptotic expansion may explode when $\beta_{\mathrm{J}} < -\frac{1}{2}$ for $q \to 0^+$ and $\alpha_{\mathrm{J}} < -\frac{1}{2}$ for $q \to 1^-$ (the distribution is either concentrated on the left-hand side or the right-hand side). Similarly for Laguerre polynomials, from (3.3) the expansion explodes when $\alpha_{\mathrm{La}} < -\frac{1}{2}$ for $q \to 0^+$.

▷ *Small quantiles* $q \to 0^+$. From Lemma 3.1, we compare $q^{\frac{2\beta_{\mathrm{J}}+1}{4(\alpha_{\mathrm{J}}+1)}}$, $q^{\frac{2\alpha_{\mathrm{La}}+1}{4(\alpha_{\mathrm{La}}+1)}}$, $q^{\frac{1}{2}}$ to understand which orthogonal polynomial gives asymptotically the smallest error. As the inequality $\frac{2\alpha_{\mathrm{La}}+1}{4(\alpha_{\mathrm{La}}+1)} < \frac{1}{2}$ is always satisfied since $\alpha_{\mathrm{La}} > -1$, we have $q^{\frac{1}{2}} < q^{\frac{2\alpha_{\mathrm{La}}+1}{4(\alpha_{\mathrm{La}}+1)}}$, and so the L^2 error for the Hermite polynomials is always smaller than that of Laguerre polynomials. The L^2 smallest error is then asymptotically attained by

- the Jacobi polynomials when $\frac{2\beta_{\mathrm{J}}+1}{4(\alpha_{\mathrm{J}}+1)} > \frac{1}{2} \iff \alpha_{\mathrm{J}} < \beta_{\mathrm{J}} - \frac{1}{2}$,
- the Hermite polynomials when $\alpha_{\mathrm{J}} > \beta_{\mathrm{J}} - \frac{1}{2}$,
- both the Hermite and Jacobi polynomials when $\alpha_{\mathrm{J}} = \beta_{\mathrm{J}} - \frac{1}{2}$.

Surprisingly, notice that the Jacobi polynomials can give the highest L^2 errors when $\frac{2\beta_{\mathrm{J}}+1}{\alpha_{\mathrm{J}}+1} < \frac{2\alpha_{\mathrm{La}}+1}{\alpha_{\mathrm{La}}+1}$.

▷ *Large quantiles* $q \to 1^-$. From (3.2) and (3.4), we note that both Laguerre and Hermite polynomials admit the same order in q. We thus need to compare $(1-q)^{\frac{2\alpha_{\mathrm{J}}+1}{4(\beta_{\mathrm{J}}+1)}}$ and $(1-q)^{\frac{1}{2}}$. The L^2 smallest error is then asymptotically attained by

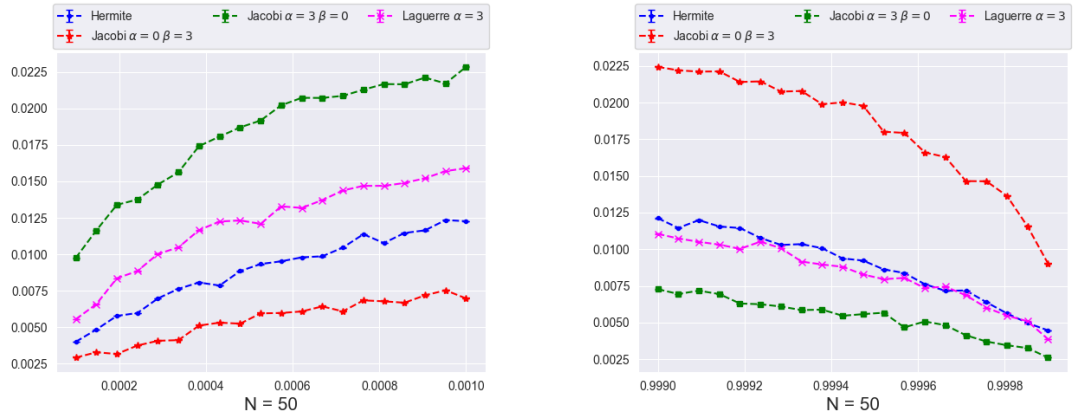
- the Jacobi polynomials when $\frac{2\alpha_{\mathrm{J}}+1}{4(\beta_{\mathrm{J}}+1)} > \frac{1}{2} \iff \beta_{\mathrm{J}} < \alpha_{\mathrm{J}} - \frac{1}{2}$,
- the Hermite and Laguerre polynomials when $\beta_{\mathrm{J}} > \alpha_{\mathrm{J}} - \frac{1}{2}$,
- all polynomials when $\beta_{\mathrm{J}} = \alpha_{\mathrm{J}} - \frac{1}{2}$ (up to some logarithmic error terms).

Unsurprisingly, from the symmetry of (2.18), the condition for the Jacobi polynomials to provide the smallest L^2 error for large quantiles corresponds to the same condition that of low quantiles by swapping α_{J} and β_{J}.

To illustrate Lemma 3.1, let us plot the L^2 error for small quantiles, i.e., $q \in \left(10^{-4}, 10^{-3}\right)$. We consider the L^2 errors for the Hermite polynomials (blue with the points), Laguerre polynomials with $\alpha_{\mathrm{La}} = 3$ (magenta with the crosses), and for Jacobi polynomials with two different sets of parameters $\alpha_{\mathrm{J}}^{(1)} = 0, \beta_{\mathrm{J}}^{(1)} = 3$ (red with the stars), $\alpha_{\mathrm{J}}^{(2)} = 3, \beta_{\mathrm{J}}^{(2)} = 0$ (green with the squares). The results are reported in Figures 5(a)-5(b). We take $N = 50$ and $M = 5 \times 10^5$ Monte Carlo samples and plot the associated confidence interval for each L^2 error.

As $\alpha_{\mathrm{J}}^{(1)} < \beta_{\mathrm{J}}^{(1)} - \frac{1}{2} \iff 0 < \frac{5}{2}$, the Jacobi polynomials with parameters $\alpha_{\mathrm{J}}^{(1)}, \beta_{\mathrm{J}}^{(1)}$ should provide the smallest error and this is what is observed in Figure 5(a). As $\alpha_{\mathrm{J}}^{(2)} > \beta_{\mathrm{J}}^{(2)} - \frac{1}{2} \iff 3 > -\frac{1}{2}$ and $\frac{2\beta_{\mathrm{J}}^{(2)}+1}{\alpha_{\mathrm{J}}^{(2)}+1} < \frac{2\alpha_{\mathrm{La}}+1}{\alpha_{\mathrm{La}}+1} \iff \frac{1}{4} < \frac{7}{4}$, the Jacobi polynomials with parameters $\alpha_{\mathrm{J}}^{(2)}, \beta_{\mathrm{J}}^{(2)}$ should give the worst L^2 error compared to the Laguerre and Hermite polynomials, and again this is what is observed in Figure 5(a). Notice also that the L^2 error for Hermite polynomials is always smaller than that of Laguerre.

For large quantiles, we take $q \in \left(1 - 10^{-3}, 1 - 10^{-4}\right)$. Conducting a similar analysis, we should observe that the Jacobi polynomials parameters $\alpha_{\mathrm{J}}^{(1)}, \beta_{\mathrm{J}}^{(1)}$ should give the worst L^2 error, the ones with $\alpha_{\mathrm{J}}^{(2)}, \beta_{\mathrm{J}}^{(2)}$ the lowest, and very close L^2 errors for both Hermite and Laguerre polynomials. This is precisely the behaviors exhibited in Figure 5(b).



(a) Small quantiles

(b) Large quantiles

Figure 5: L^2 error for the COPS w.r.t. small quantiles $q \in \left(10^{-4}, 10^{-3}\right)$ (left) and large quantiles $q \in \left(1 - 10^{-3}, 1 - 10^{-4}\right)$ (right) for a fixed $N = 50$.

The above Figure confirms again the tightness of our estimates displayed in Theorem 2.2–2.4.

4 Conclusion

We have proposed a thorough comparative study of polynomial-type chaos expansions for indicator functions $\mathbf{1}_{c \leq X}$ for a given parameter c and scalar random variable X associated with a COPS. Pointwise convergence of the Nth order chaos $\sum_{n=0}^{N} \gamma_n(c) p_n(X)$ is proved along with accurate global and local estimates for the resulting L^2 error, as a function of N and c: the tightness of error bounds is confirmed by numerous numerical experiments.

We have also shown that the optimal probabilistic transformation for $\mathbf{1}_{c \leq X}$ is achieved with Jacobi polynomials for no large α, β when c is not too extreme. For extremely low values of c, the optimal probabilistic transformation is achieved with Jacobi polynomials provided that

$\alpha < \beta - \frac{1}{2}$ or the Hermite polynomials otherwise. For extremely large values of c, it is achieved with Jacobi polynomials provided that $\beta < \alpha - \frac{1}{2}$, or with Laguerre and Hermite polynomials otherwise. This sheds a new light on which probability distribution to consider when performing a polynomial chaos expansion.

5 Proofs

Standard result. We will make extensive use of the following tail asymptotic expansion:

$$\sum_{n=N}^{+\infty} \frac{1}{n^m} \underset{N\to+\infty}{\sim} \frac{1}{m-1}\frac{1}{N^{m-1}}, \qquad \text{for all } m > 1. \tag{5.1}$$

5.1 Proof of Proposition 2.1

The $L^2(\nu)$ equality in (1.4) results from the denseness of $(p_n)_{n\in\mathbb{N}}$ in $L^2(\nu)$, implying that (1.4) holds ν almost everywhere. For the COPS, it is possible to obtain a stronger result, namely, pointwise convergence. Using asymptotic expansions in the Christoffel–Darboux representation of (1.5), Uspensky [35] managed to calculate this limit in the case of Laguerre or Hermite polynomials. In particular when $f \in L^2(\nu)$, f is absolutely integrable on any finite interval, and has bounded variation in the neighborhood of x, then

$$\sum_{n=0}^{+\infty} \gamma_n(f) p_n(x) = \frac{f(x^+) + f(x^-)}{2}.$$

Taking $f : x \mapsto \mathbf{1}_{\{c\le x\}}$, the above pointwise limit is $f(x)$ for every $x \in I_w \setminus \{c\}$, and $\frac{1}{2}$ if $x = c$ (Gibb's phenomenon). An overview of those methods can also be found in [34, 32]. Concerning Jacobi polynomials, to the best of our knowledge, there is no such a result. However, in view of [34, Theorem 9.1.2], we simply have to show that the Fourier series of the function $\varphi : \theta \in [-\pi,\pi] \mapsto (1-\cos(\theta))^{\frac{2\alpha+1}{4}} (1+\cos(\theta))^{\frac{2\beta+1}{4}} \mathbf{1}_{[c,1]}(\cos(\theta))$ converges for every θ such that $\cos(\theta) \neq c$. Since φ is locally Hölder (away from the above discontinuity point), this a consequence of the Dini criterion (see [40, Theorem 6.1., p.52]), and the pointwise convergence holds for the Jacobi polynomials for $x \in (-1,1) \setminus \{c\}$. We now focus on establishing the announced formulas for the $\gamma_n(\cdot)$.

▷ *Coefficients $\gamma_n(c)$ for the Hermite polynomials.* From (1.4), we have first

$$\gamma_0(c) = \frac{1}{h_0}\int_c^{+\infty} \mathrm{He}_0(x)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\mathrm{d}x = \Phi(-c), \quad \gamma_1(c) = \frac{1}{1!}\int_c^{+\infty} x\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\mathrm{d}x = \frac{e^{-\frac{c^2}{2}}}{\sqrt{2\pi}},$$

and for every $n \in \mathbb{N}^*$,

$$\gamma_n(c) = \frac{(-1)^n}{n!\sqrt{2\pi}}\int_c^{+\infty} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-\frac{x^2}{2}}\right)\mathrm{d}x = \frac{(-1)^{n-1}}{n!\sqrt{2\pi}}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left(e^{-\frac{x^2}{2}}\right)\Big|_{x=c} = \frac{e^{-\frac{c^2}{2}}\mathrm{He}_{n-1}(c)}{n!\sqrt{2\pi}}.$$

▷ *Coefficients $\gamma_n(c)$ for the Laguerre polynomials.* By definition,

$$\gamma_0(c) = \frac{1}{0!}\int_c^{+\infty} 1 \times \frac{x^\alpha e^{-x}}{\Gamma_{\alpha+1}}\mathrm{d}x = \mathscr{G}_{\alpha+1}(c),$$

and a direct integration gives

$$\gamma_1(c) = \frac{1}{\Gamma_{\alpha+2}}\int_c^{+\infty} (1+\alpha-x)\, x^\alpha e^{-x}\mathrm{d}x = -\frac{c^{1+\alpha}e^{-c}}{\Gamma_{\alpha+2}}.$$

From the identity (see [31, 18.9.24]),

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\alpha}e^{-x}\mathrm{L}_n^{(\alpha)}(x)\right) = (n+1)x^{\alpha-1}e^{-x}\mathrm{L}_{n+1}^{(\alpha-1)}(x), \tag{5.2}$$

we infer that

$$\gamma_n(c) = \frac{n!}{\Gamma_{n+\alpha+1}}\int_c^{+\infty}\mathrm{L}_n^{(\alpha)}(x)x^{\alpha}e^{-x}\mathrm{d}x = -\frac{(n-1)!}{\Gamma_{n+\alpha+1}}c^{\alpha+1}e^{-c}\mathrm{L}_{n-1}^{(\alpha+1)}(c).$$

▷ *Coefficients $\gamma_n(c)$ for the Jacobi polynomials.* As $\mathrm{P}_0^{(\alpha,\beta)}(x) = 1$ and $h_0 = 1$ (see Table 1 for the definition of h_n), we have first,

$$\gamma_0(c) = \frac{2^{-\alpha-\beta-1}}{\mathrm{B}_{\alpha+1,\beta+1}}\int_c^1 (1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{d}x = \mathscr{B}_{\alpha+1,\beta+1}\left(\frac{1-c}{2}\right).$$

Similarly, using that $\mathrm{P}_1^{(\alpha,\beta)}(x) = (\alpha+1) + (\alpha+\beta+2)\,\frac{(x-1)}{2}$, we obtain

$$\begin{aligned} h_1\gamma_1(c) &= (\alpha+1)\gamma_0(c) - \frac{2^{-\alpha-\beta-2}\,(\alpha+\beta+2)}{\mathrm{B}_{\alpha+1,\beta+1}}\int_c^1 (1-x)^{\alpha+1}(1+x)^{\beta}\,\mathrm{d}x \\ &= \frac{(\alpha+1)\mathrm{B}_{\alpha+1,\beta+1}\left(\frac{1-c}{2}\right) - (\alpha+\beta+2)\,\mathrm{B}_{\alpha+2,\beta+1}\left(\frac{1-c}{2}\right)}{\mathrm{B}_{\alpha+1,\beta+1}}, \end{aligned}$$

hence the expression for $\gamma_1(c)$ using that $h_1 = \frac{\Gamma_{\alpha+2}\Gamma_{\beta+2}}{(\alpha+\beta+3)\Gamma_{\alpha+\beta+2}\mathrm{B}_{\alpha+1,\beta+1}} = \frac{(\alpha+1)(\beta+1)}{\alpha+\beta+3}$. From the identity (see [31, 18.9.16]),

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{P}_n^{(\alpha,\beta)}(x)\right] = -2(n+1)\,(1-x)^{\alpha-1}(1+x)^{\beta-1}\,\mathrm{P}_{n+1}^{(\alpha-1,\beta-1)}(x), \tag{5.3}$$

we obtain:

$$\begin{aligned} \gamma_n(c) &= \frac{1}{h_n}\int_c^1 \mathrm{P}_n^{(\alpha,\beta)}(x)w(x)\mathrm{d}x \\ &= \frac{(2n+\alpha+\beta+1)\,\Gamma_{n+\alpha+\beta+1}\,(n-1)!}{2^{\alpha+\beta+2}\Gamma_{n+\alpha+1}\Gamma_{n+\beta+1}}(1-c)^{\alpha+1}\,(1+c)^{\beta+1}\,\mathrm{P}_{n-1}^{(\alpha+1,\beta+1)}(c). \end{aligned} \tag{5.4}$$

5.2 Proof of Proposition 2.2

First notice that for any classical orthogonal polynomial, the following limit holds

$$x\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[F(x)^{n+1}w(x)\right] \xrightarrow[x\to b]{} 0, \tag{5.5}$$

where the polynomial F is defined in Theorem 2.1. The limit (5.5) holds for the Hermite and Laguerre polynomials, for which $b = +\infty$, thanks to the exponential factor of their weight function $w(\cdot)$ which converges to 0 faster than any polynomial at $+\infty$. For the Jacobi polynomials, combining (5.3),

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x)^{\alpha+1}(1+x)^{\beta+1}\,\mathrm{P}_n^{(\alpha+1,\beta+1)}(x)\right] = -2(n+1)\,(1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{P}_{n+1}^{(\alpha,\beta)}(x),$$

along with Rodrigues' formula (2.2), we infer that

$$\begin{aligned} \frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left[(1-x^2)^{n+1}\,\frac{(1-x)^{\alpha}(1+x)^{\beta}}{2^{\alpha+\beta+1}\mathrm{B}_{\alpha+1,\beta+1}}\right] &= \frac{(-2)^{n+1}\,(n+1)!}{2^{\alpha+\beta+1}\mathrm{B}_{\alpha+1,\beta+1}}\,(1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{P}_{n+1}^{(\alpha,\beta)}(x) \\ &= \frac{(-2)^n\,n!}{2^{\alpha+\beta+1}\mathrm{B}_{\alpha+1,\beta+1}}\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x)^{\alpha+1}(1+x)^{\beta+1}\,\mathrm{P}_n^{(\alpha+1,\beta+1)}(x)\right]. \end{aligned}$$

Integrating the previous identity from -1 to x, and observing that the constant terms are zero on both sides, we conclude that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[F(x)^{n+1}w(x)\right] = \frac{(-2)^n\, n!}{2^{\alpha+\beta+1}\mathrm{B}_{\alpha+1,\beta+1}}\,(1-x)^{\alpha+1}\,(1+x)^{\beta+1}\,\mathrm{P}_n^{(\alpha+1,\beta+1)}(x) \xrightarrow[x\to 1]{} 0.$$

Now, from the three-term recurrence relation (2.1), we have

$$\begin{aligned}\gamma_{n+2}(c) &= \frac{1}{h_{n+2}}\mathbb{E}\left[\mathbf{1}_{c\le X}p_{n+2}(X)\right]\\ &= \frac{1}{h_{n+2}}\mathbb{E}\left[\mathbf{1}_{c\le X}\left(\left(A_{n+1}X+B_{n+1}\right)p_{n+1}(X)-C_{n+1}p_n(X)\right)\right]\\ &= B_{n+1}\frac{h_{n+1}}{h_{n+2}}\gamma_{n+1}(c) - C_{n+1}\frac{h_n}{h_{n+2}}\gamma_n(c) + A_{n+1}\frac{1}{h_{n+2}}\mathbb{E}\left[\mathbf{1}_{c\le X}Xp_{n+1}(X)\right].\end{aligned}$$

Besides, an integration by parts along with (2.2), and (5.5) give

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{c\le X}Xp_{n+1}(X)\right] &= \int_c^b xp_{n+1}(x)w(x)\mathrm{d}x\\ &= \frac{1}{\kappa_{n+1}}\int_c^b x\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left[F(x)^{n+1}w(x)\right]\mathrm{d}x\\ &= -\frac{1}{\kappa_{n+1}}\left(c\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[F(x)^{n+1}w(x)\right]\Big|_{x=c} + \int_c^b \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[F(x)^{n+1}w(x)\right]\mathrm{d}x\right).\end{aligned}$$

Now, integrating the identity (2.2), i.e.

$$\kappa_{n+1}p_{n+1}(x)w(x) = \frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left[F(x)^{n+1}w(x)\right] \tag{5.6}$$

from x to b, and using (5.5), we infer that

$$\kappa_{n+1}\underbrace{\int_x^b p_{n+1}(t)w(t)\mathrm{d}t}_{=h_{n+1}\gamma_{n+1}(x)} = -\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[F(x)^{n+1}w(x)\right].$$

All in all, we get that

$$\mathbb{E}\left[\mathbf{1}_{c\le X}Xp_{n+1}(X)\right] = h_{n+1}\left(c\gamma_{n+1}(c) + \int_c^b \gamma_{n+1}(x)\mathrm{d}x\right),$$

and obtain (2.7). Hence, to derive the recurrence relation for the COPS, one only needs to express $\int_c^b \gamma_{n+1}(x)\mathrm{d}x$ in terms of $\gamma_n(c)$ and $\gamma_{n+1}(c)$.

▷ *Hermite polynomials.* As $b = +\infty$, from $\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\frac{x^2}{2}}\mathrm{He}_n(x)\right) = -e^{-\frac{x^2}{2}}\mathrm{He}_{n+1}(x)$ (see [31, 18.9.28]), we have first

$$\int_c^{+\infty}\gamma_{n+1}(x)\mathrm{d}x = -\frac{1}{(n+1)!\sqrt{2\pi}}\int_c^{+\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\frac{x^2}{2}}\mathrm{He}_{n-1}(x)\right)\mathrm{d}x = \frac{\gamma_n(c)}{n+1}.$$

Then, from the relations (see Table 1),

$$\left(B_{n+1}+A_{n+1}c\right)\frac{h_{n+1}}{h_{n+2}} = \frac{c}{n+2}, \quad \frac{h_{n+1}}{h_{n+2}}\frac{A_{n+1}}{n+1} - C_{n+1}\frac{h_n}{h_{n+2}} = -\frac{n}{(n+1)(n+2)},$$

and (2.7), we obtain (2.8).

▷ *Laguerre polynomials.* Again $b = +\infty$ and we explicit the dependence on α and, with a slight

abuse of notation, write $\gamma_n(c,\alpha) := \gamma_n(c)$. From (2.5), (5.2), the relation $\Gamma_{x+1} = x\Gamma_x$ for $x > 0$, and the identity ([31, 18.9.14]),

$$x\mathrm{L}_{n-1}^{(\alpha+2)}(x) = -n\mathrm{L}_n^{(\alpha+1)}(x) + (n+\alpha+1)\,\mathrm{L}_{n-1}^{(\alpha+1)}(x),$$

we have

$$\begin{aligned}
\int_c^{+\infty} \gamma_{n+1}(x,\alpha)\,\mathrm{d}x &= -\frac{n!}{\Gamma_{n+\alpha+2}} \int_c^{+\infty} x^{\alpha+1}e^{-x}\mathrm{L}_n^{(\alpha+1)}(x)\mathrm{d}x \\
&= -\frac{(n-1)!}{\Gamma_{n+\alpha+2}} \int_c^{+\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\alpha+2}e^{-x}\mathrm{L}_{n-1}^{(\alpha+2)}(x)\right)\mathrm{d}x \\
&= \frac{(n-1)!}{\Gamma_{n+\alpha+2}} c^{\alpha+2}e^{-c}\mathrm{L}_{n-1}^{(\alpha+2)}(c) \\
&= \frac{(n-1)!}{\Gamma_{n+\alpha+2}} c^{\alpha+1}e^{-c}\left(-n\mathrm{L}_n^{(\alpha+1)}(c) + (n+\alpha+1)\,\mathrm{L}_{n-1}^{(\alpha+1)}(c)\right) \\
&= \gamma_{n+1}(c,\alpha) - \frac{\Gamma_{n+\alpha+1}}{\Gamma_{n+\alpha+2}}\,(n+\alpha+1)\,\gamma_n(c,\alpha) \\
&= \gamma_{n+1}(c,\alpha) - \gamma_n(c,\alpha).
\end{aligned}$$

Whence from the identities (see Table 1),

$$(B_{n+1} + A_{n+1}(1+c))\,\frac{h_{n+1}}{h_{n+2}} = \frac{2n+\alpha+2-c}{n+\alpha+2}, \qquad -\frac{h_nC_{n+1} + h_{n+1}A_{n+1}}{h_{n+2}} = -\frac{n}{n+\alpha+2},$$

and (2.7), we obtain (2.9).

▷ *Jacobi polynomials.* We explicit the dependence in both α and β, and write $\gamma_n(c,\alpha,\beta) := \gamma_n(c)$. We first define the new function

$$C(n,\alpha,\beta) := \frac{(2n+\alpha+\beta+1)\,\Gamma_{n+\alpha+\beta+1}\,(n-1)!}{2^{\alpha+\beta+2}\Gamma_{n+\alpha+1}\Gamma_{n+\beta+1}}, \tag{5.7}$$

such that, from (2.6), $\gamma_n(c,\alpha,\beta) = C(n,\alpha,\beta)(1-c)^{\alpha+1}(1+c)^{\beta+1}\mathrm{P}_{n-1}^{(\alpha+1,\beta+1)}(c)$. From the identity [31, 18.9.16],

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big((1-x)^{\alpha+2}(1+x)^{\beta+2}\mathrm{P}_{n-1}^{(\alpha+2,\beta+2)}(x)\Big) = -2n(1-x)^{\alpha+1}(1+x)^{\beta+1}\mathrm{P}_n^{(\alpha+1,\beta+1)}(x)$$

and $C(n+1,\alpha,\beta) = C(n,\alpha+1,\beta+1)\frac{4n}{n+\alpha+\beta+2}$, we obtain that

$$\begin{aligned}
\int_c^1 \gamma_{n+1}(x,\alpha,\beta)\mathrm{d}x &= C(n+1,\alpha,\beta)\int_c^1 -\frac{1}{2n}\frac{\mathrm{d}}{\mathrm{d}x}\Big((1-x)^{\alpha+2}(1+x)^{\beta+2}\mathrm{P}_{n-1}^{(\alpha+2,\beta+2)}(x)\Big)\mathrm{d}x \\
&= \frac{2}{2+n+\alpha+\beta}\gamma_n(c,\alpha+1,\beta+1).
\end{aligned} \tag{5.8}$$

Although the above identity is simple, once plugged in (2.7), it exhibits a coupling between the γ_n for different α,β which makes the computation heavier. Alternatively, we want to rewrite the last term $\gamma_n(c,\alpha+1,\beta+1)$ in terms of $\gamma_n(c,\alpha,\beta)$ and $\gamma_{n+1}(c,\alpha,\beta)$. To do so, we need to express $\mathrm{P}_{n-1}^{(\alpha+2,\beta+2)}$ in terms of $\mathrm{P}_{n-1}^{(\alpha+1,\beta+1)}$ and $\mathrm{P}_n^{(\alpha+1,\beta+1)}$. Combining [31, 18.9.15–17], we have

$$\begin{aligned}
&\frac{1}{2}\,(2n+2+\alpha+\beta)\,(n+\alpha+\beta+3)\,\left(1-x^2\right)\mathrm{P}_{n-1}^{(\alpha+2,\beta+2)}(x) \\
&= n\,(\alpha-\beta-(2n+2+\alpha+\beta)\,x)\,\mathrm{P}_n^{(\alpha+1,\beta+1)}(x) + 2\,(n+\alpha+1)\,(n+\beta+1)\,\mathrm{P}_{n-1}^{(\alpha+1,\beta+1)}(x),
\end{aligned}$$

and we obtain (after some simplifications[1]),

$$\gamma_n(c,\alpha+1,\beta+1) = \frac{(\alpha-\beta-(2n+\alpha+\beta+2)\,c)\,(n+\alpha+\beta+2)}{2\,(2n+\alpha+\beta+2)\,(n+\alpha+\beta+3)}\gamma_{n+1}(c,\alpha,\beta) + \frac{(\alpha+\beta+n+1)\,(\alpha+\beta+n+2)\,(\alpha+\beta+2n+3)}{(\alpha+\beta+n+3)\,(\alpha+\beta+2n+1)\,(2n+\alpha+\beta+2)}\gamma_n(c,\alpha,\beta).$$

Combining Table 1, (5.8), and (2.7), we obtain after some standard simplifications (2.10).

5.3 Proof of Theorem of 2.2

In light of (2.4), the L^2 error for the Hermite polynomials writes as

$$\mathcal{E}_N^{\mathrm{He}}(c) = \left(\sum_{n=N}^{+\infty} h_{n+1}\gamma_{n+1}(c)^2\right)^{\frac{1}{2}} = e^{-\frac{c^2}{4}}\left(\sum_{n=N}^{+\infty}\frac{e^{-\frac{c^2}{2}}\mathrm{He}_n^2(c)}{2\pi(n+1)!}\right)^{\frac{1}{2}}.$$

We start with some local estimates for the Hermite polynomials.

Lemma 5.1 ([34, Theorem 8.91.3]). *Let $k\in\mathbb{R}$, $c_0>0$ and $0<\eta<4$. Then, there exist positive constants B_1 and B_2 (which depend on c_0, k, and η) such that for every $n\in\mathbb{N}^*$,*

$$\max_{c_0\le c}\frac{c^k e^{-\frac{c^2}{2}}\mathrm{He}_n^2(c)}{n!} \le B_1 n^{S_1}, \quad S_1 := \max\left(\frac{k}{2}-\frac{1}{6},-\frac{1}{2}\right), \tag{5.9}$$

$$\max_{c_0\le c\le\sqrt{(4-\eta)n}}\frac{c^k e^{-\frac{c^2}{2}}\mathrm{He}_n^2(c)}{n!} \le B_2 n^{S_2}, \quad S_2 := \max\left(\frac{k}{2}-\frac{1}{2},-\frac{1}{2}\right). \tag{5.10}$$

When $|c|\approx\sqrt{4n+2}$ (transition region), it is known (see, e.g., [25]) that the function $c\mapsto \mathrm{He}_n^2(c)\,e^{-\frac{c^2}{2}}$ is close to its maximum, and is essentially of order $\mathcal{O}(n!n^{-\frac{1}{6}})$ (see (5.12)). When $|c|<\sqrt{4n+2}$ (oscillatory region) and $|c|>\sqrt{4n+2}$ (monotonic region), the Hermite polynomials behave in a different way. More precisely, we have the following estimates.

Lemma 5.2 ([6, Theorem 1, Lemma 1]). *There exist universal positive constants A,B,C,D such that for every $n\in\mathbb{N}^*$,*

$$\frac{e^{-\frac{c^2}{2}}\mathrm{He}_n^2(c)}{n!} \le \frac{A}{\sqrt{4n+2-c^2}}, \quad \text{if } |c|<\sqrt{4n+2}, \quad \text{(oscillatory region)}, \tag{5.11}$$

$$Bn^{-\frac{1}{6}} \le \sup_{c\in\mathbb{R}}\left(\frac{e^{-\frac{c^2}{2}}\mathrm{He}_n^2(c)}{n!}\right) \le Cn^{-\frac{1}{6}}, \tag{5.12}$$

$$\frac{e^{-\frac{c^2}{2}}\mathrm{He}_n^2(c)}{n!} \le \frac{Dn^{-\frac{5}{6}}}{\left(c-\sqrt{4n+2}\right)^4}, \quad \text{if } |c|>\sqrt{4n+2}, \quad \text{(monotonic region)}. \tag{5.13}$$

The estimates exposed in Lemma 5.2 for the map $c\mapsto\frac{e^{-\frac{c^2}{2}}\mathrm{He}_n^2(c)}{n!}$ are discontinuous when $c\approx\sqrt{4n+2}$ (maximum region). The strategy of the proof will consist in breaking down the sum in n for $\mathcal{E}_N^{\mathrm{He}}(c)$ into three pieces (see (5.16)) according to the monotonic, oscillatory, and maximum regions and optimizing each term.

[1] the formulas were checked with Mathematica

In the following, C denotes an independent (of c and N) positive constant that can change from one line to another. As $c \in \mathbb{R} \mapsto \frac{e^{-\frac{c^2}{2}} \mathrm{He}_n^2(c)}{(n+1)!}$ is even, we can assume (without loss of generality) that $c \geq 0$.

▷*Estimate (2.13).* Let $c \in [0, \sqrt{4N}]$. For n such that $c \leq \sqrt{4N} < \sqrt{4n+2}$, we can use (5.11) which gives

$$\begin{aligned}
\mathcal{E}_N^{\mathrm{He}}(c) e^{\frac{c^2}{4}} &\leq C \left(\sum_{n \geq N} \frac{1}{n\sqrt{4n+2-4N}} \right)^{1/2} \\
&\leq C \left(\frac{1}{\sqrt{2}N} + \int_N^{+\infty} \frac{\mathrm{d}x}{x\sqrt{4x+2-4N}} \right)^{1/2} \\
&\leq C \left(\frac{1}{N} + \frac{1}{\sqrt{2N-1}} \int_0^{+\infty} \frac{\mathrm{d}y}{1+y^2} \right)^{\frac{1}{2}} \qquad \left(\text{set } y = \frac{\sqrt{4x+2-4N}}{\sqrt{4N-2}}\right) \\
&\leq \frac{C}{N^{\frac{1}{4}}},
\end{aligned}$$

where we have used in the first line that the function $x \mapsto \frac{1}{x\sqrt{4x+2-c^2}}$ is decreasing and the integral test for convergence. Estimate (2.13) readily follows.

▷ *Estimates (2.14) and (2.15).* Suppose that $c \geq \sqrt{2N}$ and $N \geq 13$. We set $\varepsilon(c,N) := (\frac{\sqrt{N}}{c})^{\frac{1}{6}} \in (0, 2^{-\frac{1}{12}}]$ and the intervals

$$\Omega_{1,n}\left(\varepsilon(c,N)\right) := \left[\left(1+\varepsilon(c,N)\right)\sqrt{4n+2}, +\infty\right), \quad \Omega_{2,n}\left(\varepsilon(c,N)\right) := \left[0, \sqrt{4n+2}\left(1-\varepsilon(c,N)\right)\right].$$

From (5.13), if $c \in \Omega_{1,n}\left(\varepsilon(c,N)\right)$,

$$\frac{e^{-\frac{c^2}{2}} \mathrm{He}_n^2(c)}{(n+1)!} \leq \frac{C n^{-\frac{5}{6}}}{\varepsilon(c,N)^4 (4n+2)^2 n} \leq \frac{C}{\varepsilon(c,N)^4 n^{\frac{23}{6}}}, \tag{5.14}$$

and from (5.11), if $c \in \Omega_{2,n}(\varepsilon(c,N))$,

$$\frac{e^{-\frac{c^2}{2}} \mathrm{He}_n^2(c)}{(n+1)!} \leq \frac{C}{(2\varepsilon(c,N) - \varepsilon(c,N)^2)^{\frac{1}{2}} n^{\frac{3}{2}}} \leq \frac{C}{\varepsilon(c,N)^{\frac{1}{2}} n^{\frac{3}{2}}}. \tag{5.15}$$

We further define

$$n_1(c, \varepsilon(c,N)) := \left\lfloor \frac{1}{4}\left(\left(\frac{c}{1+\varepsilon(c,N)}\right)^2 - 2\right) \right\rfloor, \quad n_2(c, \varepsilon(c,N)) := \left\lfloor \frac{1}{4}\left(\left(\frac{c}{1-\varepsilon(c,N)}\right)^2 - 2\right) \right\rfloor.$$

Notice that for every c, we have $n_1(c, \varepsilon(c,N)) \leq n_2(c, \varepsilon(c,N))$. Moreover, for every $n \leq n_1(c, \varepsilon(c,N))$ (resp. $n \geq n_2(c, \varepsilon(c,N))+1$), we have $c \in \Omega_{1,n}(\varepsilon(c,N))$ (resp. $c \in \Omega_{2,n}(\varepsilon(c,N))$).

First suppose that $N \leq n_1\left(c, \varepsilon(c,N)\right) \leq n_2\left(c, \varepsilon(c,N)\right)$; in that case, the sum can be decomposed as:

$$\begin{aligned}
\sum_{n=N}^{\infty} \frac{e^{-\frac{c^2}{2}} \left(\mathrm{He}_n(c)\right)^2}{(n+1)!} &= \sum_{n=N}^{n_1(c,\varepsilon(c,N))} \frac{e^{-\frac{c^2}{2}} \left(\mathrm{He}_n(c)\right)^2}{(n+1)!} + \sum_{n=n_2(c,\varepsilon(c,N))+1}^{\infty} \frac{e^{-\frac{c^2}{2}} \left(\mathrm{He}_n(c)\right)^2}{(n+1)!} \\
&\quad + \sum_{n=n_1(c,\varepsilon(c,N))+1}^{n_2(c,\varepsilon(c,N))} \frac{e^{-\frac{c^2}{2}} \left(\mathrm{He}_n(c)\right)^2}{(n+1)!} \\
&=: R_1\left(c, \varepsilon(c,N)\right) + R_2\left(c, \varepsilon(c,N)\right) + R_{1,2}\left(c, \varepsilon(c,N)\right).
\end{aligned}$$

Combining estimates (5.14), (5.15), (5.12) together with (5.1), we obtain

$$
\begin{aligned}
R_1\left(c,\varepsilon(c,N)\right) &\leq \sum_{n=N}^{n_1(c,\varepsilon(c,N))} \frac{C}{\varepsilon(c,N)^4 n^{\frac{23}{6}}} \leq \frac{C}{\varepsilon(c,N)^4 N^{\frac{17}{6}}},\\
R_2\left(c,\varepsilon(c,N)\right) &\leq \sum_{n=n_2(c,\varepsilon(c,N))+1}^{\infty} \frac{C}{\varepsilon(c,N)^{\frac{1}{2}} n^{\frac{3}{2}}} \leq \frac{C}{\varepsilon(c,N)^{\frac{1}{2}} n_2(c,\varepsilon(c,N))^{\frac{1}{2}}},\\
R_{1,2}\left(c,\varepsilon(c,N)\right) &\leq \sum_{n=n_1(c,\varepsilon(c,N))+1}^{n_2(c,\varepsilon(c,N))} \frac{C}{n^{\frac{7}{6}}} \leq C\frac{n_2\left(c,\varepsilon(c,N)\right)-n_1\left(c,\varepsilon(c,N)\right)}{n_1\left(c,\varepsilon(c,N)\right)^{\frac{7}{6}}}.
\end{aligned}
\tag{5.16}
$$

First, there exists $C>0$ such that $n_2(c,\varepsilon(c,N)) \geq n_1(c,\varepsilon(c,N)) \geq Cc^2$ using $\varepsilon(c,N)\leq 2^{-\frac{1}{12}}$ and $c\geq\sqrt{2N}\geq\sqrt{26}$. Second, let x (resp. y) be the argument in $\lfloor\dots\rfloor$ from the definition of $n_1(c,\varepsilon(c,N))$ (resp. $n_2(c,\varepsilon(c,N))$):

$$
y-x=\frac{\varepsilon(c,N)c^2}{(1-\varepsilon(c,N)^2)^2}=\frac{N^{\frac{1}{12}}c^{\frac{11}{6}}}{(1-\varepsilon(c,N)^2)^2}\geq 1
$$

using $c\geq\sqrt{26}$, and $\varepsilon(c,N)\leq 2^{-\frac{1}{12}}$. Therefore from the inequality $\lfloor y\rfloor-\lfloor x\rfloor\leq 2(y-x)$ for any $y\geq x+1$, we have that

$$
n_2(c,\varepsilon(c,N))-n_1(c,\varepsilon(c,N))\leq 2\frac{\varepsilon(c,N)c^2}{(1-\varepsilon(c,N)^2)^2}\leq C\varepsilon(c,N)c^2.
$$

Hence there exists $C>0$ such that

$$
\begin{aligned}
\left(\sum_{n=N}^{\infty}\frac{e^{-\frac{c^2}{2}}\left(\mathrm{He}_n\left(c\right)\right)^2}{(n+1)!}\right)^{\frac{1}{2}} &\leq C\left(\frac{1}{\varepsilon(c,N)^4N^{\frac{17}{6}}}+\frac{1}{\varepsilon(c,N)^{\frac{1}{2}}c}+\frac{\varepsilon(c,N)}{c^{\frac{1}{3}}}\right)^{\frac{1}{2}}\\
&\leq C\big(\frac{c}{\sqrt{N}}\big)^{\frac{1}{3}}N^{-\frac{17}{12}}\vee\big(\frac{\sqrt{N}}{c}\big)^{\frac{11}{24}}N^{-\frac{1}{4}}\vee\big(\frac{\sqrt{N}}{c}\big)^{\frac{1}{4}}N^{-\frac{1}{12}}.
\end{aligned}
\tag{5.17}
$$

If $n_1(c,\varepsilon(c,N))+1\leq N\leq n_2(c,\varepsilon(c,N))$, then similar computations yield

$$
\begin{aligned}
\sum_{n=N}^{\infty}\frac{e^{-\frac{c^2}{2}}\left(\mathrm{He}_n\left(c\right)\right)^2}{(n+1)!} &= \sum_{n=N}^{n_2(c,\varepsilon(c,N))}\frac{e^{-\frac{c^2}{2}}\left(\mathrm{He}_n\left(c\right)\right)^2}{(n+1)!}+R_2(c,\varepsilon(c,N))\\
&\leq C\frac{n_2\left(c,\varepsilon(c,N)\right)-N+1}{N^{\frac{7}{6}}}+\frac{C}{\varepsilon(c,N)^{\frac{1}{2}}n_2(c,\varepsilon(c,N))^{\frac{1}{2}}},
\end{aligned}
$$

and finally, after simplifications, (5.17) holds also. Similarly, if $n_2(c,\varepsilon(c,N))+1\leq N$, then $\sum_{n=N}^{\infty}\frac{e^{-\frac{c^2}{2}}(\mathrm{He}_n(c))^2}{(n+1)!}\leq R_2(c,\varepsilon(c,N))\leq\frac{C}{\varepsilon(c,N)^{\frac{1}{2}}N^{\frac{1}{2}}}$, and (5.17) holds again. In other words, we have shown (5.17) for any $c\geq\sqrt{2N}$ and $N\geq 7$.

To conclude, we obtain estimates (2.14) and (2.15) observing that in the regime $c\geq\sqrt{2N}$, $\big(\frac{\sqrt{N}}{c}\big)^{\frac{11}{24}}N^{-\frac{1}{4}}\leq C\big(\frac{\sqrt{N}}{c}\big)^{\frac{1}{4}}N^{-\frac{1}{12}}$, and that $c\leq N^{\frac{39}{14}}\iff\big(\frac{c}{\sqrt{N}}\big)^{\frac{1}{3}}N^{-\frac{17}{12}}\leq\big(\frac{\sqrt{N}}{c}\big)^{\frac{1}{4}}N^{-\frac{1}{12}}$. □

5.4 Proof of Theorem of 2.3

From (2.5), the L^2 error rewrites as

$$
\mathcal{E}_N^{\mathrm{La}}(c)=\frac{1}{\Gamma_{\alpha+1}^{\frac{1}{2}}}\left(\sum_{n=N}^{+\infty}\frac{n!}{(n+1)\Gamma_{n+\alpha+2}}\left(c^{\alpha+1}e^{-c}\mathrm{L}_n^{(\alpha+1)}(c)\right)^2\right)^{\frac{1}{2}}.
$$

Again, we start with some estimates on the maximum of Laguerre polynomials.

Lemma 5.3 ([34, Theorem 8.91.2]). *Let $k \in \mathbb{R}$, $c_0 > 0$, and $0 < \eta < 4$. Then, there exist positive constants C_1 and C_2 (which depend on c_0, η, α, and k) such that for every $n \in \mathbb{N}^*$,*

$$\max_{c_0 \le c} \left| c^k e^{-\frac{c}{2}} \mathrm{L}_n^{(\alpha+1)}(c) \right|^2 \le C_1 n^{Q_1}, \quad Q_1 := \max\left(2k - \frac{2}{3}, \alpha + \frac{1}{2}\right), \tag{5.18}$$

$$\max_{c_0 \le c \le (4-\eta)n} \left| c^k e^{-\frac{c}{2}} \mathrm{L}_n^{(\alpha+1)}(c) \right|^2 \le C_2 n^{Q_2}, \quad Q_2 := \max\left(2k - 1, \alpha + \frac{1}{2}\right). \tag{5.19}$$

Lemma 5.4 ([34, Theorem 7.6.5]). *Define $M_n := \max_{0<c\le c_1}(c^{\frac{\alpha}{2}+\frac{3}{4}} e^{-\frac{c}{2}} |\mathrm{L}_n^{(\alpha+1)}(c)|)$ for a fixed $c_1 > 0$. Then, as $n \to +\infty$, we have $M_n^2 \sim C n^{\alpha+\frac{1}{2}}$ for some $C > 0$.*

From the identity $(n-1)! = \Gamma_n$ and evoking [31, 5.6.8], we have for $n \ge 1$,

$$\frac{n!}{(n+1)\Gamma_{n+\alpha+2}} \le \frac{\Gamma_n}{\Gamma_{n+\alpha+2}} \le \frac{1}{n^{\alpha+2}}$$

and consequently $\mathcal{E}_N^{\mathrm{La}}(c) \le \frac{1}{\Gamma_{\alpha+1}^{\frac{1}{2}}} \left(\sum_{n=N}^{+\infty} \frac{1}{n^{\alpha+2}} \left(c^{\alpha+1} e^{-c} \mathrm{L}_n^{(\alpha+1)}(c) \right)^2 \right)^{\frac{1}{2}}$.

▷ *Estimate (2.16) for $c \le 1$.* Combine Lemma 5.4 with $c_1 = 1$ and (5.1).

▷ *Estimate (2.16) for $1 \le c \le (4-\eta)N$.* Combining (5.19) with $k = \frac{\alpha}{2} + \frac{3}{4}$ and $c_0 = 1$ along with (5.1), there exists a universal constant $A_1 > 0$ such that $\mathcal{E}_N^{\mathrm{La}}(c) \le A_1 e^{-\frac{c}{2}} c^{\frac{\alpha}{2}+\frac{1}{4}} N^{-\frac{1}{4}}$.

▷ *Estimate (2.17).* Combine (5.18) with $k = \frac{\alpha}{2} + \frac{7}{12}$ and conclude again with (5.1).

5.5 Proof of Theorem of 2.4

▷ *L^2 error for the Jacobi polynomials.* From (2.6), the L^2 error writes as (after some easy simplifications)

$$\mathcal{E}_N^{\mathrm{J}}(c) = \frac{2^{-(\alpha+\beta+2)}}{\mathrm{B}_{\alpha+1,\beta+1}^{\frac{1}{2}}} (1-c)^{\frac{\alpha}{2}+\frac{1}{4}} (1+c)^{\frac{\beta}{2}+\frac{1}{4}}$$
$$\Big(\sum_{n=N}^{+\infty} \frac{(2n+\alpha+\beta+3)}{(n+1)(n+\alpha+\beta+2)} \frac{\Gamma_{n+1}\Gamma_{n+\alpha+\beta+3}}{\Gamma_{n+\alpha+2}\Gamma_{n+\beta+2}} \sqrt{1-c^2} (1-c)^{\alpha+1} (1+c)^{\beta+1} \big(\mathrm{P}_n^{(\alpha+1,\beta+1)}(c) \big)^2 \Big)^{\frac{1}{2}}.$$

Writing $w^{(\alpha,\beta)}(x) := w(x)$ and $h_n^{(\alpha,\beta)} := h_n$ for the weight function and L^2 norm for the Jacobi polynomials (see Table 1), it holds

$$\frac{(2n+\alpha+\beta+3)\Gamma_{n+1}\Gamma_{n+\alpha+\beta+3}}{\Gamma_{n+\alpha+2}\Gamma_{n+\beta+2}} (1-c)^{\alpha+1} (1+c)^{\beta+1} = \frac{2^{\alpha+\beta+3} w^{(\alpha+1,\beta+1)}(c)}{h_n^{(\alpha+1,\beta+1)}},$$

and using that $\frac{1}{(n+1)(n+\alpha+\beta+2)} \le \frac{1}{n^2}$, we have

$$\mathcal{E}_N^{\mathrm{J}}(c) \le \frac{2^{-\frac{\alpha+\beta+1}{2}}}{\mathrm{B}_{\alpha+1,\beta+1}^{\frac{1}{2}}} (1-c)^{\frac{\alpha}{2}+\frac{1}{4}} (1+c)^{\frac{\beta}{2}+\frac{1}{4}} \Big(\sum_{n=N}^{+\infty} \frac{1}{n^2} \sqrt{1-c^2} w^{(\alpha+1,\beta+1)}(c) \Big(\frac{\mathrm{P}_n^{(\alpha+1,\beta+1)}(c)}{\sqrt{h_n^{(\alpha+1,\beta+1)}}} \Big)^2 \Big)^{\frac{1}{2}}.$$

We conclude thanks to the inequality ([27, Theorem 1])

$$\max_{c\in[-1,1]} \sqrt{1-c^2} w^{(\alpha+1,\beta+1)}(c) \Big(\frac{\mathrm{P}_n^{(\alpha+1,\beta+1)}(c)}{\sqrt{h_n^{(\alpha+1,\beta+1)}}} \Big)^2 \le \frac{2e}{\pi} \left(2 + \sqrt{(\alpha+1)^2 + (\beta+1)^2} \right),$$

valid for all $\alpha, \beta > -1$ along with (5.1).

5.6 Proof of Lemma 3.1

▷ *Jacobi polynomials.* Introducing $y = \frac{t}{x}$, it holds

$$\mathrm{B}_{\alpha+1,\beta+1} x^{-(\alpha+1)} \mathscr{B}_{\alpha+1,\beta+1}(x) = x^{-(\alpha+1)} \int_0^x t^\alpha (1-t)^\beta \,\mathrm{d}t = \int_0^1 y^\alpha (1-xy)^\beta \,\mathrm{d}y, \tag{5.20}$$

and so we have $\mathscr{B}_{\alpha+1,\beta+1}(x) \underset{x\to 0^+}{\sim} \frac{x^{\alpha+1}}{(\alpha+1)\mathrm{B}_{\alpha+1,\beta+1}}$. Setting $x = \mathscr{B}^{-1}_{\alpha+1,\beta+1}(q)$, we easily get that

$$\mathscr{B}^{-1}_{\alpha+1,\beta+1}(q) \underset{q\to 0^+}{\sim} ((\alpha+1)\mathrm{B}_{\alpha+1,\beta+1})^{\frac{1}{\alpha+1}} \, q^{\frac{1}{\alpha+1}}.$$

Using that $c_{\mathrm{J}}(q) = 2\mathscr{B}^{-1}_{\alpha+1,\beta+1}(q) - 1$, we readily obtain (3.5). The equivalent (3.6) is obtained by swapping the role of α and β.

▷ *Hermite polynomials.* Using that $-\sqrt{2\pi} x \Phi(x) \underset{x\to-\infty}{\sim} e^{-\frac{x^2}{2}}$ ([31, 7.12.1]) and $\Phi^{-1}(q) \underset{q\to 0^+}{\sim} -\sqrt{-2\ln(q)}$ ([13, Proposition 21]), we infer that $e^{-\frac{\Phi^{-1}(q)^2}{4}} \underset{q\to 0^+}{\sim} 2^{\frac{1}{2}} \pi^{\frac{1}{4}} q^{\frac{1}{2}} |\ln(q)|^{\frac{1}{4}}$, and we readily obtain (3.1). For the case $q \to 1^-$, proceed as before using $\Phi^{-1}(1-q) = -\Phi^{-1}(q)$.

▷ *Laguerre polynomials.* For $q \to 0^+$, $\mathscr{G}^{-1}_{\alpha+1}(q) \to 0$, and $\mathscr{G}_{\alpha+1}(x) \underset{x\to 0^+}{\sim} \frac{x^{\alpha+1}}{\Gamma_{\alpha+2}}$ ([31, 8.7.1]). We then obtain $\mathscr{G}^{-1}_{\alpha+1}(q) \underset{q\to 0^+}{\sim} (q\Gamma_{\alpha+2})^{\frac{1}{\alpha+1}}$ and (3.3) readily follows.

When $q \to 1^-$, we have $\mathscr{G}^{-1}_{\alpha+1}(q) \to +\infty$. Starting from the expansion $\Gamma_{\alpha+1}(x) \underset{x\to+\infty}{\sim} x^\alpha e^{-x}$ ([31, 8.11.2]), we infer that

$$\mathscr{G}^{-1}_{\alpha+1}(q)^\alpha e^{-\mathscr{G}^{-1}_{\alpha+1}(q)} \underset{q\to 1^-}{\sim} \Gamma_{\alpha+1}\left(\mathscr{G}^{-1}_{\alpha+1}(q)\right) = \Gamma_{\alpha+1} \times (1-q),$$

and consequently, as $q \to 1^-$,

$$\mathscr{G}^{-1}_{\alpha+1}(q) \sim -\alpha W_{-1}\left(-\frac{(\Gamma_{\alpha+1}\times(1-q))^{\frac{1}{\alpha}}}{\alpha}\right) \sim -\alpha \ln\left(-\frac{(\Gamma_{\alpha+1}\times(1-q))^{\frac{1}{\alpha}}}{\alpha}\right) \sim -\ln(1-q),$$

where $W_{-1}(\cdot)$ denotes the lower branch of the Lambert function and we have used that $W_{-1}(-x) \underset{x\to 0^+}{\sim} \ln(x)$ (see [11] for a review of the Lambert function $W(\cdot)$). The expansion (3.4) for $q \to 1^-$ readily follows.

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