Brownian sheet and time inversion - From G-orbit to $L(G)$-orbit

Manon Defosseux

BROWNIAN SHEET AND TIME INVERSION FROM G-ORBIT TO $L(G)$-ORBIT

MANON DEFOSSEUX

Abstract. We have proved in a previous paper that a space-time Brownian motion conditioned to remain in a Weyl chamber associated to an affine Kac–Moody Lie algebra is distributed as the radial part process of a Brownian sheet on the compact real form of the underlying finite dimensional Lie algebra, the radial part being defined considering the coadjoint action of a loop group on the dual of a centrally extended loop algebra. We present here a very brief proof of this result based on a time inversion argument and on elementary stochastic differential calculus.

1. Introduction

We propose here a short proof of the main result of [5]. Let us briefly recall this result. For this we need to consider a connected simply connected simple compact Lie group G and its Lie algebra $\mathfrak{g}$ equipped with an invariant scalar product for the adjoint action of G on $\mathfrak{g}$. One considers a standard Brownian sheet $\{x_{s,t}, s \in [0,1], t \geq 0\}$ with values in $\mathfrak{g}$ and for each $t > 0$, the process $\{Y_{s,t}, s \in [0,1]\}$ starting from the identity element of G and satisfying the stochastic differential equation (in s)

$$t\, dY_{s,t} = Y_{s,t} \circ dx_{s,t},$$

where $\circ$ stands for the Stratonovitch integral. This is a G-valued process. The adjoint orbits in G are in correspondence with an alcove which is a fundamental domain for the action on a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ of the extended Weyl group associated the roots of G. We have proved in [5] that if for any $t > 0$ one denotes by $\mathcal{O}(Y_{1,t})$ the element in the alcove corresponding to the orbit of $Y_{1,t}$ then the random process[1]

$$\{(t, t\mathcal{O}(Y_{1,t})) : t \geq 0\}$$

is a space-time brownian motion in $\mathbb{R} \times \mathfrak{t}$ conditioned in Doob's sense to remain in a Weyl chamber which occurs in the framework of affine Kac–Moody algebras [8]. The proof of [5] rests on a Kirillov–Frenkel character formula [7] from which follows an intertwining relation between the transition probability semi-group of the Brownian sheet and the one of the conditioned process. Then a Rogers and Pitman's criteria [11] can be applied, which provides the result. The conditioned process obtained when $G = \mathrm{SU}(2)$ plays a crucial role in [2] where a Pitman type theorem is proved for a real Brownian motion in the unit interval. Time inversion is a key ingredient to get the Pitman type theorem in this case. In the present communication a new proof of the main result of [5] is proposed, which rests on such a time inversion firstly and secondly on an elementary but nice property of the Brownian sheet on $\mathfrak{g}$ and its wrapping on G.

[1] with the convention that $Y_{1,0}$ is the identity element of G

The results presented are valuable for themselves rather than for their proofs which are rudimentary. We present them in section 2 before giving the precise definitions of the objets that they involve. The rest of the communication is organized as follows. In section 3 we recall the general framework of [5]. In particular we describe the coadjoint orbits of the loop group $L(G)$ in the dual of the centrally extended loop algebra $L(\mathfrak{g})$ and the Weyl chamber associated to such an infinite dimensional Lie algebra which is an affine Kac–Moody algebra. In section 4 we define the radial process associated to the Brownian sheet on $\mathfrak{g}$ and recall the main theorem of [5]. In section 5 we define two Doob conditioned processes living respectively in an alcove or in an affine Weyl chamber, and prove that the two processes are equal up to a time inversion. Finally in section 6 we propose a brief proof of the main result of [5].

2. Statement of the results

Let us fix γ in a fixed alcove associated to G and consider $\{X^\gamma_{s,t} : s \in [0,1], t \geq 0\}$ a random sheet with values in G, such that for any $t \geq 0$,

$$\begin{cases} X^\gamma_{s,t} = X_{s,t} \circ d(x_{s,t} + \gamma s) \\ X^\gamma_{0,t} = e, \end{cases}$$

where e is the identity element of G. Then one has the three following statements, the second one being an immediate consequence of the first, and the last one being deduced from the second by a time inversion argument.

Satement 1 :
The random process $\{X^\gamma_{1,t} : t \geq 0\}$ is a standard Brownian motion on G starting from $\exp(\gamma)$.

Satement 2 :
The random process $\{\mathcal{O}(X^\gamma_{1,t}) : t \geq 0\}$ is a standard Brownian motion starting from γ conditioned to remain in the alcove.

Satement 3 :
The radial part process $\{\mathrm{rad}(t\Lambda_0 + \int_0^1 (\cdot|d(x_{s,t} + \gamma st)) : t \geq 0\}$ is a space-time Brownian motion with drift γ conditioned to remain in an affine Weyl chamber.

3. Loop group and its orbits

In this part we fix succinctely the general framework of the results. One can find more details in [5] and references therein for instance. Let G be a connected simply connected simple compact Lie group and $\mathfrak{g}$ its Lie algebra equipped with a Lie bracket denoted by $[\,\cdot\,,\cdot\,]_{\mathfrak{g}}$. We choose a maximal torus T in G and denote by $\mathfrak{t}$ its Lie algebra. By compacity we suppose without loss of generality that G is a matrix Lie group. We denote by Ad the adjoint action of G on itself or on its Lie algebra $\mathfrak{g}$ which is equipped with an $\mathrm{Ad}(G)$-invariant scalar product $(\cdot|\cdot)$. We consider the real vector space $L(\mathfrak{g})$ of smooth loops defined on the unit circle S^1 with values in $\mathfrak{g}$, S^1 being identified with $[0,1]$. We equip $L(\mathfrak{g})$ with an

$\mathrm{Ad}(G)$-invariant scalar product also denoted by $(\cdot|\cdot)$ letting

$$(\eta|\xi) = \int_0^1 (\eta(s)|\xi(s))\, ds, \quad \eta, \xi \in L(\mathfrak{g}).$$

Equipped with the Lie bracket $[\,\cdot\,,\cdot\,]_{\mathfrak{g}}$ pointwise defined, $L(\mathfrak{g})$ is a Lie algebra. We consider its central extension

$$\widetilde{L}(\mathfrak{g}) = L(\mathfrak{g}) \oplus \mathbb{R}c,$$

equipped with a Lie bracket $[\,\cdot\,,\cdot\,]$ defined by

$$[\xi + \lambda c +, \eta + \mu c] = [\xi, \eta]_{\mathfrak{g}} + (\xi'|\eta)c, \tag{1}$$

for $\xi, \eta \in L(\mathfrak{g})$, $\lambda, \mu \in \mathbb{R}$. We consider the fundamental weight Λ_0 in $\widetilde{L}(\mathfrak{g})^*$ defined by

$$\Lambda_0(L(\mathfrak{g})) = 0, \quad \Lambda_0(c) = 1,$$

the set of smooth loops $L(G)$ with values on G and its coadjoint action Ad^* on $L(\mathfrak{g})^* \oplus \mathbb{R}\Lambda_0$. It is defined by

$$\mathrm{Ad}^*(\gamma)(\phi + \tau\Lambda_0) = [\gamma.\phi - \tau(\gamma'\gamma^{-1}|\cdot)] + \tau\Lambda_0, \tag{2}$$

for $\gamma \in L(G), \phi \in L(\mathfrak{g})^*, \tau \in \mathbb{R}$, where $(\gamma.\phi)(.) = \phi(\gamma^{-1}.\gamma)$. We notice that the coordinate along Λ_0 of a linear form in $\widetilde{L}(\mathfrak{g})^*$, which is called the level of the linear form, is not affected by the coadjoint action.

Coadjoint $L(G)$-orbit. For ζ in $\widetilde{L}(\mathfrak{g})^*$ we denoted by $\widetilde{\mathcal{O}}_\zeta$ the coadjoint orbit $\mathrm{Ad}^*(L(G))\{\zeta\}$ in $\widetilde{L}(\mathfrak{g})^*$. We have recalled in [5] that roughly speaking for ξ in $\widetilde{\mathcal{O}}_\zeta$, provided that ζ has a positive level, we find $\gamma \in L(G)$ such that $\xi = \mathrm{Ad}^*(\gamma)(\zeta)$ solving a differential equation. Actually if ξ is written

$$\xi = \tau\Lambda_0 + \int_0^1 (\cdot|\dot{x}_s)\, ds, \tag{3}$$

with $\dot{x} \in L(\mathfrak{g})$ and $\tau > 0$, then ξ is in $\widetilde{\mathcal{O}}_{\tau\Lambda_0 + (a|\cdot)}$ for $a \in \mathfrak{t}$ if and only if the G-valued function $\{X_s : s \in [0,1]\}$ starting from the identity element of G and satisfying the differential equation

$$\tau dX = X dx,$$

satisfies $X_1 \in \mathrm{Ad}(G)\{\exp(a/\tau)\}$ (see [12] for details or [5] in which the results of [12] are recalled). Thus coadjoint orbits in the subspace of linear forms in $\tau\Lambda_0 + L(\mathfrak{g})^*$ written like in (3) are in one-to-one correspondence with the adjoint G-orbits in G. In order to parametrize these orbits, it is more convenient to consider the real roots of G rather than the infinitesimal ones. One can find for instance in chapters 5 and 7 of [3] definitions and properties recalled in the next two paragraphs.

Real roots. We consider the complexified Lie algebra $\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{g}$ of $\mathfrak{g}$ that we denote by $\mathfrak{g}_{\mathbb{C}}$. The set of real roots is

$$\Phi = \{\alpha \in \mathfrak{t}^* : \exists X \in \mathfrak{g}_{\mathbb{C}} \setminus \{0\}, \forall H \in \mathfrak{t}, [H, X] = 2i\pi\alpha(H)X\}.$$

Suppose that $\mathfrak{g}$ is of rank n and choose a set of simple real roots

$$\Pi = \{\alpha_k, k \in \{1, \dots, n\}\}.$$

We denote by Φ_+ the set of positive real roots. The half sum of positive real roots is denoted by ρ. Letting for $\alpha \in \Pi$,

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \forall H \in \mathfrak{t}, [H, X] = 2i\pi\alpha(H)X\},$$

the coroot $\alpha^\vee$ of $\alpha \in \Phi$ is defined as the only vector of $\mathfrak{t}$ in $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(\alpha^\vee) = 2$. For $\alpha \in \Pi$, one defines two transformations on $\mathfrak{t}$, the reflection $s_{\alpha^\vee}$ and the translation $t_{\alpha^\vee}$, letting for $x \in \mathfrak{t}$

$$s_{\alpha^\vee}(x) = x - \alpha(x)\alpha^\vee \text{ and } t_{\alpha^\vee}(x) = x + \alpha^\vee.$$

One considers the Weyl group $W^\vee$ and the group $\Gamma^\vee$ respectively generated by reflections $s_{\alpha^\vee}$ and translations $t_{\alpha^\vee}$, for $\alpha \in \Pi$, and the extended Weyl group Ω generated by $W^\vee$ and $\Gamma^\vee$. Actually Ω is the semi-direct product $W^\vee \ltimes \Gamma^\vee$. A fundamental domain for its action on $\mathfrak{t}$ is

$$A = \{x \in \mathfrak{t} : \forall\alpha \in \Phi_+,\ 0 \leq \alpha(x) \leq 1\}.$$

Adjoint G-orbit. The group G being simply connected, the conjugaison classes $G/\mathrm{Ad}(G)$ is in correspondence with the fundamental domain A. Actually for every $\mathsf{u} \in G$, there exists a unique element $x \in A$ such that $\mathsf{u} \in \mathrm{Ad}(G)\{\exp(x)\}$. For $\tau \in \mathbb{R}_+$, one defines the alcove A_τ of level τ by

$$A_\tau = \{x \in \mathfrak{t} : \forall\alpha \in \Phi_+,\ 0 \leq \alpha(x) \leq \tau\},$$

i.e. $A_\tau = \tau A$. In particular $A_1 = A$.

Alcoves and coadjoint $L(G)$-orbit. For a positive real number τ and a linear form $\xi \in \widetilde{L}(\mathfrak{g})^*$ written as in (3) there is a unique element in $a \in A_\tau$ such that

$$X_1 \in \mathrm{Ad}(G)\{\exp(a/\tau)\}$$

where $X = \{X_s : s \in [0,1]\}$ starts from the identity element e of G and satisfies

$$\tau dX = X dx.$$

Discussion above ensures that the pair (τ, a) determines the orbit of ξ. Thus coadjoint orbits in the subspace of linear forms in $\mathbb{R}_+^*\Lambda_0 + L(\mathfrak{g})^*$ written like in (3) are in one-to-one correspondence with

$$\{(\tau, a) \in \mathbb{R}_+^* \times \mathfrak{t} : a \in A_\tau\}.$$

This last domain can be identified (if we add it $(0,0)$) with a Weyl Chamber associated to an affine Kac–Moody algebra as it is explained in the following paragraph.

Affine Weyl chamber. From now on the scalar product on $\mathfrak{g}$ is normalized such that $(\theta|\theta) = 2$. We denote by θ the highest real root and we let $\alpha_0^\vee = c - \theta^\vee$. We consider

$$\widehat{\mathfrak{h}} = \mathrm{Vect}_{\mathbb{C}}\{\alpha_0^\vee, \alpha_1^\vee, \dots, \alpha_n^\vee, d\} \text{ and } \widehat{\mathfrak{h}}^* = \mathrm{Vect}_{\mathbb{C}}\{\alpha_0, \alpha_1, \dots, \alpha_n, \Lambda_0\},$$

where $\alpha_0 = \delta - \theta$ and for $i \in \{0, \dots, n\}$

$$\alpha_i(d) = \delta_{i0}, \quad \delta(\alpha_i^\vee) = 0, \quad \Lambda_0(\alpha_i^\vee) = \delta_{i0}, \quad \Lambda_0(d) = 0.$$

We let

$$\widehat{\Pi} = \{\alpha_i : i \in \{0, \dots, n\}\} \text{ and } \widehat{\Pi}^\vee = \{\alpha_i^\vee : i \in \{0, \dots, n\}\}.$$

Then $(\widehat{\mathfrak{h}}, \widehat{\Pi}, \widehat{\Pi}^\vee)$ is a realization of a generalized Cartan matrix of affine type. These objects are studied in details in [8]. The following definitions mainly come from chapters 1 and 6. We consider the restriction of $(\cdot|\cdot)$ to $\mathfrak{t}$ and extend it to $\widehat{\mathfrak{h}}$ by $\mathbb{C}-$linearity and by letting

$$(\mathbb{C}c + \mathbb{C}d|\, \mathfrak{t}) = 0, \quad (c|c) = (d|d) = 0, \quad (c|d) = 1.$$

Then the linear isomorphism

$$\begin{aligned} \nu : \widehat{\mathfrak{h}} &\to \widehat{\mathfrak{h}}^* \\ h &\mapsto (h|\cdot) \end{aligned}$$

identifies $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{h}}^*$. We still denote $(\cdot|\cdot)$ the induced bilinear form on $\widehat{\mathfrak{h}}^*$. We record that

$$(\delta|\alpha_i) = 0, \quad i = 0, \dots, n, \quad (\delta|\delta) = 0, \quad (\delta|\Lambda_0) = 1.$$

Due to the normalization we have $\nu(\theta^\vee) = \theta$ and $(\theta^\vee|\theta^\vee) = 2$. We define the affine Weyl group $\widehat{W}$ as the subgroup of $\mathrm{GL}(\widehat{\mathfrak{h}}^*)$ generated by fondamental reflections s_α, $\alpha \in \widehat{\Pi}$, defined by

$$s_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha, \quad \beta \in \widehat{\mathfrak{h}}^*.$$

The bilinear form $(\cdot|\cdot)$ is $\widehat{W}$-invariant. The affine Weyl group $\widehat{W}$ is equal to the semi-direct product $W \ltimes \Gamma$, where W is the Weyl group of G generated by s_{α_i}, $i \in \{1, \dots, n\}$, and Γ the group of translations t_α, $\alpha \in \nu(Q^\vee)$, defined by

$$t_\alpha(\lambda) = \lambda + \lambda(c)\alpha - \big[(\lambda|\alpha) + \frac{1}{2}(\alpha|\alpha)\lambda(c)\big]\delta, \quad \lambda \in \widehat{\mathfrak{h}}^*. \tag{4}$$

Identification of $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{h}}^*$ via ν allows to define an action of $\widehat{W}$ on $\widehat{\mathfrak{h}}$. One lets $wx = \nu^{-1}w\nu x$, for $w \in \widehat{W}$, $x \in \widehat{\mathfrak{h}}$. Then the action of $\widehat{W}$ on $\Lambda_0 \oplus \mathfrak{t}^* \oplus \mathbb{R}\delta/\mathbb{R}\delta$ or $d \oplus \mathfrak{t} \oplus \mathbb{R}c/\mathbb{R}c$ is identified to the one of Ω on $\mathfrak{t}$. Moreover a fondamental domain for the action of $\widehat{W}$ on the quotient space $(\mathbb{R}_+\Lambda_0 + \mathfrak{t}^* + \mathbb{R}\delta)/\mathbb{R}\delta$ is

$$\{\lambda \in \mathbb{R}\Lambda_0 \oplus \mathfrak{t}^* : \lambda(\alpha^\vee) \geq 0, \alpha \in \widehat{\Pi}\},$$

and for $\tau \geq 0$, $\tau\Lambda_0 + \phi_a$, with $\phi_a = (a|\cdot)$, is in this fundamental domain if and only if $a \in A_\tau$. Then we consider the following domain which is identified with the fundamental affine Weyl chamber viewed in the quotient space

$$C_W = \{(\tau, x) \in \mathbb{R}_+ \times \mathfrak{t} : x \in A_\tau\}.$$

4. Coadjoint $L(G)$-orbit and Brownian motion

When $\{x_s : s \in [0,1]\}$ is a continuous semi-martingale with values in $\mathfrak{g}$, then for $\tau > 0$ the stochastic differential equation

$$\tau\, dX = X \circ dx, \tag{5}$$

where $\circ$ stands for the Stratonovitch integral, has a unique solution starting from e. Such a solution is a G-valued process, that we denote by $\epsilon(\tau, x)$ [9],[10]. This is the Stratonovitch stochastic exponential of $\frac{x}{\tau}$. The previous discussion leads naturally to the following definition.

Definition 4.1. *For $\tau \in \mathbb{R}_+^*$, and $x = \{x_s : s \in [0,1]\}$ a $\mathfrak{g}$-valued continuous semi-martingale, we defines the radial part of $\tau\Lambda_0 + \int_0^1 (\cdot|dx_s)$ that we denote by $rad(\tau\Lambda_0 + \int_0^1 (.|dx_s))$ by*[2]

$$rad(\tau\Lambda_0 + \int_0^1 (\cdot|dx_s)) = (\tau, a),$$

where a is the unique element in A_τ such that

$$\epsilon(\tau, x)_1 \in Ad(G)\{\exp(a/\tau)\}.$$

We have proved in [5] the following theorem, where the conditioned space-time Brownian motion is the one defined in section 5.2. This is this theorem for which we propose a new proof.

Theorem 4.2. *If $\{x_{s,t} : s \in [0,1], t \geq 0\}$ is a Brownian sheet with values in $\mathfrak{g}$ such that for any $a, b \in \mathfrak{g}$, $s_1, s_2 \in [0,1]$, $t_1, t_2 \in \mathbb{R}_+^*$,*

$$\mathbb{E}\big((a|x_{s_1,t_1})(b|x_{s_2,t_2})\big) = \min(s_1, s_2)\min(t_1, t_2)(a|b),$$

then

$$\{rad(t\Lambda_0 + \int_0^1 (\cdot|dx_{s,t})) : t \geq 0\}$$

is a space-time Brownian motion in $\mathbb{R} \times \mathfrak{t}$ conditioned to remain in the affine Weyl chamber C_W.

5. Conditioned brownian motions

In whole the communication, when we write $f_t(x) \propto g_t(x)$ for $f_t(x), g_t(x) \in \mathbb{C}$, we mean that $f_t(x)$ and $g_t(x)$ are equal up to a multiplicative constant independent of the parameters t and x.

5.1. A Brownian motion conditioned to remain in an alcove. There is a common way to construct a Brownian motion conditioned in Doob sense to remain in an alcove, which is to consider at each time the $\mathrm{Ad}(G)$-orbit of a brownian motion in G. The brownian motion on G is left Levy process. Its transition probability densities $(\mathbf{p}_s)_{s\geq 0}$ with respect to the Haar measure on G can be expanded as a sum of characters of highest-weight complex representations of G. These representations are in correspondence with

$$P_+ = \{\lambda \in \mathfrak{t}^* : \lambda(\alpha_i^\vee) \in \mathbb{N}, i \in \{0, \dots, n\}\}.$$

[2]We do not specify in which space lives this distribution. We use this notation here just to keep track of the fact that when x is a Brownian motion the Wiener measure provides a natural measure on a coadjoint orbit in the original work of I. B. Frenkel.

One has for $s \geq 0$, $\mathsf{u}, \mathsf{v} \in G$,

$$\mathbf{p}_s(\mathsf{u}, \mathsf{v}) = \mathbf{p}_s(e, \mathsf{u}^{-1}\mathsf{v}) = \sum_{\lambda \in P_+} \mathrm{ch}_\lambda(e)\mathrm{ch}_\lambda(\mathsf{u}^{-1}\mathsf{v})e^{-\frac{s(2\pi)^2}{2}(||\lambda+\rho||^2 - ||\rho||^2)},$$

where ch_λ is the character of the irreducible representation of highest weight λ (see for instance [6]). By the Weyl character formula one has[3] for $h \in \mathfrak{t}$

$$\mathrm{ch}_\lambda(e^h) = \frac{\sum_{w \in W} \det(w) e^{2i\pi \langle w(\lambda+\rho), h\rangle}}{\sum_{w \in W} \det(w) e^{2i\pi \langle w(\rho), h\rangle}}. \tag{6}$$

We let

$$\pi(h) = \prod_{\alpha \in \Phi_+} \sin \pi\alpha(h),$$

which is the denominator in (6). Such a process starting from $\mathsf{u} \in G$ can be obtained considering a standard Brownian motion $\{x_s : s \geq 0\}$ with values in $\mathfrak{g}$, and the solution $\{X_s : s \geq 0\}$ of the stochastic differential equation

$$dX = X \circ dx$$

with initial condition $X_0 = \mathsf{u}$. Then $\{X_s : s \geq 0\}$ is a standard Brownian motion on G starting from u. If $\mathsf{u} = \exp(\gamma)$ with $\gamma \in A$ then the process $\{r_s^\gamma : s \geq 0\}$ such that for any $s \geq 0$, r_s^γ is the unique element in A such that

$$X_s \in \mathrm{Ad}(G)\{\exp(r_s^\gamma)\},$$

is a Markov process starting from γ with transition probability densities $(q_t)_{t\geq 0}$ with respect to the Haar measure on G given by

$$q_t(x, y) \propto \pi(y)^2 \sum_{\lambda \in P_+} \mathrm{ch}_\lambda(e^{-x})\mathrm{ch}_\lambda(e^y)e^{-\frac{t(2\pi)^2}{2}(||\rho+\lambda||^2 - ||\rho||^2)}, \tag{7}$$

for $t \geq 0$, $x, y \in A$. This is obtained integrating over an $\mathrm{Ad}(G)$-orbit (see (4.3.3) in [7] for instance) and using the Weyl integration formula. This Markov process is actually a Brownian motion killed on the boundary of A conditioned never to die. In fact if we denote by $(u_t)_{t\geq 0}$ the transition densities of the standard Brownian motion on $\mathfrak{t}$ killed on the boundary of A, a reflection principle gives that for $t > 0$, $x, y \in A$,

$$u_t(x, y) = \sum_{w \in \Omega} \det(w) p_t(x, w(y)), \tag{8}$$

where p_t is the standard heat kernel on $\mathfrak{t}$ and $\det(w)$ is the determinant of the linear part of w. A Poisson summation formula (see [1] for general results, and [7] or [5] for this particular case) then shows that

$$q_t(x, y) \propto \frac{\pi(y)}{\pi(x)} e^{2\pi^2(\rho|\rho)t} u_t(x, y), \tag{9}$$

which is the transition probability of the killed Brownian motion conditioned in the sense of Doob to remain in A.

[3]The presence of a factor $2i\pi$ is due to the fact that we have considered the real roots rather than the infinitesimal ones.

5.2. A space-time Brownian motion conditioned to remain in an affine Weyl chamber. We define a space-time Brownian motion conditioned to remain in an affine Weyl chamber as it has been defined in [5] and also in [2] when $G = \mathrm{SU}(2)$. It is defined as an h-process, with the help of an anti-invariant classical theta function. For $\tau \in \mathbb{R}_+^*$, $b \in \mathfrak{t}$, $a \in A_\tau$, we define $\widehat{\psi}_b(\tau, a)$ by

$$\widehat{\psi}_b(\tau, a) = \frac{1}{\pi(b)} \sum_{w \in \widehat{W}} \det(w) e^{\langle w(\tau\Lambda_0 + \phi_a), d+b\rangle}. \tag{10}$$

From now on we fix $\gamma \in A$. One considers a standard Brownian motion $\{b_t : t \geq 0\}$ with values in $\mathfrak{t}$, the space-time Brownian motion $\{B_t^\gamma = (t, b_t + \gamma t) : t \geq 0\}$, and the stopping time $T = \inf\{t \geq 0 : B_t^\gamma \notin C_W\}$. One defines a function Ψ_γ on C_W by

$$\Psi_\gamma : (t, x) \in C_W \to e^{-(\gamma|x)} \psi_\gamma(t, x). \tag{11}$$

Identity (8) and decomposition $\widehat{W} = W \ltimes \Gamma$ implies that

$$\Psi_\gamma(t, x)\pi(\gamma) \propto t^{-n/2} u_{\frac{1}{t}}(\gamma, x/t) e^{\frac{t}{2}||\gamma - x/t||^2} \tag{12}$$

Proposition 5.1. *The function Ψ_γ is a constant sign harmonic function for the killed process $\{B_{t\wedge T}^\gamma : t \geq 0\}$, vanishing on the boundary of C_W.*

Proof. The fact that Ψ_γ is harmonic and satisfies the boundary conditions is clear from (11). It is non negative by (12). □

Definition 5.2. *We define $\{A_t^\gamma = (t, a_t^\gamma) : t \geq 0\}$ as the killed process $\{B_{t\wedge T}^\gamma : t \geq 0\}$ starting from $(0,0)$ conditionned in Doob's sense not to die, via the harmonic function Ψ_γ.*

More explicitely, if we let for $t \geq 0$, $K_t^\gamma = B_{t\wedge T}^\gamma$, and $K_t^\gamma = (t, k_t^\gamma)$, then $\{A_t^\gamma = (t, a_t^\gamma) : t \geq 0\}$ is a Markov process starting from $(0,0)$ such that for $r, t > 0$, the probability density of a_{t+r}^γ given that $a_r^\gamma = x$, with $x \in A_r$, is

$$s_t^\gamma((r,x),(r+t,y)) = \frac{\Psi_\gamma(r+t,y)}{\Psi_\gamma(r,x)} w_t^\gamma((r,x),(r+t,y)), \ (r+t,y) \in C_W, \tag{13}$$

where $w_t^\gamma((r,x),(r+t,\cdot))$ is the probability density of k_{r+t}^γ given that $k_t^\gamma = x$, and the probability density of a_t^γ is given by

$$s_t^\gamma((0,0),(t,y)) = C_t \Psi_\gamma(t,y)\pi(\frac{y}{t}) e^{-\frac{1}{2t}||y-\gamma t||^2}, \quad y \in A_t, \tag{14}$$

where C_t is a normalizing constant depending on t.

5.3. The two conditioned processes and time inversion. Actually the two Doob transformations previously defined are equal up to a time inversion. We prove this property as it is done in [2] for the Brownian motion in the unit interval. The following lemma is immediately deduced from (12) and (14).

Lemma 5.3. *For $t > 0$, $x \in A$, one has*

$$s_{1/t}^\gamma((0,0),(1/t,x/t)) = q_t(\gamma, x).$$

Lemma 5.4. *For $0 < r \leq t$, $x \in A_r$, $y \in A_t$*

$$e^{-\frac{1}{2t}||y||^2} u_{\frac{1}{r}-\frac{1}{t}}(y/t, x/r) = e^{-\frac{1}{2r}||x||^2} w_{t-r}^0((r,x),(t,y)).$$

Proof. Using expression (8) and the time inversion invariance property for the standard heat kernel on $\mathfrak{t}$, one obtains that

$$e^{-\frac{1}{2t}||y||^2}u_{\frac{1}{r}-\frac{1}{t}}(y/t,x/r)=e^{-\frac{1}{2r}||x||^2}\sum_{w\in\Omega}e^{-\frac{1}{2t}(||y||^2-||tw(y/t)||^2)}p_{t-r}(x,tw(y/t)).$$

The sum on the right-hand size of the identity is exactly $w^0_{t-r}((r,x),(t,y))$ according to lemma 6.3 of [4], which achieves the proof. ☐

In the following proposition $\{r^\gamma_t : t\geq 0\}$ is the conditioned process defined in section 5.1 and $\{a^\gamma_t : t\geq 0\}$ is the one defined in section 5.2.

Proposition 5.5. *One has in distribution*

$$\{ta^\gamma_{1/t} : t\geq 0\}\stackrel{d}{=}\{r^\gamma_t : t\geq 0\}.$$

Proof. It follows immediately from the two previous lemmas and identity (12). ☐

6. A new proof of Theorem 4.2

For every $t>0$ one considers the diffusion process $\{Y^\gamma_{s,t} : s\in[0,1]\}$ starting from the identity element e of G satisfying the EDS (in s)

$$tdY^\gamma_{s,t}=Y^\gamma_{s,t}\circ d(x_{s,t}+\gamma st).$$

For $\mathsf{u}\in G$ one denotes by $\mathcal{O}(\mathsf{u})$ the unique element in A such that

$$\mathsf{u}\in \mathrm{Ad}(G)\{\exp(\mathcal{O}(\mathsf{u}))\}.$$

We have proved in [5] that the random process $\{(t,t\mathcal{O}(Y^0_{1,t})) : t\geq 0\}$ is distributed as $\{A^0_t : t\geq 0\}$. As Y^γ satisfies

$$dY^\gamma_{s,t}=Y^\gamma_{s,t}\circ d(\frac{1}{t}x_{s,t}+\gamma s),$$

and $\{\frac{1}{t}x_{s,t} : s,t>0\}\stackrel{d}{=}\{x_{s,1/t} : s,t>0\}$, one could deduce from [5], with the help of a Kirillov-Frenkel character formula from [7] and a Cameron–Martin theorem, that the result remains true for any $\gamma\in A$.

We propose here a brief proof of the theorem, which is valid for every γ. For every $t\geq 0$, one considers the diffusion process $\{X^\gamma_{s,t} : s\in[0,1]\}$ starting from $e\in G$ satisfying the stochastic differential equation (in s)

$$dX^\gamma_{s,t}=X^\gamma_{s,t}\circ d(x_{s,t}+\gamma s). \tag{15}$$

Proposition 6.1.

1. *For $t,t'\geq 0$, the random process $\{X^\gamma_{s,t+t'}(X^\gamma_{s,t})^{-1} : s\in[0,1]\}$ has the same law as $\{X^0_{s,t'} : s\in[0,1]\}$.*
2. *For $t,t'\geq 0$, the random process $\{X^\gamma_{s,t+t'}(X^\gamma_{s,t})^{-1} : s\in[0,1]\}$ is independent of $\{X^\gamma_{s,r} : s\in[0,1], r\leq t\}$.*
3. *The random process $\{X^\gamma_{1,t} : t\geq 0\}$ is a standard Brownian motion in G starting from $\exp(\gamma)$.*

Proof. For the first point, we let $Z_s = X^\gamma_{s,t+t'}(X^\gamma_{s,t})^{-1}$, $s \in [0,1]$. The process $\{(X^\gamma_{s,t})^{-1} : s \in [0,1]\}$ satisfies the EDS (in s)

$$d(X^\gamma_{s,t})^{-1} = -d(x_{s,t} + \gamma s) \circ (X^\gamma_{s,t})^{-1}$$

from which we immediately deduce that Z satisfies

$$dZ_s = Z_s \circ X^\gamma_{s,t}\, d(x_{s,t+t'} - x_{s,t})(X^\gamma_{s,t})^{-1}.$$

As $\{\int_0^s X^\gamma_{r,t}\, d(x_{r,t+t'} - x_{r,t})(X^\gamma_{r,t})^{-1} : s \in [0,1]\}$ has the same law as $\{x_{s,t'} : s \in [0,1]\}$, and is independent of $\{x_{s,r} : s \in [0,1], r \leq t\}$, one gets the first two points, which imply in particular that $\{X^\gamma_{1,t} : t \geq 0\}$ is a right Levy process. The $\mathrm{Ad}(G)$-invariance of the increments law implies that it is also a left Levy process. As for any $t > 0$, $X^0_{1,t}$ and $X^0_{t,1}$ are equal in distribution, the third point follows.

□

Proposition 6.1 has the two following corollaries, the second one being deduced from the first by proposition 5.5.

Corollary 6.2. *The random process $\{\mathcal{O}(X^\gamma_{1,t}) : t \geq 0\}$ is a standard Brownian motion starting from γ killed on the boundary of A conditioned in Doob's sense to remain in A.*

Corollary 6.3. *The random process $\{(t, t\mathcal{O}(X^\gamma_{1,1/t})) : t \geq 0\}$ has the same distribution as the conditioned process $\{A^\gamma_t : t \geq 0\}$.*

As the two processes $\{x_{s,1/t} : s, t > 0\}$ and $\{\frac{1}{t}x_{s,t} : s, t > 0\}$ are equal in distribution, Theorem 4.2 follows from corollary 6.3 with $\gamma = 0$. For any $\gamma \in A$, one has under the same hypothesis as in the theorem the following one.

Theorem 6.4. *The radial part process*

$$\{rad(t\Lambda_0 + \int_0^1 (\cdot|d(x_{s,t} + \gamma st))) : t \geq 0\}$$

is distributed as the Doob conditioned process $\{A^\gamma_t : t \geq 0\}$.

References

[1] R. Bellman, *A brief introduction to Theta functions*, Holt, Rinehart and Winston, 1961.

[2] Ph. Bougerol, M. Defosseux, *Pitman transforms and Brownian motion in the interval viewed as an affine alcove*, arXiv:1808.09182v4

[3] T. Bröcker, T. Dieck, *Representations of compact Lie groups,* Graduate texts in Mathematics, 2003.

[4] M. Defosseux, *Affine Lie algebras and conditioned space-time Brownian motions in affine Weyl chambers*. Probab. Theory Relat. Fields 165 (2015) 1–17.

[5] M. Defosseux, *Kirillov-Frenkel character formula for loop groups, radial part and Brownian sheet*, Ann. of Probab. 47 (2019) 1036–1055.

[6] H.G. Fegan, *The heat equation on a compact Lie group*. Trans. Amer. Math. Soc. 246, 339- 357 (1978)

[7] I.B. Frenkel, *Orbital theory for affine Lie algebras*, Invent. Math. 77 (1984) 301–352.

[8] V.G. Kac, *Infinite dimensional Lie algebras.* Third edition, Cambridge University Press, 1990.

[9] R.L. Karandikar , *Girsanov type formula for a Lie group valued Brownian motion,* Séminiaire de probabilités (Strasbourg), tome 17 (1983), p. 198-204.

[10] M. Hakim-Dowek, D. Lépingle, *L'exponentielle stochastqiue des groupes de Lie,* Séminiaire de probabilités (Strasbourg), tome 20 (1986), p. 352-374.
[11] J.W. Pitman and L.C.G. Rogers, *Markov functions*, Ann. Prob. 9 (1981) 573–582.
[12] A. Pressley, G. Segal, *Loop Groups*, Oxford University Press (1988)