



Spectrum of semisimple locally symmetric spaces and admissibility of spherical representations

Salah Mehdi, M Olbrich

► To cite this version:

Salah Mehdi, M Olbrich. Spectrum of semisimple locally symmetric spaces and admissibility of spherical representations. Lie Groups, Number Theory and Vertex Algebras : Conference on Representation Theory XVI, Dražen Adamović, Andrej Dujella, Antun Milas, Pavle Pandžić, Jun 2019, Dubrovnik, Croatia. pp.55-64. hal-03336291

HAL Id: hal-03336291

<https://hal.science/hal-03336291>

Submitted on 14 Sep 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

SPECTRUM OF SEMISIMPLE LOCALLY SYMMETRIC SPACES AND ADMISSIBILITY OF SPHERICAL REPRESENTATIONS

S. MEHDI AND M. OLBRICH

ABSTRACT. *We consider compact locally symmetric spaces $\Gamma \backslash G/H$ where G/H is a non-compact semisimple symmetric space and Γ is a discrete subgroup of G . We discuss some features of the joint spectrum of the (commutative) algebra $D(G/H)$ of invariant differential operators acting, as unbounded operators, on the Hilbert space $L^2(\Gamma \backslash G/H)$ of square integrable complex functions on $\Gamma \backslash G/H$. In the case of the Lorentzian symmetric space $SO_0(2, 2n)/SO_0(1, 2n)$, the representation theoretic spectrum is described explicitly. The strategy is to consider connected reductive Lie groups L acting transitively and co-compactly on G/H , a cocompact lattice $\Gamma \subset L$, and study the spectrum of the algebra $D(L/L \cap H)$ on $L^2(\Gamma \backslash L/L \cap H)$. Though the group G does not act on $L^2(\Gamma \backslash G/H)$, we explain how (not necessarily unitary) G -representations enter into the spectral decomposition of $D(G/H)$ on $L^2(\Gamma \backslash G/H)$ and why one should expect a continuous contribution to the spectrum in some cases. As a byproduct, we obtain a result on the L -admissibility of G -representations. These notes contain the statements of the main results, the proofs and the details will appear elsewhere.*

1. INTRODUCTION.

Let G be a connected non-compact semisimple real Lie group and H a connected closed subgroup of G with complexified Lie algebras \mathfrak{g} and \mathfrak{h} respectively. Suppose that G/H is a semisimple symmetric space of the non-compact type with respect to an involution σ and consider the associated decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \quad \text{where } \mathfrak{q} := \{X \in \mathfrak{g} \mid \sigma(X) = -X\}$$

Fix an invariant measure on G/H and consider the Hilbert space $L^2(G/H)$ of square integrable complex functions on G/H . The action by left translations of G on $L^2(G/H)$ defines a unitary representation whose decomposition is known as the Plancherel formula for symmetric spaces [2][3][7][20]:

$$L^2(G/H) \simeq \int_{\widehat{G}_H}^{\oplus} (W_{\tilde{\rho}, -\infty}^H)_0 \otimes W_{\rho} d\mu(\rho)$$

$$\begin{array}{ccc} \circlearrowleft & & \circlearrowleft \\ D(G/H) & \leftarrow & \mathcal{Z}(\mathfrak{g}) \end{array}$$

where W_{ρ} is a unitary irreducible representation of G , $W_{\tilde{\rho}}$ the dual representation, $W_{\tilde{\rho}, -\infty}$ the space of distribution vectors of $W_{\tilde{\rho}}$, $W_{\tilde{\rho}, -\infty}^H$ the space of H -invariants distribution vectors, $(W_{\tilde{\rho}, -\infty}^H)_0$ is a finite dimensional Hilbert space embedded in $W_{\tilde{\rho}, -\infty}^H$ whose dimension is the multiplicity of W_{ρ} in $L^2(G/H)$, \widehat{G}_H is the H -spherical dual $\{\rho \in \widehat{G} \mid W_{\tilde{\rho}, -\infty}^H \neq \{0\}\}$ and $d\mu$ the Plancherel measure. The (commutative) algebra $D(G/H)$ of G -invariant differential operators on G/H acts on $(W_{\tilde{\rho}, -\infty}^H)_0$ and the center $\mathcal{Z}(\mathfrak{g})$ of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of \mathfrak{g} acts via infinitesimal character on W_{ρ} . Except for exceptional cases where G

is E_6 , E_7 and E_8 which we will not consider here, there is a surjective map $\mathcal{Z}(\mathfrak{g}) \twoheadrightarrow D(G/H)$ [11]. In other words, harmonic analysis on G/H is closely related to the spectrum of $D(G/H)$ on $L^2(G/H)$:

$$\text{harmonic analysis on } G/H \longleftrightarrow \text{Spec}_{L^2(G/H)}(D(G/H))$$

A compact locally symmetric space is a smooth compact manifold of the form $\Gamma \backslash G/H$ where Γ is a discrete subgroup of G , in particular the natural map $G \rightarrow \Gamma \backslash G/H$ is smooth for the quotient topology. As above one may ask about the analysis on $\Gamma \backslash G/H$. Let us consider the following group case example.

Example 1.1. $G = G_1 \times G_1$, $H = \Delta(G_1 \times G_1)$ and $\Gamma = \Gamma_1 \times \{e\}$, where G_1 is a connected non-compact semisimple real Lie group, σ is the permutation $G_1 \times G_1 \rightarrow G_1 \times G_1$, $(a, b) \mapsto (b, a)$ and Γ_1 a co-compact torsion free discrete subgroup of G_1 . Then $L^2(G/H) \simeq L^2(G_1)$ is described by the Harish-Chandra Plancherel formula and

$$L^2(\Gamma \backslash G/H) \simeq L^2(\Gamma_1 \backslash G_1) \simeq \bigoplus_{\pi \in \widehat{G}_1} V_{\pi, -\infty}^{\Gamma_1} \widehat{\otimes} V_{\pi}$$

where $\dim(V_{\pi, -\infty}^{\Gamma_1}) < \infty$, $\dim(V_{\pi}) = \infty$ (except for the trivial representation π) and the set $\{\pi \in \widehat{G}_1 \mid V_{\pi, -\infty}^{\Gamma_1} \neq \{0\}\}$ is discrete. Moreover $D(G/H) \simeq \mathcal{Z}(\mathfrak{g})$ acts scalarly on π by infinitesimal characters. In particular, the spectrum of $D(G/H)$ on $L^2(\Gamma \backslash G/H)$ is discrete, consists of eigencharacters only, and (almost) all eigenspaces are infinite dimensional.

Already this example shows that analysis on $\Gamma \backslash G/H$ is often related to automorphic forms:

$$\text{analysis on } \Gamma \backslash G/H \longleftrightarrow \text{automorphic forms}$$

Problem 1.2. Describe the spectrum $\text{Spec}_{L^2(\Gamma \backslash G/H)}(D(G/H))$ of $D(G/H)$ on $L^2(\Gamma \backslash G/H)$ in general.

Issues:

- (1) When H is not compact then a discrete subgroup $\Gamma \subset G$ such that $\Gamma \backslash G/H$ is a smooth compact manifold need not exist as it is illustrated by the Calabi-Markus phenomenon for $SO_0(1, n+1)/SO_0(1, n)$ [6]. Given such a Γ , the space $\Gamma \backslash G/H$ will be called a compact *Clifford-Klein form* of G/H . One can ask for deformations of Γ such that the new quotient is still a Clifford-Klein form (see [16]).
- (2) The group G does not act on $\Gamma \backslash G/H$, and it is not clear how representations of G are involved in $\text{Spec}_{L^2(\Gamma \backslash G/H)}(D(G/H))$.
- (3) When H is not compact, the coset $\Gamma \backslash G$ is not compact and it is not clear if $L^2(\Gamma \backslash G)$ and $L^2(\Gamma \backslash G/H)$ are related.
- (4) $D(G/H)$ acts on $C^\infty(\Gamma \backslash G/H)$ and $C^\infty(\Gamma \backslash G/H)$ is dense in $L^2(\Gamma \backslash G/H)$, so $D(G/H)$ acts via unbounded operators on $L^2(\Gamma \backslash G/H)$. Some care is required with the domains of the operators.

Problem 1.2 is related with a program developed by Kobayashi to study hidden symmetries (see [14][15] and references therein). In the following we shall describe and state some of the main results we prove in [21].

2. EXISTENCE OF COMPACT CLIFFORD-KLEIN FORMS AND SPHERICAL TRIPLES.

When H is compact, the existence of a discrete subgroup Γ of G such that $\Gamma \backslash G/H$ is a smooth compact manifold is equivalent to the existence of torsion free uniform lattices in G . It turns out that torsion free uniform lattices always exist in G by a result of Borel and Harish-Chandra [5].

For H noncompact, this gives a sufficient condition for the existence of Γ , employed extensively by Kobayashi starting with [18]:

$$\left. \begin{array}{l} \text{existence of a closed connected Lie group } L \subset G \text{ such that} \\ (i) \ L \text{ is reductively embedded in } G \\ (ii) \ L \text{ acts transitively on } G/H \\ (iii) \ L \cap H \text{ is compact} \end{array} \right\} \Rightarrow \text{existence of compact Clifford-Klein forms for } G/H$$

Indeed, if Γ is a co-compact lattice in L then $\Gamma \backslash G/H$ is a compact locally symmetric space. Triples (G, H, L) satisfying (i) and (ii) above were classified, when G is simple, by Onishik in the late 1960's [23] (see also [19]). We will refer to triples (G, H, L) satisfying (i)(ii)(iii) as *transitive triples*.

In fact, the above implication remains true, if we weaken *transitively* in (ii) to *cocompactly*. However, we are able to show that this does not change anything: *cocompactly* implies *transitively* [21].

Next fix $P = MAN$ the Langlands decomposition of a minimal parabolic subgroup in G and consider a principal series representation $\text{Ind}_P^G \tau \otimes e^\nu \otimes 1$ of G , where $\tau \in \widehat{M}$ is a finite dimensional irreducible representation of M and ν is a linear form on the complexified Lie algebra of A . Then van den Ban proved in [1] that the space of H -invariants distributions vectors in $\text{Ind}_P^G \tau \otimes e^\nu \otimes 1$ is finite dimensional:

$$\dim (\text{Ind}_P^G \tau \otimes e^\nu \otimes 1)_{-\infty}^H < \infty$$

Now pick a transitive triple (G, H, L) and let $P_L = M_L A_L N_L$ be the Langlands decomposition of a minimal parabolic in L . Let $\tau_L \in \widehat{M_L}$ be a finite dimensional irreducible representation of M_L and ν_L a linear form on the complexified Lie algebra of A_L . Observe that L is not σ -stable, $L/L \cap H$ need not be a symmetric space and the space of $L \cap H$ -invariants distributions $(\text{Ind}_{P_L}^L \tau_L \otimes e^{\nu_L} \otimes 1)_{-\infty}^{L \cap H}$ could be infinite dimensional as it is the case for the triple $(G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times \{e\})$. In fact we obtain the following criterion for the space of $L \cap H$ -invariants to be finite dimensional.

Proposition 2.1. *For any $\tau_L \in \widehat{M_L}$ a finite dimensional irreducible representation of M_L and ν_L a linear form on the complexified Lie algebra of A_L , one has*

$$\dim (\text{Ind}_{P_L}^L \tau_L \otimes e^{\nu_L} \otimes 1)_{-\infty}^{L \cap H} < \infty \Leftrightarrow P_L \text{ acts transitively on } L/L \cap H.$$

Transitive triples (G, H, L) such that P_L acts transitively on $L/L \cap H$ will be called *spherical triples*. The use of the qualifier 'spherical' is motivated by the following facts:

$$\begin{array}{ccc} L/L \cap H \text{ is a spherical homogeneous space} & \Longleftrightarrow & P_L \text{ acts with an open orbit in } L/L \cap H \\ & \Longleftrightarrow & \text{if } L \cap H \text{ is compact} \quad P_L \text{ acts transitively on } L/L \cap H \end{array}$$

Example 2.2.

- (1) $(G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times \{e\})$ is not spherical.
- (2) $(G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times K_1)$ is spherical (where K_1 is maximal compact in G_1).
- (3) $(SO_0(2, 2n), SO_0(1, 2n), U(1, n))$ is spherical ($n \geq 2$).
- (4) $(SO_0(4, 3), SO_0(4, 1) \times SO(2), G_{2(2)})$ is not spherical.

3. L -ADMISSIBILITY.

Suppose $G' \subset G$ is a connected closed reductive subgroup and π is an irreducible unitary representation of G . From a general result of Mautner and Teleman, the restriction of π to G' can be decomposed as the direct integral sum of irreducible unitary representations:

$$\pi|_{G'} \simeq \int_{G'}^{\oplus} M_{\pi}(\tau) \hat{\otimes} V_{\tau} d\mu(\tau)$$

where $M_{\pi}(\tau)$ is the multiplicity (Hilbert) space.

Definition 3.1. π is G' -admissible if $\pi|_{G'}$ decomposes discretely and $\dim M_{\pi}(\tau) < \infty$

Example 3.2.

- (1) $\hat{G} \ni \pi$ is K -admissible (Harish-Chandra admissibility theorem [8]).
- (2) $G = SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ and $H = \Delta(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))$. Let $\hat{G}_H \ni \pi = \pi_1 \otimes \pi_1$ where π_1 is a holomorphic discrete series representation of $SL(2, \mathbb{R})$. From [24], $\pi|_H$ has continuous spectrum (but multiplicities are finite). Therefore π is not H -admissible.

There is a criterion for H -admissibility in terms of associated varieties due to Kobayashi [17]. We prove the following admissibility result for spherical representations associated with triples (G, H, L) .

Theorem 3.3. Suppose (G, H, L) is a spherical triple. If $\pi \in \hat{G}_H$ then π is L -admissible. Moreover, if π is not trivial then there are infinitely many summands in $\pi|_L$ all having $L \cap H$ -invariants.

4. EMBEDDING OF CASIMIR OPERATORS.

Fix a transitive triple (G, H, L) . Write $\mathcal{U}(\mathfrak{g})^H$ for the H -invariant elements in the enveloping algebra of \mathfrak{g} and $\mathcal{U}(\mathfrak{g})\mathfrak{h}$ the left $\mathcal{U}(\mathfrak{g})$ -ideal generated by \mathfrak{h} . The algebra $D(G/H)$ of G -invariant differential operators on G/H is isomorphic to the following quotient algebra:

$$D(G/H) \simeq \mathcal{U}(\mathfrak{g})^H / \mathcal{U}(\mathfrak{g})\mathfrak{h} \cap \mathcal{U}(\mathfrak{g})^H$$

It is known that since G/H is a symmetric space, $D(G/H)$ is commutative (see [10] and references therein).

By definition, if (G, H, L) is a transitive triple then one has a diffeomorphism of homogeneous spaces

$$G/H \simeq L/L \cap H$$

and an isomorphism of algebra

$$\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{h}) \otimes_{\mathcal{U}(\mathfrak{l} \cap \mathfrak{h})} \mathcal{U}(\mathfrak{l}).$$

In particular, we get an embedding of algebras

$$\iota : D(G/H) \hookrightarrow D(L/L \cap H) .$$

The algebra $D(L/L \cap H)$ need not be commutative, though it is for many transitive triples (see [12]). Of special interest in $D(G/H)$ (resp. $D(L/L \cap H)$, $D(L \cap K/L \cap H)$) is the Casimir element Ω_G (resp. Ω_L , $\Omega_{L \cap K}$). The algebras $D(L/L \cap H)$ and $D(L \cap K/L \cap H)$ are defined in an analogous way to $D(G/H)$. Observe that when G/H has rank one then $D(G/H)$ is generated by the Casimir Ω_G .

Proposition 4.1. *For each transitive triple (G, H, L) with G/H irreducible, the embedding $\iota(\Omega_G)$ is computed explicitly in terms of the decomposition of $\mathfrak{l}/\mathfrak{l} \cap \mathfrak{h}$ into $Ad(L \cap H)$ -irreducibles.*

In particular, based on the restriction to \mathfrak{l} and $\mathfrak{l} \cap \mathfrak{k}$ of the Killing form of \mathfrak{g} , we have:

Example 4.2.

- (1) $(G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times \{e\})$: $\iota(\Omega_G) = 2\Omega_L$.
- (2) $(G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times K_1)$: $\iota(\Omega_G) = 2\Omega_L - \Omega_{L \cap K}$.
- (3) $(SO_0(2, 2n), SO_0(1, 2n), U(1, n))$: $\iota(\Omega_G) = 2\Omega_L - \Omega_{L \cap K}$.
- (4) $(SO_0(4, 3), SO_0(4, 1) \times SO(2), G_{2(2)})$: $\iota(\Omega_G) = 3\Omega_L - \frac{3}{2}\Omega_{L \cap K} + 2\Omega_{\mathfrak{l} \cap \mathfrak{s} \cap \mathfrak{q}}$

In case (3), Schlichtkrull, Trapa and Vogan obtained a similar formula for $O(2n)/U(n)$ [25].

Remark 4.3. By inspection, we observe that $\iota(\Omega_G)$ involves ‘non-compact terms’ terms from $\mathfrak{l} \cap \mathfrak{q} \cap \mathfrak{s}$ only when (G, H, L) is not spherical, except in the first group case (1).

5. ON THE SPECTRUM $\text{SPEC}_{L^2(\Gamma \backslash G/H)}(D(G/H))$.

Fix a transitive triple (G, H, L) . We have the following sequence of isomorphisms:

$$\begin{array}{ccc} L^2(\Gamma \backslash G/H) \simeq L^2(\Gamma \backslash L/L \cap H) \simeq L^2(\Gamma \backslash L)^{L \cap H} \simeq \widehat{\bigoplus_{\pi \in \widehat{L}} V_{\pi, -\infty}^\Gamma \otimes V_\pi^{L \cap H}} & & \text{(Hilbert sum)} \\ \cup & \cup & \cup \\ C^\infty(\Gamma \backslash G/H) \simeq C^\infty(\Gamma \backslash L/L \cap H) \simeq C^\infty(\Gamma \backslash L)^{L \cap H} \simeq \widehat{\bigoplus_{\pi \in \widehat{L}} V_{\pi, -\infty}^\Gamma \otimes V_{\pi, \infty}^{L \cap H}} & & \text{(closure of algebraic direct sum)} \\ \circlearrowleft & & \circlearrowleft \\ D \in D(G/H) & & \iota(D) \in D(L/L \cap H) \end{array}$$

Suppose now that (G, H, L) is spherical, i.e $\dim(\text{Ind}_{P_L}^L \tau_L \otimes e^{\nu_L} \otimes 1)_{-\infty}^{L \cap H} < \infty$, for any $\tau_L \in \widehat{M_L}$ a finite dimensional irreducible representation of M_L and ν_L a linear form on the complexified Lie algebra of A_L . Then by Proposition 2.1 and Casselman embedding Theorem, we deduce that

Proposition 5.1. *For any irreducible unitary representation (π, V_π) of L , one has*

$$\dim V_{\pi, \infty}^{L \cap H} < \infty$$

This means that the spectrum of $D(G/H)$ on $C^\infty(\Gamma \backslash G/H)$ may be reduced to the diagonalization of the operator $\pi(\iota(D))$ on the (infinitely many) finite dimensional blocks $V_{\pi, \infty}^{L \cap H}$. Then we obtain the following description of the spectrum of $D(G/H)$ on $L^2(\Gamma \backslash G/H)$.

Theorem 5.2. *Suppose (G, H, L) is a spherical triple.*

- (1) *There is a direct sum decomposition of $C^\infty(\Gamma \backslash G/H)$ and $L^2(\Gamma \backslash G/H)$ into joint eigenspaces of $D(G/H)$. $\text{Spec}_{L^2(\Gamma \backslash G/H)}(D(G/H))$ consists of the corresponding eigencharacters and their possible accumulation points.*
- (2) *In the Lorentzian case $(SO_0(2, 2n), SO_0(1, 2n), U(1, n))$, $D(G/H)$ is generated by the Casimir Ω_G , the set $\{\pi \in \widehat{L} \mid V_{\pi, \infty}^{L \cap H} \neq \{0\}\}$ is computed explicitly and the representations π are identified as well as their contributions to eigenvalues for $\iota(\Omega_G)$. More precisely, if Δ denotes the Laplace operator on $\Gamma \backslash G/H$ induced by Ω_G , one has*
 - $\text{spec}(\Delta) \cap (-\infty, -n^2]$ *comes from unitary principal series and, at $-n^2$, also from limits of discrete series of L .*
 - $\text{spec}(\Delta) \cap (-n^2, 0]$ *consists of contributions from complementary series, ends of complementary series and non-integrable discrete series.*
 - $\text{spec}(\Delta) \cap (0, \infty) = \bigcup_{\ell=n+1}^{\infty} \{\ell^2 - n^2\}$ *is the contribution of integrable discrete series. The corresponding eigenspaces are infinite dimensional.*

The last assertion about the contribution of integrable discrete series with $L \cap H$ -invariants generalizes to arbitrary spherical triples (G, H, L) . These contributions combine to infinite dimensional eigenspaces of $D(G/H)$ [21]. The shortest proof of this uses Theorem 3.3 instead of the embedding

$$\iota : D(G/H) \hookrightarrow D(L/L \cap H) .$$

In view of Theorem 3.3 the result can be rephrased in terms of G -representations as follows. Each integrable discrete series for G/H contributes an infinite dimensional eigenspace of $D(G/H)$ in $L^2(\Gamma \backslash G/H)$. Compare related results obtained in [13].

6. GENERALIZED MATRIX COEFFICIENTS, UNITARITY AND CONTINUOUS SPECTRUM.

The Hilbert spaces $L^2(G/H)$ and $L^2(\Gamma \backslash G)$ are both unitary G -representations which decompose into a direct integral of irreducible ones. Though the Hilbert space $L^2(\Gamma \backslash G/H)$ is not a G -module, there is a natural way to involve representations of G .

For an admissible representation (ρ, W_ρ) of G of finite length, one may consider generalized matrix coefficients (see [9]):

$$W_{\tilde{\rho}, -\infty} \otimes W_{\rho, -\infty} \rightarrow C^{-\infty}(G), \quad \tilde{w} \otimes w \mapsto c_{\tilde{w}, w}$$

with

$$\begin{aligned} c_{\tilde{w},w}(f) &= \langle \tilde{w}, \rho(f)w \rangle \\ \langle \rho(f)w, v \rangle &= \int_G f(g) \langle \rho(g)w, v \rangle dg \quad \text{for } f \in C_c^\infty(G), v \in W_{\rho,\infty} \end{aligned}$$

Then there is a $D(G/H)$ -equivariant map (which is injective if ρ is irreducible):

$$(6.1) \quad \begin{array}{ccc} W_{\rho,-\infty}^\Gamma \otimes W_{\rho,-\infty}^H & \longrightarrow & C^{-\infty}(\Gamma \backslash G/H) \simeq \widehat{\bigoplus_{\pi \in \hat{L}} V_{\pi,-\infty}^\Gamma \otimes V_{\pi,-\infty}^{L \cap H}} \\ \circlearrowleft & & \circlearrowleft \\ D(G/H) & & D(L/L \cap H) \end{array}$$

A comparison of (6.1) with Theorem 5.2 leads, among other things, to the following funny observation. On one hand, from Theorem 5.2, there are complementary series representations π of $U(1, n)$ contributing to finite dimensional eigenspaces. On the other hand, if ρ were unitary then $\rho|_L$ would contain, by Theorem 3.3, infinitely many unitary summands having $L \cap H$ -invariants. Moreover, a result of Bergeron and Clozel [4], Thm. 6.5.1, tell us that there are indeed lattices $\Gamma \subset U(1, n)$ such that $V_{\pi,-\infty}^\Gamma \neq \{0\}$ for complementary series π with certain integral parameters. Therefore there must exist non-unitary representations ρ of G that are involved in $L^2(\Gamma \backslash G/H)$ via (6.1).

One can also try to use (6.1) to produce families of eigendistributions on $\Gamma \backslash G/H$ depending continuously on the spectral parameter. Let $0 \neq \tilde{v} \in V_{\pi,-\infty}^\Gamma$ and assume we have a continuous nonconstant family $t \mapsto \rho_t$, $t \in (a, b) \subset \mathbb{R}$ of H -spherical principal series of G with $0 \neq w_t \in W_{\rho_t,-\infty}^H$ and, the crucial ingredient, a continuous family of non-zero intertwining operators $\Psi_t \in \text{Hom}_L(W_{\rho_t,\infty}, V_{\pi,\infty})$. Then $\Psi_t^* v \in W_{\rho_t,-\infty}^\Gamma$ and

$$t \mapsto c_{\Psi_t^* v, w_t}$$

is the desired continuous family of eigendistributions. Note that the existence of such a family for $\Gamma \backslash G/H$ excludes the existence of a discrete spectral decomposition as in Theorem 5.2. Therefore such families can only exist for non-spherical triples (G, H, L) . One expects that such families are the building blocks for the continuous part of the spectral decomposition of $L^2(\Gamma \backslash G/H)$. We are able to find such families of intertwiners for some non-spherical triples, and expect their existence for all non-spherical (G, H, L) except for the group case as in Example 2.2 (1). For instance, for the non-spherical triple $(SO(8, \mathbb{C}), SO_0(1, 7), Spin(7, \mathbb{C}))$ the existence of many families Ψ_t is ensured by the work of Jan Frahm [22].

REFERENCES

- [1] E. van den Ban, *Invariant differential operators on a semisimple symmetric space and finite multiplicities in a Plancherel formula*. Ark. Mat. **25** (1987), 175–187.
- [2] E. P. van den Ban and H. Schlichtkrull, *The Plancherel decomposition for a reductive symmetric space. I. Spherical functions*. Invent. Math. **161** (2005), no. 3, 453–566.
- [3] E. P. van den Ban and H. Schlichtkrull, *The Plancherel decomposition for a reductive symmetric space. II. Representation theory*. Invent. Math. **161** (2005), no. 3, 567–628.
- [4] N. Bergeron and L. Clozel, *Spectre automorphe des variétés hyperboliques et applications topologiques*. Astérisque **303** (2005).
- [5] A. Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*. Ann. of Math. (2) **75** (1962), 485–535.

- [6] E. Calabi and L. Markus, *Relativistic space forms*. Ann. of Math. (2) **75** (1962), 63–76.
- [7] P. Delorme, *Formule de Plancherel pour les espaces symétriques réductifs*. Ann. of Math. (2) **147** (1998), no. 2, 417–452.
- [8] Harish-Chandra, *Representations of semi-simple Lie groups*, I, III, Trans. A.M.S. **75** (1953), 185–243; **76** (1954), 234–253; IV, Amer. J. Math. **77** (1955), 743–777.
- [9] H. He, *Generalized matrix coefficients for infinite dimensional unitary representations*. J. Ramanujan Math. Soc. **29** (2014), 253–272.
- [10] S. Helgason, *Groups and geometric analysis. Integral geometry, invariant differential operators, and spherical functions*. Mathematical Surveys and Monographs **83**, American Mathematical Society, Providence, RI, 2000.
- [11] S. Helgason, *Some results on invariant differential operators on symmetric spaces*. Amer. J. Math. **114** (1992), no. 4, 789–811.
- [12] F. Kassel and T. Kobayashi, *Invariant differential operators on spherical homogeneous spaces with overgroups*. J. Lie Theory **29** (2019), 663–754.
- [13] F. Kassel and T. Kobayashi, *Poincaré series for non-Riemannian locally symmetric spaces*. Adv. Math. **287** (2016), 123–236.
- [14] T. Kobayashi, *Global analysis by hidden symmetry. Representation theory, number theory, and invariant theory*. 359–397, Progr. Math. **323**, Birkhäuser/Springer, Cham, 2017.
- [15] T. Kobayashi, *Hidden symmetries and spectrum of the Laplacian on an indefinite Riemannian manifold. Spectral analysis in geometry and number theory*. 73–87, Contemp. Math. **484**, Amer. Math. Soc., Providence, RI, 2009.
- [16] T. Kobayashi, *Discontinuous groups for non-Riemannian homogeneous spaces*. Mathematics unlimited—2001 and beyond, 723–747, Springer, Berlin, 2001.
- [17] T. Kobayashi, *Discrete decomposability of the restriction of $A_q(\lambda)$ with respect to reductive subgroups. III. Restriction of Harish-Chandra modules and associated varieties*. Invent. Math. **131** (1998), no. 2, 229–256.
- [18] T. Kobayashi, *Proper action on a homogeneous space of reductive type*. Math. Ann. **285** (1989), no. 2, 249–263.
- [19] T. Kobayashi and T. Yoshino, *Compact Clifford-Klein forms of symmetric spaces - Revisited*. Pure Appl. Math. Q. **1** (2005), no. 3, part 2, 591–663.
- [20] T. Oshima and T. Matsuki, *A description of discrete series for semisimple symmetric spaces. Group representations and systems of differential equations*. (Tokyo, 1982), 331–390, Adv. Stud. Pure Math. **4**, North-Holland, Amsterdam, 1984.
- [21] S. Mehdi and M. Olbrich, *Spectrum of pseudo-Riemannian locally symmetric spaces and admissibility of spherical representations*. Preprint.
- [22] J. Möllers, *Symmetry breaking operators for strongly spherical reductive pairs and the Gross-Prasad conjecture for complex orthogonal groups*. <https://arxiv.org/abs/1705.06109> ,
- [23] A. L. Oniščik, *Decompositions of reductive Lie groups*. Math. USSR-Sb. **9** (1969), 515–554.
- [24] J. Repka, *Tensor products of unitary representations of $SL_2(\mathbb{R})$* . Bull. Amer. Math. Soc. **82** (1976), no. 6, 930–932.
- [25] H. Schlichtkrull, P. Trapa and D. A. Vogan *Laplacian on spheres*. <https://arxiv.org/abs/1803.01267>.

INSTITUT ELIE CARTAN DE LORRAINE, CNRS - UMR 7502, UNIVERSITÉ DE LORRAINE - CAMPUS DE METZ, FRANCE
 Email address: salah.mehdi@univ-lorraine.fr

FACULTÉ DES SCIENCES, DE LA TECHNOLOGIE ET DE LA COMMUNICATION , UNIVERSITÉ DU LUXEMBOURG, LUXEMBOURG
 Email address: martin.olbrich@uni.lu