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# LOCAL EXPLOSIONS AND EXTINCTION IN CONTINUOUS-STATE BRANCHING PROCESSES WITH LOGISTIC COMPETITION

CLÉMENT FOUCART

ABSTRACT. We study by duality methods the extinction and explosion times of continuous-state branching processes with logistic competition (LCSBPs) and identify the local time at  $\infty$  of the process when it is instantaneously reflected at  $\infty$ . The main idea is to introduce a certain “bidual” process  $V$  of the LCSBP  $Z$ . The latter is the Siegmund dual process of the process  $U$ , that was introduced in [Fou19] as the Laplace dual of  $Z$ . By using both dualities, we shall relate local explosions and the extinction of  $Z$  to local extinctions and the explosion of the process  $V$ . The process  $V$  being a one-dimensional diffusion on  $[0, \infty]$ , many results on diffusions can be used and transferred to  $Z$ . A concise study of Siegmund duality for regular one-dimensional diffusions is also provided.

## 1. INTRODUCTION

Continuous-state branching processes with logistic competition (LCSBP) are Markov processes that have been introduced by Lambert in [Lam05] to model the size of a population in which a self-regulation dynamics is taken into account. Those processes are valued in  $[0, \infty]$ , the one-point compactification of the half-line, and can be seen as classical branching processes on which a deterministic competition pressure between pair of individuals, parametrized by a real value  $c > 0$ , is superimposed. For instance, if the branching dynamics are given by a critical Feller diffusion, the logistic CSBP is solution to the SDE:

$$dZ_t = \sigma \sqrt{Z_t} dB_t - \frac{c}{2} Z_t^2 dt, \quad Z_0 = z \in (0, \infty),$$

for some  $\sigma > 0$ . In the general case, the diffusive part is replaced by the complete dynamics of a CSBP, see e.g. Li [Li11, Chapter 9] and Kyprianou’s book [Kyp14, Chapter 12]. The latter is governed by a Lévy-Khintchine function  $\Psi$  defined on  $\mathbb{R}_+$ , called branching mechanism. Processes with competition do not satisfy any natural branching or affine properties. It has been observed however in Foucart [Fou19] that a LCSBP  $Z$  lies in duality with a certain diffusion process  $U$  on  $[0, \infty]$ , referred to as Laplace dual of  $Z$ : namely for any  $z \in [0, \infty]$ ,  $x \in (0, \infty)$  and  $t \geq 0$ ,

$$\mathbb{E}_z[e^{-xZ_t}] = \mathbb{E}_x[e^{-zU_t}].$$

The Laplace duality above will be used as a representation of the semigroup of the LCSBP  $Z$  in terms of that of the diffusion  $U$ . Duality relationships map entrance laws of one process to exit laws of the other, see Cox and Rösler [CR84]. Such relations turn out to be useful for the study of boundary behaviors of certain processes with jumps, we refer e.g. to Foucart and Zhou [FZ21]. We highlight that in all the article we take the convention  $0 \times \infty = 0$ . Moreover the notations  $\mathbb{P}_z$  and  $\mathbb{E}_z$  stand for the law of the underlying process started from  $z$  and its expectation. We will not address here pathwise duality relationships, and we keep this notation for all processes, which can be thought as defined on different probability spaces.

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It has been established in [Fou19] that the boundary  $\infty$  is accessible for certain LCSBPs. In other words some populations with very strong reproduction can escape from self-regulation and explode despite the quadratic competition. Behaviors of  $Z$  at its boundaries 0 and  $\infty$  are intrinsically related to those of the diffusion  $U$  at  $\infty$  and 0 respectively. A logistic CSBP can have actually its boundary  $\infty$  as exit (it hits  $\infty$  and stay there), as instantaneous regular reflecting boundary (the process immediately leaves the boundary and returns to it at a set of times of zero Lebesgue measure), or as entrance (the process enters from  $\infty$  and never visits it again).

The aim of this article is to push further the study of the process  $Z$  by studying the laws of the extinction time, the first explosion time and last but not least of the local time at  $\infty$  when  $\infty$  is a regular reflecting boundary. Diffusions will again naturally come into play. We will use a second duality relationship and introduce the Siegmund dual of process  $U$ : namely the process  $V$  satisfying for any  $x, y \in (0, \infty)$  and  $t \geq 0$ ,

$$\mathbb{P}_x(U_t < y) = \mathbb{P}_y(x < V_t).$$

We summarize both dualities in the following diagram:

$$(1.1) \quad Z \begin{array}{c} \text{Laplace dual} \\ \leftarrow \rightleftarrows \end{array} U \begin{array}{c} \text{Siegmund dual} \\ \leftarrow \rightleftarrows \end{array} V.$$

In a way, we shall see how both dualities map entrance laws of  $Z$  to entrance laws of  $V$ . Indeed, combining these two dualities one shall check the following relationship between  $Z$  and  $V$ : for any  $t \geq 0$  and all  $z, x \in (0, \infty)$ ,

$$(1.2) \quad \mathbb{E}_z(e^{-xZ_t}) = \int_0^\infty ze^{-zy} \mathbb{P}_y(V_t > x) dy.$$

When  $z$  tends to  $\infty$ , the identity (1.2) yields the following link between entrance laws of  $Z$  and  $V$ :  $\mathbb{E}_\infty(e^{-xZ_t}) = \mathbb{P}_0(V_t > x)$  for  $t, x \geq 0$ . We will see how (1.2) propagates to the laws of the times of extinction and first explosion of  $Z$  and to the local time. As we shall need it, we will also provide a study of Siegmund duality of regular one-dimensional diffusions on  $(0, \infty)$  in Section 6. This will be based on other arguments than those in [CR84] and will complete results therein.

The paper is organised as follows. In Section 2, we recall fundamental elements on logistic CSBPs, how they can be constructed up to hitting their boundaries and how extended processes ly in duality with certain generalized Feller diffusions. Our main results are stated in Section 3. Section 4 sheds some light on the case without competition. The proofs are provided in Section 5 and will make use of some general results on Siegmund duality established independently in Section 6.

## 2. BACKGROUND ON CSBPs AND LCSBPs

**2.1. Construction and Lamperti's time change.** Let  $\Psi$  be a branching mechanism, namely a function of the Lévy Khintchine form :

$$(2.1) \quad \Psi(x) = -\lambda + \frac{\sigma^2}{2}x^2 + \gamma x + \int_0^\infty (e^{-xh} - 1 + xh \mathbb{1}_{\{h \leq 1\}}) \pi(dh) \quad \text{for all } x \geq 0,$$

where  $\lambda \geq 0, \sigma \geq 0, \gamma \in \mathbb{R}$  and  $\pi$  is a Lévy measure on  $(0, \infty)$  such that  $\int_0^\infty (1 \wedge x^2) \pi(dx) < \infty$ .

Denote by  $\mathcal{L}^\Psi$  the extended generator of the CSBP( $\Psi$ ). The latter acts on  $C^2$  functions as follows:

$$(2.2) \quad \mathcal{L}^\Psi f(z) = \sigma z f''(z) + \gamma z f'(z) - \lambda z f(z) + z \int_0^\infty (f(z+h) - f(z) - hf'(z) \mathbb{1}_{h \leq 1}) \pi(dh).$$

In order to take into account the competition term, the extended generator  $\mathcal{L}$  of the LCSBP( $\Psi, c$ ) is defined as follows: for any  $f \in C^2$  and  $z \in (0, \infty)$ ,

$$\mathcal{L} f(z) := \mathcal{L}^\Psi f(z) - \frac{c}{2} z^2 f'(z).$$

We define the LCSBPs with parameter  $(\Psi, c)$ , as the Markov processes solution to the following martingale problem  $(\text{MP})_Z$ :

For any  $f \in C_c^2$ , the process

$$(2.3) \quad \left( f(Z_t) - \int_0^t \mathcal{L}f(Z_s) ds, t \geq 0 \right) \text{ is a martingale.}$$

There exists a unique solution of  $(\text{MP})_Z$  with boundary  $\infty$  absorbing, we refer to Foucart [Fou19, Section 4] and recall briefly its construction. Following Lambert's idea [Lam05, Definition 3.2], a simple construction of the process absorbed when reaching its boundaries, is provided by time-changing in Lamperti's manner a generalized Ornstein-Uhlenbeck process  $(R_t, t \geq 0)$  stopped when reaching 0. This latter process is solution to the stochastic equation

$$(2.4) \quad dR_t = dX_t - \frac{c}{2} R_t dt, \quad R_0 = z, \quad \text{for all } t \leq \sigma_0$$

where  $(X_t, t \geq 0)$  is a spectrally positive Lévy process with Laplace exponent  $\Psi$  (if  $\lambda > 0$ , it jumps to  $\infty$  at an independent exponential time  $e_\lambda$ ), and where we denote by  $\sigma_0 := \inf\{t > 0 : R_t \leq 0\}$ , the first passage time below 0 of  $R$ . Define the additive functional,

$$t \mapsto \theta_t := \int_0^{t \wedge \sigma_0} \frac{ds}{R_s} \in [0, \infty]$$

and its right-inverse

$$t \mapsto C_t := \inf\{u \geq 0 : \theta_u > t\} \in [0, \infty]$$

with the usual convention  $\inf\{\emptyset\} = \infty$ . The Lamperti time-change of the stopped process  $(R_t, t \geq 0)$  is the process  $(Z_t^{\min}, t \geq 0)$  defined by

$$Z_t^{\min} = \begin{cases} R_{C_t} & 0 \leq t < \theta_\infty \\ 0 & t \geq \theta_\infty \text{ and } \sigma_0 < \infty \\ \infty & t \geq \theta_\infty \text{ and } \sigma_0 = \infty. \end{cases}$$

This process is solution to  $(\text{MP})_Z$ , see [Fou19, Lemma 4.1], and is absorbed whenever it reaches  $\infty$ . We shall refer to it as the minimal LCSBP  $(\Psi, c)$ . This is not always the only solution. We will see in the next section solutions with boundary  $\infty$  non-absorbing.

**2.2. Boundary behaviors of CSBPs and LCSBPs.** When there is no competition, i.e.  $c = 0$ , the construction above is known as the Lamperti's transformation for CSBPs. The process  $(Z_t^{\min}, t \geq 0)$  is in this case a CSBP  $(\Psi)$ , see e.g. [Kyp14, Theorem 12.2]. Call it  $(Y_t, t \geq 0)$ . It is known that the semigroup of  $(Y_t, t \geq 0)$  satisfies the identity

$$(2.5) \quad \mathbb{E}_z[e^{-xY_t}] = e^{-zu_t(x)},$$

with  $(u_t(x), t \geq 0)$  the unique solution to

$$\frac{d}{dt} u_t(x) = -\Psi(u_t(x)) \text{ with } u_0(x) = x.$$

The map  $(u_t(x), t \geq 0)$  can not hit the boundaries 0 and  $\infty$  and therefore the boundaries  $\infty$  and 0 of  $(Y_t, t \geq 0)$  are absorbing. However  $(u_t(x), t \geq 0)$  may be started from  $x = 0$  or  $x = \infty$  in certain cases. If 0 (respectively  $\infty$ ) is an entrance for the map  $(u_t, t \geq 0)$ , i.e.  $u_t(0+) > 0$  for  $t > 0$ , (respectively  $u_t(\infty) < \infty$  for  $t > 0$ ) then the CSBP  $(Y_t, t \geq 0)$  will reach  $\infty$  (respectively 0) with positive probability. The condition for explosion and extinction (i.e. accessibility of  $\infty$  and 0) of  $Y$  are thus the integral tests

$$\int_0^\infty \frac{dx}{-\Psi(x)} < \infty \text{ (Dynkin's condition) and } \int_0^\infty \frac{dx}{\Psi(x)} < \infty \text{ (Grey's condition).}$$

When there is competition, i.e.  $c > 0$ , the boundary behaviors are richer. Necessary and sufficient conditions for the minimal LCSBP  $(Z_t^{\min}, t \geq 0)$  to explode or get extinct have been found in [Fou19, Theorem 3.1]. The striking difference with CSBPs lies in the fact that in most cases when boundary  $\infty$  is accessible, one will be able to restart the LCSBP continuously from it. In a more rigorous fashion, extension of the minimal process may exist with different boundary conditions at  $\infty$ , where by extension we mean a process  $Z$ , which once stopped at its first explosion time  $\zeta_\infty := \inf\{t > 0 : Z_t = \infty\}$ , has the same law as  $(Z_t^{\min}, t \geq 0)$ .

We briefly recall the results of [Fou19, Section 3]. Let  $x_0 > 0$  be an arbitrary constant and set

$$\mathcal{E} := \int_0^{x_0} \frac{dx}{x} \exp\left(\frac{2}{c} \int_x^{x_0} \frac{\Psi(u)}{u} du\right).$$

**Theorem 2.1** (Theorems 3.3 and 3.4 in [Fou19]). *There exists a Feller<sup>1</sup> process  $(Z_t, t \geq 0)$  with no negative jumps, extending the minimal process, such that for any  $x, z \in (0, \infty]$  and  $t \geq 0$ ,*

$$(2.6) \quad \mathbb{E}_z[e^{-xZ_t}] = \mathbb{E}_x[e^{-zU_t}],$$

where  $(U_t, t \geq 0)$  is a diffusion

$$(2.7) \quad dU_t = \sqrt{cU_t}dB_t - \Psi(U_t)dt$$

with  $(B_t, t \geq 0)$  a Brownian motion and with boundary conditions given as follows:

<i>Integral condition</i>	<i>Boundary of <math>U</math></i>	<i>Boundary of <math>Z</math></i>
$\mathcal{E} = \infty$	0 exit	$\infty$ entrance
$\mathcal{E} < \infty$ & $2\lambda/c < 1$	0 regular absorbing	$\infty$ regular reflecting
$2\lambda/c \geq 1$	0 entrance	$\infty$ exit
$\int_0^\infty \frac{dx}{\Psi(x)} = \infty$	$\infty$ natural	0 natural
$\int_0^\infty \frac{dx}{\Psi(x)} < \infty$	$\infty$ entrance	0 exit

TABLE 1. Boundaries of  $U, Z$ .

*Remark 2.2.* Note that the extinction occurs in finite time if and only if Grey's condition holds, so that competition has no impact on the possibility to get extinct in finite time.

From now on we shall mainly consider the extended Feller process  $(Z_t, t \geq 0)$  and call it simply *the* LCSBP $(\Psi, c)$ . Moreover, we stress that when the boundary  $\infty$  of  $Z$  is regular reflecting,  $\infty$  is also instantaneous and regular for itself, see [Fou19, Proposition 7.9], this entails that the process has a non-degenerate local time at  $\infty$ . We refer the reader to Bertoin [Ber96, Chapter 4, Section 2] for a general construction. The process  $(Z_t, t \geq 0)$  with boundary  $\infty$  regular reflecting has been however constructed in [Fou19, Section 7] as limit of LCSBPs whose boundaries  $\infty$  are all of entrance type. In particular, the construction did not give any information on the excursions away from infinity. The duality relationship (2.6) yields actually the probability entrance law of the process started from  $\infty$  and the fact that  $\infty$  is regular reflecting. Indeed since 0 is regular absorbing for  $U$ , by letting  $z$  go to  $\infty$  for fixed  $x$ , and  $x$  go to 0 for fixed  $z$  in (2.6), we see that for all  $t \geq 0$ ,

$$\mathbb{E}_\infty[e^{-xZ_t}] = \mathbb{P}_x(\tau_0 \leq t) > 0 \text{ and } \mathbb{P}_z(Z_t < \infty) = \mathbb{E}_{0^+}[e^{-zU_t}] = 1,$$

with  $\tau_0 := \inf\{t \geq 0 : U_t = 0\}$ .

<sup>1</sup>Here Feller means that the semigroup maps continuous bounded functions on  $[0, \infty]$  into themselves

What happens in the process past explosion is therefore entirely encoded in the law of the first hitting time of 0 of  $U$ . Lastly, we stress that no duality relationship for the minimal LCSBP  $(Z_t^{\min}, t \geq 0)$  was established in [Fou19] when  $\mathcal{E} < \infty$ . This will be part of the main results.

### 3. MAIN RESULTS

Let  $(U_t, t \geq 0)$  be the diffusion solution to (2.7) with boundary 0 either exit, regular absorbing or entrance according to the behavior at  $\infty$  of  $Z$ . Its generator  $\mathcal{A}$  is defined on  $C^2$  as follows:

$$(3.1) \quad \mathcal{A}g(x) = \frac{c}{2}xg''(x) - \Psi(x)g'(x).$$

As explained in the introduction, we will use a second duality relationship: for any  $x, y \in (0, \infty)$  and  $t \geq 0$ ,

$$(3.2) \quad \mathbb{P}_x(U_t < y) = \mathbb{P}_y(x < V_t),$$

where the process  $(V_t, t \geq 0)$  is the so-called Siegmund dual diffusion of  $U$ . We first state a proposition identifying the process  $V$  and specify the correspondences between boundaries of the three processes  $U$ ,  $V$  and  $Z$ . This is a direct application of a general statement for diffusions, established in Section 6, see Theorem 6.1.

**Proposition 3.1.** *The Siegmund dual of  $(U_t, t \geq 0)$  is the diffusion  $(V_t, t \geq 0)$  solution to an SDE of the form*

$$(3.3) \quad dV_t = \sqrt{cV_t}dB_t + (c/2 + \Psi(V_t))dt, \quad V_0 = y \in (0, \infty),$$

where  $(B_t, t \geq 0)$  is a Brownian motion<sup>2</sup> and whose boundary condition at 0 and  $\infty$  are given in correspondence with that of  $U$ , in the following way:

Integral condition	Boundary of $U$	Boundary of $V$
$\mathcal{E} = \infty$	0 exit	0 entrance
$\mathcal{E} < \infty$ & $2\lambda/c < 1$	0 regular absorbing	0 regular reflecting
$2\lambda/c \geq 1$	0 entrance	0 exit
$\int_0^\infty \frac{dx}{\Psi(x)} = \infty$	$\infty$ natural	$\infty$ natural
$\int_0^\infty \frac{dx}{\Psi(x)} < \infty$	$\infty$ entrance	$\infty$ exit

TABLE 2. Boundaries of  $U, V$ .

Gathering the correspondences displayed in Tables 1 and 2, we obtain the following ones between  $V$  and  $Z$ . Notice that the boundaries 0 and  $\infty$  are exchanged but the behaviors of the processes are not anymore.

Boundary of $V$	Boundary of $Z$
0 entrance	$\infty$ entrance
0 regular reflecting	$\infty$ regular reflecting
0 exit	$\infty$ exit
$\infty$ natural	0 natural
$\infty$ exit	0 exit

TABLE 3. Boundaries of  $V, Z$ .

<sup>2</sup>We stress that the driving Brownian motions, all denoted by  $B$ , are not supposed to be the same in the stochastic equations.

Denote by  $T_y$  the first hitting time of  $y \in [0, \infty]$  of the diffusion  $(V_t, t \geq 0)$  and set  $\mathcal{G}$  its generator:

$$(3.4) \quad \mathcal{G}f(x) := \frac{c}{2}xf''(x) + \left(\frac{c}{2} + \Psi(x)\right)f'(x).$$

Then, from the general theory of one-dimensional diffusions, see e.g. Breiman [Bre92, Theorem 16.69] and Mandl [Man68], the Laplace transform of  $T_y$  is expressed, for any  $\theta > 0$ , as

$$(3.5) \quad \mathbb{E}_x[e^{-\theta T_y}] = \begin{cases} \frac{h_\theta^+(x)}{h_\theta^+(y)}, & x \leq y \\ \frac{h_\theta^-(x)}{h_\theta^-(y)}, & x \geq y, \end{cases}$$

and functions  $h_\theta^-$  and  $h_\theta^+$  are  $C^2$  and respectively decreasing and increasing solutions to the equation

$$(3.6) \quad \mathcal{G}h(x) := \frac{c}{2}xh''(x) + \left(\frac{c}{2} + \Psi(x)\right)h'(x) = \theta h(x), \text{ for all } x \in (0, \infty).$$

For any  $z \in (0, \infty)$ , we denote by  $e_z$  an exponential random variable independent of  $V$  with parameter  $z$ , and by  $T_y^{e_z}$  the first hitting time of point  $y$  by the diffusion  $V$  started from  $e_z$ .

**Theorem 3.2** (Laplace transform of the extinction time of LCSBPs). *For any  $0 < z < \infty$  and  $\theta > 0$ ,*

$$(3.7) \quad \mathbb{E}_z[e^{-\theta \zeta_0}] = \int_0^\infty ze^{-zx} \frac{h_\theta^+(x)}{h_\theta^+(\infty)} dx = \mathbb{E}[e^{-\theta T_\infty^{e_z}}] \in [0, \infty)$$

*In particular, if  $\infty$  is not absorbing for  $Z$  (i.e. if  $2\lambda/c < 1$ ) then  $\mathbb{E}_\infty[e^{-\theta \zeta_0}] = \mathbb{E}_0[e^{-\theta T_\infty}] \in (0, \infty)$ .*

*In addition, if there is no explosion (i.e.  $\mathcal{E} = \infty$ ), then*

$$(3.8) \quad \mathbb{E}_z(\zeta_0) = \int_0^\infty dx \frac{2}{cx} e^{-Q(x)} \int_0^x (1 - e^{-z\eta}) e^{Q(\eta)} d\eta < \infty,$$

*with  $Q(x) := \int_{x_0}^x \frac{2\Psi(u)}{cu} du$  and  $x_0 > 0$ .*

*Remark 3.3.* If analytically the study of the Laplace transform of  $\zeta_0$  lies into that of the second order differential equation (3.6), more probabilistically the identity (3.7) ensures that the time of extinction of the LCSBP started from  $z$  has the same law as the time of explosion of the diffusion  $V$  started from an independent exponential variable with parameter  $z$ . The problem of studying  $\zeta_0$  is thus transferred into the study of  $T_\infty$ , see e.g. Karatzas and Ruf [KR16] for a recent account about explosion times of diffusions.

*Remark 3.4.* Extinction of LCSBPs has been studied in [Lam05] under a log-moment assumption on the Lévy measure  $\pi$ . Lambert has found, amongst other things, a representation of the Laplace transform of the extinction time in terms of the implicit solution of a certain non-homogeneous Riccati equation, see [Lam05, Theorem 3.9]. Note that (3.8) agrees with Equation (9) in [Lam05, Theorem 3.9], where the parameter of competition is  $c$  instead of our  $c/2$ . Note also that  $Q$  is not necessarily finite at  $x = 0$ , it is finite if and only if the log-moment assumption  $(L) : \int^\infty \log(h)\pi(dh) < \infty$  holds.

In the next theorem we study the first explosion time of the LCSBP.

**Theorem 3.5** (Laplace transform of the first explosion time of LCSBPs).

$$(3.9) \quad \mathbb{E}_z[e^{-\theta \zeta_\infty}] = \int_0^\infty ze^{-zx} \frac{h_\theta^-(x)}{h_\theta^-(0)} dx = \mathbb{E}[e^{-\theta T_0^{e_z}}] \in [0, \infty).$$



*Remark 3.6.* As previously, we see here that  $\zeta_\infty$  under  $\mathbb{P}_z$ , has the same law as the first time of extinction (i.e. of hitting 0) of  $V$  started from an independent exponential random variable with parameter  $z$ . In particular,  $\infty$  is accessible for  $Z$  if and only if 0 is accessible for  $V$ . The condition  $\mathcal{E} < \infty$  turns out to be Feller's test for accessibility of 0 for  $V$  (that simplifies, since  $\infty$  cannot be natural). This yields also a proof for explosion of the LCSBP based on a duality argument.

*Remark 3.7.* One may wonder how Theorem 3.2 and Theorem 3.5 work in the setting of the case without competition  $c = 0$ . This is explained in Section 4.

We establish now a Laplace duality relationship for the minimal process  $Z^{\min}$ . We focus on the case  $\mathcal{E} < \infty$ , as otherwise the minimal process does not hit its boundary  $\infty$ . Moreover when  $2\lambda/c \geq 1$  since the process  $Z$  has its boundary  $\infty$  exit, it coincides with the minimal process. Only the case  $\mathcal{E} < \infty$  &  $2\lambda/c < 1$  has to be handled. In this setting the minimal process  $(Z_t^{\min}, t \geq 0)$  can be seen as the logistic CSBP with  $\infty$  regular absorbing (i.e. stopped when it hits  $\infty$ ).

**Theorem 3.8.** *Assume  $\mathcal{E} < \infty$  &  $2\lambda/c < 1$ . For any  $z \in [0, \infty]$ ,  $x \in [0, \infty]$  and  $t \geq 0$*

$$(3.10) \quad \mathbb{E}_z[e^{-xZ_t^{\min}}] = \mathbb{E}_x[e^{-zU_t^r}],$$

*with  $(U_t^r, t \geq 0)$  the diffusion solution to (2.7) with boundary 0 regular reflecting. In particular, for all  $z \in (0, \infty)$  and  $t \geq 0$ ,*

$$(3.11) \quad \mathbb{P}_z(\zeta_\infty > t) = \mathbb{E}_0[e^{-zU_t^r}].$$

*Remark 3.9.* By taking the limit as  $z$  go to  $\infty$  in (3.11), we get  $\mathbb{P}_\infty(\zeta_\infty > t) = \mathbb{P}_0(U_t^r = 0) = 0$ , since 0 is regular reflecting for  $(U_t^r, t \geq 0)$ , we recover here the fact that  $\infty$  is regular for itself.

Theorem 3.8 completes the classification of boundaries by adding to Table 1 the following line of correspondences (unaddressed in [Fou19]):

Integral condition	Boundary of $U$	Boundary of $Z$
$\mathcal{E} < \infty$ & $2\lambda/c < 1$	0 regular reflecting	$\infty$ regular absorbing

TABLE 4.

We identify now the inverse local time at  $\infty$  of the LCSBP with boundary  $\infty$  regular reflecting. Denote by  $(L_t^Z, t \geq 0)$  the local time at  $\infty$  of  $Z$  and by  $(\tau_x^Z, 0 \leq x < \xi)$  its right-continuous inverse, namely for any  $x \geq 0$ ,  $\tau_x^Z := \inf\{t \geq 0 : L_t^Z > x\}$  and  $\xi := \inf\{x \geq 0 : \tau_x^Z = \infty\} \in (0, \infty]$ . One has, see e.g. [Ber96, Theorem 4-(iii)],

$$\mathcal{I} := \overline{\{t \geq 0 : Z_t = \infty\}} = \overline{\{\tau_x^Z, 0 \leq x < \xi\}} \text{ a.s.}$$

Note that since  $\infty$  is regular reflecting, the subordinator  $\tau^Z$  has no drift. Recall also from Proposition 3.1 that 0 is regular reflecting for the bidual process  $V$  and call  $(L_t^V, t \geq 0)$  its local time at 0.

**Theorem 3.10.** *Assume  $\infty$  regular reflecting ( $\mathcal{E} < \infty$  &  $2\lambda/c < 1$ ),  $(L_t^Z, t \geq 0)$  has the same law as  $(L_t^V, t \geq 0)$  and the Laplace exponent of the inverse local time subordinator  $(\tau_x^Z, 0 \leq x < \xi)$  is  $\kappa_Z : \theta \mapsto 1/h_\theta^-(0)$ .*

*In addition,*

$$\kappa_Z(0) = 1/S_Z(0),$$

*with  $S_Z(0) := \int_0^\infty \frac{1}{c} \frac{dx}{x} e^{-\int_{x_0}^x \frac{2\Psi(y)}{cy} dy} \in (0, \infty]$ , and  $\kappa_Z(0) > 0$  (and  $\mathcal{I}$  is bounded a.s.) if and only if  $-\Psi$  is not the Laplace exponent of a subordinator, i.e.  $\Psi$  is positive in a neighbourhood of  $\infty$ .*



*Remark 3.11.* By changing the value  $x_0$  in  $S_Z(0)$ , one only multiplies it by a constant. It thus causes a deterministic linear time change of the subordinator, which does not change its range. When  $\kappa_Z(0) > 0$  (i.e. when  $-\Psi$  is not the Laplace exponent of a subordinator), the process makes an infinite excursion away from infinity. According to [Fou19, Lemma 7.7], the process converges towards 0 a.s. in its infinite excursion (and is absorbed if and only if Grey's condition holds).

Theoretically, numerous properties of local times of diffusions can be applied to the study of  $\kappa_Z$  in order for instance to represent the Lévy measure of  $\tau_Z$  or its density, see e.g. Borodin and Salminen [BS02, Chapter II, Section 4]. No explicit formula can be hoped for a general branching mechanism  $\Psi$ . However we can identify the packing and Hausdorff dimensions of  $\mathcal{I}$ .

**Corollary 3.12.** *Assume  $\mathcal{E} < \infty$  &  $\frac{2\lambda}{c} < 1$ ,*

$$\dim_P(\mathcal{I}) = \dim_H(\mathcal{I}) = 2\lambda/c \in [0, 1) \text{ a.s.}$$

*Remark 3.13.* The dimension is zero for all branching mechanism  $\Psi$  such that  $\Psi(0) = -\lambda = 0$ . The equality of the packing and Hausdorff dimensions ensures that the Laplace exponent  $\kappa_Z$  has the same lower and upper Blumenthal-Gettoor's indices, see Bertoin [Ber99, Page 41].

*Example 3.14.* (1) A specific example is given by the case  $\Psi \equiv -\lambda$  with  $\lambda > 0$ . The LCSBP  $Z$  is degenerated into a process<sup>3</sup> which decays along the deterministic drift  $-\frac{c}{2}Z_t^2 dt$  when lying in  $(0, \infty)$  and jumps from any  $z \in (0, \infty)$  to  $\infty$  at rate  $\lambda z$ . According to Theorem 2.1, if  $2\lambda/c \geq 1$  then the boundary  $\infty$  of  $Z$  is an exit and if  $2\lambda/c < 1$ , it is a regular reflecting boundary. In this setting, the diffusion  $V$  is solution to the SDE

$$dV_t = \sqrt{cV_t}dB_t + (c/2 - \lambda)dt.$$

Therefore,  $V$  is a squared Bessel diffusion with non-negative dimension, or equivalently a CSBP with immigration (CBI) with mechanisms  $(\psi, \phi)$  where  $\psi(q) = \frac{c}{2}q^2$  and  $\phi(q) = (c/2 - \lambda)q$ . According for instance to Foucart and Uribe Bravo [FUB14, Proposition 13], the inverse local time at 0 of  $V$  is a stable subordinator with index  $2\lambda/c$ : for all  $\theta \geq 0$ ,  $\kappa_V(\theta) = \theta^{\frac{2\lambda}{c}}$ . By Theorem 3.10, the inverse local time of  $Z$  at  $\infty$  is also stable with the same index, and the Hausdorff dimension of  $\mathcal{I}$  is  $2\lambda/c \in (0, 1)$ .

- (2) A simple example of LCSBP with  $\infty$  reflecting which gets extinct almost surely is the LCSBP with  $\Psi(x) = -\lambda + (\alpha - 1)x^\alpha$  for all  $x \geq 0$ , with  $d > 0$ ,  $\alpha \in (1, 2]$ . In this case the branching part of the process behaves as a critical stable one before the first jump to  $\infty$ . When  $0 < 2\lambda/c < 1$ , the process may visit  $\infty$  but  $\kappa_Z(0) > 0$  and the process gets extinct almost-surely in finite time. The bidual process is the diffusion reflected at 0 (if  $0 < 2\lambda/c < 1$ ) solution to

$$dV_t = \sqrt{cV_t}dB_t + (c/2 - \lambda + (\alpha - 1)V_t^\alpha)dt.$$

- (3) Examples of LCSBPs with  $\infty$  regular reflecting and  $\lambda = 0$  are provided by certain branching mechanisms with slowly varying property at 0, see [Fou19, Example 3.14]. For instance if  $\pi_{|(e, \infty)}(du) = \frac{\alpha}{u(\log u)^2}du$  and  $2\alpha/c < 1$  then the Tauberian and monotone density theorems, see e.g. Bingham et al. [BGT87, Theorem 1.7 and 1.7.2], give  $\Psi(x) \underset{x \rightarrow 0+}{\sim} -\alpha/\log(1/x)$ ,  $\mathcal{E} < \infty$  and by Corollary 3.12,  $\dim_H(\mathcal{I}) = 0$  a.s.

The processes  $Z$  and  $V$  will also share their long-term regime when they are not absorbed. The relationship (1.2) clearly entails that if one process is positive recurrent, so is the other. When  $-\Psi$  is the Laplace exponent of a subordinator, the LCSBP can be positive recurrent or null recurrent, see [Fou19, Theorem 3.7] for necessary and sufficient conditions. The LCSBP in Example 3.14-(1) for instance is null recurrent. We provide more details in the next theorem.

<sup>3</sup>This example is in fact a disguised diffusion; since one can interpret the jump to infinity as a killing term and again use Feller's classification, see Borodin and Salminen [BS02, Chapter 2, Section 6].

By Itô's theory of excursions, since  $Z$  and  $V$  are Feller processes with boundary  $\infty$  and 0 regular reflecting, their trajectories can be decomposed into excursions out from their boundary  $\infty$  and 0 respectively, see [Ber96, Chapter 4, Section 4]. The process  $(e_t, t \leq L_\infty^Z)$  defined by setting for all  $t > 0$ ,

$$e_t = \left( Z_{s+\tau_{t-}^Z}, s \leq \tau_t^Z - \tau_{t-}^Z \right) \text{ if } \tau_t^Z - \tau_{t-}^Z > 0 \text{ and } e_t = \partial \text{ an isolated point, otherwise,}$$

is a Poisson point process on the set of càdlàg excursions out from  $\infty$ , stopped at the first infinite excursion, with for  $\sigma$ -finite intensity measure the excursion measure  $n_Z$ . We denote an excursion of  $Z$  by  $\epsilon : (\epsilon(t), t \leq \zeta)$  with  $\zeta$  its length. Similarly, the diffusion  $V$  being instantaneously reflected at 0, has an excursion measure  $n_V$  on the set of continuous excursions out of 0. We shall denote an excursion of  $V$  by  $\omega : (\omega(t), t \leq \ell)$ , with  $\ell$  its length. Both boundary  $\infty$  and 0 being instantaneous and regular for themselves the excursion measures  $n_Z$  and  $n_V$  are infinite.

The next two results are initiating the study of the excursion measure of  $Z$ . The first states a duality relationship "inside" the excursion measures of  $Z$  and  $V$ , the second provides some information about the law of the infimum of an excursion under  $n_Z$  for LCSBPs that get extinct.

**Theorem 3.15.** *Assume  $\infty$  regular reflecting ( $\mathcal{E} < \infty$  &  $2\lambda/c < 1$ ). One has the following duality relationship: for any  $x \in [0, \infty)$  and  $q > 0$ ,*

$$(3.12) \quad n_Z \left( \int_0^\zeta e^{-qu} e^{-x\epsilon(u)} du \right) = n_V \left( \int_0^\ell e^{-qu} \mathbb{1}_{(x, \infty)}(\omega(u)) du \right).$$

Moreover,

$$(3.13) \quad n_Z \left( \int_0^\zeta e^{-x\epsilon(u)} du \right) = \int_x^\infty e^{\int_{x_0}^y \frac{2\Psi(u)}{cu} du} dy \in (0, \infty].$$

The integral at the right hand side is finite for some  $x > 0$  if and only if  $-\Psi$  is the Laplace exponent of a subordinator and at least one of the following holds

$$(3.14) \quad \lim_{u \rightarrow \infty} \frac{\Psi(u)}{u} := -\delta < 0, \pi((0, 1)) = \infty, \bar{\pi}(0) + \lambda > \frac{c}{2}.$$

where  $\delta$  is the drift of  $-\Psi$  and  $\bar{\pi}(0)$  the total mass of the Lévy measure. In this case, the first moment of the Lévy measure of  $\tau_Z$  is finite and satisfies

$$n_Z(\zeta) = \int_0^\infty e^{\int_{x_0}^y \frac{2\Psi(u)}{cu} du} dy < \infty.$$

*Remark 3.16.* The process  $Z$  is positive recurrent if and only if  $n_Z(\zeta) < \infty$ , see the end of [Ber99, Chapter 2]. The conditions for  $n_Z(\zeta) < \infty$  match therefore with those for positive recurrence found in [Fou19, Theorem 3.7]. See also Remark 3.8 in there. Moreover in case of  $n_Z(\zeta) < \infty$ , if one renormalises (3.13) by  $n_Z(\zeta)$ , we recover the Laplace transform of the stationary distribution of  $Z$ . This is a consequence of a general result representing the stationary distribution through the excursion measure, see Dellacherie et al. [DMM92, Chapter XIX.46].

**Theorem 3.17.** *Assume  $\infty$  regular reflecting ( $\mathcal{E} < \infty$  &  $2\lambda/c < 1$ ) and that  $-\Psi$  is not the Laplace exponent of a subordinator. Denote by  $I$  the infimum of an excursion of  $Z$ . Its law under  $n_Z$  is given by*

$$n_Z(I \leq a) = 1/S_Z(a),$$

with  $S_Z(a) := \int_0^\infty \frac{1}{c} \frac{dx}{x} e^{-ax} e^{-\int_{x_0}^x \frac{2\Psi(u)}{cu} du}$  for all  $a \geq 0$ .

4. A REMARK ON THE CASE  $c = 0$ 

We make a remark on Theorem 3.2 and Theorem 3.5 in the case without competition. Recall that  $\infty$  in this case is absorbing when accessible. The Laplace transforms of the times of extinction and explosion are already easily accessible from the branching property, but we will see how to understand the role of Siegmund duality in the case  $c = 0$ . When there is no competition, the Laplace dual process of the CSBP  $(Y_t, t \geq 0)$  is the deterministic map  $(u_t(x), t \geq 0)$  solution to  $du_t(x) = -\Psi(u_t(x))dt$ ,  $u_0 = x$ . Moreover, since 0 and  $\infty$  are both absorbing for  $Y$ , by letting  $x$  go to  $\infty$  and to 0 in (2.5), we get

$$\mathbb{P}_z(\zeta_0^Y \leq t) = e^{-zu_t(\infty)} \text{ and } \mathbb{P}_z(\zeta_\infty^Y > t) = e^{-zu_t(0+)},$$

where we denote by  $\zeta_0^Y$  and  $\zeta_\infty^Y$  the extinction and explosion time of  $Y$ . Let  $\rho$  be the largest root of  $\Psi$ ,  $\rho := \sup\{x > 0 : \Psi(x) \leq 0\} \geq 0$ . We look now for expressions of the Laplace transforms of the extinction and explosion times. By the change of variable  $x = u_t(\infty)$ , using the fact that  $t = \int_x^\infty \frac{du}{\Psi(u)}$ , and performing an integration by parts, we see that for any  $z \in (0, \infty)$  and  $\theta > 0$

$$(4.1) \quad \mathbb{E}_z[e^{-\theta\zeta_0^Y}] = \mathbb{P}_z(\zeta_0^Y \leq e_\theta) = \int_0^\infty \theta e^{-\theta t} e^{-zu_t(\infty)} dt = \int_\rho^\infty z e^{-xz - \theta \int_x^\infty \frac{du}{\Psi(u)}} dx,$$

and similarly with  $x = u_t(0+)$ , using that  $t = \int_0^x \frac{du}{-\Psi(u)}$ , we get

$$(4.2) \quad \mathbb{E}_z[e^{-\theta\zeta_\infty^Y}] = \mathbb{P}_z(\zeta_\infty^Y \leq e_\theta) = \int_0^\infty \theta e^{-\theta t} (1 - e^{-zu_t(0+)}) dt = \int_0^\rho z e^{-xz - \theta \int_0^x \frac{du}{-\Psi(u)}} dx.$$

To understand the link with what we did in the case with competition, note that the Siegmund dual of  $(u_t(x), t \geq 0)$  is nothing but its inverse flow, namely

$$v_t(y) := \inf\{z \geq 0 : u_t(z) > y\},$$

which is solution to the equation

$$dv_t(y) = \Psi(v_t(y))dt, v_0 = y.$$

Equation (3.6) with  $c = 0$  becomes a first-order differential equation, with a singularity at  $\rho$  when  $\rho \in (0, \infty)$ , for which we can find the explicit form of the solutions. For any fixed value  $x_0$  in  $(\rho, \infty)$ , the increasing solution on  $(x_0, \infty)$  is of the form  $h_\theta^+(x) = e^{\theta \int_{x_0}^x \frac{du}{\Psi(u)}}$  for any  $x > x_0$ . Similarly the decreasing solution on any interval  $(0, x_1)$  with  $x_1 < \rho$  is of the form  $h_\theta^-(x) = e^{\theta \int_0^x \frac{du}{\Psi(u)}}$  for any  $x < x_1$ .

By considering the solutions  $h_\theta^-$  and  $h_\theta^+$  on their maximal interval, we recover the expressions

$$\mathbb{E}_z[e^{-\theta\zeta_0^Y}] = z \int_\rho^\infty e^{-xz} \frac{h_\theta^+(x)}{h_\theta^+(\infty)} dx \text{ and } \mathbb{E}_z[e^{-\theta\zeta_\infty^Y}] = z \int_0^\rho e^{-xz} \frac{h_\theta^-(x)}{h_\theta^-(0)} dx.$$

Note that  $h_\theta^+(\infty) < \infty$  if and only if  $\int_\rho^\infty \frac{dx}{\Psi(x)} < \infty$  (Grey's condition for extinction) and  $h_\theta^-(0) < \infty$  if and only if  $\int_0^\rho \frac{dx}{-\Psi(x)} < \infty$  (Dynkin's condition for extinction).

When  $\rho = 0$  (i.e.  $\Psi'(0+) \geq 0$ ) or  $\rho = \infty$  (i.e.  $\Psi'(\infty) := \lim_{x \rightarrow \infty} \frac{\Psi(x)}{x} \leq 0$ ), one can reinterpret (4.1) and (4.2) in terms of the identities in law

$$\zeta_0^Y \stackrel{\text{law}}{=} t_\infty^{e_z}, \text{ if } \rho = 0 \text{ and } \zeta_\infty^Y \stackrel{\text{law}}{=} t_0^{e_z} \text{ if } \rho = \infty,$$

where

$$t_\infty^y = \int_y^\infty \frac{du}{\Psi(u)} = \inf\{t > 0 : v_t(y) = \infty\} \text{ and } t_0^y = \int_0^y \frac{du}{-\Psi(u)} = \inf\{t > 0 : v_t(y) = 0\}.$$

## 5. PROOFS OF THE MAIN RESULTS

**5.1. Proof of Proposition 3.1.** This is a direct application of Theorem 6.1 in the next section. The study of Feller's conditions for the classification of the boundaries 0 and  $\infty$  of the generalized Feller diffusion  $U$  can be found in [Fou19, Lemma 5.2]. We stress that when  $U$  has boundary 0 regular absorbing,  $V$  has boundary 0 regular reflecting.  $\square$

We shall now exploit the two following dualities: for all  $x, y, z \in (0, \infty)$  and  $t \geq 0$ :

$$\mathbb{E}_z[e^{-xZ_t}] \stackrel{(2.6)}{=} \mathbb{E}_x[e^{-zU_t}] \text{ and } \mathbb{P}_x(U_t < y) \stackrel{(3.2)}{=} \mathbb{P}_y(x < V_t).$$

The diffusions  $U$  and  $V$  being regular on  $(0, \infty)$ , the laws of  $U_t$  and  $V_t$  have no atom in  $(0, \infty)$  when  $t > 0$  and (3.2) holds true with large inequalities. Recall the maps  $h_\theta^+$  and  $h_\theta^-$  and (3.5).

**5.2. Proof of Theorem 3.2.** For any  $q > 0$ , we denote by  $e_q$  an exponentially distributed random variable with parameter  $q$  independent of everything else. We first established (1.2), one has by Laplace duality (2.6) and then Siegmund duality (3.2), for any  $x, z \in (0, \infty)$ ,  $t \geq 0$ :

$$\mathbb{E}_z[e^{-xZ_t}] = \mathbb{E}_x[e^{-zU_t}] = \mathbb{P}_x(e_z > U_t) = \int_0^\infty ze^{-zy} \mathbb{P}_y(V_t > x) dy.$$

By letting  $x$  go to  $\infty$ , and recalling that  $\infty$  is an absorbing boundary for process  $V$ , we get

$$\mathbb{P}_z(\zeta_0 \leq t) = \lim_{x \rightarrow \infty} \mathbb{E}_z[e^{-xZ_t}] = \int_0^\infty ze^{-zy} \mathbb{P}_y(V_t = \infty) dy = \int_0^\infty ze^{-zy} \mathbb{P}_y(T_\infty \leq t) dy.$$

Hence for any  $\theta \in (0, \infty)$

$$\mathbb{E}_z[e^{-\theta\zeta_0}] = \mathbb{P}_z(\zeta_0 \leq e_\theta) = \int_0^\infty ze^{-zy} \mathbb{P}_y(T_\infty \leq e_\theta) dy = \int_0^\infty ze^{-zy} \mathbb{E}_y[e^{-\theta T_\infty}] dy.$$

The form in (3.7) is provided by the identity for diffusions (3.5):  $\mathbb{E}_y[e^{-\theta T_\infty}] = \frac{h_\theta^+(y)}{h_\theta^+(\infty)}$ . We now study  $\mathbb{E}_z(\zeta_0)$  under the assumptions  $\mathcal{E} = \infty$  and  $\int^\infty \frac{du}{\Psi(u)} < \infty$ . Note that this entails that 0 is non-attracting for  $V$  (from Table 2, 0 is actually an entrance) and  $\infty$  is an exit for  $V$ . We need to compute  $\mathbb{E}(T_\infty^{e_z})$ . The calculation is a bit cumbersome but follows from a general result of diffusions, see [KT81, Equation (6.6), page 227]. Let  $S_V$  be the scale function and  $M_V$  the speed measure of  $V$ :

$$(5.1) \quad S'_V(y) = s_V(y) = \frac{1}{cy} e^{-\int_{x_0}^y \frac{2\Psi(u)}{cu} du},$$

$$(5.2) \quad M'_V(y) = m_V(y) = e^{\int_{x_0}^y \frac{2\Psi(u)}{cu} du}.$$

For any  $a > 0$ ,

$$\mathbb{E}_x(T_a \wedge T_\infty) = 2 \frac{S_V[a, x]}{S_V[a, \infty]} \int_x^\infty S_V(\eta, \infty) dM_V(\eta) + 2 \frac{S_V[x, \infty]}{S_V[a, \infty]} \int_a^x S_V(a, \eta) dM_V(\eta),$$

and we see that by letting  $a$  go to 0,

$$\frac{S_V[a, x]}{S_V[a, \infty]} \xrightarrow{a \rightarrow 0} 1 \text{ and } \frac{S_V[x, \infty]}{S_V[a, \infty]} \xrightarrow{a \rightarrow 0} 0.$$

Thus

$$\mathbb{E}_\eta(T_\infty) = 2 \int_\eta^\infty S_V(v, \infty) m_V(v) dv = \int_\eta^\infty dv \int_v^\infty \frac{2}{cx} e^{-\int_v^x \frac{2\Psi(u)}{cu} du} dx.$$

We obtain

$$\mathbb{E}_z(\zeta_0) = \int ze^{-z\eta} \mathbb{E}_\eta(T_\infty) d\eta = \int ze^{-z\eta} d\eta \int_\eta^\infty dv \int_v^\infty \frac{2}{cx} e^{-\int_v^x \frac{2\Psi(u)}{cu} du} dx$$

$$\begin{aligned}
&= \int_0^\infty d\eta(1 - e^{-z\eta}) \int_\eta^\infty \frac{2}{cx} e^{-\int_\eta^x \frac{2\Psi(u)}{cu} du} dx \\
&= \int_0^\infty dx \frac{2}{cx} e^{-Q(x)} \int_0^x (1 - e^{-z\eta}) e^{Q(\eta)} d\eta,
\end{aligned}$$

where in the penultimate equality we have performed an integration by parts, and in the last equality we have set  $Q(x) := \int_{x_0}^x \frac{2\Psi(u)}{cu} du$  and applied Fubini-Tonelli's theorem.  $\square$

**5.3. Proof of Theorem 3.5.** We first treat the case  $\frac{2\lambda}{c} \geq 1$  for which  $\infty$  is an exit boundary of  $Z$  and the Laplace transform of  $\zeta_\infty$  can be computed along the same lines as that of  $\zeta_0$ .

**Lemma 5.1.** *Assume  $\frac{2\lambda}{c} \geq 1$ .*

$$\mathbb{E}_z[e^{-\theta\zeta_\infty}] = \int_{(0,\infty)} ze^{-zx} \frac{h_\theta^-(x)}{h_\theta^-(0)} dx = \mathbb{E}[e^{-\theta T_0^{xz}}] \in [0, \infty).$$

*Proof.* Since  $\infty$  is an exit boundary of  $Z$ , it is an entrance boundary for  $U$  and an exit one for  $V$ . We get by letting  $x$  go to 0 in (2.6) and (3.2):

$$\mathbb{P}_z(\zeta_\infty \leq t) = \mathbb{E}_{0+}[e^{-zU_t}] = \int_0^\infty ze^{-zy} \mathbb{P}_y(V_t = 0) dy,$$

which allows us to conclude.  $\square$

When boundary  $\infty$  is regular reflecting the proof below does not work as 0 is regular absorbing for  $U$  and one can not access to the law of the explosion time directly from the semigroup. The proof in the regular reflecting case will be based on two lemmas linking invariant functions of  $Z$  to invariant measures of  $U$  and invariant functions of  $V$ .

**Lemma 5.2** (Laplace duality and increasing  $\theta$ -invariant functions). *Recall the Laplace duality relationship (2.6). Let  $\theta > 0$  and  $\mu_\theta$  be a  $\theta$ -invariant measure for the process  $U$ , i.e.*

$$\mu_\theta P_t^U = e^{\theta t} \mu_\theta,$$

where we have denoted by  $P_t^U$  the semigroup of  $U$ . Then, provided that the following function  $f_\theta^+$  is well-defined

$$(5.3) \quad f_\theta^+(z) := \int_{[0,\infty]} (1 - e^{-xz}) \mu_\theta(dx),$$

and that boundary  $\infty$  is not absorbing for  $Z$ , it is an increasing  $\theta$ -invariant function of  $Z$ , i.e.

$$P_t^Z f_\theta^+ = e^{\theta t} f_\theta^+.$$

*Remark 5.3.* Lemma 5.2 can be seen as an analogue of Foucart and Möhle [FM20, Theorem 4.1] where instead of Laplace duality, Siegmund duality is studied.

*Proof.* By Fubini-Tonelli's theorem and the Laplace duality relationship (2.6),

$$\begin{aligned}
\mathbb{E}_z(f_\theta^+(Z_t)) &= \int_{[0,\infty]} \mathbb{E}_z(1 - e^{-xZ_t}) \mu_\theta(dx) = \int_{[0,\infty]} \mathbb{E}_x(1 - e^{-zU_t}) \mu_\theta(dx) \\
&= e^{\theta t} \int_{[0,\infty]} (1 - e^{-zx}) \mu_\theta(dx) = e^{\theta t} f_\theta^+(z).
\end{aligned}$$

$\square$

We state now a lemma linking invariant measures of  $U$  to invariant functions of  $V$ .

**Lemma 5.4** (Siegmund duality and  $\theta$ -invariant measures). *Recall the Siegmund duality (3.2). Denote by  $(P_t^V, t \geq 0)$  the semigroup of the process  $(V_t, t \geq 0)$ . Let  $\theta > 0$  and  $h_\theta^-$  be a positive decreasing  $\theta$ -invariant function for  $(P_t^V, t \geq 0)$ , i.e. for any  $t \geq 0$ ,*

$$P_t^V h_\theta^- = e^{\theta t} h_\theta^-.$$

*Then, the positive Stieltjes measure  $\mu_\theta$  defined on  $(0, \infty]$  by its tail  $\mu_\theta((x, \infty]) := h_\theta^-(x)$  for all  $x \in (0, \infty]$  is a  $\theta$ -invariant measure for  $(U_t, t \geq 0)$ .*

*Proof.* By Siegmund duality (3.2) and Fubini-Tonelli's theorem

$$\int_{[0, \infty]} \mathbb{P}_x(U_t > v) d\mu_\theta(x) = \int_{[0, \infty]} \mathbb{P}_v(x > V_t) d\mu_\theta(x) = \mathbb{E}_v(h_\theta^-(V_t)) = e^{\theta t} h_\theta^-(v) = e^{\theta t} \mu_\theta((v, \infty]),$$

thus  $\mu_\theta$  is a  $\theta$ -invariant measure for  $U$ . □

We now identify with the help of the two latter lemmas an increasing  $\theta$ -invariant function of  $Z$ , this will provide the Laplace transform of the first explosion time in the regular reflecting case and Theorem 3.5 will be established.

**Lemma 5.5.** *Assume  $\mathcal{E} < \infty$  &  $\frac{2\lambda}{c} < 1$ , and recall  $h_\theta^-$  in (3.5), then the following function*

$$f_\theta^+(z) := \int_{(0, \infty)} z e^{-xz} h_\theta^-(x) dx,$$

*is a well-defined bounded increasing  $\theta$ -invariant function of  $Z$  with  $f_\theta^+(\infty) = h_\theta^-(0) < \infty$ . Moreover, for any  $\theta > 0$  and  $z \in (0, \infty)$*

$$\mathbb{E}_z[e^{-\theta \zeta_\infty}] = \frac{f_\theta^+(z)}{f_\theta^+(\infty)}.$$

*Proof.* Recall the definition of  $f_\theta^+$  in (5.3). We recall that by convention  $0 \cdot \infty = 0$  hence  $e^{-0 \cdot \infty} = 1$ . However that  $\lim_{x \rightarrow 0^+} e^{-x \cdot z} = \mathbb{1}_{\{z < \infty\}}$ . Note that the function  $h_\theta^-$  satisfying (3.5) is differentiable on  $(0, \infty)$ . Consider the measure  $\mu_\theta$  on  $(0, \infty]$ ,  $\mu_\theta(dx) := -(h_\theta^-)'(x) dx + h_\theta^-(\infty) \delta_\infty$ . By integration by parts, for any  $z \in [0, \infty]$

$$\begin{aligned} f_\theta^+(z) &= \int_{(0, \infty]} (1 - e^{-xz}) \mu_\theta(dx) \\ &= (1 - e^{-\infty \cdot z}) \mu_\theta(\{\infty\}) + \int_{(0, \infty)} (1 - e^{-xz}) \mu_\theta(dx) \\ &= \mathbb{1}_{\{z > 0\}} \mu_\theta(\{\infty\}) + \left[ -(1 - e^{-xz}) h_\theta^-(x) \right]_{x=0^+}^{x=\infty} + z \int_{(0, \infty)} e^{-xz} h_\theta^-(x) dx \\ &= \mathbb{1}_{\{z > 0\}} h_\theta^-(\infty) + h_\theta^-(0+) \mathbb{1}_{\{z = \infty\}} - h_\theta^-(\infty) \mathbb{1}_{\{z > 0\}} + z \int_{(0, \infty)} e^{-xz} h_\theta^-(x) dx \\ &= h_\theta^-(0+) \mathbb{1}_{\{z = \infty\}} + z \int_{(0, \infty)} e^{-xz} h_\theta^-(x) dx. \end{aligned}$$

Notice that the second term converges towards  $h_\theta^-(0+)$  as  $z$  goes to  $\infty$ . In particular under the assumption  $\mathcal{E} < \infty$ , boundary 0 is accessible for  $V$  and  $h_\theta^-(0+) = h_\theta^-(0) < \infty$ . Hence  $f_\theta^+(\infty) = h_\theta^-(0+) < \infty$ , and  $f_\theta^+$  is bounded.

It remains to justify the Laplace transforms displayed in (3.9). Since  $P_t^Z f_\theta^+(z) = e^{\theta t} f_\theta^+(z)$ , the process  $(e^{-\theta t} f_\theta^+(Z_t), t \geq 0)$  is a martingale. By applying the optional stopping theorem at the bounded stopping time  $t \wedge \zeta_\infty$  and then by letting  $t$  go to  $\infty$ , we get by continuity of  $f_\theta^+$ :

$$f_\theta^+(\infty) \mathbb{E}_z(e^{-\theta \zeta_\infty}) = \lim_{t \rightarrow \infty} \mathbb{E}_z(f_\theta^+(Z_{t \wedge \zeta_\infty}) e^{-\theta t \wedge \zeta_\infty}) = f_\theta^+(z).$$



The formula of the Laplace transform of the first explosion time  $\zeta$  is then obtained and one can plainly rewrite it in terms of  $T_0^{e_z}$ , the first hitting time of 0 of  $V$  started from an independent exponential random variable with parameter  $z$ .  $\square$

*Remark 5.6.* The proof above is not adapted to the case of a boundary  $\infty$  exit, since  $f_\theta^+$  cannot be  $\theta$ -invariant when  $\infty$  is absorbing, indeed this would lead to the fact that  $P_t^Z f_\theta^+(\infty) = f_\theta^+(\infty) = e^{\theta t} f_\theta^+(\infty)$  which cannot hold true for  $\theta > 0$ .

**5.4. Proof of Theorem 3.8.** The most standard method for establishing this kind of duality result is perhaps to apply Ethier-Kurtz's results, see [EK86, Theorem 4.11, page 192], or to show that  $g : x \mapsto \mathbb{E}_z(e^{-xZ_t^{\min}})$  belongs to the domain of the generator of the diffusion  $(U_t^r, t \geq 0)$ , see Jansen and Kurt [JK14, Proposition 1.2]. Showing the conditions for applying those result does not seem to be an easy task since boundary behaviors come into play. We will show the duality relationship (3.10) through another route by introducing the Siegmund dual process of  $U^r$  that we call  $V^a$  ( $a$  and  $r$  are for absorbing and reflecting), see Theorem 6.1: for any  $x, y \in (0, \infty)$

$$(5.4) \quad \mathbb{P}_x(U_t^r < y) = \mathbb{P}_y(V_t^a > x).$$

Let  $e_z$  be an exponential random variable with parameter  $z$  independent of  $U^r$ . Note that

$$\mathbb{E}_x[e^{-zU_t^r}] = \mathbb{P}_x[e_z > U_t^r] = \int_0^\infty z e^{-zy} \mathbb{P}_y(V_t^a > x) dy,$$

where  $V^a$  is the diffusion with generator  $\mathcal{G}$  and 0 is regular absorbing.

Recall  $(Z_t, t \geq 0)$  the extension of  $(Z_t^{\min}, t \geq 0)$ . We introduce the resolvent of  $Z$ ,  $\mathcal{R}_Z^q$  defined on  $C_b([0, \infty])$  the space of continuous functions on  $[0, \infty]$ . An application of the strong Markov property at time  $\zeta_\infty$  yields

$$(5.5) \quad \begin{aligned} \mathcal{R}_Z^q f(z) &:= \mathbb{E}_z \left( \int_0^\infty e^{-qt} f(Z_t) dt \right) = \mathcal{R}_{Z^{\min}}^q f(z) + \mathbb{E}_z \left( \int_{\zeta_\infty}^\infty e^{-qt} f(Z_t) dt \right) \\ &= \mathcal{R}_{Z^{\min}}^q f(z) + \mathbb{E}_z[e^{-q\zeta_\infty}] \mathcal{R}_Z^q f(\infty), \end{aligned}$$

where  $\mathcal{R}_{Z^{\min}}^q f(z)$  is the resolvent of the minimal process  $Z^{\min}$ . Let  $e_x(z) = e_z(x) = e^{-xz}$ . By the dualities with the auxiliary processes  $U$  and  $V$ , for the extended process: if  $z < \infty$  then

$$(5.6) \quad \begin{aligned} \mathcal{R}_Z^q e_x(z) &:= \int_0^\infty e^{-qt} \mathbb{E}_z(e_x(Z_t)) dt \\ &= \int_0^\infty e^{-qt} \mathbb{E}_x(e_z(U_t)) dt && \text{(by the Laplace duality (2.6))} \\ &= \int_0^\infty e^{-qt} \mathbb{P}_x(e_z > U_t) dt \\ &= \int_0^\infty dy z e^{-yz} \int_0^\infty e^{-qt} \mathbb{P}_y(V_t > x) dt && \text{(by the Siegmund duality (3.2))} \\ &= \int_0^\infty dy z e^{-yz} \mathcal{V}^q \mathbb{1}_{(x, \infty)}(y), \end{aligned}$$

where  $(V_t, t \geq 0)$  is the Siegmund dual diffusion of  $(U_t, t \geq 0)$  which is reflected at 0 and  $\mathcal{V}^q$  is its resolvent. If now  $z = \infty$ , then for any  $x > 0$

$$(5.7) \quad \begin{aligned} \mathcal{R}_Z^q e_x(\infty) &= \mathbb{E}_\infty \left( \int_0^\infty e^{-qt} e_x(Z_t) dt \right) \\ &= \int_0^\infty e^{-qt} \mathbb{P}_x(U_t^a = 0) dt = \int_0^\infty e^{-qt} \mathbb{P}_x(\tau_0 \leq t) dt \\ &= \int_0^\infty e^{-qt} \mathbb{P}_0(V_t > x) dt = \mathcal{V}^q \mathbb{1}_{(x, \infty)}(0). \end{aligned}$$



Similarly as in (5.5), one has the decomposition

$$(5.8) \quad \mathcal{V}^q f(y) = \mathcal{V}_0^q f(z) + \mathbb{E}_y[e^{-qT_0}] \mathcal{V}^q f(0)$$

with  $\mathcal{V}_0^q$  the resolvent of the process  $(V_t^a, t \geq 0)$  the minimal process with generator  $\mathcal{G}$  (i.e. the process absorbed at the boundary 0). Moreover by Theorem 3.5,  $\mathbb{E}_z[e^{-q\zeta_\infty}] = \mathbb{E}[e^{-T_0^{e^z}}]$ , by (5.5), (5.6) and (5.7), we get:

$$\begin{aligned} \mathcal{R}_{Z^{\min}}^q e_x(z) &= \int_0^\infty dy z e^{-yz} \mathcal{V}^q \mathbb{1}_{(x,\infty)}(y) - \mathbb{E}[e^{-qT_0^{e^z}}] \mathcal{V}^q \mathbb{1}_{(x,\infty)}(0) \\ &= \int_0^\infty dy z e^{-yz} (\mathcal{V}_0^q \mathbb{1}_{(x,\infty)}(y) + \mathbb{E}_y[e^{-qT_0}] \mathcal{V}^q \mathbb{1}_{(x,\infty)}(0)) - \mathbb{E}[e^{-qT_0^{e^z}}] \mathcal{V}^q \mathbb{1}_{(x,\infty)}(0) \\ &= \int_0^\infty dy z e^{-yz} \mathcal{V}_0^q \mathbb{1}_{(x,\infty)}(y) \\ &= \int_0^\infty dy z e^{-yz} \int_0^\infty e^{-qt} \mathbb{P}_y(V_t^a > x) dt \\ &= \int_0^\infty dy z e^{-yz} \int_0^\infty e^{-qt} \mathbb{P}_x(y > U_t^r) dt \quad (\text{by the Siegmund duality (5.4)}) \\ &= \mathbb{E}_x \left( \int_0^\infty e^{-qt} e^{-zU_t^r} dt \right) = \mathcal{V}^q e_z(x). \end{aligned}$$

Since  $Z^{\min}$  and  $U^r$  are Feller processes, the maps  $t \mapsto \mathbb{E}_z[e^{-xZ_t^{\min}}]$  and  $t \mapsto \mathbb{E}_x[e^{-xU_t^r}]$  are continuous and by injectivity of the Laplace transform, we get the following Laplace duality: for any  $x, z \in (0, \infty)$  and  $t \geq 0$

$$(5.9) \quad \mathbb{E}_z[e^{-xZ_t^{\min}}] = \mathbb{E}_x[e^{-zU_t^r}].$$

□

**5.5. Proof of Theorem 3.10.** We identify the Laplace exponent of the inverse local time at  $\infty$  of  $Z$ . Assume  $\mathcal{E} < \infty$  &  $2\lambda/c < 1$  so that  $V$  and  $Z$  are reflected. We first establish that the inverse local time of  $Z$  is a subordinator with Laplace exponent  $\kappa_Z : \theta \mapsto \frac{1}{f_\theta^+(\infty)}$ . This will come from a general argument. Let  $(L_t^Z, t \geq 0)$  be the local time at  $\infty$  of  $Z$  reflected at  $\infty$ . Let  $u_\theta(z) := \mathbb{E}_z \left( \int_0^\infty e^{-\theta t} dL_t^Z \right)$ . Using that  $dL_t^Z$  has for support the times at which  $Z$  takes the value  $\infty$ , and the fact that  $(L_{t+\zeta_\infty}, t \geq 0)$  under  $\mathbb{P}_z$  has the same law as  $(L_t, t \geq 0)$  under  $\mathbb{P}_\infty$  by the strong Markov property at  $\zeta_\infty$ , we get

$$u_\theta(z) = \mathbb{E}_z \left( \int_{\zeta_\infty}^\infty e^{-\theta t} dL_t^Z \right) = \mathbb{E}_z[e^{-\theta\zeta_\infty}] u_\theta(\infty),$$

and

$$\mathbb{E}_z[e^{-\theta\zeta_\infty}] = \frac{u_\theta(z)}{u_\theta(\infty)}.$$

Since  $\mathbb{E}_z[e^{-\theta\zeta_\infty}] = \frac{f_\theta^+(z)}{f_\theta^+(\infty)}$  and  $z \mapsto u_\theta(z)$  is increasing, up to a multiplicative constant  $u_\theta(\infty) = f_\theta^+(\infty)$ . It only remains to notice that

$$u_\theta(\infty) = \mathbb{E}_\infty \left[ \int_0^\infty e^{-\theta t} dL_t^Z \right] = \mathbb{E}_\infty \left[ \int_0^\infty e^{-\theta\tau_x^Z} dx \right] = \frac{1}{\kappa_Z(\theta)}$$

where we have denoted by  $(\tau_x^Z, x \geq 0)$  the inverse of the local time  $(L_t^Z, t \geq 0)$ . Same arguments entail that the inverse local time of  $V$ ,  $(\tau_x^V, x \geq 0)$ , has Laplace exponent  $\kappa_V : \theta \mapsto \frac{1}{h_\theta^-(0)}$ . According to Lemma 5.5, for all  $\theta \geq 0$ ,  $f_\theta^+(\infty) = h_\theta^-(0) = 1/\kappa_Z(\theta)$  and  $(L_t^Z, t \geq 0)$  has the same law as  $(L_t^V, t \geq 0)$ .

We now study the killing term in  $\kappa_Z$ . Denote by  $n_V$  the excursion measure of  $V$  away from point 0. It is known that the supremum  $M := \sup_{t \leq \ell} \omega(t)$  of an excursion  $\omega$  of  $V$  has “law” under the excursion measure given by

$$n_V(M > x) = \frac{1}{S_V(x)} \text{ for any } x > 0,$$

where  $S_V$  is the scale function of  $V$ . We refer e.g. to Vallois et al. [SVY07, Theorem 5-(i)] and Pitman and Yor [PY96], see also Mallein and Yor [MY16, Exercice 13.6]. In particular, the killing term in the inverse local time of  $V$  is  $\kappa_V(0) = n_V(\ell = \infty)$  where  $\{\ell = \infty\}$  is the set of excursions with infinite lifetime, i.e those which do not hit 0. Necessarily those excursions have transient paths drifting towards  $\infty$ , (otherwise, since 0 is accessible from any point in  $(0, \infty)$ , the infinite excursion of  $V$  would ultimately hit 0). Since  $\kappa_Z = \kappa_V$ , we have

$$(5.10) \quad \kappa_Z(0) = n_V(\ell = \infty) = n_V(M = \infty) = \frac{1}{S_V(\infty)}.$$

Recalling the scale function of  $V$ , see (5.1), we obtain

$$S_V(\infty) = \frac{1}{c} \int_0^\infty \frac{dx}{x} e^{-\int_{x_0}^x \frac{2\Psi(y)}{cy} dy} =: S_Z(0).$$

It remains to see that that the condition  $\Psi(x) \geq 0$  for large enough  $x$  is necessary and sufficient for  $\kappa_Z(0) > 0$ . We first show that it is sufficient. Let  $x_1 > x_0$  be such that  $\Psi(x) \geq \Psi(x_1) \geq 0$  for all  $x \geq x_1$ . The convexity of  $\Psi$  and the fact that  $\Psi(0) \leq 0$  ensure that the map  $x \mapsto \Psi(x)/x$  is nondecreasing. Therefore,  $\Psi(x)/x \geq \Psi(x_1)/x_1 \geq 0$  for all  $x \geq x_1$ . This entails

$$\int_{x_1}^\infty \frac{dx}{x} e^{-\int_{x_0}^x \frac{2\Psi(y)}{cy} dy} \leq C \int_{x_1}^\infty \frac{dx}{x} e^{-\frac{2\Psi(x_1)}{cx_1}x} < \infty$$

with a certain constant  $C > 0$ . The integrability near 0 holds by the assumption  $\mathcal{E} < \infty$ . For the necessary part, assume that  $-\Psi$  is the Laplace exponent of a subordinator, then  $\Psi(x) \leq 0$  for all  $x \geq 0$  and plainly for any  $x_1 \geq x_0$

$$\int_{x_1}^\infty \frac{dx}{x} e^{-\int_{x_0}^x \frac{2\Psi(y)}{cy} dy} \geq \int_{x_1}^\infty \frac{dx}{x} = \infty,$$

so that  $\kappa_Z(0) = 0$ . □

**5.6. Proof of Corollary 3.12.** Since the inverse local time  $(\tau_x^Z, x \geq 0)$  has the same law as that of the diffusion  $V$ , we will be able to apply some general results on diffusions. First we transfer the problem in natural scale, see e.g. Durrett [Dur96, Section 6.5, page 229]. A possible way to define the process  $(V_t, t \geq 0)$  reflected at 0 is to consider the minimal process absorbed after its first hitting time of 0, call it  $(V_t^a, t \geq 0)$ , define

$$h(y) = \frac{c}{2} S_V'(S_V^{-1}(y))^2 S_V^{-1}(y) \text{ for } y \in [0, \infty)$$

so that  $(h(V_t^a), t \geq 0)$  is a local continuous martingale; extend  $h$  on  $\mathbb{R}$  by  $h(-y) = h(y)$  for all  $y$ , let  $(X_t, t \geq 0)$  be the diffusion on  $\mathbb{R}$  in natural scale with speed density measure  $f(y) = 1/h(|y|)$  for all  $y \in \mathbb{R}$ , and then finally define  $V_t = |S_V^{-1}(X_t)|$  for all  $t \geq 0$ . This way of defining the diffusion  $V$  leads us to study the zero-set of  $X$ , which is homeomorphic to that of  $V$ . One has

$$S_V(x) := \int_0^x \frac{dz}{z} \exp\left(\frac{2}{c} \int_z^{x_0} \frac{\Psi(u)}{u} du\right) \text{ for all } x \geq 0$$

and therefore

$$h(y) = \frac{c}{2} S_V'(S_V^{-1}(y))^2 S_V^{-1}(y) = \frac{1}{2c} \frac{1}{S_V^{-1}(y)} e^{\frac{4}{c} \int_{S_V^{-1}(y)}^{x_0} \frac{\Psi(u)}{u} du}.$$

Set for all  $x \in \mathbb{R}$ ,

$$F(x) := \int_0^x f(y)dy = C \int_0^x S_V^{-1}(|y|) e^{-\frac{4}{c} \int_{S_V^{-1}(|y|)}^{\Psi(u)} \frac{du}{u}} dy.$$

Moreover for  $x \geq 0$

$$\begin{aligned} F(x) - F(-x) &= 2F(x) = C \int_0^x S_V^{-1}(y) e^{-\frac{4}{c} \int_{S_V^{-1}(y)}^{\Psi(u)} \frac{du}{u}} dy \\ &= C \int_0^{S_V^{-1}(x)} z e^{-\frac{4}{c} \int_z^{x_0} \frac{\Psi(u)}{u} du} S_V'(z) dz \\ &= C \int_0^{S_V^{-1}(x)} e^{-\frac{2}{c} \int_z^{x_0} \frac{\Psi(u)}{u} du} dz, \end{aligned}$$

where by  $C$  we represent possibly different positive constants of no relevance. We are now in the setting of Corollary 9.8 of Bertoin [Ber99] where a formula for the Hausdorff dimension of the zero-set of  $X$  is provided with the help of  $F$ . In our case, this gives

$$\dim_H(\mathcal{I}) = \sup \left\{ \rho \leq 1 : \lim_{x \rightarrow 0^+} x^{1-1/\rho} \int_0^{S_V^{-1}(x)} e^{-\frac{2}{c} \int_z^{x_0} \frac{\Psi(u)}{u} du} dz = \infty \right\} \text{ a.s.}$$

We now study  $\int_0^{S_V^{-1}(x)} e^{-\frac{2}{c} \int_z^{x_0} \frac{\Psi(u)}{u} du} dz$ . Set  $\Psi_0$  such that  $\Psi(u) = -\lambda + \Psi_0(u)$  for all  $u \geq 0$ . One has

$$e^{-\frac{2}{c} \int_z^{x_0} \frac{\Psi(u)}{u} du} = z^{-2\lambda/c} e^{-\frac{2}{c} \int_z^{x_0} \frac{\Psi_0(u)}{u} du} =: z^{-2\lambda/c} L(z).$$

Note that  $\Psi_0(u) \xrightarrow[u \rightarrow 0]{} 0$ , so that by Karamata's representation theorem, see e.g. [BGT87, Theorem 1.3.1],  $L$  is a slowly varying function at 0. One has by Karamata's theorem, see e.g. [BGT87, Proposition 1.5.8]

$$(5.11) \quad \int_0^{S_V^{-1}(x)} e^{-\frac{2}{c} \int_z^{x_0} \frac{\Psi(u)}{u} du} dz = \int_0^{S_V^{-1}(x)} z^{-2\lambda/c} L(z) dz \underset{x \rightarrow 0^+}{\sim} C S_V^{-1}(x)^{-2\lambda/c+1} L(S_V^{-1}(x))$$

and by definition of  $S_V(x)$ :

$$S_V(x) = \int_0^x dz \frac{z^{2\lambda/c-1}}{L(z)} \underset{x \rightarrow 0^+}{\sim} C \frac{x^{2\lambda/c}}{L(x)}.$$

We now divide the proof in two cases. Assume first  $\lambda > 0$ , so that  $S_V$  is regularly varying at 0 with index  $2\lambda/c$  and so is  $S_V^{-1}$  with index  $c/2\lambda$ , see [BGT87, Theorem 1.5.12] :

$$S_V^{-1}(x) \underset{x \rightarrow 0^+}{\sim} \frac{x^{c/2\lambda}}{L'(x)}.$$

Hence

$$S_V^{-1}(x)^{-2\lambda/c+1} L(S_V^{-1}(x)) = \left( \frac{x^{c/2\lambda}}{L'(x)} \right)^{-2\lambda/c+1} L \left( \frac{x^{c/2\lambda}}{L'(x)} \right) = x^{c/2\lambda-1} L''(x).$$

Therefore the condition

$$\lim_{x \rightarrow 0^+} x^{1-1/\rho+c/2-1\lambda} L''(x) = \infty$$

turns out to be true for any  $\rho < \frac{2\lambda}{c}$ . On the other hand; if  $\rho > \frac{2\lambda}{c}$ , then the power index  $c/2\lambda - 1/\rho$  becomes positive and the limit is 0. Finally almost surely

$$\dim_H(\mathcal{I}) = \frac{2\lambda}{c}.$$

Assume now  $\lambda = 0$ . The function  $S_V$  being an increasing slowly varying, its inverse  $S_V^{-1}$  is an increasing rapidly varying function at 0, see [BGT87, Theorem 2.4.7], i.e for  $t > 1$ ,

$$(5.12) \quad S_V^{-1}(x)/S_V^{-1}(tx) \xrightarrow{x \rightarrow 0^+} 0.$$

Moreover  $S_V^{-1}$  has limit 0 at 0 and by (5.12), for any  $\beta \in \mathbb{R}$ ,  $S_V^{-1}(x)x^\beta \xrightarrow{x \rightarrow 0^+} 0$ . Equation (5.11) being valid for  $\lambda = 0$ , we see that any  $\rho > 0$  satisfies

$$\lim_{x \rightarrow 0^+} x^{1-1/\rho} S_V^{-1}(x) L(S_V^{-1}(x)) = 0,$$

hence  $\dim_H(\mathcal{I}) = \sup\{\emptyset\} = 0$  almost surely. Joining the two cases, we have that almost surely

$$\dim_H(\mathcal{I}) = 2\lambda/c.$$

A similar study replacing the supremum over  $\rho$  by the infimum entails that the packing dimension  $\dim_P(\mathcal{I})$  agrees with the Hausdorff one.  $\square$

**5.7. Proof of Theorem 3.15.** We still work under the assumption  $\mathcal{E} < \infty$  &  $2\lambda/c < 1$ . Recall  $\mathcal{R}_Z^q$  the  $q$ -resolvent of the LCSBP  $Z$  with  $\infty$  regular reflecting. The excursion measures are satisfying for any  $f \in C_b([0, \infty])$  and  $q > 0$ ,

$$(5.13) \quad n_Z \left( \int_0^\zeta e^{-qu} f(\epsilon(u)) du \right) = \kappa_Z(q) \mathcal{R}_Z^q f(\infty)$$

and

$$(5.14) \quad n_V \left( \int_0^\ell e^{-qu} f(\omega(u)) du \right) = \kappa_V(q) \mathcal{R}_V^q f(0)$$

with  $\kappa_Z$  and  $\kappa_V$  the Laplace exponents of the inverse local times of  $Z$  at  $\infty$  and of  $V$  at 0. Since the latters coincide according to Theorem 3.10, we get the identity with  $f(z) = e^{-xz}$  for any  $z \in [0, \infty]$ ,

$$(5.15) \quad n_Z \left( \int_0^\zeta e^{-qt} e^{-x\epsilon(t)} dt \right) = n_V \left( \int_0^\zeta e^{-qt} \mathbb{1}_{(x, \infty)}(\omega(t)) dt \right).$$

By letting  $q$  go to 0 in (5.15), we get by monotone convergence the following identity:

$$(5.16) \quad n_Z \left( \int_0^\zeta e^{-x\epsilon(t)} dt \right) = n_V \left( \int_0^\ell \mathbb{1}_{(x, \infty)}(\omega(t)) dt \right).$$

Recall  $M_V$  the speed measure of  $V$  in (5.2) and that for any measurable positive function  $f$ , the invariant measure  $M_V$  satisfies (up to a multiplicative constant)  $\int f dM_V = n_V \left( \int_0^\ell f(\omega(t)) dt \right)$ , see [DMM92, Chapter XIX.46], we see that the left-hand side in (5.16) is

$$M_V((x, \infty)) = \int_x^\infty m_V(dv) = \int_x^\infty e^{\int_{x_0}^v \frac{2\Psi(u)}{cu} du} dv.$$

It is clearly infinite when  $-\Psi$  is not the Laplace exponent of a subordinator, as in this case  $\Psi$  is positive in a neighbourhood of  $\infty$ . When  $-\Psi$  is the Laplace exponent of a subordinator, the following necessary and condition was found in [Fou19], see Lemma 5.3-1 and its proof. Denote by  $\delta$  the drift of  $-\Psi$  and set

$$(A) \quad \delta = 0 \text{ and } \bar{\pi}(0) + \lambda \leq c/2.$$

- i) If (A) is satisfied then for all  $x \geq 0$ ,  $M_V((x, \infty)) = \infty$  and  $Z$  is null recurrent.
- ii) If (A) is not satisfied then for all  $x \geq 0$ ,  $M_V((x, \infty)) < \infty$ . (Integrability at 0 of  $m_V$  comes from the assumption  $\frac{2\lambda}{c} < 1$ ) and  $Z$  is positive recurrent.

This finishes the proof as (A) is not satisfied as soon as one of the conditions in (3.14) holds.  $\square$

*Remark 5.7.* Heuristically, when condition **(A)** holds, the jumps in the LCSBP have a so small activity that the quadratic drift has enough time to push the path close to 0. Once at a low level, the process will take an infinite mean time for exploding. This explains the null recurrence.

**5.8. Proof of Theorem 3.17.** Recall from Section 2.1 that  $(Z_t^{\min}, t \geq 0)$  has the same law as a time-changed transient generalized Ornstein-Uhlenbeck process  $(R_t, t \geq 0)$  stopped when exiting  $(0, \infty)$ . We shall first find from the time-change construction, the law of the infimum  $Z^{\min}$  started from an arbitrary  $z \in (0, \infty)$ . The Laplace transform of the first passage time below  $a$  of the process  $(R_t, t \geq 0)$ ,  $\sigma_a := \inf\{t \geq 0 : R_t \leq a\}$ , is given by

$$(5.17) \quad \mathbb{E}_z[e^{-\mu\sigma_a}] = \frac{g_\mu(z)}{g_\mu(a)},$$

with for all  $\mu > 0$  and  $x \in [0, \infty)$ ,  $g_\mu(x) := \int_0^\infty x^{2\mu/c} e^{-zx} \frac{1}{x} e^{-\int_{x_0}^x \frac{2\Psi(y)}{cy} dy} dx$ . We refer the reader to Shiga [Shi90, Theorem 3.1] and [Fou19, Equation (4.5) page 13]. One can recognize at the right of  $x^{2\mu/c} e^{-zx}$  in the integrand, the derivative of the scale function of  $V$  up to some multiplicative constant, namely

$$s_V(x) = \frac{1}{cx} e^{-\int_{x_0}^x \frac{2\Psi(u)}{cu} du}.$$

By Lamperti's time-change construction and letting  $\mu$  go to 0 in (5.17), we get

$$(5.18) \quad \mathbb{P}_z(\inf_{u \geq 0} Z_u^{\min} \leq a) = \mathbb{P}_z(\sigma_a < \infty) = \lim_{\mu \rightarrow 0} \frac{g_\mu(z)}{g_\mu(a)} = \frac{\int_0^\infty e^{-xz} s_V(x) dx}{\int_0^\infty e^{-xa} s_V(x) dx} =: \frac{S_Z(z)}{S_Z(a)}.$$

By assumption  $-\Psi$  is not the Laplace exponent of a subordinator, we have seen in the proof of Theorem 3.10 that this entails  $\int_0^\infty s_V(x) dx < \infty$ . In particular  $S_V(\infty) = \int_0^\infty s_V(x) dx = S_Z(0) < \infty$  and  $\mathbb{P}_z(\inf_{u \geq 0} Z_u^{\min} = 0) = \frac{S_Z(z)}{S_Z(0)}$ . Recall Theorem 3.10 and that we have previously established, see (5.10), that,

$$(5.19) \quad n_Z(\inf_{0 \leq s < \zeta} \epsilon(s) = 0) = \kappa_Z(0) = \frac{1}{S_V(\infty)} = \frac{1}{S_Z(0)}.$$

By the monotone convergence theorem, for any  $a \geq 0$ ,

$$(5.20) \quad n_Z(\inf_{0 \leq s < \zeta} \epsilon(s) \leq a) = \lim_{t \rightarrow 0+} n_Z(\inf_{t \leq s < \zeta} \epsilon(s) \leq a).$$

By the Markov property under the excursion measure  $n_Z$ , see e.g. [Ber96, page 117], and the identity (5.18) we see that

$$n_Z(\inf_{t \leq s < \zeta} \epsilon(s) = 0) = \int_0^\infty n_Z(\epsilon(t) \in dz) \mathbb{P}_z(\inf_{s \geq t} Z_{s-t}^{\min} = 0) = \int_0^\infty n_Z(\epsilon(t) \in dz) \frac{S_Z(z)}{S_Z(0)}.$$

Therefore by (5.19) and (5.20) with  $a = 0$ ,

$$\lim_{t \rightarrow 0+} \int_0^\infty n_Z(\epsilon(t) \in dz) S_Z(z) = 1.$$

Let  $a \geq 0$ . By following the same arguments, we finally get

$$\begin{aligned} n_Z(\inf_{0 \leq s < \zeta} \epsilon(s) \leq a) &= \lim_{t \rightarrow 0+} \int_0^\infty n_Z(\epsilon(t) \in dz) \mathbb{P}_z(\inf_{s \geq t} Z_{s-t}^{\min} \leq a) \\ &= \lim_{t \rightarrow 0+} \int_0^\infty n_Z(\epsilon(t) \in dz) \frac{S_Z(z)}{S_Z(a)} = \frac{1}{S_Z(a)}. \end{aligned}$$

□

6. ONE-DIMENSIONAL DIFFUSIONS ON  $(0, \infty)$  AND SIEGMUND DUALITY

This section deals with general one-dimensional diffusions that are regular on  $(0, \infty)$ . We study their so-called Siegmund duals. The results presented below may have independent interest than the study of LCSBPs.

Siegmund [Sie76, Theorem 1] has established that a standard positive Markov process  $U$  with boundary  $\infty$  either inaccessible (entrance or natural) or absorbing (exit or regular absorbing) admits a dual process  $V$  such that for all  $t, u, v$ ,  $\mathbb{P}_u(U_t < v) = \mathbb{P}_v(V_t > u)$  if and only if  $U$  is *stochastically monotone*, that is to say for any  $t \geq 0$  and  $y \in (0, \infty)$ , the function  $x \mapsto \mathbb{P}_x(U_t \leq y)$  is nonincreasing. We provide below a study of Siegmund duality in the framework of diffusions. Stochastic monotonicity of one-dimensional diffusions is well-known. It can be established for instance using a coupling  $(U^x, U^{x'})$  of two diffusions started from  $x$  and  $x'$  with  $x' \geq x$  and verifying  $U_t^x = U_t^{x'}$  for any time  $t \geq \tau := \inf\{t > 0 : U_t^x = U_t^{x'}\}$ . So that  $\mathbb{P}(U_t^x \leq U_t^{x'}) = 1$  and

$$\mathbb{P}(U_t^{x'} \leq z) \leq \mathbb{P}(U_t^x \leq z, U_t^{x'} \geq U_t^x) = \mathbb{P}(U_t^x \leq z).$$

The next theorem was first established for boundary 0 instantaneously reflecting by Cox and Rösler [CR84, Theorem 5]. See also Liggett [Lig05, Chapter II, Section 3], Kolokoltsov [Kol11] and Assiotis et al. [AOW19]. The proof in [CR84] is only sketched and relied on scaling limits of birth-death processes. We provide an alternative proof and complete Cox and Rösler's theorem by considering also the framework of attracting, natural, exit or entrance boundaries.

**Theorem 6.1** (Diffusions and Siegmund duality). *Let  $\sigma^2$  be a  $C^1$  strictly positive function on  $(0, \infty)$  and  $\mu$  be a continuous function on  $(0, \infty)$ . Let  $(U_t, t \geq 0)$  be a diffusion over  $(0, \infty)$  with generator*

$$\mathcal{A}f(x) := \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x)$$

such that  $\infty$  is either inaccessible (entrance or natural) or absorbing (exit or regular absorbing).

Then for any  $0 < u, v < \infty$  and any  $t \geq 0$

$$(6.1) \quad \mathbb{P}_u(U_t < v) = \mathbb{P}_v(V_t > u),$$

with  $(V_t, t \geq 0)$  the diffusion whose generator is

$$(6.2) \quad \mathcal{G}f(x) := \frac{1}{2}\sigma^2(x)f''(x) + \left(\frac{1}{2}\frac{d}{dx}\sigma^2(x) - \mu(x)\right)f'(x).$$

Moreover, the following correspondences for boundaries and longterm behaviors of  $U$  and  $V$  hold:

Feller's conditions	Boundary of $U$	Boundary of $V$
$S_U(0, x] < \infty$ & $M_U(0, x] < \infty$	0 regular	0 regular
$S_U(0, x] = \infty$ & $J_U(0) < \infty$	0 entrance	0 exit
$M_U(0, x] = \infty$ & $I_U(0) < \infty$	0 exit	0 entrance
$I_U(0) = \infty$ , $J_U(0) = \infty$	0 natural	0 natural
$S_U[0, \infty] < \infty$ , $M_U[0, \infty] = \infty$ & $I_U(\infty), I_U(0) = \infty$	$\infty$ & 0 attracting	positive recurrence

TABLE 5. Boundaries of  $U, V$ .

When both boundaries 0 of  $U$  and  $V$  are regular then necessary one is absorbing and the other is reflecting. Similar correspondences hold for the boundary  $\infty$  replacing everywhere 0 by  $\infty$ . Last, when  $\infty$  and 0 are attracting for  $U$ , the stationary law of  $V$  satisfies

$$\mathbb{P}(V_\infty > x) = \mathbb{P}_x(U_t \xrightarrow[t \rightarrow \infty]{} 0) \in (0, 1) \text{ for any } x > 0.$$

We recall the definitions of the scale function and speed measure of  $U$  and the Feller's conditions displayed in Table 5. Let  $u_0, x_0$  be arbitrary fixed points in  $(0, \infty)$ . Set  $s_U(u) := \exp\left(-\int_{u_0}^u \frac{2\mu(y)}{\sigma^2(y)} dy\right)$  and

$$(6.3) \quad S_U(x) = \int_{x_0}^x s_U(u) du = \int_{x_0}^x \exp\left(-\int_{u_0}^u \frac{2\mu(y)}{\sigma^2(y)} dy\right) du.$$

We shall also denote by  $S_U$  the Stieltjes measure associated to  $S_U$ . Let  $m_U$  be the speed density  $m_U(x) := \frac{1}{\sigma^2(x)s_U(x)}$  and

$$(6.4) \quad M_U(x) = \int_{x_0}^x m_U(y) dy = \int_{x_0}^x \frac{1}{\sigma^2(y)} \exp\left(\int_{u_0}^y \frac{2\mu(v)}{\sigma^2(v)} dv\right) dy.$$

For any  $l \in [0, \infty]$ , the integral tests  $I_U$  and  $J_U$  are defined by

$$I_U(l) := \int_l^x S_U[l, x] dM_U(x) \quad \text{and} \quad J_U(l) := \int_l^x S_U[u, x] dM_U(u).$$

The analytical classification of boundaries of  $U$  in Table 5 can be found for instance in Karlin and Taylor's book [KT81, Table 6.2, page 234].

*Remark 6.2.* We will see that up to some irrelevant multiplicative constants, we have the equalities  $S_U = M_V$ ,  $J_U = I_V$  and symmetrically with  $U$  replaced by  $V$ . Combining the two first lines of Table 5, we see that 0 is non-absorbing for  $U$  (i.e.  $J_U(0) < \infty$ , and 0 regular or entrance) if and only if 0 is accessible for  $V$  (i.e.  $I_V(0) < \infty$ , and 0 regular or exit)

*Proof.* We start by establishing that the process  $V$ , satisfying the duality relationship (6.1): for all  $s \geq 0$ ,  $\mathbb{P}_v(V_s > u) = \mathbb{P}_u(U_s < v)$  for all  $u, v \in (0, \infty)$ , is Feller. Namely for any bounded continuous function  $f$  on  $(0, \infty)$ ,  $P_t^V f(w) \xrightarrow{w \rightarrow v} P_t^V f(v)$ . It suffices to show that for all  $u, v \in (0, \infty)$ ,  $\mathbb{P}_w(V_s > u) \xrightarrow{w \rightarrow v} \mathbb{P}_v(V_s > u)$  and  $\mathbb{P}_v(V_s = u) = 0$ . On the one hand, under our assumptions, for any  $s > 0$ , the law of  $U_s$  has no atom in  $(0, \infty)$ , the map

$$v \mapsto \mathbb{P}_v(u < V_s) = \mathbb{P}_u(U_s < v),$$

is therefore continuous on  $(0, \infty)$ . On the other hand, by the strong Feller property of  $U$ , see e.g. Azencott [Aze74, Proposition 1.11],  $u \mapsto \mathbb{P}_u(U_s < v)$  is also continuous, hence for any  $u, v \in (0, \infty)$ ,

$$\mathbb{P}_v(V_s > u) = \lim_{\epsilon \rightarrow 0} \mathbb{P}_v(V_s > u + \epsilon) = \mathbb{P}_v(V_s \geq u),$$

which yields  $\mathbb{P}_v(V_s = u) = 0$ .

We now show that  $V$  has generator  $\mathcal{G}$ . We will show that  $V$  satisfies the martingale problem associated to  $(\mathcal{G}, C_c^2)$ ; namely

$$(MP)_V : \text{for any } F \in C_c^2, \text{ the process } \left( F(V_t) - \int_0^t \mathcal{G}F(V_s) ds, t \geq 0 \right) \text{ is a martingale.}$$

Our arguments are adapted from those in Bertoin and Le Gall [BLG05, Theorem 5]. We refer also to [FMM19, Section 6, page 36] where the case of branching Feller diffusions is treated.

Let  $g$  and  $f$  be two functions belonging to  $C_c^2$ . Set  $G(x) = \int_0^x g(u) du$  and  $F(x) = \int_x^\infty f(t) dt$ . By Fubini's theorem

$$\int_0^\infty \int_0^\infty g(u) f(x) \mathbb{1}_{\{x \geq u\}} du dx = \int_0^\infty g(u) F(u) du = \int_0^\infty f(x) G(x) dx,$$

and

$$\int_0^\infty f(x) \mathbb{P}_u(V_s < x) dx = \mathbb{E}_u[F(V_s)], \quad \int_0^\infty g(u) \mathbb{P}_x(U_s > u) du = \mathbb{E}_x[G(U_s)].$$



Recall  $\mathbb{P}_u(V_s < x) = \mathbb{P}_x(U_s > u)$ . Then, integrating this with respect to  $f(x)g(u)dxdu$  provides

$$\int_0^\infty dug(u)\mathbb{E}_u[F(V_s) - F(u)] = \int_0^\infty dx f(x)\mathbb{E}_x[G(U_s) - G(x)].$$

Since  $(U_s, s \geq 0)$  has generator  $\mathcal{A}$  then

$$\mathbb{E}_x[G(U_s) - G(x)] = \int_0^s \mathcal{A} P_t^U G(x) dt.$$

Hence

$$\int_0^\infty dx f(x)\mathbb{E}_x[G(U_s) - G(x)] = \int_0^\infty dx f(x) \int_0^s \mathcal{A} P_t^U G(x) dt.$$

Since  $f$  has a compact support, so does  $x \mapsto |f(x)\mathcal{A} P_t^U G(x)|$  and the function  $(t, x) \mapsto f(x)\mathcal{A} P_t^U G(x)$  is integrable on  $(0, s) \times (0, \infty)$ . Therefore, by Fubini's theorem

$$\int_0^\infty dx f(x) \int_0^s \mathcal{A} P_t^U G(x) dt = \int_0^s dt \int_0^\infty dx f(x)\mathcal{A} P_t^U G(x).$$

Set  $h(x) = P_t^U G(x)$  and  $\phi(x) = f'(x)\frac{1}{2}\sigma^2(x) + f(x)\left(\frac{1}{2}\frac{d}{dx}\sigma^2(x) - \mu(x)\right)$ . We now compute  $\int_0^\infty dx f(x)\mathcal{A} P_t^U G(x)$ . By two integration by parts

$$\begin{aligned} \int_0^\infty dx f(x)\mathcal{A} h(x) &= \int_0^\infty dx f(x) \left[ \frac{1}{2}\sigma^2(x)h''(x) + \mu(x)h'(x) \right] \\ &= \left[ f(x)\frac{1}{2}\sigma^2(x)h'(x) \right]_0^\infty - \int_0^\infty dx \left[ f'(x)\frac{1}{2}\sigma^2(x) + f(x)\frac{1}{2}\frac{d}{dx}\sigma^2(x) \right] h'(x) + \int_0^\infty dx f(x)\mu(x)h'(x) \\ &= \left[ f(x)\frac{1}{2}\sigma^2(x)h'(x) \right]_0^\infty - \int_0^\infty \phi(x)h'(x)dx. \\ &= \left[ f(x)\frac{1}{2}\sigma^2(x)h'(x) \right]_0^\infty - \phi(\infty)h(\infty) + \int_0^\infty \phi'(x)h(x)dx \\ &= \int_0^\infty \phi'(x)\mathbb{E}_x \left[ \int_0^\infty dug(u)\mathbb{1}_{\{u < U_t\}} \right] dx \text{ since } f \text{ has a compact support} \\ &= \int_0^\infty dug(u) \int_0^\infty \phi'(x)\mathbb{P}_u(V_t < x)dx \\ &= - \int_0^\infty dug(u)\mathbb{E}_u[\phi(V_t)] = - \int_0^\infty dug(u)\mathbb{E}_u[\mathcal{G}F(V_t)]. \end{aligned}$$

Therefore for any  $g \in C_c$ ,

$$\int_0^\infty dug(u)\mathbb{E}_u \left[ F(V_s) - F(u) - \int_0^s \mathcal{G}F(V_t)dt \right] = 0.$$

Thus,

$$(6.5) \quad \mathbb{E}_u \left[ F(V_s) - F(u) - \int_0^s \mathcal{G}F(V_t)dt \right] = 0 \text{ for almost all } u \in (0, \infty).$$

Since  $V$  satisfies the Feller property on  $(0, \infty)$ , the map

$$u \mapsto \mathbb{E}_u \left[ F(V_s) - F(u) - \int_0^s \mathcal{G}F(V_t)dt \right],$$

is continuous on  $(0, \infty)$  and (6.5) holds for all  $u \in (0, \infty)$ .

This entails that the process  $V$  satisfies  $(MP)_V$  for functions of the form  $F(x) := \int_x^\infty f(u)du$  with  $f \in C_c^2$ . By linearity, the martingale problem will be verified more generally for functions of the form  $F(x) := \int_x^\infty f(u)du - \int_x^\infty g(u)du$ , with  $g \in C_c^2$ , which contain all functions  $F \in C_c^2$ .

The martingale problem being well-posed for the process stopped when reaching its boundaries, see e.g. Durrett [Dur96, Section 6.1, Theorem 1.6], we therefore have established that  $V$ , up to hitting its boundaries, is a diffusion with generator  $\mathcal{G}$ .

We now explain the correspondences between types of boundaries stated in Table 5. Let  $\mu^V$  be the drift term of  $V$ , i.e.  $\mu^V(y) = \frac{1}{2} \frac{d}{dy} \sigma^2(y) - \mu(y)$ . Simple calculations provide

$$s_V(v) = \exp\left(-\int_{v_0}^v \frac{\mu^V(y)}{\sigma^2(y)/2} dy\right) = \frac{\sigma^2(v_0)}{\sigma^2(v)} \frac{1}{s_U(v)}.$$

and  $S_V(x) = \int_{x_0}^x s_V(v) dv = \sigma^2(v_0) M_U(x)$ . Similarly, one has  $m_V(x) = \frac{1}{\sigma^2(x) s_V(x)} = \frac{s_U(x)}{\sigma^2(v_0)}$  and  $M_V(x) = \frac{1}{\sigma^2(v_0)} S_U(x)$ . Up to some multiplicative constant, we get

$$I_U(l) = \int_l^x M_V(l, x) dS_V(x) = J_V(l).$$

Hence, the scale function and speed measure are exchanged by Siegmund duality, as well as Feller integral tests  $I_U$  and  $J_V$  and Table 5 follows. We recall that the condition for boundary 0 to be regular:  $S_U(0, x) < \infty$  &  $M_U(0, x) < \infty$  is equivalent to the condition  $I_U(0) < \infty$  &  $J_U(0) < \infty$ , see [KT81, Table 6.2, page 234]. The last line of Table 5 follows from the fact that the diffusion  $V$  has a finite speed measure on  $(0, \infty)$ , i.e.  $M_V[0, \infty] < \infty$ , when both  $\infty$  & 0 are attracting for  $U$ .

We now justify that if  $U$  has its boundary 0 regular absorbing then  $V$  has boundary 0 reflecting. The proof will be similar for  $\infty$  and we omit it. By the first line of Table 5, we know that if  $U$  has boundary 0 regular then so does  $V$ . If 0 is regular absorbing for  $U$ , then by the duality relationship (6.1),  $\mathbb{P}_{0+}(U_t \geq y) = \mathbb{P}_y(V_t = 0) = 0$  and therefore 0 is regular reflecting for  $V$ . Last, the fact that

$$\mathbb{P}(V_\infty > x) = \mathbb{P}_x(U_t \xrightarrow{t \rightarrow \infty} 0) = \frac{S_U(x) - S_U(0)}{S_U(\infty) - S_U(0)},$$

follows directly by taking the limit in (6.1). We recover the classic formula of the stationary distribution of  $V$  since  $S_U = M_V$ .  $\square$

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