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Rigidity of saddle loops

Daniel PANAZZOLO*, Maja RESMAN†, Loïc TEYSSIER‡

Winter 2021

Abstract

A saddle loop is a germ of a holomorphic foliation near a homoclinic saddle connection. We prove that they are classified by their Poincaré first-return map. We also prove that they are formally rigid when the Poincaré map is multivalued. Finally, we provide a list of all analytic classes of Liouville-integrable saddle loops.

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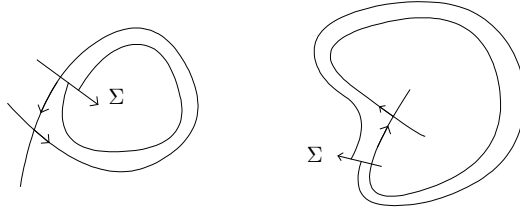
1 Introduction

Let X be a real analytic vector field defined on an open subset U of \mathbb{R}^2 and let $\Gamma \subset U$ be an invariant subset formed by a saddle point and a solution curve connecting two local branches of the stable and unstable separatrices. We will say that the triple (U, X, Γ) is a *real planar saddle loop*.

The qualitative behavior of the solution curves in the vicinity of Γ is encoded by fixing an analytic transverse section Σ through a regular point $\sigma \in \Gamma$ and considering the Poincaré return map,

$$P : (\Sigma_{\geq 0}, \sigma) \rightarrow (\Sigma_{\geq 0}, \sigma).$$

We denote by $\Sigma_{\geq 0}$ the positive part of Σ , with respect to a conveniently chosen parameterization (see figure below) and the notation (V, p) indicates some open neighborhood of a point p in a topological vector space V .



The two main goals of this article are to study the analytic classification of such maps and relate it to the classification of the germ of X along Γ . More precisely, consider two analytic planar saddle loops (U, X, Γ) , $(\tilde{U}, \tilde{X}, \tilde{\Gamma})$, and let P, \tilde{P} denote the underlying Poincaré maps. We introduce the following equivalence relations.

1. (U, X, Γ) and $(\tilde{U}, \tilde{X}, \tilde{\Gamma})$ are *analytically equivalent* if, up to changing U, \tilde{U} for smaller neighborhoods of $\Gamma, \tilde{\Gamma}$, there exists a real analytic diffeomorphism

$$\Phi : \tilde{U} \rightarrow U$$

with $\Phi(\tilde{\Gamma}) = \Gamma$, mapping the solution curves of \tilde{X} into the solution curves of X and preserving the orientation (but not necessarily the natural parameterization by the time).

2. P, \tilde{P} are *analytically conjugate* if there exists a germ of a real analytic diffeomorphism

$$\varphi : (\tilde{\Sigma}, \tilde{\sigma}) \rightarrow (\Sigma, \sigma)$$

preserving the orientation (*i.e.* mapping $\tilde{\Sigma}_{\geq 0}$ to $\Sigma_{\geq 0}$) such that $\tilde{P} = \varphi^{-1}P\varphi$.

The two main problems that we want to address are the following.

Problem 1. Describe the analytic conjugacy classes of Poincaré maps for saddle loops.

Problem 2. Show that $(U, X, \Gamma) \sim (\tilde{U}, \tilde{X}, \tilde{\Gamma})$ if and only if $P \sim \tilde{P}$.

Concerning Problem 1, we observe that related problems have been studied by P. Mardešić, D. Peran *et al.* [RM21b, RM21a, PRRS21, PRRS22, Per21]

but with respect to the action of a much larger group of germs admitting logarithmic asymptotic expansions (generalizations of the so-called *Dulac germs*). As we shall see, the results are strikingly different with respect to the present classification.

To put Problem 2 in some context, notice that, if we assume $\Gamma, \tilde{\Gamma}$ to be *periodic orbits*, the above statement is a classical result, as both (U, X, Γ) and $(\tilde{U}, \tilde{X}, \tilde{\Gamma})$ are analytically equivalent to suspensions of P and \tilde{P} (see *e.g.* [Sma63]).

In our case, due to the presence of singular points, there is no clear analogue of this construction by suspension. We take an indirect approach by complexifying the problem and working with the associated complex holomorphic foliations.

In Sections 1.1 and 1.2, we discuss separately Problem 2 and Problem 1, respectively.

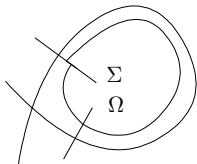
1.1 Real and complex saddle loops

Since our approach involves considering complex holomorphic foliations, one is naturally led to consider the following more general set-up. Let (S, \mathcal{G}) be a smooth holomorphic surface equipped with a singular foliation. A *saddle loop* for (S, \mathcal{G}) is defined by a saddle singularity $s \in S$ and an oriented C^1 -path $\Gamma : [-1, 1] \rightarrow S$ such that:

1. Γ is everywhere tangent to \mathcal{G} and $\lim_{t \rightarrow \pm 1} \Gamma(t) = \{s\}$;
2. there exists some $\varepsilon > 0$ such that $\Gamma|_{]-1, -1+\varepsilon[}$ and $\Gamma|_{]1-\varepsilon, 1[}$ lie on distinct local separatrices.

In other words, we assume that the two local separatrices at s lie on a common global leaf \mathcal{L}_s of \mathcal{G} and fix a path $\Gamma \subset \mathcal{L}_s$ which accumulates on s along distinct separatrices as $t \rightarrow \pm 1$. As above, we fix a local holomorphic transverse section (Σ, σ) at a point $\sigma \in \Gamma$. However, contrary to the real setting, there is no natural choice of a Poincaré first return map P , since the leaves are not 1-dimensional and there is no natural order on the multiple crossing points between leaves and transversals.

Since difficulties concentrate in the vicinity of the saddle point, it is convenient to place the transverse section (Σ, σ) near s , with base-point σ on one of the local separatrices, and introduce an auxiliary transverse section (Ω, ω) , with base-point ω lying on the other separatrix.



In this setting, we write the factorization $P = RD$, where the *regular transition*

$$R : (\Omega, \omega) \rightarrow (\Sigma, \sigma)$$

is a well-defined, injective holomorphic map obtained by lifting the sub-path of Γ linking Ω to Σ to the nearby leaves, and the so-called *corner transition*¹ D is a multivalued map establishing a point-wise correspondence between Σ and Ω . The motivation for this construction is that we can study D by using the classical Poincaré-Dulac local normal form theory.

For instance, to fix a determination of such corner transition we proceed as follows. Choose an oriented path lying on a sufficiently close regular leaf \mathcal{L} connecting a point $p \in \Sigma \cap \mathcal{L}$ to a point $q \in \Omega \cap \mathcal{L}$ (we call it a *guiding path*). By holonomy transport, this path uniquely determines a map

$$D : (\Sigma, p) \rightarrow (\Omega, q)$$

which (by a very elegant construction of Ilyashenko [Ily84]) has an holomorphic extension to a map between domains lying in the universal coverings $\widetilde{\Sigma \setminus \{\sigma\}}$ and $\widetilde{\Omega \setminus \{\omega\}}$.

It is important to emphasize that, except for some rather special cases, the Poincaré map cannot be extended as a holomorphic map on Σ itself, and the passage to the universal covering is necessary due to the intrinsic multivaluedness of P .

Of course, different choices of guiding path can lead to different determinations of D (and hence of P), and one needs to take this choice into account in the definition of equivalence between complex saddle loops. We refer to Section 3.1.6 for the details.

The above construction suggests to work with a more abstract model for a saddle loop, where one considers a pair of a germ of a complex saddle foliation \mathcal{F} in $(\mathbb{C}^2, 0)$, namely a foliation defined by a differential 1-form

$$\omega = xdy + y(\lambda - K(x, y))dx, \quad \lambda \in \mathbb{R}_{<0}, \quad K \in \mathbb{C}\{x, y\},$$

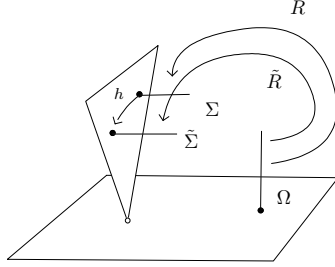
equipped with two transversals Ω, Σ through the separatrices $\{y = 0\}$ and $\{x = 0\}$, and a germ of a holomorphic map $R : (\Omega, \omega) \rightarrow (\Sigma, \sigma)$, seen intuitively as a recipe for *gluing* these transversals.

One of the important points is that, while we can always fix the position of one of these transversals, assuming for instance that $\Omega = \{x = 1\}$, the position of the other transversal Σ should be allowed to vary, while still staying inside the same equivalence class. More precisely, we identify two such gluing maps

$$R : (\Omega, 1) \rightarrow (\Sigma, \sigma), \quad \text{and} \quad \widetilde{R} : (\Omega, 1) \rightarrow (\widetilde{\Sigma}, \widetilde{\sigma})$$

if there exists a germ of a \mathcal{F} -holonomy map, $h : (\Sigma, \sigma) \rightarrow (\widetilde{\Sigma}, \widetilde{\sigma})$ such that $\widetilde{R} = hR$.

¹In many references, D is called a Dulac map, but we prefer to keep this name for a much larger class of maps which will appear very frequently in the present paper.



In other words, we define a *germ of a saddle loop* (or, shortly, a *loop germ*) as a pair $\mathbb{L} = (\mathcal{F}, \mathcal{R})$, where:

1. \mathcal{F} is a prepared germ of a complex saddle foliation (see Section 3.1);
2. \mathcal{R} is an *equivalence class* of germs of a diffeomorphism between the fixed vertical transversal $\Omega = \{x = 1\}$ and a varying horizontal transversal Σ , with respect to the equivalence relation defined above.

We will say that each germ $R \in \mathcal{R}$ is a *determination* of the regular transition. We refer to Section 3.4 for the detailed definitions and to Section 3.5 for some examples.

To a loop germ \mathbb{L} , we associate the *class of Poincaré first return maps* as the set, denoted by $\text{Poinc}(\mathcal{F}, \mathcal{R})$, of all possible compositions

$$P = RD,$$

where $R : (\Omega, 1) \rightarrow (\Sigma, \sigma)$ and $D : (\Sigma, \sigma) \rightarrow (\Omega, 1)$ are arbitrary determinations of the regular transition and corner transition, respectively.

In this new setting, the following generalization of the Problem 2 seems quite natural.

Problem 2'. Suppose that two germs of a saddle loop $\mathbb{L}, \tilde{\mathbb{L}}$ have (some determinations of) their Poincaré maps which are analytically conjugate. Does this imply an analytic equivalence between \mathbb{L} and $\tilde{\mathbb{L}}$?

More precisely, we consider the action of the group $\text{Diff}_{\text{fib}}(\mathbb{C}^2, \bar{\Delta})$ of germs of biholomorphisms defined in a neighborhood of the closed unit disk $\bar{\Delta} \times \{0\}$ in \mathbb{C}^2 , preserving the fibration $\{x = \text{cst}\}$, and we say that two loop germs $\mathbb{L} = (\mathcal{F}, \mathcal{R})$ and $\tilde{\mathbb{L}} = (\tilde{\mathcal{F}}, \tilde{\mathcal{R}})$ are $\text{Diff}_{\text{fib}}(\mathbb{C}^2, \bar{\Delta})$ -*equivalent* if there exists a germ of a x -fibred biholomorphism $\Phi \in \text{Diff}_{\text{fib}}(\mathbb{C}^2, \bar{\Delta})$ such that:

1. $\Phi(\mathcal{F}) = \tilde{\mathcal{F}}$;
2. Φ maps the class \mathcal{R} to the class $\tilde{\mathcal{R}}$.

The last requirement means that there exist suitable representatives $R \in \mathcal{R}$ and $\tilde{R} \in \tilde{\mathcal{R}}$ which are conjugate under the restriction of Φ to the appropriate transversals.

We are now ready to enunciate the two main equivalence results for loop germs.

Theorem A. *Consider two $\text{Diff}_{\text{fib}}(\mathbb{C}^2, \bar{\Delta})$ -equivalent loop germs $\mathbb{L} = (\mathcal{F}, \mathcal{R})$ and $\tilde{\mathbb{L}} = (\tilde{\mathcal{F}}, \tilde{\mathcal{R}})$. Then, there exist representatives of such germs in suitable neighborhoods of the origin in \mathbb{C}^2 such that each Poincaré map $P \in \text{Poinc}(\mathcal{F}, \mathcal{R})$ is analytically conjugate to a Poincaré map $\tilde{P} \in \text{Poinc}(\tilde{\mathcal{F}}, \tilde{\mathcal{R}})$.*

In fact, a more refined version of Theorem A, stated in Section 3.4, explicitly describes the correspondence between P and its “associate” \tilde{P} .

We also establish a converse to this statement.

Theorem B. *Let $(\tilde{\mathcal{F}}, \tilde{\mathcal{R}})$ and $(\mathcal{F}, \mathcal{R})$ be two loop germs, and suppose that there exist two Poincaré maps,*

$$P \in \text{Poinc}(\mathcal{F}, \mathcal{R}) \quad \text{and} \quad \tilde{P} \in \text{Poinc}(\tilde{\mathcal{F}}, \tilde{\mathcal{R}}),$$

which are analytically conjugate. Then, $(\tilde{\mathcal{F}}, \tilde{\mathcal{R}})$ and $(\mathcal{F}, \mathcal{R})$ are $\text{Diff}_{\text{fib}}(\mathbb{C}^2, \bar{\Delta})$ -equivalent.

The basic idea to prove the latter result is quite simple. As we shall see, the conjugacy of the Poincaré maps implies the conjugacy of the local holonomies at the saddle points. Hence, one can use the Mattei-Moussu theorem [MM80] to construct a fibered local equivalence between the foliations near the saddle points by path lifting.

The subtle point is that such equivalence does not necessarily preserve both transversals, and this is one of the reasons for having introduced the more general concept of a loop germ, thus allowing one of the transversals to *move*.

Fortunately, in the case of *real* saddle loops, the structure is more rigid. In particular, the local equivalence obtained by the Mattei-Moussu theorem respects the real structure, so one can better control how the other transversal moves under such an equivalence map. As a consequence, we obtain the following positive answer to Problem 2. Here, by the Poincaré map of a real saddle loop we mean the canonically chosen determination which preserves the real line.

Theorem C. *Two real saddle loops (U, X, Γ) , $(\tilde{U}, \tilde{X}, \tilde{\Gamma})$ are analytically equivalent if and only if the associated Poincaré maps P, \tilde{P} are analytically conjugate.*

1.2 Rigidity of Poincaré maps

We now turn to the problem of classifying Poincaré first return maps. For a loop germ \mathbb{L} as above, consider an associated Poincaré map

$$P = RD.$$

Following [II'91], we observe that the lift of P to the logarithmic chart $x = e^{-z}$ defines a *Dulac germ*: a holomorphic function p on a quadratic standard domain and having a formal asymptotic expansion in the pol-exp scale,

$$p(z) \sim az + b + \sum_{k \geq 1} P_k(z) e^{-\lambda_k z},$$

where $a > 0$, $b \in \mathbb{C}$, $P_k \in \mathbb{C}[z]$ and $\{\lambda_k\}_k$ is an increasing sequence of positive real numbers. We denote by \mathcal{D} the group of Dulac germs and by $\widehat{\mathcal{D}}$ is its *formal* counterpart, consisting in all formal pol-exp series as above. One fundamental fact, also due to Ilyashenko, is that the *Taylor map* $T : \mathcal{D} \rightarrow \widehat{\mathcal{D}}$, which associates to each Dulac germ its asymptotic expansion, is injective. We refer to Section 2.1 for detailed statements.

In order to characterize the Dulac germs originating from Poincaré maps of saddle loops, we consider the so-called *functional variation operator* $\text{var} : \mathcal{D} \rightarrow \mathcal{D}$,

$$\text{var}(d) = [\tau, d],$$

where $[a, b] = a^{-1}b^{-1}ab$ is the commutator operator and τ is the translation map $z \mapsto z + 2\pi i$ (also seen as an element of \mathcal{D}). We say that a Dulac germ $d \in \mathcal{D}$ is *unramified* (noted $d \in \mathcal{U}$) if

$$\text{var}(d) = \text{Id}, \tag{1}$$

and we say that d is *mildly ramified* (noted $d \in \mathcal{M}$) if

$$\text{var}(\text{var}(d)) = \text{Id}. \tag{2}$$

The following result establishes a correspondence between these notions and the maps considered above.

Proposition. (Dictionary).

1. A Dulac germ is the lift of an analytic germ from $\text{Diff}(\mathbb{C}, 0)$ if and only if it is unramified.
2. A Dulac germ is the lift of a Poincaré map of a germ of a complex saddle loop if and only if it is mildly ramified.

Item 1. is an immediate consequence of the definition of unramified germs (Proposition 2.2). Item 2. uses the nontrivial realization theorems proved in [MR83] and [PY94]: every mildly ramified germ f can be realized as a corner map of a saddle foliation, because its variation $\text{var}(f)$ can be realized as the holonomy of a saddle foliation \mathcal{F} computed on a horizontal transversal. By using properties of the variation operator on the spaces \mathcal{D} and \mathcal{M} , stated in Remark 2.4, one can establish that the holonomy on the vertical transversal belongs to the same conjugacy class as $\text{var}(f^{-1})$, providing a gluing germ R such that the loop germ $(\mathcal{F}, \mathcal{R})$ admits f as Poincaré map.

We observe that equations (1) and (2) also make perfect sense at the formal level, and define two subsets $\widehat{\mathcal{U}}$ and $\widehat{\mathcal{M}}$ of $\widehat{\mathcal{D}}$, so-called *unramified* and *mildly ramified* formal Dulac series. Moreover, since the Taylor map induces a group morphism between \mathcal{D} and $\widehat{\mathcal{D}}$, one has

$$\mathrm{T}(\mathcal{U}) \subset \widehat{\mathcal{U}} \quad \text{and} \quad \mathrm{T}(\mathcal{M}) \subset \widehat{\mathcal{M}}.$$

Based on the above dictionary, we can formulate our contribution to the Problem 1 stated above purely in terms of properties of \mathcal{U} , \mathcal{M} and their formal counterparts.

Theorem D. *Let $d_1, d_2 \in \mathcal{M}$ be two mildly ramified Dulac germs which are \mathcal{U} -conjugate. Then, one of the following alternatives holds:*

1. d_1 and d_2 belong to \mathcal{U} ;
2. d_1, d_2 belong to $\mathcal{M} \setminus \mathcal{U}$ and they are \mathcal{U} -conjugate.

More precisely, we prove the following *strong rigidity* property. Suppose that a formal series $\varphi \in \widehat{\mathcal{U}}$ satisfies the relation

$$d_1 = \varphi^{-1} d_2 \varphi$$

for some $d_1, d_2 \in \mathcal{M} \setminus \mathcal{U}$. Then $\varphi \in \mathcal{U}$.

We immediately obtain the following consequence.

Corollary. *Consider two Poincaré maps of saddle loop germs P_1, P_2 which are ramified, and suppose that there exists a formal diffeomorphism $\phi \in \widehat{\mathrm{Diff}}(\mathbb{C}, 0)$ conjugating P_1 to P_2 . Then ϕ converges.*

We observe that any germ of a diffeomorphism $P \in \mathrm{Diff}(\mathbb{C}, 0)$ can be realized as an *unramified* Poincaré first return map of a germ of a saddle loop. This is for instance the case of a loop germ obtained by gluing a linear 1 : 1 saddle (whose canonical corner transition map is the identity) by P . Therefore, according to the well-known theories of Birkhoff–Écalle–Voronin (resonant diffeomorphisms) and Yoccoz (quasi-resonant diffeomorphisms), one cannot expect to have a similar rigidity result if we do not assume P_1, P_2 to be ramified.

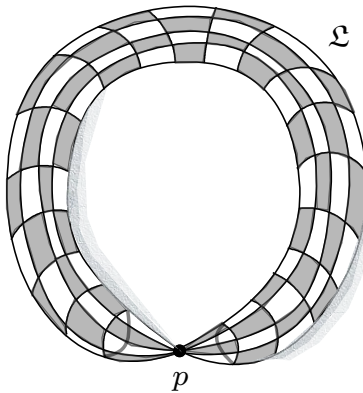
1.3 Complex saddle loops in birational models and symplectic foliations

Complex saddle loops appear quite frequently in the problem of birational classification of holomorphic foliations. For instance, suppose that a foliated surface (S, \mathcal{G}) contains a singular invariant set which is a nodal rational curve and that the nodal point is a saddle singularity for \mathcal{G} . Then (S, \mathcal{G}) contains a saddle loop.

A saddle loop also appears in the “very special foliation” described by Brunella in [Bru15, Section 4.2] as one of the class of foliations without a minimal model. We remark in passing that the existence of an invariant set of the above type

imposes strong restrictions if we assume (S, \mathcal{G}) to be *algebraic*. We refer to [Bru15, Example 7.1] for more details.

Another motivation to consider complex saddle loops comes from the related problem of classifying singular Liouville foliations of focus-focus type in 4-dimensional symplectic manifolds (M, ω) (see *e.g.* [Ngo03], [BI19], [Dui80], or [Smi14]). More precisely, one considers germs of a completely integrable singular Liouville foliation \mathcal{F} , defined in the vicinity of a singular compact leaf $\mathcal{L} \subset M$ which is homeomorphic to a *pinched torus*.



Under generic hypothesis, it follows from a theorem of Eliasson [Eli90] that one can find canonical local complex coordinates $(z, w) \in \mathbb{C}^2$ in a neighborhood of the pinch point p such that the leaves of \mathcal{F} are given as the level sets $\{zw = \text{cst}\}$. In other words, we locally obtain a saddle type singularity for a holomorphic foliation.

In this context, we can define a *symplectic* saddle loop by considering any path $\gamma \subset \mathcal{L}$ which connects p to itself and is not homotopic to the trivial path. Notice that the global foliation \mathcal{F} is not necessarily complex holomorphic (as (M, ω) is not necessarily equipped with a compatible complex structure) but we expect that our results will give new invariants for the above classification, notably if one considers non-integrable perturbations of the completely integrable case.

1.4 Integrability of loop germs in the Liouvillian class

The discussion about symplectic saddle loops naturally prompts to consider integrable settings. The usual framework to speak about integrable holomorphic foliations on complex surfaces is that of Liouvillian first integrals. A *Liouvillian function* lies in a finite tower of extensions of differential fields, starting from meromorphic germs, of the following three types: algebraic, integral or exponential. A classical reference is [Sin92].

A loop germ $(\mathcal{F}, \mathcal{R})$ is said to be *integrable* whenever there exists a (non-constant) Liouvillian first integral H of \mathcal{F} , meaning that the leaves of \mathcal{F} are included in level sets $\{H = \text{cst}\}$, that is compatible with a gluing map $R \in \mathcal{R}$.

When we say that H and R are *compatible*, we require that R preserve the “nice” transverse structure provided by H . We refer to Section 4 for a precise definition.

The theory developed by M. Berthier and F. Touzet in [BT99] classifies the saddle foliations \mathcal{F} admitting Liouvillian first integrals H . We carry out here the final step that allows us to list all compatible gluing maps $R \in \text{Diff}(\mathbb{C}, 0)$ leading to integrable loop germs.

Theorem E. *Any integrable loop germ $\mathbb{L} = (\mathcal{F}, \mathcal{R})$ is $\text{Diff}_{\text{fib}}(\mathbb{C}^2, \bar{\Delta})$ -equivalent to one appearing in the following list.*

1. \mathcal{F} is a linear $1 : \lambda$ saddle foliation, $\lambda > 0$, and $R \in \mathcal{R}$ is a linear map.
2. \mathcal{F} is a linear $1 : 1$ saddle foliation and $R \in \mathcal{R}$ is a Bernoulli diffeomorphism.
3. \mathcal{F} is a non-linear $1 : 1$ saddle foliation in Poincaré-Dulac normal form and $R \in \mathcal{R}$ embeds in a holomorphic flow.

Section 4 contains the detailed list of the model loop germs and the precise definitions of the terms involved above. For the sake of readability, let us simply say that a Bernoulli diffeomorphism is a resonant diffeomorphism whose Écalle–Voronin cocycles [Vor81, Éca75] are ramified homographies (with same ramification order), whereas R embeds in a holomorphic flow if its cocycles are all trivial.

Because integrable foliations have solvable holonomy groups, and since solvable finitely-generated subgroups of $\text{Diff}(\mathbb{C}, 0)$ are very rare, one is not surprised to encounter only so few integrable foliations. What may seem more surprising is that we only encounter Abelian holonomy groups in the list. Let us try to give an intuitive explanation of the fact. It is clear that a linear foliation can be glued by a linear map R without destroying the transverse structure (a linear map commutes with the other linear data), and in that case the holonomy group is itself linear and commutative. In contrast, the non-linear terms of R have to conciliate the different behaviors near distinct local branches of the separatrices passing through the saddle point.

On the one hand, when the local eigenratio λ differs from 1, the roles played by the branches of the local separatrices are asymmetric, which hints at why only linear gluing maps are compatible with the existence of a Liouvillian first integral. Nonlinearities in the foliation pose the same problems, and indeed a nonlinear $p : q$ saddle in Poincaré-Dulac normal form, given as the solutions to the differential equation

$$\frac{dy}{y} = \frac{1 + \mu (x^q y^p)^k}{(x^q y^p)^{k+1}} d(x^q y^p), \quad k \in \mathbb{Z}_{>0}, \mu \in \mathbb{C},$$

even when glued with $R = \text{Id}$, is *not* an integrable loop germ.

On the other hand, when $\lambda = 1$ the additional symmetry gives rise to richer outcomes. Since the holonomy group is generated by tangent to the identity

mappings, it is well-known that solvability is equivalent to commutativity. It is worth noticing, though, that not every Abelian holonomy group that arise in loop germs leads to an integrable situation. Proposition 2.11 gives the complete list of loop germs with Abelian holonomy around a saddle 1 : 1 foliation. In addition to the cases described by Theorem E, one finds the exceptional case of foliations that are formally conjugate to the Poincaré-Dulac foliation with $\mu = \frac{1}{2}$, and having both generators of the holonomy group being equal. The holonomy group is thus commutative for trivial reasons having nothing to do with the potential integrability of the underlying foliations (the realization theorem of [MR83] can be used again to prove that every candidate diffeomorphism is actually realizable in a loop germ), and it turns out that none of them correspond to an actual Liouville-integrable loop germ, except those that embed in a holomorphic flow.

1.5 Plan of the article

The main results of the article, stated in the introduction, are proved in the sections that are listed below.

Theorem A is proved in Section 3.4 as Theorem 3.36.

Theorem B is proved in Section 3.4 as Theorem 3.37.

Theorem C is proved in Section 3.6 as Theorem 3.56.

Theorem D is proved in Section 2 as Theorem 2.6 (attracting/repelling case) and Theorem 2.9 (indifferent case).

Theorem E is proved in Section 4 as Theorem 4.8.

2 Rigidity in the Dulac group

2.1 The Dulac group \mathcal{D}

Let $\Delta_c \in \mathbb{C}$, $c \geq 0$, denote the open disk of radius c centered at the origin. Following [Pi'91], we say that a subset $\Omega \subset \mathbb{C}$ is a *quadratic standard domain* if there exist 2 constants $c \geq 0$ and $d > 0$ such that $\Omega = \varphi_d(\mathbb{C} \setminus \Delta_c)$, where

$$\varphi_d(z) = z + d(z + 1)^{1/2}.$$

Let $A(\Omega)$ denote the ring of holomorphic functions on Ω . The collection (QSD, ∞) of all quadratic standard domains form a (direct) partially ordered set for the inclusion relation. The direct limit $\mathcal{Q} = \varinjlim A(\Omega)$ will be called the *ring of QSD-germs*.

We define an asymptotic partial order relation in this ring by writing $g \succ f$ (or $f = o(g)$) for two elements $f, g \in \mathcal{Q}$ if, for each $\varepsilon > 0$, there exist representatives (also denoted by f, g) defined in a common domain Ω such that $|f(z)| \leq \varepsilon|g(z)|$ for all $z \in \Omega$.

We will be mostly interested in QSD-germs having a particular type of asymptotic expansion. The *pol-exp asymptotic scale* is the collection of QSD-germs defined by

$$f_{k,\lambda} = z^k e^{-\lambda z}$$

with $(k, \lambda) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}_{\geq 0}$. Notice that this collection is totally ordered with respect to the asymptotic relation defined above, namely $f_{k,\lambda} \succ f_{l,\mu}$ if and only if $(\lambda, -k) < (\mu, -l)$ for the usual lexicographical order.

The *Dulac formal ring* $\widehat{\mathcal{AR}}$ is the set of formal sums in the pol-exp scale

$$f = \sum_{(k,\lambda)} a_{k,\lambda} f_{k,\lambda}, \quad a_{k,\lambda} \in \mathbb{C},$$

whose *support* $\text{supp}(f) = \{(k, \lambda) : a_{k,\lambda} \neq 0\}$ satisfy the following conditions:

- The *exponential coefficient set*

$$L = \{\lambda \mid \exists k : (k, \lambda) \in \text{supp}(f)\}$$

forms a discrete subset of $\mathbb{R}_{\geq 0}$.

- For each $\lambda \in L$, the set $\{k \mid (k, \lambda) \in \text{supp}(f)\}$ is finite.

In other words, the elements of $\widehat{\mathcal{AR}}$ are formal sums

$$\sum_{\lambda \in L} P_\lambda(z) e^{-\lambda z}, \quad (3)$$

with $L \subset \mathbb{R}_{\geq 0}$ a discrete subset and $\{P_\lambda\}_{\lambda \in L}$ a collection of complex polynomials.

The *Dulac ring* is the sub-ring $\mathcal{AR} \subset \mathcal{Q}$ of QSD-germs having an asymptotic expansion in $\widehat{\mathcal{AR}}$. Consider the Taylor map

$$\mathbb{T} : \mathcal{AR} \longrightarrow \widehat{\mathcal{AR}},$$

which is the ring morphism mapping each Dulac germ to its asymptotic expansion. The following result of quasi-analyticity is crucial to the theory.

Proposition 2.1 (see [II'91, §0.3, Theorem 1]). *The morphism \mathbb{T} is injective.*

The *formal Dulac group* is the subset $\widehat{\mathcal{D}} \subset \widehat{\mathcal{AR}}$ of elements $f \in \widehat{\mathcal{AR}}$ whose initial part has the form

$$f = az + b + o(1)$$

for some *real* coefficient $a > 0$ (the so-called *multiplier* of f) and some $b \in \mathbb{C}$. Notice that $\widehat{\mathcal{D}}$ forms a group with respect to the composition. To see this, it suffices to remark that the substitution $z \rightarrow az + b + z^l e^{-\mu z}$ into $e^{-\lambda z}$ can be re-expanded as:

$$e^{-\lambda(az+b+z^l e^{-\mu z})} = e^{-b\lambda} e^{-a\lambda z} \sum_{k \geq 0} \frac{1}{k!} z^{kl} e^{-k\mu z},$$

and leads to an expansion in $\widehat{\mathcal{AR}}$.

The *Dulac group* is the subset $\mathcal{D} \subset \mathcal{AR}$ of QR-germs f such that $T(f)$ belongs to $\widehat{\mathcal{D}}$. We will refer to the elements of \mathcal{D} (*resp.* $\widehat{\mathcal{D}}$) simply as Dulac germs (*resp.* formal Dulac series).

Notice that the Taylor map T restricts to an injective group morphism $T : \mathcal{D} \rightarrow \widehat{\mathcal{D}}$.

2.2 Two subgroups of \mathcal{D}

Consider the $2\pi i$ -translation map $\tau(z) = z + 2\pi i$, seen as an element of \mathcal{D} . The *functional variation* of a Dulac germ $f \in \mathcal{D}$ is defined by the commutator:

$$\text{var}(f) = [\tau, f] = \tau^{-1} f^{-1} \tau f.$$

Appendix A contains some general identities involving the commutators which will be useful in the sequel.

A germ $f \in \mathcal{D}$ is said to be *unramified* (which we write $f \in \mathcal{U}$) if $\text{var}(f) = \text{Id}$ (*i.e.* if it commutes with τ). It is said to be *mildly ramified* (which we write $f \in \mathcal{M}$) if $\text{var}^2(f) = \text{Id}$ (*i.e.* if $\text{var}(f)$ is unramified). Appendix A contains some general identities involving the commutators which will be useful in the sequel. For instance, based on the definitions given in (47) of Appendix A, we can equivalently define:

- $\mathcal{U} = \mathfrak{C}_1(\tau)$ is the subgroup of *unramified germs*,
- $\mathcal{M} = \mathfrak{C}_2(\tau)$ is the set of *mildly ramified germs*.

The nomenclature is justified by the following immediate result.

Proposition 2.2. *A Dulac germ $f \in \mathcal{D}$ is unramified if and only if there exists a germ $F \in \text{Diff}(\mathbb{C}, 0)$ such that the following diagram commutes*

$$\begin{array}{ccc} (\text{QSD}, \infty) & \xrightarrow{f} & (\text{QSD}, \infty) \\ \downarrow \Pi & & \downarrow \Pi \\ (\mathbb{C}, 0) & \xrightarrow{F} & (\mathbb{C}, 0) \end{array}$$

where $\Pi(z) = e^{-z}$. In other words, f is a lift of a holomorphic germ under the universal covering $\Pi : \mathbb{C}_{\geq 0} \rightarrow \mathbb{D}^*$.

In fact, by the uniqueness of the lift *modulo* post composition by the deck transformation τ , it follows that there exists a group morphism from \mathcal{U} to $\text{Diff}(\mathbb{C}, 0)$ whose kernel is precisely the cyclic group $\langle \tau \rangle$ generated by τ . By an abuse of notation, we will also denote this morphism

$$\Pi : \mathcal{U} \longrightarrow \text{Diff}(\mathbb{C}, 0). \quad (4)$$

On the formal side, we define, in exactly the same way, the sets

$$\widehat{\mathcal{U}} \subset \widehat{\mathcal{M}} \subset \widehat{\mathcal{D}}$$

of unramified and mildly ramified formal Dulac series. The following characterization of $\widehat{\mathcal{U}}$ will be quite useful.

Proposition 2.3. *A formal Dulac series f is in $\widehat{\mathcal{U}}$ if and only if it has a multiplier $a = 1$, its exponential coefficient set L is contained in $\mathbb{Z}_{\geq 0}$ and, for all $\lambda \geq 1$, the polynomial P_λ is of degree zero.*

Proof. It suffices to write the identity $\tau f = f\tau$ and compare the asymptotic expansions. \square

It follows that an unramified formal series f can always be written in the form

$$f = \Pi^{-1} \circ \left(\lambda z \left(1 + \sum_{k \geq 1} a_k z^k \right) \right) \circ \Pi = z - \ln(\lambda) - \ln \left(1 + \sum_{k \geq 1} a_k e^{-kz} \right)$$

where $\lambda z (1 + \sum a_k z^k)$ is an arbitrary invertible formal series (*i.e.* an element of $\widehat{\text{Diff}}(\mathbb{C}, 0)$) and $\ln(\lambda)$ is chosen *modulo* $2\pi i\mathbb{Z}$.

Recall that the Taylor expansion map T defines an embedding of \mathcal{D} into $\widehat{\mathcal{D}}$. Notice that the var operator commutes with the Taylor map T , and, therefore

$$T(\mathcal{U}) \subset \widehat{\mathcal{U}}, \quad T(\mathcal{M}) \subset \widehat{\mathcal{M}}.$$

By an abuse of notation, we will keep on writing \mathcal{U} , \mathcal{M} , \mathcal{D} to refer to their images in $\widehat{\mathcal{D}}$ under the Taylor map.

Remark 2.4. It follows from Proposition A.3 in the Appendix that the operator var establishes one-to-one correspondences

$$\text{var} : \mathcal{U} \setminus \mathcal{M} \longrightarrow \mathcal{U}, \quad \text{and} \quad \text{var} : \widehat{\mathcal{U}} \setminus \widehat{\mathcal{M}} \longrightarrow \widehat{\mathcal{U}}$$

where $H \setminus G = \{Hg : g \in G\}$ denotes the set of right cosets of a subgroup H in a group G . Using a formal iterative procedure (see Proposition B.3), it is not difficult to prove that the rightmost map is indeed a bijection. A much deeper fact, which is an immediate consequence of the results in [MR83] and [PY94], is that the leftmost map is also a bijection.

2.3 Ramified classification of attracting/repelling Dulac germs

Let us briefly discuss the ramified classification (*i.e.* up to \mathcal{D} -conjugation) of attracting/repelling Dulac germs established in [PRRS21], [PRRS22] and [Per21]. Consider a Dulac germ $f \in \mathcal{D}$ of the form

$$f = az + b + o(1).$$

We will say that f is

- *super-attracting* if $a > 1$;

- *super-repelling* if $a < 1$;
- *hyperbolically attracting* if $a = 1$ and $\operatorname{Re}(b) > 0$;
- *hyperbolically repelling* if $a = 1$ and $\operatorname{Re}(b) < 0$.

For shortness, we will simply say that f is attracting/repelling if one of the above four conditions holds, and in the complementary case

$$a = 1 \quad \text{and} \quad \operatorname{Re}(b) = 0$$

we will say that f is *indifferent*.

Proposition 2.5 (Ramified classification [Per21]). *Suppose that $f \in \mathcal{D}$ is attracting/repelling. Then, f is respectively \mathcal{D} -conjugate to*

$$g(z) = az, \quad \text{or} \quad g(z) = z + b$$

in the super attracting/repelling and hyperbolically attracting/repelling cases.

We remark that, in contrast to Proposition 2.5, the ramified classification of indifferent germs is much more involved. Indeed, up to a conjugation by a scaling map $s(z) = cz$ for some $c \in \mathbb{R}_{>0}$, we can assume that

$$b \in 2\pi i\mathbb{Z},$$

and the latter case is studied in [RM21a] and [RM21b]. More precisely, the authors embed \mathcal{D} in a larger group of germs defined on some QSD which admit *transserial* asymptotic expansions in power-logarithm scale. It is shown that there exist functional moduli similar to *Birkhoff-Écalle-Voronin cocycles* for the classification up to conjugacy.

2.4 Unramified rigidity of attracting/repelling Dulac germs

We say that a Dulac germ $f \in \mathcal{D}$ is *strongly formally \mathcal{U} -rigid* if, whenever a formal unramified series $\psi \in \widehat{\mathcal{U}}$ satisfies $\psi^{-1}f\psi \in \mathcal{D}$, then in fact

$$\psi \in \mathcal{U}.$$

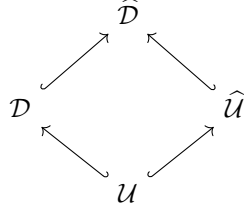
In particular, denoting by $\operatorname{Orbit}(x, G)$ the orbit of an element x in a group G under the G -action by conjugacy, we have

$$\operatorname{Orbit}(f, \widehat{\mathcal{U}}) \cap \mathcal{D} = \operatorname{Orbit}(f, \mathcal{U}).$$

Our first rigidity result is the following:

Theorem 2.6. *Suppose that $f \in \mathcal{D}$ is attracting/repelling. Then f is strongly formally \mathcal{U} -rigid.*

To prove Theorem 2.6, we start by considering the following diagram of strict group inclusions (see Section 2.2):



Lemma 2.7. $\mathcal{D} \cap \widehat{\mathcal{U}} = \mathcal{U}$.

Proof. It follows from Propositions 2.2 and 2.3 that, if a Dulac germ $f \in \mathcal{D}$ is formally unramified (*i.e.* lies in $\widehat{\mathcal{U}}$), then it is unramified. \square

Let $\text{Center}(x, G)$ denote the centralizer of an element x in a group G . An immediate consequence of Proposition 2.5 is the following lemma.

Lemma 2.8. *Let $f \in \mathcal{D}$ be attracting/repelling. Then*

$$\text{Center}(f, \widehat{\mathcal{D}}) \subset \mathcal{D}.$$

Proof. Suppose firstly that f is super attracting/repelling. By Proposition 2.5, up to conjugation inside \mathcal{D} , we can assume that $f(z) = az$. Since the conjugation with a germ az , $a > 0$, preserves the order of the monomials in the Dulac expansion, it suffices to remark that the monomials of the form $\mu z^k e^{-\lambda z}$ commuting with az must satisfy:

$$\mu a^k z^k e^{-\lambda az} = \mu a z^k e^{-\lambda z},$$

which gives $\mu a^k = \mu a$ and $\lambda az \equiv \lambda z \pmod{2\pi i \mathbb{Z}}$. We conclude that either $\mu = 0$ or $(k, \lambda) = (1, 0)$. Therefore,

$$\text{Center}(az, \widehat{\mathcal{D}}) = \{\mu z : \mu \in \mathbb{R}_{>0}\} \subset \mathcal{D}.$$

The case where f is hyperbolically attracting/repelling is similar. Using again Proposition 2.5 we assume, up to a \mathcal{D} -conjugation, that $f(z) = z + b$, with $\text{Re}(b)$ non-zero. Now, if a formal Dulac series

$$g = cz + d + \sum_{L \subset \mathbb{R}_{>0}} P_\lambda(z) e^{-\lambda z}$$

commutes with f , it necessarily follows that $c = 1$, and that each term $P_\lambda(z) e^{-\lambda z}$ satisfies

$$e^{-\lambda b} P_\lambda(z + b) e^{-\lambda z} = P_\lambda(z) e^{-\lambda z}.$$

If $P_\lambda \neq 0$, the leading terms on both sides are equal. We get a contradiction since $\text{Re}(b)$ is non-zero. Therefore, all P_λ vanish and we get:

$$\text{Center}(z, \widehat{\mathcal{D}}) = \{z + d : d \in \mathbb{C}\} \subset \mathcal{D}.$$

\square

Proof of Theorem 2.6. Let $\psi \in \widehat{\mathcal{U}}$ be a formal unramified series conjugating f to another Dulac germ $g \in \mathcal{D}$, and let $\varphi \in \mathcal{D}$ be a Dulac germ conjugating f and g , whose existence is proved in Proposition 2.5.

Then, the element $\psi^{-1}\varphi$ lies in $\text{Center}(f, \widehat{\mathcal{D}})$. By Lemma 2.8, this implies that

$$\psi^{-1}\varphi \in \mathcal{D}.$$

Therefore, ψ lies in the intersection $\mathcal{D} \cap \widehat{\mathcal{U}}$. We conclude using Lemma 2.7. \square

2.5 Unramified rigidity of indifferent Dulac germs

We now consider indifferent Dulac germs, namely, germs $f \in \mathcal{D}$ of the form

$$f(z) = z + b + o(1) \tag{5}$$

with $\text{Re}(b) = 0$.

Here, the rigidity issue is more subtle. In particular, assuming that f is *itself* an unramified germ, it follows from Proposition 2.2 that the classification up to \mathcal{U} -conjugation is precisely the same as the holomorphic classification of the projected diffeomorphism $F = \Pi(f) \in \text{Diff}(\mathbb{C}, 0)$ (see (4)), given by

$$F(Z) = e^b Z(1 + o(1)).$$

In this case, the formal and the holomorphic classification can differ spectacularly when $\text{Re}(b) = 0$. We refer the reader to the vast literature on the subject (see e.g. [MR82], [Vor81], [Yoc92]).

For the remaining of this section, we prove that indifferent Dulac germs *that are not itself unramified* are strongly rigid.

Theorem 2.9. *Let $f \in \mathcal{D}$, $f(z) = z + 2\pi i\beta + o(1)$, $\beta \in \mathbb{R}$, be an indifferent Dulac germ which is mildly ramified but not unramified. Then, f is strongly formally \mathcal{U} -rigid.*

The main ingredient of the proof is the interplay between the unramified conjugation and the variation operator. We first state and prove Lemma 2.10 and Proposition 2.11 that will be used in the proof.

We define the \mathcal{D} -variation group of a Dulac germ $f \in \mathcal{D}$ as the group

$$\text{Dvar}(f) = \langle \text{var}(f), \text{var}(f^{-1}) \rangle \subset \mathcal{D}.$$

The following result shows that $\text{Dvar}(f)$ behaves “nicely” under unramified conjugation.

Lemma 2.10. *Let g be an unramified germ. Then, for $f \in \mathcal{D}$,*

$$\text{Dvar}(g^{-1}fg) = g^{-1}\text{Dvar}(f)g.$$

Proof. It follows from formulas (50) in Appendix A that

$$\text{var}(g^{-1}fg) = g^{-1}\text{var}(f)g \quad \text{and} \quad \text{var}(g^{-1}f^{-1}g) = g^{-1}\text{var}(f^{-1})g.$$

\square

In this subsection, we assume that f is an indifferent germ lying in $\mathcal{M} \setminus \mathcal{U}$, and write

$$f = z + 2\pi i\beta + o(1) \quad (6)$$

with $\beta \in \mathbb{R}$. We consider the germs

$$G = \Pi(\text{var}(f)), \quad H = \Pi(\text{var}(f^{-1}))$$

of diffeomorphisms in $\text{Diff}(\mathbb{C}, 0)$ obtained by the projection morphism Π given in (4).

Proposition 2.11. *Let $f \in \mathcal{M} \setminus \mathcal{U}$ be indifferent, as in (6), and let $G, H \in \text{Diff}(\mathbb{C}, 0)$ be defined as above. Then, the germs G, H are k -tangent to the identity² at the same order $k \in \mathbb{Z}_{\geq 1}$. Moreover, G and H commute if and only if one of the following two situations occurs.*

- **Embedded into a flow case:** $\beta \notin \frac{1}{2k}\mathbb{Z}$ and

$$G = \exp \partial \quad \text{and} \quad H = \exp \left(-\frac{1}{\nu} \partial \right),$$

where $\nu = e^{-2\pi i k \beta}$ and, up to a conjugation in $\text{Diff}(\mathbb{C}, 0)$,

$$\partial = 2\pi i \frac{x^k}{1 + \frac{1}{1-\nu} x^k} x \frac{\partial}{\partial x}. \quad (7)$$

- **Identical variations case:** $\beta \in \frac{1}{2k} + \frac{1}{k}\mathbb{Z}$, $G = H$ and, up to a formal conjugation in $\widehat{\text{Diff}}(\mathbb{C}, 0)$, $G = H = \exp \partial$, where

$$\partial = 2\pi i \frac{x^k}{1 + \frac{1}{2} x^k} x \frac{\partial}{\partial x}.$$

Proof. Suppose first that $\beta = 0$. Then $f \in \widehat{\mathcal{D}}_{>0}$ (see the notation in Section B.1 of Appendix B). By Lemma B.1 in the Appendix, $f = \exp X$ for some (possibly formal) nilpotent derivation $X \in \mathcal{N}(\widehat{\mathcal{D}})$. By Corollary B.9, up to a simultaneous conjugation of both G and H inside $\text{Diff}(\mathbb{C}, 0)$, there exist constants $k \in \mathbb{Z}_{\geq 1}$ and $\mu \in \mathbb{C}$ such that

$$G = \exp \partial \quad \text{and} \quad H = \exp \eta,$$

where the formal derivations $\partial, \eta \in \mathcal{N}(\widehat{\text{Diff}}(\mathbb{C}, 0))$ are given by

$$\partial = \left(2\pi i \frac{x^k}{1 + \mu x^k} + o(x^{2k}) \right) x \frac{\partial}{\partial x},$$

²A tangent to identity germ $G \in \text{Diff}(\mathbb{C}, 0)$ is said to be k -tangent to identity if $G(x) = x + ax^{k+1} + o(x^{k+1})$, $a \in \mathbb{C}^\times$, as $x \rightarrow 0$, $k \in \mathbb{Z}_{\geq 1}$.

and

$$\eta = \left(-2\pi i \frac{x^k}{1 + (\mu - 1)x^k} + o(x^{2k}) \right) x \frac{\partial}{\partial x}.$$

Note that ∂, η are respectively the images of the derivations Z, W given in (55) and (56) by the map $x = e^{-z}$. Furthermore, it follows from that same Corollary B.9 that the commutator $[H, G]$ has the form

$$[H, G] = \exp \left(4\pi^2 k x^{3k} + o(x^{3k}) \right) x \frac{\partial}{\partial x},$$

and hence G and H do not commute.

Suppose now that $\beta \neq 0$. We write $f(z) = \mathbf{t}_\beta^{-1} f_0(z)$, where $f_0 = z + o(1)$ is an element of $\widehat{\mathcal{D}}_{>0}$, and

$$\mathbf{t}_\beta(z) = z - 2\pi i \beta.$$

It follows from identities (50) that

$$\text{var}(f) = \text{var}(f_0), \quad \text{and} \quad \text{var}(f^{-1}) = \text{var}(f_0^{-1})^{\mathbf{t}_\beta}.$$

Therefore, if we denote by G, H and G_0, H_0 the respective images in $\text{Diff}(\mathbb{C}, 0)$ of the $\text{var}(f), \text{var}(f^{-1})$ and $\text{var}(f_0), \text{var}(f_0^{-1})$ by the projection morphism Π , we obtain

$$G = G_0, \quad H = \mathbf{s}_B^{-1} H_0 \mathbf{s}_B,$$

where $\mathbf{s}_B(x) = Bx$ is the scaling map with ratio $B = e^{2\pi i \beta}$. By the previous paragraph, up to a conjugacy inside $\text{Diff}(\mathbb{C}, 0)$, we can assume $G = \exp \partial$ and $H = \exp \eta^B$, where

$$\eta^B = \left(-2\pi i \frac{B^k x^k}{1 + (\mu - 1) B^k x^k} + o(x^{2k}) \right) x \frac{\partial}{\partial x}$$

is the conjugate of η under \mathbf{s}_B .

Let us now suppose that G and H commute, or, equivalently, that

$$[\partial, \eta^B] = 0.$$

It follows immediately that $B^k \neq 1$, (otherwise, η^B coincides with η and does not commute with ∂ according to the previous paragraph). Moreover, it follows from the explicit characterization of the centralizer of ∂ (see *e.g.* [CM88], 1.3) that there exists some constant $t \in \mathbb{C}$ such that

$$\eta^B = t \partial.$$

By comparing coefficients on both sides, we get

$$t = -B^k, \quad \text{and} \quad B^k (\mu - 1) = \mu.$$

Since $|B| = 1$ and $B^k \neq 1$, there are two cases to consider:

1. B^k is non-real.
2. $B^k = -1$ (i.e. $\beta \in \frac{1}{2k} + \frac{1}{k}\mathbb{Z}$).

We now use Écalle's theory on the existence of iterative roots in $\text{Diff}(\mathbb{C}, 0)$ (see e.g. [Éca75]).

In case 1., since G has a non-real iterative root H , it follows from Écalle's theory that ∂ is a *convergent* derivation and that both G and H can be embedded in a common holomorphic flow. Moreover, up to conjugation inside $\text{Diff}(\mathbb{C}, 0)$, we can assume that $G = \exp \partial$ and $H = \exp(-B^k \partial)$, where

$$\partial = 2\pi i \frac{x^k}{1 + \mu x^k} x \frac{\partial}{\partial x},$$

for $\mu = \frac{B^k}{B^k - 1}$.

In case 2., B is a k^{th} -root of -1 , and we necessarily have $\eta^B = \partial$ and

$$G = H.$$

Moreover, in this case the residue of the (possibly formal) derivation ∂ is $\mu = \frac{1}{2}$.

Finally, we see that in both cases (1. and 2.) G and H indeed commute, which then proves also the converse direction. \square

Proof of Theorem 2.9. Let $\psi \in \widehat{\mathcal{U}}$ be an unramified formal series conjugating f to another indifferent Dulac germ f_0 . It follows from Lemma 2.10 that

$$\psi \text{var}(f) \psi^{-1} = \text{var}(f_0) \quad \text{and} \quad \psi \text{var}(f^{-1}) \psi^{-1} = \text{var}(f_0^{-1}).$$

Consider the respective images of these variations in $\text{Diff}(\mathbb{C}, 0)$ under the projection morphism (4),

$$G = \Pi(\text{var}(f)), \quad H = \Pi(\text{var}(f^{-1})) \quad \text{and} \quad G_0 = \Pi(\text{var}(f_0)), \quad H_0 = \Pi(\text{var}(f_0^{-1})).$$

It follows that $\Psi = \Pi(\psi) \in \widehat{\text{Diff}}(\mathbb{C}, 0)$ defines a conjugation between the subgroups

$$\langle G, H \rangle \quad \text{and} \quad \langle G_0, H_0 \rangle,$$

which are subgroups of the group $\text{Diff}_1(\mathbb{C}, 0)$ of tangent to the identity diffeomorphisms.

We now consider different cases. If either G, H do not commute or G, H commute and case (1) from Proposition 2.11 occurs, the group $\langle G, H \rangle$ is not cyclic. Hence, it follows from [CM88, Proposition 1], that Ψ converges. Therefore, ψ lies in \mathcal{U} and we conclude that f is strongly formally \mathcal{U} -rigid.

Finally, if G, H commute and case (2) from Proposition 2.11 occurs, then $G = H$, that is, $\text{var}(f) = \text{var}(f^{-1})$. By Item 3. of Lemma A.2 (with $x = y = f$), it follows that $f^2 \in \mathcal{U}$ is unramified. Let us denote by $F^2 = \Pi(f^2)$ the corresponding germ of a diffeomorphism in $\text{Diff}(\mathbb{C}, 0)$, and similarly $F_0^2 = \Pi(f_0^2)$.

It follows that the formal germ Ψ conjugates the group $\langle G, F^2 \rangle$ to $\langle G_0, F_0^2 \rangle$. According to [CM88, Proposition 1], to prove that Ψ converges, it suffices to

show that the subgroup $\langle G, F^2 \rangle \cap \text{Diff}_1(\mathbb{C}, 0)$ of the group of tangent to identity germs is not cyclic. Suppose the contrary. Then, there exists a tangent to identity germ $C \in \text{Diff}_1(\mathbb{C}, 0)$ such that

$$G = C^m \quad \text{and} \quad F^{2k} = C^n,$$

for some $m, n \in \mathbb{Z}$. Note that F^{2k} is tangent to identity (while F^2 is not necessarily), since $F(x) = e^{2\pi i \beta} x + o(x)$ and $\beta \in \frac{1}{2k} + \frac{1}{k}\mathbb{Z}$. Therefore, in the covering coordinate $x = e^{-z}$, there exists a unramified germ $c \in \mathcal{U}$ such that

$$\text{var}(f) = c^m \quad \text{and} \quad f^{2k} = c^n.$$

Using Lemma A.1 in the Appendix and the fact that $\text{var}(f) = \text{var}(f^{-1})$, we get $f \text{var}(f) = \text{var}(f)^{-1} f$. Therefore, $f \text{var}(f)^n = \text{var}(f)^{-n} f$, which finally gives:

$$f c^{mn} = c^{-mn} f.$$

In other words, $f f^{2km} = f^{-2km} f$, i.e., f^{4km} is the identity. This implies that $c = \text{Id}$, and, consequently, that $\text{var}(f)$ is the identity. This is a contradiction with the assumption in the enunciate that f is not unramified. \square

Remark 2.12. In the case 1. of the Proposition 2.11, up to unramified conjugation, f is the solution of the following implicit equation

$$e^{kf} + \frac{k}{1-\nu} f = \frac{1}{\nu} \left(e^{kz} + \frac{k}{1-\nu} z \right)$$

where $\nu = e^{-2\pi i k \beta}$ and $k \in \mathbb{Z}_{\geq 1}$. Consider the *Fatou coordinate*

$$w = \mathfrak{F}(z) = -\frac{1}{2\pi i} \left(e^{kz} + \frac{1}{1-\nu} z \right)$$

which conjugates the vector field ∂ to $\frac{\partial}{\partial w}$. Then, the elliptic function $\wp(w)$ with period lattice

$$\Lambda = \left\{ m + n \frac{1}{\nu} \mid m, n \in \mathbb{Z} \right\}$$

is a first integral for the variation group $\text{Dvar}(f)$, as the action of such group in the w -coordinate is generated by the translations $\mathbf{t}_1(w) = w + 1$ and $\mathbf{t}_{1/\nu}(w) = w + \frac{1}{\nu}$. In these coordinates, the Dulac map f assumes the simple form of a scaling

$$f(w) = \frac{-1}{\nu} w$$

which conjugates \mathbf{t}_1^{-1} to $\mathbf{t}_{1/\nu}$. Notice however that f is not in general an automorphism of the associated elliptic curve \mathbb{C}/Λ , except for some few exceptional cases (see e.g. [Mil06, Lemma 5.4])

3 Equivalence of saddle loops

3.1 Prepared saddle foliations

Let us introduce some preliminary notation. We denote by Δ_r the open disk of radius $r > 0$ in \mathbb{C} and by $\bar{\Delta}_r$ its closure. The open unit disk will be denoted simply by Δ , and we write $\bar{\Delta}^* = \bar{\Delta} \setminus \{0\}$ for the punctured unit disk. In order to simplify the notation, the closed subset of \mathbb{C}^2 given by $\bar{\Delta} \times \{0\}$ will also be denoted simply by $\bar{\Delta}$ when there is no risk of ambiguity.

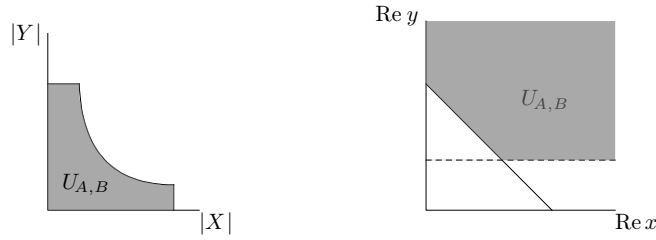
Given a positive constant $\lambda > 0$, a λ -adapted region is a subset in \mathbb{C}^2 of the form

$$U_{A,B} = \{|y||x|^\lambda \leq A, |x| \leq 1, |y| \leq B\} \quad (8)$$

for some constants $A, B > 0$. In logarithmic coordinates $x = e^{-z}, y = e^{-w}$, (8) becomes:

$$U_{A,B} = \{\operatorname{Re}(w) + \lambda \operatorname{Re}(z) \geq -\log A, \operatorname{Re}(z) \geq 0, \operatorname{Re}(w) \geq -\log B\}. \quad (9)$$

The domains $U_{A,B}$ are illustrated below.



A λ -neighborhood is an open subset $U \subset \mathbb{C}^2$ containing a λ -adapted region (for some $A, B > 0$). In the sequel, it will be convenient to work with a well-chosen prepared form for a saddle-type singularity, which we now define. We will prove in Proposition 3.4 that any saddle-type singularity can be put in such a form up to a convenient choice of local coordinates.

Definition 3.1. A *prepared saddle foliation* with eigenratio $\lambda > 0$ is a pair (U, \mathcal{F}) such that:

1. U is a λ -neighborhood such that $U \cap \{x = 0\}$ and $U \cap \{y = 0\}$ are simply connected;
2. \mathcal{F} is a singular foliation in U , defined by a holomorphic differential 1-form

$$\omega = xdy + \lambda y(1 + x^n y K(x, y))dx, \quad (10)$$

for some positive integer $n \geq \lfloor \lambda \rfloor$ and some $K \in \mathcal{O}(U)$ satisfying the bound

$$\sup_U |K| \leq \varepsilon,$$

for some constant $\varepsilon > 0$ such that $\varepsilon B < 1$.

From now on, we will denote by (x, y) the coordinates in \mathbb{C}^2 . The invariant lines $\{x = 0\}$ and $\{y = 0\}$ are respectively the *vertical* and the *horizontal* separatrix of (U, \mathcal{F}) .

It follows from the expression of ω and the choice for ε that \mathcal{F} is both transverse to the vertical fibration $\{x = \text{cst}\}$ and to the horizontal fibration $\{y = \text{cst}\}$ in the domain $U_{A,B}$.

Some of the constructions that we describe in the sequel are only valid up to restricting (U, \mathcal{F}) to some smaller λ -neighborhood. Therefore, from now on, we adopt the following *germified* point of view: we say that (U, \mathcal{F}) satisfies a certain property (P) if the property holds up to restricting the foliation to a sufficiently small λ -neighborhood.

For instance, we will tacitly identify two prepared foliations (U, \mathcal{F}) and $(\tilde{U}, \tilde{\mathcal{F}})$ if these foliations coincide in some smaller λ -neighborhood $V \subset U \cap \tilde{U}$.

Definition 3.2.

1. The *fixed base transversal* for (U, \mathcal{F}) is the germ of the vertical fiber $\{x = 1\}$ with base-point $(1, 0)$. For brevity, we will denote it simply by $(\Omega, 1)$.
2. A *floating base transversal* for (U, \mathcal{F}) is a germ of a curve (Σ, σ) with base-point $y = \sigma \in \mathbb{C}^\times$ in the vertical separatrix $\{x = 0\}$ such that $|\sigma| \leq B$, and which is both transverse to \mathcal{F} and to the vertical fibration $\{x = \text{cst}\}$. Here, $B > 0$ is any positive real number such that $U_{A,B} \subset U$ for some $A > 0$.

Remark 3.3. Equivalently, Σ is given locally as the graph of a function $y = \alpha(x)$, with $\alpha(0) = \sigma$. Notice that we do not assume Σ to be a fiber of the horizontal fibration $\{y = \text{cst}\}$. Later on, we will see the importance of considering such floating base transversals when constructing equivalences of prepared saddles.

3.1.1 Preparation of saddles in foliated surfaces

The following result shows that, up to a convenient choice of local holomorphic coordinates, a prepared form (10) can always be assumed to hold in the vicinity of a saddle type singularity.

Proposition 3.4. *Let s be a saddle type singularity of eigenratio λ contained in a holomorphic foliated surface (S, \mathcal{G}) . There exist local coordinates (x, y) defined in a neighborhood V of s mapping $(S, \mathcal{G})|_V$ to a prepared saddle of the form (10).*

Proof. It well-known (see e.g. [Lor10, Théorème 5.1.2]) that there exists local holomorphic coordinates (x, y) near s such that \mathcal{G} is locally generated by

$$\omega = xdy + y(\lambda - K(x, y))dx,$$

with K holomorphic at $(0, 0)$ and divisible by xy . Up to a scaling $x \mapsto \alpha x$, we can assume that ω is defined in a neighborhood of the closed disk $\{0\} \times \bar{\Delta}$. In

particular, there exists some $R > 1$ such that:

$$K(x, y) = \sum_{i \in \mathbb{Z}_{\geq 1}} c_i(y) x^i$$

where each c_i is a function in $\mathcal{O}(\Delta_R)$ vanishing at 0. We define the y -valuation of K as the smallest index $i \geq 1$ such that c_i is nonzero.

We now prove that, up to a further holomorphic coordinate change, we can assume that K has the y -valuation greater than or equal to $\lfloor \lambda \rfloor$ (the integer part of λ). To prove this, let us consider the dual vector field

$$\partial = \left(x \frac{\partial}{\partial x} - \lambda y \frac{\partial}{\partial y} \right) + Ky \frac{\partial}{\partial y},$$

and compute the action of the automorphism $\Phi = \exp \left((ax^i)y \frac{\partial}{\partial y} \right)$, where $a \in \mathcal{O}(\Delta_R)$ is a function of the y -variable and $i \geq 1$. The Lie bracket of $(ax^i)y \frac{\partial}{\partial y}$ with the initial diagonal term of ∂ gives

$$\begin{aligned} \left[(ax^i)y \frac{\partial}{\partial y}, \left(x \frac{\partial}{\partial x} - \lambda y \frac{\partial}{\partial y} \right) \right] &= - \left(\left(x \frac{\partial}{\partial x} - \lambda y \frac{\partial}{\partial y} \right) (ax^i) \right) y \frac{\partial}{\partial y} \\ &= (L_i(a)x^i)y \frac{\partial}{\partial y}, \end{aligned}$$

where L_i given by $L_i(a) = (\lambda a' - ia)$, $a \in \mathcal{O}(\Delta_R)$, is a linear operator in $\mathcal{O}(\Delta_R)$. Writing the Taylor expansion $a(y) = \sum a_j y^j$, we get:

$$L_i(a) = \sum_{j \in \mathbb{Z}_{\geq 0}} (\lambda j - i) a_j y^j,$$

and we conclude that the operator L_i is invertible for a fixed $i \geq 1$, provided that $i \notin \lambda \mathbb{Z}_{\geq 0}$.

Notice that:

$$\{1, \dots, \lfloor \lambda \rfloor - 1\} \cap \lambda \mathbb{Z}_{\geq 0} = \emptyset.$$

Therefore, if K has y -valuation i_0 with $i_0 \leq \lfloor \lambda \rfloor - 1$, we can eliminate its order i_0 (in x) element by applying the automorphism considered above with the conveniently chosen coefficient a .

Finally, re-scaling the coordinates one can enforce the remaining conditions of Definition 3.1. \square

3.1.2 Partial path lifting

Let us consider a prepared saddle foliation (U, \mathcal{F}) and denote by

$$\text{Fib} : U \setminus \{0\} \rightarrow W \supseteq \bar{\Delta}^*$$

the fibration defined by $\{x = \text{cst}\}$, with base $W \subseteq \{y = 0\}$ containing the punctured unit disk $\bar{\Delta}^*$. By the transversality property, we can intuitively see

each leaf of \mathcal{F} as a covering space over the base (save for $\{x = 0\}$). However, we must be careful with the fact that we only have a *partial lifting property*: not all paths in $\bar{\Delta}^*$ lift to paths in a leaf of (U, \mathcal{F}) .

To study this issue, consider a path $\Gamma \subset \bar{\Delta}^*$ with initial point x_0 , and denote by $\Gamma_{y_0} \subseteq U$ the \mathcal{F} -lift of Γ through the point $(x_0, y_0) \in \text{Fib}^{-1}(x_0)$ (if it exists). Let (x, y) be the end-point of Γ_{y_0} .

Our goal is to estimate the modulus $|y|$ in terms of $|y_0|$ and the length of Γ . It turns out to be easier to make the estimates in the covering variables (z, w) , where $x = e^{-z}$ and $y = e^{-w}$. We consider the *quasi-first integral*

$$\varphi = w + \lambda z.$$

In the (z, φ) coordinates, (10) becomes:

$$\frac{d\varphi}{dz} = -\lambda e^{-((n-\lambda)z+\varphi)} k,$$

where $k(z, w) = K(e^{-z}, e^{-w})$.

Let us choose a point $(z_0, w_0) \in \mathbb{C}^2$ with $\text{Re}(z_0) \geq 0$ and $\text{Re}(w_0) \geq 0$ which projects to (x_0, y_0) under the covering map, and let $\gamma \subset \{z : \text{Re}(z) > 0\}$ be the path with initial point z_0 which covers Γ under $x = e^{-z}$.

Proposition 3.5. *The solution of the Cauchy problem*

$$\begin{cases} \frac{d\varphi}{dz} = -\lambda e^{-((n-\lambda)z+\varphi)} k \\ \varphi_0 = w_0 + \lambda z_0 \end{cases} \quad (11)$$

along the curve γ satisfies the estimate

$$e^{\text{Re}(\varphi_0)} - \lambda \varepsilon \text{length}(\gamma) \leq e^{\text{Re}(\varphi)} \leq e^{\text{Re}(\varphi_0)} + \lambda \varepsilon \text{length}(\gamma). \quad (12)$$

Proof. We write

$$e^\varphi d\varphi = -\lambda e^{-sz} k dz,$$

where $s = n - \lambda \geq 0$. Using the estimate $|k| \leq \varepsilon$, we obtain

$$|e^{\text{Re}(\varphi)} - e^{\text{Re}(\varphi_0)}| \leq |e^\varphi - e^{\varphi_0}| \leq \lambda \varepsilon \int_\gamma e^{-s\text{Re}(z)} |dz| \leq \lambda \varepsilon \text{length}(\gamma),$$

which immediately implies (12). \square

Corollary 3.6. *Consider the solution $t \rightarrow \varphi(t)$ of the above Cauchy problem along a parameterized curve $z = \gamma(t)$, with an initial condition $(z_0, \varphi_0 = w_0 + \lambda z_0)$ such that $(z_0, w_0) \in U_{A,B}$, where $U_{A,B}$ is given in (9). Then, the pair*

$$(z(t), w(t)) = (\gamma(t), \varphi(t) - \lambda \gamma(t))$$

stays inside the region $U_{A,B}$ provided that the following inequalities hold for all values of t :

$$\begin{aligned} \text{Re}(\gamma(t)) &\geq 0, \\ e^{\text{Re}(\varphi_0)} - \lambda \varepsilon \text{length}(\gamma(t)) &\geq \max \left\{ \frac{1}{A}, \frac{1}{B} e^{\lambda \text{Re}(\gamma(t))} \right\}. \end{aligned} \quad (13)$$

Proof. It suffices to plug the lower bound for $e^{\operatorname{Re}(\varphi(t))}$ provided by the Proposition 3.5 in the three inequalities in (9) that define $U_{A,B}$. \square

Remark 3.7. Following [Lor10, Section 5.2], it is sometimes useful to have an estimate for the growth of the quasi-integral $x^\lambda y$ in terms of the length of the path Γ in the original (x, y) -coordinates. Using the same Cauchy problem (11) and the fact that $\operatorname{Re}(z) \geq 0, \operatorname{Re}(w) \geq -\log B$, and taking $n \geq 1$, we get:

$$|d\varphi| \leq \lambda \varepsilon |e^{-z} dz|.$$

Denoting $F = e^{-\varphi} = x^\lambda y$, we obtain

$$\left| \frac{dF}{F} \right| \leq \lambda \varepsilon B |dx|,$$

and, by integration,

$$e^{-\lambda \varepsilon B \operatorname{length}(\Gamma)} \leq \frac{|y||x^\lambda|}{|y_0||x_0^\lambda|} \leq e^{\lambda \varepsilon B \operatorname{length}(\Gamma)}. \quad (14)$$

We note that, for a fixed x_0 and for a fixed path Γ with base-point x_0 (and endpoint x), we can always choose y_0 sufficiently small in modulus such that

$$e^{\lambda \varepsilon B \operatorname{length}(\Gamma)} \cdot \left(\frac{|x_0|}{|x|} \right)^\lambda \cdot |y_0| \leq B \quad \text{and} \quad e^{\lambda \varepsilon B \operatorname{length}(\Gamma)} \cdot |x_0|^\lambda \cdot |y_0| \leq A. \quad (15)$$

In this case, it follows from the inequality (14) that the path Γ can be \mathcal{F} -lifted to (x_0, y_0) , staying inside the domain $U_{A,B}$ given in (8).

3.1.3 Examples of path lifting

Consider a *radial* path $\Gamma(t) = e^{-t}x_0$ with $|x_0| = 1, t \in \mathbb{R}$, and its \mathcal{F} -lift $\Gamma_{y_0}(t)$ with initial point (x_0, y_0) . The following example estimates the interval $[0, T]$ such that $\Gamma_{y_0}([0, T])$ stays inside a given domain $U_{A,B}$.

Example 3.8. Let us assume $A = B = 1$ in $U_{A,B}$ for simplicity. In the covering coordinates $x = e^{-z}, y = e^{-w}$, the path Γ corresponds to the horizontal path $\gamma(t) = z_0 + t$, with $\operatorname{Re}(z_0) = 0$ and $t \in [0, T]$. Then $\operatorname{length}(\gamma(t)) = t$ and the inequalities (13) are satisfied along this path as long as T satisfies:

$$e^{\operatorname{Re}(w_0)} \geq \max \{1, e^{\lambda T}\} + \lambda \varepsilon T.$$

Consider a *purely circular* path $\Gamma(t) = e^{-it}, t \in \mathbb{R}$, in the (x, y) coordinates. Let us estimate the number of *laps* this path can make before the modulus of $|y|$ becomes larger than 1.

Example 3.9. Let us again assume $A = B = 1$ in $U_{A,B}$ for simplicity. In the covering coordinates, the path Γ corresponds to the vertical path $\gamma(t) = it$ with endpoint $z = \gamma(T)$ and initial point $z_0 = 0$. We obtain

$$\text{length}(\gamma) = |T|.$$

Therefore, the inequalities (13) read

$$e^{\text{Re}(w_0)} - \lambda\varepsilon|T| \geq 1$$

or, equivalently,

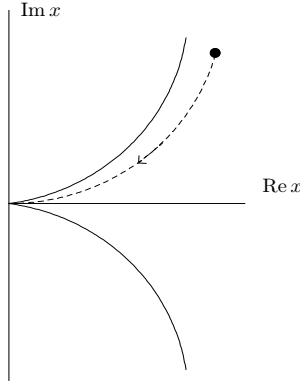
$$|T| \leq \frac{e^{\text{Re}(w_0)} - 1}{\lambda\varepsilon}$$

The following class of *exponential paths* is used in the Ilyashenko's proof of the analytic extension of the Dulac map to quadratic standard domains.

Example 3.10. Consider the family of exponential paths

$$\xi_{\alpha,C}(t) = t + iC(e^{\alpha t} - 1), \quad t \in \mathbb{R}_{\geq 0}, \quad (16)$$

with parameters $C \in \{\pm 1\}$ and $\alpha \in [0, \lambda]$.



Let us fix some parameters α, C , some $T \geq 0$, and consider the path γ going *backwards* from $z_0 = \xi_{\alpha,C}(T) = T + iC(e^{\alpha T} - 1)$ to $\xi_{\alpha,C}(0) = 0$ along the same trajectory as $\xi_{\alpha,C}$. Then, we have the estimate

$$\text{length}(\gamma) \leq \frac{2}{\alpha} e^{\alpha T}.$$

As a consequence, the inequalities (13) are satisfied (and γ stays inside $U_{A,B}$) as long as T satisfies:

$$\begin{aligned} e^{\text{Re}(w_0) + \lambda T} - \lambda\varepsilon \frac{2}{\alpha} e^{\alpha T} &\geq \max \left\{ \frac{1}{A}, \frac{1}{B} e^{\lambda T} \right\} \\ e^{\text{Re}(w_0)} &\geq \max \left\{ \frac{1}{A} e^{-\lambda T}, \frac{1}{B} \right\} + \lambda\varepsilon \frac{2}{\alpha} e^{-(\lambda - \alpha)T} \end{aligned} \quad (17)$$

which, if $e^{\text{Re}(w_0)} \geq \max \left\{ \frac{1}{A}, \frac{1}{B} \right\} + \lambda\varepsilon \frac{2}{\alpha}$, holds for all $T \geq 0$.

3.1.4 Monodromy and holonomy groupoids of a foliation

Let us recall the definitions of monodromy and holonomy groupoids for a foliation. We follow closely the exposition at [MM03, Section 5.2] and refer to this book for further details. Let (M, \mathcal{F}) be a foliated manifold. The *monodromy groupoid* $\text{Mon}(M, \mathcal{F})$ is a groupoid over M with the following arrows:

1. if $x, y \in M$ lie on the same leaf L of \mathcal{F} , then the arrows in $\text{Mon}(M, \mathcal{F})$ with source x and target y are the homotopy classes (with fixed endpoints) of paths in L from x to y ; the set of all such arrows will be denoted by $\text{Mon}(M, \mathcal{F})_{x,y}$;
2. if $x, y \in M$ lie on different leaves of \mathcal{F} then there are no arrows between them.

The groupoid operation is induced by concatenation of paths. Notice that the isotropy groups of the monodromy groupoid are the fundamental groups of the leaves (*i.e.* $\text{Mon}(M, \mathcal{F})_{x,x} = \pi_1(L, x)$).

Similarly, the *holonomy groupoid* $\text{Hol}(M, \mathcal{F})$ is the groupoid over M where the arrows between $x, y \in M$ are the classes of arrows in $\text{Mon}(M, \mathcal{F})$ *modulo* the following equivalence relation: two arrows $\gamma, \eta \in \text{Mon}(M, \mathcal{F})_{x,y}$ are equivalent if

$$\text{hol}_{\mathcal{F}, \gamma} = \text{hol}_{\mathcal{F}, \eta},$$

where the holonomy maps are seen as germs $(\Sigma, x) \rightarrow (\Omega, y)$ computed on arbitrarily fixed local transversals (Σ, x) and (Ω, y) . We will often omit the symbol \mathcal{F} in the notation when the underlying foliation is clear from the context.

Remark 3.11. Consider two transversals Σ, Σ' with the same base-point x . Then, we can unambiguously identify each holonomy map $(\Sigma, x) \rightarrow (\Omega, y)$ to a holonomy map $(\Sigma', x) \rightarrow (\Omega, y)$ by choosing an arbitrary foliated chart centered on x and identifying these two transversals *via* the associated holonomy map $h : (\Sigma, x) \rightarrow (\Sigma', x)$. It is obvious that such identification does not depend on the choice of the foliated chart. We will tacitly use such an identification from now on.

In order to keep the usual terminology, the arrows in $\text{Mon}(M, \mathcal{F})$ will be called *paths*, and the arrows in $\text{Hol}(M, \mathcal{F})$ will be called *holonomy germs*.

3.1.5 Saddle holonomies

Back to our present setting, let (U, \mathcal{F}) be a prepared saddle foliation and $(\Sigma, \sigma), (\Omega, 1)$ be two transversals as in Definition 3.2.

Definition 3.12. Let η be the circle $\{x = 0, |y| = |\sigma|\}$ and γ be the circle $\{|x| = 1, y = 0\}$, oriented positively. The germs of a map

$$\mathfrak{h}_\Sigma = \text{hol}_\eta : (\Sigma, \sigma) \longrightarrow (\Sigma, \sigma), \quad \mathfrak{h}_\Omega = \text{hol}_\gamma : (\Omega, 1) \longrightarrow (\Omega, 1),$$

will be called the *saddle holonomies* of (U, \mathcal{F}) .

Example 3.13. The linear foliation $\mathcal{F}_{1:\lambda}$ is defined by the differential 1-form

$$\omega_\lambda = \frac{dy}{y} + \lambda \frac{dx}{x} = d \log (yx^\lambda).$$

and its leaves coincide with the level sets of the Darbouxian first-integral

$$H_\lambda(x, y) = yx^\lambda.$$

Consider the transversals $\Omega = \{x = 1\}$ and $\Sigma = \{y = 1\}$, parameterized by the restriction of the ambient coordinates. Then

$$\mathfrak{h}_\Omega(y) = e^{-2i\pi\lambda}y \quad \text{and} \quad \mathfrak{h}_\Sigma(x) = e^{-2i\pi/\lambda}x.$$

The following result is an easy consequence of the definition of a prepared saddle foliation.

Lemma 3.14. *The linear foliation provides the linear part of the holonomy of a general prepared saddle foliation (U, \mathcal{F}) with same eigenratio. More precisely, for an arbitrary choice of local coordinates z and w on the transversals Σ and Ω ,*

$$\begin{aligned} \mathfrak{h}_\Sigma(z) &= e^{-2i\pi/\lambda}z + o(z), \\ \mathfrak{h}_\Omega(w) &= e^{-2i\pi\lambda}w + o(w). \end{aligned}$$

Proof. It suffices to combine (10) with an easy perturbation argument. We refer to [MM80, Section 5] for the details. \square

We now recall the following fundamental result relating the conjugacy of holonomies and the equivalence of the underlying foliations.

Theorem 3.15 (Equivalence of prepared saddles [MM80, Théorème 2]). *Let (U, \mathcal{F}) and $(\tilde{U}, \tilde{\mathcal{F}})$ be prepared saddle foliations with the same eigenratio λ and whose holonomies \mathfrak{h}_Ω and $\mathfrak{h}_{\tilde{\Omega}}$ are conjugate by a holomorphic germ*

$$\varphi : (\tilde{\Omega}, 1) \rightarrow (\Omega, 1).$$

Up to restricting \tilde{U} and U to smaller λ -neighborhoods, there exists a biholomorphic map

$$\Phi : \tilde{U} \rightarrow U$$

such that:

1. $\Phi(\tilde{\mathcal{F}}) = \mathcal{F}$
2. Φ preserves the vertical fibers $\{x = \text{cst}\}$.
3. The restriction of Φ to the transversal $\tilde{\Omega}$ coincides with φ .

Moreover, Φ is uniquely determined by conditions 1., 2. and 3.

Proof. The proof uses the path-lifting method. The only remaining technical point is to prove that Φ , as constructed in Mattei-Moussu's proof, is a map between two λ -neighborhoods. This is an immediate consequence of both the upper and lower estimates in (14) along the paths approaching the origin radially. We refer to [Lor10, Section 5.2] for the details. \square

Remark 3.16. For future use, we observe the following invariance property in Mattei-Moussu's construction. Assume that:

1. (U, \mathcal{F}) and $(\tilde{U}, \tilde{\mathcal{F}})$ are complexifications of *real analytic foliations*;
2. the conjugacy map φ is real analytic.

Then the biholomorphism Φ will also be the complexification of a real analytic map. In particular, it defines an analytic equivalence between the underlying real foliations.

3.1.6 Saddle corner transition maps and determinations

Let (U, \mathcal{F}) be a prepared saddle foliation as in Definition 3.1. Informally, a corner map establishes a leaf-wise correspondence

$$\mathcal{F} \cap \Sigma \rightarrow \mathcal{F} \cap \Omega$$

where $(\Omega, 1)$ and (Σ, σ) are respectively the fixed base transversal and a floating base transversal associated to (U, \mathcal{F}) .

This map is generally multivalued, since a leaf can intersect these transversals at multiple points. We can choose a *determination* by fixing a path in the punctured horizontal separatrix (say, connecting a point sufficiently close to the origin to 1) and considering its \mathcal{F} -lift, through the fibration $\text{Fib} = \{x = \text{cst}\}$, to a path in a leaf going from Σ to Ω . As we have already observed in Section 3.1.2, we must be careful with the fact that not every path can be \mathcal{F} -lifted.

In what follows, we use the family of exponential paths

$$\{\xi_{\alpha, C}\}_{\alpha \in [0, \lambda], C \in \{\pm 1\}},$$

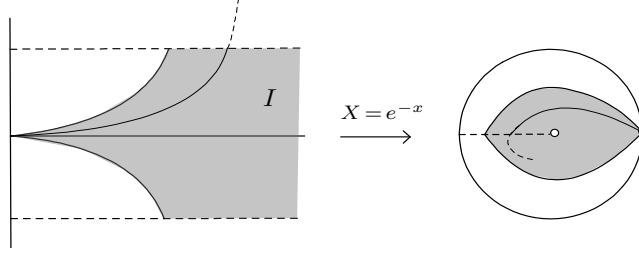
introduced in Example 3.10. Consider the restriction of the images of this family of paths to the strip $I = \{z \in \mathbb{C} : \text{Im}(z) \in]-\pi, \pi[\}$. By $I_{\alpha, C}$ we denote the time domains for these restrictions. Namely, for each parameter value α, C , we consider the interval $t \in I_{\alpha, C}$ such that $\xi_{\alpha, C}(I_{\alpha, C}) \subset I$. It can easily be computed from (16) that:

$$I_{\alpha, C} = \left[0, \frac{1}{\alpha} \log(1 + \pi) \right).$$

Now, let

$$\Xi_{\alpha, C} = e^{-\xi_{\alpha, C}|_{I_{\alpha, C}}}, \quad \alpha \in [0, \lambda), \quad C \in \{\pm 1\}, \quad (18)$$

denote the image of the restricted paths $\xi_{\alpha, C}|_{I_{\alpha, C}}$ under the covering map $x = e^{-z}$ (see Figure 3.1.6 below).



By definition, these paths $\Xi_{\alpha,C}$ are disjoint, and there exists some radius $0 < r < 1$ (depending on λ) such that their union covers the *cut disk*

$$\bar{\Delta}_r^{\text{cut}} = \bar{\Delta}_r \setminus \mathbb{R}_{\leq 0}$$

In what follows, we will denote by γ^{-1} the reverse path of a given path γ (*i.e.* γ traveled *backwards*).

The next lemma gives conditions under which the paths $\Xi_{\alpha,C}^{-1}$ can be \mathcal{F} -lifted through the points of Σ .

Lemma 3.17. *Let (\mathcal{F}, U) be a prepared saddle and let $U_{A,B} \subseteq U$, $A, B > 0$, be a λ -adapted subdomain. Suppose that the base-point σ of the floating transversal Σ satisfies the estimate*

$$|\sigma| \leq \frac{1}{\max \left\{ \frac{1}{A(1+\pi)}, \frac{1}{B} \right\} + 2\varepsilon}. \quad (19)$$

Then, there exists some radius $r > 0$ such that, for each point $x \in \bar{\Delta}_r^{\text{cut}}$, the path $\Xi_{\alpha,C}^{-1}$ (containing x) has a \mathcal{F} -lift by Fib in $U_{A,B} \subseteq U$ through the point $p = \text{Fib}^{-1}(x) \cap \Sigma$.

Proof. It is an immediate consequence of the estimates derived in Example 3.10. Note that $\xi_{\alpha,C}^{-1}(t)$ remains inside the domain I for $t \in [0, \frac{1}{\alpha} \log(1+\pi)]$. By (17) in Example 3.10, putting $T = \frac{1}{\alpha} \log(1+\pi)$, we see that $(\xi_{\alpha,C} \Big|_{I_{\alpha,C}})^{-1}$ has a \mathcal{F} -lift in $U_{A,B}$ through the point $(x_0 = \xi_{\alpha,C}(T), y_0)$ if:

$$\frac{1}{|y_0|} \geq \max \left\{ \frac{1}{A} e^{-\frac{\lambda}{\alpha} \log(1+\pi)}, \frac{1}{B} \right\} + \lambda \varepsilon \frac{2}{\alpha} e^{-\frac{\lambda-\alpha}{\alpha} \log(1+\pi)}.$$

Now, for all $\alpha \in [0, \lambda[$:

$$\frac{1}{A} e^{-\frac{\lambda}{\alpha} \log(1+\pi)} \leq \frac{1}{A(1+\pi)}, \quad \lambda \varepsilon \frac{2}{\alpha} e^{-\frac{\lambda-\alpha}{\alpha} \log(1+\pi)} \leq 2\varepsilon.$$

Therefore, if

$$|y_0| \leq \frac{1}{\max \left\{ \frac{1}{A(1+\pi)}, \frac{1}{B} \right\} + 2\varepsilon}, \quad (20)$$

all paths $(\xi_{\alpha,C}|_{I_{\alpha,C}})^{-1}$ can be \mathcal{F} -lifted through (x_0, y_0) , for every $\alpha \in [0, \lambda[$ and $C \in \{\pm 1\}$. Note that (20) implies also that $|y_0| < B$. \square

From now on, we will suppose that the base-point σ of the floating transversal (Σ, σ) of a prepared saddle (U, \mathcal{F}) always satisfies the estimate (19) for some $A, B > 0$ such that $U_{A,B} \subseteq U$. Moreover, for $r > 0$ given by Lemma 19, we define the germ of a *cut transversal* $(\Sigma^{\text{cut}}, \sigma)$ by

$$\Sigma^{\text{cut}} \subset \text{Fib}^{-1}(\bar{\Delta}_r^{\text{cut}}) \cap \Sigma. \quad (21)$$

and, based on the previous Lemma, we state the following:

Definition 3.18. The *canonical corner-transition map* associated to (U, \mathcal{F}) is the germ of a map

$$D_{[0]} : (\Sigma^{\text{cut}}, \sigma) \rightarrow (\Omega, 1)$$

which associates to each point $p \in \Sigma^{\text{cut}}$ the endpoint $D_{[0]}(p) \in \Omega$ of the \mathcal{F} -lift of $\Xi_{\alpha,C}^{-1}$ through p .

Lemma 3.17 can be further generalized to general paths with endpoint 1 and of bounded length ℓ , as we explain now.

Remark 3.19.

- Using (15) we get: for a given base-point σ of the floating transversal Σ , where $|\sigma| < B$, and $\ell > 0$, there exists a radius $r = r_{\ell,|\sigma|,A,B} > 0$ such that, for every $x \in \bar{\Delta}_r$, every path of length less than ℓ containing x and landing at 1 can be \mathcal{F} -lifted in $U_{A,B}$ through the point $p = \text{Fib}^{-1}(x) \cap \Sigma$.
- Choosing a path γ from $x \in \bar{\Delta}_r^{\text{cut}}$ to 1 of length less than ℓ , the holonomy map $\text{hol}_{\mathcal{F},\gamma}$ along γ through $p = \text{Fib}^{-1}(x) \cap \Sigma$ to Ω can be holomorphically extended to the whole cut transversal $\Sigma^{\text{cut}} = \text{Fib}^{-1}(\bar{\Delta}_r^{\text{cut}}) \cap \Sigma$. This construction produces one particular determination of the corner map of the saddle $D : (\Sigma^{\text{cut}}, \sigma) \rightarrow (\Omega, 1)$.

Remark 3.20. By Chapter 7 of [Lor10], all other determinations of the corner map for a prepared saddle (U, \mathcal{F}) are obtained from the canonical determination $D_{[0]}$ by the formula

$$D_{[n]} = \mathfrak{h}_{\Omega}^{-n} D_{[0]}, \quad n \in \mathbb{Z}, \quad (22)$$

where \mathfrak{h}_{Ω} is the holonomy of Ω . In other words, they are obtained by \mathcal{F} -lifting (to a sufficiently small germ of a floating transversal, see Remark 3.19), the concatenation of paths

$$(\Xi_{\alpha,C})^{-1} \star \gamma^{-n}$$

where $\Xi_{\alpha,C}$ are the paths used in the definition of $D_{[0]}$ and γ^n is the n^{th} composition of the path of Definition 3.12.

Notice that we obtain all determinations equally by taking the compositions

$$D_{[n]} = D_{[0]} \mathfrak{h}_{\Sigma}^n, \quad n \in \mathbb{Z}, \quad (23)$$

where \mathfrak{h}_{Σ} is the holonomy of Σ .

Remark 3.21. Assume that (U, \mathcal{F}) is a *real prepared saddle foliation*, i.e. that the differential form in Definition 3.1 is real analytic. Suppose further that the floating transversal Σ is the *complexification* of a real analytic transversal $\Sigma_{\mathbb{R}} \subset \mathbb{R}^2$. Then, by construction, the canonical determination for the corner map preserves the real axis, namely $D_{[0]}(\mathbb{R}_{>0}) \subset \mathbb{R}_{>0}$.

Let us now describe a global holomorphic extension of $D_{[0]}$ by fixing natural coordinates on the transversals. Namely, we consider the *quasi-first integral* $F = x^\lambda y$ and parameterize the transversals Ω and Σ in such a way that

$$F|_{\Omega}(y) = y, \quad \text{and} \quad F|_{\Sigma}(x) = x^\lambda$$

where x^λ is chosen with respect to the main branch of the logarithm.

In the next lemma, we study the lift of $y = D_{[0]}(x)$ through the covering maps $x = e^{-z}$, $y = e^{-w}$. Note that the lift is well-defined only up to the action of the deck transformation $\tau = \text{Id} + 2\pi i$.

Lemma 3.22. *A lift of $D_{[0]}$ in the (z, w) -coordinates defines a Dulac germ $d \in \mathcal{D}$ having an asymptotic expansion*

$$w = d(z) = \lambda z + 2\pi i k + o(1)$$

for some integer $k \in \mathbb{Z}$.

Proof. The holomorphic extension of d is simply obtained by considering the \mathcal{F} -lift to Σ of the totality of exponential paths $e^{-\xi\alpha, c}$ (i.e. no longer restricted to the strip I). The estimates in Example 3.10 show that all these paths can be lifted (in their totality), and therefore that d_0 is holomorphic on a domain of the form

$$\mathcal{E} = \{z \in \mathbb{C} : \text{Re}(z) > R, |\text{Im}(z)| < e^{\beta \text{Re}(z)}\}$$

for some $R > 0$ and some $0 < \beta < \lambda$. Such domain clearly contains a QSD. Moreover, the Cauchy problem written in Proposition 3.5 has the equivalent integral form

$$e^\varphi = e^{\varphi_0} + \int_{\gamma} F(z, \varphi) dz,$$

where $F(z, \varphi) = -\lambda e^{-((n-\lambda)z+\varphi)}$. According to the computations of Example 3.10, the integral on the right-hand side is uniformly bounded on \mathcal{E} by a constant multiplied by $e^{\beta \text{Re}(z)}$. With the choice of coordinates on the transversals described above, and passing to the covering coordinates z and w , we have $\varphi_0 = \lambda z$ and $\varphi = d(z)$. Therefore, we can write:

$$e^{d(z)} = e^{\lambda z} \left(1 + e^{(\beta-\lambda)z} f \right),$$

where f is some uniformly bounded function on \mathcal{E} . The expansion in the enunciate is obtained by taking logarithms of both sides.

Finally, by applying the truncated version of the Dulac normal form theorem to the 1-form given in Definition 3.1, we can prove (see, for instance, the *Geometric Lemma* in [II'91, section 0.3]) that d has an asymptotic expansion in the pol-exp scale, as required in the definition of Dulac germs in Section 2.1. \square

Based on this Lemma, we may fix an unambiguous choice for the lift of $D_{[0]}$ to the (z, w) coordinates. The *lifted (canonical) corner transition map* for (U, \mathcal{F}) is the unique lift $d_0 \in \mathcal{D}$ of $D_{[0]}$ having the asymptotic expansion

$$d_0(z) = \lambda z + o(1).$$

We remark that this lift is simply obtained by taking an appropriate branch of the logarithm in the map $w = -\log y$ (*i.e.* an appropriate choice of the iterate of the deck transform in the lift). Note that, by using the same deck transforms, all other determinations $D_{[k]}$, $k \in \mathbb{Z}$, lift to germs of the type

$$d_k(z) = \lambda z + 2k\pi i + o(1).$$

Remark 3.23. Notice that, by its own definition, the inverse d_0^{-1} of d_0 in \mathcal{D} corresponds to a lift of (some determination of) the inverse corner transition $\mathcal{F} \cap \Omega \rightarrow \mathcal{F} \cap \Sigma$. However, we stress the fact that it does not necessarily correspond to the canonical Dulac map that we could define by exchanging the roles of the horizontal and vertical separatrices in the construction described in Section 3.1.6.

3.1.7 Holonomies and variations of the corner transition maps

As remarked by Ilyashenko, the monodromy of the canonical corner transition map and the holonomy of a prepared saddle (U, \mathcal{F}) are related by the equation

$$D_{[0]}(e^{2\pi i} x) = \mathfrak{h}_\Omega^{-1} D_{[0]}(x) \quad (24)$$

(see *e.g.* [Lor10, Section 7.1.4]). Let us rewrite this equation in the covering coordinate $x = e^{-z}$, $y = e^{-w}$. Based on the expressions given in Lemma 3.14, we choose a lift h_Ω of \mathfrak{h}_Ω of the form

$$h_\Omega(w) = w + 2i\pi(1 - \lambda) + o(1).$$

Notice that h_Ω is an element of the Dulac group, convergent on some right-half plane $\{z \in \mathbb{C} : \operatorname{Re}(z) > c\}$ and *unramified, i.e.*

$$\operatorname{var}(h_\Omega) = \operatorname{Id},$$

where $\operatorname{var}(f) = [\tau, f]$ is the functional variation defined in Section 2.2.

We thus rewrite (24) as the following equation in the Dulac group \mathcal{D} ,

$$d_0 \tau = \tau h_\Omega^{-1} d_0.$$

In other words,

$$\operatorname{var}(d_0^{-1}) = h_\Omega^{-1}.$$

Similarly, based on Remark 3.20, we can write the relation

$$\operatorname{var}(d_0) = h_\Sigma,$$

where $h_\Sigma \in \mathcal{D}$ is the lift of the other holonomy \mathfrak{h}_Σ with the expansion

$$h_\Sigma(z) = z + 2i\pi \left(1 - \frac{1}{\lambda}\right) + o(1).$$

Remark 3.24.

1. It follows from Remark 3.20 that the lift of other determinations of the corner transition map are Dulac germs $d_n \in \mathcal{D}$ related to the canonical determination d_0 by

$$d_n = h_\Omega^{-n} d_0 = d_0 h_\Sigma^n, \quad n \in \mathbb{Z}.$$

2. The variation of a lifted corner transition map does not depend on its determination. That is, $\text{var}(d_n) = \text{var}(d_0)$, $n \in \mathbb{Z}$. This follows directly from the definition of variation operator, the above item 1. and the fact that holonomies are unramified.

3.2 Germ of a corner transition

We consider a prepared saddle foliation (U, \mathcal{F}) , where the domain U contains some fixed λ -adapted region $U_{A,B}$ (see Section 3.1). In what follows, we will work frequently with germs of a transversal. We will say that a germ of a floating transversal (Σ, σ) with base-point σ on the vertical separatrix has a *realization* in U if it extends to a holomorphic curve in $U_{A,B}$ (which we denote by the same letter Σ) such that:

1. Σ is both transversal to the foliation \mathcal{F} and the vertical fibration $\text{Fib} = \{x = \text{cst}\}$;
2. Σ is simply connected.

Recall that, for each point σ in $U_{A,B}$ lying in the vertical separatrix, there exists a standard determination of the corner transition map

$$D_{[0]} : (\Sigma^{\text{cut}}, \sigma) \rightarrow (\Omega, 1),$$

which is defined by lifting the exponential paths to an arbitrary transversal through σ (see Definition 3.18). All other determinations of the corner transition are given by the identity

$$D_{[n]} = D_{[0]} \text{hol}_\sigma^n, \quad (25)$$

where $n \in \mathbb{Z}$ and hol_σ is the holonomy map associated to the circular path $t \rightarrow (0, e^{2\pi i t} \sigma)$, $t \in [0, 1]$.

From now on, applying the identification of transversals with the same base-point (as explained in Remark 3.11), we will use the notation $D_{\sigma, [n]}$ to indicate the n^{th} determination of the germ of a corner transition associated to the *initial point* $\sigma \in U_{A,B}$. More generally, we will write simply $D_{\sigma, \star}$ to refer to one of these determinations, when it is not necessary to specify the index n .

The *corner transition class* $\text{Corner}(U, \mathcal{F})$ is the set $\{D_{\sigma, \star}\}$ of all such corner transition germs, indexed by the base-point and the corresponding determination. The groupoid of holonomy germs $\text{Hol}(U, \mathcal{F})$ acts on $\text{Corner}(U, \mathcal{F})$ by right composition. Namely, for each germ $D_2 \in \text{Corner}(U, \mathcal{F})$ with initial point σ_2

and each holonomy germ $h \in \text{Hol}(U, \mathcal{F})$ with initial point σ_1 and endpoint σ_2 , we define the action of h on D_{σ_2} by

$$D_{\sigma_1} = D_{\sigma_2} h$$

which gives a germ D_{σ_1} on $\text{Corner}(U, \mathcal{F})$ with initial point σ_1 . Notice that (25) is a particular case of this action, where σ_1 and σ_2 coincide.

3.2.1 Corner transitions determined by connecting paths

For what follows, it will be convenient to relate the corner transition germs and the germs obtained by lifting of other C^1 paths in the base, possibly different from the exponential paths used in the definition of corner transitions in Section 3.1.6.

A *connecting path* is a piecewise C^1 -path Γ in the punctured disk $\bar{\Delta}^*$ with initial point $x_0 \in \bar{\Delta}^*$ and endpoint 1.

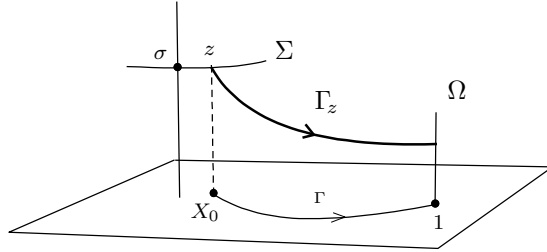
We will say that a corner transition germ $D = D_{\sigma, \star} \in \text{Corner}(U, \mathcal{F})$ based at σ is *determined* by Γ if there exists a realization of a transversal (Σ, σ) in U and an open subset $V \subset \Sigma$ such that:

1. the germ D is holomorphic at V ;
2. the path Γ has a lift Γ_z through a point $z \in V$ and

$$\text{hol}_{\Gamma_z} = D$$

(seen as germs at z of a holomorphic map from the transversal Σ to the transversal Ω).

In this case we will write $D = D_{\sigma, \Gamma}$ to indicate the connecting path which determines D .



Remark 3.25. Observe that, by construction, each corner transition $D_{\sigma, [n]}$ has the form

$$D_{\sigma, [n]} = D_{\sigma, \Gamma_n}$$

where the path $\Gamma_n = \Xi^{-1} \star \gamma^{-n}$ is obtained as the concatenation of some finite segment Ξ^{-1} of an exponential path with the $(-n)^{\text{th}}$ -iteration of the circular path $\gamma : t \rightarrow (e^{2\pi i t}, 0)$, which makes one *lap* around the boundary $S^1 = \partial \bar{\Delta}$ of the unit disk. In particular, this shows that all corner transition germs are determined by at least one connecting path.

Suppose now that two connecting paths Γ^0, Γ^1 (with same initial point x_0) are path-homotopic. Then, by the partial lifting property discussed in the previous section, for each piecewise- C^1 path-homotopy

$$H : [0, 1] \times [0, 1] \rightarrow \bar{\Delta}^*$$

such that $H(0, \cdot) = \Gamma^0(\cdot)$, $H(1, \cdot) = \Gamma^1(\cdot)$, $H(\cdot, 0) = x_0$ and $H(\cdot, 1) = 1$, there exists a disk $\mathbb{D}_H \subset \text{Fib}^{-1}(x_0)$ (lying in the fiber above the initial point and whose radius depends on both x_0 and the maximal length of paths within the homotopy H) such that for each $z \in \mathbb{D}_H$, the lifted paths $\Gamma_z^0, \Gamma_z^1 \in \text{Mon}(M, \mathcal{F})$ are homotopic³.

In particular, we obtain the following result.

Lemma 3.26. *Let Γ^0 and Γ^1 be two homotopic connecting paths. Then, there exists a radius $r > 0$ such that, for each base-point σ with $|\sigma| < r$, the following statements are equivalent:*

1. Γ^0 determines a corner transition germ based at σ ;
2. Γ^1 determines a corner transition germ based at σ .

Moreover, in this case, $D_{\sigma, \Gamma^0} = D_{\sigma, \Gamma^1}$.

Remark 3.27. The converse of this lemma does not necessarily hold.

3.2.2 Corner transitions related by the horizontal holonomy

Consider two corner transition germs $D_1 = D_{\sigma_1, \Gamma^1}$ and $D_2 = D_{\sigma_2, \Gamma^2}$ which are determined by two connecting paths Γ^1, Γ^2 . Note that Γ^1 and Γ^2 are not necessarily path-homotopic, neither have they necessarily the same initial point. Let Σ_1, Σ_2 be two respective transversals through σ_1, σ_2 and let

$$z_1 \in \Sigma_1 \quad \text{and} \quad z_2 \in \Sigma_2$$

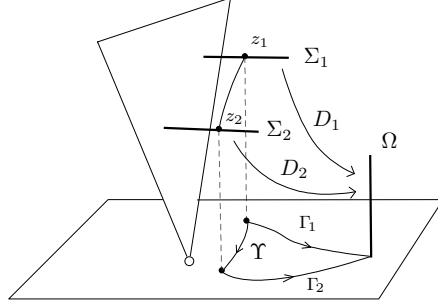
be the initial points of the corresponding lifted paths $\Gamma_{z_1}^1$ and $\Gamma_{z_2}^2$.

We will say that D_1 and D_2 are *related by the horizontal holonomy* if there exists a path $\Upsilon \subset \bar{\Delta}^*$ which admits a lift $\Upsilon_{z_1, z_2} \in \text{Mon}(U, \mathcal{F})$ in the foliation with initial point z_1 and endpoint z_2 , and satisfying

$$\text{hol}_{\Gamma_{z_1}^1} = \text{hol}_{\Gamma_{z_2}^2} \text{hol}_{\Upsilon_{z_1, z_2}},$$

where the holonomies are computed with respect to the transversals Σ_1 and Σ_2 (see figure below).

³By taking the initial point in a disk \mathbb{D}_H of sufficiently small radius, we ensure that both paths admit a complete lift starting from this initial point.



In other words, for the corresponding germs of a corner transition D_1 and D_2 , centered respectively at points z_1 and z_2 , we have:

$$D_1 = D_2 \text{hol}_{\Upsilon_{z_1, z_2}}.$$

We will say that Υ is a *transporting* path between D_1 and D_2 .

3.3 Germs of a regular transition and abstract saddle loops

Recall from the introduction that a saddle loop in a foliated surface (S, \mathcal{G}) is given by a germ of a saddle singularity and a curve Γ which is tangent to the foliation and connects the two local separatrices. The regular transition map is defined by choosing two points σ, ω on Γ near the singularity and by considering the holonomy along the sub-path $\gamma \subset \Gamma$ connecting these points.

Clearly, such a curve γ has no intrinsic meaning since we can freely move its initial point and endpoint. However, the genuine ambiguity in its choice is encoded by the local holonomy groupoid near the saddle. This motivates the definition of abstract saddle loop which we give below.

Definition 3.28. Let (U, \mathcal{F}) be a prepared saddle foliation. The *regular transition class* is the set $\text{Reg}(U, \mathcal{F})$ whose elements are germs of a diffeomorphism

$$R : (\Omega, 1) \rightarrow (\Sigma, \sigma)$$

between the fixed transversal and a transversal with base-point $\sigma \in U$ on the vertical separatrix. We say that σ is the *endpoint* of R .

Following the convention from Remark 3.11, we identify two such germs

$$R : (\Omega, 1) \rightarrow (\Sigma, \sigma), \quad \tilde{R} : (\Omega, 1) \rightarrow (\tilde{\Sigma}, \sigma)$$

with the *same* endpoint σ , if there exists a foliated chart centered on σ such that $hR = \tilde{R}$, where $h : (\Sigma, \sigma) \rightarrow (\tilde{\Sigma}, \sigma)$ is the germ of a diffeomorphism given by the chart.

The groupoid of holonomy germs $\text{Hol}(U, \mathcal{F})$ acts on $\text{Reg}(U, \mathcal{F})$ by left composition. Namely, for each germ $R_{\sigma_1} \in \text{Reg}(U)$ with endpoint σ_1 and each germ $h \in \text{Hol}(U, \mathcal{F})$ with initial point σ_1 and endpoint σ_2 , we define the action of h on R_{σ_1} by

$$(h, R_{\sigma_1}) \mapsto R_{\sigma_2} = hR_{\sigma_1} \tag{26}$$

which gives a germ $R_{\sigma_2} \in \text{Reg}(U, \mathcal{F})$ with endpoint σ_2 . Evidently, this action accounts for the change of floating transversal.

Definition 3.29. An *abstract loop* on U is a triple $(U_{A,B}, \mathcal{F}, \mathcal{R})$, where (U, \mathcal{F}) is a prepared saddle and $\mathcal{R} \subset \text{Reg}(U)$ is an orbit for the action of $\text{Hol}(U, \mathcal{F})$ defined above. In other words, an orbit \mathcal{R} is a subset of regular germs in $\text{Reg}(U, \mathcal{F})$ linked by the relation (26).

Notice that, for a fixed base-point σ , there exist at most countably many distinct germs $R \in \mathcal{R}$ with endpoint σ . Indeed, given one such germ $R_{[0]} \in \mathcal{R}$, all others are of the form

$$R_{[n]} = \text{hol}_\sigma^n R_{[0]} \quad (27)$$

for some $n \in \mathbb{Z}$, where hol_σ is the holonomy map associated to the circular path $t \mapsto (0, e^{2\pi i t} \sigma)$, $t \in [0, 1]$.

Following the convention introduced in Section 3.2 for germs of a corner transition, we will denote simply by $R_{\sigma, \star}$ a regular transition germ in \mathcal{R} with endpoint σ (whenever it is irrelevant to specify which one we consider).

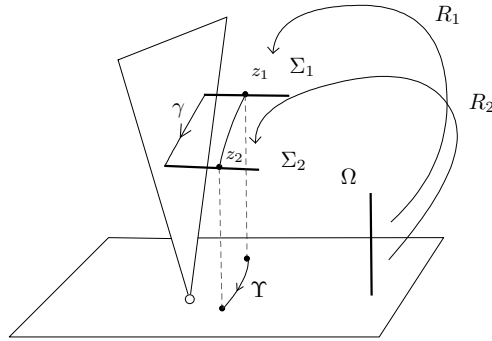
3.3.1 Regular transitions related by the horizontal holonomy

Let $(U, \mathcal{F}, \mathcal{R})$ be an abstract saddle loop and let $R_1 = R_{\sigma_1, \star}$ and $R_2 = R_{\sigma_2, \star}$ be two germs in \mathcal{R} with respective endpoints σ_1, σ_2 .

As in Section 3.2.2, we will say that R_1 and R_2 are *related by the horizontal holonomy* if there exist a path $\Upsilon \subset \bar{\Delta}^*$, two realizations of R_1, R_2 on respective transversals Σ_1, Σ_2 and two points z_1, z_2 in these transversals such that Υ lifts to a path $\Upsilon_{z_1, z_2} \in \text{Mon}(U_{A,B}, \mathcal{F})$ from z_1 to z_2 and

$$R_2 = \text{hol}_{\Upsilon_{z_1, z_2}} R_1,$$

understood as germs from z_1 to z_2 (see figure below). We will say that Υ is a *transporting path* between R_1 and R_2 .



Remark 3.30. Observe that any two germs $R_1, R_2 \in \mathcal{R}$ are always related by the horizontal holonomy. Indeed, by the definition of \mathcal{R} , there exists a path

$\gamma \in \text{Mon}(U, \mathcal{F})$ between the base-points σ_1 and σ_2 (which is *a fortiori* contained in the vertical leaf $\{x = 0\}$) such that

$$R_2 = \text{hol}_\gamma R_1$$

(as germs at σ_1 and σ_2 respectively). In order to obtain a path Υ , it suffices to choose a realization of hol_γ , R_1 and R_2 on small open neighborhoods of the transversals and choose points $z_1 \neq \sigma_1, z_2 \neq \sigma_2$ such that $\text{hol}_\gamma(z_1) = z_2$. By the uniqueness of the lift, it follows that the germ of hol_γ at z_1 is simply the holonomy associated to some path $\Upsilon_{z_1, z_2} \in \text{Mon}(U, \mathcal{F})$, which can be projected by the fibration Fib onto a path $\Upsilon \subset \Delta^*$ satisfying the above conditions.

3.3.2 Poincaré first return maps of an abstract saddle loop

Let $(U, \mathcal{F}, \mathcal{R})$ be an abstract saddle loop. Given a base-point σ on the vertical separatrix, let $R = R_{\sigma, \star} \in \text{Reg}(U, \mathcal{F})$ be a representative of \mathcal{R} with endpoint σ and let $D = D_{\sigma, \star} \in \text{Corner}(U, \mathcal{F})$ be a corner transition germ with initial point σ . The composition

$$P = RD$$

will be called a *Poincaré first return germ based at σ* .

Definition 3.31. The set of all such germs will be called the *Poincaré first return class* of the abstract loop, and denoted by $\text{Poinc}(U, \mathcal{F}, \mathcal{R})$. The factorization $P = RD$ as above will be called a *dynamical decomposition* of the germ P .

For future use, let us state the following obvious result.

Lemma 3.32. *Let (R_1, D_1) and (R_2, D_2) be elements in $\mathcal{R} \times \text{Corner}(U, \mathcal{F})$ which define respective Poincaré first return germs*

$$P_1 = R_1 D_1 \quad \text{and} \quad P_2 = R_2 D_2,$$

*based respectively at σ_1, σ_2 . Suppose that both R_1, R_2 and D_1, D_2 are related by the **same** transporting path Υ . Then P_1 and P_2 are analytically conjugate.*

Proof. By definition, there exists a realization of both R_1, D_1 on a common transversal Σ_1 and a realization of R_2, D_2 on a common transversal Σ_2 and two points $(z_1, z_2) \in \Sigma_1 \times \Sigma_2$ such that

$$R_2 = \text{hol}_{\Upsilon_{z_1, z_2}} R_1 \quad \text{and} \quad D_1 = D_2 \text{hol}_{\Upsilon_{z_1, z_2}}. \quad (28)$$

By assumption, R_1 and R_2 are analytic diffeomorphisms defined on connected domains, mapping $1 \in \Omega$ respectively to σ_1 and σ_2 . Therefore, putting $\varphi = R_2 R_1^{-1}$, the leftmost identity in (28) implies that φ defines a germ of a diffeomorphism from (Σ_1, σ_1) to (Σ_2, σ_2) such that $\varphi = \text{hol}_{\Upsilon_{z_1, z_2}}$ (seen as a germ from z_1 to z_2). Hence, by (28),

$$P_2 = R_2 D_2 = \varphi R_1 D_1 \varphi^{-1} = \varphi P_1 \varphi^{-1}.$$

□

3.4 Loop germs and their equivalence

We now *germify* the notion of abstract saddle loops by considering the basis of neighborhoods of the disk $\bar{\Delta}$ defined by the family of λ -neighborhoods.

Definition 3.33. A *loop germ* $(\mathcal{F}, \mathcal{R})$ is an equivalence class of abstract saddle loops, where two abstract loops

$$(U_1, \mathcal{F}_1, \mathcal{R}_1) \quad (U_2, \mathcal{F}_2, \mathcal{R}_2)$$

are *equivalent* if there exists a λ -adapted domain U contained in $U_1 \cap U_2$ such that

$$\mathcal{F}_1 = \mathcal{F}_2 \quad \text{on} \quad U$$

and the regular transition classes \mathcal{R}_1 and \mathcal{R}_2 coincide when restricted to U .

Accordingly, the *corner transition class* of a loop germ is the set $\text{Corner}(\mathcal{F})$ of Dulac germs which can be obtained as corner transition germs of some of its realizations (U, \mathcal{F}) . We define analogously the *Poincaré first return class* as the set $\text{Poinc}(\mathcal{F}, \mathcal{R})$ of Dulac germs which can be obtained as Poincaré first return maps of some of its realizations $(U, \mathcal{F}, \mathcal{R})$.

Consider now a germ of a fibered diffeomorphism $\Phi \in \text{Diff}_{\text{fib}}(\mathbb{C}^2, \bar{\Delta})$. The *action* of Φ on a loop germ $(\mathcal{F}, \mathcal{R})$ defines a new loop germ

$$(\tilde{\mathcal{F}}, \tilde{\mathcal{R}})$$

obtained as follows.

1. Choose representatives of both Φ and \mathcal{F} on some λ -adapted neighborhood U and define

$$\tilde{\mathcal{F}} = \Phi(\mathcal{F}).$$

2. Choose an element $R \in \mathcal{R}$, *i.e.* a germ of a regular transition

$$R : (\Omega, 1) \longrightarrow (\Sigma, \sigma),$$

and define $\tilde{\mathcal{R}}$ to be the regular transition class generated by

$$\tilde{R} = (\Phi|_{\Sigma})R(\Phi|_{\Omega})^{-1}. \tag{29}$$

Definition 3.34. Under conditions 1. and 2. above, we write $(\tilde{\mathcal{F}}, \tilde{\mathcal{R}}) = \Phi \cdot (\mathcal{F}, \mathcal{R})$, and we say that $(\tilde{\mathcal{F}}, \tilde{\mathcal{R}})$ and $(\mathcal{F}, \mathcal{R})$ are $\text{Diff}_{\text{fib}}(\mathbb{C}^2, \bar{\Delta})$ -*equivalent*.

Remark 3.35. We remark the following basic correspondence between connecting paths of corner transition maps in $\text{Diff}_{\text{fib}}(\mathbb{C}^2, \bar{\Delta})$ -equivalence. In the above notation, suppose that Φ has a realization on a domain U and let $D \in \text{Corner}(U, \mathcal{F})$ be a corner transition map defined on a transversal Σ with base-point σ . Then

$$\tilde{D} \stackrel{\text{def}}{=} (\Phi|_{\Omega})D(\Phi|_{\Sigma})^{-1} \tag{30}$$

is a corner transition germ for $\tilde{\mathcal{F}}$, defined on a transversal with base-point $\tilde{\sigma} = \Phi(\sigma)$. Moreover, if $D = D_{\sigma, \Gamma}$ is determined by \mathcal{F} -lifting a connecting path Γ then $\tilde{D} = D_{\tilde{\sigma}, \tilde{\Gamma}}$ is determined by the $\tilde{\mathcal{F}}$ -lifting of the *same* connecting path (since Φ preserves the vertical fibration).

We will say that the germs $\tilde{R} \in \tilde{\mathcal{R}}$ and $\tilde{D} \in \text{Corner}(\tilde{\mathcal{F}})$ defined by (29) and (30) are respectively the Φ -*correspondents* of $R \in \mathcal{R}$ and $D \in \text{Corner}(\mathcal{F})$. We denote this relation by $\tilde{R} = \Phi_*(R)$ and $\tilde{D} = \Phi_*(D)$.

3.4.1 Equivalence implies conjugacy of Poincaré transition germs

Our first equivalence theorem states that equivalent loop germs possess conjugate Poincaré first return classes.

Theorem 3.36. *Consider two $\text{Diff}_{\text{fib}}(\mathbb{C}^2, \bar{\Delta})$ -equivalent loop germs $(\tilde{\mathcal{F}}, \tilde{\mathcal{R}})$ and $(\mathcal{F}, \mathcal{R})$ and let Φ be a fibered diffeomorphism such that $(\mathcal{F}, \mathcal{R}) = \Phi \cdot (\tilde{\mathcal{F}}, \tilde{\mathcal{R}})$. Let*

$$P \in \text{Poinc}(\mathcal{F}, \mathcal{R}), \quad \tilde{P} \in \text{Poinc}(\tilde{\mathcal{F}}, \tilde{\mathcal{R}})$$

be two Poincaré first return germs with dynamical decompositions $P = RD$ and $\tilde{P} = \tilde{R}\tilde{D}$, such that the pairs $R_1, R_2 \in \mathcal{R}$ and $D_1, D_2 \in \text{Corner}(\mathcal{F})$ defined by

$$R_1 = R, R_2 = \Phi_*(\tilde{R}) \quad \text{and} \quad D_1 = D, D_2 = \Phi_*(\tilde{D})$$

*are related by a **same** transporting path $\Upsilon \in \bar{\Delta}^*$. Then, P and \tilde{P} are analytically conjugate.*

Proof. It follows from Lemma 3.32 that that $P = P_1 = R_1D_1$ and $P_2 = R_2D_2$ are analytically conjugate. Moreover, we have

$$P_2 = R_2D_2 = (\Phi|_{\bar{\Sigma}})\tilde{R}(\Phi|_{\Omega})^{-1}(\Phi|_{\Omega})\tilde{D}(\Phi|_{\bar{\Sigma}})^{-1} = (\Phi|_{\bar{\Sigma}})\tilde{P}(\Phi|_{\bar{\Sigma}})^{-1}$$

and, therefore, P_2 is analytically conjugate to \tilde{P} . \square

3.4.2 Conjugacy of Poincaré transition germs implies equivalence

Our next goal is to prove the converse of Theorem 3.36.

Theorem 3.37. *Let $(\tilde{\mathcal{F}}, \tilde{\mathcal{R}})$ and $(\mathcal{F}, \mathcal{R})$ be two loop germs, and suppose that there exists two Poincaré first return germs*

$$P \in \text{Poinc}(\mathcal{F}, \mathcal{R}), \quad \tilde{P} \in \text{Poinc}(\tilde{\mathcal{F}}, \tilde{\mathcal{R}})$$

which are analytically conjugate. Then, $(\tilde{\mathcal{F}}, \tilde{\mathcal{R}})$ and $(\mathcal{F}, \mathcal{R})$ are $\text{Diff}_{\text{fib}}(\mathbb{C}^2, \bar{\Delta})$ -equivalent.

Proof. Let

$$P \in \text{Poinc}(\mathcal{F}, \mathcal{R}), \quad \tilde{P} \in \text{Poinc}(\tilde{\mathcal{F}}, \tilde{\mathcal{R}}),$$

from the statement be the germs of a first return map defined on the respective transversals (Σ, σ) and $(\tilde{\Sigma}, \tilde{\sigma})$. Then, by assumption, there exists a germ of an analytic diffeomorphism φ between these transversals such that $\varphi(\sigma) = \tilde{\sigma}$ and:

$$\tilde{P} = \varphi P \varphi^{-1}.$$

Let $P = RD$, $\tilde{P} = \tilde{R}\tilde{D}$ be dynamical decompositions of P and \tilde{P} based respectively at σ and $\tilde{\sigma}$. The conjugacy relation between P and \tilde{P} gives $\tilde{R}\tilde{D} = \varphi RD\varphi^{-1}$, or, equivalently,

$$(R^{-1}\varphi^{-1}\tilde{R})\tilde{D}\varphi = D.$$

Let us consider the lift of these germs to the covering coordinates $x = e^{-z}$, $y = e^{-w}$. Then, we obtain the relation

$$v \tilde{d} u = d,$$

where the Dulac germs \tilde{d} , d are the lifts of \tilde{D} , D , and the unramified germs u, v are respectively the lifts of φ and $R^{-1}\varphi^{-1}\tilde{R}$. By applying the variation operator to both sides, the identity (50) from the Appendix A gives:

$$v \operatorname{var}(\tilde{d}) v^{-1} = \operatorname{var}(d^{-1}).$$

We now apply Theorem 3.15 to obtain a fibered germ $\Phi \in \operatorname{Diff}_{\text{fib}}(\mathbb{C}^2, \bar{\Delta})$, which maps $\tilde{\mathcal{F}}$ to \mathcal{F} (as germs, on some neighborhoods of the separatrices that contain λ -prepared domains for $\tilde{\mathcal{F}}$ resp. \mathcal{F}), such that

$$\Phi \Big|_{\Omega} = R^{-1} \circ \varphi^{-1} \circ \tilde{R}. \quad (31)$$

Without loss of generality, we may assume that the transversal $(\tilde{\Sigma}, \tilde{\sigma})$ above is chosen sufficiently close to zero, so that it lies in the domain of definition of the equivalence Φ and its image by Φ belongs to some λ -adapted region for \mathcal{F} . Indeed, instead of $\tilde{P} = \tilde{R}\tilde{D}$ on $(\tilde{\Sigma}, \tilde{\sigma})$, take another element $\tilde{P}_1 \in \operatorname{Poinc}(\tilde{\mathcal{F}}, \tilde{\mathcal{R}})$, defined on some transversal $(\tilde{\Sigma}_1, \tilde{\sigma}_1)$ sufficiently close to the origin, so that it lies in the domain of definition of Φ . Let $\tilde{P}_1 = \tilde{R}_1\tilde{D}_1$ be its dynamical decomposition at $\tilde{\sigma}_1$. We get that $\tilde{D}_1 = \tilde{D}k^{-1}$ and $\tilde{R}_1 = k\tilde{R}$, that is, $\tilde{P}_1 = k\tilde{P}k^{-1}$, for some holonomy k of $\tilde{\mathcal{F}}$. That means that $\tilde{P}_1 = k\varphi P \varphi^{-1} k^{-1}$. Applying Theorem 3.15, in the same way as above we get that there exists a diffeomorphism Φ_1 of foliations \mathcal{F} and $\tilde{\mathcal{F}}$ such that

$$\Phi_1 \Big|_{\Omega} = R^{-1} \circ \varphi^{-1} \circ k^{-1} \circ \tilde{R}_1 = R^{-1} \circ \varphi^{-1} \circ \tilde{R} = \Phi \Big|_{\Omega}.$$

By the uniqueness of the analytic extension, Φ_1 extends as Φ analytically over the transversal $(\tilde{\Sigma}_1, \tilde{\sigma}_1)$.

Let us now consider the image of the loop germ $(\tilde{\mathcal{F}}, \tilde{\mathcal{R}})$ under the action of Φ , which we denote by $(\mathcal{F}, \bar{\mathcal{R}})$. By construction, using (31), the Φ -correspondent of \tilde{R} is given by

$$\bar{R} = (\Phi|_{\tilde{\Sigma}})\tilde{R}(\Phi|_{\Omega})^{-1} = \Phi \Big|_{\tilde{\Sigma}} \circ \varphi \circ R. \quad (32)$$

What is left to prove is that $(\mathcal{F}, \overline{\mathcal{R}})$ and $(\mathcal{F}, \mathcal{R})$ define the same loop germ. That is, that there exist $\overline{R} \in \overline{\mathcal{R}}$, $R \in \mathcal{R}$ and a holonomy map h of \mathcal{F} such that $R = h\overline{R}$.

Take the decomposition $\overline{R}\overline{D}$ such that \overline{R} and \overline{D} are Φ -correspondents of \tilde{R} and \tilde{D} . Then $\overline{P} = \overline{R}\overline{D} \in \text{Poinc}(\mathcal{F}, \overline{\mathcal{R}})$, and:

$$\overline{P} = (\Phi|_{\overline{\Sigma}})\tilde{P}(\Phi|_{\overline{\Sigma}})^{-1} = (\Phi|_{\overline{\Sigma}}) \circ \varphi \circ P \circ \varphi^{-1} \circ (\Phi|_{\overline{\Sigma}})^{-1}. \quad (33)$$

Since \overline{D} and D are corner transition germs of the same foliation \mathcal{F} (defined in its λ -invariant domain), there exists a holonomy germ h such that

$$\overline{D} = Dh. \quad (34)$$

Now, putting (34) and (32) in (33), writing $P = RD$ and $\overline{P} = \overline{R}\overline{D}$, we get after simplifications that $h^{-1} = \Phi|_{\overline{\Sigma}} \circ \varphi$, that is, by (32), $\overline{R} = h^{-1}R$. \square

3.5 Geometric realizations of a loop germ

We recall from the Introduction that a saddle loop in a foliated complex analytic surface (S, \mathcal{G}) is defined by a saddle singularity $s \in S$ and an oriented C^1 -path $\Gamma \subset S$ that is tangent to \mathcal{G} and connects its local separatrices. Using Proposition 3.1.1, it is easy to prove that we can uniquely associate a loop germ $\mathbb{L} = (\mathcal{F}, \mathcal{R})$ to the data given by (S, \mathcal{G}) , s and Γ .

In this subsection, our goal is to prove the converse. Namely, we prove that any germ of a saddle loop can be *embedded* in a foliated complex analytic surface. Let $\mathbb{L} = (\mathcal{F}, \mathcal{R})$ be a loop germ, as defined in Section 3.4. A *gluing domain* for \mathbb{L} is an open neighborhood $W \subset \mathbb{C}^2$ of the horizontal unit disk $\overline{\Delta}$ whose traces on the coordinates axis, $W \cap \{y = 0\}$ and $W \cap \{x = 0\}$ are simply connected domains such that both \mathcal{F} and \mathcal{R} have realizations on W .

We recall that the latter condition means that \mathcal{F} extends to a holomorphic foliation on W and that one can choose some horizontal transverse section (Σ, σ) with base-point $(0, \sigma)$ in $\{x = 0\} \cap W$ and a diffeomorphism

$$R : (\Omega, 1) \longrightarrow (\Sigma, \sigma) \quad (35)$$

that belongs to the regular transition class \mathcal{R} (see Definition 3.29).

Remark 3.38. We will use this map to quotient W out by identifying a point of (a foliated neighborhood of) Ω with its image by R . Notice that we do not assume W to be an adapted λ -neighborhood as in Definition 3.1. In fact, we will see that W will sometimes have to be shrunk in order to guarantee that the quotient space is Hausdorff.

Let (S, \mathcal{G}) be a foliated complex analytic surface.

Definition 3.39. A (*geometric*) realization of \mathbb{L} in (S, \mathcal{G}) is given by the following data:

1. A biholomorphic map $\Psi : W \rightarrow V$ between a gluing domain $W \subset \mathbb{C}^2$ and an open subset $V \subset S$ such that

$$\Psi(\mathcal{F}|_W) = \mathcal{G}|_V$$

In particular, $s = \Psi(0)$ is a saddle singularity of \mathcal{G} .

2. An oriented curve $\gamma \subset S$ contained in a leaf of \mathcal{G} , connecting the points $\Psi((0, 1))$ to $\Psi((0, \sigma))$ and such that the associated germ of a holonomy map $\text{hol}_{\mathcal{G}, \gamma}$ between $\Psi(\Omega)$ and $\Psi(\Sigma)$ satisfies

$$\text{hol}_{\mathcal{G}, \gamma} = (\Psi|_{\Sigma})R(\Psi|_{\Omega})^{-1}$$

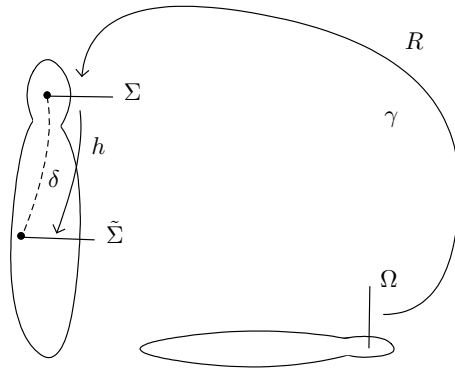
for some germ $R \in \mathcal{R}$ as in (35).

We will say Ψ is the *embedding map* and that γ is the *connecting path* of the realization.

Remark 3.40. Notice that a geometric realization is defined by choosing one representative $R \in \mathcal{R}$ of the regular transition class. However, once the embedding map $\Psi : W \rightarrow V$ and the connecting path $\gamma \subset S$ have been chosen, we can obtain realizations for different choices of representative in \mathcal{R} . More precisely, in the notation of the previous definition, let $\delta \in \text{Mon}(W, \mathcal{F})$ be a path on the vertical separatrix between σ and another point $\tilde{\sigma}$ and let $\tilde{R} \in \mathcal{R}$ be the germ from $(\Omega, 1)$ to $(\tilde{\Sigma}, \tilde{\sigma})$ defined by

$$\tilde{R} = \text{hol}_{\mathcal{F}, \delta} R$$

(see Definition 3.29). Thus condition 2. of the previous definition holds with R replaced by \tilde{R} if we change the path γ to $\gamma \star \varepsilon$, where the $\varepsilon \in \text{Mon}(S, \mathcal{G})$ is the image of δ under the induced morphism of groupoids $\Psi_* : \text{Mon}(W, \mathcal{F}) \rightarrow \text{Mon}(S, \mathcal{G})$.



Let us see some examples of a geometric realization.

Example 3.41. Consider the foliation \mathcal{C} on $\mathbb{P}^1(\mathbb{C})$ defined by the polynomial differential form

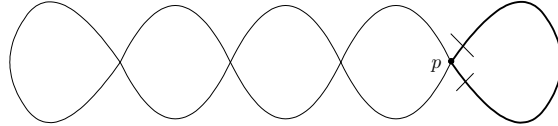
$$\omega = d(y^2 - x^2 + x^3).$$

The saddle point at the origin is linearizable, locally orbitally equivalent to the linear foliation \mathcal{F}_{lin} defined by $d(uv) = 0$. The separatrices $\{u = 0\}$ and $\{v = 0\}$ correspond to the local branches of the cubic $\Gamma = (y^2 - x^2 + x^3 = 0)$. Up to a scaling, we can assume that the image U of this orbital equivalence contains the closure of the polydisk $\Delta \times \Delta$ in the u, v variables.

Fixing the transverse sections $\Sigma = \{v = 1\}, \Omega = \{u = 1\}$, equipped with the natural parameterizations defined by the ambient coordinates, and the connecting path γ given by the real trace of Γ , we easily see that $(\mathbb{P}^1(\mathbb{C}), \mathcal{C})$ contains a realization of the loop germ

$$\mathbb{L} = (\mathcal{F}_{1:1}, \text{Id}). \quad (36)$$

Example 3.42. More generally, consider a polynomial $h \in \mathbb{R}[z, w]$ which has a simple critical point of saddle type at $p \in \mathbb{R}^2$ and such that the real algebraic curve $\Gamma_{\mathbb{R}} = h^{-1}(\{h(p)\})$ contains a smooth component in $\Gamma_{\mathbb{R}} \setminus \{p\}$ connecting p to itself. Then, this data also provides a realization of the abstract saddle loop (36) in the foliated surface $(\mathbb{P}^1(\mathbb{C}), \text{Fol}(dh))$.



Example 3.43. Consider the following perturbation of the integrable foliation given in Example 3.41,

$$\omega_{\varepsilon} = d(w^2 - z^2 + z^3) + \varepsilon \frac{(z^2 - z^3)^2}{wz} d\left(\frac{w^2}{z^2 - z^3}\right)$$

Notice that the cubic Γ is an invariant algebraic curve for all values of the parameter ε . The ratio of eigenvalues at the saddle point is given by $\lambda_{\varepsilon} = \frac{\varepsilon-1}{\varepsilon+1}$. By a suitable compactification, we can prove that the (closure of the) complex leaf containing Γ is homeomorphic to a pinched torus, and that the holonomies are periodic when λ_{ε} is positive rational. Hence that the saddle point is linearizable. We do not know if this is true for irrational values of λ_{ε} (we refer to [Bru15, Example 7.1 and Corollary 9.2] for a more general discussion on foliations tangent to virtually elliptic curves).

In the Examples 3.41–3.43, the connecting curve γ is contained in an algebraic leaf. Let us see an example where this is not the case.

Example 3.44. Consider the one-parameter family of foliations defined by

$\omega_b = P_b dy - Q_b dx$, where

$$P_b = \frac{1}{4}(2x+1)(2y+1) - b$$

$$Q_b = \frac{1}{4}(2x-1)(2y-1) - b$$

This foliation has a Darbouxian first integral $H = e^{y-x} \left(xy + \frac{1}{2}(y-x) - b - \frac{1}{4} \right)$. For each real $b < \frac{1}{4}$, such a foliation has a saddle loop inside the non-algebraic leaf given by the level set $\{H = h_c\}$, where $h_c = -\frac{1}{2} \left(\sqrt{-4b+1} - 1 \right) e^{-\sqrt{-4b+1}}$.

More generally, we can exhibit examples of non-algebraic saddle loops inside non-integrable foliations.

Example 3.45. The *Kaypten-Dulac family* is the family of foliations defined by the 6-parameter differential form $\omega_\lambda = P_\lambda dy - Q_\lambda dx$, where

$$P_\lambda = -y + \lambda_1 x - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2$$

$$Q_\lambda = x + \lambda_1 y + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2$$

For $\lambda = (0, 0, 0, 0, 0, -1)$, the form $\omega_\lambda = -d(y^2/2 + y^3/3 + x^2/2)$ is integrable and has an algebraic saddle loop. It follows from [Rou98, Section 5.3.2] that there exists a nonzero vector $(a, b, c) \in \mathbb{R}^3$ such that for all small perturbation of parameters lying inside the hypersurface

$$a\lambda_1 + b\lambda_2\lambda_4 + c\lambda_5 = 0,$$

the family still has a saddle loop. For generic values of parameters inside this hypersurface, the associated foliation cannot be integrable, since each one of the integrability strata in Kaypten-Dulac family (given by the Dulac theorem) is of codimension greater or equal than two.

All previous examples show realizations of abstract saddle loops in algebraic foliated surfaces. However, it follows⁴ from [GT10] that not every abstract saddle loop has a realization in the algebraic category. Let us prove that a realization always exists in the analytic category.

Proposition 3.46. *Each loop germ can be realized in a foliated analytic complex surface.*

Proof. Let $(\mathcal{F}, \mathcal{R})$ be a loop germ. We fix once and for all a λ -neighborhood $U \subset \mathbb{C}^2$ where \mathcal{F} is holomorphic and a germ R as in (35) which represents \mathcal{R} on U . The basic idea of proof is quite simple: we will construct the surface S by *holomorphically gluing* the transverse sections Ω and Σ through R . However, we must be careful to guarantee that the resulting topological space is Hausdorff.

⁴The argument of [GT10] translates to resonant saddle foliations at least when $\lambda = 1$ through a blow-up of the saddle-node singularity. It most probably does for other rational eigenratios.

We consider first two neighborhoods $V_1, V_2 \subset U$ of the base-points $(1, 0)$ and $(0, \sigma)$ of the transversals, equipped with appropriate rectifying coordinates $(x_1, y_1), (x_2, y_2)$. More precisely, we choose V_1, V_2 such that the next two conditions are fulfilled.

1. $V_1 \cap \{y = 0\}$ is a disk centered in $(1, 0)$ and there exists a biholomorphism $\Psi_1 : V_1 \rightarrow \Delta \times \Delta$ mapping $\mathcal{F}|_{V_1}$ to the horizontal foliation $\{y_1 = \text{cst}\}$, such that $\Psi_1(\Omega) = \Delta \times \{0\}$ and $\Psi_1(y = 0) = (y_1 = 0)$.
2. $V_2 \cap \{x = 0\}$ is a disk centered in $(0, \sigma)$ and there exists a biholomorphism $\Psi_2 : V_2 \rightarrow \Delta \times \Delta$ mapping $\mathcal{F}|_{V_2}$ to the vertical foliation $\{x_2 = \text{cst}\}$, such that $\Psi_2(\Sigma) = \{0\} \times \Delta$ and $\Psi_2(x = 0) = (x_2 = 0)$.

Up to reducing the size of such neighborhoods, we can further suppose Ψ_1, Ψ_2 have holomorphic extensions to the closures $\overline{V_1}, \overline{V_2}$ and that

$$\overline{V_1} \cap \overline{V_2} = \emptyset. \quad (37)$$

We now choose two polydisks centered on the origin

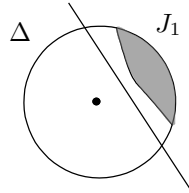
$$D_1 = \Delta_a \times \Delta_\varepsilon, \quad D_2 = \Delta_\varepsilon \times \Delta_b, \quad (38)$$

where $\varepsilon > 0$ is some small constant to be chosen later and the radius $a, b > 0$ are going to be chosen such that, intuitively, the intersections $D_1 \cap V_1$ and $D_2 \cap V_2$ are *sufficiently small*.

More precisely, we choose $0 < a < 1$ such that the region

$$J_1 = \Psi_1((\Delta_a \times \{0\}) \cap V_1)$$

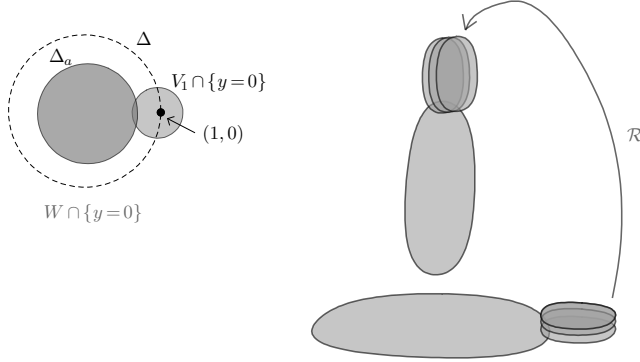
is a non-empty open subset of the unit disk whose closure can be separated from the origin by an affine line (see the figure below).



We choose similarly $0 < b < |\sigma|$ such that $J_2 = \Psi_2((\{0\} \times \Delta_b) \cap V_2)$ is a non-empty subset of Δ whose closure can be separated from the origin by an affine line. It follows that we can choose a rotation $r_\lambda(z) = \lambda z$, $|\lambda| = 1$, such that $r_\lambda(\overline{J_1}) \cap \overline{J_2} = \emptyset$.

We now modify one of the rectifying chart, say (x_1, y_1) , by post-composing the horizontal coordinate with the above rotation $x_1 \mapsto r_\lambda(x_1)$. Therefore, in these new coordinates, we obtain simply

$$\overline{J_1} \cap \overline{J_2} = \emptyset. \quad (39)$$



Up to taking $\varepsilon > 0$ sufficiently small, we can assume that the same separation property holds for the whole polydisks, namely that the sets

$$\overline{\Psi_1(D_1 \cap V_1)} \quad \text{and} \quad \overline{\Psi_2(D_2 \cap V_2)} \quad (40)$$

are disjoint. Up to further reducing ε , we can therefore suppose that the intersections

$$\overline{D_1} \cap \overline{V_2} \quad \text{and} \quad \overline{D_2} \cap \overline{V_1} \quad (41)$$

are empty.

Now, using the coordinate charts (x_1, y_1) and (x_2, y_2) given above we fix a realization $x_2 = \tilde{R}(y_1)$ of the germ R between two small simply connected neighborhoods $\tilde{\Sigma} \subset \Sigma$ and $\tilde{\Omega} \subset \Omega$ of the corresponding base-points and define a biholomorphism $G : \tilde{\Omega} \times \Delta \rightarrow \Delta \times \tilde{\Sigma}$ by

$$(x_2, y_2) = G(x_1, y_1) = (y_1, \tilde{R}(x_1)),$$

which we call the *gluing map*. Notice that, according to this definition, the restricted map $G|_{y_1=0}$ is the identity. In particular, $G|_{y_1=0}(\overline{J_1}) = \overline{J_1}$ is disjoint from $\overline{J_2}$, according to (39). Therefore, up to a further decrease of $\varepsilon > 0$, we can assume from (40) that

$$G\left(\overline{\Psi_1(D_1 \cap V_1)}\right) \quad \text{and} \quad \overline{\Psi_2(D_2 \cap V_2)} \quad (42)$$

are disjoint.

We now consider the gluing domain given by the union $W = D_1 \cup D_2 \cup (\tilde{\Omega} \times \Delta) \cup (\Delta \times \tilde{\Sigma})$ and the quotient set

$$S = W / \sim$$

where the equivalence relation \sim identifies each point p in $\tilde{\Omega} \times \Delta$ to its image point $G(p) \in \Delta \times \tilde{\Sigma}$.

It is obvious that S has a holomorphic structure inherited from the holomorphic structure on W . Moreover, if we denote by $\pi : W \rightarrow S$ the quotient

map, the image $\mathcal{G} = \pi(\mathcal{F})$ is a holomorphic foliation in S satisfying all the above requirements.

The only thing that remains to be proved is that S is Hausdorff. Or, equivalently, to show that the diagonal $\Delta(S)$ is closed in the product $S \times S$. By the above construction, this amounts to proving that the graph of the gluing map G ,

$$\text{Graph}(G) = \{(p, G(p)) : p \in (\tilde{\Omega} \times \Delta)\}$$

is a closed subset of $W \times W$. But this property is a consequence of the separation property (40).

Indeed, let us suppose by contradiction that there exists a limit point $(\bar{p}, \bar{q}) \in W \cap \partial \text{Graph}(G)$. It follows from (37) and (41) that $\bar{p} \in D_1$ and $\bar{q} \in D_2$. On the other hand, there exists a sequence

$$\{(p_n, G(p_n))\}_{n \geq 1} \subset \text{Graph}(G)$$

converging to (\bar{p}, \bar{q}) . Consider the image of this sequence using the Ψ_1 and Ψ_2 maps. Then, for all n sufficiently large, p_n lies in $\Psi_1(D_1 \cap V_1)$ and $G(p_n)$ lies in $\Psi_2(D_2 \cap V_2)$, which contradicts (42). The proposition is proved. \square

Remark 3.47. Based on the above construction, a natural question is whether one can choose S to be a Stein manifold (that can be embedded into \mathbb{C}^n for some sufficiently large n). Answering this question will be the subject of an upcoming work.

3.6 Real saddle loops

We now consider the problem of classifying real analytic saddle loops. We will say that a loop germ $\mathbb{L} = (\mathcal{F}, \mathcal{R})$ is *real* if:

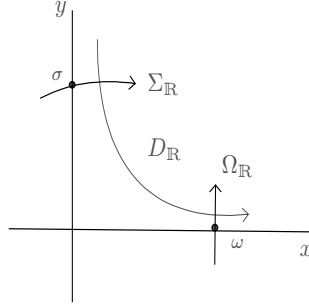
1. we can choose a generator x of \mathcal{F} which is a real analytic vector field;
2. we can choose a representative

$$R : (\Sigma, \sigma) \rightarrow (\Omega, 1)$$

that belongs to the transition class \mathcal{R} , such that Σ, Ω are complexifications of real analytic curves (which we denote by $\Sigma_{\mathbb{R}}$ and $\Omega_{\mathbb{R}}$) and $R : \Sigma_{\mathbb{R}} \rightarrow \Omega_{\mathbb{R}}$ is a real analytic map.

Let us fix one such real analytic germ R . Up to a reflection and multiplication of x by -1 , we can suppose that the configuration of the transversals and the orientation of the solution curves of x is as illustrated below.

In the picture, we indicate by an arrow the orientation of the real transversals in such a way that both ordered basis $\{x(\omega), T_{\omega}\Omega\}$ and $\{T_{\sigma}\Sigma, x(\sigma)\}$ are positively oriented (with respect to the standard orientation of \mathbb{R}^2). Based on these orientations, let us denote by $\Sigma_{\mathbb{R}>0}$ and $\Omega_{\mathbb{R}>0}$ the positive parts of these transversals (and define similarly $\Sigma_{\mathbb{R}<0}$ and $\Omega_{\mathbb{R}<0}$).



The leaf correspondence induced by the real trajectories uniquely defines a determination of the corner transition that maps $\Sigma_{\mathbb{R}>0}$ into $\Omega_{\mathbb{R}>0}$. This is precisely the canonical determination considered in Section 3.1.6 (see Remark 3.21). We call it the *real* corner transition map associated to \mathbb{L} , and write it simply D .

For later use, we also distinguish two possible cases concerning the regular real transition map R considered in the item 2. of the above definition.

Definition 3.48. We will say that R *preserves the orientation* if it maps $\Omega_{\mathbb{R}>0}$ to $\Sigma_{\mathbb{R}>0}$. We say that R *reverses the orientation* if it maps $\Omega_{\mathbb{R}>0}$ to $\Sigma_{\mathbb{R}<0}$.

Remark 3.49. If R preserves the orientation then the *real* Poincaré first return map,

$$P = RD$$

is a real analytic germ mapping $\Sigma_{\mathbb{R}>0}$ into itself.

3.6.1 Planar realizations via Morrey-Grauert embedding

Thanks to the realness assumption on \mathbb{L} , we prove a more refined version of the Proposition 3.46, showing that a *planar* realization always exists. In particular, this provides a positive answer to the question asked in Remark 3.47 for real loop germs.

We specialize Definition 3.39 to the real context.

Definition 3.50. Let $\mathbb{L} = (\mathcal{F}, \mathcal{R})$ be a real loop germ. We say that a geometric realization of \mathbb{L} in a foliated surface (S, \mathcal{G}) is *real and planar* if:

1. (S, \mathcal{G}) is a complexification of a pair (U, \mathcal{X}) formed by an open subset $U \subset \mathbb{R}^2$ equipped with a real analytic (oriented) foliation \mathcal{X} ;
2. the embedding map Ψ is the complexification of a real analytic diffeomorphism, mapping the real solution curves of \mathcal{F} to the real solution curves of \mathcal{X} (preserving the orientations);
3. the connecting path γ is real (*i.e.* contained in U).

Let us denote by $\Gamma \subset U$ the curve given by the saturation of γ by the leaves of \mathcal{X} . For shortness, we will say that the triple (U, \mathcal{X}, Γ) is a *real planar realization* of \mathbb{L} .

Remark 3.51. We notice that a real planar realization is defined by picking one representative $R \in \mathcal{R}$ of the regular transition class. As in Remark 3.40, once such a realization is obtained we can modify the representative R by pre-composing it with the holonomy of some path δ , *i.e.* by taking

$$\tilde{R} = \text{hol}_{\mathcal{F}, \delta} R$$

and modifying accordingly the connecting path γ to $\gamma \star \delta$. Notice however that since we want to preserve the real transversals, here we restrict to the case where the path δ is a segment of the *real* vertical separatrix $\{x = 0\} \subset \mathbb{R}^2$ (see Figure 1).

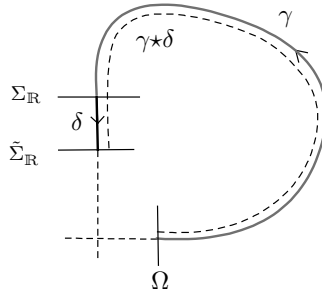


Figure 1:

The following result is essentially a consequence of the Morrey-Grauert embedding theorem and the gluing argument that we used in the proof of Proposition 3.46. A similar construction is provided in [Pi'91, section 0.3 §C].

Proposition 3.52. *Suppose that $\mathbb{L} = (\mathcal{F}, \mathcal{R})$ is real and that there exists a representative $R \in \mathcal{R}$ which preserves the orientation. Then, \mathbb{L} has a real planar realization.*

Proof. We define a gluing domain $W \subset \mathbb{C}^2$ for \mathbb{L} as in the proof of the Proposition 3.46. By construction, its real trace $W_{\mathbb{R}} = W \cap \mathbb{R}^2$ is as illustrated in Figure 2, where F_1 and F_2 are real flow boxes.

Notice that here the gluing map G can be chosen in such a way that the following additional properties hold:

1. $G(F_1) = F_2$;
2. G maps the solution curves of X on F_1 into the solution curves of X on F_2 preserving the orientation.

As a result, the glued foliation \mathcal{G} is real and preserves the natural orientation of the real solution curves. Notice that the orientation-preserving condition on R is equivalent to the condition that $S_{\mathbb{R}}$ be an orientable surface.

Claim: There exists a C^∞ diffeomorphism mapping $S_{\mathbb{R}}$ to an open subset V_0 of \mathbb{R}^2 .

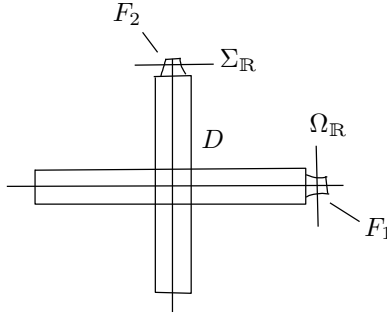


Figure 2:

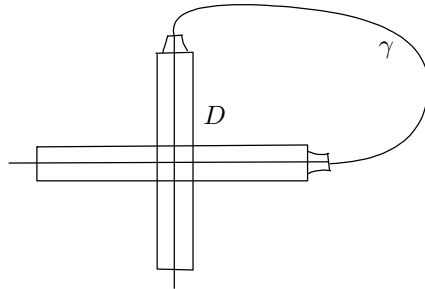


Figure 3:

Indeed, let us choose an arbitrary real smooth curve $\gamma \subset \mathbb{R}^2$ connecting F_1 to F_2 as depicted by Figure 3.

Next we use a partition of unity subordinated to the covering $D \cup (F_1 \sim_G F_2)$, and we can construct a smooth diffeomorphism Ψ_s mapping $S_{\mathbb{R}}$ to $D \cup V_0$, where V_0 is an open neighborhood γ (and such that $\Psi_s|_D = \text{Id}$). We leave the details to the reader.

On the other hand, it follows from the Morrey-Grauert embedding theorem [Gra58] that there exists a real analytic proper embedding $\varphi : S_{\mathbb{R}} \rightarrow \mathbb{R}^N$ for some sufficiently large N . Using the embedding and the Weierstrass Approximation Theorem on \mathbb{R}^N , we can find a real analytic diffeomorphism Ψ_a (arbitrarily close to Ψ_s) between $S_{\mathbb{R}}$ and an open subset $U \subset \mathbb{R}^2$. The realization of \mathbb{L} in U is obtained by taking the image \mathcal{X} of the foliation \mathcal{G} under Ψ_a . \square

Remark 3.53. Suppose that we modify the hypothesis by requiring instead that R reverse the orientation. We could construct a realization of \mathbb{L} in a Moëbius band using the same strategy as above.

3.6.2 Equivalence of planar realizations

Using the Euclidean structure of \mathbb{R}^2 , we can easily construct a global real analytic equivalence between planar realizations of the same real loop germ.

Proposition 3.54. *Let (U, \mathcal{X}, Γ) and $(\tilde{U}, \tilde{\mathcal{X}}, \tilde{\Gamma})$ be two planar realizations of the same real loop germ $\mathbb{L} = (\mathcal{F}, \mathcal{R})$. Then, up to choosing U and \tilde{U} as smaller neighborhoods of Γ and $\tilde{\Gamma}$, there exist a real analytic diffeomorphism H mapping $(\tilde{U}, \tilde{\mathcal{X}})$ to (U, \mathcal{X}) and such that $H(\tilde{\Gamma}) = \Gamma$.*

Proof. We use the Euclidean metric on \mathbb{R}^2 to define a local transverse fibration in the vicinity of $\Gamma \setminus \{s\}$, where s is the saddle point. Namely, on each point $p \in \Gamma \setminus \{s\}$, we define the fiber through p as the affine line

$$\text{Fib}^{-1}(p) = \left\{ p + u \frac{X^\perp}{\|X^\perp\|} : u \in \mathbb{R} \right\}$$

where X is an arbitrary local generator of \mathcal{X} . We define similarly a transverse fibration $\tilde{\text{Fib}}$ in the vicinity of $\tilde{\Gamma} \setminus \{\tilde{s}\}$.

Let Ψ and $\tilde{\Psi}$ be the embedding maps associated to the two realizations (see Definition 3.50). The real analytic map $\Phi = \Psi\tilde{\Psi}^{-1}$ establishes an orientation preserving equivalence between $\tilde{\mathcal{X}}$ and \mathcal{X} in corresponding neighborhoods of the saddle points \tilde{s} and s . Therefore, we need to show that this equivalence, which in principle is only defined in the vicinity of the saddles, extends analytically to a whole neighborhood of the loops.

Up to moving the transversals according to Remark 3.51, we can assume that Φ maps the transversals $\Psi(\Sigma), \Psi(\Omega)$ to $\tilde{\Psi}(\Sigma), \tilde{\Psi}(\Omega)$ respectively. Furthermore, according to the identification made in Remark 3.11, we can assume that the transversals are fibers of the respective fibrations $\text{Fib}, \tilde{\text{Fib}}$ defined above.

Note that, by the item 2. of the definition of a realization, we have

$$\text{hol}_{\mathcal{X}, \gamma} = (\Psi|_\Sigma)R(\Psi|_\Omega)^{-1}, \quad \text{and} \quad \text{hol}_{\tilde{\mathcal{X}}, \tilde{\gamma}} = (\tilde{\Psi}|_\Sigma)R(\tilde{\Psi}|_\Omega)^{-1}$$

where the connecting curve γ (*resp.* $\tilde{\gamma}$) is simply the part of Γ (*resp.* $\tilde{\Gamma}$) lying between Ω and Σ (*resp.* $\tilde{\Omega}$ and $\tilde{\Sigma}$).

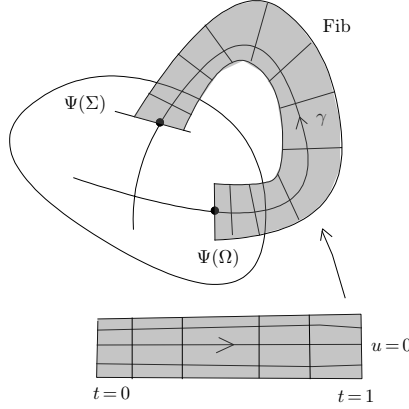
Let us fix real analytic regular parameterizations $p : [0, 1] \rightarrow \gamma$, $\tilde{p} : [0, 1] \rightarrow \tilde{\gamma}$ of these curves. We obtain diffeomorphisms

$$(t, u) \rightarrow p(t) + u \frac{X^\perp}{\|X^\perp\|}, \quad (t, u) \rightarrow \tilde{p}(t) + u \frac{\tilde{X}^\perp}{\|\tilde{X}^\perp\|}$$

between neighborhoods of $[0, 1] \times \{0\}$ in $[0, 1] \times \mathbb{R}$ and neighborhoods of γ and $\tilde{\gamma}$ in V and \tilde{V} , respectively. Let us keep the notations \mathcal{X} and $\tilde{\mathcal{X}}$ to indicate the image of the above foliations under these diffeomorphisms.

We now define an analytic diffeomorphism between two open neighborhoods of $[0, 1] \times \{0\}$ in $[0, 1] \times \mathbb{R}$, mapping $\tilde{\mathcal{X}}$ to \mathcal{X} . To wit, we set

$$\rho(t, u) = (t, \text{hol}_{\mathcal{X}, t}(\Phi|_{\tilde{\Psi}(\Omega)})\text{hol}_{\tilde{\mathcal{X}}, t}^{-1}(u))$$



where $\text{hol}_{\mathcal{X},t}$ (*resp.* $\text{hol}_{\tilde{\mathcal{X}},t}$) is the holonomy map of \mathcal{X} (*resp.* $\tilde{\mathcal{X}}$) going from the fiber $\{0\} \times \mathbb{R}$ to the fiber $\{t\} \times \mathbb{R}$.

It remains to show that the equivalence defined by ρ is compatible with the local equivalence in the neighborhood of the saddle points defined by Φ . For this, it suffices to show that

$$\rho(0, \cdot) = \Phi|_{\tilde{\Psi}(\Omega)} \quad \text{and} \quad \rho(1, \cdot) = \Phi|_{\tilde{\Psi}(\Sigma)}.$$

The first equality is obvious from the definition. For the second, we observe that, by construction,

$$\text{hol}_{\mathcal{X},1} = (\Psi|_{\Sigma})R(\Psi|_{\Omega})^{-1} \quad \text{and} \quad \text{hol}_{\tilde{\mathcal{X}},1} = (\tilde{\Psi}|_{\Sigma})R(\tilde{\Psi}|_{\Omega})^{-1}.$$

Therefore the second component of $\rho(1, \cdot)$ is given by

$$(\Psi|_{\Sigma})R(\Psi|_{\Omega})^{-1}(\Phi|_{\tilde{\Psi}(\Omega)})(\tilde{\Psi}|_{\Omega})R^{-1}(\tilde{\Psi}|_{\Sigma})^{-1} = \Phi|_{\tilde{\Psi}(\Sigma)},$$

where the middle terms simplify because $\Phi = \Psi\tilde{\Psi}^{-1}$. This concludes the proof. \square

3.6.3 \mathbb{R} -equivalence and proof of Theorem C

In the present setting, it is natural to adapt the Definition 3.34 of equivalence stated in Section 3.4 by requiring that the equivalence map preserve the reals. The group of *fibred orientation preserving real analytic diffeomorphisms* is the group $\text{Diff}_{\text{fb}}^+(\mathbb{R}^2, [-1, 1])$ of germs of the form

$$\Phi(x, y) = (x, \varphi(x, y)),$$

where φ is real analytic in a neighborhood of $[-1, 1] \times \{0\}$ and satisfies the following three conditions:

- $\varphi(x, 0) = 0$;

- $\frac{\partial \varphi}{\partial y}(x, 0) > 0$;
- φ extends holomorphically to a neighborhood of the unit closed disk $\bar{\Delta} \times \{0\}$ in \mathbb{C}^2 .

This last condition implies that $\text{Diff}_{\text{fib}}^+(\mathbb{R}^2, [-1, 1])$ is a subgroup of $\text{Diff}_{\text{fib}}(\mathbb{C}^2, \bar{\Delta})$.

Definition 3.55. Two real loop germs $\mathbb{L} = (\mathcal{F}, \mathcal{R})$ and $\tilde{\mathbb{L}} = (\tilde{\mathcal{F}}, \tilde{\mathcal{R}})$ are \mathbb{R} -equivalent if

$$(\tilde{\mathcal{F}}, \tilde{\mathcal{R}}) = \Phi \cdot (\mathcal{F}, \mathcal{R})$$

for some germ $\Phi \in \text{Diff}_{\text{fib}}^+(\mathbb{R}^2, [-1, 1])$.

The following result is a consequence of Theorems A and B for real loop germs.

Theorem 3.56. *Two real loop germs \mathbb{L} and $\tilde{\mathbb{L}}$ are \mathbb{R} -equivalent if and only if the corresponding real Poincaré first return maps P and \tilde{P} are conjugate by an orientation-preserving real analytic map.*

Proof. This enunciate is a particular case of Theorems 3.36 and 3.37 proved above. Recalling the arguments used in the proof of those results, we remark that the realness of \mathbb{L} and $\tilde{\mathbb{L}}$ leads to the following significant simplifications:

1. the real Poincaré first return maps P and \tilde{P} associated respectively to \mathbb{L} and to $\tilde{\mathbb{L}}$ do not depend on a specific choice of a connecting curve;
2. according to the Remark 3.16, the path-lifting method of Mattei-Moussu will give a fibered biholomorphism which is a complexification of a real analytic map.

We further observe that, assuming φ to preserve the orientation, the analytic equivalence obtained by the Mattei-Moussu theorem will give a germ lying in the group $\text{Diff}_{\text{fib}}^+(\mathbb{R}^2, [-1, 1])$ which establishes an equivalence between the underlying real analytic foliations. We leave the details to the reader. \square

As a consequence, we derive the proof of Theorem C.

Proof of Theorem C. The fact that equivalent real saddle loops have analytically conjugate (real) Poincaré maps is trivial since there is no ambiguity on the choice of a determination of the corner transition map.

Let us prove the converse. Let \mathbb{L} and $\tilde{\mathbb{L}}$ be the saddle loop germs associated to (U, X, Γ) and $(\tilde{U}, \tilde{X}, \tilde{\Gamma})$ respectively. Then, by Theorem 3.56, $\mathbb{L}, \tilde{\mathbb{L}}$ are \mathbb{R} -equivalent. Using Proposition 3.52, we find a common real planar realization (V, \mathcal{Y}, Ω) of both $\mathbb{L}, \tilde{\mathbb{L}}$. Proposition 3.54, establishes a local equivalence between (V, \mathcal{Y}, Ω) and (U, \mathcal{X}, Γ) and also between (V, \mathcal{Y}, Ω) and $(\tilde{U}, \tilde{\mathcal{X}}, \tilde{\Gamma})$. The result follows from the transitivity of the relation of local equivalence. \square

4 Integrability of loop germs

In the following we consider a complex saddle loop given as a loop germ $(\mathcal{F}, \mathcal{R})$ as in Section 3.4. It means that we work in a λ -neighborhood U of the unit polydisk $\bar{\Delta} \times \bar{\Delta} = \{|x| \leq 1, |y| \leq 1\}$ in \mathbb{C}^2 and consider the foliated manifold (S, \mathcal{G}) given by Proposition 3.46. We recall that we pick some representative $R \in \mathcal{R}$ to identify the leaves of \mathcal{F} near the transversals $\Sigma = \{y = 1\}$ and $\Omega = \{x = 1\}$.

A theorem of M. Singer [Sin92] guarantees that \mathcal{F} admits a Liouvillian first-integral if and only if there exists a *Godbillon-Vey sequence of length 2*, i.e. a pair (ω, η) of differential 1-forms on U such that ω is holomorphic, η is meromorphic and

$$\begin{cases} d\omega &= \eta \wedge \omega \\ d\eta &= 0 \end{cases}, \quad (43)$$

where the distribution $\ker \omega$ defines the foliation \mathcal{F} . In that case, the (generally multivalued) function obtained by integrating the closed 1-form $\exp(-\int \eta) \omega$,

$$H = \int \frac{\omega}{\exp \int \eta}, \quad (44)$$

is a Liouvillian first-integral of \mathcal{F} since $dH \wedge \omega = 0$.

The study we conduct here amounts to picking among all integrable prepared saddles \mathcal{F} those that can be glued by some germ of a biholomorphism R to obtain a loop germ $(\mathcal{F}, \mathcal{R})$, in such a way that R preserves the “niceness” of the transverse structure provided by Godbillon-Vey sequences. Because we want $H|_{\Sigma}R$ to be the restriction of a first-integral of \mathcal{F} near Ω , it must coincide with a determination of $H|_{\Omega}$. Observe that the special form (44) of the Liouvillian first-integral H forces its monodromy to be affine (since the monodromy of $\int \eta$ is additive). Therefore, we are naturally led to give the following definition of integrability for loop germs.

Definition 4.1. The loop germ $(\mathcal{F}, \mathcal{R})$ is said to be *integrable* whenever there exists a Liouvillian first integral H of \mathcal{F} which is compatible with a gluing map $R \in \mathcal{R}$:

$$\exists R \in \mathcal{R}, \alpha \in \mathbb{C}^\times, c \in \mathbb{C} : H|_{\Sigma}R = \alpha H|_{\Omega} + c. \quad (45)$$

In the rest of the section, we use the following immediate characterization of integrability.

Lemma 4.2. *A loop germ $(\mathcal{F}, \mathcal{R})$ is integrable if and only if both conditions are met:*

- \mathcal{F} is integrable with Liouvillian first-integral $H = \int \frac{\omega}{\exp \int \eta}$;
- there exists $R \in \mathcal{R}$ and $\alpha \in \mathbb{C}^\times$ such that $R^* \frac{\omega}{\exp \int \eta}|_{\Sigma} = \alpha \frac{\omega}{\exp \int \eta}|_{\Omega}$.

Here $\frac{\omega}{\exp f \eta}|_{\Sigma}$ (resp. $\frac{\omega}{\exp f \eta}|_{\Omega}$) denotes the pulled-back 1-form $\iota_j^* \frac{\omega}{\exp f \eta}$ by the respective inclusion $\iota_1 : x \mapsto (x, 1)$ and $\iota_2 : y \mapsto (1, y)$.

Proof. It suffices to differentiate both sides of (45). □

Remark 4.3. If (ω, η) is a Godbillon-Vey sequence of length 2 and u is a meromorphic function, then $(u\omega, u\eta + du)$ satisfies again (43). Said differently, we can freely choose the holomorphic 1-form ω defining the saddle foliation, and we may even choose it meromorphic if it better suits our needs (*e.g.* if it allows us to work with closed 1-forms defining the same foliation).

4.1 Integrability and holonomy

We let \mathfrak{h}_{Σ} and \mathfrak{h}_{Ω} be the holonomy mappings of \mathcal{F} as defined in Definition 3.12, and denote by $f_{\star} \in \text{Corner}(\mathcal{F}, \mathcal{R})$ a representative of the corner transition map of the saddle foliation \mathcal{F} . We recall that the holonomy group of \mathcal{G} is given by $\Pi(\text{var}(f)) = \langle R^* \mathfrak{h}_{\Sigma}, \mathfrak{h}_{\Omega} \rangle$, where $f \in \text{Poinc}(\mathcal{F}, \mathcal{R})$ is a Poincaré map of the loop germ associated to the dynamical decomposition $f = Rf_{\star}$ for some $R \in \mathcal{R}$.

Theorem 4.4 ([BCL96]). *If the foliation \mathcal{G} is integrable, then its holonomy group is solvable.*

Solvability imposes very strict limitations on the holonomy group of \mathcal{F} and is seldom achieved. Solvable subgroups of $\text{Diff}(\mathbb{C}, 0)$ were intensively studied in the 90's, hence we can use their analytical classification (we refer to the Appendix for the precise sources and results we rely on here).

If $\Pi(\text{var}(f))$, coming from some loop germ $(\mathcal{F}, \mathcal{R})$, is analytically conjugate to $\Pi(\text{var}(\tilde{f}))$, coming from $(\tilde{\mathcal{F}}, \tilde{\mathcal{R}})$, then \mathcal{F} is analytically conjugate to $\tilde{\mathcal{F}}$ (by Mattei-Moussu theorem) and, in this new \mathcal{U} -coordinate, $f = r\tilde{f}$ for some unramified germ $r \in \mathcal{U}$ commuting with a generator of the holonomy (Proposition A.4). In most situations, r must itself be a power of the holonomy, hence (by definition of \mathcal{R} and Theorem B) the loop germs $(\mathcal{F}, \mathcal{R})$ and $(\tilde{\mathcal{F}}, \tilde{\mathcal{R}})$ are equivalent. In the remaining cases, we are able to describe exactly which r correspond to integrable loop germs.

Together with the theory developed by M. Berthier and F. Touzet in [BT99], this argument allows us to give the complete list of (analytic classes of) all integrable abstract saddle loops.

4.2 Admissible gluing maps

The classification result presented here is a consequence of the following characterization of germs R appearing in Lemma 4.2 under suitable conditions. The fact that the restricted 1-forms $\frac{\omega}{\exp f \eta}|_{\Sigma}$ and $\frac{\omega}{\exp f \eta}|_{\Omega}$ meet the requirements of the proposition below will be made clear in the course of the classification.

Proposition 4.5. For a finite collection of complex numbers a, b_1, \dots, b_k and a germ A at 0 of a non-vanishing holomorphic function, define

$$F(y) = A(y) y^{-a} \prod_{0 < j \leq k} \exp(b_j y^{-j}).$$

Assume $R \in \text{Diff}(\mathbb{C}, 0)$ and $\alpha \in \mathbb{C}^\times$ are such that

$$R'(y) \times F(R(y)) = \alpha F(y).$$

Then, either R is analytically linearizable or $R'(0) \in e^{2i\pi\mathbb{Q}}$ and R is analytically conjugate to $R'(0)R_0$, where R_0 is the (tangent to the identity) holonomy of a Bernoulli differential equation computed on $\{x = 1\}$:

$$y^{k+1} dx = x(1 + ay^k + x^d y^\sigma Q(y)) dy \quad (46)$$

for some polynomial $Q(y)$ of degree at most $k-1$, some integers $d \in \mathbb{Z}_{\geq -1} \setminus \{0\}$ and $\sigma \in \mathbb{Z}_{\geq 0}$ chosen in such a way that $\sigma + ad \notin \mathbb{R}_{\leq 0}$.

Definition 4.6. A Bernoulli diffeomorphism of order k is a germ $R = \theta R_0$, where R_0 is given by the holonomy of a Bernoulli differential equation (46) as in the previous proposition and $\theta \in e^{2i\pi\mathbb{Q}}$.

Remark 4.7.

1. This case also covers the situation where R_0 embeds in a holomorphic flow ($Q = 0$).
2. Since Bernoulli equations are explicitly solvable, Bernoulli diffeomorphisms can be expressed using a quadrature.

Proof. Without loss of generality we can assume that $|R'(0)| = 1$, else R is analytically linearizable, so that $R(y) = e^{2i\pi\beta} y + o(y)$ for some $\beta \in \mathbb{R}$. The hypothesis implies $\exp((1-a)2i\pi\beta) = \alpha$ and, if $b_j \neq 0$, $j\beta \in \mathbb{Z}$. Let H be an antiderivative of F ; there exists $c \in \mathbb{C}$ such that H semi-conjugates R to the affine map $\rho : t \mapsto \alpha t + c$:

$$HR = \alpha H + c.$$

The case $\beta \in \mathbb{Q}$ has been dealt with in [BT99, Proposition 4.1 and Proposition 5.5], where they conclude $R(y) = e^{2i\pi\beta} R_0(y)$, with R_0 being the holonomy of a Bernoulli foliation. The fact that this equation can be chosen polynomial of the specified form is proved in [Tey04, Theorem 2].

We are left with the case $\beta \notin \mathbb{Q}$, so that all b_j vanish. We assume for the sake of contradiction that R is not analytically linearizable. According to the proof of Dulac-Moussu's conjecture by R. Perez-Marco [PM97, Section IV.2], there exists a forward sub-orbit $(y_n)_{n \in \mathbb{Z}_{\geq 0}} = (R^{\circ k_n}(y_0))_{n \in \mathbb{Z}_{\geq 0}}$, with $y_0 \neq 0$ arbitrarily close to 0, converging to the fixed-point 0. Using a case-by-case analysis, we show that in every possible situation this property cannot be fulfilled. The contradiction we reach is that the complete orbit $(R^{\circ n}(y_0))_{n \in \mathbb{Z}_{\geq 0}}$ converges to 0, but this cannot be the case as the sub-orbit $(R^{\circ q_n}(y_0))_{n \in \mathbb{Z}_{\geq 0}}$ converges towards y_0 when $(q_n)_n$ is the sequence of denominators of the convergents of $\lambda = \lim_{n \rightarrow +\infty} \frac{p_n}{q_n}$.

1. Suppose first $\alpha = 1$, so that $a = 1$ and R is semi-conjugate to the translation $\rho : t \mapsto t + c$ by $H(y) \sim A(0) \log y$. The limit of $|H(y)|$ as y goes to 0 is ∞ . For $H(R^{\circ n}(y_0)) = H(y_0) + nc$ to tend to ∞ it is necessary that $c \neq 0$, from which we deduce $\lim_{n \rightarrow +\infty} R^{\circ n}(y_0) = 0$.
2. Assume next that $\alpha \neq 1$. By subtracting the constant $\frac{c}{\alpha-1}$ to H we may assume without loss of generality that $c = 0$ and R is semi-conjugate to the linear map $\rho : t \mapsto \alpha t$.
 - (a) If $a \notin \mathbb{R}$, then $|\alpha| \neq 1$ and $\lim_{n \rightarrow +\infty} \alpha^n$ is either 0 or ∞ . Therefore $H(R^{\circ n}(y_0)) = \alpha^n H(y_0)$ also converges towards the same limit, because we may choose y_0 in such a way that $H(y_0) \notin \{0, \infty\}$. But this can only mean $\lim_{n \rightarrow +\infty} R^{\circ n}(y_0) = 0$.
 - (b) If $a \in \mathbb{R}$, then $|\alpha| = 1$ and $H(y)$ tends to 0 or ∞ as $y \rightarrow 0$. For all $n \in \mathbb{Z}_{\geq 0}$ we have $|H(R^{\circ k_n}(y_0))| = |H(y_0)|$, which is clearly impossible.

□

4.3 Classification of integrable loop germs

We are now ready to state the classification of integrable saddle loops. The most striking fact is that the holonomy group of an integrable foliation must be Abelian: no purely metabelian holonomy can correspond to integrable loop germs.

Theorem 4.8. *Any integrable saddle loop, given as a loop germ, is $\text{Diff}_{\text{fib}}(\mathbb{C}^2, \bar{\Delta})$ -equivalent to one of the loop germs appearing in the following list.*

Linear model $(\mathcal{F}_{1:\lambda}, \mathcal{R})$, with eigenratio $-\lambda < 0$ and $R \in \text{GL}_1(\mathbb{C})$, defined by the closed 1-form

$$\omega_{1:\lambda} = \frac{dy}{y} + \lambda \frac{dx}{x}.$$

Bernoulli model $(\mathcal{F}_{1:1}, \mathcal{R})$, where R is a Bernoulli diffeomorphism.

Poincaré-Dulac model $(\mathcal{F}_{k,\mu}, \mathcal{R})$, with eigenratio -1 and:

- $\mathcal{F}_{k,\mu}$ is the $1 : 1$ resonant saddle foliation defined by a closed 1-form

$$\omega_{k,\mu} = \frac{dy}{y} - \frac{1 + \mu u^k}{u^k} \omega_{1:1}$$

for some $k \in \mathbb{Z}_{\geq 1}$ and $\mu \in \mathbb{C}$, where $u = xy$ is the resonant monomial (so that $\omega_{1:1} = \frac{du}{u}$);

- $R = \exp \chi$ is the exponential of a derivation

$$\chi(y) = \frac{y^{k+1}}{1 + \nu y^k} \frac{\partial}{\partial y}, \quad \nu \in \mathbb{C}.$$

Proof. We start from an integrable loop germ $(\mathcal{F}, \mathcal{R})$ with gluing map R , and let $\langle R^* \mathfrak{h}_\Sigma, \mathfrak{h}_\Omega \rangle$ be its holonomy group.

1. Let us first deal with the simplest case $\lambda \notin \mathbb{Q}$. Theorem 3.1 from [BT99] states in that case that \mathcal{F} is analytically linearizable. Hence, up to considering the pulled-back loop germ as in Definition 3.34, one can suppose that $\mathcal{F} = \mathcal{F}_{1:\lambda}$ and choose $\omega = \omega_{1:\lambda}$. Because $d\omega_{1:\lambda} = 0 = \eta \wedge \omega_{1:\lambda}$, we deduce that $\eta = a\omega_{1:\lambda}$ for some meromorphic germ a . Plugging this identity into $d\eta = 0$ we infer that a is a meromorphic first-integral of $\omega_{1:\lambda}$, hence a constant $a \in \mathbb{C}$. Therefore $\exp \int \eta = cx^{\lambda a} y^a$ for some $c \in \mathbb{C}^\times$. Next we apply Lemma 4.2: there exists $\alpha \in \mathbb{C}^\times$ for which R must solve

$$\lambda \frac{R'}{R^{1+\lambda a}}(y) = \alpha \frac{1}{y^{1+a}}.$$

Because $R(y) = \gamma y(1 + O(y))$ for some $\gamma \neq 0$ we deduce $a = 0$ and $\lambda = \alpha$. Finally $R \in \text{GL}_1(\mathbb{C})$.

2. Assume next $\lambda = \frac{p}{q} \in \mathbb{Q}_{>0} \setminus \{1\}$. Following Proposition C.1 from the Appendix, and after a convenient analytic change of coordinates, we can assume that \mathfrak{h}_Ω is linear. It stems from Mattei-Moussu theorem that \mathcal{F} can be chosen as the linear foliation $\mathcal{F}_{1:\lambda}$, that is $\omega = \omega_{1:\lambda}$. We let $u = x^p y^q$ be the resonant monomial so that $\omega_{1:\lambda} = \frac{du}{qu}$ and every meromorphic first-integral of $\omega_{1:\lambda}$ factors meromorphically through u [MM80]. Following the same line of reasoning as for 1., we have $\eta = G(u)\omega_{1:\lambda}$ for some meromorphic germ G . We compute directly

$$F(u) = \exp \int \eta = A(u) u^{-a} \prod_{0 < j \leq k} \exp(b_j u^{-j}) \quad , \quad a, b_1, \dots, b_k \in \mathbb{C},$$

where $A(0) \neq 0$. Thanks to Lemma 4.2, we know there exists $\alpha \in \mathbb{C}^\times$ such that

$$\lambda \frac{R'}{R \times F(R^p)} = \alpha \frac{1}{y F(y^q)}.$$

Because $p \neq q$ and $R \in \text{Diff}(\mathbb{C}, 0)$, the function F must be constant: we recover the case studied in 1., yielding a linear model $R \in \text{GL}_1(\mathbb{C})$. Observe in particular that the only possible alternative described by Proposition C.1 is $\varepsilon = 0$.

3. Let us continue the classification with the case $\lambda = 1$. The symmetry offered by the variables x and y explains why this case is so rich. Let

us first consider the case of a foliation \mathcal{F} whose formal Poincaré-Dulac normal form is linear: it is actually analytically linearizable and we may assume that $\omega = \omega_{1:1}$ and $\mathcal{F} = \mathcal{F}_{1:1}$. We let $u = xy$ be the resonant monomial so that $\omega_{1:1} = \frac{du}{u}$ and every meromorphic first-integral of $\omega_{1:1}$ factors meromorphically through u .

In that case the holonomy group is trivial, the corner transition map f_* is the identity and the Poincaré map f coincides with $\Pi^{-1}(R)$. According to Theorem B, the analytical class of R determines that of $(\mathcal{F}, \mathcal{R})$. Moreover, an equivalence $\varphi^*R = \tilde{R}$ between two such analytic germs induces a fibered isotropy of \mathcal{F} through $(x, y) \mapsto \left(x, y \frac{\varphi(u)}{u}\right)$, sending u to $\varphi(u)$. Hence, we can change R for any element \tilde{R} of its conjugacy class while preserving at the same time the linearity of \mathcal{F} and the analytical class of $(\mathcal{F}, \mathcal{R})$.

We have again $\eta = G(u)\omega_{1:1}$ for some meromorphic germ G . Thanks to Lemma 4.2, the germ R and the multivalued function $\exp \int \eta$ meet the requirements of Proposition 4.5. R is thus either analytically conjugate to the rotation $e^{2i\pi\beta}\text{Id}$, giving a linear model, or to $e^{2i\pi\beta}R_0$ (only if $\beta \in \mathbb{Q}$), where R_0 is a Bernoulli diffeomorphism of order $|m|$, giving a Bernoulli model.

4. Let us conclude our proof when $\lambda = 1$ and the Poincaré-Dulac normal form of \mathcal{F} is nonlinear. Each generator \mathfrak{h}_j of the holonomy group is tangent to the identity, hence $\Pi(\text{var}(f))$ is solvable if and only if it is Abelian [CM88]. Proposition 2.11 describes the only two situations that may happen.

(a) In convenient analytic coordinates, $\mathfrak{h}_\Omega = \exp \partial$ and $R^*\mathfrak{h}_\Sigma = \exp(c\partial)$ are embedded in the same holomorphic flow. Then \mathcal{F} is analytically conjugate to its Poincaré-Dulac normal form $\mathcal{F}_{k,\mu}$. Since $\mathfrak{h}_\Sigma = \exp(c'\partial)$ for some other $c' \in \mathbb{C}$, it follows that R also embeds in the same holomorphic flow, yielding a Poincaré-Dulac model.

(b) $R^*\mathfrak{h}_\Sigma = \mathfrak{h}_\Omega$ and $\mu = \frac{1}{2}$. According to [BT99, Proposition 6.2], the foliation \mathcal{F} is analytically conjugate to a foliation defined by $\omega = \omega_{k,\mu} + P(u)du$ for some analytic germ P and $d \in \mathbb{Z}_{\geq -1} \setminus \{0\}$, which can actually be chosen as a polynomial [Tey04, Theorem 2] like the one in Proposition 4.5. Therefore \mathfrak{h}_Σ and \mathfrak{h}_Ω are Bernoulli diffeomorphisms. If moreover $P = 0$, then we obtain again a Poincaré-Dulac model. Let us prove that this is the only possible outcome by assuming $P \neq 0$ in the sequel.

Requesting that $\eta = a\omega + bdu$ satisfy $d\omega = \eta \wedge \omega$ yields $b = dy^d u^\sigma P(u)$, and requesting that it be closed yields $da \wedge \omega = b(a-d)dx \wedge dy$. Therefore $S = a-d$ is a separatrix of \mathcal{F} , hence $S = y^n u^m s$ for some holomorphic function s (which is either 0 or does not vanish at 0) and some $n, m \in \mathbb{Z}$. Indeed, the only branches of a separatrix of \mathcal{F} are $\{x=0\}$ and $\{y=0\}$.

Let us first prove that $s = 0$ by contraposition. Since $\frac{d(y^n u^m)}{y^n u^m} \wedge \omega =$

$y(n(1 + \mu u^k) + mu^k) dx \wedge dy$, the non-vanishing holomorphic germ s solves the cohomological equation

$$X \cdot \log s = y^{d+1} u^{k+1} P(u) (d - n) - y(n(1 + \mu u^k) + mu^k),$$

where $X(x, y) = xu^k \frac{\partial}{\partial x} + (1 + \mu u^k + u^{k+1} y^d P(u)) \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right)$ is the vector field dual to $yu^{k+1}\omega$. The only way to reach the terms $y(n(1 + \mu u^k) + mu^k)$ is through $X \cdot (y\phi(u))$ for some holomorphic ϕ . Observe that $d \neq 0$, hence $y^{d+1}u^{k+1}P(u)$ does not mix up with the other terms, and ϕ must solve

$$u^{k+1}\phi'(u) + (1 + \mu u^k)\phi(u) = -n(1 + \mu u^k) + mu^k.$$

An immediate formal computation on $\phi(u) = \sum_{j \geq 0} \phi_j u^j$ gives the only possible solution

$$\begin{aligned} \phi_0 &= -n, \quad \phi_j = 0 \text{ for } 0 < j < k, \\ \phi_k &= m \quad \text{and } \phi_{j+1} = (j - \mu)\phi_j \text{ for } k \leq j. \end{aligned}$$

The only way for this power series to have a positive radius of convergence is to be a finite sum, so that $\mu \in \mathbb{Z}_{\geq k}$ and in particular $\mu \neq \frac{1}{2}$. So far we have proved $a = d$ and $b = dy^d P(u)$, from which we deduce that $\eta = a\omega_{k,\mu}$ and $\exp \int \eta = y^d u^{-d\mu} \exp \frac{d}{ku^k}$. Consequently, the condition expressed by Lemma 4.2 amounts to

$$\frac{1 + \frac{1}{2}R^k + R^{k+1}P(R)}{R^{1+k-d/2} \exp \frac{dR^{-k}}{k}} R' = \alpha \frac{1 - \frac{1}{2}y^k + y^{d+k+1}P(y)}{y^{1+k+d/2} \exp \frac{dy^{-k}}{k}}.$$

Since $d \neq 0$ there is no solution belonging to $\text{Diff}(\mathbb{C}, 0)$, as can be seen by equating the respective first-order contributions of $R' \times R^{-1-k+d/2}$ and $\alpha y^{-1-k-d/2}$. This completes the classification. \square

To conclude this classification, let us give an example of a Bernoulli loop germ.

Example 4.9 (Bernoulli model). The linear foliation $\mathcal{F}_{1:1}$ with Godbillon-Vey sequence

$$\omega = \omega_{1:1} \quad \text{and} \quad \eta = \frac{1+u}{u^2} du, \quad u = xy,$$

corresponds to the Liouvillian first integral

$$H = \int \frac{\exp \frac{1}{u}}{u^2} du = -\exp \frac{1}{u}.$$

To obtain an associated loop germ with gluing map R , one must solve the equation $HR = \alpha H + c$ for complex constants α and c , that is:

$$\exp \frac{1}{R} = \alpha \exp \frac{1}{y} - c.$$

Every admissible gluing germ R for $c = 0$ has the form

$$R(y) = \frac{y}{1 + y \log \alpha}.$$

For $c \neq 0$ one must take R as the time- $\log \alpha$ flow of the vector field $\frac{u^2}{1+cu} \frac{\partial}{\partial u}$.

A Some general commutator identities

Given a group G , we define the *commutator* of two elements $x, y \in G$ by

$$[x, y] = x^{-1}y^{-1}xy.$$

For the sake of brevity, we write the conjugation of x by y simply as

$$x^y = y^{-1}xy.$$

Let us list some well-known identities

$$\begin{aligned} \text{(C0)} \quad & [x, y] = x^{-1}x^y \\ \text{(C1)} \quad & [x, y]^{-1} = [y, x] \\ \text{(C2)} \quad & [x, y]^z = [x^z, y^z] \\ \text{(C3)} \quad & [x, y^{-1}] = [y, x]^{y^{-1}} \text{ and } [x^{-1}, y] = [y, x]^{x^{-1}} \\ \text{(C4)} \quad & [x, yz] = [x, z][x, y]^z \text{ and } [xy, z] = [x, z]^y[y, z] \end{aligned}$$

Let $\tau \in G$ be fixed. The τ -*functional variation* (or simply the *variation*) $\text{var} : G \rightarrow G$ is defined as the commutator

$$\text{var}(x) = [\tau, x].$$

Consider the sets

$$\mathfrak{C}_0(\tau) = \{\text{Id}\}, \quad \mathfrak{C}_k(\tau) = \{x \in G : \text{var}(f) \in \mathfrak{C}_{k-1}(\tau)\}, \quad k \geq 1, \quad (47)$$

where Id denotes the identity element. Notice that $\mathfrak{C}_1(\tau)$ is a subgroup of G , since it is precisely the centralizer of τ in G . This property does not hold for $\mathfrak{C}_k(\tau)$ for $k \geq 2$ since the var operator is not a morphism of groups.

Lemma A.1. *Let $x \in G$. Then*

$$\text{var}(x^{-1})^x = \text{var}(x)^{-1}. \quad (48)$$

Proof. Using (C1) and (C3), we have:

$$\text{var}(x^{-1}) = [\tau, x^{-1}] = [x^{-1}, \tau]^{-1} = \left([\tau, x]^{x^{-1}} \right)^{-1} = (\text{var}(x)^{-1})^{x^{-1}}.$$

□

We will be mostly interested in $\mathfrak{C}_2(\tau)$, formed by the set of group elements x such that $[\tau, [\tau, x]] = \text{Id}$. We now show that such a set is closed under inversion and a partial composition operation.

Lemma A.2. *Let $x, y \in \mathfrak{C}_2(\tau)$. Then:*

1. $x^{-1} \in \mathfrak{C}_2(\tau)$;
2. $\text{var}(x)$ and $\text{var}(y^{-1})$ commute if and only if xy belong to $\mathfrak{C}_2(\tau)$;
3. $\text{var}(x) = \text{var}(y^{-1})$ if and only if xy belongs to $\mathfrak{C}_1(\tau)$.

Proof.

1. If we denote $z = \text{var}(x)^{-1}$ then, by (48), it suffices to prove that $z^{x^{-1}}$ is in $\mathfrak{C}_1(\tau)$. We compute its variation

$$\text{var}(z^{x^{-1}}) = [\tau, z^{x^{-1}}] = [\tau^x, z]^{x^{-1}} = [\tau z^{-1}, z]^{x^{-1}} = \text{Id},$$

where we have used the identity (C0) to write $\tau^x = \tau z^{-1}$ and the fact that z commutes with τ to obtain the last equality.

2. Using identity (C4) and (48), we write:

$$\text{var}(xy) = [\tau, xy] = [\tau, y][\tau, x]^y = \text{var}(y)\text{var}(x)^y = (\text{var}(y^{-1})^{-1}\text{var}(x))^y. \quad (49)$$

Put $z = \text{var}(x)$ and $w = \text{var}(y^{-1})$. Then:

$$[\tau, \text{var}(xy)] = [\tau, (w^{-1}z)^y] = [\tau^{y^{-1}}, w^{-1}z]^y = [\tau w, zw^{-1}]^y.$$

Using the fact that z and w commute with τ , we obtain

$$[\tau w, zw^{-1}] = [z, w^{-1}]^\tau.$$

3. is an immediate consequence of (49).

□

Given an element $x \in \mathfrak{C}_2(\tau)$, both the left coset $x\mathfrak{C}_1(\tau)$ and the right coset $\mathfrak{C}_1(\tau)x$ are entirely contained in $\mathfrak{C}_2(\tau)$. More precisely, for any $z \in \mathfrak{C}_1(\tau)$, a simple computation shows that

$$\text{var}(xz) = \text{var}(x)^z, \quad \text{and} \quad \text{var}(zx) = \text{var}(x). \quad (50)$$

It follows that the right coset $\mathfrak{C}_1(\tau)x$ is contained in the set $\text{var}^{-1}(\text{var}(x))$. We prove that in fact

$$\mathfrak{C}_1(\tau)x = \text{var}^{-1}(\text{var}(x)).$$

Indeed, suppose that $y \in \mathfrak{C}_2(\tau)$ is such that $\text{var}(x) = \text{var}(y)$. Then, using (C4) we obtain

$$\text{var}(yx^{-1}) = [\tau, yx^{-1}] = [\tau, x^{-1}] [\tau, y]^{x^{-1}} = \text{var}(x^{-1})\text{var}(x)^{x^{-1}} = \text{Id},$$

where the last equality follows from (48). Therefore, $y = zx$ for some $z \in \mathfrak{C}_1(\tau)$. This discussion can be restated as follows.

Proposition A.3. *The operator var defines an embedding*

$$\text{var} : \mathfrak{C}_1(\tau) \setminus \mathfrak{C}_2(\tau) \longrightarrow \mathfrak{C}_1(\tau)$$

(which is not necessarily surjective).

Let $x, y \in \mathfrak{C}_2(\tau)$ be such that

$$\text{var}(x) = \text{var}(y), \quad \text{and} \quad \text{var}(x^{-1}) = \text{var}(y^{-1}) \quad (51)$$

Then, by the previous proposition, the left-hand equality gives $y = zx$, for some $z \in \mathfrak{C}_1(\tau)$. Inserting this in the right-hand equality and using (50), we obtain

$$\text{var}(x^{-1}) = \text{var}(y^{-1}) = \text{var}(x^{-1}z^{-1}) = \text{var}(x^{-1})^{z^{-1}},$$

which shows that z belongs to the centralizer of $\text{var}(x^{-1})$ in $\mathfrak{C}_1(\tau)$. We have thus proved the following result.

Proposition A.4. *Let $x, y \in \mathfrak{C}_2(\tau)$. Then, the relations (51) hold if and only if*

$$y = cx,$$

for some c in the centralizer of $\text{var}(x^{-1})$.

A simple computation shows that the centralizers of $\text{var}(x^{-1})$ and $\text{var}(x)$ are conjugate by x . Therefore, assuming (51), we can equally write $y = xc$ for some c in the centralizer of $\text{var}(x)$.

B Derivations of the formal Dulac ring and normal forms

B.1 Derivations

We consider the set of differential operators

$$Pe^{-\lambda z} \frac{\partial}{\partial z},$$

for $(P, \lambda) \in \mathbb{C}[z] \times \mathbb{R}_{>0}$, seen as derivations in the ring \mathcal{Q} of QSD-germs. This set is totally ordered with respect to the \succ relation defined in Section 2.1. Furthermore, it is closed under the usual Lie bracket, since we have:

$$\left[P e^{-\lambda z} \frac{\partial}{\partial z}, Q e^{-\mu z} \frac{\partial}{\partial z} \right] = \left(P \partial_\mu(Q) - \partial_\lambda(P) Q \right) e^{-(\lambda+\mu)z} \frac{\partial}{\partial z}, \quad (52)$$

where ∂_λ is the differential operator $\frac{\partial}{\partial z} - \lambda \text{Id}$.

A *nilpotent derivation* x of the formal Dulac ring $\widehat{\mathcal{AR}}$ is an asymptotically ordered sum of the above differential operators,

$$X = \sum_{\lambda \in L} P_\lambda e^{-\lambda z} \frac{\partial}{\partial z},$$

where $L \subset \mathbb{R}_{>0}$ is a discrete subset. The set of such derivations forms a Lie algebra $\mathbb{N}(\widehat{\mathcal{D}})$ under the bracket defined in (52).

The nilpotency condition guarantees that the usual exponential series

$$\exp X = \text{Id} + X + \frac{1}{2} X^2 + \dots$$

is summable, and that $z \mapsto (\exp X)(z)$ is an element of the Dulac ring.

We now establish a correspondence between the group of formal Dulac series with multiplier 1 and the nilpotent derivations. We denote by $\widehat{\mathcal{D}}_{>0}$ the normal subgroup of formal Dulac series having an asymptotic expansion of the form

$$f = z + o(1),$$

and by $\mathbb{N}(\widehat{\mathcal{D}})$ we denote the set of all nilpotent derivations of the formal Dulac ring $\widehat{\mathcal{AR}}$. More specifically, given a $\lambda \in \mathbb{R}_{>0}$, we say that a formal Dulac series $f \in \widehat{\mathcal{D}}$ is *λ -tangent to identity* if it has an expansion of the form:

$$f = z + P(z) e^{-\lambda z} + o(e^{-\lambda z}),$$

for some polynomial P (possibly zero). The set of λ -tangent to identity formal series is a normal subgroup of $\widehat{\mathcal{D}}_{>0}$, which we denote by $\widehat{\mathcal{D}}_{\geq \lambda}$. Similarly, we say that a nilpotent derivation X is *λ -flat* if it can be written as:

$$X = P(z) e^{-\lambda z} \frac{\partial}{\partial z} + o(e^{-\lambda z}),$$

for some polynomial P (possibly zero). The sets of all such derivations form a filtered collection of Lie ideals:

$$\left[\mathbb{N}(\widehat{\mathcal{D}})_{\geq \lambda}, \mathbb{N}(\widehat{\mathcal{D}})_{\geq \mu} \right] \subset \mathbb{N}(\widehat{\mathcal{D}})_{\geq \lambda + \mu},$$

as can be easily seen by the formula (52) for the Lie bracket.

In what follows, we identify each formal Dulac series f with the corresponding *automorphism* of the formal Dulac ring defined by

$$p \in \widehat{\mathcal{AR}} \mapsto p \circ f$$

(which we will also denote by f). The following result has an immediate proof.

Lemma B.1. *The formal exponential series establishes a bijection*

$$\exp : \mathbb{N}(\widehat{\mathcal{D}}) \longrightarrow \widehat{\mathcal{D}}_{>0},$$

which maps $\mathbb{N}(\widehat{\mathcal{D}})_{\geq \lambda}$ to $\widehat{\mathcal{D}}_{\geq \lambda}$ for each $\lambda > 0$.

The inverse map is given by the formal logarithmic series, which we denote by \log .

B.2 BCH and variation operator

By Lemma B.1, we can uniquely write an element $f \in \widehat{\mathcal{D}}_{>0}$ as $f = \exp X$ for some nilpotent derivation X . The variation operator takes the form:

$$\text{var}(f) = \text{var}(\exp X) = (\exp -X^\tau)(\exp X),$$

where $X^\tau = X\tau^{-1}$ denotes the conjugation of X by the translation $\tau(z) = z + 2\pi i$. In particular, if X is one of the basis elements $P(z)e^{-\lambda z} \frac{\partial}{\partial z}$,

$$X^\tau = P(z - 2\pi i)e^{-\lambda(z-2\pi i)} \frac{\partial}{\partial z}.$$

If X is λ -flat, then the same identity holds for X^τ . Let us denote by $\text{lvar}(X)$ the unique derivation Z such that

$$\exp Z = \text{var}(\exp X).$$

In other words, $\text{lvar} = \log \circ \text{var} \circ \exp$. Using the usual Baker-Campbell-Hausdorff expansion, we have:

$$\text{lvar}(X) = X - X^\tau + \frac{1}{2}[X, X^\tau] + \dots, \quad (53)$$

where the omitted terms are higher order commutators that vanish if $[X, X^\tau] = 0$. Moreover, for $X \in \mathbb{N}(\widehat{\mathcal{D}})_{\geq \lambda}$, the omitted terms belongs to $\mathbb{N}(\widehat{\mathcal{D}})_{\geq 3\lambda}$. Finally, note that $\text{var}(\exp X) = \text{Id}$ if and only if $\text{lvar}(X) = 0$.

Proposition B.2. *Let $f = \exp X$, for some nilpotent derivation*

$$X = \sum_{\lambda \in L} P_\lambda e^{-\lambda z} \frac{\partial}{\partial z}.$$

Then the following three statements are equivalent:

1. $\text{var}(f) = \text{Id}$ (i.e. f is unramified);
2. $X = X^\tau$;
3. $L \subset \mathbb{Z}_{\geq 1}$ and all polynomials P_λ are constants.

Proof. As remarked above, $\text{var}(f) = \text{Id}$ if and only if $\text{lvar}(X) = 0$. Using the asymptotic expansion given by the BCH formula (53), this is equivalent to saying that $X = X^\tau$. Hence, 1. and 2. are equivalent.

In order to prove the equivalence of 2. and 3., it suffices to observe that the conjugation of X by τ preserves the order of the asymptotic terms in the expansion of X . In other words, the equality $X = X^\tau$ is equivalent to requesting

$$\left(P_\lambda e^{-\lambda z} \frac{\partial}{\partial z}\right)^\tau = P_\lambda e^{-\lambda z} \frac{\partial}{\partial z}$$

for each $\lambda \in L$. This holds if and only if P_λ lies in the kernel of the linear operator $\mathbb{C}[z] \rightarrow \mathbb{C}[z]$ defined by

$$P(z) \mapsto e^{2\pi i \lambda} P(z - 2\pi i) - P(z)$$

This is equivalent to saying that λ is an integer and that P_λ is of degree 0. \square

By analogy with the notations introduced in Section 2.2, we say that a nilpotent derivation X is:

- *unramified* if $\text{lvar}(X) = 0$;
- *mildly ramified* if $\text{lvar}(X)$ is unramified.

Notice that these conditions are equivalent to $\exp X \in \mathcal{U}$ and $\exp X \in \mathcal{M}$, respectively.

B.3 Computing lvar^{-1} and normal forms

Let Z be an unramified derivation. We are interested in determining another nilpotent derivation X such that

$$Z = \text{lvar}(X).$$

Consider the difference operator $\Delta : \mathbb{C}[z] \rightarrow \mathbb{C}[z]$ defined by

$$(\Delta P)(z) = P(z) - P(z - 2\pi i)$$

or equivalently, $\Delta P = P - P\tau^{-1}$. It is easy to see that the sequence of polynomials

$$\mathbf{P}_0 = 1, \quad \mathbf{P}_1 = \frac{1}{2\pi i} z, \quad \mathbf{P}_k = \frac{1}{(2\pi i)^k} z(z + 2\pi i) \cdots (z + 2\pi i(k - 1)) \quad (54)$$

satisfies $\Delta \mathbf{P}_k = k \mathbf{P}_{k-1}$. In particular, \mathbf{P}_0 generates the kernel of Δ .

Proposition B.3. *Let Z be an unramified nilpotent derivation. There exist a nilpotent derivation X such that*

$$Z = \text{lvar}(X).$$

Moreover, X is uniquely determined modulo the choice of a section of the cokernel of Δ .

Proof. Let us write the expansion of Z as

$$Z = \sum_{k \geq 1} c_k e^{-kz} \frac{\partial}{\partial z},$$

where c_k are complex numbers. We now look for an expansion of X in the form

$$X = \sum_{k \geq 1} P_k e^{-kz} \frac{\partial}{\partial z}.$$

It follows from $Z = \text{lvar}(X)$ and the BCH formula (53) that the polynomials P_k can be recursively determined by the formula

$$\Delta(P_k) = c_k + R,$$

where R depends only on previously determined polynomials P_ℓ for $0 < \ell < k$. In other words, P_k is uniquely determined from this formula *modulo* an additive constant. \square

Remark B.4. This algorithm can also be used to compute $\text{lvar}^{-1}(Z)$ for arbitrary (*i.e.* not necessarily unramified) nilpotent derivations

$$Z = \sum_L Q_\lambda e^{-\lambda z} \frac{\partial}{\partial z}.$$

In this case, we eventually need to consider the twisted difference operators $\Delta^\lambda P = P(z) - e^{2\pi i \lambda} P(z - 2\pi i)$, which are *invertible* for $\lambda \notin \mathbb{Z}$. In particular, if the additive semi-group generated by the exponent set L is disjoint from $\mathbb{Z}_{\geq 0}$, then $\text{lvar}^{-1}(Z)$ is uniquely determined.

We reprove the following normal form result.

Proposition B.5. *Let Z be a nilpotent, unramified derivation with an expansion*

$$Z = c_k e^{-kz} \frac{\partial}{\partial z} + o(e^{-kz}),$$

such that c_k is nonzero. Then, there exists a formal unramified series $g \in \widehat{\mathcal{U}}$ which conjugates Z to a derivation of the form

$$-2\pi i \frac{e^{-kz}}{1 + \mu e^{-kz}} \frac{\partial}{\partial z} + o(e^{-3kz}),$$

where $k \in \mathbb{Z}_{\geq 1}$ and $\mu \in \mathbb{C}$ (called the residue of Z) are formal invariants.

Remark B.6. It is well-known that actually $g \in \mathcal{U}$. This stems from the fact that only a finite number of reduction steps are applied.

Proof. First of all, we conjugate Z by a germ of the form $g_0(z) = z + b$, $b \in \mathbb{C}$, to obtain

$$Z^{g_0} = c_k e^{-kb} e^{-kz} \frac{\partial}{\partial z} + o(e^{-kz}).$$

Therefore, up to a convenient choice of b , we can assume that $c_k = -2\pi i$. Then, considering the Lie bracket identity

$$\left[e^{-jb} \frac{\partial}{\partial z}, e^{-kz} \frac{\partial}{\partial z} \right] = (j - k) e^{-(k+j)z} \frac{\partial}{\partial z},$$

we observe that we can eliminate all terms $\{c_{k+j}\}_{1 \leq j < k-1}$, by successive conjugations by germs of the form

$$g_j = \exp \left(b_j e^{-jb} \frac{\partial}{\partial z} \right), \quad 1 \leq j < k-1.$$

The remaining term $j = k$ corresponds to a resonance. It is an invariant for $\widehat{\mathcal{U}}$ -conjugation. \square

Remark B.7. Although not needed in the sequel, observe that we further reduce the above expression by a formal series $g \in \widehat{\mathcal{U}}$ in order to eliminate all $o(e^{-3kz})$ terms and get simply

$$-2\pi i \frac{e^{-kz}}{1 + \mu e^{-kz}} \frac{\partial}{\partial z}.$$

In fact, each $\widehat{\mathcal{U}}$ -conjugacy class contains a unique representative of this form.

Theorem B.8. *Let X be a mildly ramified (but not unramified) nilpotent derivation. Then there exists a formal unramified series $g \in \mathcal{U}$ which conjugates X to the form*

$$-(z + a) e^{-kz} \frac{\partial}{\partial z} + \left(\left(\mu - \frac{1}{2} \right) z + b \right) e^{-2kz} \frac{\partial}{\partial z} + o(e^{-2kz})$$

for some $k \in \mathbb{Z}_{\geq 1}$ and $a, b, \mu \in \mathbb{C}$.

Proof. By the assumption, the derivation $Z = \text{lvar}(X)$ is a nontrivial unramified derivation. It follows from Proposition B.5 that there exists a formal unramified series g such that

$$Z^g = -2\pi i \frac{e^{-kz}}{1 + \mu e^{-kz}} \frac{\partial}{\partial z} + o(e^{-3kz}).$$

It suffices to show that $X^g = \text{lvar}^{-1}(Z^g)$ has the desired form.

In fact, it suffices to look at the initial steps of the algorithm described in the proof of Proposition B.3. Using only the first two terms in the expansion

$$Z = X - X^\tau + \frac{1}{2}[X, X^\tau] + \dots,$$

it follows that X^g has an expansion of the form

$$X^g = Pe^{-kz} \frac{\partial}{\partial z} + Qe^{-2kz} \frac{\partial}{\partial z} + o(e^{-2kz}),$$

where P and Q satisfy the following polynomial equations:

$$\Delta P = -2\pi i \quad \text{and} \quad \Delta Q + \frac{1}{2}(P(P\tau^{-1})' - P\tau^{-1}P') = 2\pi i\mu.$$

The first equation has general solution $P = -(z + a)$ (with an arbitrary $a \in \mathbb{C}$). Then

$$\frac{1}{2}(P(P\tau^{-1})' - P\tau^{-1}P') = \frac{1}{2}((z + a) - (z + a - 2\pi i)) = i\pi,$$

and we conclude that $Q = (\mu - \frac{1}{2})z + b$, for an arbitrary $b \in \mathbb{C}$. \square

Corollary B.9. *Let X be as in the enunciate of the Theorem. Then, up to unramified conjugation, we have*

$$\text{lvar}(X) = -2\pi i \frac{e^{-kz}}{1 + \mu e^{-kz}} \frac{\partial}{\partial z} + o(e^{-2kz}) \quad (55)$$

and

$$\text{lvar}(-X) = 2\pi i \frac{e^{-kz}}{1 + (\mu - 1)e^{-kz}} \frac{\partial}{\partial z} + o(e^{-2kz}). \quad (56)$$

As a consequence, denoting $Z = \text{lvar}(X)$ and $W = \text{lvar}(-X)$, we get:

$$[W, Z] = (4\pi^2 k e^{-3kz} + o(e^{-3kz})) \frac{\partial}{\partial z}.$$

Proof. This is a simple computation using the formulas given above. \square

C Mildly ramified germs with rational multiplier

Let $f \in \mathcal{M}$ be a mildly ramified germ with linear part $\frac{p}{q}z$, $(p, q) = 1$. As usual, we denote by $\Pi(\text{var}(f)) < \text{Diff}(\mathbb{C}, 0)$ the group generated by the images of $\text{var}(f)$ and $\text{var}(f^{-1})$ under the morphism $\Pi : \mathcal{U} \rightarrow \text{Diff}(\mathbb{C}, 0)$.

Although not needed for the proof of Theorem D, the integrability result of Section 4 uses some finer properties of $\text{var}(f)$, which we summarize in the next proposition.

Proposition C.1. *The group $\Pi(\text{var}(f))$ is formally rigid. Moreover, if it is solvable, then it is analytically conjugate to*

$$\left\langle e^{2\pi i q/p} x, \frac{e^{2\pi i p/q} x}{(1 + \varepsilon x^k)^{1/k}} \right\rangle,$$

where k is some positive integer not in $p\mathbb{Z} \cup q\mathbb{Z}$ and $\varepsilon \in \{0, 1\}$.

Remark C.2. Both generators of the group are analytically linearizable, but in general not in the same coordinates. The group itself is linearizable if and only if it is Abelian, which happens exactly when $\varepsilon = 0$.

Proof. Up to a conjugation inside \mathcal{U} , we can suppose that

$$f = \frac{p}{q}z + o(1)$$

and the variation $g = \text{var}(f)$ has the form

$$g = t_{2\pi i(\frac{q}{p}-1)}\tilde{g}$$

where $t_c(z) = z + c$ is the translation map and \tilde{g} is a tangent to identity unramified germ. We split the proof into two cases.

Case 1. Up to a conjugation inside $\widehat{\mathcal{U}}$, we can write $g = \text{var}(f)$ in prepared form

$$g = t_{2\pi i(\frac{q}{p}-1)} \exp\left(-2\pi i e^{-kpz} \frac{\partial}{\partial z} + o(e^{-kpz})\right)$$

Using the fact that $h = \text{var}(f^{-1}) = f \text{var}(f)^{-1} f^{-1}$, we obtain

$$h = t_{2\pi i(\frac{p}{q}-1)} \exp\left(-2\pi i e^{-kqz} \frac{\partial}{\partial z} + o(e^{-kqz})\right)$$

Therefore, $G = \Pi(g)$ and $H = \Pi(h)$ are such that G^p and H^q do not commute. We conclude that the group $\langle G, H \rangle$ is not solvable.

Case 2. It remains to consider the case $g = \text{var}(f)$ when it is \mathcal{U} -linearizable, that is (after convenient conjugation):

$$g = t_{2\pi i(\frac{q}{p}-1)}$$

and, according to Proposition A.1, the Dulac germ has the form $f = r s_{p/q}$ for some tangent to identity unramified germ r . As a consequence, $h = \text{var}(f^{-1})$ has the form

$$r t_{2\pi i(\frac{p}{q}-1)} r^{-1}.$$

Therefore f is the Poincaré map of a model $(\mathcal{F}_{p:q}, R)$, where $\mathcal{F}_{p:q}$ is the germ of a foliation generated by the linear differential 1-form $\omega_{p:q} = d(x^q y^p)$, and $R \in \text{Diff}(\mathbb{C}, 0)$ is an arbitrary holomorphic germ. In particular, the variation group is generated by two germs of a diffeomorphism G, H with respective linear parts $e^{2\pi i q/p} x$ and $e^{2\pi i p/q} x$, and such that

$$G^p = H^q = \text{Id}.$$

We now use a result of [LM94], which establishes the following trichotomy:

- Case 2.a. Either $\langle G, H \rangle$ is non-solvable, or
Case 2.b. $\langle G, H \rangle$ is Abelian and analytically linearizable, or
Case 2.c. $\langle G, H \rangle$ is metabelian and, in appropriate formal coordinates, has the form

$$\langle G, H \rangle = \left\langle \alpha x, \frac{\beta x}{(1+x^k)^{1/k}} \right\rangle$$

where we note $\alpha = e^{2\pi i q/p}, \beta = e^{2\pi i p/q}$ and k is some positive integer in $\mathbb{Z}_{\geq 1} \setminus \{p\mathbb{Z} \cup q\mathbb{Z}\}$

Proof based on [LM94]. According to [LM94, Proposition I.1], in the metabelian case we can write in convenient formal coordinates

$$G = s_\alpha \exp\left(ax^k x \frac{\partial}{\partial x}\right), H = s_\beta \exp\left(bx^k x \frac{\partial}{\partial x}\right),$$

where $s_\lambda(z) = \lambda z$ is the scaling map. We now impose the conditions $H^q = G^p = \text{Id}$ and $[G, H] \neq \text{Id}$.

First of all,

$$\begin{aligned} \text{Id} &= H^q \\ &= \left(s_\beta \exp\left(bx^k x \frac{\partial}{\partial x}\right)\right)^q \\ &= s_\beta \exp\left(bx^k x \frac{\partial}{\partial x}\right) s_\beta \exp\left(bx^k x \frac{\partial}{\partial x}\right) \cdots s_\beta \exp\left(bx^k x \frac{\partial}{\partial x}\right) \\ &= \exp\left(\left(\beta^{k(1-q)} + \beta^{k(2-q)} + \cdots + 1\right) bx^k x \frac{\partial}{\partial x}\right), \end{aligned}$$

which implies that k is not a multiple of q . Similarly, from $G^p = \text{Id}$ we get that $k \notin p\mathbb{Z}$. A simple computation finally gives

$$[G, H] = \exp\left(\left(a(\beta^{-k} - 1) - b(\alpha^{-k} - 1)\right) x^k x \frac{\partial}{\partial x}\right),$$

which implies that $a(\beta^{-k} - 1) + b(1 - \alpha^{-k}) \neq 0$. Notice that

$$\exp\left(-\lambda x^k x \frac{\partial}{\partial x}\right) g \exp\left(\lambda x^k x \frac{\partial}{\partial x}\right) = s_\alpha \exp\left(a + \lambda(1 - \alpha^{-k}) x^k x \frac{\partial}{\partial x}\right).$$

Therefore, upon conjugating by $\exp\left(\lambda x^k x \frac{\partial}{\partial x}\right)$ with λ such that

$$a + \lambda(1 - \alpha^{-k}) = 0$$

and by a further scaling, we obtain

$$G = \alpha x \quad \text{and} \quad H = \frac{\beta x}{(1+x^k)^{1/k}}.$$

□

Finally, in order to prove the formal rigidity, it suffices to consider the case where $\text{Var}(f)$ is metabelian, because non-solvable groups are rigid, and we just proved that Abelian groups are analytically linearizable. According to [CM88, Section 1, Remarque 1], the group $\langle G, H \rangle$ is exceptional (*i.e.* not rigid) if and only if

$$e^{2\pi i \frac{q}{p}} = e^{i\pi \frac{n_1}{k}}, \quad e^{2\pi i \frac{p}{q}} = e^{i\pi \frac{n_2}{k}}$$

with n_1, n_2 odd integers. In other words,

$$\frac{q}{p} \equiv \frac{n_1}{2k} \pmod{\mathbb{Z}} \quad \text{and} \quad \frac{p}{q} \equiv \frac{n_2}{2k} \pmod{\mathbb{Z}}$$

But this is impossible since it would imply that both q and p are even numbers. Therefore, $\langle G, H \rangle$ is always rigid, and in particular is *analytically* conjugate to $\left\langle \alpha x, \frac{\beta x}{(1+x^k)^{1/k}} \right\rangle$. \square

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