

ON POISSON TRANSFORMS FOR DIFFERENTIAL FORMS ON REAL HYPERBOLIC SPACES

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ABSTRACT. We study the Poisson transform for differential forms on the real hyperbolic space \mathbb{H}^n . For $1 < r < \infty$, we prove that the Poisson transform is a topological isomorphism from the space of L^r differential q -forms on the boundary $\partial\mathbb{H}^n$ onto a Hardy-type subspace of p -eigenforms of the Hodge-de Rham Laplacian on \mathbb{H}^n , for $0 \leq p < \frac{n-1}{2}$ and $q \in \{p-1, p\}$.

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1. INTRODUCTION

Let G/K be a Riemannian symmetric space of non-compact type. For each parabolic subgroup P of G there exists a natural Poisson transform from the space of C^∞ -functions on G/P to space of analytic functions on G/K .

When the parabolic P is minimal, one of the main problem stated by Helgason [12] claims that all eigenfunctions of G -invariant differential operators on G/K are obtained as Poisson transforms of hyperfunctions on the Furstenberg boundary G/P . This conjecture was proved by Helgason when G/K is of rank one, and in full generality by Kashiwara *et al.* [16]. Since then, this problem has received a lot of attention by many people in different settings (see, e.g., [2–6, 14, 18, 19, 23, 26]).

A natural extension of this problem is to investigate the analogous of Helgason's claim for Poisson transforms for homogeneous vector bundles over G/K (see,

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e.g., [8, 9, 11, 15, 20–22, 25, 28, 32]). One of the most interesting vector bundles is the bundle of differential forms on G/K . In this paper we consider the vector bundle of differential forms on the real hyperbolic space.

Let $\mathbb{H}^n = G/K$ be the real hyperbolic space realized as the open unit ball in \mathbb{R}^n , where $G = \mathrm{SO}_o(n, 1)$ and $K \simeq \mathrm{SO}(n)$. Its boundary $\partial\mathbb{H}^n$ is the unit sphere \mathbb{S}^{n-1} . As a homogeneous space, we have $\partial\mathbb{H}^n = G/P$, where $P = MAN$. Here $M \simeq \mathrm{SO}(n-1)$, $A \simeq \mathbb{R}$ and $N \simeq \mathbb{R}^{n-1}$.

For $0 \leq p \leq n$, let τ_p be the p -th exterior power of the coadjoint representation of K on $V_{\tau_p} = \Lambda^p \mathbb{C}^n$. Then the space $C^\infty(\Lambda^p \mathbb{H}^n)$ of smooth p -forms on \mathbb{H}^n can be identified with the space $C^\infty(G/K; \tau_p)$ of V_{τ_p} -valued smooth functions on G that are right covariant of type τ_p .

Throughout this paper we will assume that $0 \leq p < \frac{n-1}{2}$ (for this choice of p see Section 2). Then the decomposition of τ_p restricted to M is $\tau_{p|M} = \sigma_{p-1} \oplus \sigma_p$, where σ_q is q -th exterior power of the coadjoint representation of M on $V_{\sigma_q} = \Lambda^q \mathbb{C}^{n-1}$, with $q \in \{p-1, p\}$.

Let \mathfrak{a} be the Lie algebra of A , and identify its complexified dual $\mathfrak{a}_{\mathbb{C}}^*$ with \mathbb{C} . For $\lambda \in \mathbb{C}$, we consider the irreducible representation $\sigma_{q,\lambda}$ of $P = MAN$ given by $\sigma_{q,\lambda}(ma_t n) = \sigma_q(m) e^{(\rho-i\lambda)t}$, where $\rho = \frac{n-1}{2}$. Let $E_{q,\lambda}$ be the corresponding homogeneous vector bundle over $\partial\mathbb{H}^n$. We identify its space of hyperfunction sections with the space $C^{-\omega}(G/P; \sigma_{q,\lambda})$ of all V_{σ_q} -valued hyperfunctions f on G such that

$$f(gma_t n) = e^{(i\lambda-\rho)t} \sigma_q(m^{-1}) f(g) \quad \forall g \in G, \forall m \in M, \forall n \in N, \forall a_t \in A.$$

For $q \in \{p-1, p\}$, let ι_q^p be the natural embedding of V_{σ_q} into V_{τ_p} . Notice that $\iota_q^p \in \mathrm{Hom}_M(V_{\sigma_q}, V_{\tau_p})$. Then we can define a Poisson transform

$$\mathcal{P}_{q,\lambda}^p : C^{-\omega}(G/P; \sigma_{q,\lambda}) \rightarrow C^\infty(\Lambda^p \mathbb{H}^n)$$

by

$$\mathcal{P}_{q,\lambda}^p f(g) = \sqrt{\frac{\dim \tau_p}{\dim \sigma_q}} \int_K \tau_p(k) \iota_q^p(f(gk)) dk, \quad g \in G.$$

We mention that $E_{p,\lambda}$ can be seen as the vector bundle $G \times_P V_{\sigma_p} \otimes \mathcal{E}[\rho - i\lambda]$, where σ_p is extended to a representation of P , and $\mathcal{E}[\rho - i\lambda]$ is the density line bundle over the character $ma_t n \mapsto e^{(\rho-i\lambda)t}$ of P . Sections of the above bundle are q -hyperforms with value in $\mathcal{E}[\rho - i\lambda]$. In view of this observation, $\mathcal{P}_{p,\lambda}^p = \sqrt{\frac{\dim \tau_p}{\dim \sigma_p}} \Phi_p^{\rho-i\lambda}$, where $\Phi_p^{\rho-i\lambda}$ is the Poisson transform considered in [8].

Let $\Delta = d d^* + d^* d$ be the Hodge-de Rham Laplacian, where $d : C^\infty(\Lambda^p \mathbb{H}^n) \rightarrow C^\infty(\Lambda^{p+1} \mathbb{H}^n)$ is the differential and d^* is the codifferential (the adjoint of d which is defined by the hyperbolic metric).

For $\lambda \in \mathbb{C}$, denote by $C_{q,\lambda}^\infty(\Lambda^p \mathbb{H}^n)$ the space of all $\omega \in C^\infty(\Lambda^p \mathbb{H}^n)$ which are closed if $q = p-1$ and co-closed if $q = p$, with the additional condition $\Delta \omega = (\lambda^2 + (\rho - q)^2) \omega$. It was proved in [9], that for $0 \leq p < (n-1)/2$, the Poisson transforms $\mathcal{P}_{q,\lambda}^p$, $q = p-1, p$ provide the following isomorphisms:

$$(i) \quad \mathcal{P}_{p,\lambda}^p : C^{-\omega}(G/P; \sigma_{p,\lambda}) \rightarrow C_{p,\lambda}^\infty(\Lambda^p \mathbb{H}^n) \text{ iff } i\lambda \notin \{-\rho + p\} \cup (\mathbb{Z}_{\leq 0} - \rho),$$

and

(ii) $\mathcal{P}_{p-1,\lambda}^p : C^{-\omega}(G/P; \sigma_{p-1,\lambda}) \rightarrow C_{p-1,\lambda}^\infty(\Lambda^p \mathbb{H}^n)$ iff $i\lambda \notin \{\rho - p + 1\} \cup (\mathbb{Z}_{\leq 0} - \rho)$.

Now, let $C^{-\omega}(K/M; \sigma_q)$ be the space of V_{σ_q} -valued hyperfunctions f on K satisfying $f(km) = \sigma_q(m^{-1})f(k)$, for all $k \in K, m \in M$. By the Iwasawa decomposition, the restriction map $f \mapsto f|_K$ gives an isomorphism from $C^{-\omega}(G/P; \sigma_{q,\lambda})$ onto $C^{-\omega}(K/M; \sigma_q)$. Via this isomorphism we can define the Poisson transform from $C^{-\omega}(K/M; \sigma_q)$ into $C_{q,\lambda}^\infty(\Lambda^p \mathbb{H}^n)$. To state our main result, let us introduce further notation.

For $1 < r < \infty$, let $L^r(K/M; \sigma_q)$ be the space of V_{σ_q} -valued functions f on K which are covariant of type σ_q , and such that

$$\|f\|_{L^r(K/M; \sigma_q)} = \left(\int_K \|f(k)\|_{\Lambda^q \mathbb{C}^{n-1}}^r dk \right)^{\frac{1}{r}} < \infty.$$

The space $L^r(K/M; \sigma_q)$ is identified with the space of L^r differential q -forms on the boundary $\partial \mathbb{H}^n = K/M$. From above, it follows that the Poisson transform $\mathcal{P}_{q,\lambda}^p$ maps $L^r(K/M; \sigma_q)$ into the space $C_{q,\lambda}^\infty(\Lambda^p \mathbb{H}^n)$.

The goal of this paper is to characterize those eigenforms in $C_{q,\lambda}^\infty(\Lambda^p \mathbb{H}^n)$ which are Poisson transforms of elements in $L^r(K/M; \sigma_q)$, for $1 < r < \infty$. To this end we introduce the Hardy type space $\mathcal{E}_{q,\lambda}^r(G/K; \tau_p)$ of all F in $C_{q,\lambda}^\infty(\Lambda^p \mathbb{H}^n)$ such that

$$\|F\|_{\mathcal{E}_{q,\lambda}^r} := \sup_{t>0} e^{(\rho - \Re(i\lambda))t} \left(\int_K \|F(ka_t)\|_{\Lambda^p \mathbb{C}^n}^r dk \right)^{\frac{1}{r}} < \infty,$$

where we have identified $C^\infty(\Lambda^p \mathbb{H}^n)$ with $C^\infty(G/K; \tau_p)$.

We pin down that throughout the paper we will often view p -forms in $C^\infty(\Lambda^p \mathbb{H}^n)$ as functions in $C^\infty(G/K; \tau_p)$ and vice-versa.

Our main result is the following:

Theorem A (see Theorem 6.1). *Let $0 \leq p < (n-1)/2$ be an integer and $q \in \{p-1, p\}$. Assume $\lambda \in \mathbb{C}$ such that*

$$\begin{cases} \Re(i\lambda) > 0 & \text{if } q = p, \\ \Re(i\lambda) > 0 \text{ and } i\lambda \neq \rho - p + 1 & \text{if } q = p - 1. \end{cases}$$

The Poisson transform $\mathcal{P}_{q,\lambda}^p$ is a topological isomorphism of the space $L^r(K/M; \sigma_q)$ onto the space $\mathcal{E}_{q,\lambda}^r(G/K; \tau_p)$. Moreover, there exists a positive constant γ_λ such that

$$|c_q(\lambda, p)| \|f\|_{L^r(K/M; \sigma_q)} \leq \sqrt{\frac{\dim \sigma_q}{\dim \tau_p}} \|\mathcal{P}_{q,\lambda}^p f\|_{\mathcal{E}_{q,\lambda}^r} \leq \gamma_\lambda \|f\|_{L^r(K/M; \sigma_q)},$$

for every $f \in L^r(K/M; \sigma_q)$.

Above, $c_q(\lambda, p)$ ($q = p-1, p$) denote the scalar components of the vector-valued Harish-Chandra c -function $\mathbf{c}(\lambda, p)$. We refer the reader to (4.2) for the integral representation of $\mathbf{c}(\lambda, p)$. The explicit expressions of $c_q(\lambda, p)$ will be given in Proposition 4.6.

As an immediate consequence of Theorem A we obtain when $q = p$ and $i\lambda = \rho - p$ (the harmonic case) a characterization of co-closed harmonic p -forms, see Corollary 6.2. Furthermore, if in addition $p = 0$, we recover the classical fact

that the Poisson transform is an isometric isomorphism from $L^r(\partial\mathbb{H}^n)$ onto the Hardy-harmonic space on \mathbb{H}^n (see [27]).

Our strategy in proving Theorem A is to begin with the case $r = 2$. The most difficult part is to prove the sufficiency condition. Let us give a short outline of its proof. Let $F \in \mathcal{E}_{q,\lambda}^2(G/K; \tau_p)$, then we show the existence of a functional T on $C^\infty(G/P; \sigma_{q,\bar{\lambda}})$ such that $F = \widetilde{\mathcal{P}}_{q,\lambda}^p(T)$ ¹ (Proposition 5.1). To prove that T is indeed in L^2 we need to establish the asymptotic behavior of certain Eisenstein type integrals (see (5.8), (5.9)). To this end we prove a Fatou-type theorem (Theorem 4.3),

$$\lim_{t \rightarrow \infty} e^{(\rho - i\lambda)t} \mathcal{P}_{q,\lambda}^p f(ka_t) = \sqrt{\frac{\dim \tau_p}{\dim \sigma_q}} \mathbf{c}(\lambda, p) l_q^p(f(k)),$$

in $L^r(K, \Lambda^p \mathbb{C}^n)$, for every $f \in L^r(K/M; \sigma_q)$.

Let us mention that instead of Proposition 5.1 we might use the result of Gaillard, stated in Proposition 3.2 below, to ensure the existence of a hyperform $f \in C^{-\omega}(G/P; \sigma_{q,\lambda})$ such that $F = \mathcal{P}_{q,\lambda}^p f$. We would prefer to keep our argument because it is potentially useful in studying Poisson transform on vector bundles over symmetric spaces of non-compact type.

To establish Theorem A for every $1 < r < \infty$, we prove that any $F \in \mathcal{E}_{q,\lambda}^r(G/K; \tau_p)$ can be approximated by a sequence $(F_m)_m$ in $\mathcal{E}_{q,\lambda}^2(G/K; \tau_p)$. Using the first part of our result which corresponds to $r = 2$, we can deduce that there exists $f_m \in L^2(K/M; \sigma_q)$ such that $F_m = \mathcal{P}_{q,\lambda}^p(f_m)$. By an L^2 -inversion formula of the Poisson transform (Theorem 5.5) we conclude that f_m is indeed in $L^r(K/M; \sigma_q)$. Henceforth the linear form

$$T_m(\varphi) = \int_K \langle f_m(k), \varphi(k) \rangle_{\Lambda^q \mathbb{C}^{n-1}} dk,$$

is uniformly bounded on $L^s(K/M; \sigma_q)$, with $\frac{1}{r} + \frac{1}{s} = 1$. Thanks to Banach-Alaouglu-Bourbaki theorem, there exists a subsequence of bounded operators $(T_{m_j})_j$ which converges to a bounded operator T under the weak- \star topology. Thus by Riesz representation theorem, we conclude that there exists $f \in L^r(K/M; \sigma_q)$ such that $F = \mathcal{P}_{q,\lambda}^p f$.

The paper is organized as follows. Section 2 contains notations and background material for later use. In particular we recall some materials on differential forms on \mathbb{H}^n and $\partial\mathbb{H}^n = \mathbb{S}^{n-1}$ as sections of specific vector bundles. Section 3 is devoted to the definition of the Poisson transform $\mathcal{P}_{q,\lambda}^p$ on the space of differential forms on \mathbb{S}^{n-1} . In Section 4 we prove a Fatou type theorem for $\mathcal{P}_{q,\lambda}^p$, which will be of particular use to find the explicit expression of the Harish-Chandra c -function appearing in Theorem A. The Fatou type theorem will essentially play a crucial role in Section 5 where we prove Theorem A for the case $r = 2$. Section 5 contains also an L^2 -inversion formula for the Poisson transform. These results will allow us in Section 6 to prove Theorem A for every $1 < r < \infty$.

¹The parameter λ in Yang [32] corresponds in our notation to $i\lambda$.

2. BACKGROUND

2.1. The real hyperbolic space. Let $\mathbb{H}^n = \mathbb{H}^n(\mathbb{R})$ be the real hyperbolic space of dimension $n \geq 2$ realized as the open unit ball \mathbb{B}^n in \mathbb{R}^n . Let $G = \text{SO}_o(n, 1)$ be the connected component of the identity of the group of all linear transforms of \mathbb{R}^{n+1} with determinant 1 keeping invariant the Lorentzian quadratic form

$$[\mathbf{x}, \mathbf{x}] = x_1^2 + \cdots + x_n^2 - x_{n+1}^2, \quad \mathbf{x} = (x_1, \cdots, x_n, x_{n+1}).$$

Then the group G acts transitively on $\overline{\mathbb{B}^n}$ by fractional transformations and as a homogeneous space we have the identification $\mathbb{H}^n = G/K$, where $K = \text{SO}(n)$, the isotropy subgroup of $\mathbf{0} \in \mathbb{B}^n$, is a maximal compact subgroup of G .

Let $\mathfrak{g} = \mathfrak{so}(n, 1)$ and $\mathfrak{k} = \mathfrak{so}(n)$ be the Lie algebras of G and K , respectively. Let as usual $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of \mathfrak{g} . The subspace \mathfrak{p} is identified with the tangent space $T_{\mathbf{o}}(G/K) \simeq \mathbb{R}^n$ of $G/K = \mathbb{H}^n$ at the origin $\mathbf{o} = eK$.

Put

$$H_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathfrak{p},$$

then $\mathfrak{a} = \mathbb{R}H_0$ is a maximal abelian subspace of \mathfrak{p} , and the corresponding analytic Lie subgroup A of G is parametrized by

$$a_t = \exp(tH_0) = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}, \quad t \in \mathbb{R}.$$

Let

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & y & 0 \\ -y^T & 0_{n-1} & y^T \\ 0 & y & 0 \end{pmatrix}, y \in \mathbb{R}^{n-1} \right\} \simeq \mathbb{R}^{n-1},$$

so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is the Iwasawa decomposition of \mathfrak{g} . Here y^T stands for the transpose of a vector $y \in \mathbb{R}^{n-1}$.

Let $N = \exp(\mathfrak{n})$ be the connected Lie subgroup of G having \mathfrak{n} as Lie algebra. According to the Iwasawa decomposition $G = KAN$, every element $g \in G$ can be uniquely written as

$$g = \kappa(g)e^{H(g)}n,$$

where $\kappa(g) \in K$, $H(g) \in \mathfrak{a}$ and $n \in N$.

Let ρ be the half sum of positive roots of $(\mathfrak{g}, \mathfrak{a})$. Then $\rho(H_0) = \frac{n-1}{2}$ and we will write $\rho = \rho(H_0)$.

Let $P = MAN$ be the standard minimal parabolic subgroup of G , where M is the centralizer of A in K given by

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1 \end{pmatrix} : m \in \text{SO}(n-1) \right\} \simeq \text{SO}(n-1).$$

Then $G/P = K/M$ may be identified with the unit sphere \mathbb{S}^{n-1} in \mathbb{R}^n .

2.2. Differential forms on \mathbb{H}^n and \mathbb{S}^{n-1} . Let $\langle \cdot, \cdot \rangle$ be the standard Euclidean scalar product in \mathbb{R}^n . Let (e_1, e_2, \dots, e_n) be the standard orthonormal basis of \mathbb{R}^n and denote $(e_1^*, e_2^*, \dots, e_n^*)$ its dual basis.

For an integer p with $0 \leq p \leq n$, let $\Lambda^p(\mathbb{C}^n)^* = \Lambda^p(\mathbb{R}^n)^* \otimes \mathbb{C}$ be the space of complex-valued alternating multilinear p -forms on \mathbb{R}^n . A basis of $\Lambda^p(\mathbb{C}^n)^*$ is given by set of

$$e_I^* := e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* \quad \text{where} \quad \begin{cases} I = \{i_1, \dots, i_p\}, \\ 1 \leq i_1 < \cdots < i_p \leq n. \end{cases}$$

The interior product $\iota_v \omega$ of a p -form ω with a vector $v \in \mathbb{R}^n$ is the $(p-1)$ -form defined on the given basis by

$$\iota_{e_j}(e_{i_1}^* \wedge \cdots \wedge e_{i_p}^*) = \begin{cases} 0 & \text{if } j \neq \text{any } i_r \\ (-1)^{r-1} e_{i_1}^* \wedge \cdots \wedge \widehat{e_{i_r}^*} \wedge \cdots \wedge e_{i_p}^* & \text{if } j = i_r \end{cases}$$

where $\widehat{}$ over $e_{i_r}^*$ means that it is deleted from the exterior product.

For a given $v \in \mathbb{R}^n$, the exterior product $\varepsilon_v \omega$ of a p -form ω with the linear form v^* is the $(p+1)$ -form defined by

$$\varepsilon_v \omega = v^* \wedge \omega.$$

For the reader's convenience and to keep the notations simple, we will identify $(\mathbb{C}^n)^*$ with \mathbb{C}^n and $\Lambda^p(\mathbb{C}^n)^*$ with $\Lambda^p \mathbb{C}^n$.

We define an inner product $\langle \cdot, \cdot \rangle_{\Lambda^p \mathbb{C}^n}$ on $\Lambda^p \mathbb{C}^n$ as an extension of the one on \mathbb{C}^n by setting

$$\langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p \rangle_{\Lambda^p \mathbb{C}^n} = \mathbf{det}(\langle v_i, w_j \rangle)_{i,j}. \quad (2.1)$$

It is easy to show that the basis of $\Lambda^p \mathbb{C}^n$ consisting of the p -vectors $e_I := e_{i_1} \wedge \cdots \wedge e_{i_p}$ (where $I = \{i_1, \dots, i_p\}$, with $1 \leq i_1 < \cdots < i_p \leq n$) is an orthonormal basis of $\Lambda^p \mathbb{C}^n$ with respect to (2.1). We have further the useful identity

$$\langle \iota_v \omega, \xi \rangle_{\Lambda^{p-1} \mathbb{C}^n} = \langle \omega, \varepsilon_v \xi \rangle_{\Lambda^p \mathbb{C}^n}, \quad v \in \mathbb{R}^n, \omega \in \Lambda^p \mathbb{C}^n, \xi \in \Lambda^{p-1} \mathbb{C}^n. \quad (2.2)$$

For $0 \leq p \leq n$, we let τ_p to be the p -exterior product $\Lambda^p \text{Ad}^*$ of the coadjoint representation of $K = \text{SO}(n)$ on $\mathfrak{p}_{\mathbb{C}}^*$. Its representation space being $V_{\tau_p} := \Lambda^p(\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}})^* \simeq \Lambda^p \mathbb{C}^n$. Notice that τ_p is unitary with respect to the inner product (2.1), and is equivalent to the standard representation of K on $\Lambda^p \mathbb{C}^n$. By [7] or [13], the representation τ_p is irreducible for $p \neq \frac{n}{2}$ (n even), while $\tau_{\frac{n}{2}} = \tau_{\frac{n}{2}}^+ \oplus \tau_{\frac{n}{2}}^-$. The two factors $\tau_{\frac{n}{2}}^{\pm}$ being irreducible, inequivalent and act on the following eigenspaces of the Hodge star operator \star ,

$$\Lambda_{\frac{n}{2}}^{\pm} \mathbb{C}^n = \{w \in \Lambda^{\frac{n}{2}} \mathbb{C}^n : \star w = \mu_{\pm} w\},$$

where $\mu_{\pm} = \pm 1$ if $\frac{n}{2}$ is even and $\mu_{\pm} = \pm i$ if $\frac{n}{2}$ is odd. Since the Hodge operator \star induces the equivalence $\tau_p \simeq \tau_{n-p}$, we will restrict our attention to the case $0 \leq p < \frac{n}{2}$, without loss of generality.

For $0 \leq q \leq n-1$, let σ_q be the standard representation of $M \simeq \text{SO}(n-1)$ on $V_{\sigma_q} = \Lambda^q \mathbb{C}^{n-1}$. It is an irreducible representation for $q \neq \frac{n-1}{2}$, and as before $\sigma_{\frac{n-1}{2}} = \sigma_{\frac{n-1}{2}}^+ \oplus \sigma_{\frac{n-1}{2}}^-$.

Lemma 2.1 (See, e.g. [1, 13]). *Let $\tau_{p|_M}$ be the restriction of τ_p to $M \simeq \text{SO}(n-1)$. Then $\tau_{p|_M}$ decomposes into inequivalent factors as follow :*

- 1) For $p = 0$, $\tau_{p|_M} = \sigma_p$.
- 2) For $0 < p < \frac{n-1}{2}$,
 $\tau_{p|_M} = \sigma_{p-1} \oplus \sigma_p$ with

$$\Lambda^p \mathbb{C}^n = e_1 \wedge \Lambda^{p-1} \mathbb{C}^{n-1} \oplus \Lambda^p \mathbb{C}^{n-1} \simeq \Lambda^{p-1} \mathbb{C}^{n-1} \oplus \Lambda^p \mathbb{C}^{n-1}. \quad (2.3)$$

- 3) For $p = \frac{n-1}{2}$, $\tau_{p|_M} = \sigma_{p-1} \oplus \sigma_p^+ \oplus \sigma_p^-$.
- 4) For $p = \frac{n}{2}$, $\tau_{p|_M} = 2\sigma_{p-1} \sim 2\sigma_p$.

Henceforth, we will assume along this paper that $0 \leq p < \frac{n-1}{2}$ (we say p generic).

Remark 2.2. (1) In the decomposition (2.3) we have identified \mathbb{C}^{n-1} with $\text{span}\{e_2, \dots, e_n\}$. The isomorphism (2.3) follows from the $\text{SO}(n-1)$ -equivariance of the decomposition

$$\omega = e_1 \wedge \omega' + \omega'' \quad \text{with } \omega' \in \Lambda^{p-1} \mathbb{C}^{n-1} \text{ and } \omega'' \in \Lambda^p \mathbb{C}^{n-1}$$

for any $\omega \in \Lambda^p \mathbb{C}^n$.

(2) The scalar products on $\Lambda^q \mathbb{C}^{n-1}$, $q \in \{p-1, p\}$, are induced from the one on $\Lambda^p \mathbb{C}^n$ defined in (2.1).

(3) For $q \in \{p-1, p\}$, we will consider the following natural isometric embedding

$$\iota_q^p: V_{\sigma_q} = \Lambda^q \mathbb{C}^{n-1} \rightarrow V_{\tau_p} = \Lambda^p \mathbb{C}^n. \quad (2.4)$$

Notice that $\iota_q^p \in \text{Hom}_M(V_{\sigma_q}, V_{\tau_p})$ and it is given by

$$\begin{aligned} \iota_{p-1}^p: \Lambda^{p-1} \mathbb{C}^{n-1} &\rightarrow e_1 \wedge \Lambda^{p-1} \mathbb{C}^{n-1} \oplus \Lambda^p \mathbb{C}^{n-1} \\ \xi &\mapsto e_1 \wedge \xi + 0 \end{aligned}$$

and

$$\begin{aligned} \iota_p^p: \Lambda^p \mathbb{C}^{n-1} &\rightarrow e_1 \wedge \Lambda^{p-1} \mathbb{C}^{n-1} \oplus \Lambda^p \mathbb{C}^{n-1} \\ \xi &\mapsto 0 + \xi \end{aligned}$$

In particular, for any $\omega, \omega' \in \Lambda^{p-1} \mathbb{C}^{n-1}$,

$$\langle \omega, \omega' \rangle_{\Lambda^{p-1} \mathbb{C}^{n-1}} = \langle e_1 \wedge \omega, e_1 \wedge \omega' \rangle_{\Lambda^p \mathbb{C}^n}$$

(4) For $q \in \{p-1, p\}$, let π_p^q denotes the natural projection

$$\pi_p^q: V_{\tau_p} \rightarrow V_{\sigma_q}.$$

Then one can see from (2.2) that $(\pi_p^q)^* = \iota_q^p$.

3. POISSON TRANSFORM ON DIFFERENTIAL FORMS

In this section we shall define the Poisson transform for differential forms on $\partial \mathbb{H}^n$. We will follow the definition of Okamoto [21], see also Minemura [20], Yang [32], Juhl [15], Van der ven [28], Olbrich [22] and Pedon [24, 25]. There is also another approach to define the differential forms-valued Poisson transforms initiated by Gaillard [8] and generalized by Harrach [11].

Let $G \times_K V_{\tau_p}$ be the homogeneous vector bundle over G/K associated with τ_p . The space of its smooth sections is identified with

$$C^\infty(G/K; \tau_p) = \{f: G \rightarrow V_{\tau_p} \text{ smooth} \mid f(gk) = \tau_p(k^{-1})f(g) \forall g \in G, \forall k \in K\}.$$

As a homogeneous vector bundle, we have $\Lambda^p \mathbb{H}^n := \Lambda^p T_{\mathbb{C}}^* \mathbb{H}^n = G \times_K V_{\tau_p}$ and therefore we identify the space $C^\infty(\Lambda^p \mathbb{H}^n)$ of its smooth sections (*i.e.*, smooth differential p -forms on \mathbb{H}^n) with the space $C^\infty(G/K; \tau_p)$.

Consider the exterior differentiation operator $d: C^\infty(\Lambda^p \mathbb{H}^n) \rightarrow C^\infty(\Lambda^{p+1} \mathbb{H}^n)$ and the co-differentiation $d^* = (-1)^{n(p+1)+1} \star d \star: C^\infty(\Lambda^p \mathbb{H}^n) \rightarrow C^\infty(\Lambda^{p-1} \mathbb{H}^n)$. Let $\Delta = dd^* + d^*d$ be the Hodge-de Rham Laplacian on $C^\infty(\Lambda \mathbb{H}^n)$. Let $\mathbf{D}(\Lambda^p \mathbb{H}^n)$ be the algebra of G -invariant differential operators acting on $C^\infty(\Lambda^p \mathbb{H}^n)$. Its known by [10] that for generic p , $\mathbf{D}(\Lambda^p \mathbb{H}^n)$ is a commutative algebra generated by dd^* and d^*d .

Next, we shall describe the eigenforms for differential operators in $\mathbf{D}(\Lambda^p \mathbb{H}^n)$ by means of Poisson transforms.

For $q \in \{p-1, p\}$ and $\lambda \in \mathbb{C}$, we consider the following irreducible representation of $P = MAN$,

$$\sigma_{q,\lambda}: ma_t n \mapsto \sigma_q(m) e^{(\rho-i\lambda)t}$$

Let $E_{q,\lambda}$ be the homogeneous vector bundle over $\partial \mathbb{H}^n$ corresponding to $\sigma_{q,\lambda}$. We denote by $C^{-\omega}(\partial \mathbb{H}^n; E_{q,\lambda})$ the space of its hyperfunction sections and we identify it with the space $C^{-\omega}(G/P; \sigma_{q,\lambda})$ of V_q -valued hyperfunctions ϕ on G such that

$$f(gma_t n) = e^{(i\lambda-\rho)t} \sigma_q(m^{-1}) f(g)$$

for all $g \in G, m \in M, n \in N, a_t \in A$. Then, define the Poisson transform

$$\mathcal{P}_{q,\lambda}^p: C^{-\omega}(G/P; \sigma_{q,\lambda}) \rightarrow C^\infty(\Lambda^p \mathbb{H}^n)$$

by

$$\mathcal{P}_{q,\lambda}^p f(g) = c_{p,q} \int_K \tau_p(k) \iota_q^p(f(gk)) dk, \quad g \in G,$$

where ι_q^p is the embedding given by (2.4), dk denotes the normalized Haar measure on K , and where the constant factor $c_{p,q}$ is given by

$$c_{p,q} = \sqrt{\frac{\dim \tau_p}{\dim \sigma_q}} = \begin{cases} \sqrt{\frac{n}{n-p}} & \text{if } q = p, \\ \sqrt{\frac{n}{p}} & \text{if } q = p-1. \end{cases} \quad (3.1)$$

Let us mention that for $q = p$, $E_{p,\lambda}$ can be seen as the vector bundle $G \times_P V_{\sigma_p} \otimes \mathcal{E}[\rho - i\lambda]$, where σ_p is extended to a representation of P and $\mathcal{E}[\rho - i\lambda]$ is the density line bundle associated to the character $ma_t n \mapsto e^{(\rho-i\lambda)t}$ of P . Then $C^{-\omega}(\partial \mathbb{H}^n; E_{p,\lambda})$ can be viewed as the space of p -hyperforms on $\partial \mathbb{H}^n$ with value in $\mathcal{E}[\rho - i\lambda]$. In view of this observation, $\mathcal{P}_{p,\lambda}^p = c_{p,p} \Phi_p^{\rho-i\lambda}$, where $\Phi_p^{\rho-i\lambda}$ is the Poisson transform considered in [8]. When $i\lambda = \rho - p$ (which corresponds to the harmonic case, see below) the space $C^{-\omega}(\partial \mathbb{H}^n; E_{p,-i(\rho-p)})$ consists of p -hyperforms with value in $\mathcal{E}[p]$.

By the Iwasawa decomposition, the restriction map of $f \mapsto f|_K$ gives an isomorphism from $C^{-\omega}(G/P; \sigma_{q,\lambda})$ onto the space $C^{-\omega}(K/M; \sigma_q)$ of V_q -valued hyperfunctions f on K satisfying $f(km) = \sigma_q(m^{-1})f(k)$, for all $k \in K, m \in M$.

In this compact model, the Poisson transform

$$\mathcal{P}_{q,\lambda}^p: C^{-\omega}(K/M; \sigma_q) \rightarrow C^\infty(\Lambda^p \mathbb{H}^n)$$

takes the form

$$\mathcal{P}_{q,\lambda}^p f(g) = c_{p,q} \int_K e^{-(i\lambda + \rho)H(g^{-1}k)} \tau_p(\kappa(g^{-1}k)) \iota_q^p(f(gk)) dk, \quad g \in G.$$

Below, we shall give the explicit action of the algebra $\mathbf{D}(\Lambda^p \mathbb{H}^n)$ on the Poisson transform of elements in $C^{-\omega}(K/M; \sigma_q)$. The following result is due to Gaillard [8, 9], see also Pedon [25].

Proposition 3.1. *For $f \in C^{-\omega}(K/M; \sigma_q)$ with $q \in \{p-1, p\}$, we have*

$$\begin{aligned} d^* \mathcal{P}_{p,\lambda}^p(f) &= 0, & d \mathcal{P}_{p-1,\lambda}^p(f) &= 0, \\ d^* d \mathcal{P}_{p,\lambda}^p(f) &= (\lambda^2 + (\rho - p)^2) \mathcal{P}_{p,\lambda}^p(f), & dd^* \mathcal{P}_{p-1,\lambda}^p(f) &= (\lambda^2 + (\rho - p + 1)^2) \mathcal{P}_{p-1,\lambda}^p(f). \end{aligned}$$

For a character $\chi: \mathbf{D}(\Lambda^p \mathbb{H}^n) \rightarrow \mathbb{C}$, let $\mathcal{E}_\chi(\Lambda^p \mathbb{H}^n)$ be the corresponding eigenspace,

$$\mathcal{E}_\chi(\Lambda^p \mathbb{H}^n) := \{f \in C^\infty(\Lambda^p \mathbb{H}^n) \mid Df = \chi(D)f, \forall D \in \mathbf{D}(\Lambda^p \mathbb{H}^n)\}.$$

Put $\chi(\Delta) = \gamma$ and suppose $\gamma \neq 0$. Similarly, denote $\chi(dd^*) = \gamma_1$ and $\chi(d^*d) = \gamma_2$. Consider the eigenspace

$$\mathcal{E}_\gamma(\Lambda^p \mathbb{H}^n) := \{f \in C^\infty(\Lambda^p \mathbb{H}^n) \mid \Delta f = \gamma f\}.$$

Since $(d^*d)(dd^*) = 0$, we have $\gamma_1 \gamma_2 = 0$. As $\gamma \neq 0$ and $\gamma = \gamma_1 + \gamma_2$, therefore, we have either $(\gamma_1 = 0$ and $\gamma_2 = \gamma)$ or $(\gamma_2 = 0$ and $\gamma_1 = \gamma)$. We denote χ by χ_1 in the first case and by χ_2 the second case. Thus,

$$\mathcal{E}_\gamma(\Lambda^p \mathbb{H}^n) = \mathcal{E}_{\chi_1}(\Lambda^p \mathbb{H}^n) \oplus \mathcal{E}_{\chi_2}(\Lambda^p \mathbb{H}^n).$$

In view of Proposition 3.1, we deduce that $\gamma_1 = \lambda^2 + (\rho - p + 1)^2$, $\gamma_2 = \lambda^2 + (\rho - p)^2$ and

$$\begin{aligned} \mathcal{E}_{\chi_1}(\Lambda^p \mathbb{H}^n) &= \left\{ f \in C^\infty(\Lambda^p \mathbb{H}^n) \mid \begin{cases} \Delta f &= (\lambda^2 + (\rho - p)^2) f \\ d^* f &= 0 \end{cases} \right\}, \\ \mathcal{E}_{\chi_2}(\Lambda^p \mathbb{H}^n) &= \left\{ f \in C^\infty(\Lambda^p \mathbb{H}^n) \mid \begin{cases} \Delta f &= (\lambda^2 + (\rho - p + 1)^2) f \\ df &= 0 \end{cases} \right\}. \end{aligned}$$

Under the identification $C^\infty(\Lambda^p \mathbb{H}^n) \simeq C^\infty(G/K; \tau_p)$, we let D , D^* and $-C$ to be the counterpart of d , d^* and Δ acting on $C^\infty(G/K; \tau_p)$, given by

$$D = \sum_j X_j \varepsilon_{X_j}, \quad D^* = - \sum_j X_j \iota_{X_j}, \quad C = \sum_j X_j^2 - \sum_j Y_j^2, \quad (3.2)$$

where (X_i) and (Y_i) are orthonormal ² bases of \mathfrak{p} and \mathfrak{k} respectively. Thus, the spaces $\mathcal{E}_{\chi_1}(\Lambda^p \mathbb{H}^n)$ and $\mathcal{E}_{\chi_2}(\Lambda^p \mathbb{H}^n)$ are identified respectively with

²with respect to the normalized Killing form $\frac{1}{2(n-1)}B$

$$\mathcal{E}_{p,\lambda}(G/K; \tau_p) = \left\{ f \in C^\infty(G/K; \tau_p) \mid \begin{cases} \mathcal{C}f & = -(\lambda^2 + (\rho - p)^2)f \\ D^*f & = 0 \end{cases} \right\}, \quad (3.3)$$

$$\mathcal{E}_{p-1,\lambda}(G/K; \tau_p) = \left\{ f \in C^\infty(G/K; \tau_p) \mid \begin{cases} \mathcal{C}f & = -(\lambda^2 + (\rho - p + 1)^2)f \\ Df & = 0 \end{cases} \right\}. \quad (3.4)$$

Notice that \mathcal{C} is the Casimir operator of \mathfrak{g} acting on $C^\infty(G/K; \tau_p)$.

Proposition 3.2 (see [9]). *Let $0 \leq p < (n-1)/2$, $q \in \{p-1, p\}$ and let $\lambda \in \mathbb{C}$ such that*

$$\begin{cases} i\lambda \notin \{-\rho + p\} \cup (\mathbb{Z}_{\leq 0} - \rho) & \text{if } q = p, \\ i\lambda \notin \{\rho - p + 1\} \cup (\mathbb{Z}_{\leq 0} - \rho) & \text{if } q = p - 1. \end{cases}$$

The Poisson transform $\mathcal{P}_{q,\lambda}^p$ is a topological isomorphism from the space $C^{-\omega}(K/M; \sigma_q)$ onto the space $\mathcal{E}_{q,\lambda}(G/K; \tau_p)$.

We point out that the above statement was stated in [15] for $q = p$ and n even.

For $1 < r < \infty$, we denote by $L^r(K/M; \sigma_q)$ the space of $\Lambda^q \mathbb{C}^{n-1}$ -valued functions on K which are covariant of type σ_q , i.e.,

$$f(km) = \sigma_q(m^{-1})f(k), \quad \forall k \in K, \forall m \in M,$$

and such that

$$\|f\|_{L^r(K/M; \sigma_q)} := \left(\int_K \|f(k)\|_{\Lambda^q \mathbb{C}^{n-1}}^r dk \right)^{\frac{1}{r}} < \infty.$$

Note that, for any $F: K \rightarrow \Lambda^k \mathbb{C}^N$ we have

$$\left\| \int_K F(k) dk \right\|_{\Lambda^k \mathbb{C}^N} \leq \int_K \|F(k)\|_{\Lambda^k \mathbb{C}^N} dk. \quad (3.5)$$

From above, it follows that the Poisson transform $\mathcal{P}_{q,\lambda}^p$ maps $L^r(K/M; \sigma_q)$ into $\mathcal{E}_{q,\lambda}(G/K; \tau_p)$. Our aim is to characterize the exact image of the space $L^r(K/M; \sigma_q)$ by the Poisson transform $\mathcal{P}_{q,\lambda}^p$ for generic p and $q \in \{p-1, p\}$.

4. FATOU-TYPE THEOREM AND THE HARISH-CHANDRA c -FUNCTION

For $\lambda \in \mathbb{C}$, generic p , and $q \in \{p-1, p\}$, we define for $1 < r < \infty$, the space $\mathcal{E}_{q,\lambda}^r(G/K; \tau_p)$ to be the subspace of all F in $\mathcal{E}_{q,\lambda}(G/K; \tau_p)$ for which

$$\|F\|_{\mathcal{E}_{q,\lambda}^r} := \sup_{t>0} e^{(\rho - \Re(i\lambda))t} \left(\int_K \|F(ka_t)\|_{\Lambda^q \mathbb{C}^n}^r dk \right)^{\frac{1}{r}}$$

is finite.

Proposition 4.1. *For every $\lambda \in \mathbb{C}$ with $\Re(i\lambda) > 0$, there exists a positive constant γ_λ such that, for any $f \in L^r(K/M; \sigma_q)$ we have*

$$\left(\int_K \|\mathcal{P}_{q,\lambda}^p f(ka_t)\|_{\Lambda^q \mathbb{C}^n}^r dk \right)^{1/r} \leq \gamma_\lambda c_{p,q} e^{(\Re(i\lambda) - \rho)t} \|f\|_{L^r(K/M; \sigma_q)}. \quad (4.1)$$

Proof. By (3.5) we have

$$\begin{aligned} & \| \mathcal{P}_{q,\lambda}^p f(ka_t) \|_{\Lambda^p \mathbb{C}^n} \\ & \leq c_{p,q} \int_K e^{-(\Re(i\lambda)+\rho)H(a_t^{-1}k^{-1}h)} \| \tau_p(\kappa(a_t^{-1}k^{-1}h)) \iota_q^p(f(h)) \|_{\Lambda^p \mathbb{C}^n} dh \\ & \leq c_{p,q} \int_K e^{-(\Re(i\lambda)+\rho)H(a_t^{-1}k^{-1}h)} \| \iota_q^p(f(h)) \|_{\Lambda^p \mathbb{C}^n} dh, \end{aligned}$$

where the last inequality follows from the unitarity of τ_p . Since ι_q^p is an isometric embedding, we can deduce that

$$\begin{aligned} \| \mathcal{P}_{q,\lambda}^p f(ka_t) \|_{\Lambda^p \mathbb{C}^n} & \leq c_{p,q} \int_K e^{-(\Re(i\lambda)+\rho)H(a_t^{-1}k^{-1}h)} \| f(h) \|_{\Lambda^q \mathbb{C}^{n-1}} dh \\ & = c_{p,q} e_{\lambda,t}(\cdot) * \| f(\cdot) \|_{\Lambda^q \mathbb{C}^{n-1}}(k), \end{aligned}$$

where $e_{\lambda,t}(k) = e^{-(\Re(i\lambda)+\rho)H(a_t^{-1}k^{-1})}$, and $*$ is the convolution over K . Therefore, by Young's inequality, we obtain

$$\left(\int_K \| \mathcal{P}_{q,\lambda}^p f(ka_t) \|_{\Lambda^p \mathbb{C}^n}^r dk \right)^{1/r} \leq c_{p,q} \| e_{\lambda,t} \|_{L^1(K/M; \sigma_q)} \| f \|_{L^r(K/M; \sigma_q)}.$$

Further,

$$\| e_{\lambda,t} \|_{L^1(K/M; \sigma_q)} = \int_K e^{-(\Re(i\lambda)+\rho)H(a_t^{-1}k^{-1})} dk = \phi_{-i\Re(i\lambda)}^{(\rho-\frac{1}{2}, -\frac{1}{2})}(t),$$

where $\phi_{\nu}^{(\alpha, \beta)}$ is the Jacobi function, see (4.7). Since $\Re(i\lambda) > 0$, by (4.8) we have

$$\phi_{-i\Re(i\lambda)}^{(\rho-\frac{1}{2}, -\frac{1}{2})}(t) = e^{(\Re(i\lambda)-\rho)t} \left(c_{\rho-\frac{1}{2}, -\frac{1}{2}}(-i\Re(i\lambda)) + o(1) \right) \text{ as } t \rightarrow \infty,$$

where $c_{\rho-\frac{1}{2}, -\frac{1}{2}}(-i\Re(i\lambda))$ is given by (4.9). This proves the estimate (4.1) and consequently that the Poisson transform is continuous from $L^r(K/M; \sigma_q)$ into $\mathcal{E}_{q,\lambda}^r(G/K; \tau_p)$. \square

Let $\bar{N} = \theta(N)$, where θ is the Cartan involution of G . For $\lambda \in \mathbb{C}$ and $0 \leq p < \frac{n-1}{2}$, define the generalized Harish-Chandra c -function by

$$\mathbf{c}(\lambda, p) = \int_{\bar{N}} e^{-(i\lambda+\rho)H(\bar{n})} \tau_p(\kappa(\bar{n})) d\bar{n} \in \text{End}(\Lambda^p \mathbb{C}^n). \quad (4.2)$$

Here $d\bar{n}$ is the Haar measure on \bar{N} with the normalization

$$\int_{\bar{N}} e^{-2\rho(H(\bar{n}))} d\bar{n} = 1.$$

The integral (4.2) converges for λ such that $\Re(i\lambda) > 0$ and has a meromorphic continuation to \mathbb{C} (see, e.g. [31]). Since the restriction $\mathbf{c}(\lambda, p)|_{V_{\sigma_q}}$ commutes with σ_q , then by Schur's lemma, there exists a complex scalar $c_q(\lambda, p)$ such that $\mathbf{c}(\lambda, p)|_{V_{\sigma_q}} = c_q(\lambda, p) \text{Id}_{\Lambda^q \mathbb{C}^{n-1}}$. Therefore,

$$\mathbf{c}(\lambda, p) = c_{p-1}(\lambda, p) \text{Id}_{\Lambda^{p-1} \mathbb{C}^{n-1}} + c_p(\lambda, p) \text{Id}_{\Lambda^p \mathbb{C}^{n-1}}. \quad (4.3)$$

In [29], an explicit expression of $c_{p-1}(\lambda, p)$ and $c_p(\lambda, p)$ are given by a direct computation of the integral (4.2). However, below in Proposition 4.6, we will recover their expressions by using a different approach.

The following lemma is needed for later use.

Lemma 4.2. (1) For every $v \in V_{\sigma_q}$,

$$\|\mathbf{c}(\lambda, p)\iota_q^p(v)\|_{\Lambda^p\mathbb{C}^n} = |c_q(\lambda, p)| \|v\|_{\Lambda^q\mathbb{C}^{n-1}}. \quad (4.4)$$

(2) For every linear operator L form a vector space V to V_{σ_q} ,

$$\|\mathbf{c}(\lambda, p)\iota_q^p L\|_{\text{HS}} = |c_q(\lambda, p)| \|L\|_{\text{HS}}, \quad (4.5)$$

Proof. Using Remark 2.2, the first statement follows directly from

$$\mathbf{c}(\lambda, p)\iota_q^p(v) = \begin{cases} c_{p-1}(\lambda, p)e_1 \wedge v, & q = p-1 \\ c_p(\lambda, p)v, & q = p. \end{cases}$$

On the other hand,

$$\begin{aligned} \|\mathbf{c}(\lambda, p)\iota_q^p L\|_{\text{HS}}^2 &= \mathbf{tr}((\mathbf{c}(\lambda, p)\iota_q^p L)^*(\mathbf{c}(\lambda, p)\iota_q^p L)) \\ &= \mathbf{tr}(L^*(\pi_p^q \mathbf{c}(\lambda, p)^* \mathbf{c}(\lambda, p)\iota_q^p)L). \end{aligned}$$

Notice that $\pi_p^q \mathbf{c}(\lambda, p)^* \mathbf{c}(\lambda, p)\iota_q^p \in \text{End}_M(V_{\sigma_q})$, (hence is scalar). By (4.3), we deduce that

$$\mathbf{c}(\lambda, p)^* \mathbf{c}(\lambda, p) = \begin{pmatrix} |c_{p-1}(\lambda, p)|^2 \text{Id}_{\Lambda^{p-1}\mathbb{C}^{n-1}} & 0 \\ 0 & |c_p(\lambda, p)|^2 \text{Id}_{\Lambda^p\mathbb{C}^{n-1}} \end{pmatrix}.$$

Thus $\pi_p^q \mathbf{c}(\lambda, p)^* \mathbf{c}(\lambda, p)\iota_q^p = |c_q(\lambda, p)|^2 \text{Id}_{\Lambda^q\mathbb{C}^{n-1}}$, and this proves the second statement. \square

Theorem 4.3. Let $\lambda \in \mathbb{C}$ such that $\Re(i\lambda) > 0$. Then

$$\lim_{t \rightarrow \infty} e^{(\rho-i\lambda)t} \mathcal{P}_{q,\lambda}^p f(ka_t) = c_{p,q} \mathbf{c}(\lambda, p)\iota_q^p(f(k)),$$

(i) uniformly for $f \in C^\infty(K/M; \sigma_q)$,

(ii) in the $L^r(K; \Lambda^p\mathbb{C}^n)$ -sents, for every $f \in L^r(K/M; \sigma_q)$.

Proof. The statement (i) has been proved earlier, see for instance [28] and [32].

(ii) Let $f \in L^r(K/M; \sigma_q)$ and $\varepsilon > 0$. By density argument, there exists a K -finite vector φ in $C^\infty(K/M; \sigma_q)$ such that $\|f - \varphi\|_{L^r(K/M; \sigma_q)} < \varepsilon$. Put $p_\lambda^t(f)(k) = \mathcal{P}_{q,\lambda}^p f(ka_t)$, then

$$\begin{aligned} \|e^{-(i\lambda-\rho)t} p_\lambda^t(f)(k) - c_{p,q} \mathbf{c}(\lambda, p)\iota_q^p f(k)\|_{\Lambda^p\mathbb{C}^n}^r &\leq \|e^{-(i\lambda-\rho)t} p_\lambda^t(f - \varphi)(k)\|_{\Lambda^p\mathbb{C}^n}^r \\ &\quad + \|e^{-(i\lambda-\rho)t} p_\lambda^t(\varphi)(k) - c_{p,q} \mathbf{c}(\lambda, p)\iota_q^p \varphi(k)\|_{\Lambda^p\mathbb{C}^n}^r \\ &\quad + c_{pq}^r \|\mathbf{c}(\lambda, p)\iota_q^p \varphi(k) - \mathbf{c}(\lambda, p)\iota_q^p f(k)\|_{\Lambda^p\mathbb{C}^n}^r. \end{aligned}$$

From Proposition 4.1 we obtain

$$\int_K \|e^{-(i\lambda-\rho)t} p_\lambda^t(f - \varphi)(k)\|_{\Lambda^p\mathbb{C}^n}^r dk \leq \gamma_\lambda^r c_{p,q}^r \|f - \varphi\|_{L^r(K/M; \sigma_q)}^r,$$

and from part (i) above it follows that

$$\lim_{t \rightarrow \infty} \int_K \|e^{-(i\lambda-\rho)t} p_\lambda^t(\varphi)(k) - c_{p,q} \mathbf{c}(\lambda, p)\iota_q^p \varphi(k)\|_{\Lambda^p\mathbb{C}^n} dk = 0.$$

Further, according to (4.4) we obtain

$$\int_K \|c(\lambda, p)t_q^p \varphi(k) - c(\lambda, p)t_q^p f(k)\|_{\Lambda^p \mathbb{C}^n}^r dk \leq |c_q(\lambda, p)|^r \|f - \varphi\|_{L^r(K/M; \sigma_q)}^r.$$

In conclusion we have

$$\lim_{t \rightarrow \infty} \int_K \|e^{-(i\lambda - \rho)t} p_\lambda^t(f)(k) - c_{p,q} c(\lambda, p)t_q^p f(k)\|_{\Lambda^p \mathbb{C}^n}^r dk \leq \varepsilon^r c_{p,q}^r (\gamma_\lambda^r + |c_q(\lambda, p)|^r),$$

and this proves the desired statement. \square

The following inequalities are crucial.

Proposition 4.4. *For every $\lambda \in \mathbb{C}$ such that $\Re(i\lambda) > 0$, there exists a positive constant γ_λ such that for all $f \in L^r(K/M; \sigma_q)$, $1 < r < \infty$, we have*

$$c_{p,q} |c_q(\lambda, p)| \|f\|_{L^r(K/M; \sigma_q)} \leq \|\mathcal{P}_{q,\lambda}^p f\|_{\mathcal{E}_{q,\lambda}^r} \leq c_{p,q} \gamma_\lambda \|f\|_{L^r(K/M; \sigma_q)}. \quad (4.6)$$

Proof. The right-hand side inequality is noting but the estimate (4.1). For the left-hand side inequality, by Theorem 4.3[(ii)], there exists a sequence $(t_j)_j$ with $t_j \rightarrow \infty$ such that

$$\lim_{j \rightarrow \infty} \|e^{(\rho - i\lambda)t_j} \mathcal{P}_{q,\lambda}^p f(ka_{t_j})\|_{\Lambda^p \mathbb{C}^n} = \|c_{p,q} c(\lambda, p)t_q^p(f(k))\|_{\Lambda^p \mathbb{C}^n}$$

almost every where in K . Consequently, by the classical Fatou theorem and (4.4) we get

$$c_{p,q}^r |c_q(\lambda, p)|^r \int_K \|f(k)\|_{\Lambda^q \mathbb{C}^{n-1}}^r dk \leq \sup_j e^{r\Re(\rho - i\lambda)t_j} \int_K \|p_\lambda^{t_j}(f)(k)\|_{\Lambda^p \mathbb{C}^n}^r dk,$$

which implies

$$c_{p,q} |c_q(\lambda, p)| \|f\|_{L^r(K/M; \sigma_q)} \leq \|\mathcal{P}_{q,\lambda}^p f\|_{\mathcal{E}_{q,\lambda}^r}.$$

\square

In the rest of this section we will see how the asymptotic behavior formula given in Theorem 4.3 will allows us to give explicitly the Harish-Chandra c -function

$$c(\lambda, p) = \int_{\bar{N}} e^{-(i\lambda + \rho)H(\bar{n})} \tau_p(\kappa(\bar{n})) d\bar{n}.$$

To this aim, recall the Jacobi functions, see, e.g. [17],

$$\phi_\lambda^{(\alpha, \beta)}(t) = {}_2F_1 \left(\frac{i\lambda + \alpha + \beta + 1}{2}, \frac{-i\lambda + \alpha + \beta + 1}{2}; \alpha + 1; -\sinh^2 t \right), \quad (4.7)$$

with $\Re(\alpha + 1) > 0$ and ${}_2F_1$ is the classical hypergeometric function. We shall need the following asymptotic behavior of Jacobi functions,

$$\phi_\lambda^{(\alpha, \beta)}(t) = e^{(i\lambda - \alpha - \beta - 1)t} (c_{\alpha, \beta}(\lambda) + o(1)) \text{ as } t \rightarrow \infty \quad (4.8)$$

for $\Re(i\lambda) > 0$, where

$$c_{\alpha, \beta}(\lambda) = \frac{2^{\alpha + \beta + 1 - i\lambda} \Gamma(\alpha + 1) \Gamma(i\lambda)}{\Gamma\left(\frac{i\lambda + \alpha + \beta + 1}{2}\right) \Gamma\left(\frac{i\lambda + \alpha - \beta + 1}{2}\right)}. \quad (4.9)$$

A continuous function $F: G \rightarrow \text{End}(V_{\tau_p})$ is called elementary τ_p -spherical if F satisfies

$$(i) \text{ (}\tau_p\text{-radial function) } F(k_1 g k_2) = \tau_p(k_2)^{-1} F(g) \tau_p(k_1^{-1}), \quad \forall g \in G, \forall k_1, k_2 \in K,$$

(ii) F is a joint-eigenfunction of all $D \in \mathbf{D}(G/K; \tau_p)$ with $F(e) = \text{Id}$.

A τ_p -radial function $F: G \rightarrow \text{End}(V_{\tau_p})$ (i.e. satisfying (i)) is determined by its restriction $F|_A$ to the subgroup A of G . Since A and M commute, $F|_A$ becomes an M -morphism of $V_{\tau_p} = \Lambda^p \mathbb{C}^n$. Now, in the generic case, $\tau_p|_M$ is multiplicity free, therefore by Schur's lemma, $F|_A$ is scalar on each M -irreducible component $V_{\sigma_p} = \Lambda^p \mathbb{C}^{n-1}$ and $V_{\sigma_{p-1}} = \Lambda^{p-1} \mathbb{C}^{n-1}$. Thus

$$F|_A(a_t) = f_{p-1}(t) \text{Id}_{\Lambda^{p-1} \mathbb{C}^{n-1}} + f_p(t) \text{Id}_{\Lambda^p \mathbb{C}^{n-1}},$$

the coefficients f_{p-1} and f_p are called the scalar components of F .

For $\lambda \in \mathbb{C}$, we define the Eisenstein integral $\Phi_q^p(\lambda, g) \in \text{End}(V_{\tau_p})$ by

$$\Phi_q^p(\lambda, g) = c_{p,q}^2 \int_K e^{-(i\lambda + \rho)H(g^{-1}k)} \tau_p(\kappa(g^{-1}k)) t_q^p(\pi_p^q(\tau_p(k)^{-1})) dk. \quad (4.10)$$

Proposition 4.5 (see [25, Theorem 5.4]). *Assume that $0 \leq p < \frac{n-1}{2}$.*

(1) *The set $\{\Phi_q^p(\lambda, \cdot), q = p-1, p; \lambda \in \mathbb{C} \setminus \{\pm 1\}\}$ exhausts the class of τ_p -elementary spherical functions.*

(2) *The scalar components $\varphi_{q,p-1}(\lambda, t)$, $\varphi_{q,p}(\lambda, t)$ of $\Phi_q^p(\lambda, a_t)$ are given by*

$$\Phi_p^p(\lambda, a_t) : \begin{cases} \varphi_{p,p-1}(\lambda, t) = \phi_\lambda^{(\frac{n}{2}, -\frac{1}{2})}(t), \\ \varphi_{p,p}(\lambda, t) = \frac{n}{n-p} \phi_\lambda^{(\frac{n}{2}-1, -\frac{1}{2})}(t) - \frac{p}{n-p} (\cosh t) \phi_\lambda^{(\frac{n}{2}, -\frac{1}{2})}(t), \end{cases} \quad (4.11)$$

and

$$\Phi_{p-1}^p(\lambda, a_t) : \begin{cases} \varphi_{p-1,p-1}(\lambda, t) = \frac{n}{p} \phi_\lambda^{(\frac{n}{2}-1, -\frac{1}{2})}(t) - \frac{n-p}{p} (\cosh t) \phi_\lambda^{(\frac{n}{2}, -\frac{1}{2})}(t), \\ \varphi_{p-1,p}(\lambda, t) = \phi_\lambda^{(\frac{n}{2}, -\frac{1}{2})}(t). \end{cases} \quad (4.12)$$

For $q \in \{p-1, p\}$, let us introduce the notation $\rho_q = \rho - q = \frac{n-1}{2} - q$.

Proposition 4.6. *Let $\lambda \in \mathbb{C}$ such that $\Re(i\lambda) > 0$. The generalized Harish-Chandra c -function is given by*

$$\mathbf{c}(\lambda, p) = c_{p-1}(\lambda, p) \text{Id}_{\Lambda^{p-1} \mathbb{C}^{n-1}} + c_p(\lambda, p) \text{Id}_{\Lambda^p \mathbb{C}^{n-1}},$$

where the scalar coefficients are explicitly given by

$$c_{p-1}(\lambda, p) = \frac{i\lambda - \rho_{p-1}}{i\lambda + \rho} c(\lambda),$$

and

$$c_p(\lambda, p) = \frac{i\lambda + \rho_p}{i\lambda + \rho} c(\lambda),$$

with

$$c(\lambda) = 2^{\rho - i\lambda} \frac{\Gamma(i\lambda) \Gamma(\rho + \frac{1}{2})}{\Gamma(\frac{i\lambda + \rho}{2}) \Gamma(\frac{i\lambda + \rho + 1}{2})}.$$

Proof. Let $\lambda \in \mathbb{C}$ such that $\Re(i\lambda) > 0$. Since

$$\Phi_q^p(\lambda, ka_t) = \mathcal{P}_{q,\lambda}^p \left(c_{p,q} \pi_p^q(\tau(k^{-1})) \right) (a_t),$$

Theorem 4.3 implies

$$\Phi_q^p(\lambda, a_t) = c_{p,q}^2 \mathbf{c}(\lambda, p) e^{(i\lambda - \rho)t} (\pi_p^q + o(1)) \quad \text{as } t \rightarrow \infty, \quad (4.13)$$

with

$$c_{p,q}^2 = \begin{cases} \frac{n}{n-p} & \text{if } q = p, \\ \frac{n}{p} & \text{if } q = p - 1. \end{cases}$$

Let us first consider the case $q = p$. Using the asymptotic behavior of Jacobi functions (4.8) together with the relation

$$c_{\frac{n}{2}, -\frac{1}{2}}(\lambda) = \frac{2n}{i\lambda + \rho} c_{\frac{n}{2}-1, -\frac{1}{2}}(\lambda),$$

we obtain

$$\begin{aligned} \varphi_{p,p}(\lambda, t) & \underset{t \rightarrow \infty}{=} \frac{1}{n-p} e^{(i\lambda - \rho)t} \left(n c_{\frac{n}{2}-1, -\frac{1}{2}}(\lambda) - \frac{p}{2} c_{\frac{n}{2}, -\frac{1}{2}}(\lambda) + o(1) \right), \\ & \underset{t \rightarrow \infty}{=} e^{(i\lambda - \rho)t} \frac{n}{n-p} c_{\frac{n}{2}-1, -\frac{1}{2}}(\lambda) \left(\frac{i\lambda + \rho - p}{i\lambda + \rho} + o(1) \right). \end{aligned}$$

Similarly, we get

$$\varphi_{p,p-1}(\lambda, t) \underset{t \rightarrow \infty}{=} e^{(i\lambda - \rho - 1)t} \left(c_{\frac{n}{2}, -\frac{1}{2}}(\lambda) + o(1) \right).$$

Thus

$$\begin{aligned} \Phi_p^p(\lambda, a_t) & = e^{(i\lambda - \rho - 1)t} \left(c_{\frac{n}{2}, -\frac{1}{2}}(\lambda) + o(1) \right) \text{Id}_{\Lambda^{p-1}\mathbb{C}^{n-1}} \\ & + e^{(i\lambda - \rho)t} \frac{n}{n-p} c_{\frac{n}{2}-1, -\frac{1}{2}}(\lambda) \left(\frac{i\lambda + \rho - p}{i\lambda + \rho} + o(1) \right) \text{Id}_{\Lambda^p\mathbb{C}^{n-1}}, \end{aligned}$$

from which we deduce that

$$\lim_{t \rightarrow \infty} e^{(\rho - i\lambda)t} \Phi_p^p(\lambda, a_t) = \frac{n}{n-p} \left(\frac{i\lambda + \rho - p}{i\lambda + \rho} \right) c_{\frac{n}{2}-1, -\frac{1}{2}}(\lambda) \text{Id}_{\Lambda^p\mathbb{C}^{n-1}}. \quad (4.14)$$

Finally, by identification of (4.13) and (4.14) it follows that

$$c_p(\lambda, p) = \frac{i\lambda + \rho - p}{i\lambda + \rho} c_{\frac{n}{2}-1, -\frac{1}{2}}(\lambda) = \frac{i\lambda + \rho - p}{i\lambda + \rho} c(\lambda).$$

Similarly, for $q = p - 1$ we can prove that

$$\lim_{t \rightarrow \infty} e^{(i\lambda - \rho)t} \Phi_{p-1}^p(\lambda, a_t) = \frac{n}{p} \left(\frac{i\lambda - \rho + p - 1}{i\lambda + \rho} \right) c_{\frac{n}{2}-1, -\frac{1}{2}}(\lambda) \text{Id}_{\Lambda^{p-1}\mathbb{C}^{n-1}},$$

from which we deduce that

$$c_{p-1}(\lambda, p) = \frac{i\lambda - \rho + p - 1}{i\lambda + \rho} c_{\frac{n}{2}-1, -\frac{1}{2}}(\lambda) = \frac{i\lambda - \rho + p - 1}{i\lambda + \rho} c(\lambda).$$

□

5. THE L^2 -RANGE OF THE POISSON TRANSFORM

Recall that our main goal is to characterize the image of the space $L^r(K/M; \sigma_q)$ under the Poisson transform $\mathcal{P}_{q,\lambda}^p$, for $1 < r < \infty$. To do so, we will start with the case $r = 2$.

Fix $\sigma_q \in \widehat{M}$ acting on the space $V_{\sigma_q} = \Lambda^q \mathbb{C}^{n-1}$ of dimension d_{σ_q} . To simplify notations, we will write sometimes (σ, V_σ) instead of (σ_q, V_{σ_q}) .

Let (δ, V_δ) be an element in $\widehat{K}(\sigma)$, where $\widehat{K}(\sigma) \subset \widehat{K}$ denotes the subset of those classes containing σ upon restriction to K . It follows from Frobenius reciprocity theorem together with [13] that σ occurs in $\delta|_M$ with multiplicity one and therefore $\dim \text{Hom}_M(V_\delta, V_\sigma) = 1$. We choose the orthogonal projection $P_\delta : V_\delta \rightarrow V_\sigma$ as a generator of $\text{Hom}_M(V_\delta, V_\sigma)$.

let $(v_j)_{j=1}^{d_\delta}$ be an orthonormal basis for V_δ , where $d_\delta = \dim V_\delta$. Then the functions

$$k \mapsto \phi_j^\delta(k) = P_\delta(\delta(k^{-1})v_j), \quad 1 \leq j \leq d_\delta, \quad \delta \in \widehat{K}(\sigma)$$

define an orthogonal basis of the space $L^2(K/M; \sigma_q)$, see, e.g. [30]. Thus, the Fourier expansion of every $f \in L^2(K/M; \sigma_q)$ is given by

$$f(k) = \sum_{\delta \in \widehat{K}(\sigma)} \sum_{j=1}^{d_\delta} a_j^\delta \phi_j^\delta(k),$$

with

$$\|f\|_{L^2(K/M; \sigma)}^2 = \sum_{\delta \in \widehat{K}(\sigma)} \frac{d_\delta}{d_\sigma} \sum_{j=1}^{d_\delta} |a_j^\delta|^2. \quad (5.1)$$

Next, we will prove a general result giving the Poisson integral representation of a joint eigensections of the algebra $\mathbf{D}(G/K; \tau_p)$ of G -invariant differential operators acting on $C^\infty(G/K; \tau_p)$.

By a functional on $E_{q,\lambda} = G \times_P V_{\sigma_q}$ we shall mean a linear form T on $C^\infty(G/P; \sigma_{q,\bar{\lambda}})$. For a such functional T , we define $\widetilde{\mathcal{P}}_{q,\lambda}^p(T)$ by

$$\langle v, \widetilde{\mathcal{P}}_{q,\lambda}^p T(g) \rangle_{\Lambda^p \mathbb{C}^n} = c_{p,q}(T, \pi_p^q L_g \Phi_\lambda v), \quad \forall v \in \Lambda^p \mathbb{C}^n \quad (5.2)$$

where L_g is the left regular action, and $\Phi_\lambda : G \rightarrow \text{End}(V_{\tau_p})$ is given by

$$\Phi_\lambda(g) = e^{(i\bar{\lambda} - \rho)H(g)} \tau_p^{-1}(\kappa(g)). \quad (5.3)$$

Notice that $\Phi_\lambda(g^{-1}k)^* = P_{q,\lambda}^p(g, k)$, where $P_{q,\lambda}^p : G \times K \rightarrow \text{End}(V_{\tau_p})$ is the Poisson kernel given by

$$P_{q,\lambda}^p(g, k) = e^{-(i\lambda + \rho)H(g^{-1}k)} \tau_p(\kappa(g^{-1}k)). \quad (5.4)$$

If $T = T_f$ is a functional given by $f \in C^\infty(G/P; \sigma_{q,\lambda})$, then

$$\widetilde{\mathcal{P}}_{q,\lambda}^p(T_f) = \mathcal{P}_{q,\lambda}^p(f). \quad (5.5)$$

Indeed,

$$\begin{aligned}
 \langle v, \widetilde{\mathcal{P}}_{q,\lambda}^p T_f(g) \rangle_{\Lambda^p \mathbb{C}^n} &= c_{p,q} (T, \pi_p^q L_g \Phi_\lambda v), \\
 &= c_{p,q} \int_K \langle f(k), \pi_p^q L_g \Phi_\lambda(k) v \rangle_{\Lambda^q \mathbb{C}^{n-1}} dk, \\
 &= c_{p,q} \int_K \langle f(k), \pi_p^q \Phi_\lambda(g^{-1}k) v \rangle_{\Lambda^q \mathbb{C}^{n-1}} dk, \\
 &= c_{p,q} \int_K \langle \Phi_\lambda^*(g^{-1}k) \iota_q^p f(k), v \rangle_{\Lambda^p \mathbb{C}^n} dk, \\
 &= c_{p,q} \int_K \langle P_{q,\lambda}^p(g, k) \iota_q^p f(k), v \rangle_{\Lambda^p \mathbb{C}^n} dk, \\
 &= \langle v, \mathcal{P}_{q,\lambda}^p f(g) \rangle_{\Lambda^p \mathbb{C}^n}.
 \end{aligned}$$

Proposition 5.1. *For every eigensection F of $\mathbf{D}(G/K; \tau_p)$, there exists a functional T on $C^\infty(G/P; \sigma_{q,\bar{\lambda}})$ such that $F = \widetilde{\mathcal{P}}_{q,\lambda}^p T$.*

Proof. Let F be an arbitrary joint eigensection of all $D \in \mathbf{D}(G/K; \tau_p)$. Then F has an expansion

$$F(g) = \sum_{\delta \in \widehat{K}(\sigma)} F_\delta(g)$$

in $C^\infty(G/K; \tau_p)$. Since F_δ is K -finite of type δ , then, by [32, Corollary 10.8], there exists a K -finite vector f_δ in $C^\infty(G/P; \sigma_{q,\lambda})$ such that $F_\delta = \mathcal{P}_{q,\lambda}^p f_\delta$. We have

$$f_\delta(k) = \sum_{j=1}^{d_\delta} a_j^\delta P_\delta(\delta(k^{-1})v_j).$$

Define a functional T by

$$(T, \varphi) = \sum_{\delta \in \widehat{K}(\sigma)} \sum_{j=1}^{d_\delta} \overline{a_j^\delta} \int_K \langle \varphi(k), P_\delta(\delta(k^{-1})v_j) \rangle_{\Lambda^q \mathbb{C}^{n-1}} dk, \quad (5.6)$$

for all $\varphi \in C^\infty(G/P; \sigma_{q,\bar{\lambda}})$ for which the above sum converges. Choose φ in (5.6) to be $\varphi: k \mapsto c_{p,q} \pi_p^q(\Phi_\lambda(g^{-1}k)w)$ with $w \in V_{\tau_p}$, then we get

$$\begin{aligned}
(T, \varphi) &= c_{p,q} \sum_{\delta \in \widehat{K}(\sigma)} \sum_{j=1}^{d_\delta} \overline{a_j^\delta} \int_K \langle \pi_p^q \Phi_\lambda(g^{-1}k)w, P_\delta(\delta(k^{-1})v_j) \rangle_{\Lambda^q \mathbb{C}^{n-1}} dk, \\
&= c_{p,q} \sum_{\delta} \sum_j \overline{a_j^\delta} \int_K \langle w, \Phi_\lambda(g^{-1}k)^* (\pi_p^q)^* P_\delta(\delta(k^{-1})v_j) \rangle_{\Lambda^p \mathbb{C}^n} dk, \\
&= c_{p,q} \sum_{\delta} \sum_j \overline{a_j^\delta} \int_K \langle w, e^{-(i\lambda+\rho)H(g^{-1}k)} \tau_p(\kappa(g^{-1}k)) \iota_q^p P_\delta(\delta(k^{-1})v_j) \rangle_{\Lambda^p \mathbb{C}^n} dk, \\
&= \langle w, \sum_{\delta} c_{p,q} \int_K e^{-(i\lambda+\rho)H(g^{-1}k)} \tau_p(\kappa(g^{-1}k)) \iota_q^p \sum_j a_j^\delta P_\delta(\delta(k^{-1})v_j) dk \rangle_{\Lambda^p \mathbb{C}^n}, \\
&= \langle w, \sum_{\delta \in \widehat{K}(\sigma)} \mathcal{P}_{\lambda,p}^q f_\delta(g) \rangle_{\Lambda^p \mathbb{C}^n}, \\
&= \langle w, F(g) \rangle_{\Lambda^p \mathbb{C}^n}.
\end{aligned}$$

On the other hand, by the definition (5.2) of the Poisson transform on functionals, we have

$$(T, c_{p,q} \pi_p^q(L_g \Phi_\lambda w)) = \langle w, \widetilde{\mathcal{P}_{q,\lambda}^p T(g)} \rangle_{\Lambda^p \mathbb{C}^n},$$

from which we deduce that $F(g) = \widetilde{\mathcal{P}_{q,\lambda}^p T(g)}$, since the vector w is arbitrary. \square

Theorem 5.2. *Assume that $\lambda \in \mathbb{C}$ such that*

$$\begin{cases} \Re(i\lambda) > 0 & \text{if } q = p, \\ \Re(i\lambda) > 0 \text{ and } i\lambda \neq \rho - p + 1 & \text{if } q = p - 1. \end{cases} \quad (5.7)$$

The Poisson transform $\mathcal{P}_{q,\lambda}^p$ is a topological isomorphism from the space $L^2(K/M; \sigma_q)$ onto the space $\mathcal{E}_{q,\lambda}^2(G/K; \tau_p)$. Moreover, there exists a positive constant γ_λ such that

$$c_{p,q} |c_q(\lambda, p)| \|f\|_{L^2(K/M; \sigma_q)} \leq \|\mathcal{P}_{q,\lambda}^p f\|_{\mathcal{E}_{q,\lambda}^2} \leq c_{p,q} \gamma_\lambda \|f\|_{L^2(K/M; \sigma_q)},$$

for every $f \in L^2(K/M; \sigma_q)$.

Proof. On one hand, by Proposition 3.2 and Proposition 4.4 it follows that $\mathcal{P}_{q,\lambda}^p$ is a continuous map from $L^2(K/M; \sigma_q)$ into $\mathcal{E}_{q,\lambda}^2(G/K; \tau_p)$.

On the other hand, for $F \in \mathcal{E}_{q,\lambda}^2(G/K; \tau_p)$, by Proposition 5.1, there exists a functional T on $C^\infty(G/P; \sigma_{q,\bar{\lambda}})$ defined by (5.6) such that $F = \widetilde{\mathcal{P}_{q,\lambda}^p T}$. From the proof of Proposition 5.1, it follows that

$$F(g) = c_{p,q} \sum_{\delta \in \widehat{K}(\sigma)} \sum_{j=1}^{d_\delta} a_j^\delta \int_K e^{-(i\lambda+\rho)H(g^{-1}k)} \tau_p(\kappa(g^{-1}k)) \iota_q^p P_\delta(\delta(k^{-1})v_j) dk.$$

Define $\Phi_{\lambda,\delta}$ by

$$\Phi_{\lambda,\delta}(g)(v) = c_{p,q} \int_K e^{-(i\lambda+\rho)H(g^{-1}k)} \tau_p(\kappa(g^{-1}k)) \iota_q^p P_\delta(\delta(k^{-1})v) dk, \quad (5.8)$$

for $g \in G$ and $v \in V_\delta$. Clearly $\Phi_{\lambda,\delta}(k_1 g k_2) = \tau_p(k_2^{-1})\Phi_{\lambda,\delta}(g)\delta(k_1^{-1})$ for every $g \in G$ and $k_1, k_2 \in K$. Further

$$\int_K \langle F(ka_t), F(ka_t) \rangle_{\Lambda^p \mathbb{C}^n} dk = \sum_{\delta, \delta'} \sum_{j, \ell} a_j^\delta \overline{a_\ell^{\delta'}} \int_K \langle \Phi_{\lambda,\delta}(ka_t)v_j, \Phi_{\lambda,\delta'}(ka_t)v_\ell \rangle_{\Lambda^p \mathbb{C}^n} dk.$$

By the covariance property and Schur's lemma, we obtain

$$\begin{aligned} \int_K \langle \Phi_{\lambda,\delta}(ka_t)v_j, \Phi_{\lambda,\delta'}(ka)v_\ell \rangle_{\Lambda^p \mathbb{C}^n} dk &= \int_K \langle \Phi_{\lambda,\delta'}(a_t)^* \Phi_{\lambda,\delta}(a_t) \delta(k^{-1})v_j, \delta'(k^{-1})v_\ell \rangle_{V_\delta} dk \\ &= \begin{cases} 0 & \text{if } \delta' \not\sim \delta \\ \frac{1}{d_\delta} \mathbf{tr}(\Phi_{\lambda,\delta}(a_t)^* \Phi_{\lambda,\delta}(a_t)) \langle v_j, v_\ell \rangle_{V_\delta} & \text{otherwise} \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} \int_K \|F(ka_t)\|_{\Lambda^p \mathbb{C}^n}^2 dk &= \sum_{\delta \in \widehat{K}(\sigma)} \frac{1}{d_\delta} \sum_{j=1}^{d_\delta} |a_j^\delta|^2 \mathbf{tr}(\Phi_{\lambda,\delta}(a_t)^* \Phi_{\lambda,\delta}(a_t)), \\ &= \sum_{\delta} \frac{1}{d_\delta} \|\Phi_{\lambda,\delta}(a_t)\|_{\text{HS}}^2 \sum_j |a_j^\delta|^2, \end{aligned}$$

where $\|\cdot\|_{\text{HS}}$ is the Hilbert-Schmidt norm. Hence, for a finite subset $\Lambda \subset \widehat{K}(\sigma)$ we get

$$\begin{aligned} \sum_{\delta \in \Lambda} \frac{1}{d_\delta} \sum_j \|a_j^\delta e^{(\rho-i\lambda)t} \Phi_{\lambda,\delta}(a_t)\|_{\text{HS}}^2 &\leq \sup_{t>0} e^{2(\rho-\Re(i\lambda))t} \int_K \|F(ka_t)\|_{\Lambda^p \mathbb{C}^n}^2 dk, \\ &= \|F\|_{\mathcal{E}_{2,\lambda}^2}^2. \end{aligned}$$

Under the assumption (5.7) we may use Theorem 4.3, *i.e.*,

$$\lim_{t \rightarrow \infty} e^{(\rho-i\lambda)t} \Phi_{\lambda,\delta}(a_t) = c_{p,q} \mathbf{c}(\lambda, p) t_q^p P_\delta, \quad (5.9)$$

and (4.5) to obtain

$$c_{p,q}^2 |c_q(\lambda, p)|^2 \sum_{\delta \in \Lambda} \frac{1}{d_\delta} \sum_j \|a_j^\delta P_\delta\|_{\text{HS}}^2 \leq \|F\|_{\mathcal{E}_{2,\lambda}^2}^2.$$

That is

$$c_{p,q}^2 |c_q(\lambda, p)|^2 \sum_{\delta \in \Lambda} \frac{1}{d_\delta} \sum_j d_\sigma |a_j^\delta|^2 \leq \|F\|_{\mathcal{E}_{2,\lambda}^2}^2.$$

Since the subset $\Lambda \subset \widehat{K}(\sigma)$ is arbitrary, it follows that

$$c_{p,q}^2 |c_q(\lambda, p)|^2 \sum_{\delta \in \widehat{K}(\sigma)} \frac{d_\sigma}{d_\delta} \sum_j |a_j^\delta|^2 \leq \|F\|_{\mathcal{E}_{2,\lambda}^2}^2 < \infty.$$

This shows that the functional $T(k) \sim \sum_{\delta \in \widehat{K}(\sigma)} \sum_{j=1}^{d_\delta} a_j^\delta P_\delta \delta(k^{-1})v_j$ defines a function $f \in L^2(K/M; \sigma)$ and by (5.5), we deduce that $F = \mathcal{P}_{q,\lambda}^p f$ with

$$c_{p,q} |c_q(\lambda, p)| \|f\|_{L^2(K/M; \sigma)} \leq \|\mathcal{P}_{q,\lambda}^p f\|_{\mathcal{E}_{2,\lambda}^2}.$$

□

Lemma 5.3. *We have*

$$\sup_{t>0} e^{(\rho-\Re(i\lambda))t} \|\Phi_{\lambda,\delta}(a_t)\|_{\text{HS}} \leq \gamma_\lambda c_{p,q} \|P_\delta\|_{\text{HS}} = \gamma_\lambda c_{p,q} \sqrt{d_\sigma}.$$

Proof. By Proposition 4.1 we have

$$\sup_{t>0} e^{(\rho-\Re(i\lambda))t} \left(\int_K \|\mathcal{P}_{q,\lambda}^p(P_\delta(\delta^{-1}(\cdot)v))(ka_t)\|_{\Lambda^p\mathbb{C}^n}^2 dk \right)^{1/2} \leq \gamma_\lambda c_{p,q} \|P_\delta(\delta^{-1}(\cdot)v)\|_{L^2(K/M;\sigma_q)}.$$

Since $\mathcal{P}_{q,\lambda}^p(P_\delta(\delta^{-1}(\cdot)v))(ka_t) = \Phi_{\lambda,\delta}(ka_t)(v)$, we get

$$\begin{aligned} \int_K \|\mathcal{P}_{q,\lambda}^p(P_\delta(\delta^{-1}(\cdot)v))(ka_t)\|_{\Lambda^p\mathbb{C}^n}^2 dk &= \int_K \langle \Phi_{\lambda,\delta}(a_t)\delta(k^{-1})v, \Phi_{\lambda,\delta}(a_t)\delta(k^{-1})v \rangle_{\Lambda^p\mathbb{C}^n} dk, \\ &= \frac{1}{d_\delta} \text{tr}(\Phi_{\lambda,\delta}(a_t)^* \Phi_{\lambda,\delta}(a_t)) \|v\|_{V_\delta}^2, \\ &= \frac{1}{d_\delta} \|\Phi_{\lambda,\delta}(a_t)\|_{\text{HS}}^2 \|v\|_{V_\delta}^2. \end{aligned}$$

Now the desired inequality follows from

$$\|P_\delta(\delta^{-1}(\cdot)v)\|_{L^2(K/M;\sigma_q)}^2 = \frac{d_\sigma}{d_\delta} \|v\|_{V_\delta}^2.$$

□

Lemma 5.4. *We have*

$$\lim_{t \rightarrow \infty} e^{2(\rho-\Re(i\lambda))t} \|\Phi_{\lambda,\delta}(a_t)\|_{\text{HS}}^2 = c_{p,q}^2 |c_q(\lambda, p)|^2 d_{\sigma_q},$$

where $c_q(\lambda, p)$ is the scalar component of $\mathbf{c}(\lambda, p)$ on $V_{\sigma_q} = \Lambda^q\mathbb{C}^{n-1}$.

Proof. Recall that $\Phi_{\lambda,\delta}(a_t) = \mathcal{P}_{q,\lambda}^p(P_\delta(\delta^{-1}(\cdot)))(a_t)$. Then

$$\begin{aligned} e^{2(\rho-\Re(i\lambda))t} \|\Phi_{\lambda,\delta}(a_t)\|_{\text{HS}}^2 &= \sum_{j=1}^{d_\delta} \|e^{(\rho-\Re(i\lambda))t} \Phi_{\lambda,\delta}(a_t)v_j\|_{\Lambda^p\mathbb{C}^n}^2, \\ &= \sum_{j=1}^{d_\delta} \|e^{(\rho-\Re(i\lambda))t} \mathcal{P}_{q,\lambda}^p(P_\delta(\delta^{-1}(\cdot)v_j))(a_t)\|_{\Lambda^p\mathbb{C}^n}^2. \end{aligned}$$

Using Theorem 4.3 and (4.5), we obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{2(\rho-\Re(i\lambda))t} \|\Phi_{\lambda,\delta}(a_t)\|_{\text{HS}}^2 &= c_{p,q}^2 \sum_{j=1}^{d_\delta} \langle \mathbf{c}(\lambda, p) i_q^p P_\delta v_j, \mathbf{c}(\lambda, p) i_q^p P_\delta v_j \rangle_{\Lambda^p\mathbb{C}^n}. \\ &= c_{p,q}^2 \|\mathbf{c}(\lambda, p) i_q^p P_\delta\|_{\text{HS}}^2 \\ &= c_{p,q}^2 |c_q(\lambda, p)|^2 d_{\sigma_q}. \end{aligned}$$

□

Theorem 5.5 (Inversion formula). *Assume $\lambda \in \mathbb{C}$ such that*

$$\begin{cases} \Re(i\lambda) > 0 & \text{if } q = p, \\ \Re(i\lambda) > 0 \text{ and } i\lambda \neq \rho - p + 1 & \text{if } q = p - 1. \end{cases}$$

Let $F \in \mathcal{E}_{q,\lambda}^2(G/K; \tau_p)$ and let $f \in L^2(K/M; \sigma_q)$ be its boundary value. Then the following inversion formula holds in $L^2(K/M; \sigma_q)$

$$f(k) = c_{p,q}^{-1} |c_q(\lambda, p)|^{-2} \lim_{t \rightarrow \infty} e^{2(\rho - \Re(i\lambda))t} \pi_p^q \left(\int_K P_{q,\lambda}^p(ha_t, k) * F(ha_t) dh \right),$$

where $P_{q,\lambda}^p(\cdot, \cdot)$ is the Poisson kernel given in (5.4).

Proof. Let $F \in \mathcal{E}_{q,\lambda}^2(G/K; \tau_p)$. By Theorem 5.2, there exists a unique $f \in L^2(K/M; \sigma_q)$ such that $F = \mathcal{P}_{q,\lambda}^p f$. Write

$$f(k) = \sum_{\delta \in \hat{K}(\sigma_q)} \sum_{j=1}^{d_\delta} a_j^\delta P_\delta(\delta(k^{-1})) v_j.$$

Then

$$F(ka_t) = \sum_{\delta} \sum_j a_j^\delta \Phi_{\lambda,\delta}(a_t) \delta(k^{-1}) v_j,$$

and therefore

$$\int_K \|F(ka_t)\|_{\Lambda^p \mathbb{C}^n}^2 dk = \sum_{\delta} \sum_j \frac{|a_j^\delta|^2}{d_\delta} \|\Phi_{\lambda,\delta}(a_t)\|_{\text{HS}}^2.$$

From Lemma 5.4 we deduce

$$\lim_{t \rightarrow \infty} e^{2(\rho - \Re(i\lambda))t} \int_K \|\mathcal{P}_{q,\lambda}^p f(ka_t)\|_{\Lambda^p \mathbb{C}^n}^2 dk = c_{p,q}^2 |c_q(\lambda, p)|^2 \|f\|_{L^2(K/M; \sigma_q)}^2,$$

which implies

$$\lim_{t \rightarrow \infty} (g_t, \varphi)_{L^2(K/M; \sigma_q)} = (f, \varphi)_{L^2(K/M; \sigma_q)}, \quad \forall \varphi \in L^2(K/M; \sigma_q),$$

where g_t is the V_{σ_q} -valued function defined by

$$g_t(k) = c_{p,q}^{-1} |c_q(\lambda, p)|^{-2} e^{2(\rho - \Re(i\lambda))t} \pi_p^q \int_K P_{q,\lambda}^p(ha_t, k) * F(ha_t) dh.$$

To obtain the inversion formula, it is only required to show that

$$\lim_{t \rightarrow \infty} \|g_t\|_{L^2(K/M; \sigma_q)} = \|f\|_{L^2(K/M; \sigma_q)}.$$

To do so, let us first compute the Fourier coefficients $c_j^\delta(g_t)$ of g_t :

$$\begin{aligned} c_j^\delta(g_t) &= \frac{d_\delta}{d_\sigma} \int_K \langle g_t(k), P_\delta \delta(k^{-1}) v_j \rangle_{\Lambda^q \mathbb{C}^{n-1}} dk \\ &= c_{p,q}^{-1} |c_q(\lambda, p)|^{-2} e^{2(\rho - \Re(i\lambda))t} \\ &\times \frac{d_\delta}{d_\sigma} \sum_{\delta', \ell} a_{\ell'}^\delta \int_K \langle \pi_p^q \int_K P_{q,\lambda}^p(ha_t, k) * \Phi_{\lambda,\delta'}(a_t) \delta'(h^{-1}) v_\ell dh, P_\delta \delta(k^{-1}) v_j \rangle_{\Lambda^q \mathbb{C}^{n-1}} dk. \end{aligned}$$

Since $(\pi_p^q)^* = \iota_q^p$, we get

$$\begin{aligned} c_j^\delta(g_t) &= c_{p,q}^{-1} |c_q(\lambda, p)|^{-2} e^{2(\rho - \Re(i\lambda))t} \\ &\quad \times \frac{d_\delta}{d_\sigma} \sum_{\delta', \ell} a_\ell^{\delta'} \int_K \int_K \langle \Phi_{\lambda, \delta'}(a_t) \delta'(h^{-1}) v_\ell, P_{q, \lambda}^p(h a_t, k) \iota_q^p P_\delta \delta(k^{-1}) v_j \rangle_{\Lambda^p \mathbb{C}^n} dh dk, \end{aligned}$$

As $\int_K P_{q, \lambda}^p(h a_t, k) \iota_q^p P_\delta \delta(k^{-1}) dk = c_{p,q}^{-1} \Phi_{\lambda, \delta}(h a_t)$, we obtain

$$\begin{aligned} c_j^\delta(g_t) &= c_{p,q}^{-2} |c_q(\lambda, p)|^{-2} e^{2(\rho - \Re(i\lambda))t} \\ &\quad \times \frac{d_\delta}{d_\sigma} \sum_{\delta', \ell} a_\ell^{\delta'} \int_K \langle \Phi_{\lambda, \delta'}(a_t) \delta'(h^{-1}) v_\ell, \Phi_{\lambda, \delta}(a_t) \delta(h^{-1}) v_j \rangle_{\Lambda^p \mathbb{C}^n} dh, \\ &= c_{p,q}^{-2} |c_q(\lambda, p)|^{-2} e^{2(\rho - \Re(i\lambda))t} \\ &\quad \times \frac{d_\delta}{d_\sigma} \sum_{\delta', \ell} a_\ell^{\delta'} \int_K \langle \delta(h) \Phi_{\lambda, \delta}(a_t)^* \Phi_{\lambda, \delta'}(a_t) \delta'(h^{-1}) v_\ell, v_j \rangle_{\Lambda^p \mathbb{C}^n} dh. \end{aligned}$$

By the Schur lemma, we get

$$\begin{aligned} c_j^\delta(g_t) &= c_{p,q}^{-2} |c_q(\lambda, p)|^{-2} e^{2(\rho - \Re(i\lambda))t} \\ &\quad \times \frac{d_\delta}{d_\sigma} \sum_\ell a_\ell^\delta \int_K \frac{1}{d_\delta} \mathbf{tr}(\Phi_{\lambda, \delta}(a_t)^* \Phi_{\lambda, \delta}(a_t)) \langle v_\ell, v_j \rangle_{V_\delta} dh, \\ &= c_{p,q}^{-2} |c_q(\lambda, p)|^{-2} e^{2(\rho - \Re(i\lambda))t} \frac{1}{d_\sigma} a_j^\delta \|\Phi_{\lambda, \delta}(a_t)\|_{\text{HS}}^2. \end{aligned}$$

From all the above computations, we conclude that,

$$\|g_t\|_{L^2(K/M; \sigma)}^2 = \left(e^{2(\rho - \Re(i\lambda))t} |c_{p,q} c_q(\lambda, p)|^{-2} \right)^2 \sum_\delta \frac{d_\sigma}{d_\delta} \sum_j \frac{1}{d_\sigma^2} |a_j^\delta|^2 \|\Phi_{\lambda, \delta}(a_t)\|_{\text{HS}}^4,$$

and by Lemma 5.4 we get

$$\lim_{t \rightarrow \infty} \|g_t\|_{L^2(K/M; \sigma)}^2 = \sum_\delta \frac{d_\sigma}{d_\delta} \sum_j |a_j^\delta|^2 = \|f\|_{L^2(K/M; \sigma)}^2.$$

To finish the proof, we have to justify that we can reverse $\lim_{t \rightarrow \infty}$ and \sum_δ by proving that the serie

$$\sum_{\delta \in \widehat{K}(\sigma)} \frac{1}{d_\delta} \sum_{j=1}^{d_\delta} |a_j^\delta|^2 \|\Phi_{\lambda, \delta}(a_t)\|_{\text{HS}}^4,$$

is uniformly convergent. This follows easily from Lemma 5.3. \square

6. THE L^r -RANGE OF THE POISSON TRANSFORM

In this section we shall generalize Theorem 5.2 to $L^r(K/M; \sigma_q)$ with $1 < r < \infty$.

Theorem 6.1. *Let $0 \leq p < (n-1)/2$, and $\lambda \in \mathbb{C}$ such that*

$$\begin{cases} \Re(i\lambda) > 0 & \text{if } q = p, \\ \Re(i\lambda) > 0 \text{ and } i\lambda \neq \rho - p + 1 & \text{if } q = p - 1. \end{cases}$$

For $1 < r < \infty$, the Poisson transform $\mathcal{P}_{q,\lambda}^p$ is a topological isomorphism from the space $L^r(K/M; \sigma_q)$ onto the space $\mathcal{E}_{q,\lambda}^r(G/K; \tau_p)$. Moreover, there exists a positive constant γ_λ such that

$$c_{p,q}|c_q(\lambda, p)| \|f\|_{L^r(K/M; \sigma_q)} \leq \|\mathcal{P}_{q,\lambda}^p f\|_{\mathcal{E}_{q,\lambda}^r} \leq c_{p,q} \gamma_\lambda \|f\|_{L^r(K/M; \sigma_q)},$$

for every $f \in L^r(K/M; \sigma_q)$.

Proof. The necessary condition follows from Proposition 3.2 and Proposition 4.4. For the sufficiency condition, let $F \in \mathcal{E}_{q,\lambda}^r(G/K; \tau_p)$ and write $F(g) = \sum_i F_i(g)u_i$ where $(u_i)_i$ is an orthonormal basis of $\Lambda^p \mathbb{C}^n$. Fix $(\chi_m)_m$ to be an approximation of the identity in $C^\infty(K)$ and let $F_{i,m}(g) = \int_K \chi_m(k) F_i(k^{-1}g) dk$. Then $(F_{i,m})_m$ converges point-wise to F_i . Define $F_m : G \rightarrow \Lambda^p \mathbb{C}^n$ by $F_m(g) = \sum_i F_{i,m}(g)u_i$. Then

$$\begin{aligned} F_m(g) &= \sum_i \left(\int_K \chi_m(k) F_i(k^{-1}g) dk \right) u_i, \\ &= \int_K \chi_m(k) \sum_i F_i(k^{-1}g) u_i dk, \\ &= \int_K \chi_m(k) F(k^{-1}g) dk. \end{aligned}$$

We have $\|F_m(g) - F(g)\|_{\Lambda^p \mathbb{C}^n}^2 \xrightarrow{m \rightarrow \infty} 0$ and since the operators \mathcal{C} , D and D^* in (3.2) are K -invariant, then $F_m \in \mathcal{E}_{q,\lambda}^r(G/K; \tau_p)$ for every m . Further,

$$\begin{aligned} F_m(ka_t) &= \int_K \chi_m(h) F(h^{-1}ka_t) dh, \\ &= (\chi_m * F^t)(k), \end{aligned}$$

where $F^t : K \rightarrow \Lambda^p \mathbb{C}^n$ is defined for any $t > 0$ by $F^t(k) = F(ka_t)$ for every F . By (3.5) we have

$$\|(\chi_m * F^t)(k)\|_{\Lambda^p \mathbb{C}^n} \leq \int_K |\chi_m(h)| \|F^t(h^{-1}k)\|_{\Lambda^p \mathbb{C}^n} dh,$$

that is

$$\|F_m^t(k)\|_{\Lambda^p \mathbb{C}^n} \leq (|\chi_m(\cdot)| * \|F^t(\cdot)\|_{\Lambda^p \mathbb{C}^n})(k).$$

Therefore

$$\|F_m^t\|_{L^r(K; \Lambda^p \mathbb{C}^n)} \leq \| |\chi_m(\cdot)| * \|F^t(\cdot)\|_{\Lambda^p \mathbb{C}^n} \|_{L^r(K)}.$$

By Young's inequalities we obtain

$$\begin{aligned} \|F_m^t\|_{L^r(K; \Lambda^p \mathbb{C}^n)} &\leq \|\chi_m\|_{L^1(K)} \| \|F^t(\cdot)\|_{\Lambda^p \mathbb{C}^n} \|_{L^r(K)}, \\ &= \|F^t\|_{L^r(K; \Lambda^p \mathbb{C}^n)}, \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} \|F_m^t\|_{L^2(K; \Lambda^p \mathbb{C}^n)} &\leq \|\chi_m\|_{L^2(K)} \| \|F^t(\cdot)\|_{\Lambda^p \mathbb{C}^n} \|_{L^1(K)}, \\ &= \|\chi_m\|_{L^2(K)} \|F^t\|_{L^r(K; \Lambda^p \mathbb{C}^n)}. \end{aligned} \tag{6.2}$$

The inequality (6.2) implies

$$\sup_{t>0} e^{(\rho-\Re(i\lambda))t} \left(\int_K \|F_m(ka_t)\|_{\Lambda^p \mathbb{C}^n}^2 \right)^{1/2} \leq \|\chi_m\|_{L^2(K)} \|F\|_{\mathcal{E}_{q,\lambda}^r} < \infty.$$

Hence, for each m , $F_m \in \mathcal{E}_{q,\lambda}^2(G/K; \tau_p)$ and from Theorem 5.2 it follows that there exists $f_m \in L^2(K/M; \sigma_q)$ such that $F_m = \mathcal{P}_{q,\lambda}^p f_m$. To prove that $f_m \in L^r(K/M; \sigma_q)$ we will follow the same method as in [5]. According to Theorem 5.5 we have, for any $\varphi \in C^\infty(K/M; \sigma_q)$,

$$\int_K \langle f_m(k), \varphi(k) \rangle_{\Lambda^q \mathbb{C}^{n-1}} dk = \lim_{t \rightarrow \infty} \int_K \langle g_m^t(k), \varphi(k) \rangle_{\Lambda^q \mathbb{C}^{n-1}} dk,$$

where

$$g_m^t(k) := c_{p,q}^{-2} |c_q(\lambda, p)|^{-2} e^{2(\rho-\Re(i\lambda))t} \pi_p^q \int_K P_\lambda(ha_t, k)^* F_m(ha_t) dh.$$

Further,

$$\begin{aligned} & \int_K \langle g_m^t(k), \varphi(k) \rangle_{\Lambda^q \mathbb{C}^{n-1}} dk \\ &= c_{p,q}^{-2} |c_q(\lambda, p)|^{-2} e^{2(\rho-\Re(i\lambda))t} \int_K \langle \pi_p^q \int_K P_\lambda(ha_t, k)^* F_m(ha_t) dh, \varphi(k) \rangle_{\Lambda^q \mathbb{C}^{n-1}} dk, \\ &= c_{p,q}^{-2} |c_q(\lambda, p)|^{-2} e^{2(\rho-\Re(i\lambda))t} \int_K \int_K \langle F_m(ha_t), P_\lambda(ha_t, k) i_q^p \varphi(k) \rangle_{\Lambda^p \mathbb{C}^n} dk dh, \\ &= c_{p,q}^{-3} |c_q(\lambda, p)|^{-2} e^{2(\rho-\Re(i\lambda))t} \int_K \langle F_m(ha_t), (\mathcal{P}_{q,\lambda}^p \varphi)(ha_t) \rangle_{\Lambda^p \mathbb{C}^n} dh. \end{aligned}$$

It follows that

$$\begin{aligned} & \left| \int_K \langle g_m^t(k), \varphi(k) \rangle_{\Lambda^q \mathbb{C}^{n-1}} dk \right| \\ & \leq c_{p,q}^{-3} |c_q(\lambda, p)|^{-2} e^{2(\rho-\Re(i\lambda))t} \int_K \|F_m(ha_t)\|_{\Lambda^p \mathbb{C}^n} \|\mathcal{P}_{q,\lambda}^p \varphi(ha_t)\|_{\Lambda^p \mathbb{C}^n} dh. \end{aligned}$$

By Hölder's inequality (with $\frac{1}{r} + \frac{1}{s} = 1$), we deduce

$$\begin{aligned} & \left| \int_K \langle g_m^t(k), \varphi(k) \rangle_{\Lambda^q \mathbb{C}^{n-1}} dk \right| \\ & \leq c_{p,q}^{-3} |c_q(\lambda, p)|^{-2} e^{2(\rho-\Re(i\lambda))t} \|F_m^t\|_{L^r(K; \Lambda^p \mathbb{C}^n)} \|(\mathcal{P}_{q,\lambda}^p \varphi)^t\|_{L^s(K; \Lambda^p \mathbb{C}^n)}, \end{aligned}$$

where $(\mathcal{P}_{q,\lambda}^p \varphi)^t(k) = (\mathcal{P}_{q,\lambda}^p \varphi)(ka_t)$. Using (6.1) and Proposition 4.1 we get

$$\begin{aligned} \left| \int_K \langle f_m(k), \varphi(k) \rangle_{\Lambda^q \mathbb{C}^{n-1}} dk \right| & \leq \gamma \lambda c_{p,q}^{-2} |c_q(\lambda, p)|^{-2} \|F_m\|_{\mathcal{E}_{q,\lambda}^r} \|\varphi\|_{L^s(K/M; \sigma_q)}, \\ & \leq \gamma \lambda c_{p,q}^{-2} |c_q(\lambda, p)|^{-2} \|F\|_{\mathcal{E}_{q,\lambda}^r} \|\varphi\|_{L^s(K/M; \sigma_q)}. \end{aligned}$$

By taking the supremum over all $\varphi \in C^\infty(K/M; \sigma_q)$ with $\|\varphi\|_{L^s(K/M; \sigma_q)} = 1$ we obtain

$$\|f_m\|_{L^r(K/M; \sigma_q)} \leq \gamma \lambda c_{p,q}^{-2} |c_q(\lambda, p)|^{-2} \|F\|_{\mathcal{E}_{q,\lambda}^r},$$

which implies f_m , initially belongs to $L^2(K/M; \sigma_q)$, is in fact in $L^r(K/M; \sigma_q)$.

For every m , define the linear form T_m on $L^s(K/M; \sigma_q)$ by

$$T_m(\varphi) = \int_K \langle f_m(k), \varphi(k) \rangle_{\Lambda^q \mathbb{C}^{n-1}} dk.$$

Clearly, T_m is continuous and

$$|T_m(\varphi)| \leq \gamma_\lambda c_{p,q}^{-2} |c_q(\lambda, p)|^{-2} \|F\|_{\mathcal{E}_{q,\lambda}^r} \|\varphi\|_{L^s(K/M; \sigma_q)}.$$

This shows that $(T_m)_m$ is uniformly bounded in $L^s(K/M; \sigma_q)$, with

$$\sup_m \|T_m\|_{\text{op}} \leq \gamma_\lambda c_{p,q}^{-2} |c_q(\lambda, p)|^{-2} \|F\|_{\mathcal{E}_{q,\lambda}^r}.$$

The Banach-Alaouglu-Bourbaki theorem will then ensures the existence of a subsequence of bounded operators (T_{m_j}) which converges to a bounded operator T under the weak-* topology, with

$$\|T\|_{\text{op}} \leq \gamma_\lambda c_{p,q}^{-2} |c_q(\lambda, p)|^{-2} \|F\|_{\mathcal{E}_{q,\lambda}^r}.$$

Thus, Riesz's representation theorem guarantees the existence of a unique $f \in L^r(K/M; \sigma_q)$ such that

$$T(\varphi) = \int_K \langle \varphi(k), f(k) \rangle_{\Lambda^q \mathbb{C}^{n-1}} dk.$$

By means of the Poisson kernel (5.4), we consider the test function $\varphi_g(k) = \mathcal{P}_{q,\lambda}^p(g, k)v$ with $v \in \Lambda^p \mathbb{C}^n$, then

$$T(\varphi_g) = \langle v, \mathcal{P}_{q,\lambda}^p f(g) \rangle_{\Lambda^p \mathbb{C}^n}.$$

On the other hand

$$T_{m_j}(\varphi_g) = \langle v, \mathcal{P}_{q,\lambda}^p f_{m_j}(g) \rangle_{\Lambda^p \mathbb{C}^n} = \langle v, F_{m_j}(g) \rangle_{\Lambda^p \mathbb{C}^n}.$$

Taking the limit of the above identity when $j \rightarrow \infty$ we conclude that $F(g) = \mathcal{P}_{q,\lambda}^p f(g)$ for every $g \in G$. \square

As an immediate consequence of Theorem 6.1 we obtain the following characterization of co-closed harmonic p -forms on \mathbb{H}^n :

Corollary 6.2. *Let p be an integer with $0 \leq p < (n-1)/2$. For $1 < r < \infty$, the Poisson transform $\mathcal{P}_{p,i(p-\rho)}^p$ is a topological isomorphism from the space $L^r(K/M; \sigma_p)$ onto the space $\mathcal{E}_{p,i(p-\rho)}^r(G/K; \tau_p)$. Moreover, for every $f \in L^r(K/M; \sigma_p)$ the following estimates hold,*

$$\frac{2(\rho-p)}{2\rho-p} c_p(\rho) \|f\|_{L^r(K/M; \sigma_p)} \leq \|\mathcal{P}_{p,i(p-\rho)}^p f\|_{\mathcal{E}_{p,i(p-\rho)}^r} \leq c_p(\rho) \|f\|_{L^r(K/M; \sigma_p)},$$

where

$$c_p(\rho) = c_{p,p} \frac{2^p \Gamma(\rho + \frac{1}{2}) \Gamma(\rho - p)}{\Gamma(\rho - \frac{p}{2}) \Gamma(\rho - \frac{p}{2} + \frac{1}{2})}.$$

In the case where $p = 0$, we recover the classical fact that the Poisson transform is an isometric isomorphism from $L^r(\partial\mathbb{H}^n)$ onto the Hardy-harmonic space on \mathbb{H}^n (see [27]).

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