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**CMB quadrupole suppression. I. Initial conditions of inflationary perturbations**D. Boyanovsky,<sup>1,2,3,\*</sup> H. J. de Vega,<sup>1,2,3,†</sup> and N. G. Sanchez<sup>2,‡</sup><sup>1</sup>*Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260, USA*<sup>2</sup>*Observatoire de Paris, LERMA, Laboratoire Associé au CNRS UMR 8112, 61, Avenue de l'Observatoire, 75014 Paris, France*<sup>3</sup>*LPTHE, Université Pierre et Marie Curie (Paris VI) et Denis Diderot (Paris VII),**Laboratoire Associé au CNRS UMR 7589, Tour 24, 5ème. étage, 4, Place Jussieu, 75252 Paris, Cedex 05, France*

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We investigate the issue of initial conditions of curvature and tensor perturbations at the beginning of slow roll inflation and their effect on the power spectra. Renormalizability and small backreaction constrain the high  $k$  behavior of the Bogoliubov coefficients that define these initial conditions. We introduce a *transfer function*  $D(k)$  which encodes the effect of generic initial conditions on the power spectra. The constraint from renormalizability and small backreaction entails that  $D(k) \lesssim \mu^2/k^2$  for large  $k$ , implying that observable effects from initial conditions are more prominent in the *low* multipoles. This behavior affects the CMB quadrupole by the observed amount  $\sim 10\%$ – $20\%$  when  $\mu$  is of the order of the energy scale of inflation. The effects on high  $l$ -multipoles are suppressed by a factor  $\sim 1/l^2$  due to the falloff of  $D(k)$  for large wave vectors  $k$ . We show that the determination of generic initial conditions for the fluctuations is equivalent to the scattering problem by a potential  $\mathcal{V}(\eta)$  localized just prior to the slow roll stage. Such potential leads to a transfer function  $D(k)$  which automatically obeys the renormalizability and small backreaction constraints. We find that an attractive potential  $\mathcal{V}(\eta)$  yields a *suppression* of the lower CMB multipoles. Both for curvature and tensor modes, the quadrupole suppression depends only on the energy scale of  $\mathcal{V}(\eta)$ , and on the time interval where  $\mathcal{V}(\eta)$  is nonzero. A suppression of the quadrupole for curvature perturbations consistent with the data is obtained when the scale of the potential is of the order of  $k_Q^2$  where  $k_Q$  is the wave vector whose physical wavelength is the Hubble radius today.

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**I. INTRODUCTION**

Inflation is a central part of early Universe cosmology originally introduced to explain several shortcomings of the standard big bang cosmology [1–5] and at the same time it provides a mechanism for generating scalar (density) and tensor (gravitational wave) perturbations [6–10]. A distinct aspect of inflationary perturbations is that metric perturbations are generated by quantum fluctuations of the scalar field(s) that drive inflation. After their wavelength becomes larger than the Hubble radius, these fluctuations are amplified and grow, becoming classical and decoupling from causal microphysical processes. Upon reentering the horizon, during the matter era, these classical perturbations seed the inhomogeneities which generate structure upon gravitational collapse [6–10].

Most inflationary models predict fairly generic features: a Gaussian, nearly scale invariant spectrum of (mostly) adiabatic scalar and tensor primordial fluctuations, making the inflationary paradigm fairly robust. The confirmation of many of the predictions of inflation by current high precision observations places inflationary cosmology on solid grounds.

The Gaussian, adiabatic, and nearly scale invariant spectrum of primordial fluctuations provide an excellent fit to the highly precise wealth of data provided by the Wilkinson Microwave Anisotropy Probe (WMAP) [11–

18]. Perhaps the most striking validation of inflation as a mechanism for generating *superhorizon* ('acausal') fluctuations is the anticorrelation peak in the temperature-polarization (TE) angular power spectrum at  $l \sim 150$  corresponding to superhorizon scales [14,15].

The power spectra for scalar curvature and tensor (gravitational wave) quantum fluctuations generated during the inflationary stage determine the angular power spectrum of cosmic microwave background (CMB) anisotropies. Their initial conditions are usually chosen as Bunch-Davies conditions [19], which fix the asymptotic behavior for large negative conformal time  $\eta$  to be the same as in Minkowski space-time in terms of positive frequencies. The Bunch-Davies states transform as irreps under the maximal symmetry group  $O(4, 1)$  of de Sitter space-time. (Other initial states were also considered [20]). The requirement that the energy momentum tensor be renormalizable constrains the UV asymptotic behavior of the Bogoliubov coefficients that encode different initial conditions [21].

The possibility that more precise observations of the anisotropies in the CMB may probe physical aspects of the initial conditions of quantum fluctuations motivated a substantial effort to study different initial conditions and their potential observational consequences [22]. The remarkable high quality data and the exhaustive analysis of the 3 yr results from WMAP [16] reveal that outlying points and wiggles near the acoustic peaks in earlier data have all but disappeared thus rendering much less statistical significance to potential observables from "transplanckian" effects in the CMB [22] on small angular scales.

\*Electronic address: boyan@pitt.edu

†Electronic address: devega@lpthe.jussieu.fr

‡Electronic address: Norma.Sanchez@obspm.fr

On the other hand, while there are no statistically significant departures from the slow roll inflationary scenario at small angular scales ( $l \gtrsim 100$ ), the third year WMAP data again confirms the surprisingly low quadrupole  $C_2$  [16–18] and suggests that it cannot be completely explained by galactic foreground contamination. The low value of the quadrupole has been an intriguing feature on large angular scales since first observed by COBE/DMR [23], and confirmed by the WMAP data [11–18]. The observation of a low quadrupole [24] and the surprising alignment of quadrupole and octupole [24,25] sparked many different proposals for their explanation [26]. More recently the robustness of these features in the low multipoles to foreground contamination has been studied [27] with the suggestion [28] that these may originate in extended (large scale) foregrounds perhaps generated by Sunayev-Zeldovich (SZ) distortions by hot electrons in the local supercluster.

### The goals and main results

In this article we address the issue of the initial conditions of the fluctuations and the effects they imprint on the primordial spectra of curvature and tensor perturbations within the *effective* field theory of inflation [29]. We show that initial conditions consistent with renormalizability and small backreaction influence mainly the low CMB multipoles. In particular we find a *suppression* of the CMB quadrupole consistent with the current observations. Furthermore, we formulate the problem of the initial conditions in terms of a potential that affects the evolution of scalar and tensor fluctuations *prior to slow roll*. In a companion article [30] we show that this potential is a generic feature of a brief *fast roll* stage prior to slow roll inflation, and study its observational consequences as a suppression in the CMB quadrupole for temperature and tensor modes. We highlight that these results are derived within the context of the effective theory of inflation [29,31]. Namely, we provide a consistent assessment of the initial conditions at the energy scale of inflation which is the grand unification scale ( $\sim 10^{16}$  GeV), without the *need* to advocate transplanckian physics or extra assumptions as an explanation for non-Bunch-Davies (BD) initial conditions. As described in detail below, non-BD initial conditions can be consistently incorporated within the effective field theory valid at the inflation scale.

The goal of this article is to study the effects on the power spectra of curvature and gravitational wave perturbations of initial conditions consistent with the criteria of renormalizability of the gauge invariant energy momentum tensor and negligible backreaction. These general initial conditions are related to the Bunch-Davies initial conditions by a Bogoliubov transformation. The renormalizability criteria constrains the high  $k$  behavior of the Bogoliubov coefficients [21]. We show that these constraints imply that observable effects from initial condi-

tions are more pronounced in the *low multipoles*, namely, in the region of the angular power spectra corresponding to the Sachs-Wolfe plateau. Our main results are summarized as follows:

- (i) We introduce a *transfer function for initial conditions*  $D(k)$  which encodes the effect of general initial conditions on the power spectra. The constraint from renormalizability and small backreaction entail that  $D(k) = \mathcal{O}(\mu^2/k^2)$  for large  $k$ . We show that this behavior naturally yields an observable correction to the quadrupole. This correction can account for the suppression of the quadrupole by the observed amount  $\sim 10\%$ – $20\%$  when the high energy tail of the initial conditions is set by the inflation scale. The corrections to higher  $l$  multipoles are suppressed by a factor  $\sim 1/l^2$  and therefore they are not observable within the present data.
- (ii) The equation for the fluctuations can be interpreted as a one-dimensional Schrödinger equation with a (conformal) time dependent potential. We argue that this potential features two distinct parts: (i) the slow roll part  $[(\nu^2 - 1/4)/\eta^2]$  which is repulsive, (like a repulsive potential barrier,  $\eta$  being the conformal time,  $\nu$  being  $3/2$  plus slow roll parameters), and (ii) a different part  $\mathcal{V}(\eta)$  with support before slow roll starts. The potential  $\mathcal{V}(\eta)$  vanishes in the slow roll stage, hence it does not affect the dynamics during this stage, but its presence imprints the physical initial conditions to the fluctuations in the slow roll stage both for metric and tensor perturbations.
- (iii) We demonstrate that the problem of setting generic initial conditions in the fluctuations equation is equivalent to the scattering problem by a potential. Thus, by implementing the powerful methods of scattering theory we show that the potential  $\mathcal{V}(\eta)$  yields initial conditions on the fluctuations for the beginning of slow roll whose large  $k$  behavior is consistent with renormalizability. We describe the potential  $\mathcal{V}(\eta)$  in a general manner and establish that an *attractive* potential  $\mathcal{V}(\eta)$  leads to an observable suppression of the quadrupole.
- (iv) We find that the effects on the power spectrum are robust and only depend on the strength and width of the potential  $\mathcal{V}(\eta)$ , namely, on the energy scale of  $\mathcal{V}(\eta)$ , which is the inflation scale, and on the time interval where  $\mathcal{V}(\eta)$  is nonzero.
- (v) Our analysis applies both to the curvature as well as the tensor fluctuations. Therefore, we predict that the initial conditions for slow roll also affect the quadrupole for  $B$ -modes.

We show in the companion article [30] that the potential  $\mathcal{V}(\eta)$  is quite generic and originates in a stage of *fast roll* inflaton dynamics. This is an early stage during which the inflaton varies rapidly, slowing down to merge with the slow roll stage.

## II. INITIAL CONDITIONS AND THE ENERGY MOMENTUM TENSOR OF SCALAR AND TENSOR PERTURBATIONS

The effective field theory of slow roll inflation has two main ingredients: the classical Friedmann equations in terms of a *classical* part of the energy momentum tensor described by a homogeneous and isotropic condensate, and a quantum part. The latter features scalar fluctuations determined by a gauge invariant combination of the scalar field (inflaton) and metric fluctuations, and a tensor component, gravitational waves. A consistency condition for this description is that the contributions from the fluctuations to the energy momentum tensor be much smaller than those from the homogeneous and isotropic condensate. The effective field theory must include renormalization counterterms so that it is insensitive to the possible ultraviolet singularities of the short wavelength fluctuations. Different initial conditions on the mode functions of the quantum fluctuations yield different values for their contribution to the energy momentum tensor. Different initial conditions on the mode functions of the quantum fluctuations yield different values for the energy momentum tensor.

Criteria for acceptable initial conditions must include the following: (i) backreaction effects from the quantum fluctuations should not modify the inflationary dynamics described by the inflaton, (ii) the ultraviolet counterterms that renormalize the energy momentum tensor should not depend on the particular choice of initial conditions, namely, different initial conditions *should not* introduce new ultraviolet divergences: a single renormalization scheme, independent of initial conditions, should render the energy momentum tensor UV finite. This set of criteria imply that the ultraviolet allowed states have their large  $k$  behavior constrained up to the fourth order in  $1/k$  [21]. In Ref. [21] only the energy momentum tensor of *inflaton* fluctuations was considered. However, the fluctuations of the inflaton field are *not* gauge invariant, and in order to establish a set of criteria for UV allowed initial states in a gauge invariant manner, we must study the full gauge invariant energy momentum of scalar and tensor fluctuations.

### A. Scalar perturbations

The gauge invariant energy momentum tensor for quadratic scalar metric fluctuations has been obtained in Ref. [32] where the reader is referred to for details. Its form simplifies in longitudinal gauge, and in cosmic time it is given by [32]

$$\begin{aligned} \langle T_{00} \rangle = & M_{\text{Pl}}^2 \left[ 12H \langle \psi \dot{\psi} \rangle - 3 \langle (\dot{\psi})^2 \rangle + \frac{9}{a^2(t)} \langle (\nabla \psi)^2 \rangle \right] + \frac{1}{2} \\ & \times \langle (\dot{\phi})^2 \rangle + \frac{\langle (\nabla \phi)^2 \rangle}{2a^2(t)} + \frac{V''(\Phi)}{2} \langle \phi^2 \rangle + 2V'(\Phi) \langle \psi \phi \rangle, \end{aligned} \quad (2.1)$$

where  $\Phi(t)$  stands for the zero mode of the inflaton field,  $\phi(t, \vec{x})$  for the inflaton fluctuations around  $\Phi(t)$ ,  $\psi(t, \vec{x})$  is the longitudinal gauge form of the Bardeen potential, and the dots stand for derivatives with respect to cosmic time. During inflation the Newtonian potential and the Bardeen potential are the same in the longitudinal gauge [7,9] and this property has been used in the above expression.

In longitudinal gauge, the equations of motion in cosmic time for the Fourier modes are [7,9]

$$\begin{aligned} \ddot{\psi}_{\vec{k}} + \left( H - 2 \frac{\ddot{\Phi}}{\dot{\Phi}} \right) \dot{\psi}_{\vec{k}} + \left[ 2 \left( \dot{H} - 2H \frac{\ddot{\Phi}}{\dot{\Phi}} \right) + \frac{k^2}{a^2(t)} \right] \psi_{\vec{k}} &= 0, \\ \ddot{\phi}_{\vec{k}} + 3H \dot{\phi}_{\vec{k}} + \left[ V''(\Phi) + \frac{k^2}{a^2(t)} \right] \phi_{\vec{k}} + 2V'(\Phi) \dot{\psi}_{\vec{k}} - 4\dot{\Phi} \dot{\psi}_{\vec{k}} &= 0, \end{aligned} \quad (2.2)$$

with the constraint equation

$$\dot{\psi}_{\vec{k}} + H \psi_{\vec{k}} = \frac{1}{2M_{\text{Pl}}^2} \phi_{\vec{k}} \dot{\Phi}. \quad (2.3)$$

Initial conditions on the mode functions of the quantum fluctuations correspond to an initial value problem at a fixed time hypersurface. For modes of cosmological relevance this time slice at which the initial conditions are established is such that these modes are subhorizon. Therefore, we must focus on the contribution to the energy momentum tensor from subhorizon fluctuations, and, in particular, in the large momentum region to assess the criteria for UV allowed states.

For subhorizon modes with wave vectors  $k \gg a(t)H$ , the solutions of the Eq. (2.2) are [7]

$$\psi_{\vec{k}}(t) \approx e^{\pm ik\eta} \Rightarrow \dot{\psi}_{\vec{k}}(t) \sim \frac{ik}{a(t)} \psi_{\vec{k}}(t). \quad (2.4)$$

For  $k \gg a(t)H$  the constraint equation (2.3) entails that [32]

$$\psi_{\vec{k}}(t) \approx \frac{ia(t)}{2M_{\text{Pl}}^2 k} \dot{\Phi} \phi_{\vec{k}}. \quad (2.5)$$

In slow roll,

$$\begin{aligned} \dot{\Phi} &= -\frac{V'(\Phi)}{3H} \left[ 1 + \mathcal{O}\left(\frac{1}{N}\right) \right] \\ &= -HM_{\text{Pl}} \sqrt{2\epsilon_v} \left[ 1 + \mathcal{O}\left(\frac{1}{N}\right) \right], \end{aligned} \quad (2.6)$$

where the slow roll parameters  $\epsilon_v$ ,  $\eta_v$  are of the order  $1/N$  [29],

$$\begin{aligned} \epsilon_v &= \frac{M_{\text{Pl}}^2}{2} \left[ \frac{V'(\Phi)}{V(\Phi)} \right]^2 = \mathcal{O}\left(\frac{1}{N}\right), \\ \eta_v &= M_{\text{Pl}}^2 \frac{V''(\Phi)}{V(\Phi)} = \mathcal{O}\left(\frac{1}{N}\right), \end{aligned} \quad (2.7)$$

and  $N \sim 55$  stands for the number of e-folds from horizon

exit until the end of inflation. Therefore, for subhorizon modes,

$$\psi_{\vec{k}}(t) \approx -i\sqrt{2\epsilon_v} \left( \frac{Ha(t)}{k} \right) \frac{\phi_{\vec{k}}}{2M_{\text{Pl}}}. \quad (2.8)$$

These identities, valid in the limit  $k \gg a(t)H$  allow to obtain an estimate for the different contributions to  $T_{00}$ . The first line of Eq. (2.1), namely, the contribution from the Newtonian potential mode with comoving wave vector  $k$  is

$$\langle T_{00}^{(\psi)} \rangle \approx 6\epsilon_v H^2 \langle (\phi_{\vec{k}})^2 \rangle. \quad (2.9)$$

The first three terms in the second line of Eq. (2.1) (the quadratic contribution from the scalar field fluctuations) is

$$\langle T_{00}^\phi \rangle \approx \left( \frac{k}{a(t)} \right)^2 \langle (\phi_{\vec{k}})^2 \rangle, \quad (2.10)$$

and the crossed term is:

$$V'(\Phi) \langle \psi_k \phi_k \rangle \approx \epsilon_v H^2 \left( \frac{a(t)H}{k} \right) \langle (\phi_{\vec{k}})^2 \rangle. \quad (2.11)$$

Therefore, in slow roll,  $\epsilon_v, \eta_v \ll 1$  and for subhorizon modes  $k \gtrsim a(t)H$ , the leading contribution to the energy momentum tensor for the scalar fluctuations is given by the contribution from the inflaton fluctuations, namely

$$\langle T_{00} \rangle \approx \frac{1}{2} \langle (\dot{\phi})^2 \rangle + \frac{\langle (\nabla \phi)^2 \rangle}{2a^2(t)} + \frac{V''(\Phi)}{2} \langle \phi^2 \rangle. \quad (2.12)$$

Furthermore, in terms of the slow roll parameter  $\eta_v$ ,  $V'' = 3\eta_v H^2$  and for subhorizon wave vectors with  $k \gg a(t)H$  the last term in Eq. (2.12) is subdominant and will be neglected. Hence, the contribution to the energy momentum tensor from subhorizon fluctuations during the slow roll stage is determined by the subhorizon quantum fluctuations of the inflaton and given by

$$\langle T_{00} \rangle \approx \frac{1}{2} \langle (\dot{\phi})^2 \rangle + \frac{\langle (\nabla \phi)^2 \rangle}{2a^2(t)}. \quad (2.13)$$

This analysis allows us to connect with the results in Ref. [21] for inflaton fluctuations.

The inflaton fluctuation obeys the equation of motion

$$\ddot{\phi}_{\vec{k}} + 3H\dot{\phi}_{\vec{k}} + \left[ 3H^2\eta_v + \frac{k^2}{a^2(t)} \right] \phi_{\vec{k}} = 0. \quad (2.14)$$

In what follows it is convenient to pass to conformal time in terms of which, the homogeneous and isotropic Friedmann-Robertson-Walker (FRW) metric is determined by

$$ds^2 = dt^2 - a^2(t)(d\vec{x})^2 = C^2(\eta)[d\eta^2 - (d\vec{x})^2], \quad (2.15)$$

where  $t$  and  $\eta$  stand for cosmic and conformal time, respectively. During slow roll

$$C(\eta) = -\frac{1}{\eta H(1 - \epsilon_v)}. \quad (2.16)$$

In conformal time  $\eta$  the solution of Eq. (2.14) is given by

$$\phi_{\vec{k}}(\eta) = \frac{1}{C(\eta)} [\alpha_{\vec{k}} S_\phi(k, \eta) + \alpha_{-\vec{k}}^\dagger S_\phi^*(k, \eta)], \quad (2.17)$$

where the mode functions  $S_\phi(k, \eta)$  are solutions of

$$\left[ \frac{d^2}{d\eta^2} + k^2 + M_\phi^2 C^2(\eta) - \frac{C''(\eta)}{C(\eta)} \right] S_\phi(k, \eta) = 0, \quad (2.18)$$

here,

$$M_\phi^2 = V''(\Phi) = 3H^2\eta_v, \quad (2.19)$$

and prime stands for derivative with respect to the conformal time. Using Eqs. (2.7) and (2.16) during slow roll, this equation simplifies to

$$\left[ \frac{d^2}{d\eta^2} + k^2 - \frac{\nu_\phi^2 - \frac{1}{4}}{\eta^2} \right] S_\phi(k, \eta) = 0, \quad (2.20)$$

where for inflaton fluctuations during slow roll

$$\nu_\phi = \frac{3}{2} + \epsilon_v - \eta_v + \mathcal{O}\left(\frac{1}{N^2}\right). \quad (2.21)$$

The operators  $\alpha_{\vec{k}}, \alpha_{\vec{k}}^\dagger$  in Eq. (2.17) obey the usual canonical commutation relations.

## B. Tensor perturbations

Tensor perturbations (gravitational waves) are gauge invariant. The expectation value of their energy momentum pseudotensor in a quantum state has been obtained in Ref. [32] (see also Ref. [33]) and is given by

$$\langle T_{00}^{(T)} \rangle = M_{\text{Pl}}^2 \left\{ H \langle \dot{h}_{kl} \dot{h}_{kl} \rangle + \frac{1}{8} \left[ \langle \dot{h}_{kl} \dot{h}_{kl} \rangle + \frac{1}{a^2(t)} \langle \nabla h_{kl} \nabla h_{kl} \rangle \right] \right\}, \quad (2.22)$$

where the dot stands for the derivative with respect to cosmic time. The equations of motion for the spatial Fourier transform of the dimensionless tensor field  $h$  are

$$\ddot{h}_{kl}(\vec{k}) + 3H\dot{h}_{kl}(\vec{k}) + \frac{k^2}{a^2(t)} h_{kl}(\vec{k}) = 0. \quad (2.23)$$

Tensor perturbations correspond to minimally coupled massless fields with two physical polarizations. Passing to conformal time the spatial Fourier transform of the quantum fields are written as [8,10]

$$h_{ij}(\vec{k}, \eta) = \frac{1}{C(\eta)M_{\text{Pl}}\sqrt{2}} \sum_{\lambda=\times,+} \epsilon_{ij}(\lambda, \vec{k}) [\alpha_{\lambda, \vec{k}} S_T(k; \eta) + \alpha_{\lambda, \vec{k}}^\dagger S_T^*(k; \eta)], \quad (2.24)$$

where  $\lambda$  labels the two standard transverse and traceless polarizations  $\times$  and  $+$ . The operators  $\alpha_{\lambda, \vec{k}}, \alpha_{\lambda, \vec{k}}^\dagger$  obey the usual canonical commutation relations, and  $\epsilon_{ij}(\lambda, \vec{k})$  are



the two independent traceless-transverse tensors constructed from the two independent polarization vectors transverse to  $\hat{\mathbf{k}}$ , chosen to be real and normalized such that  $\epsilon_j^i(\lambda, \vec{k})\epsilon_k^j(\lambda', \vec{k}) = \delta_k^i \delta_{\lambda, \lambda'}$ .

The mode functions  $S_T(k; \eta)$  obey the differential equation

$$S_T''(k; \eta) + \left[ k^2 - \frac{C''(\eta)}{C(\eta)} \right] S_T(k; \eta) = 0, \quad (2.25)$$

where in the slow roll regime,

$$\frac{C''(\eta)}{C(\eta)} = \frac{\nu_T^2 - \frac{1}{4}}{\eta^2}; \quad \nu_T = \frac{3}{2} + \epsilon_v + \mathcal{O}\left(\frac{1}{N^2}\right). \quad (2.26)$$

Thus, to leading order in slow roll the mode functions for gravitational waves obey the same equations of motion as for scalar fields but with vanishing mass, namely, setting  $\eta_v = 0$ .

### C. Initial conditions

We treat both scalar and tensor perturbations on the same footing by focusing on mode functions solutions of the general equation

$$\left[ \frac{d^2}{d\eta^2} + k^2 - \frac{\nu^2 - \frac{1}{4}}{\eta^2} \right] S(k, \eta) = 0. \quad (2.27)$$

For general initial conditions we write

$$S(k; \eta) = A(k)g_\nu(k; \eta) + B(k)f_\nu(k; \eta), \quad (2.28)$$

where two linearly independent solutions of Eq. (2.27) are

$$g_\nu(k; \eta) = \frac{1}{2}i^{\nu+(1/2)}\sqrt{-\pi\eta}H_\nu^{(1)}(-k\eta), \quad (2.29)$$

$$f_\nu(k; \eta) = [g_\nu(k; \eta)]^*, \quad (2.30)$$

where  $H_\nu^{(1)}(z)$  are Hankel functions. These solutions are normalized so that their Wronskian is given by

$$\begin{aligned} W[g_\nu(k; \eta), f_\nu(k; \eta)] &= g_\nu'(k; \eta)f_\nu(k; \eta) \\ &\quad - g_\nu(k; \eta)f_\nu'(k; \eta) \\ &= -i \end{aligned} \quad (2.31)$$

(here prime stands for the derivative with respect to the conformal time). For the specific cases of scalar or tensor perturbations, the mode functions and coefficients  $A(k)$ ,  $B(k)$  will feature a subscript index  $\phi$ ,  $T$ , respectively.

For wave vectors deep inside the Hubble radius  $|k\eta| \gg 1$ , the mode functions for arbitrary  $\nu$  have the Bunch-Davies asymptotic behavior

$$g_\nu(k; \eta) \xrightarrow{\eta \rightarrow -\infty} \frac{1}{\sqrt{2k}} e^{-ik\eta}, \quad f_\nu(k; \eta) \xrightarrow{\eta \rightarrow -\infty} \frac{1}{\sqrt{2k}} e^{ik\eta}, \quad (2.32)$$

and for  $\eta \rightarrow 0^-$ , the mode functions behave as:

$$g_\nu(k; \eta) \xrightarrow{\eta \rightarrow 0^-} \frac{\Gamma(\nu)}{\sqrt{2\pi k}} \left( \frac{2}{ik\eta} \right)^{\nu-(1/2)}. \quad (2.33)$$

The complex conjugate formula holds for  $f_\nu(k; \eta)$ .

In particular, in the scale invariant case  $\nu = \frac{3}{2}$  which is the leading order in the slow roll expansion, the mode functions equations (2.29) simplify to

$$g_{3/2}(k; \eta) = \frac{e^{-ik\eta}}{\sqrt{2k}} \left[ 1 - \frac{i}{k\eta} \right]. \quad (2.34)$$

The coefficients  $A(k)$ ,  $B(k)$  for the general solution equation (2.28) are determined by an initial condition on the mode functions  $S(k; \eta)$  at a given initial conformal time  $\bar{\eta}$ , namely

$$B(k) = -i[g_\nu(k; \bar{\eta})S'(k; \bar{\eta}) - g_\nu'(k; \bar{\eta})S(k; \bar{\eta})], \quad (2.35)$$

$$A(k) = -i[f_\nu(k; \bar{\eta})S(k; \bar{\eta}) - f_\nu'(k; \bar{\eta})S'(k; \bar{\eta})]. \quad (2.36)$$

Canonical commutation relations for the Heisenberg fields entail that

$$|A(k)|^2 - |B(k)|^2 = 1. \quad (2.37)$$

The  $S$ -vacuum state  $|0\rangle_S$  is annihilated by the operators  $\alpha_{\vec{k}}$  associated with the modes  $S(k; \eta)$ . A different choice of the coefficients  $A(k)$ ,  $B(k)$  determines different choices of vacua, the Bunch-Davies vacuum corresponds to  $A(k) = 1$ ,  $B(k) = 0$ . An illuminating representation of these coefficients can be gleaned by computing the expectation value of the number operator in the Bunch-Davies vacuum. Consider the expansion of the fluctuation field both in terms of Bunch-Davies modes  $g_\nu(k; \eta)$  and in terms of the general modes  $S(k; \eta)$ , for example, for the scalar field  $\phi$  (similarly for tensor fields with a subscript  $T$  and corresponding normalization)

$$\begin{aligned} \phi_{\vec{k}}(\eta) &= \frac{1}{C(\eta)} [a_{\vec{k}} g_{\nu_\phi}(k; \eta) + a_{-\vec{k}}^* g_{\nu_\phi}^*(k; \eta)] \\ &\equiv \frac{1}{C(\eta)} [\alpha_{\vec{k}} S_\phi(k; \eta) + \alpha_{-\vec{k}}^\dagger S_\phi^*(k; \eta)], \end{aligned} \quad (2.38)$$

the creation and annihilation operators are related by a Bogoliubov transformation

$$\begin{aligned} \alpha_{\vec{k}}^\dagger &= A_\phi(k) a_{\vec{k}}^\dagger - B_\phi(k) a_{-\vec{k}}, \\ \alpha_{\vec{k}} &= A_\phi^*(k) a_{\vec{k}} - B_\phi^*(k) a_{-\vec{k}}^\dagger. \end{aligned} \quad (2.39)$$

The Bunch-Davies vacuum  $|0\rangle_{BD}$  is annihilated by  $a_{\vec{k}}$ , hence we find the expectation value

$${}_{BD}\langle 0 | \alpha_{\vec{k}}^\dagger \alpha_{\vec{k}} | 0 \rangle_{BD} = |B_\phi(k)|^2 = N_\phi(k), \quad (2.40)$$

where  $N_\phi(k)$  is interpreted as the number of  $S$ -vacuum particles in the Bunch-Davies vacuum. In combination with the constraint equation (2.37) the above result suggests the following representation for the coefficients  $A(k)$ ,  $B(k)$

$$A_\phi(k) = \sqrt{1 + N_\phi(k)} e^{i\theta_A(k)}; \quad B_\phi(k) = \sqrt{N_\phi(k)} e^{i\theta_B(k)}, \quad (2.41)$$

where  $N_\phi(k)$ ,  $\theta_{A,B}(k)$  are real. The only relevant phase is the difference

$$\theta_k = \theta_B(k) - \theta_A(k). \quad (2.42)$$

Notice that we provide the initial conditions at a given conformal time  $\bar{\eta}$  which is obviously the same for all  $k$ -modes. This is the consistent manner to define the initial value problem (or Cauchy problem) for the fluctuations. This is different from what is often done in the literature when an *ad hoc* dependence on  $k$  is given to  $\bar{\eta}$  [22].

#### D. The transfer function of initial conditions and its asymptotic behavior

For gauge invariant scalar perturbations, the analysis leading to Eq. (2.13) indicates that in order to study the energy momentum tensor for general initial conditions it is enough to consider the leading order in the slow roll expansion. Consistently with neglecting the contributions from the Newtonian potential as well as the term proportional to  $V''[\Phi]$  for the inflaton fluctuations, we set  $\nu = 3/2$  in the expression for the mode functions equation (2.29). This simplification results in considering the scalar field fluctuations as massless and minimally coupled to gravity.

The energy density in the vacuum state defined by the new initial conditions is

$$\rho = {}_S\langle 0|T_{00}|0\rangle_S. \quad (2.43)$$

The renormalized energy density from the fluctuations of the inflaton field is found to be [21]

$$\rho = \rho^{BD} + I_1 + I_2, \quad (2.44)$$

where  $\rho^{BD}$  corresponds to the Bunch-Davies vacuum initial conditions  $N_k = 0$  and

$$I_1 = \frac{1}{2\pi^2} \int_0^\infty dk k^2 \{N_\phi(k) |\dot{F}(k, \eta)|^2 + \sqrt{N_\phi(k)[1 + N_\phi(k)]} \text{Re}[e^{-i\theta_k} (\dot{F}(k, \eta))^2]\}, \quad (2.45)$$

$$I_2 = \frac{1}{2\pi^2} \int_0^\infty dk k^2 \frac{k^2}{a^2} \{N_\phi(k) |F(k, \eta)|^2 + \sqrt{N_\phi(k)[1 + N_\phi(k)]} \text{Re}[e^{-i\theta_k} (F(k, \eta))^2]\}, \quad (2.46)$$

where  $F(k, \eta)$  is given in terms of the Bunch-Davies mode function equation (2.34) for  $\nu = 3/2$  as

$$F(k, \eta) = (-H\eta) g_{3/2}(k, \eta) = \frac{H}{\sqrt{2}k^3} e^{-ik\eta} (i - k\eta). \quad (2.47)$$

The power spectrum of the inflaton fluctuations is given by

[4,8],

$$P_\phi(k, t) = {}_S\langle 0|\phi_k(\eta)|^2|0\rangle_S = P_\phi^{BD}(k, t) + \frac{k^3}{\pi^2} \{N_\phi(k) |F(k, \eta)|^2 + \sqrt{N_\phi(k)[1 + N_\phi(k)]} \times \text{Re}[e^{-i\theta_k} (F(k, \eta))^2]\}, \quad (2.48)$$

where we used Eq. (2.38) and

$$P_\phi^{BD}(k, t) = \frac{k^3}{2\pi^2} |F(k, \eta)|^2. \quad (2.49)$$

We find

$$I_1 = \frac{(H\eta)^4}{(2\pi)^2} \int_0^\infty dk k^3 \{N_\phi(k) - \sqrt{N_\phi(k)[1 + N_\phi(k)]} \cos[2k\eta + \theta_k]\}, \quad (2.50)$$

$$I_2 = \frac{(H^2\eta)^2}{(2\pi)^2} \int_0^\infty dk k \{N_\phi(k)(1 + k^2\eta^2) - \sqrt{N_\phi(k)[1 + N_\phi(k)]} [(1 - k^2\eta^2) \cos[2k\eta + \theta_k] + 2k\eta \sin[2k\eta + \theta_k]]\},$$

$$P_\phi(k, t) = \left(\frac{H}{2\pi}\right)^2 \{(1 + k^2\eta^2)[1 + 2N_\phi(k)] - 2\sqrt{N_\phi(k)[1 + N_\phi(k)]} [(1 - k^2\eta^2) \times \cos[2k\eta + \theta_k] + 2k\eta \sin[2k\eta + \theta_k]]\}. \quad (2.51)$$

Evaluating the power spectrum after horizon crossing  $|k\eta| \ll 1$ , yields

$$\frac{P_\phi}{P_\phi^{BD}} \Big|_{|k\eta| \ll 1} = 1 + D_\phi(k), \quad (2.52)$$

where we have introduced the *transfer function for initial conditions*

$$D_\phi(k) = 2|B_\phi(k)|^2 - 2\text{Re}[A_\phi(k)B_\phi^*(k)] = 2N_\phi(k) - 2\sqrt{N_\phi(k)[1 + N_\phi(k)]} \cos\theta_k. \quad (2.53)$$

The integrals  $I_{1,2}$  are finite provided that asymptotically for  $k \rightarrow \infty$  the occupation numbers behave as

$$N_\phi(k) = \mathcal{O}\left(\frac{1}{k^{4+\delta}}\right), \quad (2.54)$$

with  $\delta > 0$ . Namely, the finiteness of the energy momentum tensor constrains the asymptotic behavior of the occupation numbers to vanish faster than  $\frac{1}{k^4}$  for  $k \rightarrow \infty$  [21]. Of course, this asymptotic condition leaves a large freedom on the occupation numbers  $N_k$ .

We systematically impose the constraint equation (2.54) which guarantees the finiteness of the energy momentum

tensor. This is not always the case for initial conditions considered in the literature (see [22]).

Let us establish a bound on the large momentum behavior of  $N_k$  inserting the asymptotic behavior

$$N_k = N_\mu \left( \frac{\mu}{k} \right)^{4+\delta}, \quad (2.55)$$

with  $0 < \delta \ll 1$  in the integrals  $I_{1,2}$ . Namely, assuming that the integrals are dominated by the region of high momenta  $k/H \gg 1$  and that the occupation number attains the largest possible values consistent with ultraviolet finiteness [Eq. (2.54)]. We observe that  $k|\eta| \gg 1$  in the early stages of inflation for large  $k$ , and that the maximum contribution from these integrals are at early time  $\eta \sim -1/H$ . Hence, the oscillatory terms in  $I_1, I_2$  average out and we have from Eqs. (2.50) and (2.51),

$$I_1 \sim I_2 \sim \frac{N_\mu}{(2\pi)^2} \frac{\mu^4}{\delta}. \quad (2.56)$$

The contribution from the fluctuations to the energy momentum tensor does not lead to large backreaction effects affecting the inflationary dynamics provided that  $I_1, I_2 \ll H^2 M_{\text{Pl}}^2$ , which yields

$$N_\mu \ll 2\pi^2 \frac{H^2 M_{\text{Pl}}^2}{\mu^4} \delta. \quad (2.57)$$

Equation (2.57) provides an occupation number distribution exhibiting the largest asymptotic value compatible with a UV finite energy momentum tensor. This maximal occupation number distribution falls off for  $k \rightarrow \infty$  with the minimal acceptable power tail exponent  $4 + \delta$  with  $\delta \ll 1$ .

Gravitons are massless particles with two independent polarizations, therefore their energy momentum tensor is given by Eq. (2.13) times a factor two. The first term in the energy momentum pseudotensor for gravitational waves Eq. (2.22) features only one time derivative, which results in only one factor  $k$  for large momenta, whereas the terms with two time or spatial derivatives yield  $k^2$ . Therefore, the first term is subdominant in the ultraviolet and the short wavelength contribution to the energy momentum (pseudo) tensor of gravitational waves is the same as that for a free massless scalar field, up to a factor 2 from the physical polarization states [33,34]. Therefore we can directly extend the results obtained above for scalar fluctuations to the case of tensor fluctuations.

Small backreaction effects from the fluctuations is a necessary consistency condition for the validity of the inflationary scenario. In addition, the condition that different initial states *should not affect* the renormalization aspects of the energy momentum tensor is a consistency condition on the renormalizability of the effective field theory of inflation: the theory should be insensitive to the short distance physics for *any* initial conditions. These criteria lead to the following important consequences:

- (i) If  $\mu \sim M_{\text{Pl}}$  then  $N_\mu \lesssim H^2/M_{\text{Pl}}^2 \ll 1$  because  $H/M_{\text{Pl}} \ll 1$  in the effective field theory expansion and the effect of initial conditions becomes negligible.
- (ii) For  $\mu \sim \sqrt{HM_{\text{Pl}}} \sim V^{1/4}(\Phi)$ , namely  $\mu$  of the order of the scale of inflation during the slow roll stage, then  $N_\mu \lesssim 1$ . For example for  $\delta \sim 0.01$  one obtains  $N_\mu \sim 0.1$ . If  $\mu \ll \sqrt{HM_{\text{Pl}}}$ , for example  $\mu \sim H$ , the bound equation (2.57) is rather loose allowing a wide range of  $N_\mu$  with potentially appreciable effects.
- (iii) The condition that the occupation number falls off faster than  $1/k^4$  for a large wave vector, implies that the possible effects from different initial conditions are more prominent for the smaller wave vectors, those that exited the Hubble radius the *earliest*. For cosmologically relevant wave vectors, these are those that crossed about 55 e-folds before the end of inflation. Today those wave vectors correspond to the present Hubble scale, hence the low multipoles in the CMB.

We conclude that consistent with renormalizability and small backreaction there may be a substantial effect from the initial conditions when the characteristic scale  $\mu \leq \sqrt{HM_{\text{Pl}}}$ . The rapid falloff of the occupation numbers  $N_\phi(k)$  for subhorizon wavelengths and the backreaction constraint equation (2.57) entails that for these modes the transfer function equation (2.53) for initial conditions simplifies to

$$D_\phi(k)^{N_\phi(k) \ll 1} = -2\sqrt{N_\phi(k)} \cos\theta_k, \quad (2.58)$$

and that the *smaller values* of  $k$  yield the *larger corrections from initial conditions*. The result Eq. (2.58) suggests a suppression of the power spectrum for  $\cos\theta_k > 0$ . These observations will be crucial below when we study the effect of initial conditions on the multipoles of the CMB.

While this discussion focused on the fluctuations of the inflaton, they are directly applicable to the case of gauge invariant perturbations studied below.

### III. EFFECTS OF THE INITIAL CONDITIONS ON THE CURVATURE PERTURBATIONS

In the previous section we focused on the backreaction effects from initial conditions beginning with the gauge invariant energy momentum tensor for scalar and tensor perturbations. Since the fluctuation modes are initialized on a fixed time hypersurface while their wavelengths are well inside the Hubble radius, we established a correspondence with Ref. [21] which refers solely to the quantum fluctuations of the inflaton field. The effect of different initial conditions is encoded in the Bogoliubov coefficients, and, in particular, in the occupation numbers  $N_k$  and the phases  $\theta_k$ . Ultraviolet allowed initial conditions require that  $N_k$  diminishes faster than  $1/k^4$  for asymptoti-



cally large momenta. Small backreaction effects require in general that  $N_k \ll 1$ .

In this section we study UV allowed initial conditions on the quantum fluctuations associated with gauge invariant variables which determine the power spectra of observables. We focus on both scalar and tensor perturbations.

### A. Effects of initial conditions on the power spectrum

The gauge invariant curvature perturbation of the co-moving hypersurfaces is given in terms of the Newtonian potential ( $\psi$ ) and inflaton fluctuation ( $\phi$ ) by [7,8]

$$\mathcal{R} = -\psi - \frac{H}{\dot{\Phi}} \phi, \quad (3.1)$$

where  $\dot{\Phi}$  stands for the derivative of the inflaton field  $\Phi$  with respect to the cosmic time  $t$ .

It is convenient to introduce the gauge invariant potential [7],

$$u(\vec{x}, t) = -z\mathcal{R}(\vec{x}, t), \quad (3.2)$$

where

$$z = a(t) \frac{\dot{\Phi}}{H}. \quad (3.3)$$

The spatial Fourier transform of the gauge invariant field  $u(\vec{x}, t)$  is expanded in terms of conformal time mode functions and creation and annihilation operators as follows [7,8]

$$u(\vec{k}, \eta) = \alpha_{\mathcal{R}}(k)S_{\mathcal{R}}(k; \eta) + \alpha_{\mathcal{R}}^\dagger(k)S_{\mathcal{R}}^*(k; \eta), \quad (3.4)$$

where the vacuum state is annihilated by the operators  $\alpha_{\mathcal{R}}(k)$  and the mode functions are solutions of the equation

$$\left[ \frac{d^2}{d\eta^2} + k^2 - \frac{1}{z} \frac{d^2 z}{d\eta^2} \right] S_{\mathcal{R}}(k; \eta) = 0. \quad (3.5)$$

During slow roll and to leading order in slow roll variables

$$\begin{aligned} \frac{1}{z} \frac{d^2 z}{d\eta^2} &= \frac{2}{\eta^2} \left[ 1 + \frac{3}{2}(3\epsilon_v - \eta_v) \right] = \frac{\nu_{\mathcal{R}}^2 - \frac{1}{4}}{\eta^2}, \\ \nu_{\mathcal{R}} &= \frac{3}{2} + 3\epsilon_v - \eta_v + \mathcal{O}\left(\frac{1}{N^2}\right). \end{aligned} \quad (3.6)$$

The general solution of Eq. (3.5) in the slow roll regime is given by

$$S_{\mathcal{R}}(k; \eta) = A_{\mathcal{R}}(k)g_{\nu_{\mathcal{R}}}(\eta) + B_{\mathcal{R}}(k)g_{\nu_{\mathcal{R}}}^*(\eta), \quad (3.7)$$

where the function  $g_\nu(\eta)$  is given by Eq. (2.29), the Bogoliubov coefficients obey the relation (2.37) and can be written in terms of the occupation number of Bunch-Davies particles as in Eq. (2.41) with the label  $\mathcal{R}$  replacing  $\phi$ .

The power spectrum of curvature perturbations in the state with general initial conditions is given by [4,8]

$$P_{\mathcal{R}}(k) \stackrel{\eta \rightarrow 0^-}{=} \frac{k^3}{2\pi^2} \left| \frac{S_{\mathcal{R}}(k; \eta)}{z} \right|^2. \quad (3.8)$$

From Eq. (3.6) we see that in the slow roll regime  $z$  behaves as

$$z(\eta) = \frac{z_0}{(-k_0 \eta)^{\nu_{\mathcal{R}} - (1/2)}}, \quad (3.9)$$

where  $z_0$  is the value of  $z$  when a reference scale  $k_0$  exits the horizon. Combining this result with the small  $\eta$  limit Eq. (2.33) we find from Eqs. (3.8) and (3.9),

$$P_{\mathcal{R}}(k) = P_{\mathcal{R}}^{BD}(k)[1 + D_{\mathcal{R}}(k)], \quad (3.10)$$

where we introduced the transfer function for the initial conditions of curvature perturbations:

$$\begin{aligned} D_{\mathcal{R}}(k) &= 2|B_{\mathcal{R}}(k)|^2 - 2\text{Re}[A_{\mathcal{R}}(k)B_{\mathcal{R}}^*(k)i^{2\nu_{\mathcal{R}}-3}] \\ &= 2N_{\mathcal{R}}(k) - 2\sqrt{N_{\mathcal{R}}(k)[1 + N_{\mathcal{R}}(k)]} \\ &\quad \times \cos[\theta_k - \pi(\nu_{\mathcal{R}} - \frac{3}{2})] \end{aligned} \quad (3.11)$$

and

$$P_{\mathcal{R}}^{BD}(k) = \left( \frac{k}{2k_0} \right)^{3-2\nu_{\mathcal{R}}} \frac{\Gamma^2(\nu_{\mathcal{R}})}{\pi^3} \left( \frac{kH}{a(t)\dot{\Phi}} \right)_0^2. \quad (3.12)$$

The index 0 refers to the time where the reference scale  $k_0$  exits the horizon. In terms of the slow roll parameter  $\epsilon_v$  this expression simplifies to the usual result [4,8]

$$P_{\mathcal{R}}^{BD}(k) = \left( \frac{k}{2k_0} \right)^{n_s-1} \frac{\Gamma^2(\nu_{\mathcal{R}})}{\pi^3} \frac{H^2}{2\epsilon_v M_{\text{Pl}}^2} \equiv \mathcal{A}_{\mathcal{R}}^2 \left( \frac{k}{k_0} \right)^{n_s-1}, \quad (3.13)$$

where the amplitude is given by

$$\begin{aligned} \mathcal{A}_{\mathcal{R}}^2 &= \frac{1}{8\pi^2\epsilon_v} \left( \frac{H}{M_{\text{Pl}}} \right)^2 \left\{ 1 + (3\epsilon_v - \eta_v) \left[ \ln 4 + \psi\left(\frac{3}{2}\right) \right] \right. \\ &\quad \left. + \mathcal{O}\left(\frac{1}{N^2}\right) \right\}, \end{aligned} \quad (3.14)$$

$n_s - 1 = 3 - 2\nu_{\mathcal{R}} = 2\eta_v - 6\epsilon_v$  in the slow roll regime,  $\psi(z)$  is the digamma function and  $\psi(\frac{3}{2}) = -1.463\,510\,026\dots$  As shown above, for wave vectors of cosmological relevance,  $N_{\mathcal{R}}(k) \ll 1$ , hence to lowest order in slow roll ( $2\nu_{\mathcal{R}} = 3$ ), the transfer function for initial conditions simplifies to

$$D_{\mathcal{R}}(k) \stackrel{N_{\mathcal{R}}(k) \ll 1}{=} -2\sqrt{N_{\mathcal{R}}(k)} \cos\theta_k. \quad (3.15)$$

Therefore, for a positive  $\cos\theta_k$ , we have a *negative*  $D_{\mathcal{R}}(k)$ . That is, the initial conditions Eq. (3.11) suppress the power in such case.

### B. Tensor perturbations (gravitational waves)

The quantization of tensor fluctuations for general initial conditions has been discussed in Sec. II B.

Following the same steps as in Sec. III A we find the power spectrum of gravitational waves to be [4,8]

$$P_T(k) \stackrel{\eta \rightarrow 0^-}{=} \frac{k^3}{2\pi^2} \left| \frac{S_T(k; \eta)}{C(\eta)} \right|^2 = P_T^{BD}(k)[1 + D_T(k)], \quad (3.16)$$

where the transfer function for the initial conditions of tensor perturbations is

$$\begin{aligned} D_T(k) &= 2|B_T(k)|^2 - 2\text{Re}[A_T(k)B_T^*(k)i^{2\nu_T-3}] \\ &= 2N_T(k) - 2\sqrt{N_T(k)[1 + N_T(k)]} \\ &\quad \times \cos\left[\theta_k - \pi\left(\nu_T - \frac{3}{2}\right)\right] \end{aligned} \quad (3.17)$$

and

$$P_T^{BD}(k) = \mathcal{A}_T^2 \left(\frac{k}{k_0}\right)^{n_T}, \quad (3.18)$$

with

$$n_T = -2\epsilon_v, \quad \frac{\mathcal{A}_T^2}{\mathcal{A}_R^2} = r = 16\epsilon_v. \quad (3.19)$$

The contribution from gravitational waves to the energy momentum (pseudo) tensor is gauge invariant and up to a factor of 2 from the polarizations is exactly of the form Eq. (2.13) with  $\phi$  replaced by  $h$  [34]. Thus, the constraint on the occupation number Eqs. (2.55), (2.56), and (2.57) from the analysis of the backreaction and renormalizability translate *directly* to the case of gravitational waves for the occupation number  $N_T(k)$ .

This implies that corrections to the power spectrum of tensor modes from initial conditions are substantial if  $\mu$ , the asymptotic  $k$  scale of  $N_T(k)$ , is  $\mu \leq \sqrt{HM_{\text{Pl}}}$ , that is of the order of the inflation scale [see discussion below Eq. (2.55)]. We get from Eq. (3.17) for  $N_T(k) \ll 1$  and to leading order in slow roll,

$$D_T(k) \stackrel{N_T(k) \ll 1}{=} -2\sqrt{N_T(k)} \cos\theta_k. \quad (3.20)$$

Again, for a positive  $\cos\theta_k$ , we have a negative  $D_T(k)$ . That is, the initial conditions suppress the tensor power spectrum in such case.

#### IV. THE EFFECT OF INITIAL CONDITIONS ON THE LOW MULTIPOLES OF THE CMB

We have shown above that the fast falloff of the occupation number  $N(k)$  (for the corresponding perturbation) entails that initial conditions can only provide substantial corrections for perturbation modes whose wave vectors crossed out of the Hubble radius *early* during inflation. In turn, today these wave vectors correspond to scales of the order of the Hubble radius, namely, to the low multipoles in the CMB.

In the region of the Sachs-Wolfe plateau for  $l \lesssim 30$ , the matter-radiation transfer function can be set equal to unity and the  $C_l$ 's are given by [7–10]

$$C_l = \frac{4\pi}{9} \int_0^\infty \frac{dk}{k} P_X(k) \{j_l[k(\eta_{\text{tod}} - \eta_{\text{LSS}})]\}^2, \quad (4.1)$$

where  $P_X$  is the power spectrum of the corresponding perturbation,  $X = \mathcal{R}$  for curvature perturbations and  $X = T$  for tensor perturbations,  $j_l(x)$  are spherical Bessel functions [35], and

$$\eta_{\text{tod}} - \eta_{\text{LSS}} = \frac{1}{a_0 H_0} \int_{1/1+z_{\text{LSS}}}^1 \frac{dx}{x[\frac{\Omega_M}{x} + \Omega_\Lambda x^2]^{1/2}}, \quad (4.2)$$

is the comoving distance between today and the last scattering surface (LSS). In the above expression we consider that the dark energy component is described by a cosmological constant. Taking  $\Omega_M = 0.28$ ,  $\Omega_\Lambda = 0.72$ ,  $z_{\text{LSS}} = 1100$  we find

$$\eta_{\text{tod}} - \eta_{\text{LSS}} \sim \frac{3.3}{a_0 H_0}. \quad (4.3)$$

Notice that  $k/[a_0 H_0] \sim d_H/\lambda_{\text{phys}}(t_0)$  is the ratio between today's Hubble radius and the physical wavelength. The power spectra for curvature ( $\mathcal{R}$ ) or gravitational wave ( $T$ ) perturbations are of the form given by Eqs. (3.10), (3.13), (3.16), and (3.18),

$$P_X = \mathcal{A}_X^2 \left(\frac{k}{k_0}\right)^{n_s-1} [1 + D_X(k)], \quad (4.4)$$

with  $n_s = n_{\mathcal{R}}$  for curvature perturbations,  $n_s = 1 + n_T$  for tensor perturbations, and  $k_0 \sim a_0 H_0$  is a pivot scale. Then, from Eq. (2.52), the relative change  $\Delta C_l$  in the  $C_l$ 's due to the effect of generic initial conditions (generic vacua), is given by

$$C_l \equiv C_l^{BD} + \Delta C_l, \quad \frac{\Delta C_l}{C_l} = \frac{\int_0^\infty D_X(\kappa x) f_l(x) dx}{\int_0^\infty f_l(x) dx}, \quad (4.5)$$

where  $x = k/\kappa$  and

$$\kappa \equiv a_0 H_0 / 3.3, \quad (4.6)$$

$D(\kappa x)$  is the transfer function of initial conditions for the corresponding perturbation and

$$f_l(x) = x^{n_s-2} [j_l(x)]^2. \quad (4.7)$$

We now focus on curvature perturbations since these are directly related to the temperature fluctuations [15]. For  $n_s = 0.96$  [16], the functions  $f_l(x)$  are strongly peaked at  $x \sim l$ . Therefore,  $\frac{\Delta C_l}{C_l}$  is dominated by wave numbers  $k \sim l\kappa$ .

Low multipoles  $l$  correspond to wavelengths *today* of the order of the Hubble radius. These wavelengths crossed the Hubble radius about 55 e-folds before the end of inflation. Therefore, since inflation lasted a total number of e-folds  $N_{\text{total}} \gtrsim 60$ , these wave vectors were subhorizon during the

first few e-folds, namely, during the slow roll stage  $k \gg H$ . As already discussed, let us take for these wave vectors the occupation number  $N_k \ll 1$  as given by the asymptotic expression

$$N_k = N_\mu \left( \frac{\mu}{k} \right)^{4+\delta}; \quad 0 < \delta \ll 1 \quad (4.8)$$

and assume that the angles  $\theta_k$  are slowly varying functions of  $k$  in the region of  $k$  corresponding to *today's Hubble radius* so that  $\cos\theta_k \approx \overline{\cos\theta}$ . Then, we find that the fractional change in the coefficients  $C_l$  is given by

$$\frac{\Delta C_l}{C_l} \approx -2\sqrt{N_\mu} \left( \frac{3.3\mu}{a_0 H_0} \right)^{2+(\delta/2)} \frac{I_l(n_s - 2 - \frac{\delta}{2})}{I_l(n_s)}, \quad (4.9)$$

where

$$I_l(n_s) = \frac{1}{2^{3-n_s}} \frac{\Gamma(3-n_s)\Gamma(\frac{2l+1-3+n_s+1}{2})}{\Gamma^2(\frac{4-n_s}{2})\Gamma(\frac{2l+1+3-n_s+1}{2})}. \quad (4.10)$$

To obtain an estimate of the corrections, we take the values  $n_s = 1$ ,  $\delta = 0$  and find

$$\frac{\Delta C_l}{C_l} \approx -\frac{4}{3}\sqrt{N_\mu} \left( \frac{3.3\mu}{a_0 H_0} \right)^2 \frac{\overline{\cos\theta}}{(l-1)(l+2)}. \quad (4.11)$$

The  $\sim 1/l^2$  behavior is a result of the  $1/k^2$  falloff of  $D(k)$ , a consequence of the renormalizability condition on the occupation number. For the quadrupole, the relevant wave vectors correspond to  $x \sim 2$ , namely  $k_Q \sim a_0 H_0$ . It is convenient to write

$$k_Q = a_{sr} H_i = a_0 H_0, \quad (4.12)$$

where  $a_{sr}$  and  $H_i$  are the scale factor and the Hubble parameter during the slow roll stage of inflation when the wavelength corresponding to *today's* Hubble radius exits the horizon.

Hence, when the scale  $\mu$  in the asymptotic form of the occupation number Eq. (4.8) is of the order of the largest scale of cosmological relevance today, one has

$$\frac{\mu}{a_0 H_0} \sim 1, \quad (4.13)$$

and, for example, with  $N_\mu \sim 0.01$  we find that the fractional change in the quadrupole is given by:

$$\frac{\Delta C_2}{C_2} \sim -0.3\overline{\cos\theta}, \quad (4.14)$$

namely a suppression of the order of  $\sim 10\%$  in the quadrupole provided that  $\overline{\cos\theta} \sim 1$ . This corresponds to  $\mu$  of the order of the scale of inflation during the slow roll stage [see Eq. (4.12)].

We emphasize that these are general arguments based on the criteria of renormalizability and small backreaction which initial conditions must fulfill.

In a companion article [30] we show that these initial conditions are effectively realized in inflationary dynamics. There we show that a short stage just prior to the onset of slow roll inflation and in which the inflaton field evolves *fast*, imprints initial conditions on the curvature perturbations corresponding to the above analysis.

## V. INITIAL CONDITIONS AS THE SCATTERING BY A POTENTIAL

In the previous sections we have systematically analyzed the consequences of generic initial conditions different from Bunch-Davies, and which are UV allowed and consistent with small backreaction effects. We now provide a novel explanation of the origin of these initial conditions.

The mode equations (2.27) have the form of the Schrödinger equation in one dimension suggesting to consider them more generally as a *potential scattering problem*. The Eqs. (2.18), (2.25), and (2.27) can be written in the form

$$\left[ \frac{d^2}{d\eta^2} + k^2 - W(\eta) \right] S(k; \eta) = 0 \quad (5.1)$$

as a Schrödinger equation, with  $\eta$  the coordinate,  $k^2$  the energy, and  $W(\eta)$  a potential that depends on the coordinate  $\eta$ . In the cases under consideration

$$W(\eta) = \begin{cases} z''/z & \text{for curvature perturbations} \\ C''/C & \text{for tensor perturbations} \end{cases}. \quad (5.2)$$

It is convenient to explicitly separate the behavior of  $W(\eta)$  during the slow roll stage by writing

$$W(\eta) = \mathcal{V}(\eta) + \frac{\nu^2 - \frac{1}{4}}{\eta^2}, \quad (5.3)$$

where  $\nu = \frac{3}{2} + \mathcal{O}(\frac{1}{N})$  [see Eqs. (2.21), (2.26), and (3.6)]. Consider the potential  $\mathcal{V}(\eta)$  localized in a region of conformal time *prior* to the slow roll stage and which vanishes during the slow roll inflationary stage (during which cosmologically relevant modes cross out of the Hubble radius). Namely, a potential with the following properties:

$$\mathcal{V}(\eta) = \begin{cases} \neq 0 & \text{for } -\infty < \eta < \bar{\eta} \\ 0 & \text{for } \bar{\eta} < \eta \end{cases}, \quad (5.4)$$

where  $\bar{\eta}$  is the time when slow roll starts.

With Bunch-Davies (outgoing) boundary conditions,

$$S(k; \eta) \xrightarrow{\eta \rightarrow -\infty} e^{-ik\eta}/\sqrt{2k},$$

the solution of Eq. (5.1) for  $W(\eta)$  given by Eqs. (5.3) and (5.4) is

$$S(k; \eta) = \begin{cases} A(k)g_\nu(k) + B(k)g_\nu^*(k), & \text{for } \eta > \bar{\eta} \\ \frac{e^{-ik\eta}}{\sqrt{2k}}, & \text{for } \eta \rightarrow -\infty \end{cases}. \quad (5.5)$$

The coefficients  $A(k)$ ,  $B(k)$  are obtained by matching the wave function  $S(k; \eta)$  and its derivative at  $\bar{\eta}$ .  $A(k)$  and  $B(k)$  are simply related to the usual transmission and reflection coefficients of the scattering by a potential (see below).

We formally consider the conformal time starting at  $\eta = -\infty$ . However, the inflationary dynamics of the Universe presumably starts at some negative value  $\eta_0 < \bar{\eta}$ .

In this article we study the general consequences of such potential, deferring to a companion article [30], a comprehensive analytic and numerical study that shows that an *attractive* potential of the form of Eq. (5.4) originates naturally within the effective field theory of inflation from a brief fast roll stage just prior to slow roll.

### A. The effect of the potential $\mathcal{V}(\eta)$ as a change in the initial conditions

In summary, the equations for the quantum fluctuations are

$$\left[ \frac{d^2}{d\eta^2} + k^2 - \frac{\nu^2 - \frac{1}{4}}{\eta^2} - \mathcal{V}(\eta) \right] S(k; \eta) = 0. \quad (5.6)$$

As discussed above the potential  $\mathcal{V}(\eta)$  describes the deviation from the slow roll dynamics during a (brief) stage prior to slow roll and is vanishingly small for  $\eta > \bar{\eta}$ , where  $\bar{\eta}$  denotes the beginning of the slow roll stage during which modes of cosmological relevance today exit the Hubble radius.

The retarded Green's function  $G_k(\eta, \eta')$  of Eq. (5.6) for  $\mathcal{V}(\eta) \equiv 0$  obeys

$$\left[ \frac{d^2}{d\eta^2} + k^2 - \frac{\nu^2 - \frac{1}{4}}{\eta^2} \right] G_k(\eta, \eta') = \delta(\eta - \eta'); \quad (5.7)$$

$$G_k(\eta, \eta') = 0 \quad \text{for } \eta' > \eta,$$

it is given by

$$G_k(\eta, \eta') = i[g_\nu(k; \eta)g_\nu^*(k; \eta') - g_\nu(k; \eta')g_\nu^*(k; \eta)] \times \Theta(\eta - \eta'), \quad (5.8)$$

where  $g_\nu(k; \eta)$  is given by Eq. (2.29).

The solution of the mode equation (5.6) can be written as an integral equation using the Green's function equation (5.8)

$$S(k; \eta) = g_\nu(k; \eta) + \int_{-\infty}^0 G_k(\eta, \eta') \mathcal{V}(\eta') S(k; \eta') d\eta'. \quad (5.9)$$

This is the Lippmann-Schwinger equation familiar in potential scattering theory. Inserting Eq. (5.8) into Eq. (5.9) yields

$$S(k; \eta) = g_\nu(k; \eta) + i g_\nu(k; \eta)$$

$$\times \int_{-\infty}^{\eta} g_\nu^*(k; \eta') \mathcal{V}(\eta') S(k; \eta') d\eta' - i g_\nu^*(k; \eta) \times \int_{-\infty}^{\eta} g_\nu(k; \eta') \mathcal{V}(\eta') S(k; \eta') d\eta'. \quad (5.10)$$

This solution has the Bunch-Davies asymptotic condition

$$S(k; \eta \rightarrow -\infty) = \frac{e^{-ik\eta}}{\sqrt{2k}}. \quad (5.11)$$

Since  $\mathcal{V}(\eta)$  vanishes for  $\eta > \bar{\eta}$ , the mode functions  $S(k; \eta)$  for  $\eta > \bar{\eta}$  can be written as a linear combination of the mode functions  $g_\nu(k; \eta)$  and  $g_\nu^*(k; \eta)$  as follows,

$$S(k; \eta) = A(k)g_\nu(k; \eta) + B(k)g_\nu^*(k; \eta), \quad \eta > \bar{\eta}, \quad (5.12)$$

where the coefficients  $A(k)$  and  $B(k)$  can be read from Eq. (5.10),

$$A(k) = 1 + i \int_{-\infty}^0 g_\nu^*(k; \eta) \mathcal{V}(\eta) S(k; \eta) d\eta, \quad (5.13)$$

$$B(k) = -i \int_{-\infty}^0 g_\nu(k; \eta) \mathcal{V}(\eta) S(k; \eta) d\eta.$$

The constancy of the Wronskian  $W[g_\nu(\eta), g_\nu^*(\eta)] = -i$  and Eq. (5.12) imply the relation

$$|A(k)|^2 - |B(k)|^2 = 1.$$

It is clear that the action of a potential  $\mathcal{V}(\eta)$  that vanishes for  $\eta > \bar{\eta}$  is equivalent to setting initial conditions Eqs. (5.12) and (5.13) on the mode functions at  $\eta = \bar{\eta}$  which subsequently evolve during the slow roll stage in which  $\mathcal{V}(\eta) = 0$ . This is one of our main observations.

The integral equation can be solved iteratively in a perturbative expansion if the potential  $\mathcal{V}(\eta)$  is small when compared to  $k^2 - (\nu^2 - 1/4)/\eta^2$ . In such case, we can use for the coefficients  $A(k)$ ,  $B(k)$  the first approximation obtained by replacing  $S(k; \eta')$  by  $g_\nu(k; \eta')$  in the integral equation (5.13). This is the Born approximation, in which

$$A(k) = 1 + i \int_{-\infty}^0 \mathcal{V}(\eta) |g_\nu(k; \eta)|^2 d\eta, \quad (5.14)$$

$$B(k) = -i \int_{-\infty}^0 \mathcal{V}(\eta) g_\nu^2(k; \eta) d\eta.$$

These simple expressions are very illuminating. For asymptotically large  $k$  Eq. (2.32) for the mode functions can be used, and if the potential  $\mathcal{V}(\eta)$  is differentiable and of compact support, an integration by parts yields

$$B(k) \stackrel{k \rightarrow \infty}{\approx} -\frac{1}{4k^2} \int_{-\infty}^0 e^{-2ik\eta} \mathcal{V}'(\eta) d\eta, \quad (5.15)$$

where the prime stands for derivative with respect to  $\eta$ . Therefore, according to the Riemann-Lebesgue lemma,  $N_k = |B(k)|^2 \lesssim 1/k^4$  for large  $k$  and UV convergence in

the integrals for the energy momentum tensor is guaranteed. Hence, an immediate consequence of the explanation of the initial conditions as a scattering problem with a localized potential is that these initial conditions are *automatically* ultraviolet allowed.

To illustrate the main aspects and highlight the main consequences, we consider now two simple potential models for  $\mathcal{V}(\eta)$  localized at  $\eta = \eta_0 < \bar{\eta}$  and characterized by a strength  $v_0$  and width  $\Delta$ . The first one has an exponential profile and the second is a square well. We solve the first one in the Born approximation while the second one is exactly solvable. These exact results agree remarkably well with the simpler Born approximation. Thus, the *exactly solvable* example supports the reliability of the Born approximation in the present framework.

### B. Born approximation

As is clear from the integral equation (5.13) the occupation number  $N(k)$  grows with the strength of the potential  $\mathcal{V}(\eta)$ . Moreover, as shown above, negligible backreaction requires  $N(k) \ll 1$  for wave vectors of cosmological relevance. This is the regime where the Born approximation equation (5.14) is reliable.

To leading order in slow roll we consider the scale invariant case,  $\nu = \frac{3}{2}$  with the mode functions given by Eq. (2.34), and model a potential localized at a time scale  $\eta_0$  by

$$\mathcal{V}(\eta) = \frac{v_0}{\sqrt{\pi}} e^{-[(\eta - \eta_0)/\Delta]^2}, \quad (5.16)$$

where  $\bar{\eta} = \eta_0 + \Delta$ . The localization length must be  $|\Delta| \ll |\eta_0|$  such that the potential must not influence the dynamics during slow roll, and  $v_0$  must be small for the Born approximation to be valid. More precisely  $|v_0 \Delta| \ll k$ , as seen below. Under these conditions we find,

$$A(k) = 1 + i \frac{v_0 |\Delta|}{2k} \left[ 1 + \frac{1}{(k\eta_0)^2} + \mathcal{O}\left(\left|\frac{\Delta}{\eta_0}\right|^2\right) \right], \quad (5.17)$$

$$B(k) = -i \frac{v_0 |\Delta|}{2k} e^{-(k\Delta)^2} e^{-2ik\eta_0} \left[ 1 - \frac{1}{(k\eta_0)^2} - \frac{2i}{(k\eta_0)} + \mathcal{O}\left(\left|\frac{\Delta}{\eta_0}\right|^2\right) \right]. \quad (5.18)$$

To lowest order in  $v_0$  the number of produced BD modes  $N_k$  and the transfer function  $D(k)$  are given by

$$N_k = \left( \frac{v_0 |\Delta|}{2k} \right)^2 e^{-2(k\Delta)^2} \left[ 1 + \frac{1}{(k\eta_0)^2} \right]^2, \quad (5.19)$$

and

$$D(k) = -\frac{v_0 |\Delta|}{k} e^{-(k\Delta)^2} \left[ \sin(2k|\eta_0|) \left( 1 - \frac{1}{(k\eta_0)^2} \right) + \frac{2 \cos(2k|\eta_0|)}{k|\eta_0|} \right], \quad (5.20)$$

respectively. The particle number  $N_k$  clearly falls off faster than any power at large  $k$ , thereby ensuring the ultraviolet convergence of the integrals in the energy momentum tensor.

This example reveals that a potential that is localized near a (conformal) time scale  $\eta_0$  results in a *phase difference*  $\sim e^{-2ik\eta_0}$  between the Bogoliubov coefficients  $A(k)$ ,  $B(k)$ . This is a general result that stems directly from the general expressions for these coefficients Eq. (5.14) and that in turn results in the oscillatory component of the transfer function  $D(k)$ .

In the integral equation (4.5) that yields the coefficients  $\Delta C_l/C_l$ , the transfer function  $D(k)$  multiplies a function that is strongly peaked at  $x \sim l$ , namely, for momenta  $k \sim l\kappa$ . Therefore, if  $k|\eta_0| \sim l\kappa|\eta_0| \gg 1$ , the rapid oscillations in  $D(k)$  average out in the integrand, resulting in a vanishing contribution to the  $\Delta C_l/C_l$ 's. Hence, there are significant contributions to  $\Delta C_l/C_l$  only when  $l\kappa|\eta_0| \sim 1$ . For the quadrupole this corresponds to  $[a_0 H_0 |\eta_0|] \sim 1$ .

The potential  $\mathcal{V}(\eta)$  acts prior to the slow roll stage during which cosmologically relevant modes cross the Hubble radius. For the corrections to the low multipoles to be substantial, the condition for wave vectors corresponding to the Hubble radius today is  $k \sim a_0 H_0$  and  $k|\eta_0| \sim 1$ . The conclusion is that modifications to the low multipoles arise from a potential  $\mathcal{V}(\eta)$  localized just prior to horizon crossing of the modes whose wavelengths correspond to the Hubble radius today. It is also clear that the corrections for higher wave vectors are strongly suppressed because of the rapid falloff of  $B(k)$ .

Furthermore, in the Born approximation the sign of the correction  $\Delta C_l/C_l$  is determined by the sign of the potential. In the example given above with the potential equation (5.16), it is given by the sign of  $v_0$ , negative (positive)  $v_0$  corresponding to an attractive (repulsive) potential. Figure 1 shows the quadrupole correction  $\Delta C_2/C_2$  determined by the transfer function equation (5.20) for an attractive potential ( $v_0 = -|v_0|$ ) clearly revealing a suppression for  $\kappa|\eta_0| \sim 1$ .

The corrections for the higher multipoles are substantially smaller, vanishing very rapidly for  $l \geq 3$  as shown in Fig. 2 for the attractive potential.

### C. Exactly solvable potential

The Born approximation is the first order in perturbation theory and is valid in the regime  $k \gg |v_0|$ . A simple and exactly solvable example is the square well potential

$$\mathcal{V}(\eta) = \begin{cases} -|v_0| & \text{for } \eta_1 \leq \eta \leq \eta_2, \\ 0 & \text{otherwise} \end{cases}, \quad (5.21)$$

where  $\eta_1 \equiv \eta_0 - \frac{\Delta}{2}$  and  $\eta_2 \equiv \eta_0 + \frac{\Delta}{2}$ ,  $\Delta > 0$  is the width of the potential well and  $-|v_0|$  its depth. The case of a potential barrier is obtained by the replacement  $|v_0| \rightarrow -|v_0|$ .



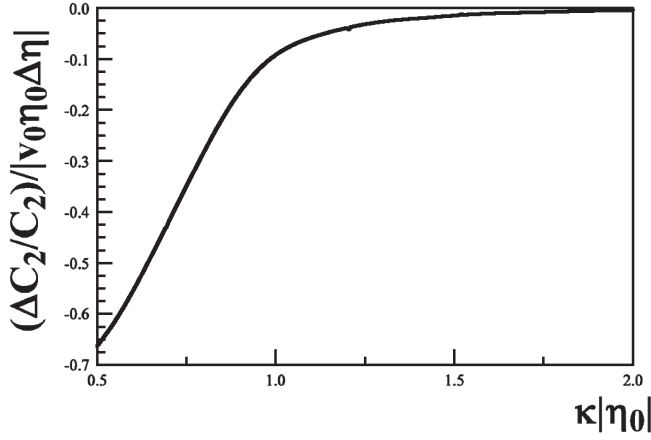


FIG. 1. The quadrupole correction  $(\Delta C_2/C_2)/|v_0\eta_0\Delta\eta|$  vs  $\kappa|\eta_0|$  in the Born approximation for an attractive potential of the form (5.16), for  $|\Delta/\eta_0| = 0.01, 0.05, 0.1$ . Here,  $\kappa = \frac{a_0 H_0}{3.3}$ . It clearly reveals a suppression for  $\kappa\eta_0 \sim 1$ , that is for the modes whose wavelengths correspond to the Hubble radius today.

We compute the mode functions by matching the functions and derivatives as in the familiar case of the step-potential in the one-dimensional Schrödinger equation, and again to lowest order in slow roll we consider the scale invariant case  $\nu = 3/2$ . The wave function is given by

$$S(k, \eta) = \begin{cases} g(k\eta), & \eta < \eta_1 \\ E(k)g(q\eta) + Q(k)g^*(q\eta), & \eta_1 \leq \eta \leq \eta_2 \\ A(k)g(k\eta) + B(k)g^*(k\eta), & \eta_2 < \eta < 0 \end{cases} \quad (5.22)$$

where

$$g(k\eta) = e^{-ik\eta} \left[ 1 - \frac{i}{k\eta} \right]; \quad q = \sqrt{k^2 + |v_0|}. \quad (5.23)$$

Matching functions and derivatives at  $\eta = \eta_1$  and  $\eta =$

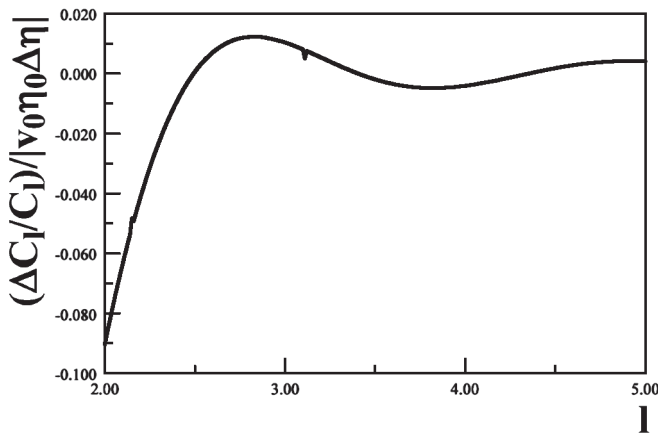


FIG. 2. The corrections  $(\Delta C_l/C_l)/|v_0\eta_0\Delta\eta|$  vs  $l$  in the Born approximation for an attractive potential of the form (5.16), for  $\kappa|\eta_0| = 1$ ;  $|\Delta/\eta_0| = 0.01$ . The corrections for the higher multipoles are substantially smaller than the quadrupole correction, vanishing very rapidly for  $l \geq 3$ .

$\eta_2$ , we find

$$A(k) = \frac{e^{ik\Delta}}{4kq} \left\{ e^{-iq\Delta}(k+q)^2 \left[ 1 + i \frac{k-q}{kq\eta_1} \right] \left[ 1 - i \frac{k-q}{kq\eta_2} \right] - e^{iq\Delta}(k-q)^2 \left[ 1 - i \frac{k+q}{kq\eta_1} \right] \left[ 1 + i \frac{k+q}{kq\eta_2} \right] \right\},$$

$$B(k) = -|v_0| \frac{e^{-2ik\eta_0}}{4kq} \left\{ e^{-iq\Delta} \left[ 1 + i \frac{k-q}{kq\eta_1} \right] \left[ 1 - i \frac{k+q}{kq\eta_2} \right] - e^{iq\Delta} \left[ 1 - i \frac{k+q}{kq\eta_1} \right] \left[ 1 + i \frac{k-q}{kq\eta_2} \right] \right\}. \quad (5.24)$$

It is straightforward to check the unitarity condition  $|A(k)|^2 - |B(k)|^2 = 1$ . We consider an arbitrary depth  $|v_0|$  to include nonperturbative aspects, but focus on the case of a narrow well for which

$$\left| \frac{\Delta}{\eta_0} \right| \ll 1. \quad (5.25)$$

The number of particles created by the pre-slow roll stage is

$$N_k = |B(k)|^2 = \frac{(v_0\eta_0^2)^2}{4(k\eta_0)^2} \left| \frac{\Delta}{\eta_0} \right|^2 \left\{ \left[ \frac{\sin q\Delta}{q\Delta} \left( 1 - \frac{|v_0|}{(kq\eta_0)^2} \right) - \frac{\cos q\Delta}{(q\eta_0)^2} \right]^2 + \left[ \frac{2}{k\eta_0} \frac{\sin q\Delta}{q\Delta} \right]^2 \right\}. \quad (5.26)$$

For arbitrary strength of the potential  $v_0$ , a small number of produced particles requires a narrow width (5.25).

For  $k \gg |v_0|$  the number of particles is

$$N_k \xrightarrow{k \rightarrow \infty} \frac{v_0^2}{8k^4}. \quad (5.27)$$

Moreover, a continuous and differentiable potential  $\mathcal{V}(\eta)$  with smooth edges will yield a  $N_k$  vanishing faster than  $1/k^4$  for large  $k$  since the Fourier transform of a continuous and differentiable function of compact support falls off exponentially at large  $k$ . Therefore, the asymptotic behavior for a general continuous potential with a typical scale  $v_0$  is  $N_k < v_0^2/k^4$  and the ultraviolet finiteness of the energy momentum tensor is guaranteed [21].

To leading order in the 'narrow width' approximation the transfer function is

$$D(k) = \frac{|v_0|\eta_0^2}{k|\eta_0|} \left| \frac{\Delta}{\eta_0} \right| \left\{ \sin(2k|\eta_0|) \left[ \frac{\sin q\Delta}{q\Delta} \left( 1 - \frac{|v_0|\eta_0^2}{(kq\eta_0^2)^2} \right) - \frac{\cos q\Delta}{(q\eta_0)^2} \right] + 2 \frac{\cos 2k\eta_0}{k|\eta_0|} \frac{\sin q\Delta}{q\Delta} \right\}, \quad (5.28)$$

where we have written the transfer function explicitly in terms of the relevant dimensionless combinations of parameters  $v_0$ ,  $\Delta$ , and  $\eta_0$ .

We have performed an exhaustive numerical study of the corrections to the  $C'_l$ s in a wide range of the dimensionless parameters  $\kappa|\eta_0|$ ,  $|v_0|\eta_0^2$ , and  $|\Delta/\eta_0|$  where  $\kappa$  is defined in Eq. (4.6), with the conclusion that for an attractive poten-

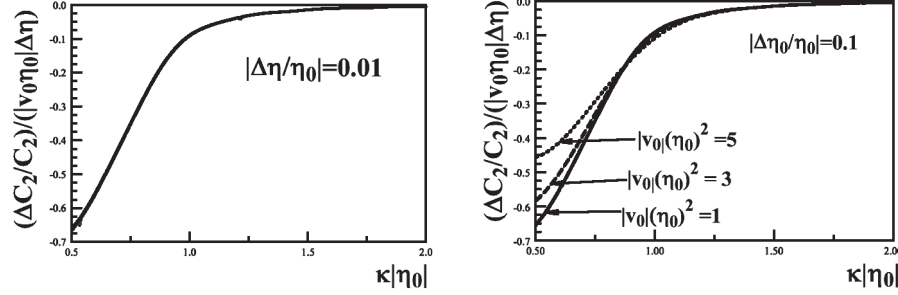


FIG. 3.  $(\Delta C_2/C_2)/(v_0\eta_0\Delta)$  vs  $\kappa|\eta_0|$  for the square well potential, for  $|v_0|\eta_0^2 = 1, 3, 5$ ;  $|\Delta/\eta_0| = 0.01$  (left panel) and  $|\Delta/\eta_0| = 0.1$  (right panel). There is a substantial suppression of the quadrupole when the potential is localized at a time scale  $\eta_0 \sim 1/[a_0 H_0]$ . This time scale is approximately 55 e-folds before the end of inflation when the wavelengths corresponding to today's Hubble radius exited the horizon.

tial there is a substantial suppression of the quadrupole for  $\kappa|\eta_0| \sim 1$  and that the corrections for higher multipoles are negligibly small and observationally irrelevant since these fall off as  $1/l^2$ , hence much smaller than the irreducible cosmic variance.

Figure 3 displays the changes in the quadrupole for various representative values of  $|v_0|\eta_0^2$  for the cases  $|\Delta/\eta_0| = 0.01, 0.1$ , respectively. Figure 4 shows that the suppression of the higher multipoles  $l > 2$  is negligibly small. It is clear from these figures that there is a substantial suppression of the quadrupole if the potential is localized at a time scale  $|\eta_0| \sim 1/[a_0 H_0]$ . This time scale approximately coincides with 55 e-folds before the end of inflation when the wavelengths corresponding to today's Hubble radius exited the horizon.

Furthermore, for localization scales of the potential  $0.05 \leq |\Delta/\eta_0| \leq 0.1$ , a 10%–20% suppression of the quadrupole is obtained for  $|v_0| \sim \eta_0^{-2}$ . Therefore a substantial suppression of the quadrupole is explained quite naturally within the effective field theory of inflation with a pre-slow roll potential of scale  $v_0 \sim \eta_0^{-2}$ .

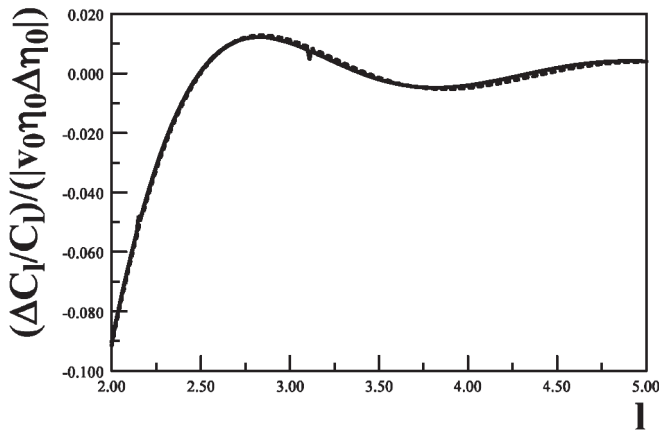


FIG. 4.  $(\Delta C_l/C_l)/(v_0\eta_0\Delta)$  vs  $l$  for the square well potential, for  $|v_0|\eta_0^2 = 1$ ;  $\kappa|\eta_0| = 1$  and  $|\Delta/\eta_0| = 0.01, 0.1$ . We see that the suppression of higher multipoles are negligibly small (they fall off as  $1/l^2$ ) and observationally irrelevant.

These exact results agree remarkably well with the simpler Born approximation within the range of parameters consistent with a small number of particles as required by the small backreaction condition. Therefore, this exactly solvable example lends support to the statement that the Born approximation is indeed robust and describes remarkably well the main corrections from the potential  $\mathcal{V}(\eta)$ .

These results apply equally well to curvature and tensor perturbations. Therefore, this analysis leads to the *prediction* that the quadrupole of *tensor* perturbations will also feature a suppression.

## VI. CONCLUSIONS

In this article we studied the effect of initial conditions on metric and tensor perturbations with emphasis on the observational consequences of initial conditions consistent with renormalizability and small backreaction. Generalized initial conditions for the mode functions of gauge invariant perturbations are encoded in Bogoliubov coefficients, or equivalently in distribution functions of Bunch-Davies quanta. We begun the study by clarifying the constraint on the Bogoliubov coefficients from the general restrictions of renormalizability and negligible backreaction on the energy momentum tensor of gauge invariant perturbations. These general criteria constrain the asymptotic behavior for large wave vectors of the Bogoliubov coefficients up to  $1/k^4$ . We find that the modifications to the power spectra of gauge invariant perturbations due to general initial conditions are encoded in a *transfer function for initial conditions*  $D(k)$ . Our main results are summarized as follows:

- (i) General arguments based on the asymptotic behavior of the Bogoliubov coefficients at large wave vector show that only the *low* multipoles, those in the Sachs-Wolfe plateau, are sensitive to initial conditions allowed by renormalizability and small backreaction. Effects upon high multipoles are strongly suppressed by the rapid falloff of the Bogoliubov coefficients at large wave vectors  $k$ . We compute the change in the quadrupole due to generic initial con-

- ditions described by the asymptotic limit of the Bogoliubov coefficients. A substantial change of the order 10%–20% on the CMB quadrupole is found when the momentum scale at which the asymptotic behavior sets corresponds to the physical wavelength of the order of the Hubble radius *today*.
- (ii) We show that mode functions with general initial conditions determined before the slow roll stage are equivalent to those that result from scattering by the potential  $\mathcal{V}(\eta)$  arising from the cosmological evolution just prior to the onset of slow roll. The influence of initial conditions upon the power spectra of curvature and tensor perturbations is encoded in a transfer function  $D(k)$ .
  - (iii) Implementing methods from scattering theory, we develop the formulation of initial conditions arising from scattering by a potential  $\mathcal{V}(\eta)$  and obtain the transfer function of initial conditions  $D(k)$  in terms of this potential. The changes in the low multipoles are studied both in the Born approximation and in an exact solvable case for  $\mathcal{V}(\eta)$ , with complete agreement between the results in both cases. The transfer function for initial conditions is shown to have the asymptotic large  $k$  behavior consistent with renormalizability and negligible backreaction.
  - (iv) Furthermore, this study reveals that attractive potentials lead to a suppression of the quadrupole with a value consistent with the WMAP data if the potential is localized at a time scale  $\eta_0$  very near the scale at

which the wavelength corresponding to the Hubble radius today exits the horizon during inflation, with a strength  $\mathcal{V}(\eta_0) \sim 1/\eta_0^2$ . The change in the  $l$ -multipole falls off as  $1/l^2$  as a consequence of the falloff of the Bogoliubov coefficients for large  $k$ . This entails that only the quadrupole features an observable suppression, while the corrections in higher multipoles are not observable with the present data.

- (v) Our study applies to curvature and tensor perturbations, hence we predict a suppression of quadrupole for  $B$ -modes for an attractive potential localized prior to slow roll.

In the companion article [30] we show that the potential  $\mathcal{V}(\eta)$  which determines the initial conditions for the fluctuations in the slow roll stage is a general feature of a stage of fast roll inflaton dynamics followed by slow roll. Under general circumstances this potential turns out to be attractive and results in a suppression on the CMB quadrupole of the order  $\sim 10\%$ – $20\%$  consistent with the observations.

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