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## TWO-TRACK DEPICTIONS OF LEIBNIZ'S FICTIONS

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ABSTRACT. Leibniz described imaginary roots, negatives, and infinitesimals as useful fictions. But did he view such ‘impossible’ numbers as mathematical entities? Alice and Bob take on the labyrinth of the current Leibniz scholarship.

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### 1. WHAT WAS LEIBNIZ’S TAKE ON ‘IMPOSSIBLE’ NUMBERS?

Leibniz described imaginary roots, negatives, and infinitesimals as useful fictions. But did he view such ‘impossible’ numbers as mathematical entities? Did he envision a violation of the Archimedean axiom? And what were his ‘bounded infinities’? Can a person of infinite age have been born? Did mathematical existence have comparable meaning to Leibniz as to Hilbert?

Numbers that we take for granted today, such as negatives, irrationals, imaginary roots, and infinitesimals, go beyond the conceptual

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world of Greek mathematicians.<sup>1</sup> In a sense such numbers are impossible, or fictional.

Many 17th century pioneers saw their task as either explaining, or expanding upon, Greek mathematics. A typical case is Fermat's reconstruction of Apollonius' *Plane Loci* (see [30], [12] for a discussion). In particular, they had to justify the status of certain 'fictions' admitting no geometric representation. Throughout his mathematical career, Leibniz argued for the virtues of expanding the scope of 'quantity' to include negatives, imaginaries, and infinitesimals. It is, however, not always easy to discern the precise nature of Leibniz's attitude toward expanding the conceptual resources of mathematics.

Leibniz saw Galileo's paradox of the infinite<sup>2</sup> as an indication that the concept of an *infinite whole* is contradictory because it contradicts the part-whole principle.<sup>3</sup> Leibniz's view is in sharp contrast with the modern one, comfortable with the concept of an infinite cardinality. Leibniz often used 'infinite number' in the sense of what we would refer to today as *infinite cardinality* (contradictory in Leibniz's view), indicating that he used the term 'infinite number' in a generalized sense. When he spoke of the reciprocals of infinitesimals used in his calculus, he tended to use either 'infinite quantity' or *infinitem terminatum* (bounded infinity) rather than 'infinite number', though occasionally he used the latter term as well, as when he defined an infinitesimal as an "infinitely small fraction, or one whose denominator is an infinite number."<sup>4</sup>

In his 1683 *Elementa Nova matheseos universalis*, Leibniz explained that some mathematical operations cannot be performed in actuality, but nonetheless one can exhibit 'a construction in our characters' (*in nostris characteribus* [16, p. 520]), meaning that one can carry out a formal calculation, such as those with imaginary roots. Just as Leibniz is pushing the envelope by extending the domain of quantities to include inassignable ones, he is pushing the envelope by extending the

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<sup>1</sup>Some prominent 20th century scholars were still opposed to, or uncomfortable with, both irrationals and infinitesimals. Thus, Errett Bishop opposed both the classical development of the real numbers and the use of infinitesimals in teaching calculus [5]. For a discussion, see [11], [9], [10], [31].

<sup>2</sup>Galileo observed that the natural numbers admit a one-to-one correspondence with the squares.

<sup>3</sup>The principle asserts that a (proper) part is smaller than the whole. The principle goes back to Euclid (*Elements* I, common notion 8).

<sup>4</sup>"fraction infiniment petite, ou dont le denominateur soit un nombre infini" [20, p. 93].

meaning of ‘construction’ to include a mental operation using ‘our characters’.

Leibniz referred to infinitesimals as *fictional entities*. But what is the precise meaning of that expression? The crux of the matter is whether Leibniz viewed infinitesimals as mathematical entities.

Alice holds that the Leibnizian term ‘infinitesimal’ does not refer to a mathematical entity,<sup>5</sup> and sees Leibniz’s expression *fictional entities* as including terms that only seem to refer to mathematical entities but in actuality do not. Alice reads the epithet *fictional* as undermining the noun *entity*. Furthermore, to Alice ‘infinitesimals’ do not refer to mathematical entities because such would be inconsistent, i.e., contradictory, and more specifically contrary to the part-whole axiom.

Bob holds that infinitesimals *are* mathematical entities, and interprets the expression *fictional entities* as describing entities of a special kind, namely *fictional*. Bob reads the epithet *fictional* not as undermining but as delimiting the meaning of *entity*, and views these mathematical entities as consistent, as any mathematical entity would have to be; in particular, they do not contradict the part-whole axiom. Bob holds that their fictionality references the fact that they are merely accidentally impossible (in accordance with the Leibnizian philosophy of knowledge) but nonetheless consistent, and therefore legitimate mathematical entities in Leibniz’s view.

## 2. LAW AND *fictio juris*

Did Leibniz view the term ‘infinitesimal’ as tied up with contradiction? Alice cites as evidence the fact that Leibniz sometimes used contradictory notions in jurisprudence.<sup>6</sup>

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<sup>5</sup>“[Numbers or ideal entities] are entities that are referred to. Fictions, on the other hand, are not entities to which we refer. *They are not abstract entities*” [8, p. 100] (emphasis added).

“Leibniz conceived of infinitely small as *compendia cogitandi* for proofs and discovery and *not as genuine mathematical entities*” [28, p. 360] (emphasis added).

“[R]eference to the infinite and infinitely small *does not amount to the acceptance of genuine infinite entities*, but is a ‘way of speaking’ referring ultimately to the only existing mathematical quantities, that is, finite quantities” [29, p. 441] (emphasis added).

<sup>6</sup>“[E]ven though its concept [infinitesimal] may contain a contradiction, it can nevertheless be used to discover truths, provided a demonstration can (in principle) be given to show that its being used according to some definite rules will avoid contradiction. This strategy of using ‘fictions’ is not limited to mathematics and was very widespread in Law, the discipline which Leibniz first learned as a student, where it took the form of the *fictio juris*” ([29, p. 407]; emphasis in the original).

Bob argues that jurisprudence fails to provide convincing evidence as far as Leibniz’s mathematical practice is concerned. Bob holds that non-contradiction was the very foundation of the mathematical method for Leibniz<sup>7</sup> (see also Section 8), barring any inference from legal usage.

### 3. REFERENCE TO VIOLATION OF EUCLID V.4

Infinitesimals, as usually conceived, involve a violation of the Archimedean property. One can therefore ask whether Leibniz ever alluded to such a violation in writing. In fact, Leibniz wrote in a 14/24 June 1695 letter to l’Hospital:

I use the term *incomparable magnitudes* to refer to [magnitudes] of which one multiplied by any finite number whatsoever, will be unable to exceed the other, in the same way [adopted by] Euclid in the fifth definition of the fifth book [of *The Elements*]. [17]

In modern editions of *The Elements*, the notion of comparability appears in Book V, Definition 4.<sup>8</sup> A similar discussion of incomparability in the context of Euclid’s definition appears in a 1695 publication of Leibniz’s [18] in response to Nieuwentijt’s criticism.

Alice reads the Leibnizian reference to Euclid’s Definition V.4, and the violation thereof by infinitesimals when compared to ordinary magnitudes, as merely a ‘nominal definition’.<sup>9</sup> Alice quotes Leibniz to the effect that nominal definitions could harbor contradictions. Alice holds that the true meaning of infinitesimals resides in the Archimedean exhaustion-style unwrapping of ostensibly infinitesimal arguments.<sup>10</sup>

Bob argues that Archimedean paraphrases in exhaustion style constitute an *alternative* method rather than an unwrapping of the infinitesimal method.<sup>11</sup> Bob notes that, while Leibniz warned that nominal definitions *may* harbor contradictions, there is no indication that they

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<sup>7</sup>See [4, § 3.4].

<sup>8</sup>“Magnitudes which when multiplied can exceed one another are said to have a ratio to one another” [Transl. Mueller]. A system of magnitudes satisfying Euclid V def. 4 is said to be Archimedean, in connection with the so-called Archimedean axiom; in modern notation: “for every  $A$  and every  $B$ , a multiple  $nA$  of  $A$  exists such that  $nA > B$ ”.

<sup>9</sup>“If one wants to infer existence, one cannot just rely on the nominal definition of ‘incomparables’ (as not respecting the definition of Archimedean quantities)” [29, p. 433].

<sup>10</sup>“The strict proof operating only with assignable quantities justifies proceeding by simply appealing to the fact that  $dv$  is incomparable with respect to  $v$ : in keeping with the *Archimedean axiom*, it can be made so small as to render any error in neglecting it smaller than any given” [2, pp. 567–568].

<sup>11</sup>See [4, § 1.3].

*must* do so; hence, regardless of whether one interprets the violation of Euclid's Definition V.4 as a 'nominal' move, infinitesimals can still be consistent mathematical entities.

#### 4. FICTIONS, USEFUL FICTIONS, AND WELL-FOUNDED FICTIONS

Do fictions involve contradictions? Some Leibnizian texts shed light on the matter. In 1674, Leibniz analyzed the area under the hyperbola, and concluded that

“the infinite is not a whole, but only a fiction, since otherwise the part would be equal to the whole” [22], A VII 3, 468; october 1674.

Alice quotes this text as evidence that Leibniz uses the term *fiction* to refer to a contradictory infinite whole.<sup>12</sup>

Bob points out that, while Leibniz does use the term *fiction* in this analysis of an infinite whole, he never refers to such contradictory notions as either *useful* or *well-founded* fictions; meanwhile, Leibniz does describe infinitesimals as both useful fictions and well-founded fictions.<sup>13</sup> Furthermore, Leibniz did not actually write that an infinite whole was a fiction, contrary to Alice's inference. Leibniz wrote that ‘the infinite is not a whole, but only a fiction’. That is not the same as saying that an infinite whole is a fiction. Therefore the inference from the 1674 passage is inconclusive.

#### 5. INFINITE CARDINALITIES AND INFINITE QUANTITIES

Alice and Bob have argued about both the meaning of ‘infinite number’ in Leibniz, and his distinction between *infinita terminata* (bounded infinities) and *infinita interminata* (unbounded infinities). One of the main sources for this Leibnizian distinction is his *De Quadratura Arithmetica* [25].

Leibniz's writings contain many speculations about the paradoxical behavior of the *infinita terminata*. For example, Leibniz mentioned the

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<sup>12</sup>The Leibnizian passage is quoted as evidence in [2] and [29] as follows:

“Even though this establishes the *fictional nature of such infinite wholes*, however, this does not mean that one cannot calculate with them; only, the viability of the resulting calculation is contingent on the provision of a demonstration” ([2, p. 557]; emphasis added);

“Here, the infinite area is that between the hyperbola and its asymptote (bounded on one side), and Leibniz argues that since taking it as a true whole leads to contradiction with the axiom that the whole is greater than its (proper) part, it should instead be regarded as a *fiction*” ([29, p. 405]; emphasis added).

<sup>13</sup>See [14], [32], [3].

allegory of somebody of infinite age who nonetheless was born; somebody who lives infinitely many years and yet dies [1, p. 51]. According to Leibniz, the kind of infinite quantities one obtains by inverting infinitesimals is *infinita terminata*, as in the example of an infinite-sided polygon. Bob argues that these ideas seem difficult today because of the prevalence of a post-Weierstrassian mindset in traditional mathematical training.<sup>14</sup>

Alice quotes passages where Leibniz argues that infinite wholes are contradictory because contrary to the part-whole axiom.<sup>15</sup><sup>16</sup> Alice holds that *infinite number* necessarily means *infinite whole*, infinitesimals are their inverses, and therefore all are contradictory.

Bob analyzes the Leibnizian distinction between bounded infinity and unbounded infinity, and points out that the latter is akin to cardinality.<sup>17</sup> The former are the inverses of infinitesimals, constitute a notion distinct from cardinalities, and involve no contradiction.<sup>18</sup> Bob holds that the expression *infinite number* in Leibniz is ambiguous and could refer either to cardinalities (contradicting the part-whole axiom), or to (noncontradictory) *infinita terminata*. For a modern illustration of *infinita terminata* see Section 6.

## 6. BOUNDED INFINITIES FROM LEIBNIZ TO SKOLEM

We provide a modern formalisation of Leibniz’s *infinita terminata* in terms of the extensions of  $\mathbb{N}$  developed by Skolem in 1933 [33]. Such an extension, say  $M$ , satisfies the axioms of Peano Arithmetic (and in this sense is indistinguishable from  $\mathbb{N}$ ). Yet  $M$  is a *proper* extension, of which  $\mathbb{N}$  is an initial segment. Such models are sometimes referred to as nonstandard models of arithmetic; see e.g., [15]. Each element of the complement

$$M \setminus \mathbb{N}$$

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<sup>14</sup>See Section 6 for Skolem’s formalisation of the idea of an infinite integer.

<sup>15</sup>“[Leibniz] argued in some critical comments on Galileo’s *Discorsi* in 1672 that the part-whole axiom must be upheld even in the infinite. It follows that it is impossible to regard ‘all the numbers’ and ‘all the square numbers’ as true wholes, since then the latter would be a proper part of the former, and yet equal to it, yielding a contradiction” [29, pp. 405–406].

<sup>16</sup>Arguably, Leibniz in fact possessed the means to see that the part-whole axiom and the existence of infinite wholes are not incompatible [34].

<sup>17</sup>See [4, § 2.2].

<sup>18</sup>“[U]nlike the infinite number or the number of all numbers, for Leibniz infinitary concepts do not imply any contradiction, although they may imply paradoxical consequences” [6, Section 7].

is greater than each element of  $\mathbb{N}$  and in this sense can be said to be infinite.

Notice that, depending on the background logical system, one can view Skolem's extensions as either 'potentially' or 'actually' infinite (of course in the former case neither  $\mathbb{N}$  nor  $M$  exists as a completed whole). The sense in which elements of  $M \setminus \mathbb{N}$  are 'infinite' is unrelated to the Aristotelian distinction. An element of  $M \setminus \mathbb{N}$  provides a modern formalisation of the *infinita terminata*.

## 7. LEIBNIZ'S REBUTTAL OF BERNOULLI'S INFERENCE FROM SERIES

In a 24 february/6 march 1699 letter to Bernoulli [19], Leibniz noted that the infinity of terms in a geometric progression does not prove the existence of infinitesimals:

You do not reply to the reason which I have proposed for the view that, given infinitely many terms, it does not follow that there must also be an infinitesimal term. This reason is that we can conceive an infinite series consisting merely of finite terms or of terms ordered in a decreasing geometric progression. I concede the infinite plurality of terms, but this plurality itself does not constitute a number or a single whole. [23, p. 514]

Leibniz used the distinction between a plurality and an infinite whole to refute Bernoulli's attempted inference from the existence of infinite series to the existence of infinitesimals, and reiterated his position against viewing an infinite plurality as a whole (see Section 5).

Alice argues that Leibniz's exchange with Bernoulli about infinite series shows that Leibniz viewed infinitesimals and infinite quantities as contradictory.<sup>19</sup>

Bob notes that Leibniz stresses the distinction between infinite cardinality and infinite quantity (reciprocal of infinitesimals). Bob argues that the exchange with Bernoulli precisely refutes Alice's attempt to blend infinite cardinality and infinite quantity so as to deduce the inconsistency of infinitesimals. Bob holds that Leibniz didn't blend cardinality and quantity: only Alice did. Leibniz, on the contrary, emphasized the distinction in order to refute Bernoulli's inference. Bob holds that Leibniz's rebuttal of Bernoulli's inference does a serviceable

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<sup>19</sup>“This remains Leibniz's position into his maturity and both arguments are to be found, for example, in the correspondence with Bernoulli in 1698 . . . That is, he held that the part-whole axiom is constitutive of quantity, so that the concept of an infinite quantity, such as an infinite number or an infinite whole, involves a contradiction” [29, p. 406].



job of refuting Alice’s inference concerning a purported inconsistency of fictional entities, as well.

## 8. MATHEMATICAL POSSIBILITY

Among Leibniz’s preparatory material for his *Characteristica Universalis*, we find the following definition of possible, dating approximately from 1678:

A possible thing is that which does not imply a contradiction.<sup>20</sup>

The same definition appears in many writings, as for instance the 24 february/6 march 1699 letter to Bernoulli analyzed in Section 7, where Leibniz wrote:

Possible things are those which do not imply a contradiction.<sup>21</sup>

If even in the broader framework of the *Characteristica Universalis*, a thing is possible as soon as it causes no contradiction, then certainly in the narrower mathematical context, the absence of contradiction is sufficient to guarantee that the thing is possible. And in fact, that Leibniz meant the principle of non-contradiction to apply to mathematics is evident from his second letter to Clarke, from 1715:

The great foundation of mathematics is the *principle of contradiction or identity*, that is, that a proposition cannot be true and false at the same time, and that therefore  $A$  is  $A$  and cannot be not  $A$ . This single principle is sufficient to demonstrate every part of arithmetic and geometry, that is, all mathematical principles.<sup>22</sup> [26, p. 7]

In itself, the identification of possibility with the principle of non-contradiction is not a novelty, as already in his *Summa theologiae*, Thomas Aquinas clearly explained the major consequences of this assumption:

But what implies contradiction is not submitted to divine omnipotence, because it cannot bear the qualification of possible.<sup>23</sup>

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<sup>20</sup>“Possibile est quod non implicat contradictionem” A VI-2 p. 495. The definition is an addition made in 1678 to a text dating from 1671–1672 [ibid., p. 487].

<sup>21</sup>“Possibilia sunt quae non implicant contradictionem” [19].

<sup>22</sup>“Le grand fondement des Mathematiques est le *Principe de la Contradiction*, ou de l’*Identité*, c’est à dire, qu’une Enontiation ne sauroit etre vraye et fausse en même temps, et qu’ainsi  $A$  est  $A$ , et ne sauroit etre non  $A$ . Et ce seul principe suffit pour demontrer toute l’Arithmetique et toute la Geometrie, c’est à dire tous les Principes Mathematiques” (Leibniz [21], 7:355–356).

<sup>23</sup>“Ea vero quae contradictionem implicant, sub divina omnipotentia non continentur, quia non possunt habere possibilium rationem” (*Summa theologiae*, I, q. 25, a. 3).

But while in the Middle Ages possibility, and hence non-contradiction, was deemed to be a necessary condition for the existence of an entity, but not a sufficient one (not every possibility is actualized), Bob argues that in Leibniz's mathematics the condition is also sufficient: mathematical existence is equivalent to mathematical possibility, and the latter is wholly determined by a (global) principle of non-contradiction. Of course, this is not the case in physics, so that Leibniz can introduce the notion of accidental impossibilities, namely notions that are possible – and hence they exist in mathematics – but not necessarily instantiated in *rerum natura*. Accordingly, Leibniz held a non-contradiction view of mathematical existence that can be seen as an early antecedent of Hilbert's Formalism.<sup>24</sup> Bob argues that, to be usable in mathematics, a concept must first and foremost be non-contradictory, and that Leibniz's letter undercuts Alice's claim that Leibniz viewed infinitesimals as contradictory.

### 9. A-TRACK AND B-TRACK

Alice (A) and Bob (B) represent a pair of rival depictions in the scholarly debate concerning the interpretation of Leibniz's fictional quantities such as infinitesimals and their reciprocals. A similar debate exists with regard to Cauchy's infinitesimal analysis [13].

On the A-track reading, these quantities, just like infinite wholes violating the part-whole axiom, were contradictory concepts; the expression *fictional entities* describing them harbors a contradiction. Consequently, this reading denies that infinitesimals were the very basis of the Calculus; formulations that use them were merely figures of speech, abbreviating the Archimedean unwrappings thereof.

On the B-track reading, what Leibniz viewed as contradictory were only infinite wholes (involving a contradiction with the part-whole axiom), but not infinite and infinitesimal quantities. The latter were useful and well-founded fictions involving a violation of the Archimedean property. Their legitimacy as mathematical entities stemmed from their consistency, in an early anticipation of Hilbert's Formalism.

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<sup>24</sup>This observation was first made by Dietrich Mahnke, writing contemporaneously with the development of Hilbert's formalism. See e.g., [27, pp. 284–287].

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