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# AZUMAYA ALGEBRAS AND OBSTRUCTIONS TO QUADRATIC PAIRS OVER A SCHEME

PHILIPPE GILLE, ERHARD NEHER, AND CAMERON RUETHER

**Abstract:** We investigate quadratic pairs for Azumaya algebras with involutions over a base scheme  $S$  as defined by Calmès and Fasel, generalizing the case of quadratic pairs on central simple algebras over a field (Knus, Merkurjev, Rost, Tignol). We describe a cohomological obstruction for an Azumaya algebra over  $S$  with orthogonal involution to admit a quadratic pair and provide a classification of all quadratic pairs it admits. When  $S$  is affine this obstruction vanishes, however it is non-trivial in general. In particular, we construct explicit examples with non-trivial obstructions.

**Keywords:** Azumaya Algebras, Involutions, Quadratic Forms, Quadratic Pairs.

*MSC 2020:* Primary 11E81. Secondary 11E39, 14F20, 16H05, 20G10, 20G35.

## INTRODUCTION

In the classical theory of quadratic forms over a field  $\mathbb{F}$  of characteristic not 2, there is an equivalence between symmetric bilinear forms and quadratic forms. Given a quadratic form  $q$  one considers its polar symmetric bilinear form  $b_q(x, y) = q(x + y) - q(x) - q(y)$ , and given a symmetric bilinear form  $b$  one considers a quadratic form  $q_b(x) = b(x, x)$ . Then,  $b(x, y) = \frac{1}{2}b_{q_b}(x, y)$  and so every symmetric bilinear form arises as the polar of a quadratic form. This correspondence between quadratic forms and symmetric bilinear forms breaks down when the base field is of characteristic 2. There exist symmetric bilinear forms which are not the polar of a quadratic form. For example, on the  $\mathbb{F}$ -vector space  $\mathbb{F}$  the polar of any quadratic form vanishes, while  $b(x, y) = xy$  is a non-zero symmetric bilinear form. Quirks of characteristic 2 rear their head in other places as well. Consider a central simple algebra with  $\mathbb{F}$ -linear involution  $(A, \sigma)$ . There exists a splitting

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field extension  $\mathbb{F} \hookrightarrow \mathbb{K}$  such that  $A \otimes_{\mathbb{F}} \mathbb{K} \cong M_n(\mathbb{K})$ . The involution  $\sigma \otimes 1$  on  $M_n(\mathbb{K})$  is then the adjoint involution of some bilinear form  $b$ . We follow the convention of Calmès and Fasel [CF] and say that  $\sigma$  is *orthogonal* if  $b$  is symmetric,  $\sigma$  is *weakly symplectic* if  $b$  is skew-symmetric, and  $\sigma$  is *symplectic* if  $b$  is alternating, i.e.,  $b(x, x) = 0$  for all  $x$ . These notions do not depend on the splitting field. In characteristic 2, symmetric and skew-symmetric are the same notion, and so an orthogonal involution is simultaneously a weakly-symplectic involution, and vice versa. Another quirk of characteristic 2 is the following. If  $\sigma$  is an orthogonal involution which is not symplectic, the associated orthogonal group scheme  $\mathbf{O}(A, \sigma)$  is not smooth, [KMRT, Remark 23.5], and if  $\sigma$  is also symplectic then the orthogonal group scheme is of type  $C$ . Semisimple groups of type  $D$  are seemingly missing. All semisimple groups of type  $D_n$  with  $n \neq 4$  have been constructed by Tits in [Ti] using a more general notion of quadratic form. His approach was generalized by the authors of The Book of Involutions [KMRT] who introduced the more flexible concept of a *quadratic pair*.

**Definition.** Let  $A$  be a central simple  $\mathbb{F}$ -algebra. A *quadratic pair* on  $A$  is a pair  $(\sigma, f)$  where  $\sigma$  is an orthogonal involution and  $f$  is a linear function  $f: \text{Sym}(A, \sigma) \rightarrow \mathbb{F}$  on the symmetric elements of  $A$  (those  $a \in A$  satisfying  $\sigma(a) = a$ ), such that for all  $a \in A$ ,

$$f(a + \sigma(a)) = \text{Trd}_A(a).$$

Here  $\text{Trd}_A$  denotes the reduced trace of  $A$ , a notation we will also use later for Azumaya algebras over a scheme. For example, any  $\ell \in A$  satisfying  $\ell + \sigma(\ell) = 1$  gives rise to a quadratic pair  $(\sigma, f)$  by putting  $f = \text{Trd}_A(\ell \_ )|_{\text{Sym}(A, \sigma)}$ . Conversely, any quadratic pair  $(\sigma, f)$  is of this type by [KMRT, 5.7].

If  $b$  is a regular symmetric bilinear form on a vector space  $V$  with adjoint involution  $\sigma$ , then each way  $(\text{End}(V), \sigma)$  can be extended to a quadratic pair  $(\text{End}(V), \sigma, f)$  corresponds to a quadratic form whose polar form is  $b$ , [KMRT, 5.11]. Further, for any central simple algebra with quadratic pair, the orthogonal group  $\mathbf{O}(A, \sigma, f)$  ([KMRT, 23.B]) is a smooth semisimple group of type  $D$ . Moreover, every semisimple adjoint or simply connected algebraic group of type  $D_n$  with  $n \neq 4$  can be described in terms of central simple  $\mathbb{F}$ -algebras with a quadratic pair, [KMRT, 26.15].

The notion of quadratic pairs was extended by Calmès and Fasel [CF] from the setting of central simple algebras over a field, to the setting of Azumaya algebras over a scheme  $S$ . In their setting, objects such as algebras, groups, bilinear forms, etc., are sheaves or maps of sheaves on the ringed site  $(\mathfrak{Sch}_S, \mathcal{O})$  where  $\mathfrak{Sch}_S$  is the (big) fppf site of schemes over  $S$ , and  $\mathcal{O}$  is the global sections functor. The spirit of the original theory persists in this generalization. Assume  $b$  is a regular symmetric bilinear form on a locally free  $\mathcal{O}$ -module of finite positive rank  $\mathcal{M}$ , and let  $\sigma$  be its adjoint involution on  $\text{End}_{\mathcal{O}}(\mathcal{M})$ . Then, each way  $\text{End}_{\mathcal{O}}(\mathcal{M})$  can be equipped with a quadratic pair  $(\text{End}_{\mathcal{O}}(\mathcal{M}), \sigma, f)$  corresponds to a quadratic form  $q_f$  whose polar

form is  $b$ . The groups associated to quadratic pairs, discussed in [CF, 4.4], are also the appropriate notion of semisimple groups of type  $D$  over a scheme, [CF, 8.8].

In this paper we continue the study of Azumaya algebras and quadratic pairs over an arbitrary base scheme  $S$ . In particular, we do not assume that 2 is invertible over  $S$ . We begin with a preliminary section which reviews the technicalities of our sheaf point of view. Then, we discuss a globalized version of the Skolem-Noether theorem which describes isomorphisms of Azumaya algebras. Section 3 reviews the notion of bilinear and quadratic forms on  $\mathcal{O}$ -modules, in particular introducing the adjoint anti-automorphism associated with a regular bilinear form. Such adjoint anti-automorphisms are actually involutions when the bilinear form is suitably symmetric.

Section 4 defines quadratic pairs, gives examples, and describes procedures for constructing quadratic pairs. In the case when  $\sigma$  is adjoint to a regular symmetric bilinear form  $b$ , Proposition 4.4 connects the existence of a quadratic pair on  $(\text{End}_{\mathcal{O}}(\mathcal{M}), \sigma)$  to the existence of a quadratic form whose polar form is  $b$ . The second half of Section 4 provides a classification of quadratic pairs  $(\mathcal{A}, \sigma, f)$  which extend a given Azumaya  $\mathcal{O}$ -algebra with orthogonal involution  $(\mathcal{A}, \sigma)$ . We denote by  $\text{Symd}_{\mathcal{A}, \sigma}$  the sheaf-theoretic image of  $\text{Id} + \sigma: \mathcal{A} \rightarrow \mathcal{A}$ , called the sheaf of *symmetrized elements*, and we denote by  $\text{Alt}_{\mathcal{A}, \sigma}$  the sheaf-theoretic image of  $\text{Id} - \sigma: \mathcal{A} \rightarrow \mathcal{A}$ , called the sheaf of *alternating elements*. The map  $\text{Id} + \sigma$  then descends to a well defined map  $\xi: \mathcal{A}/\text{Alt}_{\mathcal{A}, \sigma} \rightarrow \text{Symd}_{\mathcal{A}, \sigma}$ . Lemma 4.14 shows that a necessary condition for  $(\mathcal{A}, \sigma)$  to extend to a quadratic triple is that  $1_{\mathcal{A}} \in \text{Symd}_{\mathcal{A}, \sigma}(S)$ . We call  $\sigma$  a *locally quadratic involution* when this is satisfied. We then give the following classification in Theorem 4.18.

**Theorem.** *Let  $(\mathcal{A}, \sigma)$  be an Azumaya  $\mathcal{O}$ -algebra with locally quadratic involution. Then, there is a bijection*

$$\left\{ \begin{array}{l} f \text{ such that } (\mathcal{A}, \sigma, f) \\ \text{is a quadratic triple} \end{array} \right\} \leftrightarrow \xi(S)^{-1}(1_{\mathcal{A}}).$$

Section 5 is the generalization of [KMRT, 5.18, 5.20], which deals with quadratic pairs on tensor products of algebras with involution.

The final two sections, 6 and 7, address cohomological obstructions which prevent locally quadratic involutions from being extended to quadratic triples. For a locally quadratic involution we define the *weak obstruction*  $\omega(\mathcal{A}, \sigma) \in H^1(S, \text{Skew}_{\mathcal{A}, \sigma}/\text{Alt}_{\mathcal{A}, \sigma})$ , and the *strong obstruction*  $\Omega(\mathcal{A}, \sigma) \in H^1(S, \text{Skew}_{\mathcal{A}, \sigma})$ . These obstructions encode whether  $(\mathcal{A}, \sigma)$  can be equipped with a quadratic pair, and if it can, whether that pair can be described by a single element in  $\mathcal{A}(S)$ . More precisely, we show in Theorem 6.2:

**Theorem.** *Let  $(\mathcal{A}, \sigma)$  be an Azumaya  $\mathcal{O}$ -algebra with a locally quadratic involution.*

- (i) *There exists a linear map  $f: \mathcal{A} \rightarrow \mathcal{O}$  such that  $(\mathcal{A}, \sigma, f|_{\text{Sym}_{\mathcal{A}, \sigma}})$  is a quadratic triple if and only if  $\Omega(\mathcal{A}, \sigma) = 0$ . In this case  $f = \text{Trd}_{\mathcal{A}}(\ell_{\_})$  for an element  $\ell \in \mathcal{A}(S)$  with  $\ell + \sigma(\ell) = 1$ .*

- (ii) *There exists a linear map  $f: \text{Sym}_{\mathcal{A},\sigma} \rightarrow \mathcal{O}$  such that  $(\mathcal{A}, \sigma, f)$  is a quadratic triple if and only if  $\omega(\mathcal{A}, \sigma) = 0$ .*

These obstructions capture global phenomenon, since locally, for instance over an affine scheme, both obstructions are zero. Section 6 concludes by discussing whether a given quadratic triple  $(\mathcal{A}, \sigma, f)$  can be described as  $f = \text{Trd}_{\mathcal{A}}(\ell\_)$  for an element  $\ell \in \mathcal{A}(S)$ . This is also detected with a cohomological obstruction  $c(f) \in H^1(S, \text{Alt}_{\mathcal{A},\sigma})$ , and the element  $\ell$  exists when  $c(f) = 0$ . Finally, Section 7 contains various examples of Azumaya algebras with locally quadratic involutions such that their obstructions are non-trivial. In section 7.1 we construct an Azumaya algebra with quadratic pair, and therefore trivial weak obstruction, which we show in Lemma 7.2 has non-trivial strong obstruction. Additionally, in section 7.4 we construct a quaternion algebra with orthogonal involution which by Lemma 7.5 has non-trivial weak obstruction, and therefore also non-trivial strong obstruction.

In the sequel to this paper, [GNR], we will use the results of this paper to establish an equivalence between the groupoids of adjoint semisimple groups of type  $A_1 \times A_1$  and of type  $D_2$  over a scheme  $S$ , extending [KMRT, 15.7].

## 1. PRELIMINARIES

Throughout this paper  $S$  is a scheme with structure sheaf  $\mathcal{O}_S$ . Following the style of [CF], we consider “objects over  $S$ ” as sheaves on the category of schemes over  $S$  equipped with the fppf topology. Below, we explain this viewpoint for the notions that are most important for this paper.

**1.1. The Site  $\mathfrak{Sch}_S$ .** Let  $\mathfrak{Sch}_S$  be the big fppf site of  $S$  as in [SGA3, Exposé IV]. Recall that its objects are schemes with structure morphism  $T \rightarrow S$  and a cover of some  $T \in \mathfrak{Sch}_S$  is a family  $\{T_i \rightarrow T\}_{i \in I}$  of flat morphisms which are locally of finite presentation and are jointly surjective. When considering a cover, we will use the notation  $T_{ij} = T_i \times_T T_j$  for  $i, j \in I$ . When a scheme is affine we will usually denote it by  $U$ , and so an affine cover will usually be written  $\{U_i \rightarrow T\}_{i \in I}$  with  $U_{ij} = U_i \times_T U_j$  for  $i, j \in I$ . We warn that  $U_{ij}$  may not be affine since  $T$  need not be separated.

**1.2. Remark.** In [St, Tag 021S], the Stacks project authors introduce “a” big fppf site of  $S$ , instead of “the”. They do so because of set theoretic nuances avoided in [SGA3] through the use of universes. We also avoid such difficulties and work with “the” big fppf site.

If  $\mathcal{F}$  is any sheaf on  $\mathfrak{Sch}_S$  and  $T \in \mathfrak{Sch}_S$ , we denote by  $\mathcal{F}|_T$  the restriction of the sheaf to the site  $\mathfrak{Sch}_S/T = \mathfrak{Sch}_T$ . We use the same notation for the restriction of elements, namely if  $f: V \rightarrow T$  is a morphism in  $\mathfrak{Sch}_S$  and  $t \in \mathcal{F}(T)$ , then we write  $t|_V = \mathcal{F}(f)(t)$ . This overlap of notation is borrowed from [St]. Since these operations apply only to sheaves or sections respectively, the meaning of restriction will be clear from the context.

**1.3. Cohomologies.** We will make use of the first non-abelian cohomology set for sheaves groups on  $\mathfrak{Sch}_S$ . However, there are various relevant cohomology theories whose  $H^1$  are isomorphic. We clarify this now.

First, recall from the discussion before [M, III.4.5] that for a sheaf of groups  $\mathcal{G}: \mathfrak{Sch}_S \rightarrow \mathfrak{Grp}$  and  $T \in \mathfrak{Sch}_S$ , the first Čech cohomology is

$$\check{H}^1(T, \mathcal{G}) = \operatorname{colim}_{\mathcal{U}=\{T_i \rightarrow T\}_{i \in I}} \check{H}^1(\mathcal{U}, \mathcal{G})$$

where the colimit is over fppf covers of  $T$ . The Čech cohomology relative a cover is

$$\check{H}^1(\mathcal{U}, \mathcal{G}) = \{ (g_{ij})_{i,j \in I} \in \prod_{i,j \in I} \mathcal{G}(T_{ij}) \mid (g_{ij}|_{T_{ijk}})(g_{jk}|_{T_{ijk}}) = (g_{ik}|_{T_{ijk}}) \} / \sim$$

where the equivalence relation  $\sim$  says that cocycles  $(g_{ij})_{i,j \in I}$  and  $(g'_{ij})_{i,j \in I}$  are cohomologous if there exists  $(h_k) \in \prod_{k \in I} \mathcal{G}(T_k)$  such that we have  $g'_{ij} = (h_i|_{T_{ij}})g_{ij}(h_j|_{T_{ij}})^{-1}$ . This is a pointed set with distinguished element  $(1)_{i,j \in I}$ . The following lemma describes the connecting morphism for this cohomology.

**1.4. Lemma** ([M, III.4.5]). *Let  $1 \rightarrow \mathcal{G}' \rightarrow \mathcal{G} \xrightarrow{\pi} \mathcal{G}'' \rightarrow 1$  be an exact sequence of sheaves of groups on  $\mathfrak{Sch}_S$ . Then, there is an exact sequence of pointed sets*

$$1 \rightarrow \mathcal{G}'(S) \rightarrow \mathcal{G}(S) \rightarrow \mathcal{G}''(S) \xrightarrow{\check{\delta}} \check{H}^1(S, \mathcal{G}') \rightarrow \check{H}^1(S, \mathcal{G}) \rightarrow \check{H}^1(S, \mathcal{G}'')$$

where  $\check{\delta}(g'') = [((g_i|_{T_{ij}})^{-1}(g_j|_{T_{ij}}))_{i,j \in I}]$  for a section  $g'' \in \mathcal{G}''(S)$  and any cover  $\mathcal{U} = \{T_i \rightarrow S\}_{i \in I}$  and elements  $g_i \in \mathcal{G}(T_i)$  such that  $\pi(g_i) = g''|_{T_i}$ .

When this is applied to an exact sequence of abelian sheaves  $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$  we will write  $\check{\delta}(f'') = [(f_j|_{T_{ij}} - f_i|_{T_{ij}})_{i,j \in I}]$ .

Second, we have the non-abelian cohomology of Giraud. Definition [Gir, 2.4.2] interpreted in our setting defines  $H^1(S, \mathcal{G})$  to be the set of isomorphism classes of  $\mathcal{G}$ -torsors for the fppf site on  $\mathfrak{Sch}_S$ . It is also a pointed set with base point given by the class of the trivial torsor. There is similarly a connecting morphism for this cohomology.

**1.5. Lemma** ([Gir, 3.1.3, 3.3.1]). *Let  $1 \rightarrow \mathcal{G}' \rightarrow \mathcal{G} \xrightarrow{\pi} \mathcal{G}'' \rightarrow 1$  be an exact sequence of sheaves of groups on  $\mathfrak{Sch}_S$ . Then, there is an exact sequence of pointed sets*

$$1 \rightarrow \mathcal{G}'(S) \rightarrow \mathcal{G}(S) \rightarrow \mathcal{G}''(S) \xrightarrow{\delta} H^1(S, \mathcal{G}') \rightarrow H^1(S, \mathcal{G}) \rightarrow H^1(S, \mathcal{G}'')$$

where  $\delta(g'') = [\pi^{-1}(g'')]$  for any  $g'' \in \mathcal{G}''(S)$ . Here, by  $\pi^{-1}(g'')$  we mean the subsheaf of  $\mathcal{G}$  given by

$$\begin{aligned} \pi^{-1}(g'') &: \mathfrak{Sch}_S \mapsto \mathfrak{Sets} \\ T &\mapsto \pi(T)^{-1}(g''|_T). \end{aligned}$$

Since  $\pi^{-1}(g'')$  is a coset of the subgroup  $\mathcal{G}'$  in the group  $\mathcal{G}$ , there is a natural simply transitive action of  $\mathcal{G}'$  on  $\pi^{-1}(g'')$ . By surjectivity of  $\mathcal{G} \rightarrow \mathcal{G}''$ , there will be a cover over which  $\pi^{-1}(g'')$  is non-empty. Therefore, it is in fact a  $\mathcal{G}'$ -torsor and so its isomorphism class is in  $H^1(S, \mathcal{G}')$  as claimed.

Despite not being a long exact cohomology sequence in the true sense since the groups may be non-abelian, we will nonetheless refer to the exact sequences in Lemmas 1.4 and 1.5 as long exact sequences.

By [Gir, III.3.6.4] (or by [M, III.4.6] where the inverse map is constructed) there is an isomorphism of pointed sets  $\check{H}^1(S, \mathcal{G}) \xrightarrow{\sim} H^1(S, \mathcal{G})$ , and it is easy to see that this isomorphism is compatible with the connecting morphisms described above. In particular the diagram

$$\begin{array}{ccc} \mathcal{G}''(S) & \xrightarrow{\check{\delta}} & \check{H}^1(S, \mathcal{G}) \\ \parallel & & \downarrow \wr \\ \mathcal{G}''(S) & \xrightarrow{\delta} & H^1(S, \mathcal{G}) \end{array}$$

commutes. Because of this, we will simply use  $\delta$  to denote both connecting morphisms.

Finally, we note that when dealing with an abelian sheaf  $\mathcal{F}: \mathfrak{Sch}_S \rightarrow \mathfrak{Ab}$ , there is also the derived functor cohomology as in [St, 01FT], which we denote  $H_{\text{fppf}}^1(S, \mathcal{F})$ . Occasionally we will refer to  $H_{\text{Zar}}^1(S, \mathcal{F})$  which is the derived functor cohomology with respect to the Zariski site on  $\mathfrak{Sch}_S$ . By [M, III, 2.10], there is an isomorphism  $H_{\text{fppf}}^1(T, \mathcal{F}) \cong \check{H}^1(T, \mathcal{F})$ .

**1.6.  $\mathcal{O}$ -Modules.** By [St, Tag 03DU], the contravariant functor

$$\begin{aligned} \mathcal{O}: \mathfrak{Sch}_S &\rightarrow \mathfrak{Rings} \\ T &\mapsto \mathcal{O}_T(T) \end{aligned}$$

where  $\mathfrak{Rings}$  is the category of commutative rings, is an fppf-sheaf on  $\mathfrak{Sch}_S$ , making  $(\mathfrak{Sch}_S, \mathcal{O})$  a ringed site in the sense of [St, Tag 03AD]. We call  $\mathcal{O}$  the *structure sheaf*. If  $T \rightarrow S$  is an open immersion, then  $\mathcal{O}(T) = \mathcal{O}_S(T)$ . From [St, Tag 03CW], an  $\mathcal{O}$ -module is then an fppf-sheaf  $\mathcal{M}: \mathfrak{Sch}_S \rightarrow \mathfrak{Ab}$  together with a map of sheaves

$$\mathcal{O} \times \mathcal{M} \rightarrow \mathcal{M}$$

such that for each  $T \in \mathfrak{Sch}_S$ , the map  $\mathcal{O}(T) \times \mathcal{M}(T) \rightarrow \mathcal{M}(T)$  makes  $\mathcal{M}(T)$  a usual  $\mathcal{O}(T)$ -module. Given two  $\mathcal{O}$ -modules  $\mathcal{M}$  and  $\mathcal{N}$ , their internal homomorphism functor

$$\begin{aligned} \text{Hom}_{\mathcal{O}}(\mathcal{M}, \mathcal{N}): \mathfrak{Sch}_S &\rightarrow \mathfrak{Ab} \\ T &\mapsto \text{Hom}_{\mathcal{O}|_T}(\mathcal{M}|_T, \mathcal{N}|_T) \end{aligned}$$

is another  $\mathcal{O}$ -module by [St, 03EM].

We refer to [St, Tags 03DE, 03DL] for definitions of various properties of  $\mathcal{O}$ -modules. In particular, we will make use of modules which are locally free and/or of finite type, therefore we present these definitions for the convenience of the reader.

An  $\mathcal{O}$ -module  $\mathcal{M}$  is called *locally free* if for all  $T \in \mathfrak{Sch}_S$ , there is a covering  $\{T_i \rightarrow T\}_{i \in I}$  such that for each  $i \in I$ , the restriction  $\mathcal{M}|_{T_i}$  is a free  $\mathcal{O}|_{T_i}$ -module.

Explicitly,  $\mathcal{M}|_{T_i} \cong \bigoplus_{j \in J_i} \mathcal{O}|_{T_i}$  for some index set  $J_i$ . If all  $J_i$  have the same cardinality then  $\mathcal{M}$  has *constant rank*  $|J_i|$ . A locally free  $\mathcal{O}$ -module of constant rank 1 is called a *line bundle*. Isomorphism classes of line bundles form a group under tensor product, denoted  $\text{Pic}(S)$ . Since the (additive) category of line bundles is equivalent to the (additive) category of  $\mathbb{G}_m$ -torsors over  $S$  [CF, prop. 2.4.3.1], the group  $\text{Pic}(S)$  is isomorphic to the group  $H^1(S, \mathbb{G}_m)$ . It is also isomorphic to the group  $H_{\text{Zar}}^1(S, \mathbb{G}_m)$  because  $\mathbb{G}_m$ -torsors are locally trivial for the Zariski topology.

An  $\mathcal{O}$ -module  $\mathcal{M}$  is called *of finite type* if for all  $T \in \mathfrak{Sch}_S$ , there is a covering  $\{T_i \rightarrow T\}_{i \in I}$  such that for each  $i \in I$ , the restriction  $\mathcal{M}|_{T_i}$  is an  $\mathcal{O}|_{T_i}$ -module which is generated by finitely many global sections. That is, there exists a surjection of  $\mathcal{O}|_{T_i}$ -modules  $\mathcal{O}^{\oplus n_i} \rightarrow \mathcal{M}$  for some  $n_i \in \mathbb{N}$ .

An  $\mathcal{O}$ -module  $\mathcal{M}$  is called *of finite presentation* if for all  $T \in \mathfrak{Sch}_S$ , there is a covering  $\{T_i \rightarrow T\}_{i \in I}$  such that for each  $i \in I$ , the restriction  $\mathcal{M}|_{T_i}$  is an  $\mathcal{O}|_{T_i}$ -module which has a finite global presentation. That is, there exists an exact sequence of  $\mathcal{O}|_{T_i}$ -modules

$$\bigoplus_{j \in J_i} \mathcal{O}|_{T_i} \rightarrow \bigoplus_{k \in K_i} \mathcal{O}|_{T_i} \rightarrow \mathcal{E}|_{T_i} \rightarrow 0$$

for some finite index sets  $J_i$  and  $K_i$ .

We will use the terminology *finite locally free* to mean locally free and of finite type. Since  $S \in \mathfrak{Sch}_S$  is a final object, [St, Tag 03DN] applies and it suffices for us to check local conditions, such as the three detailed above or quasi-coherence detailed below, for an fppf-covering of  $S$ .

We associate with an endomorphism  $\sigma$  of an  $\mathcal{O}$ -module  $\mathcal{M}$  satisfying  $\sigma^2 = \text{Id}_{\mathcal{M}}$  the following  $\mathcal{O}$ -modules:

$$\begin{aligned} \text{Sym}_{\mathcal{M}, \sigma} &= \text{Ker}(\text{Id} - \sigma) \quad (\text{symmetric elements}), \\ \text{Alt}_{\mathcal{M}, \sigma} &= \text{Im}(\text{Id} - \sigma) \quad (\text{alternating elements}), \\ \text{Skew}_{\mathcal{M}, \sigma} &= \text{Ker}(\text{Id} + \sigma) \quad (\text{skew-symmetric elements}), \\ \text{Symd}_{\mathcal{M}, \sigma} &= \text{Im}(\text{Id} + \sigma) \quad (\text{symmetrized elements}), \end{aligned}$$

where  $\text{Im}(\_)$  is the image fppf-sheaf. Since  $\text{Alt}_{\mathcal{M}, \sigma} \subseteq \text{Skew}_{\mathcal{M}, \sigma}$ , the map  $\text{Id} + \sigma$  descends to a well defined map  $\xi: \mathcal{M}/\text{Alt}_{\mathcal{M}, \sigma} \rightarrow \text{Symd}_{\mathcal{M}, \sigma}$ . We then have a large diagram with exact rows and columns which we reference later in Sections 4 and 6.

$$(1.6.1) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{Alt}_{\mathcal{M}, \sigma} & \xlongequal{\quad} & \text{Alt}_{\mathcal{M}, \sigma} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Skew}_{\mathcal{M}, \sigma} & \longrightarrow & \mathcal{M} & \xrightarrow{\text{Id} + \sigma} & \text{Symd}_{\mathcal{M}, \sigma} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \text{Skew}_{\mathcal{M}, \sigma} / \text{Alt}_{\mathcal{M}, \sigma} & \longrightarrow & \mathcal{M} / \text{Alt}_{\mathcal{M}, \sigma} & \xrightarrow{\xi} & \text{Symd}_{\mathcal{M}, \sigma} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$



**1.7. Quasi-coherent Modules.** From [St, Tag 03DK], an  $\mathcal{O}$ -module  $\mathcal{E}$  is called *quasi-coherent* if for all  $T \in \mathfrak{Sch}_S$  there is a covering  $\{T_i \rightarrow T\}_{i \in I}$  such that for each  $i \in I$ ,  $\mathcal{E}|_{T_i}$  has a global presentation. That is there is an exact sequence of  $\mathcal{O}|_{T_i}$ -modules

$$\bigoplus_{j \in J_i} \mathcal{O}|_{T_i} \rightarrow \bigoplus_{k \in K_i} \mathcal{O}|_{T_i} \rightarrow \mathcal{E}|_{T_i} \rightarrow 0$$

for some index sets  $J_i$  and  $K_i$ . Thanks to [St, Tag 03OJ], in our context the classical notion of a quasi-coherent sheaf on  $S$  and the notion of a quasi-coherent  $\mathcal{O}$ -module are essentially equivalent: given a quasi-coherent  $\mathcal{O}$ -module  $\mathcal{E}$ , there exists a classical quasi-coherent sheaf  $E$  on  $S$  such that for  $T \in \mathfrak{Sch}_S$  with structure morphism  $f: T \rightarrow S$ , we have  $\mathcal{E}(T) = f^*(E)(T)$ . In fact,  $E$  is simply the restriction of  $\mathcal{E}$  to open subschemes of  $S$ , i.e., the small Zariski site, and so we use the notation  $\mathcal{E}_{\text{small}} = E$ . Conversely, any quasi-coherent sheaf  $E$  on  $S$  gives rise to a unique quasi-coherent  $\mathcal{O}$ -module  $\mathcal{E}$  satisfying  $\mathcal{E}(T) = f^*(E)(T)$ , and we use the notation  $E_{\text{fppf}} = \mathcal{E}$ . We of course have that  $(\mathcal{E}_{\text{small}})_{\text{fppf}} = \mathcal{E}$  and  $(E_{\text{fppf}})_{\text{small}} = E$ .

A convenient property of quasi-coherent  $\mathcal{O}$ -modules is that their first cohomology vanishes on affine schemes.

**1.8. Lemma.** *Let  $\mathcal{E}$  be a quasi-coherent  $\mathcal{O}$ -module. If  $U \in \mathfrak{Sch}_S$  is an affine scheme, then  $H_{\text{fppf}}^1(U, \mathcal{E}) = 0$ .*

*Proof.* According to [St, Tag 03P2],  $H_{\text{fppf}}^1(U, \mathcal{E})$  is the Zariski  $H^1$  of the quasi-coherent sheaf  $\mathcal{E}_{\text{small}}$ , and so vanishes by [St, Tag 01XB] since  $U$  is affine.  $\square$

We also consider the dual of the quasi-coherent  $\mathcal{O}$ -module  $\mathcal{E}$ , defined to be  $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{O})$ . The functor  $\mathcal{E}^\vee$  is represented by the affine  $S$ -scheme  $\mathbf{V}(\mathcal{E}) = \text{Spec}(\text{Sym}^\bullet(\mathcal{E}_{\text{small}}))$  by [SGA3, I, 4.6.3.1]. If  $\mathcal{E}_{\text{small}}$  is of finite type, then  $\mathbf{V}(\mathcal{E})$  is of finite type as an  $S$ -scheme, [EGA-I, I, 9.4.11].

If  $\mathcal{E}$  is finite locally free, then so is  $\text{End}(\mathcal{E}) = \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{E}) \cong \mathcal{E}^\vee \otimes_{\mathcal{O}} \mathcal{E}$ . As in [EGA-I, I, 9.6.2], we view its associated scheme  $\mathbf{V}(\text{End}(\mathcal{E}))$  as an  $S$ -scheme of unital associative algebras. Moreover, the functor

$$\begin{aligned} \text{GL}(\mathcal{E}): \mathfrak{Sch}_S &\rightarrow \mathfrak{Grp} \\ T &\mapsto \text{Aut}_{\mathcal{O}|_T}(\mathcal{E}|_T) \end{aligned}$$

is representable by an open  $S$ -subscheme of  $\mathbf{V}(\text{End}(\mathcal{E}))$ , denoted  $\mathbf{GL}(\mathcal{E})$ , [EGA-I, I, 9.6.4].

Given an  $\mathcal{O}$ -module  $\mathcal{M}$  and  $\sigma \in \text{End}_{\mathcal{O}}(\mathcal{M})$  of order 2, the  $\mathcal{O}$ -modules  $\text{Sym}_{\mathcal{M}, \sigma}$ ,  $\text{Alt}_{\mathcal{M}, \sigma}$ ,  $\text{Skew}_{\mathcal{M}, \sigma}$ , and  $\text{Symd}_{\mathcal{M}, \sigma}$  are quasi-coherent whenever  $\mathcal{M}$  is because then they are kernels and images of a map between quasi-coherent sheaves. This is shown in [GW, 7.19] for quasi-coherent sheaves on  $S$ , and the result generalizes to quasi-coherent  $\mathcal{O}$ -modules by considering an affine cover of  $S$  and using the “fppf  $\sim$  small” correspondence explained above.

1.9.  **$\mathcal{O}$ -Algebras.** An  $\mathcal{O}$ -module  $\mathcal{B}: \mathfrak{Sch}_S \rightarrow \mathbf{nc}\text{-Rings}$  from  $\mathfrak{Sch}_S$  to the category of all (not necessarily commutative) rings such that each  $\mathcal{B}(T)$  is a  $\mathcal{O}(T)$ -algebra will be called an  $\mathcal{O}$ -algebra. We call it unital, associative, commutative, etc., if each  $\mathcal{B}(T)$  is so. If  $\mathcal{B}$  is a unital associative  $\mathcal{O}$ -algebra which is finite locally free, then the functor of invertible elements

$$\begin{aligned} \mathbf{GL}_{1,\mathcal{B}}: \mathfrak{Sch}_S &\rightarrow \mathfrak{Grp} \\ T &\mapsto \mathcal{B}(T)^\times \end{aligned}$$

is representable by an affine  $S$ -group scheme [CF, 2.4.2.1]. A section  $u \in \mathbf{GL}_{1,\mathcal{B}}(S)$  induces an inner automorphism of  $\mathcal{B}$ , denoted  $\text{Inn}(u)$  which is given on  $\mathcal{B}(T)$  by  $b \mapsto u|_T \cdot b \cdot u|_T^{-1}$ . If  $\mathcal{B}$  is a separable  $\mathcal{O}$ -algebra which is locally free of finite type, then  $\mathbf{GL}_{1,\mathcal{B}}$  is a reductive  $S$ -group scheme [CF, 3.1.0.50]. We also use the notation  $\mathbf{GL}_{1,\mathcal{O}} = \mathcal{O}^\times = \mathbb{G}_m$ .

1.10. **Azumaya Algebras.** A key object of our interest is an Azumaya  $\mathcal{O}$ -algebra. First, we recall that over a commutative ring  $R$ , an Azumaya  $R$ -algebra is a central separable  $R$ -algebra. Equivalently, the sandwich map

$$\begin{aligned} \text{Sand}: A \otimes_R A^{\text{op}} &\rightarrow \text{End}_R(A) \\ a \otimes b &\mapsto (x \mapsto axb) \end{aligned}$$

is an isomorphism. For separable and Azumaya algebras over rings (equivalently, over affine schemes) we refer to [Fo] or [Knu, III §5]. Following [Gro, 5.1], we consider an *Azumaya  $\mathcal{O}$ -algebra*, or simply an *Azumaya algebra*, to be a finite locally free  $\mathcal{O}$ -algebra  $\mathcal{A}$  such that the sandwich morphism

$$\text{Sand}: \mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}^{\text{op}} \rightarrow \text{End}_{\mathcal{O}}(\mathcal{A})$$

is an isomorphism. This is equivalent to the definition of [CF, 2.5.3.4], which asks that  $\mathcal{A}$  be finite locally free and that for all affine schemes  $U \in \mathfrak{Sch}_S$ , we have that  $\mathcal{A}(U)$  is an Azumaya  $\mathcal{O}(U)$ -algebra.

By [CF, 2.5.3.6], the rank, viewed as a locally constant integer valued function, of an Azumaya algebra is always square. Following [CF, 2.5.3.7] we call the square root of an Azumaya algebra's rank the *degree* of the algebra. It is a locally constant integer valued function.

Azumaya algebras of the form  $\text{End}_{\mathcal{O}}(\mathcal{M})$ , where  $\mathcal{M}$  is a locally free  $\mathcal{O}$ -module of finite positive rank, will be called *neutral* algebras.

In addition, we have the functor of norm 1 elements

$$\begin{aligned} \mathbf{SL}_{1,\mathcal{A}}: \mathfrak{Sch}_S &\rightarrow \mathfrak{Grp} \\ T &\mapsto \{x \in \mathcal{A}(T)^\times \mid \text{Nrd}_{\mathcal{A}}(x) = 1\}, \end{aligned}$$

where  $\text{Nrd}_{\mathcal{A}}$  is the reduced norm as in [CF, 2.5.3.17]. This functor is representable by a closed subgroup scheme of  $\mathbf{GL}_{1,\mathcal{A}}$  by [CF, 3.5.0.92].

**1.11. Morita Correspondence.** We review Morita theory in the setting of this paper, following [KS, §19.5]. Over affine schemes, this is classical, see for example [Bas, II] or [B:A3, §6].

The following are equivalent for  $\mathcal{O}$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  and an  $(\mathcal{A}, \mathcal{B})$ -bimodule  $\mathcal{P}$ :

- (i)  $\mathcal{P}$  is faithfully flat and of finite presentation as an  $\mathcal{A}$ -module and  $\mathcal{B}^{\text{op}} \xrightarrow{\sim} \text{End}_{\mathcal{A}}(\mathcal{P})$  under the canonical map.
- (ii)  $\mathcal{P}$  is faithfully flat and of finite presentation as a  $\mathcal{B}$ -module and  $\mathcal{A} \xrightarrow{\sim} \text{End}_{\mathcal{B}^{\text{op}}}(\mathcal{P})$  under the canonical map.
- (iii) There exists a  $(\mathcal{B}, \mathcal{A})$ -bimodule  $\mathcal{Q}$  such that  $\mathcal{P} \otimes_{\mathcal{O}} \mathcal{Q} \cong \mathcal{A}$  and  $\mathcal{Q} \otimes_{\mathcal{O}} \mathcal{P} \cong \mathcal{B}$  as  $(\mathcal{A}, \mathcal{A})$ - respectively  $(\mathcal{B}, \mathcal{B})$ -bimodules.
- (iv) For the  $(\mathcal{B}, \mathcal{A})$ -bimodule  $\mathcal{Q}_0 = \text{Hom}_{\mathcal{A}}(\mathcal{P}, \mathcal{A})$ , the canonical morphisms  $\mathcal{P} \otimes_{\mathcal{O}} \mathcal{Q}_0 \xrightarrow{\sim} \mathcal{A}$  and  $\mathcal{Q}_0 \otimes_{\mathcal{O}} \mathcal{P} \xrightarrow{\sim} \mathcal{B}$  are  $(\mathcal{A}, \mathcal{A})$ - respectively  $(\mathcal{B}, \mathcal{B})$ -bimodule isomorphisms.
- (v)  $\mathcal{M} \mapsto \mathcal{P} \otimes_{\mathcal{B}} \mathcal{M}$  induces an equivalence  $\mathcal{B}\text{-mod} \xrightarrow{\sim} \mathcal{A}\text{-mod}$  of stacks.
- (vi)  $\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{B}} \mathcal{P}$  induces an equivalence  $\text{mod-}\mathcal{A} \xrightarrow{\sim} \text{mod-}\mathcal{B}$  of stacks.
- (vii)  $\mathcal{M} \mapsto \text{Hom}_{\mathcal{A}}(\mathcal{P}, \mathcal{M})$  induces an equivalence  $\mathcal{A}\text{-mod} \xrightarrow{\sim} \mathcal{B}\text{-mod}$  of stacks.

In case (i), we have  $\mathcal{Q} \cong \text{Hom}_{\mathcal{A}}(\mathcal{P}, \mathcal{A}) \cong \text{Hom}_{\mathcal{B}}(\mathcal{P}, \mathcal{B})$  as  $(\mathcal{A}, \mathcal{B})$ -bimodules. Moreover, any equivalence  $\text{mod-}\mathcal{A} \xrightarrow{\sim} \text{mod-}\mathcal{B}$  of stacks has the form (vi) for some  $(\mathcal{A}, \mathcal{B})$ -bimodule satisfying (i); similarly for (vii). Following the classical terminology, we call  $\mathcal{A}$  and  $\mathcal{B}$  *Morita-equivalent* if the conditions (i)–(vii) hold, and we call  $\mathcal{P}$  satisfying (i) an *invertible*  $(\mathcal{A}, \mathcal{B})$ -bimodule; analogously for  $\mathcal{Q}$ .

**Example.** Let  $\mathcal{P}$  be a faithfully flat  $\mathcal{O}$ -module of finite presentation. Then  $\mathcal{A} = \text{End}_{\mathcal{O}}(\mathcal{P})$  and  $\mathcal{B} = \mathcal{O}$  are Morita-equivalent, for example by (ii).

## 2. ISOMORPHISMS OF AZUMAYA ALGEBRAS

**2.1. The Skolem-Noether Theorem.** We globalize here [KO, §IV.3]. Let  $\mathcal{A}$  be an Azumaya  $\mathcal{O}$ -algebra, the map  $\mathbf{GL}_{1, \mathcal{A}} \rightarrow \mathbf{Aut}_{\mathcal{A}}$  defined by inner automorphisms induce an isomorphism  $\mathbf{GL}_{1, \mathcal{A}}/\mathbb{G}_m \xrightarrow{\sim} \mathbf{Aut}_{\mathcal{A}}$  [CF, prop. 2.4.4.3] hence an exact sequence of  $S$ -group schemes

$$1 \rightarrow \mathbb{G}_m \rightarrow \mathbf{GL}_{1, \mathcal{A}} \rightarrow \mathbf{Aut}_{\mathcal{A}} \rightarrow 1.$$

**2.2. Lemma.** *Assume that  $\mathcal{A}$  is of degree  $d$ . Then the sequence above fits in a commutative diagram of  $S$ -group schemes with exact rows*

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu_d & \longrightarrow & \mathbf{SL}_{1, \mathcal{A}} & \longrightarrow & \mathbf{Aut}_{\mathcal{A}} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \text{Id} \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathbf{GL}_{1, \mathcal{A}} & \longrightarrow & \mathbf{Aut}_{\mathcal{A}} \longrightarrow 1. \end{array}$$

*Proof.* The commutativity is clear so the statement is the fact that the top horizontal sequence is exact as sequence of flat sheaves over  $S$ . This is established in [CF, Rem. 3.5.0.94].  $\square$

By taking the  $S$ -points we get an exact sequence of pointed sets

$$1 \rightarrow \mathbb{G}_m(S) \rightarrow \mathbf{GL}_{1,\mathcal{A}}(S) \rightarrow \mathbf{Aut}_{\mathcal{A}}(S) \xrightarrow{\mathcal{J}} \mathrm{Pic}(S) \rightarrow H^1(S, \mathbf{GL}_{1,\mathcal{A}})$$

(note  $\mathrm{Pic}(S) = H^1(S, \mathbb{G}_m)$ ) where  $\mathcal{J}$  is the characteristic map [Gir, III, 3.4.3], described in [RZ, Theorem 7] in a concrete way. This description is as follows.

**2.3. Proposition** (Skolem-Noether). *Let  $\mathcal{A}$  be an Azumaya  $\mathcal{O}$ -algebra, and let  $\varphi$  be an automorphism of  $\mathcal{A}$ . We define an  $\mathcal{O}$ -module  $\mathcal{I}_\varphi$  by setting*

$$\mathcal{I}_\varphi(T) = \{x \in \mathcal{A}(T) \mid \varphi(a)x|_V = x|_V a \text{ for all } V \in \mathfrak{Sch}_T \text{ and } a \in \mathcal{A}(V)\}$$

for  $T \in \mathfrak{Sch}_S$ , with the natural restriction maps. Then  $\mathcal{I}_\varphi$  is an invertible  $\mathcal{O}$ -module, i.e., a locally projective  $\mathcal{O}$ -module of rank 1. Moreover, the following hold.

- (i)  $\varphi$  is an inner automorphism if and only if  $\mathcal{I}_\varphi$  is free.
- (ii) If  $\psi$  is another automorphism of  $\mathcal{A}$ , then  $\mathcal{I}_{\psi \circ \varphi} \cong \mathcal{I}_\psi \otimes_{\mathcal{O}} \mathcal{I}_\varphi$ .
- (iii) The characteristic map  $\mathcal{J}: \mathbf{Aut}_{\mathcal{A}}(S) \rightarrow \mathrm{Pic}(S)$  is given by  $\mathcal{J}(\psi) = \mathcal{I}_\psi$ .
- (iv) Denoting by  $\mathrm{Inn}(\mathcal{A})$  the inner automorphism group sheaf of  $\mathcal{A}$ , the characteristic map  $\mathcal{J}: \varphi \mapsto \mathcal{I}_\varphi$  gives rise to an embedding

$$\mathbf{Aut}_{\mathcal{A}}(S)/\mathrm{Inn}_{\mathcal{A}}(S) \xrightarrow{\bar{\mathcal{J}}} \mathrm{Pic}(S)$$

and  $\mathrm{Im}(\bar{\mathcal{J}}) = \{[I] \in \mathrm{Pic}(S) \mid \mathcal{A} \otimes_{\mathcal{O}} I \cong \mathcal{A} \text{ as left } \mathcal{A}\text{-modules}\}$ .

- (v) If  $\mathcal{A}$  has constant degree  $d$ , the automorphism  $\varphi^d$  is inner.

*Proof.* Since  $\mathcal{A}$  is quasi-coherent, over an affine scheme  $U \in \mathfrak{Sch}_S$  we have that

$$\mathcal{I}_\varphi(U) = \{x \in \mathcal{A}(U) \mid \varphi(a)x = xa \text{ for all } a \in \mathcal{A}(U)\}.$$

It is shown in [RZ, §3] that  $\mathcal{I}_\varphi(U)$  is a line bundle over  $U$ , and therefore by taking an affine cover of  $S$  we see that  $\mathcal{I}_\varphi$  is indeed a line bundle.

To see (i) we adapt the proof of [RZ, §3 Lemma 5] to our context. Not many changes are needed. First, assume that  $\varphi$  is an inner automorphism corresponding to a section  $u \in \mathcal{A}(S)^\times$ . Then, for some  $T \in \mathfrak{Sch}_S$  we have

$$\begin{aligned} \mathcal{I}_\varphi(T) &= \{x \in \mathcal{A}(T) \mid u|_V a u|_V^{-1} x|_V = x|_V a, \forall V \in \mathfrak{Sch}_T, a \in \mathcal{A}(V)\} \\ &= \{x \in \mathcal{A}(T) \mid a(u^{-1}x)|_V = (u^{-1}x)|_V a, \forall V \in \mathfrak{Sch}_T, a \in \mathcal{A}(V)\} \end{aligned}$$

Hence if  $x \in \mathcal{I}_\varphi(T)$ , then  $u^{-1}x \in \mathcal{O}(T)$ . In particular, this gives an isomorphism  $\mathcal{I}_\varphi \xrightarrow{\sim} \mathcal{O}$  by sending  $x \mapsto u^{-1}x$ , and so we see  $\mathcal{I}_\varphi$  is a free  $\mathcal{O}$ -module. Conversely, assume that  $\mathcal{I}_\varphi$  is a free  $\mathcal{O}$ -module, generated over  $\mathcal{O}$  by an element  $u \in \mathcal{I}_\varphi(S) \subset \mathcal{A}(S)$ . The map  $\mathcal{I}_\varphi \otimes_{\mathcal{O}} \mathcal{A} \rightarrow \mathcal{A}$  given by  $i \otimes a \mapsto ia$  is locally an isomorphism as per [RZ], and therefore is an isomorphism. Hence we have

$$u \cdot \mathcal{A}(S) = \mathcal{I}_\varphi(S)\mathcal{A}(S) = \mathcal{A}(S),$$

and so  $u \in \mathcal{A}(S)^\times$ . Then, by the definition of  $\mathcal{I}_\varphi$ , for  $T \in \mathfrak{Sch}_S$  and a section  $a \in \mathcal{A}(T)$ ,

$$\varphi(a)u = ua \Rightarrow \varphi(a) = uau^{-1}$$

and so  $\varphi$  is the inner automorphism associated to  $u$ .

The statements (ii), (iii), and the first claim of (iv) are shown over an affine scheme in [KO, IV, §1.1.2], and therefore they hold globally as well. To check the second claim of (iv), we note that, as shown above, there is an isomorphism  $\mathcal{I}_\varphi \otimes_{\mathcal{O}} \mathcal{A} \rightarrow \mathcal{A}$  and so

$$\mathrm{Im}(\bar{\mathcal{J}}) \subseteq \{[I] \in \mathrm{Pic}(S) \mid \mathcal{A} \otimes_{\mathcal{O}} I \cong \mathcal{A} \text{ as left } \mathcal{A}\text{-modules}\}.$$

To show the reverse inclusion, we adapt the construction from [KO, IV, §1.1.2] to our setting. Again, no fundamental changes are needed. Let  $I$  be a line bundle such that there is an isomorphism  $f: \mathcal{A} \otimes_{\mathcal{O}} I \rightarrow \mathcal{A}$  of left  $\mathcal{A}$ -modules. Since there is an isomorphism

$$\begin{aligned} \gamma: \mathcal{A} &\xrightarrow{\sim} \mathcal{E}nd_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{O}} I, \mathcal{A} \otimes_{\mathcal{O}} I)^{\mathrm{op}} \\ a &\mapsto (\gamma_a: x \otimes i \mapsto xa \otimes i), \end{aligned}$$

we may make the following definition. For  $T \in \mathfrak{Sch}_S$  and  $b \in \mathcal{A}(T)$ , the map

$$\begin{aligned} (\mathcal{A} \otimes_{\mathcal{O}} I)|_T &\rightarrow (\mathcal{A} \otimes_{\mathcal{O}} I)|_T \\ x &\mapsto f^{-1}(f(x)b) \end{aligned}$$

is left  $\mathcal{A}|_T$ -linear, and so there exists  $\varphi(b) \in \mathcal{A}(T)$  such that  $f^{-1}(f(x)b) = \gamma_{\varphi(b)}(x)$ . This defines an automorphism  $\varphi: \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ . It is then straightforward to check that  $f$  induces an inclusion

$$f: I \cong 1 \otimes_{\mathcal{O}} I \rightarrow \mathcal{I}_\varphi,$$

which by [KO] is an isomorphism over affine schemes, and hence is an isomorphism. Therefore  $[I] = [\mathcal{I}_\varphi] \in \mathrm{Im}(\bar{\mathcal{J}})$  as desired.

Over an affine scheme, (v) is proven in [KO, IV, 3.1]. Here we give a general cohomological proof. Assume furthermore that  $\mathcal{A}$  is of degree  $d$  and consider the commutative diagram of exact sequences of  $S$ -group schemes from Lemma 2.2,

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu_d & \longrightarrow & \mathrm{SL}_{1,\mathcal{A}} & \longrightarrow & \mathrm{Aut}_{\mathcal{A}} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \mathrm{Id} \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathrm{GL}_{1,\mathcal{A}} & \longrightarrow & \mathrm{Aut}_{\mathcal{A}} \longrightarrow 1. \end{array}$$

For  $f \in \mathrm{Aut}_{\mathcal{A}}(S)$ , we get an obstruction  $\varphi_d(f) \in H^1(S, \mu_d)$  such that  $\varphi(f)$  is the image of  $\varphi_d(f)$  under the group homomorphism  $H^1(S, \mu_d) \rightarrow H^1(S, \mathbb{G}_m)$ . We then obtain an embedding

$$\mathrm{Aut}_{\mathcal{A}}(S)/\mathrm{Inn}_{\mathcal{A}}(S) \hookrightarrow {}_d\mathrm{Pic}(S),$$

where  ${}_d\mathrm{Pic}(S)$  denotes the  $d$ -torsion elements in  $\mathrm{Pic}(S)$ . Therefore  $\varphi^d \mapsto 0 \in {}_d\mathrm{Pic}(S)$  and hence is an inner automorphism as desired.  $\square$

In particular, if  $\mathcal{A}$  is of constant degree  $d$  and  ${}_d\mathrm{Pic}(S) = 0$ , we see that all automorphisms of  $\mathcal{A}$  are inner. In the general case, since any Azumaya algebra is locally of constant degree, we have the following corollary.

2.4. **Corollary.** *If  $\text{Pic}(S)$  is torsion free, then all automorphisms of  $\mathcal{A}$  are inner.*

For neutral Azumaya algebras, Proposition 2.3 can be generalized to a description of isomorphisms.

2.5. **Lemma** (Isomorphisms of neutral Azumaya algebras). *Let  $\mathcal{M}$  and  $\mathcal{N}$  be locally free  $\mathcal{O}$ -modules of finite positive rank. Put  $\mathcal{A} = \text{End}_{\mathcal{O}}(\mathcal{M})$ , a neutral Azumaya  $\mathcal{O}$ -algebra.*

(a) *Let  $\mathcal{L}$  be an invertible  $\mathcal{O}$ -module and let  $\rho: \mathcal{M} \otimes_{\mathcal{O}} \mathcal{L} \xrightarrow{\sim} \mathcal{N}$  be an  $\mathcal{O}$ -module isomorphism.*

(i) *Then the map  $\varphi: \text{End}_{\mathcal{O}}(\mathcal{M}) \xrightarrow{\sim} \text{End}_{\mathcal{O}}(\mathcal{N})$ , given by*

$$(2.5.1) \quad \varphi(a) = \rho \circ (a \otimes \text{Id}_{\mathcal{L}}) \circ \rho^{-1},$$

*is an isomorphism of Azumaya  $\mathcal{O}$ -algebras.*

(ii) *Letting  $\mathcal{A}$  act canonically on the first factor of  $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}$  and on  $\mathcal{N}$  through  $\varphi$ , the map  $\rho$  becomes an isomorphism of left  $\mathcal{A}$ -modules.*

(iii) *If  $(\mathcal{L}', \rho')$  is another pair as in the assumption of (a), then  $\mathcal{L} \cong \mathcal{L}'$  as  $\mathcal{O}$ -modules.*

(b) *Conversely, let  $\varphi: \text{End}_{\mathcal{O}}(\mathcal{M}) \xrightarrow{\sim} \text{End}_{\mathcal{O}}(\mathcal{N})$  be an isomorphism of Azumaya  $\mathcal{O}$ -algebras.*

(i) *Denoting by  ${}_{\varphi}\mathcal{N}$  the left  $\mathcal{A}$ -module obtained from  $\mathcal{N}$  by letting  $\mathcal{A}$  act through  $\varphi$ , the  $\mathcal{O}$ -module  $\mathcal{L}_{\varphi} = \text{Hom}_{\mathcal{A}}(\mathcal{M}, {}_{\varphi}\mathcal{N})$  is invertible.*

(ii) *Letting  $\mathcal{A}$  act on the first factor of  $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}$ , the natural evaluation map*

$$\rho_{\varphi}: \mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}_{\varphi} \xrightarrow{\sim} {}_{\varphi}\mathcal{N}$$

*is an isomorphism of left  $\mathcal{A}$ -modules.*

(iii) *The isomorphism  $\varphi$  satisfies (2.5.1) with respect to the pair  $(\mathcal{L}_{\varphi}, \rho_{\varphi})$ .*

*Proof.* In (a) the parts (i) and (ii) are obvious. In (iii) observe that  $\rho^{-1} \circ \rho'$  is an  $\mathcal{A}$ -module isomorphism of  $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}$ . Hence, by Morita correspondence as in section 1.11, the  $\mathcal{O}$ -modules  $\mathcal{L}$  and  $\mathcal{L}'$  are isomorphic.

As before in Proposition 2.3, the proof of (b) easily reduces to the case when  $S = \text{Spec}(R)$  and the  $R$ -modules  $M$  and  $N$  are faithfully flat and finitely presented. In this case, the lemma is for example shown in [Knu, III, 8.2.1]. We include a proof with some more details than in [Knu].

Morita theory implies that  $\mathcal{M}$  and  ${}_{\varphi}\mathcal{N}$  are invertible  $(\mathcal{A}, \mathcal{O})$ -bimodules, while  $\mathcal{M}^* = \text{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$  is an invertible  $(\mathcal{O}, \mathcal{A})$ -bimodule. Hence  $\mathcal{L}_{\varphi} \cong \mathcal{M}^* \otimes_{\mathcal{A}} ({}_{\varphi}\mathcal{N})$  is an invertible  $(\mathcal{O}, \mathcal{O})$ -bimodule, which is the same as an invertible  $\mathcal{O}$ -module. The map  $\rho_{\varphi}$  is the composition of the canonical isomorphisms  $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{M}^* \otimes_{\mathcal{A}} ({}_{\varphi}\mathcal{N}) \cong \mathcal{A} \otimes_{\mathcal{A}} ({}_{\varphi}\mathcal{N}) \cong {}_{\varphi}\mathcal{N}$ .  $\square$

**Remark.** In case  $\mathcal{M} = \mathcal{N}$ , the invertible  $\mathcal{O}$ -module  $\mathcal{L}_{\varphi}$  becomes the invertible  $\mathcal{O}$ -module  $\mathcal{I}_{\varphi}$  of Proposition 2.3.

## 3. AZUMAYA ALGEBRAS WITH INVOLUTIONS

**3.1. Basic Concepts of Bilinear Forms.** We recall some concepts from [CF, 2.6]. A *bilinear form* is a pair  $(\mathcal{M}, b)$  consisting of a finite locally free  $\mathcal{O}$ -module  $\mathcal{M}$  and a bilinear morphism  $b: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{O}$  of  $\mathcal{O}$ -modules, equivalently, a morphism  $\mathcal{M} \rightarrow \mathcal{M}^\vee$  of  $\mathcal{O}$ -modules. Given  $b$ , the associated map  $\phi_b: \mathcal{M} \rightarrow \mathcal{M}^\vee$  is defined over  $T \in \mathfrak{Sch}_S$  on  $m \in \mathcal{M}(T)$  to be the map which is

$$\begin{aligned} \phi_b(m): \mathcal{M}(V) &\rightarrow \mathcal{O}(V) \\ m' &\mapsto b(m|_V, m') \end{aligned}$$

over  $V \in \mathfrak{Sch}_T$ . We summarize this by writing  $\phi_b(m) = b(m, \_)$ . A bilinear form  $(\mathcal{M}, b)$  is called *regular* if  $\phi_b$  is an isomorphism. If for all  $T \in \mathfrak{Sch}_S$  and all sections  $x, y \in \mathcal{M}(T)$  we have  $b(x, y) = b(y, x)$  we call  $b$  *symmetric*, if  $b(x, y) = -b(y, x)$  we call  $b$  *skew-symmetric*, and if  $b(x, x) = 0$  we call  $b$  *alternating*.

We associate with a regular bilinear form  $(\mathcal{M}, b)$  the so-called *adjoint anti-automorphism*  $\eta_b: \text{End}_{\mathcal{O}}(\mathcal{M}) \rightarrow \text{End}_{\mathcal{O}}(\mathcal{M})$ , defined on  $a \in \text{End}_{\mathcal{O}}(\mathcal{M})(T)$  by

$$\eta_b(a) := \phi_b|_T^{-1} \circ a^\vee \circ \phi_b|_T: \mathcal{M}|_T \xrightarrow{\phi_b|_T} \mathcal{M}^\vee|_T \xrightarrow{a^\vee} \mathcal{M}^\vee|_T \xrightarrow{\phi_b|_T^{-1}} \mathcal{M}|_T$$

where  $a^\vee$  is the dual endomorphism of  $a$ . Thus, on sections  $m_1, m_2 \in \mathcal{M}(V)$  for  $V \in \mathfrak{Sch}_T$  and  $a \in \text{End}_{\mathcal{O}}(\mathcal{M})(T)$  we have

$$(3.1.1) \quad b(\eta_b(a)(m_1), m_2) = b(m_1, a(m_2)).$$

If  $\mathcal{M}$  is locally free of finite positive rank,  $\eta_b$  is an involution if and only if there exists  $\varepsilon \in \mu_2(S)$  such that  $b(m_1, m_2) = \varepsilon|_T \cdot b(m_2, m_1)$  for all  $T \in \mathfrak{Sch}_S$  and  $m_1, m_2 \in \mathcal{M}(T)$ . In this case,  $\varepsilon$  is unique and we call it the *type* of  $\eta_b$ .

Let  $(\mathcal{M}, b)$  be a regular bilinear form. Since  $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{M}^\vee \cong \text{End}_{\mathcal{O}}(\mathcal{M})$  we have an  $\mathcal{O}$ -module isomorphism

$$(3.1.2) \quad \varphi_b: \mathcal{M} \otimes_{\mathcal{O}} \mathcal{M} \xrightarrow{\sim} \text{End}_{\mathcal{O}}(\mathcal{M}),$$

defined by  $m_1 \otimes m_2 \mapsto (m \mapsto m_1|_V \cdot b(m_2|_V, m))$  for sections  $m_1, m_2$  over  $T \in \mathfrak{Sch}_S$  and  $m$  over  $V \in \mathfrak{Sch}_T$ . If  $\eta_b$  is an involution of type  $\varepsilon$ , its pull-back to  $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{M}$  is the “ $\varepsilon$ -switch”:

$$(3.1.3) \quad \eta_b(\varphi_b(m_1 \otimes m_2)) = \varepsilon|_T \cdot \varphi_b(m_2 \otimes m_1).$$

A *quadratic form* is a pair  $(\mathcal{M}, q)$  where  $\mathcal{M}$  is a finite locally free  $\mathcal{O}$ -module,  $q: \mathcal{M} \rightarrow \mathcal{O}$  is quadratic with respect to scalar multiplication, and the associated polar form  $b_q: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{O}$ , defined on sections by  $b_q(m_1, m_2) = q(m_1 + m_2) - q(m_1) - q(m_2)$ , is bilinear. We say that  $(\mathcal{M}, q)$  is *regular* if  $(\mathcal{M}, b_q)$  is so.

**3.2. Involutions of Azumaya algebras.** We use the conventions of [CF], which are slightly different from those of [KMRT] in the case of base fields. The advantage of Calmès-Fasel’s definitions are that they behave well under arbitrary base change.

Let  $\mathcal{A}$  be an Azumaya  $\mathcal{O}$ -algebra. An *involution of the first kind* is an anti-isomorphism  $\sigma: \mathcal{A} \rightarrow \mathcal{A}$  of Azumaya algebras which satisfies  $\sigma^2 = \text{Id}_{\mathcal{A}}$ . We say

that two involutions  $\sigma$  and  $\sigma'$  are *conjugate* if there exists  $\phi \in \text{Aut}_{\mathcal{O}}(\mathcal{A})$  such that  $\sigma' = \phi \circ \sigma \circ \phi^{-1}$ . In that case we have  $\text{Sym}_{\mathcal{A},\sigma'} = \phi(\text{Sym}_{\mathcal{A},\sigma})$  and similarly for the other  $\mathcal{O}$ -modules  $\text{Alt}_{\mathcal{A},\sigma}$ ,  $\text{Skew}_{\mathcal{A},\sigma}$  and  $\text{Symd}_{\mathcal{A},\sigma}$ .

An Azumaya algebra with involution of the first kind  $(\mathcal{A}, \sigma)$  is étale-locally isomorphic to  $(\text{End}_{\mathcal{O}}(\mathcal{M}), \eta_b)$  for some regular bilinear form  $(\mathcal{M}, b)$  [CF, 2.7.0.25]. Thus, since the localization of  $\sigma$  is still an involution, the bilinear form  $b$  is  $\varepsilon$ -symmetric for some  $\varepsilon \in \mu_2(S)$ . It is known that  $\varepsilon$  only depends on  $\sigma$ . Following [CF, 2.7.0.26] (and not [KMRT]), we call  $\sigma$  *orthogonal* if  $\varepsilon = 1$  (so  $b$  is symmetric), we call it *weakly symplectic* if  $\varepsilon = -1$  (so  $b$  is skew-symmetric), and we call it *symplectic* if  $\varepsilon = -1$  and  $b$  is alternating. These notions are well-defined, stable under base change, and are local for the fppf topology on  $\mathfrak{Sch}_S$ .

Let  $(\mathcal{A}, \sigma)$  be an Azumaya  $\mathcal{O}$ -algebra with orthogonal involution. We mention some basic facts about the submodules of  $\mathcal{A}$  related to  $\sigma$ . We call a submodule  $\mathcal{M} \subset \mathcal{A}$  a *direct summand* of  $\mathcal{A}$  if there exists another module  $\mathcal{N}$  such that  $\mathcal{A} = \mathcal{M} \oplus \mathcal{N}$ . We say that  $\mathcal{M}$  is *locally a direct summand* if there exists a cover  $\{T_i \rightarrow S\}_{i \in I}$  such that each  $\mathcal{M}|_{T_i}$  is a direct summand of  $\mathcal{A}|_{T_i}$ . Some of the results of Lemma 3.3 below are proved in [CF, 2.7.0.29(1), (2)] under the assumption that  $b$  is the polar of a regular quadratic form, an assumption that is not needed for the result below.

**3.3. Lemma.** *Let  $(\mathcal{A}, \sigma)$  be an Azumaya  $\mathcal{O}$ -algebra with orthogonal involution  $\sigma$ .*

- (i) *If  $S$  is affine, then  $\text{Sym}_{\mathcal{A},\sigma}$  and  $\text{Alt}_{\mathcal{A},\sigma}$  are direct summands of  $\mathcal{A}$ . In particular,  $\text{Sym}_{\mathcal{A},\sigma}(S)$  and  $\text{Alt}_{\mathcal{A},\sigma}(S)$  are direct summands of  $\mathcal{A}(S)$  and hence are finite projective  $\mathcal{O}(S)$ -modules.*
- (ii)  *$\text{Sym}_{\mathcal{A},\sigma}$  and  $\text{Alt}_{\mathcal{A},\sigma}$  are locally direct summands of  $\mathcal{A}$  and hence are finite locally free  $\mathcal{O}$ -modules.*
- (iii) *If  $\mathcal{A}$  is locally free of constant rank  $n^2 \in \mathbb{N}_+$ , then  $\text{Sym}_{\mathcal{A},\sigma}$  and  $\text{Alt}_{\mathcal{A},\sigma}$  have rank  $n(n+1)/2$  and  $n(n-1)/2$  respectively.*

*Proof.* (i): Over an affine scheme we are considering the case of an Azumaya algebra with orthogonal involution  $(A, \sigma)$  over a ring  $R$ . Being finite projective can be checked after a faithfully flat extension, and by [EGA-I, 0, 6.7.5] this is also true for the property of being a direct summand. Therefore, by [CF, 2.7.0.25] we can assume that  $(A, \sigma) \cong (\text{End}_R(M), \eta_b)$  where  $M$  is free of finite rank and  $b: M \times M \rightarrow R$  is a regular symmetric bilinear form. Thus, as  $R$ -modules with involution we have  $(A, \sigma) \cong (M^{\otimes 2}, \tau)$  where  $\tau$  is the switch. Let  $\{m_1, \dots, m_n\}$  be a basis of  $M$ . Then  $m_{ij} = m_i \otimes m_j$  is a basis of  $M^{\otimes 2}$ . It follows that

$$\{m_{ii} \mid 1 \leq i \leq n\} \cup \{m_{ij} + m_{ji} \mid 1 \leq i < j \leq n\}$$

is a basis of  $\text{Sym}(M^{\otimes 2}, \tau)$ , while

$$\{m_{ij} - m_{ji} \mid 1 \leq i < j \leq n\}$$

is a basis for  $\text{Alt}(M^{\otimes 2}, \tau)$ . Now, observe that  $\text{Sym}(M^{\otimes 2}, \tau)$  is complemented by  $\text{Span}_R(\{m_{ij} \mid 1 \leq i < j \leq n\})$ , while  $\text{Alt}(M^{\otimes 2}, \tau)$  is complemented by

$$\text{Span}_R(\{m_{ii} \mid 1 \leq i \leq n\} \cup \{m_{ij} \mid 1 \leq i < j \leq n\}),$$



and so both are direct summands of  $A$ .

(ii): This follows from (i) by considering an affine cover of  $S$ .

(iii): If  $\mathcal{A}$  is of constant rank  $n^2$ , then locally we are in the case of the proof of (i) above where the basis  $\{m_1, \dots, m_n\}$  of  $M$  has  $n$  elements. Therefore, the basis given for  $\text{Sym}(A, \sigma)$  has  $n(n+1)/2$  elements, and the basis given for  $\text{Alt}(A, \sigma)$  has  $n(n-1)/2$  elements. Thus the ranks of  $\text{Sym}_{\mathcal{A}, \sigma}$  and  $\text{Alt}_{\mathcal{A}, \sigma}$  are as claimed.  $\square$

In particular, since the above result tells us they are finite locally free,  $\text{Sym}_{\mathcal{A}, \sigma}$  and  $\text{Alt}_{\mathcal{A}, \sigma}$  are represented by vector group schemes. The reader will have noticed that the proof of Lemma 3.3 works for an involution of any type  $\varepsilon \in \mu_2(S)$ . For example, when  $\sigma$  is weakly symplectic and  $\varepsilon = -1$ , the regular bilinear form  $b: M \times M \rightarrow R$  will be skew symmetric and induce an isomorphism  $(A, \sigma) \cong (M^{\otimes 2}, -\tau)$  of  $R$ -modules with involution. In this case, the same roles are played by  $\text{Skew}_{\mathcal{A}, \sigma}$  and  $\text{Synd}_{\mathcal{A}, \sigma}$  instead.

**3.4. Corollary.** *Let  $(\mathcal{A}, \sigma)$  be an Azumaya  $\mathcal{O}$ -algebra with weakly symplectic involution  $\sigma$ .*

- (i) *If  $S$  is affine, then  $\text{Skew}_{\mathcal{A}, \sigma}$  and  $\text{Synd}_{\mathcal{A}, \sigma}$  are direct summands of  $\mathcal{A}$ . In particular,  $\text{Skew}_{\mathcal{A}, \sigma}(S)$  and  $\text{Synd}_{\mathcal{A}, \sigma}(S)$  are direct summands of  $\mathcal{A}(S)$  and hence are finite projective  $\mathcal{O}(S)$ -modules.*
- (ii)  *$\text{Skew}_{\mathcal{A}, \sigma}$  and  $\text{Synd}_{\mathcal{A}, \sigma}$  are locally direct summands of  $\mathcal{A}$  and hence are finite locally free  $\mathcal{O}$ -modules.*
- (iii) *If  $\mathcal{A}$  is locally free of constant rank  $n^2 \in \mathbb{N}_+$ , then  $\text{Skew}_{\mathcal{A}, \sigma}$  and  $\text{Synd}_{\mathcal{A}, \sigma}$  have rank  $n(n+1)/2$  and  $n(n-1)/2$  respectively.*

These submodule pairs,  $\text{Sym}_{\mathcal{A}, \sigma}$  and  $\text{Alt}_{\mathcal{A}, \sigma}$  when  $\sigma$  is orthogonal, and  $\text{Skew}_{\mathcal{A}, \sigma}$  and  $\text{Synd}_{\mathcal{A}, \sigma}$  when  $\sigma$  is weakly symplectic, enjoy convenient orthogonality properties with respect to the trace form. For a submodule  $\mathcal{M} \subset \mathcal{A}$ , its orthogonal complement is another  $\mathcal{O}$ -module  $\mathcal{M}^\perp: \mathfrak{S}\text{ch}_S \rightarrow \mathfrak{A}\text{b}$  given over  $T \in \mathfrak{S}\text{ch}_S$  by

$$\mathcal{M}^\perp(T) = \{a \in \mathcal{A}(T) \mid \text{Trd}_{\mathcal{A}}((a|_V)m) = 0, \forall V \in \mathfrak{S}\text{ch}_T, \forall m \in \mathcal{M}(V)\}.$$

**3.5. Lemma.** *Let  $\mathcal{A}$  be an Azumaya algebra and let  $\mathcal{M} \subseteq \mathcal{A}$  be an  $\mathcal{O}$ -submodule which is finite locally free. Then,*

- (i)  *$\mathcal{M}^\perp$  is quasi-coherent, and*
- (ii)  *$\mathcal{M}^\perp$  is of finite type.*

*Proof.* First, choose a cover  $\{T_i \rightarrow S\}_{i \in I}$  over which each  $\mathcal{M}|_{T_i}$  is free of rank  $n_i$ . Over  $T_i$ , let  $\{m_1, \dots, m_{n_i}\} \in \mathcal{M}(T_i)$  be a free basis, then  $\mathcal{M}^\perp|_{T_i}$  is the kernel of the  $\mathcal{O}|_{T_i}$ -module morphism

$$\begin{aligned} \mathcal{A}|_{T_i} &\rightarrow \mathcal{A}|_{T_i}^{n_i} \\ a &\mapsto (\text{Trd}_{\mathcal{A}}(am_1), \dots, \text{Trd}_{\mathcal{A}}(am_{n_i})). \end{aligned}$$

Both claims then follow over  $T_i$  from [St, Tag 01BY (1), (2)], which applies here because  $\mathcal{A}|_{T_i}$  is quasi-coherent, see Section 1.7. Then, since quasi-coherence and being of finite type are local properties, we are done.  $\square$

**3.6. Lemma.** *Let  $\mathcal{A}$  be an Azumaya  $\mathcal{O}$ -algebra. Then the following hold.*

- (i) *The trace form  $(x, y) \mapsto \text{Trd}_{\mathcal{A}}(xy)$  is a regular symmetric bilinear form on  $\mathcal{A}$ .*
- (ii) *Let  $\sigma$  be an involution of the first kind on  $\mathcal{A}$ . Then,*

$$\begin{aligned} \text{Trd}_{\mathcal{A}}(\text{Sym}_{\mathcal{A},\sigma} \cdot \text{Alt}_{\mathcal{A},\sigma}) &= 0, \text{ and} \\ \text{Trd}_{\mathcal{A}}(\text{Skew}_{\mathcal{A},\sigma} \cdot \text{Symd}_{\mathcal{A},\sigma}) &= 0, \end{aligned}$$

*by which we mean, for  $T \in \mathfrak{S}\mathfrak{ch}_S$  and sections  $x \in \text{Sym}_{\mathcal{A},\sigma}(T)$ ,  $y \in \text{Alt}_{\mathcal{A},\sigma}(T)$  we have  $\text{Trd}_{\mathcal{A}}(xy) = 0$ , and similarly for the second statement.*

- (iii) *If  $\sigma$  is an orthogonal involution on  $\mathcal{A}$ , then*

$$\text{Sym}_{\mathcal{A},\sigma}^{\perp} = \text{Alt}_{\mathcal{A},\sigma} \text{ and } \text{Alt}_{\mathcal{A},\sigma}^{\perp} = \text{Sym}_{\mathcal{A},\sigma}.$$

- (iv) *If  $\sigma$  is a weakly symplectic involution on  $\mathcal{A}$ , then*

$$\text{Symd}_{\mathcal{A},\sigma}^{\perp} = \text{Skew}_{\mathcal{A},\sigma} \text{ and } \text{Skew}_{\mathcal{A},\sigma}^{\perp} = \text{Symd}_{\mathcal{A},\sigma}.$$

*Proof.* (i): This is [CF, 2.5.3.16] (the statement in [CF] is only for even degree algebras, but the proof holds for algebras of any degree).

(ii): Let  $T \in \mathfrak{S}\mathfrak{ch}_S$  and consider  $x \in \text{Sym}_{\mathcal{A},\sigma}(T)$  and  $y \in \text{Alt}_{\mathcal{A},\sigma}(T)$ . After passing to a cover of  $T$  if needed, we may assume  $y = a - \sigma(a)$  for some  $a \in \mathcal{A}(T)$ . Thus, we have

$$\begin{aligned} \text{Trd}_{\mathcal{A}}(x(a - \sigma(a))) &= \text{Trd}_{\mathcal{A}}(xa) - \text{Trd}_{\mathcal{A}}(x\sigma(a)) = \text{Trd}_{\mathcal{A}}(xa) - \text{Trd}_{\mathcal{A}}(a\sigma(x)) \\ &= \text{Trd}_{\mathcal{A}}(xa) - \text{Trd}_{\mathcal{A}}(ax) = 0. \end{aligned}$$

The second statement is seen similarly.

(iii): The first statement is [CF, 2.7.0.29(3)], and because of (ii), a symmetric argument produces the second statement.

(iv): In the case when  $\sigma$  is weakly symplectic,  $\text{Skew}_{\mathcal{A},\sigma}$  and  $\text{Symd}_{\mathcal{A},\sigma}$  are finite locally free  $\mathcal{O}$ -modules by Corollary 3.4(ii), which also means that  $\text{Skew}_{\mathcal{A},\sigma}^{\perp}$  and  $\text{Symd}_{\mathcal{A},\sigma}^{\perp}$  are quasi-coherent of finite type by Lemma 3.5. Therefore, we may argue as done in [CF] for the case above. From (ii) we have that

$$\text{Symd}_{\mathcal{A},\sigma} \subseteq \text{Skew}_{\mathcal{A},\sigma}^{\perp}.$$

Over residue fields this will be an equality for dimension reasons because the dimensions of  $\text{Skew}_{\mathcal{A},\sigma}$  and  $\text{Symd}_{\mathcal{A},\sigma}$  will be the same as the ranks given in Corollary 3.4(iii). Thus by Nakayama's lemma, we will have equality over local rings, and thus

$$\text{Symd}_{\mathcal{A},\sigma} = \text{Skew}_{\mathcal{A},\sigma}^{\perp}$$

globally. A symmetric argument shows the second claim.  $\square$

The trace can also be used to define pairings on  $\text{Symd}_{\mathcal{A},\sigma}$  and  $\text{Alt}_{\mathcal{A},\sigma}$  by extending a construction provided by the exercise [KMRT, II, exercise 15].

**3.7. Lemma.** *Let  $(\mathcal{A}, \sigma)$  be an Azumaya  $\mathcal{O}$ -algebra with orthogonal involution.*

- (i) *There is a unique symmetric bilinear form  $b_+ : \mathbf{Sym}_{\mathcal{A},\sigma} \times \mathbf{Sym}_{\mathcal{A},\sigma} \rightarrow \mathcal{O}$  such that for each affine scheme  $U \in \mathfrak{Sch}_S$  and  $x, y \in \mathcal{A}(U)$ , we have*

$$b_+((x + \sigma(x)), (y + \sigma(y))) = \mathrm{Trd}_{\mathcal{A}}((x + \sigma(x))y).$$

- (ii) *There is a unique symmetric bilinear form  $b_- : \mathbf{Alt}_{\mathcal{A},\sigma} \times \mathbf{Alt}_{\mathcal{A},\sigma} \rightarrow \mathcal{O}$  such that for each affine scheme  $U \in \mathfrak{Sch}_S$  and  $x, y \in \mathcal{A}(U)$ , we have*

$$b_-((x - \sigma(x)), (y - \sigma(y))) = \mathrm{Trd}_{\mathcal{A}}((x - \sigma(x))y).$$

*In addition, this bilinear form is regular.*

*Proof.* (i): We observe first that for each affine scheme  $U$  over  $S$  and  $x, y \in \mathcal{A}(U)$ , we have  $\mathrm{Trd}_{\mathcal{A}}((x + \sigma(x))y) = \mathrm{Trd}_{\mathcal{A}}(x(y + \sigma(y)))$  since  $\mathrm{Trd}_{\mathcal{A}}$  is invariant under  $\sigma$ . This implies that  $b_+$  is well-defined and also that it is symmetric.

(ii): The fact that  $b_-$  is well-defined and symmetric is similar to (i). To see that it is regular, we have to show that the induced map  $\mathbf{Alt}_{\mathcal{A},\sigma} \rightarrow \mathbf{Alt}_{\mathcal{A},\sigma}^{\vee}$  is an isomorphism. Since  $\mathbf{Alt}_{\mathcal{A},\sigma}$  is locally free of finite rank, the Nakayama lemma reduces us to the field case. For all  $x \in \mathbf{Alt}(A, \sigma)$  and  $y \in A$ , we have  $b_-(x(y - \sigma(y))) = \mathrm{Trd}_A(xy)$ . Since  $\mathrm{Trd}_A$  is regular, the kernel of  $b_-$  is zero. Thus  $b_-$  is regular.  $\square$

**3.8. Remark.** The proof of (ii) does not apply in case (i) because  $\mathbf{Sym}_{\mathcal{A},\sigma}$  is not locally free of finite rank. However,  $b_+$  is regular over fields.

**3.9. Lemma.** *Let  $(\mathcal{A}, \sigma)$  be an Azumaya  $\mathcal{O}$ -algebra with an orthogonal involution.*

- (i) *Let  $u \in \mathbf{GL}_{1,\mathcal{A}}(S)$  and let  $\mathrm{Inn}(u)$  be the inner automorphism of  $\mathcal{A}$  induced by  $u$ . Then  $\mathrm{Inn}(u) \circ \sigma$  is an involution of  $\mathcal{A}$  of the first kind. It is orthogonal if and only if  $u \in \mathbf{Sym}_{\mathcal{A},\sigma}(S)$ .*
- (ii) *Assume that the torsion subgroup  $\mathrm{Pic}(S)_{\mathrm{tors}} = 0$ . Then for any orthogonal involution  $\sigma'$  of  $\mathcal{A}$  there exists an invertible  $u \in \mathbf{Sym}_{\mathcal{A},\sigma}(S)$  such that  $\sigma' = \mathrm{Inn}(u) \circ \sigma$ . Furthermore,  $u$  is unique up to an element of  $\mathcal{O}^{\times}(S)$ .*

*Proof.* (i): The first statement is obvious. For the proof of the second we can assume that  $S$  is affine. In this case our claim follows from [Knu, III, (8.1.3)(2)].

(ii): Observe that  $\sigma' \circ \sigma$  is an automorphism of  $\mathcal{A}$ , which is inner by Corollary 2.4, i.e., there exists  $u \in \mathcal{A}(S)^{\times}$  such that  $\sigma' = \mathrm{Inn}(u) \circ \sigma$ . That  $u \in \mathbf{Sym}_{\mathcal{A},\sigma}(S)$  follows from (i). If  $\sigma'$  can also be written as  $\sigma' = \mathrm{Inn}(v) \circ \sigma$  for some invertible  $v \in \mathcal{A}(S)^{\times}$ , then  $v^{-1}u$  is central, proving the last claim.  $\square$

## 4. QUADRATIC PAIRS

**4.1. Definition** (Quadratic pairs). Let  $(\mathcal{A}, \sigma)$  be an Azumaya  $\mathcal{O}$ -algebra with orthogonal involution. Following [KMRT, §5B] and [CF, 2.7.0.30], we call  $(\sigma, f)$  a *quadratic pair* if  $f : \mathbf{Sym}_{\mathcal{A},\sigma} \rightarrow \mathcal{O}$  is a linear form satisfying

$$(4.1.1) \quad f(a + \sigma(a)) = \mathrm{Trd}_{\mathcal{A}}(a)$$

for all  $T \in \mathfrak{Sch}_S$  and every  $a \in \mathcal{A}(T)$ . In this case, we call  $(\mathcal{A}, \sigma, f)$  a *quadratic triple*.

Two quadratic triples  $(\mathcal{A}, \sigma, f)$  and  $(\mathcal{A}', \sigma', f')$  are called *isomorphic* if there exists an isomorphism  $\phi: \mathcal{A} \rightarrow \mathcal{A}'$  of  $\mathcal{O}$ -algebras such that  $\sigma' = \phi \circ \sigma \circ \phi^{-1}$  and  $f' = f \circ \phi^{-1}$  (this is well-defined since  $\mathcal{S}ym_{\mathcal{A}', \sigma'} = \phi(\mathcal{S}ym_{\mathcal{A}, \sigma})$ ). The notion of a quadratic triple is stable under arbitrary base change.

**4.2. Remark.** If  $(\mathcal{A}, \sigma, f)$  is a quadratic triple, then  $f(T) \neq 0$  for all  $T \in \mathfrak{S}ch_S$ , i.e.  $f$  is non-zero everywhere. This follows from Lemma 3.6(i), which guarantees that there is an element  $a \in \mathcal{A}(T)$  with non-zero trace, and then

$$f(a + \sigma(a)) = \text{Trd}_{\mathcal{A}}(a) \neq 0.$$

**4.3. Examples.** Let  $(\mathcal{A}, \sigma)$  be an Azumaya  $\mathcal{O}$ -algebra with orthogonal involution. (a) If  $(\sigma, f)$  is a quadratic pair, then for all  $T \in \mathfrak{S}ch_S$  and  $s \in \mathcal{S}ym_{\mathcal{A}, \sigma}(T)$  we have that

$$2f(s) = f(s + \sigma(s)) = \text{Trd}_{\mathcal{A}}(s).$$

Therefore, when  $2 \in \mathcal{O}^\times$  there exists a unique  $f$  such that  $(\sigma, f)$  is a quadratic pair, namely  $f(s) = \frac{1}{2} \text{Trd}_{\mathcal{A}}(s)$ .

(b) Suppose  $S = \text{Spec}(R)$  where  $R$  is an integral domain whose field of fractions  $K$  has characteristic different from 2. If  $(\mathcal{A}, \sigma)$  is an Azumaya  $\mathcal{O}$ -algebra with orthogonal involution, then there exists at most one linear form  $f$  making  $(\sigma, f)$  a quadratic pair on  $\mathcal{A}$ . Indeed, this follows from uniqueness in (a) after base change from  $S$  to  $\text{Spec}(K)$ .

(c) Given an element  $\ell \in \mathcal{A}(S)$  with  $\ell + \sigma(\ell) = 1$ , we define  $f: \mathcal{S}ym_{\mathcal{A}, \sigma} \rightarrow \mathcal{O}$  over  $T \in \mathfrak{S}ch_S$  as

$$\begin{aligned} f(T): \mathcal{S}ym_{\mathcal{A}, \sigma}(T) &\rightarrow \mathcal{O}(T) \\ s &\mapsto \text{Trd}_{\mathcal{A}}(\ell|_T \cdot s). \end{aligned}$$

This map is clearly linear, and for an element  $a + \sigma(a) \in \mathcal{A}(T)$  we obtain

$$f(a + \sigma(a)) = \text{Trd}_{\mathcal{A}}(\ell|_T \cdot (a + \sigma(a))) = \text{Trd}_{\mathcal{A}}((\ell|_T + \sigma(\ell|_T)) \cdot a) = \text{Trd}_{\mathcal{A}}(a),$$

and so  $(\mathcal{A}, \sigma, f)$  is a quadratic triple. By Lemma 3.6(iii), the function  $f$  is determined by  $\ell$  up to the addition of an element from  $\mathcal{A}lt_{\mathcal{A}, \sigma}(S)$ , i.e.,  $\ell_1 - \ell_2 \in \mathcal{A}lt_{\mathcal{A}, \sigma}(S)$  if and only if  $\text{Trd}_{\mathcal{A}}(\ell_1 \_\_)$  agrees with  $\text{Trd}_{\mathcal{A}}(\ell_2 \_\_)$  on  $\mathcal{S}ym_{\mathcal{A}, \sigma}$ .

More examples are be discussed below in Examples 4.5 and 4.17.

The existence of quadratic pairs is related to quadratic forms. We describe this in the following Lemma 4.4 for the neutral Azumaya algebra  $\text{End}_{\mathcal{O}}(\mathcal{M})$ , generalizing [KMRT, 5.11] for  $S = \text{Spec}(R)$  where  $R$  is a field. Part (i) of 4.4 is stated in [CF, 2.7.0.31].

**4.4. Proposition.** *Let  $\mathcal{M}$  be a locally faithfully projective  $\mathcal{O}$ -module with a regular symmetric bilinear form  $b: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{O}$ . Thus,  $\text{End}_{\mathcal{O}}(\mathcal{M})$  is an Azumaya  $\mathcal{O}$ -algebra with orthogonal adjoint involution  $\eta_b$  by Section 3. One knows that*

$$\varphi_b: \mathcal{M} \otimes_{\mathcal{O}} \mathcal{M} \xrightarrow{\sim} \text{End}_{\mathcal{O}}(\mathcal{M}),$$

defined on sections by  $m_1 \otimes m_2 \mapsto (m \mapsto m_1 b(m_2, m))$ , is an isomorphism of  $\mathcal{O}$ -modules.

(i) Assume  $b = b_q$  for a regular quadratic form  $q$ . Then, there exists a unique linear form  $f_q: \text{Sym}_{\text{End}_{\mathcal{O}}(\mathcal{M}), \eta_b} \rightarrow \mathcal{O}$  such that

$$(4.4.1) \quad (f_q \circ \varphi_b)(m \otimes m) = q(m)$$

holds for all  $T \in \mathfrak{Sch}_S$  and sections  $m \in \mathcal{M}(T)$ . The pair  $(\eta_b, f_q)$  is a quadratic pair.

(ii) Assume  $(\eta_b, f)$  is a quadratic pair for some  $f$ . Then, there exists a unique quadratic form  $q_f: \mathcal{M} \rightarrow \mathcal{O}$ , defined over  $T \in \mathfrak{Sch}_S$  on sections  $m \in \mathcal{M}(T)$  by

$$(4.4.2) \quad q_f(m) = (f \circ \varphi_b)(m \otimes m),$$

whose polar form is  $b$ .

(iii) The involution  $\eta_b$  is part of an orthogonal pair on  $\text{End}_{\mathcal{O}}(\mathcal{M})$  if and only if  $b$  is the polar form of a regular quadratic form.

*Proof.* Since in each statement a global function is given, it suffices to check the claims locally. It is therefore no harm to assume  $S = \text{Spec}(R)$  and to prove the lemma in the setting of a faithfully projective  $R$ -module  $M$  with a regular symmetric bilinear form  $b: M \times M \rightarrow R$ .

(i): We assume that  $b = b_q$  for a regular quadratic form  $q: M \rightarrow R$  and need to prove that there exists a unique linear form  $f_q: \text{Sym}(\text{End}_R(M), \eta_b) \rightarrow R$  satisfying (4.4.1).

The formula  $\eta_b(\varphi_b(m_1 \otimes m_2)) = \varphi_b(m_2 \otimes m_1)$  is immediate from the definitions. Hence  $\varphi_b$  induces an isomorphism between the symmetric elements of  $(\text{End}_R(M), \eta_b)$  and those of  $(M^{\otimes 2}, \tau)$  where  $\tau$  is the “switch”. We claim

$$(4.4.3) \quad \{x \in M \otimes_R M \mid \tau(x) = x\} = \text{Span}_R\{m \otimes m \mid m \in M\}.$$

Indeed, it is obvious that the right-hand side is contained in the left-hand side. Thus, to prove equality we may localize in a maximal ideal, equivalently, we may assume that  $R$  is a local ring and so  $M$  is free. This allows us to consider a free basis  $\{m_1, \dots, m_n\}$  and express  $x \in M^{\otimes 2}$  as  $x = \sum_{i,j=1}^n c_{ij} m_i \otimes m_j$  for some coefficients  $c_{ij} \in R$ . If  $\tau(x) = x$  then we must have that  $c_{ij} = c_{ji}$ , and so we may write

$$x = \sum_{i=1}^n c_{ii} m_i \otimes m_i + \sum_{\substack{i,j=1 \\ i < j}}^n c_{ij} (m_i \otimes m_j + m_j \otimes m_i).$$

We see this belongs to the right side of the equality above by noting that

$$m_i \otimes m_j + m_j \otimes m_i = (m_i + m_j) \otimes (m_i + m_j) - (m_i \otimes m_i) - (m_j \otimes m_j).$$

We are now left with proving the existence of a linear form

$$\tilde{f}: \text{Span}_R\{m \otimes m \mid m \in M\} \rightarrow R, \quad m \otimes m \mapsto q(m).$$

If  $\tilde{f}$  exists as a set map, it is linear and unique by (4.4.3). Thus, the point is to prove that  $\tilde{f}$  is well-defined, i.e.,

$$(4.4.4) \quad \sum_i m_i \otimes m_i = 0 \implies \sum_i q(m_i) = 0.$$

To show (4.4.4) we can again argue by localization and therefore assume that  $M$  is free. But if  $M$  is free, we can use the argument in the proof of [KMRT, 5.11] establishing the existence of  $f$  over fields.

(ii): We assume that  $(\eta_b, f)$  is a quadratic pair for the Azumaya  $R$ -algebra  $\text{End}_R(M)$  and must show that then  $q_f: M \rightarrow R$ , defined by (4.4.2) is a quadratic form whose polar form is  $b$ . To do so, we can use the proof of [KMRT, 5.11], which carries over without any change.

(iii): This summarizes (i) and (ii). □

**4.5. Examples.** We give some further examples. In particular we outline the *split examples* in all degrees.

(a) In the setting of Proposition 4.4, a quadratic pair  $(\eta_b, f)$  forces  $b = b_q$  for a regular quadratic form  $q$ . Hence, if  $\mathcal{M}$  has odd rank, then necessarily  $2 \in \mathcal{O}^\times$ .

(b) Let  $(\mathcal{M}_0, q_0) = (\mathcal{O}^{2n}, q_0)$  where  $q_0$  is the split hyperbolic form on  $\mathcal{O}^{2n}$  defined on sections by

$$q_0(x_1, \dots, x_{2n}) = \sum_{i=1}^n x_{2i-1}x_{2i}.$$

It is regular. We let  $\eta_0$  be the associated orthogonal involution of  $\mathcal{A}_0 = \text{End}_{\mathcal{O}}(\mathcal{M}_0) = \text{M}_{2n}(\mathcal{O})$ , and denote by  $f_0$  the unique linear form on  $\text{Sym}_{\mathcal{A}_0, \eta_0}$  satisfying (4.4.1). It is computed explicitly in the example after [CF, 2.7.0.31]. We will refer to  $(\mathcal{A}_0, \eta_0, f_0)$  as the *split example* (in degree  $2n$ ). We point out that there may be many different linear forms  $f$  making  $(\mathcal{A}_0, \eta_0, f)$  a quadratic triple.

(c) Assume  $2 \in \mathcal{O}^\times$  and let  $(\mathcal{M}_0, q_0) = (\mathcal{O}^{2n+1}, q_0)$  where  $q_0$  is the split hyperbolic form on  $\mathcal{O}^{2n}$  defined on sections by

$$q_0(x_1, \dots, x_{2n+1}) = \left( \sum_{i=1}^n x_{2i-1}x_{2i} \right) + x_{2n+1}^2$$

It is regular since 2 is invertible. We let  $\eta_0$  be the associated orthogonal involution of  $\mathcal{A}_0 = \text{End}_{\mathcal{O}}(\mathcal{M}_0) = \text{M}_{2n+1}(\mathcal{O})$ . It has a unique linear form on  $\text{Sym}_{\mathcal{A}_0, \eta_0}$ , namely  $f_0 = \frac{1}{2} \text{Trd}_{\mathcal{A}_0}$  by Example 4.3(a), such that  $(\mathcal{A}_0, \sigma_0, f_0)$  is a quadratic triple. We also refer to this as the *split example* (in degree  $2n + 1$ ).

(d) Other examples are given in [KMRT, 5.12 and 5.13].

**4.6. Corollary.** [CF, 2.7.0.32] *Every quadratic triple  $(\mathcal{A}, \sigma, f)$  is split étale-locally.*

*Proof.* By first considering the cover by connected components, we may assume that  $\mathcal{A}$  is of constant degree  $n$ . If  $n$  is odd, then by Example 4.5(a) above and Example 4.3(a), we must have  $2 \in \mathcal{O}^\times$  and  $f = \frac{1}{2} \text{Trd}_{\mathcal{A}, \sigma}$ . Then there exists an étale cover over which  $\sigma$  and  $\eta_0$  are isomorphic, which is sufficient since those isomorphisms will identify  $f$  and  $f_0$  due to uniqueness.

Now assume  $n$  is even. We have already started with [CF, 2.7.0.25]: every quadratic triple is étale-locally of the form  $(\text{End}_{\mathcal{O}}(\mathcal{M}), \eta_b, f)$  considered in Proposition 4.4. By Proposition 4.4(ii), the regular bilinear form of that lemma is the polar form of a regular quadratic form  $q: \mathcal{M} \rightarrow \mathcal{O}$ , given by (4.4.2). By [CF, 2.6.1.13], after a second étale extension, we can then assume that  $(\mathcal{M}, q) = (\mathcal{M}_0, q_0)$  as in Example 4.5(b), and that the linear form is the one of Proposition 4.4(i), i.e., that the quadratic triple is split.  $\square$

**4.7. Remark.** Corollary 4.6 shows that, if one wishes, it is sufficient to work with the étale topology on the category  $\mathfrak{Sch}_S$ . In this paper, our arguments do not use particular properties of the fppf topology which are not shared with the étale topology, and so our results still hold in that setting.

**4.8. Corollary.** *Let  $(\mathcal{A}, \sigma, f)$  be a quadratic triple. Assume  $\mathcal{A}$  is of constant rank  $n^2 \in \mathbb{N}$ . Then, we have*

$$f(1_{\mathcal{A}}) = \frac{n}{2} \in \mathcal{O}(S).$$

*Proof.* Since  $\mathcal{A}$  is of constant rank, there will exist a cover  $\{T_i \rightarrow S\}_{i \in I}$  such that  $\mathcal{A}|_{T_i} \cong M_n(\mathcal{O}|_{T_i})$  for all  $i \in I$ . Evaluating locally, where  $1_{\mathcal{A}}|_{T_i}$  is the identity matrix, we have that  $2f|_{T_i}(1_{\mathcal{A}}|_{T_i}) = \text{Trd}_{M_n(\mathcal{O}|_{T_i})}(1_{\mathcal{A}}|_{T_i}) = n$ , and hence

$$2f(1_{\mathcal{A}}) = n \in \mathcal{O}(S)$$

globally as well. If  $n$  is odd then  $2 \in \mathcal{O}^\times$  and so  $f(1_{\mathcal{A}}) = \frac{n}{2}$  makes sense. If  $n = 2m$  is even, we need to argue that  $f(1_{\mathcal{A}}) = m$ . We may assume by Corollary 4.6 that in the above argument, the cover was chosen such that it splits the quadratic triple, and so our local evaluation may be done on the split example. In the calculations of  $f_0$  following [CF, 2.7.0.31] the authors show that  $f_0(E_{ii} + E_{i+1, i+1}) = 1$ , and therefore it follows immediately that  $f_0(1) = m$ .  $\square$

**4.9. Example.** In contrast to Example 4.3(a), if  $2 \notin \mathcal{O}^\times$ , then not every orthogonal  $\sigma$  is part of a quadratic pair. For example, this is so for  $S = \text{Spec}(R)$  with  $2R = 0 \neq R$ , and the Azumaya  $\mathcal{O}$ -algebra  $\mathcal{A} = M_n(\mathcal{O})$  equipped with  $\tau =$  the transpose involution (note that  $a + \sigma(a)$  has 0-diagonal for any  $a \in M_n(\mathcal{O}(T))$  and  $T \in \mathfrak{Sch}_S$ ). But there exists orthogonal involutions  $\sigma$  of  $M_n(\mathcal{O})$  which are part of an orthogonal pair  $(\sigma, f)$ , see [KMRT, 5.9].

**4.10. Classification of Quadratic Triples.** In light of Example 4.9 above, we seek to characterize which orthogonal involutions can participate in a quadratic pair. Furthermore, for a given Azumaya  $\mathcal{O}$ -algebra with orthogonal involution  $(\mathcal{A}, \sigma)$ , we wish to classify all possible such  $f$  which make  $(\mathcal{A}, \sigma, f)$  a quadratic triple. We first provide this classification over affine schemes, beginning with the following.

Given a quadratic triple  $(\mathcal{A}, \sigma, f)$ , it is of the form of Example 4.3(c) above exactly when  $f$  can be extended to a linear form on all of  $\mathcal{A}$ .

**4.11. Lemma.** *Let  $(\mathcal{A}, \sigma, f)$  be a quadratic triple. Then, the following are equivalent.*

- (i) *There exists a linear form  $f' : \mathcal{A} \rightarrow \mathcal{O}$  such that  $f'|_{\text{Sym}_{\mathcal{A},\sigma}} = f$ ,*
- (ii) *There exists  $\ell \in \mathcal{A}(S)$  with  $\ell + \sigma(\ell) = 1$  such that for all  $T \in \mathfrak{Sch}_S$  and  $s \in \text{Sym}_{\mathcal{A},\sigma}(T)$ , we have*

$$f(s) = \text{Trd}_{\mathcal{A}}(\ell|_T \cdot s).$$

*Proof.* (ii) $\Rightarrow$ (i): This implication is clear, see Example 4.3(c).

(i) $\Rightarrow$ (ii): We follow the second half of the proof of [CF, 4.2.0.12]. The map  $f'$  is of the form  $f' = \text{Trd}_{\mathcal{A}}(\ell \_)$  for some  $\ell \in \mathcal{A}(S)$  due to regularity of the trace form by Lemma 3.6(i). Then, for any  $a \in \mathcal{A}(S)$  we have  $\text{Trd}_{\mathcal{A}}(\ell\sigma(a)) = \text{Trd}_{\mathcal{A}}(a\sigma(\ell)) = \text{Trd}_{\mathcal{A}}(\sigma(\ell)a)$  and so

$$\text{Trd}_{\mathcal{A}}(a) = f'(a + \sigma(a)) = \text{Trd}_{\mathcal{A}}(\ell(a + \sigma(a))) = \text{Trd}_{\mathcal{A}}((\ell + \sigma(\ell))a).$$

Hence,  $\ell + \sigma(\ell) = 1$ , once again by regularity of the trace form.  $\square$

When  $S$  is affine, the construction of Example 4.3(c) produces all quadratic triples and we have the following classification.

**4.12. Proposition** (Affine Classification). *Let  $S$  be an affine scheme and let  $(\mathcal{A}, \sigma)$  be an Azumaya  $\mathcal{O}$ -algebra with orthogonal involution. Consider  $\xi : \mathcal{A}/\text{Alt}_{\mathcal{A},\sigma} \rightarrow \text{Symd}_{\mathcal{A},\sigma}$  from (1.6.1).*

- (i) *Suppose  $(\mathcal{A}, \sigma, f)$  is a quadratic triple. Then there exists  $\ell_f \in \mathcal{A}(S)$  such that  $\ell_f + \sigma(\ell_f) = 1_{\mathcal{A}}$ , and for all  $T \in \mathfrak{Sch}_S$  and  $s \in \text{Sym}_{\mathcal{A},\sigma}(T)$ , we have*

$$f(s) = \text{Trd}_{\mathcal{A}}(\ell_f|_T \cdot s).$$

*In particular,  $\ell_f + \sigma(\ell_f) = 1_{\mathcal{A}}$  means that  $1_{\mathcal{A}} \in \text{Symd}_{\mathcal{A},\sigma}(S)$ .*

- (ii) *Assume that  $1_{\mathcal{A}} \in \text{Symd}_{\mathcal{A},\sigma}(S)$ . Then,  $\xi(S)^{-1}(1_{\mathcal{A}})$  is a non-empty subset of  $(\mathcal{A}/\text{Alt}_{\mathcal{A},\sigma})(S)$  and the map of sets*

$$\left\{ \begin{array}{l} f \text{ such that } (\mathcal{A}, \sigma, f) \\ \text{is a quadratic triple} \end{array} \right\} \rightarrow \xi(S)^{-1}(1_{\mathcal{A}})$$

$$f \mapsto \overline{\ell_f}$$

*is a bijection. In particular, for a given  $f$  the element  $\ell_f$  is uniquely determined up to addition by an element of  $\text{Alt}_{\mathcal{A},\sigma}(S)$ .*

*Proof.* (i): Since  $S$  is affine, by Lemma 3.3 we know  $\text{Sym}_{\mathcal{A},\sigma}$  is a direct summand of  $\mathcal{A}$ . Therefore, for any quadratic triple  $(\mathcal{A}, \sigma, f)$ , the function  $f$  may be extended to  $\mathcal{A}$ . The result then follows from Lemma 4.11.

(ii): First we show  $\xi(S)^{-1}(1_{\mathcal{A}})$  is non-empty. We have a short exact sequence

$$0 \rightarrow \text{Skew}_{\mathcal{A},\sigma} \rightarrow \mathcal{A} \xrightarrow{\text{Id} + \sigma} \text{Symd}_{\mathcal{A},\sigma} \rightarrow 0$$

with associated long exact sequence

$$0 \rightarrow \text{Skew}_{\mathcal{A},\sigma}(S) \rightarrow \mathcal{A}(S) \rightarrow \text{Symd}_{\mathcal{A},\sigma}(S) \rightarrow H_{\text{fppf}}^1(S, \text{Alt}_{\mathcal{A},\sigma}) \rightarrow \dots$$



Since  $S$  is affine and  $\mathcal{S}kew_{\mathcal{A},\sigma}$  is quasi-coherent, applying Lemma 1.8 gives that  $H_{\text{fppf}}^1(S, \mathcal{S}kew_{\mathcal{A},\sigma}) = 0$  and so  $\mathcal{A}(S) \rightarrow \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$  is surjective. Hence  $\xi(S)^{-1}(1_{\mathcal{A}}) \neq \emptyset$ .

Next we show that the map is bijective as claimed. Lemma 1.8 similarly shows that  $H_{\text{fppf}}^1(S, \mathcal{A}lt_{\mathcal{A},\sigma}) = 0$  and so there is a surjection  $\mathcal{A}(S) \rightarrow (\mathcal{A}/\mathcal{A}lt_{\mathcal{A},\sigma})(S) \cong \mathcal{A}(S)/\mathcal{A}lt_{\mathcal{A},\sigma}(S)$ .

Let  $(\mathcal{A}, \sigma, f_1)$  and  $(\mathcal{A}, \sigma, f_2)$  be two quadratic triples. By (i) we have elements  $\ell_{f_1}, \ell_{f_2} \in \mathcal{A}(S)$  defining the linear forms. If  $\overline{\ell_{f_1}} = \overline{\ell_{f_2}}$  then  $\ell_{f_1} - \ell_{f_2} \in \mathcal{A}lt_{\mathcal{A},\sigma}(S)$ , and so by Lemma 3.6(iii) the maps  $\text{Trd}_{\mathcal{A}}(\ell_{f_1}\_)$  and  $\text{Trd}_{\mathcal{A}}(\ell_{f_2}\_)$  agree on  $\mathcal{S}ym_{\mathcal{A},\sigma}$ , i.e.,  $f_1 = f_2$  and so the map in question is injective.

Since  $\mathcal{A}(S) \rightarrow (\mathcal{A}/\mathcal{A}lt_{\mathcal{A},\sigma})(S)$  is surjective and by the commutativity of (1.6.1), every element of  $\xi(S)^{-1}(1_{\mathcal{A}})$  is of the form  $\overline{\ell}$  for some  $\ell \in \mathcal{A}(S)$  with  $\ell + \sigma(\ell) = 1$ . We may then use Example 4.3(c) to see that  $f = \text{Trd}_{\mathcal{A}}(\ell\_)|_{\mathcal{S}ym_{\mathcal{A},\sigma}}$  extends  $(\mathcal{A}, \sigma)$  to a quadratic triple. The process in (i) then produces a possibly different  $\ell_f \in \mathcal{A}(S)$  such that  $f = \text{Trd}_{\mathcal{A}}(\ell_f\_)|_{\mathcal{S}ym_{\mathcal{A},\sigma}}$  also. Since this is the same function on  $\mathcal{S}ym_{\mathcal{A},\sigma}$ , we use Lemma 3.6(iii) again to see that  $\ell - \ell_f \in \mathcal{A}lt_{\mathcal{A},\sigma}(S)$  and so  $\overline{\ell} = \overline{\ell_f}$ . Therefore the map in question is surjective and we are done.  $\square$

**4.13. Remark.** When  $S = \text{Spec}(R)$  is an affine scheme we are dealing with an Azumaya  $R$ -algebra  $A = \mathcal{A}(S)$  and an orthogonal involution  $\sigma(S)$  on  $A$ , which by abuse of notation we also denote  $\sigma$ . Then,  $\text{Sym}(A, \sigma) = \mathcal{S}ym_{\mathcal{A},\sigma}(S)$  and a quadratic triple is given by an  $R$ -linear map  $f: \text{Sym}(A, \sigma) \rightarrow R$  satisfying  $f(a + \sigma(a)) = \text{Trd}_A(a)$  for all  $a \in A$ . By quasi-coherence we have that

$$\begin{aligned} \text{Alt}(A, \sigma) &= \mathcal{A}lt_{\mathcal{A},\sigma}(S) = \{a - \sigma(a) \mid a \in A\}, \\ \text{Symd}(A, \sigma) &= \mathcal{S}ymd_{\mathcal{A},\sigma}(S) = \{a + \sigma(a) \mid a \in A\}, \end{aligned}$$

and furthermore  $(\mathcal{A}/\mathcal{A}lt_{\mathcal{A},\sigma})(S) = A/\text{Alt}(A, \sigma)$ . The map  $\xi$  of (1.6.1) behaves over  $S$  as

$$\begin{aligned} \xi: A/\text{Alt}(A, \sigma) &\rightarrow \text{Symd}(A, \sigma) \\ \overline{a} &\mapsto a + \sigma(a) \end{aligned}$$

which is a well-defined linear map. Then Proposition 4.12 says that any quadratic triple  $(A, \sigma, f)$  is of the form  $f = \text{Trd}_A(\ell\_)$  for some  $\ell \in A$  with  $\ell + \sigma(\ell) = 1$ . Furthermore, having  $1 \in \text{Symd}(A, \sigma)$  is sufficient to extend  $(A, \sigma)$  to a quadratic triple since the preimage  $\xi^{-1}(1)$  is non-empty. In this case, all such linear forms extending  $(A, \sigma)$  are classified by  $\xi^{-1}(1) \subseteq A/\text{Alt}(A, \sigma)$ . Over fields this recovers the classification (for fixed  $\sigma$ ) given in [KMRT, §5].

By considering an affine open cover, Proposition 4.12(i) recovers [CF, 4.2.0.12], which states that Zariski locally a quadratic triple is of the form in Example 4.3(c).

Given an Azumaya  $\mathcal{O}$ -algebra with orthogonal involution  $(\mathcal{A}, \sigma)$ , Proposition 4.12 showed that when  $S$  is affine, the condition  $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$  is necessary and

sufficient for there to be a quadratic triple extending  $(\mathcal{A}, \sigma)$ . We show now that this condition is necessary in general, however Lemma 7.5 shows that it is no longer sufficient.

**4.14. Lemma.** *Let  $(\mathcal{A}, \sigma, f)$  be a quadratic triple. Then,  $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A}, \sigma}(S)$ .*

*Proof.* We know by Proposition 4.12(i) that for an affine cover  $\{U_i \rightarrow S\}_{i \in I}$  of  $S$  we have that

$$f|_{U_i} = \text{Trd}_{\mathcal{A}}(\ell_i \_)$$

for some  $\ell_i \in \mathcal{A}(U_i)$  with  $\ell_i + \sigma(\ell_i) = 1_{\mathcal{A}(U_i)}$ . In particular,  $1_{\mathcal{A}(U_i)} \in \mathcal{S}ymd_{\mathcal{A}, \sigma}(U_i)$ . Since the elements  $1_{\mathcal{A}(U_i)}$  clearly agree on overlaps of the cover  $\{U_i \rightarrow S\}_{i \in I}$  and  $\mathcal{S}ymd_{\mathcal{A}, \sigma}$  is a sheaf, we obtain that  $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A}, \sigma}(S)$ .  $\square$

**4.15. Lemma.** *Let  $(\mathcal{A}, \sigma)$  be an Azumaya  $\mathcal{O}$ -algebra with orthogonal involution. Then the following are equivalent.*

- (i)  $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A}, \sigma}(S)$
- (ii) For all affine  $U \in \mathfrak{S}ch_S$ , there exists  $f_U: \mathcal{S}ym_{\mathcal{A}|_U, \sigma|_U} \rightarrow \mathcal{O}|_U$  such that  $(\mathcal{A}|_U, \sigma|_U, f_U)$  is a quadratic triple.
- (iii) For all affine open subschemes  $U \subseteq S$ , there exists  $f_U: \mathcal{S}ym_{\mathcal{A}|_U, \sigma|_U} \rightarrow \mathcal{O}|_U$  such that  $(\mathcal{A}|_U, \sigma|_U, f_U)$  is a quadratic triple.
- (iv) There exists an fppf cover  $\{T_i \rightarrow S\}_{i \in I}$  such that for each  $i \in I$ , there exists  $f_i: \mathcal{S}ym_{\mathcal{A}|_{T_i}, \sigma|_{T_i}} \rightarrow \mathcal{O}|_{T_i}$  such that  $(\mathcal{A}|_{T_i}, \sigma|_{T_i}, f_i)$  is a quadratic triple.
- (v) There exists an fppf cover  $\{T_i \rightarrow S\}_{i \in I}$  and regular quadratic forms  $(\mathcal{M}_i, q_i)$  over  $\mathcal{O}|_{T_i}$  such that for each  $i \in I$ ,

$$(\mathcal{A}|_{T_i}, \sigma|_{T_i}) \cong (\text{End}_{\mathcal{O}|_{T_i}}(\mathcal{M}_i), \eta_q)$$

where  $\eta_q$  is the adjoint involution of the polar bilinear form of  $q$ .

*Proof.* We argue that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i), and then that (iv) $\Leftrightarrow$ (v).

(i) $\Rightarrow$ (ii): Assume  $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A}, \sigma}(S)$ . Then, for an affine  $U \in \mathfrak{S}ch_S$ , by restriction we have that  $1_{\mathcal{A}(U)} \in \mathcal{S}ymd_{\mathcal{A}, \sigma}(U)$ . Now consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{S}kew_{\mathcal{A}, \sigma} \longrightarrow \mathcal{A} \longrightarrow \mathcal{S}ymd_{\mathcal{A}, \sigma} \longrightarrow 0$$

and its associated cohomology sequence

$$\dots \longrightarrow \mathcal{A}(U) \longrightarrow \mathcal{S}ymd_{\mathcal{A}, \sigma}(U) \longrightarrow H_{\text{fppf}}^1(U, \mathcal{S}kew_{\mathcal{A}, \sigma}) \longrightarrow \dots$$

By Lemma 1.8 we have that  $H_{\text{fppf}}^1(U, \mathcal{S}kew_{\mathcal{A}, \sigma}) = 0$  since  $U$  is affine and  $\mathcal{S}kew_{\mathcal{A}, \sigma}$  is quasi-coherent. It follows that  $1 + \sigma: \mathcal{A}(U) \rightarrow \mathcal{S}ymd_{\mathcal{A}, \sigma}(U)$  is surjective and thus there exists  $\ell \in \mathcal{A}(U)$  such that  $\ell + \sigma(\ell) = 1_{\mathcal{A}(U)}$ . We may then construct our desired quadratic triple as in Example 4.3(c).

(ii) $\Rightarrow$ (iii) is obvious, and (iii) $\Rightarrow$ (iv) by taking an affine open cover of  $S$ .

(iv) $\Rightarrow$ (i): Let such a cover be given. By Lemma 4.14 we have  $1_{\mathcal{A}(T_i)} \in \mathcal{S}ymd_{\mathcal{A}, \sigma}(T_i)$  for all  $i \in I$ . These agree on overlaps and so  $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A}, \sigma}(S)$  by the sheaf property.

(iv) $\Rightarrow$ (v): Since  $(\mathcal{A}, \sigma)$  is locally isomorphic to a neutral algebra, we may assume by refining the given cover that  $(\mathcal{A}|_{T_i}, \sigma|_{T_i}, f_i) = (\text{End}_{\mathcal{O}}(\mathcal{M}_i), \sigma|_{T_i}, f_i)$  for some  $\mathcal{O}|_{T_i}$ -modules  $\mathcal{M}_i$ . There are then quadratic forms  $q_i: \mathcal{M}_i \rightarrow \mathcal{O}|_{T_i}$  whose polar bilinear form has adjoint involution  $\sigma|_{T_i}$  by Proposition 4.4(ii).

(v) $\Rightarrow$ (iv): This is Proposition 4.4(i).  $\square$

**4.16. Definition** (Locally Quadratic Involution). We call an orthogonal involution on an Azumaya  $\mathcal{O}$ -algebra *locally quadratic* if it satisfies the equivalent conditions of Lemma 4.15.

We now give two more examples of constructions which produce quadratic triples.

**4.17. Examples.** Let  $(\mathcal{A}, \sigma)$  be an Azumaya  $\mathcal{O}$ -algebra with orthogonal involution.

(a) Given an fppf cover  $\{T_i \rightarrow S\}_{i \in I}$  and elements  $\ell_i \in \mathcal{A}(T_i)$  satisfying  $\ell_i + \sigma(\ell_i) = 1$  and  $\ell_i|_{T_{ij}} - \ell_j|_{T_{ij}} \in \text{Alt}_{\mathcal{A}, \sigma}(T_{ij})$  for all  $i, j \in I$ , we can construct a unique quadratic triple  $(\mathcal{A}, \sigma, f)$  such that  $f|_{T_i} = \text{Trd}_{\mathcal{A}}(\ell_i \_)$  holds for all  $i \in I$ . To do so, locally define maps  $f_i: \text{Sym}_{\mathcal{A}, \sigma}|_{T_i} \rightarrow \mathcal{O}|_{T_i}$  by restricting the linear forms  $\text{Trd}_{\mathcal{A}}(\ell_i \_)$  to  $\text{Sym}_{\mathcal{A}, \sigma}|_{T_i}$ . By assumption and Lemma 3.6(ii), these local maps agree on overlaps of the cover  $\{T_i \rightarrow S\}_{i \in I}$  and therefore glue together to define a global linear form  $f: \text{Sym}_{\mathcal{A}, \sigma} \rightarrow \mathcal{O}$ . The fact that  $(\mathcal{A}, \sigma, f)$  is a quadratic triple can be checked locally on the  $T_i$ , where it follows from Example 4.3(c).

(b) Let  $\sigma$  be a locally quadratic involution. Let  $\xi: \mathcal{A}/\text{Alt}_{\mathcal{A}, \sigma} \rightarrow \text{Symd}_{\mathcal{A}, \sigma}$  be the map of (1.6.1). Given an element  $\lambda \in \xi(S)^{-1}(1_{\mathcal{A}}) \subseteq (\mathcal{A}/\text{Alt}_{\mathcal{A}, \sigma})(S)$ , we define a quadratic triple  $(\mathcal{A}, \sigma, f_{\lambda})$  as follows. There will exist a cover  $\{T_i \rightarrow S\}_{i \in I}$  and elements  $\ell_i \in \mathcal{A}(T_i)$  such that  $\bar{\ell}_i = \lambda|_{T_i}$ . By the commutativity of (1.6.1) and the fact that  $\xi(\lambda|_{T_i}) = 1$ , each  $\ell_i$  will satisfy  $\ell_i + \sigma(\ell_i) = 1$ . Since  $(\lambda|_{T_i})|_{T_{ij}} = (\lambda|_{T_j})|_{T_{ij}}$ , we will have that  $\ell_i|_{T_{ij}} - \ell_j|_{T_{ij}} \in \text{Alt}_{\mathcal{A}, \sigma}(T_{ij})$  and hence (a) above applies to construct  $f_{\lambda}$ . Since we can always compare over a common refinement, it is clear that  $f_{\lambda}$  is independent of which cover was chosen.

We are now able to show that all quadratic triples arise from the construction of Examples 4.17(b). This gives a classification of all quadratic triples which extend a given locally quadratic involution. In contrast to Proposition 4.12(ii), in the following Theorem the set  $\xi(S)^{-1}(1_{\mathcal{A}})$  may be empty, see Lemma 7.5.

**4.18. Theorem** (Classification). *Let  $(\mathcal{A}, \sigma)$  be an Azumaya  $\mathcal{O}$ -algebra with locally quadratic involution. Let  $\xi: \mathcal{A}/\text{Alt}_{\mathcal{A}, \sigma} \rightarrow \text{Symd}_{\mathcal{A}, \sigma}$  be the map of (1.6.1).*

- (i) *Suppose  $(\mathcal{A}, \sigma, f)$  is a quadratic triple. Then, there exists  $\lambda_f \in \xi(S)^{-1}(1_{\mathcal{A}})$  such that  $f$  is the linear form  $f_{\lambda_f}$  of Example 4.17(b).*
- (ii) *The map of sets*

$$\left\{ \begin{array}{l} f \text{ such that } (\mathcal{A}, \sigma, f) \\ \text{is a quadratic triple} \end{array} \right\} \rightarrow \xi(S)^{-1}(1_{\mathcal{A}})$$

$$f \mapsto \lambda_f$$

*is a bijection.*

*Proof.* Both claims follow from Proposition 4.12 and the fact that the functor

$$\begin{aligned} \xi^{-1}(1_{\mathcal{A}}): \mathfrak{Sch}_S &\rightarrow \mathfrak{Sets} \\ T &\mapsto \xi(T)^{-1}(1_{\mathcal{A}|T}) \end{aligned}$$

is a subsheaf of  $\mathcal{A}/\mathcal{Alt}_{\mathcal{A},\sigma}$ . In detail:

(i): Let  $(\mathcal{A}, \sigma, f)$  be a quadratic triple. Let  $\{U_i \rightarrow S\}_{i \in I}$  be an affine cover of  $S$ . Then  $(\mathcal{A}|_{U_i}, \sigma|_{U_i}, f|_{U_i})$  is a quadratic triple over an affine scheme, and so by Proposition 4.12 there exists an  $\ell_i \in \mathcal{A}(U_i)$  such that  $\overline{\ell}_i \in \xi^{-1}(1_{\mathcal{A}})(U_i)$  is the unique element corresponding to  $f|_{U_i}$ . Since  $(f|_{U_i})|_{U_{ij}} = (f|_{U_j})|_{U_{ij}}$  we have that  $\ell_i|_{U_{ij}} - \ell_j|_{U_{ij}} \in \mathcal{Alt}_{\mathcal{A},\sigma}(U_{ij})$  and so  $\overline{\ell}_i|_{U_{ij}} = \overline{\ell}_j|_{U_{ij}}$ . Therefore, these sections glue together into an element  $\lambda_f \in \xi^{-1}(1_{\mathcal{A}})(S)$ . It is then clear that  $f = f_{\lambda_f}$  since  $f|_{U_i} = \text{Trd}_{\mathcal{A}}(\lambda_f|_{U_i} \cdot \_ ) = f_{\lambda_f}|_{U_i}$  for all  $i \in I$  by construction.

(ii): Bijectivity is clear from the proof of (i) since the correspondence  $f|_{U_i} \rightarrow \lambda_f|_{U_i}$  will be bijective over each  $U_i$  in the chosen affine cover.  $\square$

**4.19. Remark.** We may equivalently view Theorem 4.18 as saying that there is an isomorphism of sheaves between

$$\begin{aligned} \mathcal{F}: \mathfrak{Sch}_S &\rightarrow \mathfrak{Sets} \\ T &\mapsto \left\{ \begin{array}{l} f \text{ such that } (\mathcal{A}|_T, \sigma|_T, f) \\ \text{is a quadratic triple over } T \end{array} \right\} \end{aligned}$$

and the sheaf  $\xi^{-1}(1_{\mathcal{A}})$ . Taking part of (1.6.1), we have an exact sequence

$$0 \rightarrow \mathcal{Skew}_{\mathcal{A},\sigma}/\mathcal{Alt}_{\mathcal{A},\sigma} \rightarrow \mathcal{A}/\mathcal{Alt}_{\mathcal{A},\sigma} \xrightarrow{\xi} \mathcal{Symd}_{\mathcal{A},\sigma} \rightarrow 0$$

with corresponding long exact sequence

$$\dots \rightarrow (\mathcal{A}/\mathcal{Alt}_{\mathcal{A},\sigma})(S) \xrightarrow{\xi(S)} \mathcal{Symd}_{\mathcal{A},\sigma}(S) \xrightarrow{\delta'} H^1(S, \mathcal{Skew}_{\mathcal{A},\sigma}/\mathcal{Alt}_{\mathcal{A},\sigma}) \rightarrow \dots$$

By Lemma 1.5, the isomorphism class of  $\xi^{-1}(1_{\mathcal{A}})$  is  $\delta'(1_{\mathcal{A}}) \in H^1(S, \mathcal{Skew}_{\mathcal{A},\sigma}/\mathcal{Alt}_{\mathcal{A},\sigma})$ . In Definition 6.1 below we call this element the weak obstruction.

## 5. QUADRATIC PAIRS ON TENSOR PRODUCTS

Given two Azumaya  $\mathcal{O}$ -algebras  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , their tensor product  $\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2$  is again an Azumaya  $\mathcal{O}$ -algebra. If in addition these algebras are equipped with involutions  $\sigma_1$  and  $\sigma_2$  respectively, then  $(\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2, \sigma_1 \otimes \sigma_2)$  is an algebra with involution. Throughout this section we will use the notation

$$(\mathcal{A}, \sigma) := (\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2, \sigma_1 \otimes \sigma_2).$$

Let  $\varepsilon_1, \varepsilon_2 \in \mu_2(S)$  be the types of  $\sigma_1$  and  $\sigma_2$ . Since the type of an involution can be determined locally, we may apply [Knu, 8.1.3(1)] which applies over affine schemes, to see that the type of  $\sigma_1 \otimes \sigma_2$  will be the product  $\varepsilon_1 \varepsilon_2$ . For our purposes we want  $\sigma_1 \otimes \sigma_2$  to be orthogonal, so we will focus on two cases: when  $\sigma_i$  are both orthogonal ( $\varepsilon_i = 1$ ), and when they are both symplectic (in which case  $\varepsilon_i = -1$ ).

In preparation, we generalize [KMRT, 5.17] to our setting. We use the notation  $\mathcal{F}^\sharp$  to denote the fppf sheafification of a presheaf  $\mathcal{F}$  as in [St, Tag 03NS].

**5.1. Lemma.** *Let  $(\mathcal{A}_1, \sigma_1)$  and  $(\mathcal{A}_2, \sigma_2)$  be two Azumaya  $\mathcal{O}$ -algebras with involution. If  $\sigma_i$  are both orthogonal, then*

$$(i) \quad \mathbf{Sym}_{\mathcal{A}, \sigma} = (\mathbf{Symd}_{\mathcal{A}, \sigma} + (\mathbf{Sym}_{\mathcal{A}_1, \sigma_1} \otimes_{\mathcal{O}} \mathbf{Sym}_{\mathcal{A}_2, \sigma_2}))^\sharp,$$

*If instead  $\sigma_i$  are both weakly symplectic, then*

$$(ii) \quad \mathbf{Sym}_{\mathcal{A}, \sigma} = (\mathbf{Symd}_{\mathcal{A}, \sigma} + (\mathbf{Skew}_{\mathcal{A}_1, \sigma_1} \otimes_{\mathcal{O}} \mathbf{Skew}_{\mathcal{A}_2, \sigma_2}))^\sharp.$$

*Proof.* (i): If there exists an fppf cover  $\{T_i \rightarrow S\}_{i \in I}$  such that the restrictions of each sheaf are equal, then the sheaves are equal globally. Since  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are Azumaya  $\mathcal{O}$ -algebras, there exists a sufficiently fine affine cover  $\{U_i \rightarrow S\}_{i \in I}$  such that  $(\mathcal{A}_1, \sigma_1)|_{U_i} \cong (\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{M}_1, \tau)$  and  $(\mathcal{A}_2, \sigma_2)|_{U_i} \cong (\mathcal{M}_2 \otimes_{\mathcal{O}} \mathcal{M}_2, \tau)$  for free  $\mathcal{O}|_{U_i}$ -modules of finite rank  $\mathcal{M}_1$  and  $\mathcal{M}_2$  with the switch involution  $\tau$  by (3.1.2). Since the component sheaves are quasi-coherent and the  $U_i$  are affine, we have

$$\begin{aligned} & (\mathbf{Symd}_{\mathcal{A}, \sigma} + (\mathbf{Sym}_{\mathcal{A}_1, \sigma_1} \otimes_{\mathcal{O}} \mathbf{Sym}_{\mathcal{A}_2, \sigma_2}))^\sharp|_{U_i} \\ &= \mathbf{Symd}_{\mathcal{A}, \sigma}|_{U_i} + (\mathbf{Sym}_{\mathcal{A}_1, \sigma_1} \otimes_{\mathcal{O}} \mathbf{Sym}_{\mathcal{A}_2, \sigma_2})|_{U_i} \\ &= \mathbf{Symd}_{\mathcal{A}|_{U_i}, \sigma|_{U_i}} + (\mathbf{Sym}_{\mathcal{A}_1|_{U_i}, \sigma_1|_{U_i}} \otimes_{\mathcal{O}|_{U_i}} \mathbf{Sym}_{\mathcal{A}_2|_{U_i}, \sigma_2|_{U_i}}) \end{aligned}$$

Quasi-coherence also means that this sheaf is determined by the  $\mathcal{O}(U_i)$ -module

$$\begin{aligned} & \mathbf{Symd}_{\mathcal{A}, \sigma}(U_i) + (\mathbf{Sym}_{\mathcal{A}_1, \sigma_1}(U_i) \otimes_{\mathcal{O}(U_i)} \mathbf{Sym}_{\mathcal{A}_2, \sigma_2}(U_i)) \\ &= \mathbf{Symd}((\mathcal{M}_1(U_i) \otimes_{\mathcal{O}(U_i)} \mathcal{M}_1(U_i)) \otimes_{\mathcal{O}(U_i)} (\mathcal{M}_2(U_i) \otimes_{\mathcal{O}(U_i)} \mathcal{M}_2(U_i)), \tau \otimes \tau) \\ & \quad + \mathbf{Sym}(\mathcal{M}_1(U_i) \otimes_{\mathcal{O}(U_i)} \mathcal{M}_1(U_i), \tau) + \mathbf{Sym}(\mathcal{M}_2(U_i) \otimes_{\mathcal{O}(U_i)} \mathcal{M}_2(U_i), \tau). \end{aligned}$$

and  $\mathbf{Sym}_{\mathcal{A}, \sigma}|_{U_i}$  is determined by the  $\mathcal{O}(U_i)$ -module

$$\begin{aligned} & \mathbf{Sym}(\mathcal{A}(U_i), \sigma(U_i)) \\ &= \mathbf{Sym}((\mathcal{M}_1(U_i) \otimes_{\mathcal{O}(U_i)} \mathcal{M}_1(U_i)) \otimes_{\mathcal{O}(U_i)} (\mathcal{M}_2(U_i) \otimes_{\mathcal{O}(U_i)} \mathcal{M}_2(U_i)), \tau \otimes \tau). \end{aligned}$$

Therefore it is sufficient to consider the case of  $M_1$  and  $M_2$  being free modules of finite rank over a ring  $R$ , and prove that

$$\begin{aligned} & \mathbf{Sym}((M_1 \otimes_R M_1) \otimes_R (M_2 \otimes_R M_2), \tau \otimes \tau) \\ &= \mathbf{Symd}((M_1 \otimes_R M_1) \otimes_R (M_2 \otimes_R M_2), \tau \otimes \tau) \\ & \quad + \mathbf{Sym}(M_1 \otimes_R M_1, \tau) \otimes_R \mathbf{Sym}(M_2 \otimes_R M_2, \tau) \end{aligned}$$

Clearly the second module is included in the first. Now, let  $\{m_1, \dots, m_a\}$  and  $\{n_1, \dots, n_b\}$  be free bases of  $M_1$  and  $M_2$  respectively. Then  $\mathbf{Sym}((M_1 \otimes_R M_1) \otimes_R (M_2 \otimes_R M_2), \tau \otimes \tau)$  has a basis consisting of the elements

$$\begin{aligned} & (m_i \otimes m_i) \otimes (n_k \otimes n_k), \\ & (m_i \otimes m_j) \otimes (n_k \otimes n_l) + (m_j \otimes m_i) \otimes (n_l \otimes n_k) \end{aligned}$$

for  $i, j \in \{1, \dots, a\}$ ,  $k, l \in \{1, \dots, b\}$  with either  $i \neq j$  or  $k \neq l$ . The first type of basis element is contained in  $\text{Sym}(M_1 \otimes_R M_1, \tau) \otimes_R \text{Sym}(M_2 \otimes_R M_2, \tau)$ , and the second type is contained in  $\text{Symd}((M_1 \otimes_R M_1) \otimes_R (M_2 \otimes_R M_2), \tau \otimes \tau)$ , so therefore the converse inclusion holds as well, and claim (i) is proved.

(ii): Since  $\sigma_i$  are now weakly symplectic, i.e., of type  $-1$ , we will have that  $(\mathcal{A}_j, \sigma_j)|_{U_i} \cong (\mathcal{M}_j \otimes_{\mathcal{O}} \mathcal{M}_j, -\tau)$  for  $j = 1, 2$ . Accounting for the facts that  $(-\tau) \otimes (-\tau) = \tau \otimes \tau$ ,

$$\begin{aligned} \text{Skew}(M_j \otimes_R M_j, -\tau) &= \text{Sym}(M_j \otimes_R M_j, \tau), \text{ and} \\ \text{Alt}(M_j \otimes_R M_j, -\tau) &= \text{Symd}(M_j \otimes_R M_j, \tau), \end{aligned}$$

all other details are the same as in (i).  $\square$

**5.2. Remark.** With the same methods as above one can also generalize [KMRT, 5.16]: If  $\sigma_i$  are both orthogonal, then

$$\begin{aligned} \text{(i)} \quad \text{Symd}_{\mathcal{A}, \sigma} \cap (\text{Sym}_{\mathcal{A}_1, \sigma_1} \otimes_{\mathcal{O}} \text{Sym}_{\mathcal{A}_2, \sigma_2}) \\ = \left( (\text{Symd}_{\mathcal{A}_1, \sigma_1} \otimes_{\mathcal{O}} \text{Sym}_{\mathcal{A}_2, \sigma_2}) + (\text{Sym}_{\mathcal{A}_1, \sigma_1} \otimes_{\mathcal{O}} \text{Symd}_{\mathcal{A}_2, \sigma_2}) \right)^{\sharp}. \end{aligned}$$

and if  $\sigma_i$  are both weakly symplectic, then

$$\begin{aligned} \text{(ii)} \quad \text{Symd}_{\mathcal{A}, \sigma} \cap (\text{Skew}_{\mathcal{A}_1, \sigma_1} \otimes_{\mathcal{O}} \text{Skew}_{\mathcal{A}_2, \sigma_2}) \\ = \left( (\text{Alt}_{\mathcal{A}_1, \sigma_1} \otimes_{\mathcal{O}} \text{Skew}_{\mathcal{A}_2, \sigma_2}) + (\text{Skew}_{\mathcal{A}_1, \sigma_1} \otimes_{\mathcal{O}} \text{Alt}_{\mathcal{A}_2, \sigma_2}) \right)^{\sharp}. \end{aligned}$$

We warn that the naive generalization of [KMRT, 5.15], which states that for central simple algebras  $(A_1, \sigma_1)$  and  $(A_2, \sigma_2)$  with involutions of the first kind over a field  $\mathbb{F}$  of characteristic 2 we have

$$\begin{aligned} \text{Symd}(A_1, \sigma_1) \otimes_{\mathbb{F}} \text{Symd}(A_2, \sigma_2) \\ = (\text{Symd}(A_1, \sigma_1) \otimes_{\mathbb{F}} \text{Sym}(A_2, \sigma_2)) \cap (\text{Sym}(A_1, \sigma_1) \otimes_{\mathbb{F}} \text{Symd}(A_2, \sigma_2)), \end{aligned}$$

does not hold in our generality. It fails when 2 is neither invertible nor zero. For example, take  $S = \text{Spec}(\mathbb{Z})$  and consider the Azumaya algebra  $\mathcal{A} = M_2(\mathcal{O})$  with the split orthogonal involution  $\eta_0$  in degree 2 from Example 4.5(b). In particular,

$$\eta_0 \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} d & b \\ c & a \end{bmatrix}.$$

Since all relevant  $\mathcal{O}$ -modules are quasi-coherent and  $S$  is affine, we may deal with the  $\mathbb{Z}$ -module with involution  $(A, \eta_0) = (M_2(\mathbb{Z}), \eta_0)$ . Then

$$\text{Sym}(A, \eta_0) = \left\{ \begin{bmatrix} a & b \\ c & a \end{bmatrix} \right\} \text{ and } \text{Symd}(A, \eta_0) = \left\{ \begin{bmatrix} a & 2b \\ 2c & a \end{bmatrix} \right\}.$$

Therefore,

$$\text{Symd}(A, \eta_0) \otimes_{\mathbb{Z}} \text{Symd}(A, \eta_0) = \left\{ \begin{bmatrix} a & 2b & 2c & 4d \\ 2e & a & 4f & 2c \\ 2g & 4h & a & 2b \\ 4i & 2g & 2e & a \end{bmatrix} \right\}.$$

while

$$\begin{aligned} & (\mathrm{Symd}(A, \eta_0) \otimes_{\mathbb{Z}} \mathrm{Sym}(A, \eta_0)) \cap (\mathrm{Sym}(A, \eta_0) \otimes_{\mathbb{Z}} \mathrm{Symd}(A, \eta_0)) \\ &= \left\{ \begin{bmatrix} a & 2b & 2c & 2d \\ 2e & a & 2f & 2c \\ 2g & 2h & a & 2b \\ 2i & 2g & 2e & a \end{bmatrix} \right\} \end{aligned}$$

and so these modules are not equal.

Lemma 5.1 will be used similarly to how [KMRT, 5.16, 5.17] are used to prove uniqueness claims about quadratic pairs on tensor products. If the algebra with involution  $(\mathcal{A}, \sigma)$  extends to a quadratic triple with some linear form  $f$ , then the behaviour of  $f$  is prescribed on  $\mathrm{Symd}_{\mathcal{A}, \sigma}$ . Therefore, by Lemma 5.1 such  $f$  will be uniquely determined by its behaviour on  $\mathrm{Sym}_{\mathcal{A}_1, \sigma_1} \otimes_{\mathcal{O}} \mathrm{Sym}_{\mathcal{A}_2, \sigma_2}$ . We use this in the next two propositions, generalizing [KMRT, 5.18, 5.20].

**5.3. Proposition.** *Let  $(\mathcal{A}_1, \sigma_1, f_1)$  be a quadratic triple and let  $(\mathcal{A}_2, \sigma_2)$  be an Azumaya  $\mathcal{O}$ -algebra with orthogonal involution. Then there exists a unique quadratic triple  $(\mathcal{A}, \sigma, f_{1*})$  such that, for sections  $s_i \in \mathrm{Sym}_{\mathcal{A}_i, \sigma_i}$  we have*

$$(5.3.1) \quad f_{1*}(s_1 \otimes s_2) = f_1(s_1) \mathrm{Trd}_{\mathcal{A}_2}(s_2).$$

*Proof.* Consider an affine cover  $\{U_i \rightarrow S\}_{i \in I}$  of  $S$ . We know by Proposition 4.12(i) that each  $f|_{U_i}$  is described by an  $\ell_i \in \mathcal{A}_1(U_i)$  with  $\ell_i + \sigma_1(\ell_i) = 1$ . Then

$$(\ell_i \otimes 1) + (\sigma_1 \otimes \sigma_2)(\ell_i \otimes 1) = (\ell_i + \sigma_1(\ell_i)) \otimes 1 = 1 \otimes 1 = 1$$

and since  $\ell_i|_{U_{ij}} - \ell_j|_{U_{ij}} \in \mathrm{Alt}_{\mathcal{A}_1, \sigma_1}(U_{ij})$ , we also have that

$$(\ell_i \otimes 1)|_{U_{ij}} - (\ell_j \otimes 1)|_{U_{ij}} = (\ell_i|_{U_{ij}} - \ell_j|_{U_{ij}}) \otimes 1 \in \mathrm{Alt}_{\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2, \sigma_1 \otimes \sigma_2}(U_{ij}).$$

Therefore, we may use the construction of Example 4.17(a) with the elements  $\ell_i \otimes 1 \in \mathcal{A}(U_i)$  to define a global  $f_{1*}$ . It remains to show that this  $f_{1*}$  behaves as claimed in (5.3.1) on sections  $s_i \in \mathrm{Sym}_{\mathcal{A}_i, \sigma_i}(T)$  for some  $T \in \mathfrak{S}ch_S$ . Consider such sections  $s_1$  and  $s_2$ . With respect to the cover  $\{T_i := U_i \times_S T \rightarrow T\}_{i \in I}$ , our new linear form  $f_{1*}$  will be locally described by the elements  $(\ell_i \otimes 1)|_{T_i}$ . So, locally we have

$$\begin{aligned} f_{1*,i}((s_1 \otimes s_2)|_{T_i}) &= \mathrm{Trd}_{\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2}((\ell_i \otimes 1)|_{T_i}(s_1 \otimes s_2)|_{T_i}) \\ &= \mathrm{Trd}_{\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2}((\ell_i|_{T_i} \cdot s_1|_{T_i}) \otimes s_2|_{T_i}) \\ &= \mathrm{Trd}_{\mathcal{A}_1}(\ell_i|_{T_i} \cdot s_1|_{T_i}) \mathrm{Trd}_{\mathcal{A}_2}(s_2|_{T_i}) \\ &= f_1|_{T_i}(s_1|_{T_i}) \mathrm{Trd}_{\mathcal{A}_2}(s_2|_{T_i}) \end{aligned}$$

and hence gluing yields  $f_{1*}(s_1 \otimes s_2) = f_1(s_1) \mathrm{Trd}_{\mathcal{A}_2}(s_2)$  as desired.  $\square$

**5.4. Example.** Here we generalize [KMRT, Example 5.19]. On one hand, let  $(\mathcal{M}_1, q_1)$  be a regular quadratic form and let  $(\mathrm{End}_{\mathcal{O}}(\mathcal{M}_1), \sigma_1, f_{q_1})$  be the quadratic pair defined in Proposition 4.4(i). On the other hand, let  $(\mathcal{M}_2, b_2)$  be a regular symmetric bilinear form on a locally free  $\mathcal{O}$ -module of finite positive rank  $\mathcal{M}_2$ . We

consider the tensor product quadratic form  $q$  on  $\mathcal{M} = \mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{M}_2$  as defined by Sah [Sa, Thm. 1] in the ring case. This  $q$  is regular, and so we consider the attached quadratic pair  $(\text{End}_{\mathcal{O}}(\mathcal{M}, \sigma, f_q))$ . The proof of [KMRT, Example 5.19] shows that there is an isomorphism

$$(\text{End}_{\mathcal{O}}(\mathcal{M}_1), \sigma_1, f_{q_1}) \otimes (\text{End}_{\mathcal{O}}(\mathcal{M}_2), \sigma_{b_2}) \xrightarrow{\sim} (\text{End}_{\mathcal{O}}(\mathcal{M}), \sigma, f_q)$$

where the tensor product stands for the construction of Proposition 5.3.

Turning our attention to the weakly symplectic case, it turns out that two weakly symplectic involutions may not have a quadratic triple structure on their tensor product, see Example 5.7 below. To obtain a result analogous to [KMRT, 5.20] we will assume we have two symplectic involutions, i.e., where the types are  $\varepsilon_i = -1$  and the local bilinear forms are alternating. This is due in part to the following lemma.

**5.5. Lemma.** *Let  $(\mathcal{A}, \sigma)$  be an Azumaya  $\mathcal{O}$ -algebra with a weakly symplectic involution. Then, the following are equivalent.*

- (i)  $\sigma$  is symplectic.
- (ii)  $\text{Trd}_{\mathcal{A}}(s) = 0$  for all sections  $s \in \text{Skew}_{\mathcal{A}, \sigma}$ .
- (iii)  $1_{\mathcal{A}} \in \text{Symd}_{\mathcal{A}, \sigma}(S)$ .

*Proof.* Since  $(\mathcal{A}, \sigma)$  is weakly symplectic, there exists an fppf cover  $\{T_i \rightarrow S\}_{i \in I}$  over which

$$(\mathcal{M}_i \otimes_{\mathcal{O}|_{T_i}} \mathcal{M}_i, -\tau) \xrightarrow{\varphi_b} (\text{End}_{\mathcal{O}|_{T_i}}(\mathcal{M}_i), \eta_b) \cong (\mathcal{A}, \sigma)|_{T_i}$$

where  $\tau$  is the switch involution and  $\varphi_b$  is the isomorphism (3.1.2) for a regular skew-symmetric bilinear form  $b$ .

(i)  $\Leftrightarrow$  (ii): For any  $T \in \mathfrak{Sch}_S$  and a section  $s \in \text{Skew}_{\mathcal{A}, \sigma}(T)$ , we may compute locally with respect to the cover  $\{T_i \times_S T \rightarrow T\}_{i \in I}$  where the section will be a linear combination of elements of the form

$$\begin{aligned} & \varphi_b(m \otimes m) \\ & \varphi_b(m \otimes n) - \eta_b(\varphi_b(m \otimes n)) \end{aligned}$$

for  $m, n \in \mathcal{M}_i(T_i \times_S T)$ . Since we have

$$\text{Trd}_{\text{End}_{\mathcal{O}|_{T_i}}(\mathcal{M}_i)}(\varphi_b(m \otimes n)) = b(m, n),$$

the trace of the second type of element is clearly 0, while the trace vanishes for all elements of the first type if and only if  $b$  is alternating. Hence,  $\text{Trd}_{\mathcal{A}}(s) = 0$  for all sections  $s \in \text{Skew}_{\mathcal{A}, \sigma}$  if and only if the underlying bilinear forms are alternating, i.e.,  $\sigma$  is symplectic.

(ii)  $\Leftrightarrow$  (iii): By Lemma 3.6(iv) we know  $\text{Skew}_{\mathcal{A}, \sigma}^{\perp} = \text{Symd}_{\mathcal{A}, \sigma}$  and  $\text{Symd}_{\mathcal{A}, \sigma}^{\perp} = \text{Skew}_{\mathcal{A}, \sigma}$ , from which the equivalence is immediate.  $\square$

Similar to the orthogonal case, having  $1_{\mathcal{A}} \in \text{Symd}_{\mathcal{A}, \sigma}$  will grant us local  $\ell_i$  with  $\ell_i + \sigma(\ell_i) = 1$  from which we can build a quadratic triple. The following result generalizes [KMRT, 5.20].



**5.6. Proposition.** *Let  $(\mathcal{A}_1, \sigma_1)$  and  $(\mathcal{A}_2, \sigma_2)$  be two Azumaya  $\mathcal{O}$ -algebras with symplectic involutions. Then there exists a unique quadratic triple  $(\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2, \sigma_1 \otimes \sigma_2, f_{\otimes})$  such that, for sections  $s_i \in \mathit{Skew}_{\mathcal{A}_i, \sigma_i}$  we have*

$$f_{\otimes}(s_1 \otimes s_2) = 0.$$

*Proof.* We first establish uniqueness. Let  $f, f' : \mathit{Sym}_{\mathcal{A}, \sigma} \rightarrow \mathcal{O}$  be two solutions. Then  $f - f'$  vanishes on  $\mathit{Symd}_{\mathcal{A}, \sigma}$  and on  $\mathit{Skew}_{\mathcal{A}_1, \sigma_1} \otimes \mathit{Skew}_{\mathcal{A}_2, \sigma_2}$ . According to Lemma 5.1(ii),  $\mathit{Symd}_{\mathcal{A}, \sigma}$  and  $\mathit{Skew}_{\mathcal{A}_1, \sigma_1} \otimes \mathit{Skew}_{\mathcal{A}_1, \sigma_1}$  generate  $\mathit{Symd}_{\mathcal{A}, \sigma}$  so that  $f' = f$ .

We now establish existence. Since  $(\mathcal{A}_1, \sigma_1)$  is symplectic we have  $1_{\mathcal{A}_1} \in \mathit{Symd}_{\mathcal{A}_1, \sigma_1}$  by Lemma 5.5, and so there is an fppf cover  $\{T_i \rightarrow S\}_{i \in I}$  such that for each  $i \in I$  there exists  $\ell_i \in \mathcal{A}_1(T_i)$  with  $\ell_i + \sigma_1(\ell_i) = 1$ . Then we will have

$$(\ell_i \otimes 1) + (\sigma_1 \otimes \sigma_2)(\ell_i \otimes 1) = 1,$$

so to use the construction of Examples 4.17(a) we need to check that  $(\ell_i \otimes 1)|_{T_{ij}} - (\ell_j \otimes 1)|_{T_{ij}} \in \mathit{Alt}_{\mathcal{A}, \sigma}(T_{ij})$ . By Lemma 3.6(ii) and Lemma 5.1(i), it is sufficient to show that these elements are orthogonal to sections of the form  $a + \sigma(a)$  for any  $a \in \mathcal{A}|_{T_{ij}}$ , and  $s_1 \otimes s_2$  for sections  $s_i \in \mathit{Skew}(\mathcal{A}_i, \sigma_i)|_{T_{ij}}$ . For the first type, we have

$$\mathrm{Trd}_{\mathcal{A}}((\ell_i \otimes 1)|_{T_{ij}}(a + \sigma(a))) = \mathrm{Trd}_{\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2}(a) = \mathrm{Trd}_{\mathcal{A}}((\ell_j \otimes 1)|_{T_{ij}}(a + \sigma(a)))$$

and so

$$\mathrm{Trd}_{\mathcal{A}}((\ell_i \otimes 1)|_{T_{ij}} - \ell_j \otimes 1|_{T_{ij}})(a + \sigma(a)) = 0.$$

For the second type, we have

$$\mathrm{Trd}_{\mathcal{A}}((\ell_i \otimes 1)|_{T_{ij}}(s_1 \otimes s_2)) = \mathrm{Trd}_{\mathcal{A}_1}(\ell_i|_{T_{ij}} s_1) \mathrm{Trd}_{\mathcal{A}_2}(s_2) = 0$$

since  $\mathrm{Trd}_{\mathcal{A}_2}(s_2) = 0$  by Lemma 5.5(ii). Similarly  $\mathrm{Trd}_{\mathcal{A}}((\ell_j \otimes 1)|_{T_{ij}}(s_1 \otimes s_2)) = 0$  and so  $(\ell_i \otimes 1)|_{T_{ij}} - (\ell_j \otimes 1)|_{T_{ij}}$  is orthogonal to  $s_1 \otimes s_2$  as well. Thus there exists a global  $f_{\otimes} : \mathit{Sym}_{\mathcal{A}, \sigma} \rightarrow \mathcal{O}$  constructed as in Example 4.17(a). The above local calculations also imply that  $f_{\otimes}(s_1 \otimes s_2) = 0$  for sections  $s_1 \in \mathit{Skew}_{\mathcal{A}_1, \sigma_1}$  and  $s_2 \in \mathit{Skew}_{\mathcal{A}_2, \sigma_2}$ , and so we have constructed the unique quadratic triple we desired.  $\square$

Note that both  $(\mathcal{A}_1, \sigma_1)$  and  $(\mathcal{A}_2, \sigma_2)$  were required to be symplectic in the above proposition. We needed  $(\mathcal{A}_1, \sigma_1)$  to be symplectic to have  $1_{\mathcal{A}_1} \in \mathit{Symd}_{\mathcal{A}_1, \sigma_1}$ , and  $(\mathcal{A}_2, \sigma_2)$  to be symplectic to have  $\mathrm{Trd}_{\mathcal{A}_2}(s_2) = 0$ . If  $(\mathcal{A}_2, \sigma_2)$  were only weakly symplectic, then we could construct the local maps  $f_{\otimes, i}$  just the same, but it is not clear if the elements  $(\ell_i \otimes 1)|_{T_{ij}} - (\ell_j \otimes 1)|_{T_{ij}}$  belong to  $\mathit{Alt}_{\mathcal{A}, \sigma}(T_{ij})$ , and so there may be examples with no quadratic triple on the tensor product.

If both involutions are only weakly symplectic there is almost no hope, even over affine schemes as in the following example.

**5.7. Example.** Let  $S = \mathrm{Spec}(\mathbb{Z}/2\mathbb{Z})$ . Since we are over an affine scheme we may simply work with Azumaya algebras over  $R = \mathbb{Z}/2\mathbb{Z}$ . Consider the Azumaya algebra  $M_2(R)$  with involution  $\sigma(B) = u^{-1}B^T u$  for

$$u = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

This is weakly symplectic since  $u^T = -u$ , but not symplectic (for example by Lemma 5.5). Note that

$$s = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in \text{Skew}(M_2(R), \sigma)$$

is a skew-symmetric element. If there were a quadratic triple  $(M_2(R) \otimes_R M_2(R), \sigma \otimes \sigma, f)$ , then necessarily

$$2f(s \otimes s) = \text{Trd}_{M_2(R) \otimes_R M_2(R)}(s \otimes s) = \text{Trd}_{M_2(R)}(s)^2 = 1^2 = 1,$$

however there is no  $x \in \mathbb{Z}/2\mathbb{Z}$  such that  $2x = 1$ . Therefore no such  $f$  exists. Since we are over an affine scheme this means that  $\sigma \otimes \sigma$  fails to even be locally quadratic.

## 6. OBSTRUCTIONS TO QUADRATIC PAIRS

In this section we introduce cohomological obstructions which prevent an Azumaya  $\mathcal{O}$ -algebra with locally quadratic involution  $(\mathcal{A}, \sigma)$  from being extended to a quadratic triple, either outright or where the triple has certain properties. We begin by considering portions of various long exact cohomology sequences arising from (1.6.1). In particular we have the following diagram

$$(6.0.1) \quad \begin{array}{ccccc} \mathcal{A}(S) & \xrightarrow{\text{Id} + \sigma} & \text{Symd}_{\mathcal{A}, \sigma}(S) & \xrightarrow{\delta} & \check{H}^1(S, \text{Skew}_{\mathcal{A}, \sigma}) \\ \downarrow & & \parallel & & \downarrow \\ (\mathcal{A}/\text{Alt}_{\mathcal{A}, \sigma})(S) & \xrightarrow{\xi(S)} & \text{Symd}_{\mathcal{A}, \sigma}(S) & \xrightarrow{\delta'} & \check{H}^1(S, \text{Skew}_{\mathcal{A}, \sigma}/\text{Alt}_{\mathcal{A}, \sigma}) \\ \downarrow c & & & & \\ \check{H}^1(S, \text{Alt}_{\mathcal{A}, \sigma}) & & & & \end{array}$$

with exact left column and exact rows.

**6.1. Definition** (Strong and Weak Obstructions). Let  $(\mathcal{A}, \sigma)$  be an Azumaya  $\mathcal{O}$ -algebra with locally quadratic involution. Then  $1_{\mathcal{A}} \in \text{Symd}_{\mathcal{A}, \sigma}(S)$  and we call  $\Omega(\mathcal{A}, \sigma) = \delta(1_{\mathcal{A}})$  the *strong obstruction* and call  $\omega(\mathcal{A}, \sigma) = \delta'(1_{\mathcal{A}})$  the *weak obstruction*.

The strong and weak obstructions prevent the existence of a quadratic triple involving  $\sigma$  in the following way.

**6.2. Theorem.** *Let  $(\mathcal{A}, \sigma)$  be an Azumaya  $\mathcal{O}$ -algebra with a locally quadratic involution. Then,*

- (i) *There exists a linear map  $f: \mathcal{A} \rightarrow \mathcal{O}$  such that  $(\mathcal{A}, \sigma, f|_{\text{Sym}_{\mathcal{A}, \sigma}})$  is a quadratic triple if and only if  $\Omega(\mathcal{A}, \sigma) = 0$ . In this case  $f = \text{Trd}_{\mathcal{A}}(\ell_{\underline{\quad}})$  for an element  $\ell \in \mathcal{A}(S)$  with  $\ell + \sigma(\ell) = 1$ .*
- (ii) *There exists a linear map  $f: \text{Sym}_{\mathcal{A}, \sigma} \rightarrow \mathcal{O}$  such that  $(\mathcal{A}, \sigma, f)$  is a quadratic triple if and only if  $\omega(\mathcal{A}, \sigma) = 0$ .*

*Proof.* (i): Assume we have  $f: \mathcal{A} \rightarrow \mathcal{O}$  such that  $(\mathcal{A}, \sigma, f|_{\mathcal{S}ym_{\mathcal{A},\sigma}})$  is a quadratic triple. Then trivially  $f|_{\mathcal{S}ym_{\mathcal{A},\sigma}}$  can be extended to  $\mathcal{A}$ , and so by Lemma 4.11 there exists an  $\ell \in \mathcal{A}(S)$  such that  $\ell + \sigma(\ell) = 1_{\mathcal{A}}$ . Thus  $1_{\mathcal{A}}$  is in the image of  $\mathcal{A}(S) \rightarrow \mathcal{S}ym_{\mathcal{A},\sigma}(S)$  in the long exact cohomology sequence, and so  $\Omega(\mathcal{A}, \sigma) = 0$ . Conversely, if  $\Omega(\mathcal{A}, \sigma) = 0$  then  $1_{\mathcal{A}}$  is in the same image and so we obtain such an  $\ell$ . Using Example 4.3(c) we can construct  $f: \mathcal{A} \rightarrow \mathcal{O}$  with the desired property.

(ii): Since the rows in (6.0.1) are exact,  $\omega(\mathcal{A}, \sigma) = 0$  if and only if  $\xi(S)^{-1}(1_{\mathcal{A}})$  is non-empty. By Theorem 4.18, this means  $\omega(\mathcal{A}, \sigma) = 0$  if and only if  $(\mathcal{A}, \sigma)$  can be extended to a quadratic triple.  $\square$

By Remark 4.19,  $\omega(\mathcal{A}, \sigma)$  is the isomorphism class of the  $\mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}lt_{\mathcal{A},\sigma}$ -torsor  $\xi^{-1}(1_{\mathcal{A}})$ . Another view of Theorem 6.2(ii) is that  $\omega(\mathcal{A}, \sigma) = 0$  if and only if this torsor is trivial. A torsor is trivial if and only if it has a global section and global sections of  $\xi^{-1}(1_{\mathcal{A}})$  correspond to forms  $f$  which make  $(\mathcal{A}, \sigma, f)$  a quadratic triple.

For examples of locally quadratic involutions with non-trivial strong or weak obstructions, see section 7.

**6.3. Remark.** Since  $2_{\mathcal{A}} = 1_{\mathcal{A}} + \sigma(1_{\mathcal{A}})$  is always in the image of  $\mathcal{A}(S) \rightarrow \mathcal{S}ym_{\mathcal{A},\sigma}(S)$ , it follows that  $\delta(2_{\mathcal{A}}) = 2\Omega(\mathcal{A}, \sigma) = 0$  so that  $2\omega(\mathcal{A}, \sigma) = 0$  also. In particular, if 2 is invertible over  $S$  we recover the fact that  $(\mathcal{A}, \sigma)$  always extends to a quadratic triple.

We will make use of the following description in order to perform computations with  $\Omega(\mathcal{A}, \sigma)$ .

**6.4. Lemma.** *Let  $(\mathcal{A}, \sigma)$  be an Azumaya  $\mathcal{O}$ -algebra with locally quadratic involution. The strong obstruction takes the form*

$$\Omega(\mathcal{A}, \sigma) = \left[ (\ell_j|_{T_{ij}} - \ell_i|_{T_{ij}})_{i,j \in I} \right] \in \check{H}^1(S, \mathcal{S}kew_{\mathcal{A},\sigma})$$

for any cover  $\{T_i \rightarrow S\}_{i \in I}$  and elements  $\ell_i \in \mathcal{A}(T_i)$  such that  $\ell_i + \sigma(\ell_i) = 1$ .

*Proof.* This follows immediately from Lemma 1.4.  $\square$

**6.5. Remarks.** (a) By Lemma 4.15, if we wish we may represent  $\Omega(\mathcal{A}, \sigma)$  as in Lemma 6.4 with an affine open cover  $\{U_i \rightarrow S\}_{i \in I}$ .

(b) Since the weak obstruction  $\omega(\mathcal{A}, \sigma)$  is simply an image of the strong obstruction, it will be represented similarly by the class

$$\left[ \overline{(\ell_j|_{T_{ij}} - \ell_i|_{T_{ij}})}_{i,j \in I} \right] \in \check{H}^1(S, \mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}lt_{\mathcal{A},\sigma})$$

for a cover  $\{T_i \rightarrow S\}_{i \in I}$  and elements  $\ell_i \in \mathcal{A}(T_i)$  with  $\ell_i + \sigma(\ell_i) = 1$ , where the overline denotes the image in  $(\mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}lt_{\mathcal{A},\sigma})(T_{ij})$ . This cover can of course also be taken to be an affine open cover if desired.

**6.6. Alternate Obstructions to Extend a Form.** Let  $(\mathcal{A}, \sigma, f)$  be a quadratic triple. Lemma 4.11 above states that  $f$  extends from  $\mathcal{S}ym_{\mathcal{A},\sigma}$  to  $\mathcal{A}$  if and only if  $f$  arises from some  $\ell \in \mathcal{A}(S)$  satisfying  $\ell + \sigma(\ell) = 1$ . The cohomology theory of

quasi-coherent sheaves provides an obstruction to such an extension, in particular an obstruction  $c(f) \in \check{H}^1(S, \mathcal{A}lt_{\mathcal{A}, \sigma})$  as follows. By Theorem 4.18, the form  $f$  corresponds to an element  $\lambda_f \in (\mathcal{A}/\mathcal{A}lt_{\mathcal{A}, \sigma})(S)$ , and we may consider its image under the map  $c$  in (6.0.1). We set  $c(f) = c(\lambda_f) \in \check{H}^1(S, \mathcal{A}lt_{\mathcal{A}, \sigma})$ . Clearly this obstruction prevents  $\lambda_f = \bar{\ell}$  for some  $\ell \in \mathcal{A}(S)$ , which would be the case exactly when  $f$  can be extended to all of  $\mathcal{A}$ .

Consider an affine covering  $\{U_i \rightarrow S\}_{i \in I}$  of  $S$ . By Proposition 4.12(i), each restriction  $f|_{U_i}$  will be of the form  $\text{Trd}_{\mathcal{A}|_{U_i}}(\ell_i)$  for an  $\ell_i \in \mathcal{A}(U_i)$  with  $\bar{\ell}_i = \lambda_f|_{U_i}$ . Lemma 1.4 then tells us that the 1-cocycle  $(\ell_j|_{U_{ij}} - \ell_i|_{U_{ij}})_{i,j \in I}$  represents the cohomological obstruction  $c(f)$ .

However, we also obtain another obstruction in the following way. We have an exact sequence of sheaves

$$0 \rightarrow \mathcal{S}ym_{\mathcal{A}, \sigma} \rightarrow \mathcal{A} \rightarrow \mathcal{A}lt_{\mathcal{A}, \sigma} \rightarrow 0$$

which, since the sheaves are finite locally free, we may dualize to obtain another exact sequence,

$$0 \rightarrow \mathcal{A}lt_{\mathcal{A}, \sigma}^{\vee} \rightarrow \mathcal{A}^{\vee} \rightarrow \mathcal{S}ym_{\mathcal{A}, \sigma}^{\vee} \rightarrow 0.$$

A portion of the long exact cohomology sequence associated with this exact sequence is

$$\dots \rightarrow \mathcal{A}^{\vee} \rightarrow \mathcal{S}ym_{\mathcal{A}, \sigma}^{\vee} \rightarrow \check{H}^1(S, \mathcal{A}lt_{\mathcal{A}, \sigma}^{\vee}) \rightarrow \dots$$

Since  $f \in \mathcal{S}ym_{\mathcal{A}, \sigma}^{\vee}(S)$ , we then get a class  $c'(f) \in \check{H}^1(S, \mathcal{A}lt_{\mathcal{A}, \sigma}^{\vee})$  which is the obstruction to extend  $f$  to  $\mathcal{A}$ . Not surprisingly, we can compare the two obstructions.

**6.7. Lemma.** *Let  $(\mathcal{A}, \sigma, f)$  be a quadratic triple, and let  $\widehat{b}_-: \mathcal{A}lt_{\mathcal{A}, \sigma} \xrightarrow{\sim} \mathcal{A}lt_{\mathcal{A}, \sigma}^{\vee}$  be the isomorphism associated to the regular bilinear form  $b_-$  provided by Lemma 3.7(ii). Then,*

- (i)  $c'(f) = \widehat{b}_-(c(f))$ .
- (ii) *The image of  $c(f)$  under the map  $\check{H}^1(S, \mathcal{A}lt_{\mathcal{A}, \sigma}) \rightarrow \check{H}^1(S, \mathcal{S}kew_{\mathcal{A}, \sigma})$  induced by the inclusion  $\mathcal{A}lt_{\mathcal{A}, \sigma} \rightarrow \mathcal{S}kew_{\mathcal{A}, \sigma}$ , is  $\Omega(\mathcal{A}, \sigma)$ .*

*Proof.* (i): Take an affine open covering  $\{U_i \rightarrow S\}_{i \in I}$  of  $S$ . Then  $f|_{U_i}(s) = \text{Trd}_{\mathcal{A}}(\ell_i s)$  for some  $\ell_i \in \mathcal{A}(U_i)$  satisfying  $\ell_i + \sigma(\ell_i) = 1$  by Proposition 4.12(i). By definition  $c(f)$  is the class of the 1-cocycle  $c_{ij} = \ell_j|_{U_{ij}} - \ell_i|_{U_{ij}}$  with values in  $\mathcal{A}lt_{\mathcal{A}, \sigma}$ . We define  $\widetilde{f}_i(a) = \text{Trd}_{\mathcal{A}}(\ell_i a)$  on  $\mathcal{A}|_{U_i}$  for  $i \in I$ .

On the other hand,  $c'(f)$  is the class of the 1-cocycle  $c'_{ij} = (\widetilde{f}_j)|_{U_{ij}} - (\widetilde{f}_i)|_{U_{ij}}$  with values in  $\mathcal{A}lt_{\mathcal{A}, \sigma}^{\vee}$ .

We need to check that  $c'_{ij} = \widehat{b}_-(c_{ij})$  for all  $i, j \in I$ , or equivalently that  $c'_{ij} = b_-(c_{ij}, \_)$  over  $U_{ij}$ . Up to localization to an affine open subset  $V \subseteq U_{ij}$ , we can

deal with elements of the form  $x = x' - \sigma(x')$  for some  $x' \in \mathcal{A}(V)$ . It follows that

$$\begin{aligned} c'_{ij}(x) &= (\tilde{f}_j)|_{U_{ij}}(x' - \sigma(x')) - (\tilde{f}_i)|_{U_{ij}}(x' - \sigma(x')) \\ &= \mathrm{Trd}_{\mathcal{A}}(\ell_j|_{U_{ij}}(x' - \sigma(x')) - \ell_i|_{U_{ij}}(x' - \sigma(x'))) \\ &= \mathrm{Trd}_{\mathcal{A}}((\ell_j|_{U_{ij}} - \ell_i|_{U_{ij}})(x' - \sigma(x'))) \\ &= b_-(c_{ij}, x) \end{aligned}$$

as desired.

(ii): We have that  $1_{\mathcal{A}(U_i)} = \ell_i + \sigma(\ell_i)$  for each  $i$ . By Lemma 6.4, the strong obstruction  $\Omega(\mathcal{A}, \sigma)$  is represented by the 1-cocycle  $\ell_j|_{U_j} - \ell_i|_{U_{ij}}$  with values in  $\mathit{Skew}_{\mathcal{A}, \sigma}$ . We conclude that the image of  $c(f)$  by the map  $\check{H}^1(S, \mathit{Alt}_{\mathcal{A}, \sigma}) \rightarrow \check{H}^1(S, \mathit{Skew}_{\mathcal{A}, \sigma})$  is  $\Omega(\mathcal{A}, \sigma)$ .  $\square$

**6.8. Obstructions for Tensor Products.** Consider Azumaya algebras  $(\mathcal{A}_1, \sigma_1)$  and  $(\mathcal{A}_2, \sigma_2)$  either both with orthogonal or both with symplectic involutions. Their tensor product  $(\mathcal{A}, \sigma) = (\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2, \sigma_1 \otimes \sigma_2)$  is an Azumaya algebra with orthogonal involution. There is a natural map

$$\begin{aligned} \mathcal{A}_1 \times \mathcal{A}_2 &\rightarrow \mathcal{A} \\ (a_1, a_2) &\mapsto a_1 \otimes a_2 \end{aligned}$$

which restricts to a morphism

$$\mathit{Skew}_{\mathcal{A}_1, \sigma_1} \times \mathit{Sym}_{\mathcal{A}_2, \sigma_2} \rightarrow \mathit{Skew}_{\mathcal{A}, \sigma}$$

which in turn induces a morphism

$$(\mathit{Skew}_{\mathcal{A}_1, \sigma_1} / \mathit{Alt}_{\mathcal{A}_1, \sigma_1}) \times \mathit{Sym}_{\mathcal{A}_2, \sigma_2} \rightarrow \mathit{Skew}_{\mathcal{A}, \sigma} / \mathit{Alt}_{\mathcal{A}, \sigma}.$$

We will make use of these tensor morphisms throughout this section to investigate the strong and weak obstructions for  $(\mathcal{A}, \sigma)$ . When  $\sigma_i$  are both orthogonal we will assume  $\sigma_1$  is locally quadratic and we will relate  $\Omega(\mathcal{A}, \sigma)$  to  $\Omega(\mathcal{A}_1, \sigma_1)$ . When  $\sigma_i$  are both symplectic we also have  $1_{\mathcal{A}_1} \in \mathit{Symd}_{\mathcal{A}_1, \sigma_1}$  by Lemma 5.5 and a connecting morphism  $\delta_1: \mathit{Symd}_{\mathcal{A}_1, \sigma_1} \rightarrow \check{H}^1(S, \mathit{Skew}_{\mathcal{A}_1, \sigma_1})$ . In this case we will relate  $\Omega(\mathcal{A}, \sigma)$  to  $\delta_1(1_{\mathcal{A}_1})$ , which is only technically a strong obstruction when  $2 = 0 \in \mathcal{O}$  and so  $\sigma_1$  would be simultaneously locally quadratic as well as symplectic.

**6.9. Lemma.** *Let  $(\mathcal{A}_1, \sigma_1)$  and  $(\mathcal{A}_2, \sigma_2)$  be two Azumaya  $\mathcal{O}$ -algebra with orthogonal involution. Let  $(\mathcal{A}, \sigma)$  be their tensor product. Assume that  $(\mathcal{A}_1, \sigma_1)$  is locally quadratic. Then, we have the following.*

- (i)  $(\mathcal{A}, \sigma)$  is locally quadratic.
- (ii) We have

$$\Omega(\mathcal{A}, \sigma) = \Omega(\mathcal{A}_1, \sigma_1) \cup 1_{\mathcal{A}_2} \in \check{H}^1(S, \mathit{Skew}_{\mathcal{A}, \sigma})$$

where  $1_{\mathcal{A}_2} \in \mathit{Sym}_{\mathcal{A}_2, \sigma_2}(S) = H^0(S, \mathit{Sym}_{\mathcal{A}_2, \sigma_2})$  and the cup-product arises from the morphism

$$\mathit{Skew}_{\mathcal{A}_1, \sigma_1} \times \mathit{Sym}_{\mathcal{A}_2, \sigma_2} \rightarrow \mathit{Skew}_{\mathcal{A}, \sigma}.$$

(iii) We have  $\omega(\mathcal{A}, \sigma) = \omega(\mathcal{A}_1, \sigma_1) \cup 1_{\mathcal{A}_2} \in \check{H}^1(S, \mathcal{S}kew_{\mathcal{A}, \sigma} / \mathcal{A}lt_{\mathcal{A}, \sigma})$  where the cup-product arises from the morphism

$$(\mathcal{S}kew_{\mathcal{A}_1, \sigma_1} / \mathcal{A}lt_{\mathcal{A}_1, \sigma_1}) \times \mathcal{S}ym_{\mathcal{A}_2, \sigma_2} \rightarrow \mathcal{S}kew_{\mathcal{A}, \sigma} / \mathcal{A}lt_{\mathcal{A}, \sigma}.$$

*Proof.* (i): The statement is local and follows then from the construction of tensor products in Proposition 5.3.

(ii): The map  $\otimes 1_{\mathcal{A}_2}: \mathcal{A}_1 \rightarrow \mathcal{A}$ ,  $a_1 \mapsto a_1 \otimes 1_{\mathcal{A}_2}$ , restricts to both  $\mathcal{S}kew_{\mathcal{A}_1, \sigma_1} \rightarrow \mathcal{S}kew_{\mathcal{A}, \sigma}$  and  $\mathcal{S}ymd_{\mathcal{A}_1, \sigma_1} \rightarrow \mathcal{S}ymd_{\mathcal{A}, \sigma}$ . Therefore, we have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{S}kew_{\mathcal{A}_1, \sigma_1} & \longrightarrow & \mathcal{A}_1 & \xrightarrow{1+\sigma_1} & \mathcal{S}ymd_{\mathcal{A}_1, \sigma_1} \longrightarrow 0. \\ & & \downarrow \otimes 1_{\mathcal{A}_2} & & \downarrow \otimes 1_{\mathcal{A}_2} & & \downarrow \otimes 1_{\mathcal{A}_2} \\ 0 & \longrightarrow & \mathcal{S}kew_{\mathcal{A}, \sigma} & \longrightarrow & \mathcal{A} & \xrightarrow{1+\sigma} & \mathcal{S}ymd_{\mathcal{A}, \sigma} \longrightarrow 0. \end{array}$$

whose rows are exact sequences of  $\mathcal{O}$ -modules. We claim this diagram commutes. Commutativity of the left square is clear, so we check commutativity of the right square with a computation. Let  $a \in \mathcal{A}_1$ , then

$$\begin{aligned} (1 + \sigma) \circ (\otimes 1_{\mathcal{A}_2})(a) &= (1 + \sigma)(a \otimes 1_{\mathcal{A}_2}) = a \otimes 1_{\mathcal{A}_2} + (\sigma_1 \otimes \sigma_2)(a \otimes 1_{\mathcal{A}_2}) \\ &= a \otimes 1_{\mathcal{A}_2} + \sigma_1(a) \otimes 1_{\mathcal{A}_2} = (a + \sigma_1(a)) \otimes 1_{\mathcal{A}_2} = (\otimes 1_{\mathcal{A}_2}) \circ (1 + \sigma_1)(a). \end{aligned}$$

Thus, we get another commutative diagram involving boundary maps,

$$\begin{array}{ccc} \mathcal{S}ymd_{\mathcal{A}, \sigma_1}(S) & \xrightarrow{\delta_1} & \check{H}^1(S, \mathcal{S}kew_{\mathcal{A}_1, \sigma_1}) \\ \downarrow \otimes 1_{\mathcal{A}_2} & & \downarrow \cup 1_{\mathcal{A}_2} \\ \mathcal{S}ymd_{\mathcal{A}, \sigma}(S) & \xrightarrow{\delta} & \check{H}^1(S, \mathcal{S}kew_{\mathcal{A}, \sigma}). \end{array}$$

Since  $1_{\mathcal{A}} = 1_{\mathcal{A}_1} \otimes 1_{\mathcal{A}_2}$ , the commutativity of the diagram implies that  $\delta(1_{\mathcal{A}}) = \delta_1(1_{\mathcal{A}_1}) \cup 1_{\mathcal{A}_2}$  whence the desired formula

$$\Omega(\mathcal{A}, \sigma) = \Omega(\mathcal{A}_1, \sigma_1) \cup [1_{\mathcal{A}_2}] \in \check{H}^1(S, \mathcal{S}kew_{\mathcal{A}, \sigma}).$$

(iii): This follows from (ii) and the commutativity of the square

$$\begin{array}{ccc} \mathcal{S}kew_{\mathcal{A}_1, \sigma_1} \times \mathcal{S}ym_{\mathcal{A}_2, \sigma_2} & \longrightarrow & \mathcal{S}kew_{\mathcal{A}, \sigma} \\ \downarrow \pi_1 \times \text{Id} & & \downarrow \pi \\ (\mathcal{S}kew_{\mathcal{A}_1, \sigma_1} / \mathcal{A}lt_{\mathcal{A}_1, \sigma_1}) \times \mathcal{S}ym_{\mathcal{A}_2, \sigma_2} & \longrightarrow & \mathcal{S}kew_{\mathcal{A}, \sigma} / \mathcal{A}lt_{\mathcal{A}, \sigma}. \end{array}$$

where the horizontal maps are the tensor morphisms.  $\square$

**6.10. Remark.** Lemma 6.9(iii) shows that if  $(\mathcal{A}_1, \sigma_1)$  is extendable to a quadratic pair, then so is  $(\mathcal{A}, \sigma)$ . We actually already knew this from Proposition 5.3. We shall see later in Proposition 7.7 that the converse is false.

A consequence of Proposition 5.6 is the following.

**6.11. Corollary.** *Let  $(\mathcal{A}_1, \sigma_1)$  and  $(\mathcal{A}_2, \sigma_2)$  be two Azumaya  $\mathcal{O}$ -algebras with symplectic involutions. Let  $(\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2, \sigma_1 \otimes \sigma_2)$  be the corresponding Azumaya  $\mathcal{O}$ -algebra with orthogonal involution. Then  $\omega(\mathcal{A}, \sigma) = 0$ .*

**6.12. Remark.** In the setting of Corollary 6.11, Remark 7.3 provides an explicit example such that  $\Omega(\mathcal{A}, \sigma) \neq 0$ .

The case of the strong obstruction resulting from two symplectic involutions is similar to Lemma 6.9(ii).

**6.13. Lemma.** *Let  $(\mathcal{A}_1, \sigma_1)$  and  $(\mathcal{A}_2, \sigma_2)$  be two Azumaya  $\mathcal{O}$ -algebras with symplectic involutions. Let  $(\mathcal{A}, \sigma) = (\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{A}_2, \sigma_1 \otimes \sigma_2)$  be the tensor Azumaya  $\mathcal{O}$ -algebra with orthogonal involution. According to Lemma 5.5, we have  $1_{\mathcal{A}_i} \in \text{Symd}_{\mathcal{A}_i, \sigma_i}(S)$  for  $i = 1, 2$ . Consider the boundary map  $\delta_1: \text{Symd}_{\mathcal{A}_1, \sigma_1} \rightarrow \check{H}^1(S, \text{Skew}_{\mathcal{A}_1, \sigma_1})$  arising from the exact sequence of  $\mathcal{O}$ -modules  $0 \rightarrow \text{Skew}_{\mathcal{A}\sigma} \rightarrow \mathcal{A} \rightarrow \text{Symd}_{\mathcal{A}, \sigma} \rightarrow 0$ . Then we have*

$$\Omega(\mathcal{A}, \sigma) = \delta_1(1_{\mathcal{A}_1}) \cup 1_{\mathcal{A}_2}$$

where  $1_{\mathcal{A}_2} \in \text{Symd}_{\mathcal{A}_2, \sigma_2}(S) = H^0(S, \text{Sym}_{\mathcal{A}_2, \sigma_2})$  and the cup-product arises from the tensor morphism  $\text{Skew}_{\mathcal{A}_1, \sigma_1} \times \text{Sym}_{\mathcal{A}_2, \sigma_2} \rightarrow \text{Skew}_{\mathcal{A}, \sigma}$ .

*Proof.* There is an fppf cover  $\{T_i \rightarrow S\}_{i \in I}$  such that for each  $i \in I$  there exists  $\ell_i \in \mathcal{A}_1(T_i)$  with  $\ell_i + \sigma_1(\ell_i) = 1$ . Then we will have

$$(\ell_i \otimes 1) + (\sigma_1 \otimes \sigma_2)(\ell_i \otimes 1) = 1,$$

According to Lemma 6.4,  $\Omega(\mathcal{A}, \sigma)$  is represented by the 1-cocycle

$$(\ell_i \otimes 1)|_{T_{ij}} - (\ell_j \otimes 1)|_{T_{ij}} \in \text{Skew}_{\mathcal{A}, \sigma}(T_{ij})$$

which also represents the class  $\Omega(\mathcal{A}_1, \sigma_1) \cup 1_{\mathcal{A}_2}$ . □

## 7. EXAMPLES OF NON-TRIVIAL OBSTRUCTIONS

We use here Brion's theory of homogeneous torsors over abelian varieties [Br1, Br2, Br3] which extends that of Mukai for vector bundles [Mu]. All concepts not defined below can be found in a standard textbook on elliptic curves such as [KM].

**7.1. Non-trivial Strong Obstruction.** Let  $k$  be an algebraically closed field of characteristic 2, and let  $S = E$  be an ordinary elliptic curve over  $k$ . Since  $E$  is ordinary, the 2-torsion points are  $E[2] \cong \mu_2 \times_k \mathbb{Z}/2\mathbb{Z}$ . We set  $E' = E$  and consider the  $\mu_2 \times_k \mathbb{Z}/2\mathbb{Z}$ -torsor  $E' \rightarrow E$  arising from multiplication by 2, where  $\mu_2 \times_k \mathbb{Z}/2\mathbb{Z}$  acts on  $E'$  by addition. Writing  $\mu_2$  multiplicatively and  $\mathbb{Z}/2\mathbb{Z}$  additively, we have an injection of group schemes  $i: \mu_2 \times_k \mathbb{Z}/2\mathbb{Z} \hookrightarrow \text{PGL}_2$  given by

$$(\varepsilon, 0) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix} \text{ and } (1, 1) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which permits us to define a  $\mathrm{PGL}_2$ -torsor  $P = E' \wedge^{\mu_2 \times_k \mathbb{Z}/2\mathbb{Z}} \mathrm{PGL}_2$  over  $E$ . Note that the isomorphism class of  $P$  is the image of the isomorphism class of  $E'$  under the induced map on cohomology

$$i_*: H^1(E, \mu_2 \times_k \mathbb{Z}/2\mathbb{Z}) \rightarrow H^1(E, \mathrm{PGL}_2).$$

The twist of  $M_2(\mathcal{O})$  by  $P$  is a quaternion  $\mathcal{O}$ -algebra  $\mathcal{Q}$ . We can describe this quaternion algebra explicitly using a Čech 1-cocycle in  $\mathrm{PGL}_2$  with respect to the cover  $\{E' \rightarrow E\}$ , which is an fppf cover by [KM, 2.3], i.e., an element  $\phi \in \mathrm{PGL}_2(E' \times_E E')$  satisfying

$$p_{23}^*(\phi|_{E' \times_E E' \times_E E'}) \circ p_{12}^*(\phi|_{E' \times_E E' \times_E E'}) = p_{13}^*(\phi|_{E' \times_E E' \times_E E'})$$

where  $p_{ij}: E' \times_E E' \times_E E' \rightarrow E' \times_E E'$  is the projection which keeps the  $i^{\mathrm{th}}$  and  $j^{\mathrm{th}}$  factors. The cocycle condition above also means that such a  $\phi$  will be a gluing datum as in [St, Tag 04TP]. To give this cocycle, we first find the global sections of  $E'$  and  $E' \times_E E'$ . We have that  $\mathcal{O}(E') = k$ , and since  $E' \times_E E' \cong E' \times_k (\mu_2 \times_k \mathbb{Z}/2\mathbb{Z})$  we also have

$$\mathcal{O}(E' \times_E E') \cong (k[x]/\langle x^2 - 1 \rangle)^2$$

because  $\mu_2 \times_k \mathbb{Z}/2\mathbb{Z}$  is represented as an affine  $k$ -group scheme by  $(k[x]/\langle x^2 - 1 \rangle)^2$ . Note that the two projections  $p_i: E' \times_E E' \rightarrow E'$  have the same associated  $k$ -algebra morphisms

$$\begin{aligned} p_1^*(E') = p_2^*(E'): k &\rightarrow (k[x]/\langle x^2 - 1 \rangle)^2 \\ c &\mapsto (c, c). \end{aligned}$$

The torsor  $E' \rightarrow E$  corresponds to a cocycle in  $\mu_2 \times_k \mathbb{Z}/2\mathbb{Z}$ , and one can compute that this cocycle is

$$\mathrm{Id} \in \mathrm{Hom}((k[x]/\langle x^2 - 1 \rangle)^2, (k[x]/\langle x^2 - 1 \rangle)^2) = (\mu_2 \times_k \mathbb{Z}/2\mathbb{Z})(E' \times_E E').$$

By interpreting the image of  $x$  as the element in  $\mu_2$  and the image of  $(1, 0)$  as the element in  $\mathbb{Z}/2\mathbb{Z}$ , we view this as being the element

$$((x, x), (1, 0)) \in (\mu_2 \times_k \mathbb{Z}/2\mathbb{Z})((k[x]/\langle x^2 - 1 \rangle)^2).$$

This then maps to the cocycle

$$\phi = \mathrm{Inn} \left( \begin{bmatrix} (0, 1) & (1, 0) \\ (x, 0) & (0, x) \end{bmatrix} \right) \in \mathrm{PGL}_2(E' \times_E E').$$

which we use to twist  $M_2(\mathcal{O})$ . The twisted algebra is the result of gluing two copies of  $M_2(\mathcal{O})|_{E'}$  using  $\phi$ , and so by [St, Tag 04TR] we obtain the quaternion algebra  $\mathcal{Q}$ . For  $T \in \mathfrak{Sch}_E$ , we have

$$\mathcal{Q}(T) = \{B \in M_2(\mathcal{O}(T \times_E E')) \mid \phi(B|_{T \times_E E' \times_E E'}) = B|_{T \times_E E' \times_E E'}\}.$$

Since  $k$  has characteristic 2, the canonical quaternion involution on  $M_2(\mathcal{O})$  is the split orthogonal involution in degree 2 of Example 4.5(b),

$$\eta_0: \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} d & b \\ c & a \end{bmatrix}.$$



The canonical involution on  $\mathcal{Q}$ , denoted by  $\theta$ , is the descent of  $\eta_0$  and hence  $(\mathcal{Q}, \theta)$  is a quaternion algebra with orthogonal involution.

Let  $n$  be a positive integer and now work over the abelian  $k$ -variety  $S = E^n$ . We define the Azumaya algebra

$$\mathcal{A} = p_1^*(\mathcal{Q}) \otimes_{\mathcal{O}} \cdots \otimes_{\mathcal{O}} p_n^*(\mathcal{Q})$$

where  $p_i: E^n \rightarrow E$  is the  $i^{\text{th}}$  projection and  $p_i^*$  the pullback of sheaves. It comes with the tensor product involution  $\sigma = p_1^*(\theta) \otimes \cdots \otimes p_n^*(\theta)$ , which is orthogonal. Alternatively,  $\mathcal{A}$  may be viewed as the following twist of  $M_{2^n}(\mathcal{O})$ . Set  $S' = S$  and view it as a scheme over  $S$  with respect to the multiplication by 2 map. This makes  $S' \rightarrow S$  a  $(\mu_2 \times_k \mathbb{Z}/2\mathbb{Z})^n$ -torsor and  $\{S' \rightarrow S\}$  an fppf cover. Since we are looking for a cocycle in  $\text{PGL}_{2^n}(S' \times_S S')$  we note that  $\mathcal{O}(S) = k$ , and since  $S' \times_S S' \cong (E' \times_E E')^n$  we have

$$\mathcal{O}(S' \times_S S') = \left( (k[x]/\langle x^2 - 1 \rangle)^2 \right)^{\otimes n}.$$

where the tensor product is over  $k$ . Using the decomposition

$$M_{2^n}(S' \times_S S') \cong M_2((k[x]/\langle x^2 - 1 \rangle)^2) \otimes_k \cdots \otimes_k M_2((k[x]/\langle x^2 - 1 \rangle)^2)$$

we have  $\phi \otimes \cdots \otimes \phi \in \text{PGL}_{2^n}(S' \times_S S')$ , and this is our desired cocycle. In this view, the tensor product involution on  $\mathcal{A}$  is the descent of the involution  $\eta' = \eta_0 \otimes \cdots \otimes \eta_0$  on  $M_{2^n}(k) \cong (M_2(k))^{\otimes n}$ .

**7.2. Lemma.** *Consider the Azumaya algebra with involution  $(\mathcal{A}, \sigma)$  defined above.*

- (i)  $(\mathcal{A}, \sigma)$  can be extended to a quadratic triple. In particular,  $(\mathcal{A}, \sigma)$  is locally quadratic with trivial weak obstruction.
- (ii)  $\mathcal{A}(S) \cong k$ .
- (iii) The strong obstruction  $\Omega(\mathcal{A}, \sigma)$  is non-trivial.

*Proof.* (i): We begin by considering the linear form

$$f': \text{Sym}_{M_{2^n}(\mathcal{O}), \sigma_{2^n}} \rightarrow \mathcal{O}$$

$$B \mapsto \text{Trd}_{M_{2^n}(\mathcal{O})} \left( \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes I_{2^{n-1}} \right) \cdot B \right)$$

where  $I_{2^{n-1}}$  denotes the  $2^{n-1} \times 2^{n-1}$  identity matrix. This is an instance of the construction of Example 4.3(c), and so  $(M_{2^n}(\mathcal{O}), \sigma_{2^n}, f')$  is a quadratic triple. The form  $f'|_{S'}$  will descend to a suitable linear form on  $\text{Sym}_{\mathcal{A}, \sigma}$  if  $f'|_{S' \times_S S'} \circ (\phi \otimes \cdots \otimes \phi) = f'|_{S' \times_S S'}$ . We verify this with the following computation, which uses the fact that

$\phi^2 = \text{Id}$ .

$$\begin{aligned} & \text{Trd}_{\mathcal{M}_{2^n}(\mathcal{O})} \left( \left( \begin{bmatrix} (1,1) & 0 \\ 0 & 0 \end{bmatrix} \otimes I_{2^{n-1}} \right) \cdot (\phi \otimes \dots \otimes \phi)(\underline{\quad}) \right) \\ &= \text{Trd}_{\mathcal{M}_{2^n}(\mathcal{O})} \left( (\phi \otimes \dots \otimes \phi) \left( \begin{bmatrix} (1,1) & 0 \\ 0 & 0 \end{bmatrix} \otimes I_{2^{n-1}} \right) \cdot \underline{\quad} \right) \\ &= \text{Trd}_{\mathcal{M}_{2^n}(\mathcal{O})} \left( \left( \begin{bmatrix} (0,1) & 0 \\ 0 & (1,0) \end{bmatrix} \otimes I_{2^{n-1}} \right) \cdot \underline{\quad} \right). \end{aligned}$$

Then, because

$$\left( \begin{bmatrix} (1,1) & 0 \\ 0 & 0 \end{bmatrix} \otimes I_{2^{n-1}} \right) - \left( \begin{bmatrix} (0,1) & 0 \\ 0 & (1,0) \end{bmatrix} \otimes I_{2^{n-1}} \right) = \begin{bmatrix} (1,0) & 0 \\ 0 & (1,0) \end{bmatrix} \otimes I_{2^{n-1}}$$

which is an element of  $\mathcal{A}lt_{\mathcal{M}_{2^n}(\mathcal{O}), \eta'}(S' \times_S S')$ , we know by Example 4.3(c) that

$$f'|_{S' \times_S S'} \circ (\phi \otimes \dots \otimes \phi) = f'|_{S' \times_S S'}.$$

Therefore  $f'|_{S'}$  descends, and there exists a linear form  $f: \text{Sym}_{\mathcal{A}, \sigma} \rightarrow \mathcal{O}$  such that  $(\mathcal{A}, \sigma, f)$  is a quadratic triple.

(ii): By construction we have that

$$\mathcal{A}(S) = \{B \in \mathcal{M}_{2^n}(k) \mid (\phi \otimes \dots \otimes \phi)(B|_{S' \times_S S'}) = B|_{S' \times_S S'}\}.$$

Given  $B \in \mathcal{A}(S)$ , we may write  $B$  uniquely as

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes B_1 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes B_2 + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes B_3 + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes B_4$$

with  $B_i \in \mathcal{M}_{2^{n-1}}(k)$ , and then setting  $B|_{S' \times_S S'} = \overline{B}$  we have

$$\begin{aligned} \overline{B} &= \begin{bmatrix} (1,1) & 0 \\ 0 & 0 \end{bmatrix} \otimes \overline{B}_1 + \begin{bmatrix} 0 & (1,1) \\ 0 & 0 \end{bmatrix} \otimes \overline{B}_2 \\ &+ \begin{bmatrix} 0 & 0 \\ (1,1) & 0 \end{bmatrix} \otimes \overline{B}_3 + \begin{bmatrix} 0 & 0 \\ 0 & (1,1) \end{bmatrix} \otimes \overline{B}_4. \end{aligned}$$

Now we apply  $\phi \otimes \dots \otimes \phi$ , setting  $\overline{\phi} = \phi^{\otimes n-1}$ .

$$\begin{aligned} (\phi \otimes \dots \otimes \phi)(\overline{B}) &= \begin{bmatrix} (0,1) & 0 \\ 0 & (1,0) \end{bmatrix} \otimes \overline{\phi}(\overline{B}_1) + \begin{bmatrix} 0 & (0,x) \\ (x,0) & 0 \end{bmatrix} \otimes \overline{\phi}(\overline{B}_2) \\ &+ \begin{bmatrix} 0 & (x,0) \\ (0,x) & 0 \end{bmatrix} \otimes \overline{\phi}(\overline{B}_3) + \begin{bmatrix} (1,0) & 0 \\ 0 & (0,1) \end{bmatrix} \otimes \overline{\phi}(\overline{B}_4). \end{aligned}$$

Since this is equal to  $\overline{B}$ , linear independence then requires that

$$\begin{aligned} \overline{B}_1 &= \overline{\phi}(\overline{B}_4) = \overline{B}_4 = \overline{\phi}(\overline{B}_1), \text{ and} \\ \overline{B}_2 &= \overline{B}_3 = \overline{\phi}(\overline{B}_2) = \overline{\phi}(\overline{B}_3) = 0 \end{aligned}$$

and so we can conclude that  $\overline{B} = I_2 \otimes \overline{B'}$  for some  $B' \in M_{2^{n-1}}(k)$  such that  $(\phi \otimes \dots \otimes \phi)(\overline{B'}) = \overline{B'}$ , now with only  $n - 1$  tensor factors. Hence, by induction we need only address the case of two  $2 \times 2$  matrices. There we have that

$$\phi(\overline{B}) = \begin{bmatrix} (b_4, b_1) & (b_3x, b_2x) \\ (b_2x, b_3x) & (b_1, b_4) \end{bmatrix} = \begin{bmatrix} (b_1, b_1) & (b_2, b_2) \\ (b_3, b_3) & (b_4, b_4) \end{bmatrix} = \overline{B}.$$

Since  $b_i \in k$ , this can only happen when  $B = aI_2$  for some  $a \in k$ . Therefore, overall

$$\mathcal{A}(S) = \{I_{2^{n-1}} \otimes aI_2 \mid a \in k\} \cong k.$$

(iii): The exact sequence

$$0 \rightarrow \mathit{Skew}_{\mathcal{A}, \sigma} \rightarrow \mathcal{A} \rightarrow \mathit{Symd}_{\mathcal{A}, \sigma} \rightarrow 0$$

gives rise to the exact sequence

$$(7.2.1) \quad 0 \rightarrow \mathit{Skew}_{\mathcal{A}, \sigma}(S) \rightarrow \mathcal{A}(S) \rightarrow \mathit{Symd}_{\mathcal{A}, \sigma}(S) \xrightarrow{\delta} \check{H}^1(S, \mathit{Skew}_{\mathcal{A}, \sigma})$$

by taking global sections. Since  $(\mathcal{A}, \sigma)$  is locally quadratic, we have  $1 \in \mathit{Symd}_{\mathcal{A}, \sigma}(S)$  and we recall that  $\Omega(\mathcal{A}, \sigma) = \delta(1)$  by Definition 6.1. Since  $\mathcal{A}(S) = k$  and  $\sigma(S)$  is  $k$ -linear, all elements of  $\mathcal{A}(S)$  are symmetric, and since  $k$  is characteristic 2, all elements of  $\mathcal{A}(S)$  are also skew-symmetric. Hence,  $\mathit{Skew}_{\mathcal{A}, \sigma}(S) = k$  also. Therefore, the sequence (7.2.1) is of the form

$$0 \rightarrow k \rightarrow k \rightarrow \mathit{Symd}_{\mathcal{A}, \sigma}(S) \xrightarrow{\delta} \check{H}^1(S, \mathit{Skew}_{\mathcal{A}, \sigma})$$

and we obtain that  $\Omega(\mathcal{A}, \sigma) = \delta(1) \neq 0$ .  $\square$

**7.3. Remark.** Note that for  $n = 2$ ,  $(\mathcal{A}, \sigma)$  is a tensor product of two Azumaya algebras with symplectic involutions, and so Corollary 6.11 provides another proof that the weak obstruction is zero in Lemma 7.2(iii). However, the important point is that the strong obstruction does not vanish.

**7.4. Non-trivial Weak Obstruction.** We continue working over an algebraically closed base field  $k$  of characteristic 2. According to a result by Serre [Se, prop. 15], for each finite group  $\Gamma$  there exists a Galois  $\Gamma$ -cover  $Y \rightarrow S$  as in [St, Tag 03SF] such that  $S$  and  $Y$  are connected smooth projective  $k$ -varieties. We use Serre's results with the group  $\Gamma = \mathrm{PGL}_2(\mathbb{F}_4)$  to obtain a  $\Gamma$ -cover  $\pi: Y \rightarrow S$ . We take  $S$  to be our base scheme. We denote by  $\Gamma_k$  the constant group scheme associated to  $\Gamma$ . It is affine and represented by  $k^{|\Gamma|}$  with componentwise multiplication. We write this algebra as

$$k^{|\Gamma|} = \{(c_g)_{g \in \Gamma} \mid c_g \in k\}.$$

The map  $Y \rightarrow S$  is then a  $\Gamma_k$ -torsor and  $\{Y \rightarrow S\}$  is an fppf cover. Since  $\Gamma_k$  embeds in  $\mathrm{PGL}_2$ , which we view as a group scheme over  $k$ , we can define the  $\mathrm{PGL}_2$ -torsor  $P = Y \wedge^{\Gamma_k} \mathrm{PGL}_2$  over  $S$ . The twist of  $M_2(\mathcal{O})$  by  $P$  is a quaternion  $\mathcal{O}$ -algebra  $\mathcal{Q}$ , which of course is not the same  $\mathcal{Q}$  as in section 7.1. Here as well we may describe  $\mathcal{Q}$  explicitly using cocycles. As before, we first describe the global sections of  $Y$

and  $Y \times_S Y$ . We know that  $\mathcal{O}(Y) \cong k$ , and since  $Y \rightarrow S$  is a Galois extension with Galois group  $\Gamma_k$ , we have  $Y \times_S Y \cong Y \times_k \Gamma_k$  and so

$$\mathcal{O}(Y \times_S Y) \cong k^{|\Gamma|}.$$

Therefore, we have that

$$M_2(\mathcal{O}(Y \times_S Y)) = M_2(k^{|\Gamma|}) \cong M_2(k)^{|\Gamma|}.$$

Identifying  $\Gamma_k$  with its embedding in  $\mathrm{PGL}_2$ , we have the element

$$(g)_{g \in \Gamma} \in \mathrm{PGL}_2(Y \times_S Y)$$

which we use as our cocycle. Thus, again by [St, Tag 04TR] the algebra  $\mathcal{Q}$  is described over  $T \in \mathfrak{Sch}_S$  by

$$\mathcal{Q}(T) = \{B \in M_2(\mathcal{O}(T \times_S Y)) \mid g(B) = B, \forall g \in \Gamma\}.$$

Or, more concisely,  $\mathcal{Q}(T) = M_2(\mathcal{O}(T \times_S Y))^{\Gamma_k}$  are the fixed points. The canonical involution  $\eta_0$  on  $M_2(\mathcal{O})$  of Example 4.5(b) is orthogonal and descends to an orthogonal involution  $\theta$  on  $\mathcal{Q}$ , which is the canonical involution on  $\mathcal{Q}$ .

**7.5. Lemma.**  *$(\mathcal{Q}, \theta)$  is locally quadratic, but cannot be extended to a quadratic triple. Thus,  $\omega(\mathcal{Q}, \theta) \neq 0 \in \check{H}^1(S, \mathrm{Skew}_{\mathcal{A}, \sigma} / \mathrm{Alt}_{\mathcal{A}, \sigma})$ , i.e., the weak obstruction is non-trivial.*

*Proof.* Since  $(\mathcal{Q}, \theta)$  is a twisted form of  $M_2(\mathcal{O})$  which splits over  $\{Y \rightarrow S\}$ , we have that

$$(\mathcal{Q}, \theta)|_Y \cong (M_2(\mathcal{O}), \eta_0)|_Y.$$

Since we have the split quadratic triple  $(M_2(\mathcal{O}), \eta_0, f_0)|_Y$  of Example 4.5(b) and  $\{Y \rightarrow S\}$  is an fppf cover, we obtain that  $(\mathcal{Q}, \theta)$  is locally quadratic by Lemma 4.15(iv).

Now, assume that we can extend  $(\mathcal{Q}, \theta)$  to a quadratic triple  $(\mathcal{Q}, \theta, f)$  over  $S$ . The linear map  $f$  then fits into the following commutative diagram with exact rows.

$$\begin{array}{ccccc} \mathrm{Sym}_{\mathcal{Q}, \theta}(S) & \longrightarrow & \mathrm{Sym}_{M_2(\mathcal{O}), \eta_0}(Y)^2 & \xrightarrow{\pi_1} & \mathrm{Sym}_{M_2(\mathcal{O}), \eta_0}(Y \times_S Y) \\ \downarrow f(S) & & \downarrow f(Y) \times f(Y) & & \downarrow f(Y \times_S Y) \\ k & \longrightarrow & k \times k & \xrightarrow{\pi_2} & k^{|\Gamma|} \end{array}$$

where  $\pi_1(B_1, B_2) = (g(B_1) - B_2)_{g \in \Gamma}$  and  $\pi_2(c_1, c_2) = (c_1 - c_2)_{g \in \Gamma}$ . Commutativity of the diagram then enforces that

$$f(Y \times_S Y)((g(B_1) - B_2)_{g \in \Gamma}) = (f(Y)(B_1) - f(Y)(B_2))_{g \in \Gamma}$$

which is equivalent to

$$(f(Y)(g(B_1)))_{g \in \Gamma} = (f(Y)(B_1))_{g \in \Gamma}$$

and hence  $f(Y)$  must be  $\Gamma$ -equivariant. We now argue that this means  $f(Y)$  must be zero. Certainly,  $f(Y)$  is of the form

$$f\left(\begin{bmatrix} a & b \\ c & a \end{bmatrix}\right) = au + bv + cw$$

for some  $u, v, w \in k = \mathcal{O}(Y)$ . By  $\Gamma$ -equivariance we obtain

$$w = f\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = v,$$

as well as

$$w = f\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = u + v + w$$

which implies that  $u + v = 0$ , i.e.,  $u = v$ . Therefore we must have  $u = v = w$ . Finally, consider  $0, 1 \neq \lambda \in \mathbb{F}_4$ . We also have

$$w = f\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 0 & 0 \\ \lambda & 0 \end{bmatrix}\right) = \lambda w$$

and so  $u = v = w = 0$ , meaning  $f(Y) \equiv 0$ . However, this contradicts Remark 4.2, and therefore no such  $f$  extending  $(\mathcal{Q}, \theta)$  can exist.  $\square$

**7.6. Remark.** We also notice that the trace induces an isomorphism

$$H_0(\Gamma, M_2(k)) \xrightarrow{\sim} k.$$

Since  $\text{Sym}(M_2(k), \eta_0)$  has dimension 3,  $H_0(\Gamma, M_2(k))$  has dimension  $\leq 1$ . The trace induces a surjective map  $H_0(\Gamma, M_2(k)) \rightarrow k$ , it is an isomorphism by counting dimensions.

Enlarging the example of section 7.4 is quite surprising. Unlike in section 7.1, we cannot use the tensor product construction to produce Azumaya algebras with involution of larger degree which have non-trivial weak obstruction.

**7.7. Proposition.** *Let  $n \geq 1$  and define on the  $k$ -variety  $S^n$  the following Azumaya algebra with orthogonal involution of degree  $2^n$*

$$(\mathcal{A}_n, \sigma_n) = p_1^*(\mathcal{Q}, \theta) \otimes_{\mathcal{O}_{|S^n}} \cdots \otimes_{\mathcal{O}_{|S^n}} p_n^*(\mathcal{Q}, \theta).$$

*Then  $(\mathcal{A}_n, \sigma_n)$  is locally quadratic for any  $n \geq 1$ , and  $(\mathcal{A}_n, \sigma_n)$  can be extended to a quadratic triple if and only if  $n \geq 2$ . Equivalently,  $\omega(\mathcal{A}_n, \sigma_n) \neq 0$  if and only if  $n = 1$ .*

*Proof.* Lemma 7.5 is the case  $n = 1$  and shows in particular that  $(\mathcal{Q}, \theta)$  is locally quadratic. Since  $k$  is characteristic 2, this means by Lemma 5.5 that  $\theta$  is also symplectic. Therefore  $p_i^*(\mathcal{Q}, \theta)$  have symplectic involutions and so when  $n = 2$ , Proposition 5.6 shows that  $p_1^*(\mathcal{Q}, \theta) \otimes_{\mathcal{O}} p_2^*(\mathcal{Q}, \theta)$  can be extended to a quadratic triple. Then Proposition 5.3 handles the cases of  $n \geq 3$ .  $\square$

## REFERENCES

- [Bas] H. Bass, *Lectures on topics in algebraic K-theory*, Notes by Amit Roy, Tata Institute of Fundamental Research Lectures on Mathematics, No. 41 (1967), Bombay.
- [B:A3] N. Bourbaki, *Algèbre*, Ch. 8, second revised edition of the 1958 edition, Springer, Berlin 2012.
- [Br1] M. Brion, *Homogeneous bundles over abelian varieties*, J. Ramanujan Math. Soc. **27** (2012), 91-118.
- [Br2] M. Brion, *Homogeneous projective bundles over abelian varieties* Algebra Number Theory **7** (2013), 2475-2510.
- [Br3] M. Brion, *Homogeneous vector bundles over abelian varieties via representation theory*, Representation Theory **20** (2020), 85-114.
- [CF] B. Calmès, J. Fasel, *Groupes classiques*, Autour des schémas en groupes, vol II, Panoramas et Synthèses **46** (2015), 1-133.
- [EGA-I] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique: I. Le langage des schémas*, Grundlehren der Mathematischen Wissenschaften **166** (2nd ed.). Berlin; New York: Springer-Verlag, 1971.
- [Fo] T. J. Ford, *Separable Algebras*, Graduate Studies in Mathematics **183**, Amer. Math. Soc, Provident, RI, (2017).
- [Gir] J. Giraud, *Cohomologie non abélienne*, Die Grundlehren der Mathematischen Wissenschaften, Band 179, Berlin-Heidelberg-New York: Springer-Verlag, 1971.
- [GNR] P. Gille, E. Neher, C. Ruether, *The Norm Functor over Schemes*, in preparation.
- [Gro] A. Grothendieck, *Le Groupe de Brauer I*, Sémin. Bourbaki, 1964/65, no 290.
- [GW] U. Görtz and T. Wedhorn, *Algebraic Geometry I*, second edition, Springer Fachmedien Wiesbaden, 2020.
- [Knu] M.-A. Knus, *Quadratic and Hermitian Forms over Rings*, Grundlehren der mathematischen Wissenschaften **294** (1991), Springer.
- [KM] N.M. Katz, B. Mazur, *Arithmetic Moduli of Elliptic Curves*, Annals of Mathematics Studies **108**, Princeton University Press (1985).
- [KMRT] M.-A. Knus, A. Merkurjev, M. Rost and J.-P. Tignol, *The Book of Involutions*, American Mathematical Society Colloquium Publications **44**, American Mathematical Society, Providence, RI (1998).
- [KO] M.-A. Knus, M. Ojanguren, *Théorie de la Descente et Algèbres d'Azumaya*, Lecture Notes in Mathematics **389** (1974), Springer.
- [KS] M. Kashiwara, P. Schapira, *Categories and Sheaves*, Grundlehren der mathematischen Wissenschaften **332** (2006), Springer.
- [M] J.-S. Milne, *Étale Cohomology*, Princeton University Press, 1980.
- [Mu] S. Mukai, *Semi-homogeneous vector bundles on an abelian variety*, J. Math. Kyoto Univ. **18** (1978) 239-272.  
cours Université Pierre et Marie Curie, 2011-2012.
- [RZ] A. Rosenberg and D. Zelinsky, *Automorphisms of separable algebras*, Pacific J. Math. **11** (1961), 1109–1117.
- [Sa] C. H. Sah, *Symmetric bilinear forms and quadratic forms*, J. Algebra **20** (1972), 144-160.
- [Se] J.-P. Serre, *Sur la topologie des variétés algébriques en caractéristique p*, Symposium Internacional de Topologia Algebraica, Universidad Nacional Autonoma de Mexico, 1958, 24-53.
- [SGA3] *Séminaire de Géométrie algébrique de l'I.H.E.S., 1963-1964, schémas en groupes, dirigé par M. Demazure et A. Grothendieck*, Lecture Notes in Math. 151-153. Springer (1970).
- [St] The Stacks Project Authors, *Stacks project*, <http://stacks.math.columbia.edu/>
- [Ti] J. Tits, *Formes Quadratiques, Groupes Orthogonaux et algèbres de Clifford*, Invent. Math. **5**, 19-41 (1968).

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