On the infimum of the absolute value of successive derivatives of a real function defined on a bounded interval

Michel Balazard

▶ **To cite this version:**

Michel Balazard. On the infimum of the absolute value of successive derivatives of a real function defined on a bounded interval. Number Theory. Dedicated to the memory of Eduard Wirsing, Springer, In press. hal-03845088v1

HAL Id: hal-03845088

https://cnrs.hal.science/hal-03845088v1

Submitted on 9 Nov 2022 (v1), last revised 20 Nov 2023 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On the infimum of the absolute value of successive derivatives of a real function defined on a bounded interval

Michel Balazard

November 9, 2022

Abstract

A study of the greatest possible ratio of the smallest absolute value of a higher derivative of some function, defined on a bounded interval, to the L^p-norm of the function.

Keywords

Chebyshev polynomials, Legendre polynomials, extremal problems, inequalities for derivatives
MSC classification: 26D10, 41A10

To the memory of Eduard Wirsing,
master of analysis,
and of its applications to number theory.

1 Introduction

Let n be a positive integer, $I = [a, b]$ a bounded segment of the real line, of length $L = b - a$. Define $\mathcal{D}^n(I)$ as the set of real functions f defined on I, with successive derivatives $f^{(k)}$ defined and continuous on I for $0 \leqslant k \leqslant n-1$, and $f^{(n)}$ defined on $\mathring{I} =]a, b[$. We will use the notation

$$m_n(f) = \inf_{a<t<b} |f^{(n)}(t)|.$$

Let p be a positive real number, or ∞.

The problem addressed in this article is that of determining the best constant $C^* = C^*(n, p, I)$ in the inequality

$$m_n(f) \leqslant C^* \|f\|_p \quad (f \in \mathcal{D}^n(I)),$$

where

$$\|f\|_p = \Big(\int_a^b |f(t)|^p \, dt \Big)^{1/p},$$

with the usual convention when $p = \infty$, here: $\|f\|_\infty = \max |f|$.

This problem has been posed by Kwong and Zettl in their 1992 Lecture Notes [11] (see Lemma 1.1, p. 6). They give upper bounds for $C^*(n, p, I)$, but their reasoning and results are erroneous. In her 1993 PhD Thesis [5], Huang has pointed out that this problem is equivalent to a classical problem in the theory of polynomial approximation: that of determining the minimal L^p-norm of a monic polynomial of given degree on a given bounded interval. Our purpose in this text is to give a new proof of the equivalence, and to list the consequences of the known results about this extremal problem for the evaluation of $C^*(n, p, I)$.

2 First observations

2.1 Homogeneity

Defining $g(u) = f(a + uL)$ for $f \in \mathcal{D}^n(I)$ and $0 \leqslant u \leqslant 1$, one has

$$g \in \mathcal{D}^n([0,1]) \quad ; \quad g^{(n)}(u) = L^n f^{(n)}(a + uL) \quad (0 < u < 1) \quad ; \quad \|g\|_p = L^{-1/p} \|f\|_p .$$

Hence,

$$C^*(n, p, I) = C^*(n, p, [0,1])\, L^{-n-1/p}, \tag{1}$$

and one is left with determining $C^*(n, p, [0,1]) = C(n, p)$, or in fact $C^*(n, p, I)$ for any fixed, chosen segment I. We will see that $I = [-1, 1]$ is particularly convenient.

2.2 An extremal problem

One has

$$\begin{aligned} C^*(n, p, I) &= \sup\{m_n(f) / \|f\|_p , \ f \in \mathcal{D}^n(I), \ m_n(f) \neq 0\} \\ &= \sup\{m_n(f) / \|f\|_p , \ f \in \mathcal{D}^n(I), \ m_n(f) = \lambda\} \quad \text{(for every } \lambda > 0) \\ &= \lambda / D^*(n, p, \lambda, I), \end{aligned}$$

where

$$\begin{aligned} D^*(n, p, \lambda, I) &= \inf\{\|f\|_p , \ f \in \mathcal{D}^n(I), \ m_n(f) = \lambda\} \\ &= \inf\{\|f\|_p , \ f \in \mathcal{D}^n(I), \ m_n(f) \geqslant \lambda\}, \end{aligned}$$

the last equality being true since $D^*(n, p, \mu, I) = \frac{\mu}{\lambda} D^*(n, p, \lambda, I) \geqslant D^*(n, p, \lambda, I)$ if $\mu \geqslant \lambda$.

Also, since a derivative has the intermediate value property (cf. [3], pp. 109-110), the inequality $m_n(f) \geqslant \lambda > 0$ implies that $f^{(n)}$ has constant sign on I, so that

$$D^*(n,p,\lambda,I) = \inf\{\|f\|_p\,,\ f \in \mathcal{D}^n(I),\ f^{(n)}(t) \geqslant \lambda \text{ for } a < t < b\}.$$

Thus, determining $C^*(n,p,I)$ is equivalent to minimizing $\|f\|_p$ for $f \in \mathcal{D}^n(I)$ with the constraint $f^{(n)}(t) \geqslant \lambda > 0$ for $a < t < b$. We will denote this extremal problem by $\mathcal{E}^*(n,p,\lambda,I)$.

3 The relevance of monic polynomials

Let $\mathcal{P}_n$ be the set of monic polynomials of degree n, with real coefficients, identified with the set of the corresponding polynomial functions on I, which is a subset of $\mathcal{D}^n(I)$. Since $m_n(f) = n!$ for $f \in \mathcal{P}_n$, one has

$$D^*(n,p,n!,I) \leqslant D^{**}(n,p,I), \tag{2}$$

where

$$D^{**}(n,p,I) = \inf\{\|Q\|_p\,,\ Q \in \mathcal{P}_n\}.$$

A basic fact in the study of the extremal problem $\mathcal{E}^*(n,p,\lambda,I)$ is that (2) is in fact an equality.

Proposition 1 *For all n,p,I, one has $D^*(n,p,n!,I) = D^{**}(n,p,I)$.*

It follows from this proposition that $C^*(n,p,I) = n!/D^{**}(n,p,I)$ and, by (1),

$$C(n,p) = L^{n+1/p} n!/D^{**}(n,p,I). \tag{3}$$

Let us review the history of Proposition 1.

For $p = \infty$, it is a corollary to a theorem of S. N. Bernstein from 1937. Denoting by $E_k(f)$ the distance (for the uniform norm on I) between f and the set of polynomials of degree at most k, he proved in particular that

$$E_{n-1}(f_0) > E_{n-1}(f_1) \quad (f_0, f_1 \in \mathcal{D}^n(I)),$$

provided that the inequality $f_0^{(n)}(\xi) > |f_1^{(n)}(\xi)|$ is valid for every $\xi \in \mathring{I}$ (cf. [2], p. 48, inequalities (47bis)-(48bis)). Proposition 1 follows by taking $f_1(x) = x^n$ and $f_0(x) = \lambda f(x)$, where f is a generic element of $\mathcal{D}^n(I)$ such that $f^{(n)}(t) \geqslant n!$ for $a < t < b$, and $\lambda > 1$, then letting $\lambda \to 1$.

This theorem of Bernstein was generalized by Tsenov in 1951 to the case of the L^p-norm on I, where $p \geqslant 1$ (cf. [15], Theorem 4, p. 477), thus providing a proof of Proposition 1 for $p \geqslant 1$. The case $0 < p < 1$ was left open by Tsenov.

The study of the extremal problem $\mathcal{E}^*(n,p,\lambda,I)$ was one of the themes of the 1993 PhD thesis of Xiaoming Huang [5]. In Lemma 2.0.7, pp. 9-10, she gave another proof (due to Saff)

of Proposition 1 in the case $p = \infty$. For $1 \leqslant p < \infty$, she gave a proof of Proposition 1 which is unfortunately incomplete (cf. [5], pp. 28-30). Again, the case $0 < p < 1$ was left open.

We present now a self-contained proof of Proposition 1, valid for $0 < p \leqslant \infty$. As it proceeds by induction on n, we will need the following classical-looking division lemma, for which we could not locate a reference (compare with [16] or [13]).

Proposition 2 *Let $n \geqslant 2$ and $f \in \mathcal{D}^n(I)$. Let $c \in [a, b]$. Put*

$$g(x) = \begin{cases} \dfrac{f(x) - f(c)}{x - c} & (x \in I,\ x \neq c) \\ f'(c) & (x = c). \end{cases} \tag{4}$$

Then $g \in \mathcal{D}^{n-1}(I)$. For every $x \in]a, b[$, one has

$$g^{(n-1)}(x) = \frac{f^{(n)}(\xi)}{n},$$

where $\xi \in]a, b[$.

Proof

Since f' is continuous, one has

$$g(x) = \int_0^1 f'\big(c + t(x - c)\big)\, dt \quad (x \in I).$$

Using the rule of differentiation under the integration sign, one sees that g is $n - 2$ times differentiable on I, with

$$g^{(n-2)}(x) = \int_0^1 t^{n-2} f^{(n-1)}\big(c + t(x - c)\big)\, dt \quad (x \in I).$$

As $f^{(n-1)}$ is continuous on I, this formula yields the continuity of $g^{(n-2)}$ on I.

The function g is n times differentiable on $\mathring{I} \setminus \{c\}$ (this set is just $\mathring{I}$ if $c = a$ or $c = b$), being a quotient of n times differentiable functions, with non-vanishing denominator. In the case $a < c < b$, we have now to check that g is $n - 1$ times differentiable at the point c.

The function $f^{(n-1)}$ being continuous on I and differentiable at the point c, there exists a function $\varepsilon(h)$, defined and continuous on the segment $[a - c, b - c]$ (the interior of which contains 0), vanishing for $h = 0$, such that

$$f^{(n-1)}(c + h) = f^{(n-1)}(c) + h f^{(n)}(c) + h\varepsilon(h) \quad (a \leqslant c + h \leqslant b).$$

Hence,

$$\begin{aligned} g^{(n-2)}(x) &= \int_0^1 t^{n-2} f^{(n-1)}\big(c + t(x-c)\big)\, dt \\ &= \int_0^1 t^{n-2}\Big(f^{(n-1)}(c) + t(x-c)f^{(n)}(c) + t(x-c)\varepsilon\big(t(x-c)\big)\Big)\, dt \\ &= \frac{f^{(n-1)}(c)}{n-1} + \frac{f^{(n)}(c)}{n}(x-c) + (x-c)\int_0^1 t^{n-1}\varepsilon\big(t(x-c)\big)\Big)\, dt \end{aligned}$$

When x tends to c, the last integral tends to 0, so that the function $g^{(n-2)}$ is differentiable at the point c, with

$$g^{(n-1)}(c) = \frac{f^{(n)}(c)}{n}\cdot$$

If $x \in \mathring{I} \setminus \{c\}$, one may use the general Leibniz rule and Taylor's theorem with the Lagrange form of the remainder in order to compute $g^{(n-1)}(x)$:

$$\begin{aligned} g^{(n-1)}(x) &= \frac{d^{n-1}}{dx^{n-1}}\Big(\big(f(x)-f(c)\big)\cdot\frac{1}{x-c}\Big) \\ &= \big(f(x)-f(c)\big)\cdot\frac{(-1)^{n-1}(n-1)!}{(x-c)^n} + \sum_{k=1}^{n-1}\binom{n-1}{k} f^{(k)}(x)\cdot\frac{(-1)^{n-1-k}(n-1-k)!}{(x-c)^{n-k}} \\ &= \frac{(n-1)!}{(c-x)^n}\Big(f(c) - f(x) - \sum_{k=1}^{n-1}\frac{f^{(k)}(x)}{k!}(c-x)^k\Big) \\ &= \frac{(n-1)!}{(c-x)^n}\cdot\frac{f^{(n)}(\xi)}{n!}(c-x)^n \quad \text{(where } \xi \text{ belongs to the open interval bounded by } c \text{ and } x\text{)} \\ &= \frac{f^{(n)}(\xi)}{n}\cdot \end{aligned}$$

□

In the next proposition, we stress the main element of our proof of Proposition 1, namely the fact that the condition $f^{(n)} \geqslant n!$, for some $f \in \mathcal{D}^n(I)$, implies that the absolute value of f dominates the absolute value of some monic polynomial of degree n.

Proposition 3 *Let $n \geqslant 1$ and $f \in \mathcal{D}^n(I)$ such that $f^{(n)}(x) \geqslant n!$ for every $x \in]a,b[$.*

Then there exists a monic polynomial P of degree n, with all its zeros in I, such that the inequality $|f(x)| \geqslant |P(x)|$ is valid for every $x \in I$.

Moreover, if $|f(x)| = |Q(x)|$ for every $x \in I$, where Q is a monic polynomial of degree n with real coefficients, then $f(x) = Q(x)$ for every $x \in I$.

Proof

The assertion about the zeros may be obtained *a posteriori*, by replacing the zeros of P by their projections on I. The following proof leads directly to a polynomial P with all zeros in I.

We use induction on n.

For $n = 1$, the function f is continuous on $[a, b]$, differentiable on $]a, b[$, with $f'(x) \geqslant 1$ for $a < x < b$.

If $f(a) \geqslant 0$, one has, for $a < x \leqslant b$, $f(x) = f(a) + (x - a)f'(\xi)$ (where $a < \xi < x$), thus $f(x) \geqslant x - a$. Hence, one has $|f(x)| \geqslant |x - a|$ for every $x \in I$.

If $f(b) \leqslant 0$, one proves similarly that $|f(x)| \geqslant |x - b|$ for every $x \in I$.

If $f(a) < 0 < f(b)$, there exists $c \in]a, b[$ such that $f(c) = 0$. One has then, for every $x \in I$,

$$f(x) = f(x) - f(c) = (x - c)f'(\xi) \quad (\text{where } a < \xi < b).$$

Hence $|f(x)| \geqslant |x - c|$ for every $x \in I$, and the result is proven for $n = 1$.

Let now $n \geqslant 2$, and suppose that the result is valid with $n - 1$ instead of n. Let $f \in \mathcal{D}^n(I)$ such that $f^{(n)}(x) \geqslant n!$ for every $x \in]a, b[$.

If f vanishes at some point $c \in I$, it follows from Proposition 2 that the function g defined on I by

$$g(x) = \begin{cases} \dfrac{f(x)}{x - c} & (x \in I,\, x \neq c) \\ f'(c) & (x = c), \end{cases}$$

belongs to $\mathcal{D}^{n-1}(I)$ and that, for every $x \in]a, b[$, one has

$$g^{(n-1)}(x) = \frac{f^{(n)}(\xi)}{n},$$

where $\xi \in]a, b[$, thus $g^{(n-1)}(x) \geqslant (n - 1)!$. By the induction hypothesis, there exists a monic polynomial Q of degree $n - 1$, with all its roots in I, such that $|g(x)| \geqslant |Q(x)|$ for every $x \in I$. Hence, one has the inequality $|f(x)| \geqslant |P(x)|$ for every $x \in I$, where $P(x) = (x - c)Q(x)$ is a monic polynomial of degree n, with all its roots in I.

If $f > 0$, it reaches a minimum at some point $c \in I$. Again, it follows from Proposition 2 that the function g defined on I by (4) satisfies the required hypothesis for degree $n - 1$. Thus there exists a monic polynomial Q of degree $n - 1$, with all its roots in I, such that $|g(x)| \geqslant |Q(x)|$ for every $x \in I$. Hence, one has the inequality

$$f(x) - f(c) = |f(x) - f(c)| \geqslant |P(x)| \quad (x \in I),$$

where $P(x) = (x - c)Q(x)$. It follows that

$$|f(x)| = f(x) \geqslant f(c) + |P(x)| > |P(x)| \quad (x \in I)$$

If $f < 0$, the reasoning is similar by considering a point $c \in I$ where f reaches a maximum.

Let us prove the last assertion. The hypothesis $|f| = |P|$ is equivalent to the equality $f^2 = P^2$, that is $(f - P)(f + P) = 0$. The set $E = \{x \in I,\ f(x) + P(x) = 0\}$ has empty interior, since $f^{(n)}(x) + P^{(n)}(x) = 0$ on every open subinterval of E, whereas $f^{(n)}(x) + P^{(n)}(x) \geqslant 2n!$ on $\mathring{I}$. The set $I \setminus E$ is therefore dense in I ; its elements x all verify $f(x) = P(x)$, hence $f = P$ on I by continuity. □

Proposition 1 is an immediate corollary of Proposition 3: by taking f and P as stated there, one has $|f(x)| \geqslant |P(x)|$ for every $x \in I$, so that

$$\int_a^b |f(x)|^p \, dx \geqslant \int_a^b |P(x)|^p \, dx, \tag{5}$$

for every $p > 0$ (for $p = \infty$: $\max |f| \geqslant \max |P|$).

Moreover, if $p < \infty$, equality in (5) implies that $|f| = |P|$ on I, hence $f = P$.

In other words, if $0 < p < \infty$, the extremal problem $\mathcal{E}^*(n, p, n!, I)$ has exactly the same solutions (value of the infimum and extremal functions) as the problem $\mathcal{E}^{**}(n, p, I)$ obtained by considering only monic polynomials of degree n, which one may even take with all their roots in I.

For $p = \infty$, our reasoning does not prove that an extremal function for $\mathcal{E}^*(n, p, n!, I)$ (if it exists) must be a polynomial. This is true anyway, as proved by Huang in [5], pp. 10-13.

4 Extremal polynomials

One may now use the results of the well developed theory of the extremal problem $\mathcal{E}^{**}(n, p, I)$ for polynomials. Thus, since the integral

$$\int_a^b |(x - x_1) \cdots (x - x_n)|^p \, dx \quad (x_1, \ldots, x_n \in I)$$

(or the value $\max_{x \in I} |(x - x_1) \cdots (x - x_n)|$) is a continuous function of $(x_1, \ldots, x_n)$, the compactness of I^n yields the existence of an extremal (polynomial) function for $\mathcal{E}^{**}(n, p, I)$, hence for $\mathcal{E}^*(n, p, n!, I)$.

It is a known fact that the polynomial extremal problem $\mathcal{E}^{**}(n, p, I)$ has a unique solution for all $p \in]0, \infty]$, but there is no proof valid uniformly for all values of p.

• For $p = \infty$, uniqueness was proved by Young in 1907 (cf. [18], Theorem 5, p. 340)) and follows from the general theory of uniform approximation (cf. [12], Theorem 1.8, p. 28).

• For $1 < p < \infty$, as proved by Jackson in 1921 (cf. [7], §6, pp. 121-122), this is a consequence of the strict convexity of the space $L^p(I)$.

• For $p = 1$, this is also due to Jackson in 1921 (cf. [6], §4, pp. 323-326).

• For $0 < p < 1$, the uniqueness of the extremal polynomial was proved in 1988 by Kroó and Saff (cf. [10], Theorem 2, p. 184). Their proof uses the uniqueness property for $p = 1$ and the implicit function theorem.

We will denote by $T_{n,p,I}$ the unique solution of the extremal problem $\mathcal{E}^{**}(n, p, I)$. Uniqueness gives immediately the relation

$$T_{n,p,I}(a + b - x) = (-1)^n T_{n,p,I}(x) \quad (x \in \mathbb{R}).$$

Another property of these polynomials is the fact that all their roots are simple. For $p = 1$, this fact was proved by Korkine and Zolotareff in 1873 (cf. [8], pp. 339-340), before their explicit determination of the extremal polynomial (see §5.4 below), and their proof extends, *mutatis mutandis*, to the case $1 < p < \infty$. For $p = \infty$, this is a property of the Chebyshev polynomials of the first kind (see §5.2 below). Lastly, for $0 < p < 1$, this was proved by Kroó and Saff in [10], p. 187.

Define $T_{n,p} = T_{n,p,[-1,1]}$, and write $n = 2k + \varepsilon$, where $k \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$. It follows from the mentioned results that

$$T_{n,p}(x) = x^{\varepsilon}(x^2 - x_{n,1}(p)^2) \cdots (x^2 - x_{n,k}(p)^2) \quad (x \in \mathbb{R}), \tag{6}$$

where

$$0 < x_{n,1}(p) < \cdots < x_{n,k}(p) \leqslant 1.$$

Kroó, Peherstorfer and Saff have conjectured that all the $x_{n,k}$ are increasing functions of p (cf. [9], p. 656, and [10], p. 192).

5 Results on $C(n, p)$

5.1 The case $n = 1$

The value $n = 1$ is the only one for which $C(n, p)$ is explicitly known for all p.

Proposition 4 *One has $C(1, p) = 2(p + 1)^{1/p}$ for $0 < p < \infty$, and $C(1, \infty) = 2$.*

Proof

By (6), one has $T_{1,p}(x) = x$, so that, for $0 < p < \infty$,

$$D^{**}(1, p, [-1, 1]) = \Big(\int_{-1}^{1} |t|^p \, dt \Big)^{1/p} = \big(2/(p + 1)\big)^{1/p},$$

and, by (3),

$$C(1, p) = 2^{1+1/p}/D^{**}(1, p, [-1, 1]) = 2(p + 1)^{1/p}. \qquad \square$$

Note that the Lemma 1.1, p. 6 of [11], asserts that $C(1, p) \leqslant 2 \cdot 3^{1/p}$ for $p \geqslant 2$, and that bound is $< 2(p + 1)^{1/p}$ for $p > 2$.

5.2 The case $p = \infty$

This is the classical case, solved by Chebyshev in 1853 by introducing the polynomials T_n defined by the relation $T_n(\cos t) = \cos nt$ (now called Chebyshev polynomial of the first kind): the unique solution of the extremal problem $\mathcal{E}^{**}(n, \infty, [-1, 1])$ is $2^{1-n}T_n$. Let us record a short proof of this fact.

Take $I = [-1, 1]$ and suppose that P is a monic polynomial of degree n satisfying the inequality $\|P\|_\infty \leqslant \left\|2^{1-n}T_n\right\|_\infty = 2^{1-n}$. Then, for $\lambda > 1$ the polynomial

$$Q_\lambda = \lambda 2^{1-n}T_n - P$$

is of degree n, with leading coefficient $\lambda - 1$. Moreover, it satisfies

$$(-1)^k Q_\lambda(\cos k\pi/n) = \lambda 2^{1-n} - (-1)^k P(\cos k\pi/n) > 0 \quad (k = 0, \ldots, n)$$

By the intermediate value property, Q_λ has at least n distinct roots, hence exactly n, and these roots, say $x_1, \ldots, x_n$, have absolute value not larger than 1. Hence,

$$|Q_\lambda(x)| = (\lambda - 1)\,|(x - x_1)\cdots(x - x_n)| \leqslant (\lambda - 1)(1 + |x|)^n \quad (x \in \mathbb{R}).$$

When $\lambda \to 1$, $Q_\lambda(x)$ tends to 0 for every real x, which means that $P = 2^{1-n}T_n$.

One deduces from this theorem the value of $C(n, \infty)$. One has

$$D^{**}(n, \infty, [-1, 1]) = \max_{|x|\leqslant 1} \left|2^{1-n}T_n(x)\right| = 2^{1-n},$$

hence

$$C(n, \infty) = 2^n \cdot n!/D^{**}(n, \infty, [-1, 1]) = 2^{2n-1}n! \tag{7}$$

(compare with the upper bound $C(n, \infty) \leqslant 2^{n(n+1)/2}n^n$ of [4], 3 (a), p. 185). This result is essentially due to Bernstein (1912, cf. [1], p. 65).

Qualitatively, the result expressed by (7) was nicely described by Soula in [14], p. 86, as follows.

> Bernstein's principle: the minimum of the absolute value of the n-th derivative of an n times differentiable function and the maximum of the absolute value of the n-th derivative of an analytic function have similar orders of magnitude.

5.3 The case $p = 2$

In this case, the extremal problem $\mathcal{E}^{**}(n, 2, [-1, 1])$ is an instance of the general problem of computing the orthogonal projection of an element of a Hilbert space onto a finite dimensional subspace. Here, the Hilbert space is $L^2(-1, 1)$, the element is the monomial function x^n, and

the subspace is the set of polynomial functions of degree less than n. The solution follows from the theory of orthogonal polynomials: the extremal polynomial for $\mathcal{E}^{**}(n, 2, [-1, 1])$ is

$$\frac{2^n (n!)^2}{(2n)!} P_n(x) \quad (|x| \leqslant 1),$$

where P_n is the n-th Legendre polynomial, defined by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

Hence,

$$D^{**}(n, 2, [-1, 1]) = \frac{2^n (n!)^2}{(2n)!} \|P_n\|_2 = \frac{2^n (n!)^2}{(2n)!} \sqrt{\frac{2}{2n+1}},$$

(see [17], §15·14, p. 305) and

$$C(n, 2) = 2^{n+\frac{1}{2}} \cdot n! / D^{**}(n, 2, [-1, 1]) = \frac{(2n)!}{n!} \sqrt{2n+1}, \tag{8}$$

a result given by Soula in 1932 (cf. [14], pp. 87-88).

5.4 The case $p = 1$

The problem $\mathcal{E}^{**}(n, 1, [-1, 1])$ was solved by Korkine and Zolotareff in [8]: the extremal polynomial is $2^{-n} U_n(x)$, where U_n is the n-th Chebyshev polynomial of the second kind, defined by the relation $U_n(\cos t) = \sin(n+1)t / \sin t$.

Therefore, one has

$$\begin{aligned} D^{**}(n, 1, [-1, 1]) &= 2^{-n} \int_{-1}^{1} |U_n(x)| \, dx = 2^{-n} \int_0^{\pi} |U_n(\cos t)| \, \sin t \, dt \\ &= 2^{-n} \int_0^{\pi} |\sin(n+1)t| \, dt = 2^{-n} \int_0^{\pi} \sin u \, du \\ &= 2^{1-n}, \end{aligned}$$

and

$$C(n, 1) = 2^{n+1} \cdot n! / D^{**}(n, 1, [-1, 1]) = 2^{2n} n!. \tag{9}$$

5.5 Bounds for $C(n, p)$

We begin with a simple monotony result.

Proposition 5 *For every positive integer n, the function $p \mapsto C(n, p)$ is decreasing on the interval $0 < p \leqslant \infty$.*

Proof

Let $I = [0,1]$. Equivalently, we will see that the function $p \mapsto D^{**}(n,p,I)$ is increasing. This is due to the fact that, for a fixed $f \in L^\infty(I)$ such that $|f|$ is not equal almost everywhere to a constant, the function $p \mapsto \|f\|_p$ is increasing (a consequence of Hölder's inequality). Thus, for every $Q \in \mathcal{P}_n$ and $0 < p < p' \leqslant \infty$,

$$\|Q\|_{p'} > \|Q\|_p \geqslant D^{**}(n,p,I),$$

which implies that $D^{**}(n,p',I) > D^{**}(n,p,I)$. □

In particular, (7) and (9) yield the inequalities

$$2^{2n-1}n! < C(n,p) < 2^{2n}n! \quad (1 < p < \infty).$$

The next proposition implies that the limit of $C(n,p)$ when p tends to 0 is $(2e)^n n!$.

Proposition 6 *For every positive integer n and every positive real number p, one has*

$$2^n(1+np)^{1/p}n! \leqslant C(n,p) \leqslant (2e)^n n!$$

Proof

Equivalently, we will prove that

$$(2e)^{-n} \leqslant D^{**}(n,p,I) \leqslant 2^{-n}(1+np)^{-1/p}, \tag{10}$$

where $I = [0,1]$.

Let $Q(t) = (t-x_1)\cdots(t-x_n)$, where $0 \leqslant x_1, \ldots, x_n \leqslant 1$. One has

$$\begin{aligned}
\ln \|Q\|_p &= \frac{1}{p} \ln \int_0^1 |Q(t)|^p \, dt \\
&\geqslant \frac{1}{p} \int_0^1 \ln \big(|Q(t)|^p \big) \, dt \quad \text{(by Jensen's inequality)} \\
&= \int_0^1 \ln |Q(t)| \, dt \\
&= \sum_{k=1}^n \int_0^1 \ln |t - x_k| \, dt.
\end{aligned}$$

Now,

$$\int_0^1 \ln |t-x| \, dt = (1-x)\ln(1-x) + x \ln x - 1 \quad (0 \leqslant x \leqslant 1),$$

attains its minimal value, namely $-1 - \ln 2$, when $x = 1/2$. This implies the first inequality of (10).

To prove the second inequality of (10), we just compute $\|Q\|_p^p$ when $Q(t) = (t - 1/2)^n$:

$$\int_0^1 |t - 1/2|^{np}\, dt = 2\frac{(1/2)^{np+1}}{np+1}. \qquad \square$$

For $0 < p < 1$, we can also prove the following result.

Proposition 7 *Let n be a positive integer, and p such that $0 < p < 1$. One has*

$$1 \leqslant \frac{C(n,p)}{2^{2n}n!} \leqslant \frac{1}{2}(8/\pi)^{1/p}.$$

Proof

The first inequality is just $C(n,1) \leqslant C(n,p)$.

To prove the second inequality, let r and s such that $1 < s < 2$ and $r^{-1} + s^{-1} = 1$. Define

$$I_1(s) = \int_{-1}^1 \frac{dt}{(1-t^2)^{s/2}}$$
$$I_2(s) = \int_{-1}^1 |t|^{(s-1)/s}\, \frac{dt}{\sqrt{1-t^2}}.$$

The integrals $I_1(s)$ and $I_2(s)$ may be computed, using the eulerian identity

$$\int_0^1 t^{x-1}(1-t)^{y-1}\, dt = \mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (x > 0,\ y > 0).$$

The results are

$$I_1(s) = 2^{1-s}\frac{\Gamma(1-\frac{s}{2})^2}{\Gamma(2-s)}$$
$$I_2(s) = \frac{\Gamma\big(1-\frac{1}{2s}\big)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2}-\frac{1}{2s})}.$$

Now, let $Q \in \mathcal{P}_n$ and put $p' = p/r$. By Hölder's inequality, one has

$$\int_{-1}^1 |Q(t)|^{p'}\, \frac{dt}{\sqrt{1-t^2}} \leqslant \Big(\int_{-1}^1 |Q(t)|^{p'r}\, dt\Big)^{1/r}\Big(\int_{-1}^1 \frac{dt}{(1-t^2)^{s/2}}\Big)^{1/s}$$
$$= \|Q\|_p^{p'}\, I_1(s)^{1/s}.$$

It was proved by Kroó and Saff (cf. [10], pp. 182-183) that

$$\begin{aligned}
2^{(n-1)p'} \int_{-1}^{1} |Q(t)|^{p'} \ \frac{dt}{\sqrt{1-t^2}} &\geqslant \int_{-1}^{1} |T_n(t)|^{p'} \ \frac{dt}{\sqrt{1-t^2}} \\
&= \int_0^{\pi} |\cos nu|^{p'} \ du = \int_0^{\pi} |\cos u|^{p'} \ du \\
&= \int_{-1}^{1} |t|^{p'} \ \frac{dt}{\sqrt{1-t^2}} \\
&\geqslant \int_{-1}^{1} |t|^{1/r} \ \frac{dt}{\sqrt{1-t^2}} \quad \text{(one has } p' = p/r < 1/r\text{)} \\
&= I_2(s).
\end{aligned}$$

Therefore, with $I = [-1, 1]$,

$$\|Q\|_p \geqslant 2^{1-n} I_2(s)^{1/p'} I_1(s)^{-1/p's} = 2^{1-n} A(s)^{1/p} \quad (1 < s < 2), \tag{11}$$

where

$$A(s) = I_2(s)^{s/(s-1)} I_1(s)^{-1/(s-1)}.$$

Hence

$$A(s) = 2\Big(\frac{\Gamma\big(1 - \frac{1}{2s}\big)^s \Gamma(\frac{1}{2})^s \Gamma(2-s)}{\Gamma(1-\frac{s}{2})^2 \Gamma\big(\frac{3}{2} - \frac{1}{2s}\big)^s}\Big)^{1/(s-1)} \quad (1 < s < 2).$$

Putting $f(s) = \ln \Gamma(s)$, one has

$$\ln A(s) = \ln 2 + \frac{sf(1-1/2s) + sf(1/2) + f(2-s) - 2f(1-s/2) - sf(3/2-1/2s)}{s-1}.$$

When s tends to 1, the last fraction tends to

$$\ln \pi + \frac{3}{2}\psi(1/2) - \frac{3}{2}\psi(1) = \ln \pi - 3 \ln 2,$$

with the usual notation $f' = \Gamma'/\Gamma = \psi$. It follows that

$$A(s) \to \frac{\pi}{4} \quad (s \to 1).$$

Together with (11), this gives the inequality

$$D^{**}(n, p, [-1, 1]) \geqslant 2^{1-n} (\pi/4)^{1/p}$$

and (3) now implies

$$C(n, p) \leqslant 2^{2n-1} n! (8/\pi)^{1/p}. \qquad \square$$

We now prove an inequality involving three values of the function C.

Proposition 8 *Let p, q, r be positive real numbers such that*

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r}\cdot$$

Let m and n be positive integers. Then,

$$\frac{C(m+n,p)}{(m+n)!} \geqslant \frac{C(m,q)}{m!} \cdot \frac{C(n,r)}{n!}\cdot$$

Proof

Equivalently, by (3), one has to prove that

$$D^{**}(m+n,p,I) \leqslant D^{**}(m,q,I) \cdot D^{**}(n,r,I),$$

where I is a segment of the real line.

In fact, if $P \in \mathcal{P}_m$ and $Q \in \mathcal{P}_n$, then$PQ \in \mathcal{P}_{m+n}$ hence

$$D^{**}(m+n,p,I)^p \leqslant \int_I |P(t)Q(t)|^p \, dt \leqslant \Big(\int_I |P(t)|^q \, dt\Big)^{p/q} \cdot \Big(\int_I |Q(t)|^r \, dt\Big)^{p/r}$$

by the definition of $D^{**}(m+n,p,I)$ and Hölder's inequality. The greatest lower bound of the last term, when P runs over $\mathcal{P}_m$ and Q runs over $\mathcal{P}_n$, is

$$D^{**}(m,q,I)^p \cdot D^{**}(n,r,I)^p.$$

The result follows. □

5.6 An open question

Finally, observing that

$$C(n,2) \sim \sqrt{\frac{2}{\pi}} \cdot 2^{2n} n! \quad (n \to \infty),$$

(an exercise on Stirling's formula from (8)), we ask the following question.

> Is it true that, for every $p > 0$, the quantity $2^{-2n} C(n,p)/n!$ tends to a limit when n tends to infinity?

References

[1] S. Bernstein – "Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné.", *Mém. Cl. Sci. Acad. Roy. Belg.* **IV** (1912), no. 1-104.

[2] S. N. Bernstein – *Extremal properties of polynomials and the best approximation of continuous functions of a single real variable. part i*, in russian éd., G. R. O . L, Leningrad, Moscow, 1937.

[3] G. Darboux – "Mémoire sur les fonctions discontinues.", *Ann. de l'Éc. Norm. (2)* **4** (1875), p. 57–112.

[4] J. Dieudonné – *Foundations of modern analysis*, Pure and Applied Mathematics, vol. 10, Academic Press, New York-London, 1969.

[5] X. Huang – "On extremal properties of algebraic polynomials", Thèse, The Ohio State University, 1993.

[6] D. Jackson – "Note on a class of polynomials of approximation.", *Trans. Am. Math. Soc.* **22** (1921), p. 320–326.

[7] — , "On functions of closest approximation.", *Trans. Am. Math. Soc.* **22** (1921), p. 117–128.

[8] A. Korkine et G. Zolotareff – "Sur un certain minimum", *Nouv. Ann.* **12** (1873), p. 337–356.

[9] A. Kroó et F. Peherstorfer – "On the zeros of polynomials of minimal L_p-norm", *Proc. Am. Math. Soc.* **101** (1987), p. 652–656.

[10] A. Kroó et E. B. Saff – "On polynomials of minimal L_q-deviation, $0 < q < 1$", *J. Lond. Math. Soc., II. Ser.* **37** (1988), p. 182–192.

[11] M. K. Kwong et A. Zettl – *Norm inequalities for derivatives and differences*, Lecture Notes in Mathematics, vol. 1536, Springer, 1992.

[12] T. J. Rivlin – *An introduction to the approximation of functions. Corr. reprint of the 1969 orig*, Dover Publications Inc., Mineola, NY, 1981.

[13] L. Schoenfeld – "On the differentiability of indeterminate quotients", *Math. Mag.* **41** (1968), p. 152–155.

[14] J. Soula – "Sur une inégalité verifiee par une fonction et sa dérivée d'ordre n", *Mathematica, Cluj* **6** (1932), p. 86–88.

[15] I. V. Tsenov – "On a question of the approximation of functions by polynomials", *Mat. Sb., Nov. Ser.* **28** (1951), p. 473–478 (Russian).

[16] H. Whitney – "Differentiability of the remainder term in Taylor's formula", *Duke Math. J.* **10** (1943), p. 153–158.

[17] E. T. Whittaker et G. N. Watson – *A course of modern analysis*, 4th éd., Cambridge University Press, 1927.

[18] J. W. Young – "General theory of approximation by functions involving a given number of arbitrary parameters.", *Trans. Am. Math. Soc.* **8** (1907), p. 331–344.

BALAZARD, Michel
Institut de Mathématiques de Marseille (I2M)
CNRS, Aix Marseille Université
Marseille, France
e-mail address: balazard@math.cnrs.fr