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# A short proof of quantitative stability for the Heisenberg-Pauli-Weyl inequality

Max Fathi

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## Abstract

We give a simple proof of quantitative stability for the Heisenberg-Pauli-Weyl inequality in  $L^2$  distance, with explicit constants.

Our goal in this note is to give a simple, constructive proof of the quantitative HPW inequality recently obtained in [8]. We shall obtain the following statement:

**Theorem 1.** *Let  $u$  be a  $W^{1,2}$  function on  $\mathbb{R}^d$ . Then*

$$\left( \int |x|^2 u^2 dx \right) \left( \int |\nabla u|^2 dx \right) \geq \frac{d^2}{4} \left( \int u^2 dx \right)^2 \\ + \frac{1}{4} \left( \int u^2 dx \right) d(u, G)^2 + \frac{1}{16} d(u, G)^4$$

where  $d(u, G)$  is the  $L^2$  distance to the set of Gaussian functions, that is

$$d(u, G)^2 := \inf \left\{ \int \left( u - c e^{-\lambda|x|^2} \right)^2 dx; \quad c \in \mathbb{R}, \lambda > 0 \right\}.$$

This actually slightly improves on the main result of [8], since we identify explicit constants, and in particular show that the prefactor in the last term does not depend on the dimension. The source of this difference is that, unlike [8], we do not use concentration-compactness, and instead rely on direct estimates via classical Gaussian functional inequalities. The proof will identify a specific Gaussian distribution as a competitor, and the distance to the set of Gaussian measures can be replaced by the  $L^2$  distance to the Gaussian density with specific values of the parameters, that depend on the mass and the second moment. As discussed in [8], the one-dimensional case goes back to [3].

The exponent 2 for the first reminder term is optimal, as can be checked by linearizing the inequality around a Gaussian measure. Since the constant  $1/4$  in front of the first term is not sharp, the second reminder term in  $d(u, G)^4$  cannot be interpreted as a deficit estimate for an improved inequality. Its inclusion here does not have any particular motivation, beyond making the comparison with the

results of [8] easier, since that work also included a quartic reminder term, with dimension-dependent prefactor.

From the homogeneity of the inequality, as well as the invariance by the transform  $u(x) \rightarrow \lambda^{-d}u(\lambda x)$ , we can assume without loss of generality that

$$\int u^2 dx = 1; \quad \int |x|^2 u^2 dx = d. \quad (1)$$

Under this scaling,  $u^2$  can be interpreted as the probability density of a distribution with normalized second moment. In which case the HPW inequality can be interpreted as a lower bound on the Fisher information of the distribution

$$I(u^2) = 4 \int |\nabla u|^2 dx = \int |\nabla \log u^2| u^2 dx.$$

We keep the factor 4 in the definition of Fisher information to keep notations coherent with the information theory literature. The HPW inequality can then be rewritten as

$$I(u^2) \geq d \quad (2)$$

for all probability densities satisfying the second moment scaling. So it is the same as the classical result of information theory that under a second moment constraint, Gaussian distributions minimize Fisher information. In this form, improving this inequality is a classical topic in information theory, going back to Stam's inequality [9], also known as the isoperimetric inequality for entropies. We refer to [4] and references therein for more about information-theoretic inequalities in this context, and their connection with uncertainty principles. Stam's inequality is also equivalent to Gross's Gaussian logarithmic Sobolev inequality [7], for which stability has been a topic of recent interest, see [2, 6, 5] and references therein.

In the formulation (2), the proof of the HPW inequality is almost immediate. Since the second moment is normalized, it is natural to introduce the relative Fisher information to the standard Gaussian distribution, whose density will be denoted by  $\gamma(x) = (2\pi)^{-d/2} \exp(-|x|^2/2)$ . We have

$$\begin{aligned} 0 \leq I_\gamma(u^2) &:= \int |\nabla \log u^2 + x|^2 u^2 dx \\ &= I(u^2) + 2 \int (x \cdot \nabla \log u^2) u^2 dx + \int |x|^2 u^2 dx \\ &= I(u^2) - d \end{aligned} \quad (3)$$

where we used an integration by parts and the scaling of the second moment to get the last identity.

Hence, to get a deficit estimate, one should seek to bound from below the relative Fisher information  $I_\gamma(u^2)$  (which is itself already a natural notion of proximity for probability densities). Note that from the classical Talagrand and logarithmic Sobolev inequality, it is also bounded from below by the  $L^2$  Wasserstein distance to the Gaussian, and relative entropy.

To get the lower bound in Theorem 1 from (3), we shall use the Gaussian Poincaré inequality

$$\forall f : \mathbb{R}^d \longrightarrow \mathbb{R} \quad \text{Var}_\gamma(f) \leq \int |\nabla f|^2 d\gamma,$$

where  $\text{Var}_\gamma(f)$  stands for the variance of the function  $f$  under the Gaussian distribution, that is

$$\text{Var}_\gamma(f) := \int f^2 d\gamma - \left( \int f d\gamma \right)^2.$$

The Gaussian Poincaré inequality is a classical tool in Gaussian analysis, and we refer to [1, Proposition 4.1.1] for a proof. We have

$$\begin{aligned} I_\gamma(u^2) &= \int |\nabla \log u^2 + x|^2 u^2 dx = 4 \int |\nabla(u\gamma^{-1/2})|^2 \gamma(x) dx \\ &\geq 4 \text{Var}_\gamma(u\gamma^{-1/2}) = 4 \int u^2 dx - 4 \left( \int u\gamma^{1/2} dx \right)^2 \\ &= 4 - \left( \int (u - \gamma^{1/2})^2 dx - 2 \right)^2 \\ &= 4 \int (u - \gamma^{1/2})^2 dx - \left( \int (u - \gamma^{1/2})^2 dx \right)^2 \\ &\geq \int (u - \gamma^{1/2})^2 dx + \frac{1}{4} \left( \int (u - \gamma^{1/2})^2 dx \right)^2 \end{aligned}$$

where for the last inequality we used the fact that  $\int (u - \gamma^{1/2})^2 dx \leq 2$  and the inequality  $4x - x^2 \geq x + x^2/4$  for  $x \leq 2$ . This concludes the proof of Theorem 1.

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