HAL open science

A short proof of quantitative stability for the Heisenberg–Pauli–Weyl inequality

Max Fathi

▶ To cite this version:

Max Fathi. A short proof of quantitative stability for the Heisenberg–Pauli–Weyl inequality. Nonlinear Analysis: Theory, Methods and Applications, 2021, 210, pp.112403. 10.1016/j.na.2021.112403 . hal-03845799

HAL Id: hal-03845799

https://cnrs.hal.science/hal-03845799

Submitted on 9 May 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

Version of Record: https://www.sciencedirect.com/science/article/pii/S0362546X21001000
Manuscript_9d5d6a7c90c68382db24965bcd424fdf

A short proof of quantitative stability for the Heisenberg-Pauli-Weyl inequality

Max Fathi

April 27, 2021

Abstract

We give a simple proof of quantitative stability for the Heisenberg-Pauli-Weyl inequality in L^2 distance, with explicit constants.

Our goal in this note is to give a simple, constructive proof of the quantitative HPW inequality recently obtained in [8]. We shall obtain the following statement:

Theorem 1. *Let u be a $W^{1,2}$ function on $\mathbb{R}^d$. Then*

$$\left(\int |x|^2 u^2 dx\right)\left(\int |\nabla u|^2 dx\right) \geq \frac{d^2}{4}\left(\int u^2 dx\right)^2$$
$$+\frac{1}{4}\left(\int u^2 dx\right) d(u,G)^2 + \frac{1}{16} d(u,G)^4$$

where $d(u,G)$ is the L^2 distance to the set of Gaussian functions, that is

$$d(u,G)^2 := \inf\left\{\int \left(u - ce^{-\lambda|x|^2}\right)^2 dx; \quad c \in \mathbb{R}, \lambda > 0\right\}.$$

This actually slightly improves on the main result of [8], since we identify explicit constants, and in particular show that the prefactor in the last term does not depend on the dimension. The source of this difference is that, unlike [8], we do not use concentration-compactness, and instead rely on direct estimates via classical Gaussian functional inequalities. The proof will identify a specific Gaussian distribution as a competitor, and the distance to the set of Gaussian measures can be replaced by the L^2 distance to the Gaussian density with specific values of the parameters, that depend on the mass and the second moment. As discussed in [8], the one-dimensional case goes back to [3].

The exponent 2 for the first reminder term is optimal, as can be checked by linearizing the inequality around a Gaussian measure. Since the constant 1/4 in front of the first term is not sharp, the second reminder term in $d(u,G)^4$ cannot be interpreted as a deficit estimate for an improved inequality. Its inclusion here does not have any particular motivation, beyond making the comparison with the

© 2021 published by Elsevier. This manuscript is made available under the CC BY NC user license
https://creativecommons.org/licenses/by-nc/4.0/

results of [8] easier, since that work also included a quartic reminder term, with dimension-dependent prefactor.

From the homogeneity of the inequality, as well as the invariance by the transform $u(x) \longrightarrow \lambda^{-d} u(\lambda x)$, we can assume without loss of generality that

$$\int u^2 dx = 1; \quad \int |x|^2 u^2 dx = d. \tag{1}$$

Under this scaling, u^2 can be interpreted as the probability density of a distribution with normalized second moment. In which case the HPW inequality can be interpreted as a lower bound on the Fisher information of the distribution

$$I(u^2) = 4 \int |\nabla u|^2 dx = \int |\nabla \log u^2| u^2 dx.$$

We keep the factor 4 in the definition of Fisher information to keep notations coherent with the information theory literature. The HPW inequality can then be rewritten as

$$I(u^2) \geq d \tag{2}$$

for all probability densities satisfying the second moment scaling. So it is the same as the classical result of information theory that under a second moment constraint, Gaussian distributions minimize Fisher information. In this form, improving this inequality is a classical topic in information theory, going back to Stam's inequality [9], also known as the isoperimetric inequality for entropies. We refer to [4] and references therein for more about information-theoretic inequalities in this context, and their connection with uncertainty principles. Stam's inequality is also equivalent to Gross's Gaussian logarithmic Sobolev inequality [7], for which stability has been a topic of recent interest, see [2, 6, 5] and references therein.

In the formulation (2), the proof of the HPW inequality is almost immediate. Since the second moment is normalized, it is natural to introduce the relative Fisher information to the standard Gaussian distribution, whose density will be denoted by $\gamma(x) = (2\pi)^{-d/2} \exp(-|x|^2/2)$. We have

$$\begin{aligned} 0 \leq I_\gamma(u^2) &:= \int |\nabla \log u^2 + x|^2 u^2 dx \\ &= I(u^2) + 2 \int (x \cdot \nabla \log u^2) u^2 dx + \int |x|^2 u^2 dx \\ &= I(u^2) - d \end{aligned} \tag{3}$$

where we used an integration by parts and the scaling of the second moment to get the last identity.

Hence, to get a deficit estimate, one should seek to bound from below the relative Fisher information $I_\gamma(u^2)$ (which is itself already a natural notion of proximity for probability densities). Note that from the classical Talagrand and logarithmic Sobolev inequality, it is also bounded from below by the L^2 Wasserstein distance to the Gaussian, and relative entropy.

To get the lower bound in Theorem 1 from (3), we shall use the Gaussian Poincaré inequality

$$\forall f : \mathbb{R}^d \longrightarrow \mathbb{R} \quad \mathrm{Var}_\gamma(f) \leq \int |\nabla f|^2 d\gamma,$$

where $\mathrm{Var}_\gamma(f)$ stands for the variance of the function f under the Gaussian distribution, that is

$$\mathrm{Var}_\gamma(f) := \int f^2 d\gamma - \left(\int f d\gamma\right)^2.$$

The Gaussian Poincaré inequality is a classical tool in Gaussian analysis, and we refer to [1, Proposition 4.1.1] for a proof. We have

$$\begin{aligned}
I_\gamma(u^2) &= \int |\nabla \log u^2 + x|^2 u^2 dx = 4 \int |\nabla (u\gamma^{-1/2})|^2 \gamma(x) dx \\
&\geq 4\,\mathrm{Var}_\gamma(u\gamma^{-1/2}) = 4 \int u^2 dx - 4 \left(\int u\gamma^{1/2} dx\right)^2 \\
&= 4 - \left(\int (u - \gamma^{1/2})^2 dx - 2\right)^2 \\
&= 4 \int (u - \gamma^{1/2})^2 dx - \left(\int (u - \gamma^{1/2})^2 dx\right)^2 \\
&\geq \int (u - \gamma^{1/2})^2 dx + \frac{1}{4} \left(\int (u - \gamma^{1/2})^2 dx\right)^2
\end{aligned}$$

where for the last inequality we used the fact that $\int (u - \gamma^{1/2})^2 dx \leq 2$ and the inequality $4x - x^2 \geq x + x^2/4$ for $x \leq 2$. This concludes the proof of Theorem 1.

Acknowledgments: The author was supported by the Projects MESA (ANR-18-CE40-006) and EFI (ANR-17-CE40-0030) of the French National Research Agency (ANR). I thank the anonymous referee for valuable comments.

References

[1] D. Bakry, I. Gentil and M. Ledoux, Analysis and Geometry of Markov Diffusion Operators. *Grundlehren der Mathematischen Wissenschaften* 348. Cham: Springer.

[2] S. G. Bobkov, N. Gozlan, C. Roberto and P.-M. Samson. Bounds on the deficit in the logarithmic Sobolev inequality. *J. Funct. Anal.* 267 (11) (2014) 4110–4138.

[3] N. G. de Bruijn, Uncertainty principles in Fourier analysis. 1967 Inequalities (Proc. Sympos. Wright-Patterson Air Force Base, Ohio, 1965) pp. 57–71 Academic Press, New York

[4] A. Dembo, T. M. Cover and J. A. Thomas, Information-theoretic inequalities. *IEEE Trans. Inform. Theory* 37 (1991), no. 6, 1501–1518.

[5] R. Eldan, J. Lehec and Y. Shenfeld, Stability of the logarithmic Sobolev inequality via the Föllmer process. *Ann. Inst. Henri Poincaré Probab. Stat.* 56 (2020), no. 3, 2253–2269.

[6] M. Fathi, E. Indrei and M. Ledoux. Quantitative logarithmic Sobolev inequalities and stability estimates. *Discrete Contin. Dyn. Syst.* 36 (12) (2016) 6835–6853.

[7] L. Gross, Logarithmic Sobolev inequalities. *Amer. J. Math.* 97 (1975), no. 4, 1061–1083.

[8] S. McCurdy and R. Venkatraman, Quantitative Stability for the Heisenberg-Pauli-Weyl inequality. *Nonlinear Analysis*, 202, 2021.

[9] A. J. Stam, Some inequalities satisfied by the quantities of information of Fisher and Shannon. *Information and Control* 2 (1959), 101–112.