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Cosmology with a dynamical dimension of spacetime

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Abstract

The four-dimensional nature of physical spacetime and the possible existence of hidden dimensions are long-standing and central questions in modern scientific research. They are associated with major theoretical challenges in physics such as the unification of fundamental interactions or the birth and evolution of the universe. In his pioneering work, Ehrenfest concluded that three spatial dimensions are necessary and sufficient for classical fundamental laws of physics to agree with experimental and observational results. Here, we extend Ehrenfest's approach by asking how this particular dimension is selected. To this purpose, we postulate that the current dimension of our physical world results from a dynamical evolution of the spacetime dimension of an expanding universe. Starting from this assumption, we develop a model within the framework of “classical” general relativity. First, we use the generalized action of gravity to any dimension of spacetime and give dimensional arguments for the existence of a physical length scale that characterizes the “spatial” extension of “hidden” components of the metric. Next, we solve a generalized cosmological model in which the dimension $n(t)$ of spacetime is allowed to jump incrementally. Within this approach and under minimal assumptions, we show the existence of a single possible scenario of an expanding universe with a space-time dimension that increases incrementally from $n(t) = 2$ to an upper limit $n(t) = 4$. We conclude our study by discussing the results as well as the relevance and possible drawbacks of our approach.

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I. INTRODUCTION

Each of us has wondered at least once why do we live in three-dimensional space and why time always seems to flow in the same direction. These questions are not of philosophical nature only, but arise in different areas of theoretical physics, whether at the level of fundamental interactions up to cosmological scales. The literature on this subject is abundant and covers a wide spectrum ranging from purely scientific aspects to popular science, which makes any attempt at an exhaustive introduction a heavy task. For this reason, we will briefly retrace the stroll in this fascinating quest by following a narrative inspired by [1, 2].

Our world is made up of physical objects that we usually describe using physical laws derived from Newtonian dynamics which suggests that the usual three spatial components and the temporal one are independent entities. In 1917, Ehrenfest wrote a paper entitled *In what way does it become manifest in the fundamental laws of physics that space has three dimensions?* [3, 4]. He noted that if the Newtonian-Keplerian problem is generalized to any integer dimension d , one obtains stable, bound non-colliding orbits if and only if $d \leq 3$. Moreover, if one requires that the gravitational potential should vanish at infinity, dimensions $d < 3$ are excluded. Ehrenfest concluded that a three-dimensional space is perfect for describing a world in which the two previous conditions are satisfied. Instead of considering space and time independently, Einstein (1905) proposed a unified view of spacetime of dimension $n = d + 1$. As part of his special theory of relativity [5], he postulated to explain how light moving at a constant speed relative to all observers, can best be expressed in four dimensions by combining spatial and temporal coordinates. A decade later in his general theory of relativity, he took up this concept and described gravity using a dynamical four-dimensional model of spacetime [5].

The question of spacetime dimensionality has often intersected with another major challenge in theoretical physics which is unification of fundamental interactions within the universe [1, 6–8]. This quest started with Maxwell (1865) who unified electricity and magnetism within the same formalism and showed that light stems from a four-dimensional potential field [5]. Since then, tremendous effort has been deployed to unite the electromagnetic force with strong nuclear force, weak nuclear force and gravity. The first scheme to incorporate gravity was developed by Kaluza (1921) and Klein (1926) who proposed to unify electromagnetism and gravity by extending general relativity by an extra dimension [1]. Since the 1970's, Kaluza-Klein theory has seen a revival through to the emergence of theories based on the idea that the fundamental components of nature are higher-dimensional vibrating strands of energy [7, 8].

Here we revisit Ehrenfest’s proposal and ask “How come $d = 3$ (or $n = 4$) was selected?” To answer this question, we approach the problem within the framework of general relativity but always from a “classical” perspective. We postulate that the current dimension of our physical world results from a dynamical evolution of the spacetime dimension of an expanding universe. To exploit this idea, we start from the generalized Einstein-Hilbert action to any integer dimension of spacetime. Then, without resorting to a compactification-type argument but using only dimensional considerations, we show that such a definition of the action must *de facto* involve a single additional length scale. We apply this generalized formulation to the Friedmann-Lemaître-Robertson-Walker (FRLW) cosmological model and show that our initial assumption reduces the possibilities on the dynamics, curvature and constituents of the universe to almost a single scenario. Most of the computations related to this part are standard in the field and they are reproduced for consistency of reasoning. Finally, we conclude our study with a discussion listing the relevance of our approach, its possible shortcomings, the main results compared to those prevailing in literature, as well as the possible extensions of the initial proposal.

II. GRAVITATION WITH A DYNAMICAL SPACETIME

A. Einstein-Hilbert Action of a multi-dimensional spacetime

Consider a general multidimensional pseudo-Riemannian metric γ built upon a n -dimensional Lorentzian spacetime metric $^{(n)}g$, with $n \geq 2$ spanning integer values only, and infinite number of Euclidian-like diagonal components such that the distance element ds of a worldline is given by

$$ds^2 = \gamma_{MN} dx^M dx^N = ^{(n)}g_{\mu\nu} dx^\mu dx^\nu - \sum_{i=n}^{\infty} dz^i dz^i. \quad (1)$$

Here, $^{(n)}g_{\mu\nu}$ depends on the coordinates x^μ only, with $0 \leq \mu \leq n-1$ and x^0 is the single time-like coordinate. We will refer to x^μ as the active coordinates and z^i ($i \geq n$) as the passive ones. In other words, we assume that the physical (observable) distance derives from the renormalised distance element $ds^2 + \sum_{i=n}^{\infty} dz^i dz^i$, which can be assumed as the projection of the distance element on the sub-manifold defined by the metric $^{(n)}g$. We define the action $^{(n)}S_g$ of the gravitational field associated to the metric γ as a generalisation of Einstein-Hilbert action to any spacetime dimension

$$^{(n)}S_g = -\kappa \int_V dV \sqrt{|\gamma|} ^{(n)}\mathcal{L}\{^{(n)}g, \Phi\}, \quad (2)$$

where κ is a physical constant, $^{(n)}\mathcal{L}$ is the Lagrangian that depends explicitly on the metric $^{(n)}g$ and possible other field(s) Φ and dV is the volume element of the infinite dimensional manifold

associated to the metric γ . Here, similarly to $^{(n)}g$, we assume that Φ depends only on the active coordinates x^μ .

To be physically consistent, the definition of the action given by Eq. (2) should be universal independently of the partition of the hyperspace in active and passive elements. Therefore, κ must be a single physical constant in the sense that it should not depend on n . Using the metric given by Eq. (1) and the assumption that $^{(n)}\mathcal{L}$ depends on the active coordinates only, one can rewrite Eq. (2) as

$$^{(n)}S_g = -\kappa \left[\prod_{i=n}^{\infty} \int dz^i \right] \int_{V_n} dV_n \sqrt{|^{(n)}g|} ^{(n)}\mathcal{L}\{^{(n)}g, \Phi\} , \quad (3)$$

where dV_n is the volume element the active part of the manifold associated to the metric $^{(n)}g$. Recall that the physical dimensions of the Lagrangian $^{(n)}\mathcal{L}$ and the action $^{(n)}S_g$ are $[L]^{-2}$ and $[M][L]^2[T]^{-1}$, respectively [5]. Therefore, using the constant c^3/G (c is the speed of light and G is Newton's gravitational constant) whose physical dimensions are $[M][T]^{-1}$, one can write

$$\kappa \prod_{i=n}^{\infty} \int dz^i = \frac{c^3 \ell^{4-n}}{16\pi G} , \quad (4)$$

where ℓ is a physical length scale which at this stage may depend on n . The constant $c^3/(16\pi G)$ and the exponent $(4-n)$ guarantee that we recover the well-established case $n = 4$. Eq. (4) should be satisfied for all $n \geq 2$ with κ is independent on n . The latter constraint is satisfied if and only if

$$\ell^{n-4} \prod_{i=n}^{\infty} \int dz^i = \prod_{i=4}^{\infty} \int dz^i . \quad (5)$$

Since the passive components of the metric γ are indistinguishable, Eq. (5) should hold for all $n \geq 2$ and for any random permutation of the coordinates z^i . A solution of Eq. (5) that satisfies these constraints is readily given by

$$\int dz^i = \ell , \quad (6)$$

for all $i \geq 2$, with ℓ a *single* physical length scale independent of the actual active dimension n . The result that the integral over z^i is finite imposes that the passive dimensions of γ should be compact, in the sense that they should have a finite extension ℓ . Consequently, the generalisation of the action of the gravitational field to any spacetime dimension introduces a single unknown constant ℓ characterising the size of the hyperspace in the directions of the passive coordinates of the general metric γ . It is tempting to identify ℓ as the Planck length, however our approach does not invoke any quantum property. Therefore, at this stage, the (order of) magnitude of ℓ is still arbitrary.

B. Spacetime dimension as a dynamical variable

The question of the selection of the current physical spacetime dimension clearly underlies our reasoning. Up to now, the active spacetime dimension n is an arbitrary variable, as it may be constant, dynamical or even nonuniform throughout the hyperspace. In the following, we postulate that the current dimension was reached through a dynamical evolution of spacetime dimension of an expanding universe. More precisely, we explore the possibility that the active dimension of the universe uniformly and incrementally evolved to reach the current state $n = 4$. To this purpose, we define the gravitational action of the universe as

$$S_g = \sum_{n \geq 2} {}^{(n)}S_g = -\frac{c^3}{16\pi G} \sum_{n \geq 2} \ell^{4-n} \int_{V_n} dV_n \sqrt{|{}^{(n)}g|} {}^{(n)}\mathcal{L}\{g, \Phi\} , \quad (7)$$

where Lagrangian density ${}^{(n)}\mathcal{L}$ is given by

$${}^{(n)}\mathcal{L}\{g, \Phi\} = \begin{cases} {}^{(n)}\mathcal{R}\Phi + \lambda \partial^\mu \Phi \partial_\mu \Phi , & n = 2 \\ {}^{(n)}\mathcal{R} , & n \geq 3 \end{cases} . \quad (8)$$

Here ${}^{(n)}\mathcal{R}$ is the Ricci curvature of the metric given by Eq. (1), λ is a dimensionless constant and Φ is the so-called dilaton field. For $n \geq 3$, the Lagrangian is identically given by the curvature ${}^{(n)}\mathcal{R}$, generalising Einstein's gravity to any dimension $n \geq 3$. The case $n = 2$ is particular as the definition of the action should be modified. Here, we use Jackiw-Teitelboim (JT) gravity model in which the Lagrangian is minimally modified by introducing an additional dimensionless field Φ and a parameter λ [9, 10]. The relevance of JT gravity with respect to other models in literature [11, 12] is beyond the present study. Our objective in the following is to test the assumption of a dimension of universe that is allowed to evolve within a “classical” cosmological model.

III. APPLICATION TO A STANDARD COSMOLOGICAL MODEL

Our generalization involves a multidimensional universe: a dynamical n -dimensional spacetime with $n \geq 2$, plus infinite number of passive dimensions of typical extension ℓ each. Furthermore, we assume a priori a homogeneous, isotropic, positive-curvature expanding universe; the necessity and relevance of these assumptions to our model will be discussed in the following. Hence, the generalised FRLW metric is given by

$$ds^2 = N_n(\tau)^2 d\tau^2 - a_n(\tau)^2 d\Omega_{n-1}^2 - \sum_{i=n}^{\infty} dz^i dz^i , \quad (9)$$

where N_n is the lapse function, a_n is the scale factor of the n -hypersphere and $d\Omega_{n-1}^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \dots + (\sin^2\theta_1 \times \dots \times \sin^2\theta_{n-2}) d\theta_{n-1}^2$ is the Euclidian metric of unit sphere in $(n-1)$ dimensions. We choose the convention that both τ and a_n have physical dimensions of length and that N_n is dimensionless. We shall now compute the Ricci curvature ${}^{(n)}\mathcal{R}$ for the metric given by Eq. (9). Because the passive dimensions $i \geq n$ are all Euclidian-like, they do not contribute to ${}^{(n)}\mathcal{R}$. Therefore, we only need to carry out computations for the active components of the metric and the result is given by [13, 14]

$${}^{(n)}\mathcal{R} = (n-1) \left(\frac{2}{N_n(\tau)} \frac{d}{d\tau} \left(\frac{\dot{a}_n(\tau)}{N_n(\tau)a_n(\tau)} \right) + n \left(\frac{\dot{a}_n(\tau)}{N_n(\tau)a_n(\tau)} \right)^2 + \frac{n-2}{a_n(\tau)^2} \right), \quad (10)$$

where the dot denotes differentiation with respect to τ . Since the Lagrangian is independent of the spatial coordinates, the integration of the spatial parts of S_g in Eq. (7) yields

$$S_g = -\frac{3\pi c^3}{4G} \sum_{n \geq 2} \alpha_n \int_{\tau_n^-}^{\tau_n^+} d\tau N_n(\tau) a_n(\tau)^{n-1} {}^{(n)}\mathcal{L}\{g, \Phi\}, \quad (11)$$

where α_n is a constant given by

$$\alpha_n = \frac{\ell^{4-n}}{6 \pi^{(2-\frac{n}{2})} \Gamma(\frac{n}{2})}, \quad (12)$$

and τ_n^\pm delimit the time domain of the era with an active dimension $n(\tau)$. It is noteworthy that the integration over all spatial coordinates can be performed solely for a finite closed (positive-curvature) universe. Indeed, this is the *only* possible FRLW geometry that allows for a finite action and thus, for the sub-manifolds to interact at their boundaries.

Now, we consider the action of the source term generating the gravitational field. Here, we assume that the universe is matter dominated at all stages with an action S_M given by

$$S_M = Mc \sum_{n \geq 2} \int_{\tau_n^-}^{\tau_n^+} d\tau N_n(\tau), \quad (13)$$

where M is the total mass of the universe. The case of purely radiation dominated universe is reported in the Appendix. Thus the total action $S = S_g + S_M$ for the multidimensional universe under consideration can be written as follow

$$S = \sum_{n \geq 2} S_n = \frac{3\pi c^3}{4G} \sum_{n \geq 2} \int_{\tau_n^-}^{\tau_n^+} d\tau \left(-\alpha_n N_n(\tau) a_n(\tau)^{n-1} {}^{(n)}\mathcal{L}\{g, \Phi\} + R_M N_n(\tau) \right), \quad (14)$$

where the length scale

$$R_M = \frac{4GM}{3\pi c^2}, \quad (15)$$

is a physical parameter associated with the constituents of the universe. Notice that in the following, we will use unconventional parameters and scalings which are justified by the assumption of a closed universe. However, one can easily reinterpret all results using standard cosmological quantities.

Let us explore two possible scenarios for an expanding universe with a dynamical active dimension. We postulate that the current spacetime dimension $n = 4$ was reached either by incrementally decreasing or increasing the active dimension from respectively $n = \infty$ or $n = 2$. We will assume that the incremental decrease (resp. increase) of the active dimension occurs by satisfying continuity conditions on the metric components such that

$$\begin{aligned} a_n(\tau_n^+) &= a_{n+1}(\tau_{n+1}^-) && \text{for an increasing } n(\tau) , \\ a_n(\tau_n^+) &= a_{n-1}(\tau_{n-1}^-) && \text{for a decreasing } n(\tau) . \end{aligned} \quad (16)$$

The transition between eras of active dimension n and $n + 1$ is supposed to take place while preserving the isotropy and the homogeneity of the universe at each era. Moreover, the boundary condition given by Eq. (16) assumes that this mechanism is associated with either a dimensional collapse (compactification) or a dimensional deployment (unfolding) of a single metric component while keeping the scale factor continuous. The latter assumption is clearly debatable and its relevance is discussed in the conclusion.

In addition to Eq. (16), the minimisation of the total action S given by Eq. (14) involves boundary terms at times $\tau = \tau_n^\pm$ that bound each spacetime era. The cancellation of these contributions leads to an additional set of continuity conditions between the metric components of the different manifolds. In the following, we conveniently scale the action in (14) by the coefficient $\frac{3\pi c^3}{4G}$. Moreover, as the Lagrangian ${}^{(n)}\mathcal{L}\{g, \Phi\}$ is given by different expressions for $n = 2$ and $n \geq 3$, we will analyse these two cases separately.

A. Case $n \geq 3$

When the active dimension satisfies $n \geq 3$, the action given by Eq. (14) can be rewritten as

$$S_n = \int_{\tau_n^-}^{\tau_n^+} d\tau \left(\mathcal{L}_n^0 + \frac{d}{d\tau} \mathcal{F}_n \right) , \quad (17)$$

where

$$\mathcal{L}_n^0 \equiv -(n-1)(n-2)\alpha_n a_n(\tau)^{n-3} N_n(\tau) \left[1 - \left(\frac{\dot{a}_n(\tau)}{N_n(\tau)} \right)^2 \right] + R_M N_n(\tau) , \quad (18a)$$

$$\mathcal{F}_n \equiv -2(n-1)\alpha_n a_n(\tau)^{n-2} \frac{\dot{a}_n(\tau)}{N_n(\tau)} . \quad (18b)$$

The minimisation of the action (17) with respect to variations of $a_n(\tau)$ and $N_n(\tau)$ yields two Euler-Lagrange equations and boundary terms at $\tau = \tau_n^\pm$. The former are given by

$$\frac{\partial \mathcal{L}_n^0}{\partial a_n} - \frac{d}{d\tau} \left(\frac{\partial \mathcal{L}_n^0}{\partial \dot{a}_n} \right) = 0, \quad (19a)$$

$$\frac{\partial \mathcal{L}_n^0}{\partial N_n} - \frac{d}{d\tau} \left(\frac{\partial \mathcal{L}_n^0}{\partial \dot{N}_n} \right) = 0, \quad (19b)$$

and the latter are given by

$$[\mathcal{B}_n]_{\tau_n^-}^{\tau_n^+} = \left[\left(\frac{\partial \mathcal{F}_n}{\partial a_n} + \frac{\partial \mathcal{L}_n^0}{\partial \dot{a}_n} \right) \delta a - \frac{\partial \mathcal{F}_n}{\partial \dot{a}_n} \delta \dot{a}_n + \left(\frac{\partial \mathcal{F}_n}{\partial N_n} + \frac{\partial \mathcal{L}_n^0}{\partial \dot{N}_n} \right) \delta N_n - \frac{\partial \mathcal{F}_n}{\partial \dot{N}_n} \delta \dot{N}_n \right]_{\tau_n^-}^{\tau_n^+}. \quad (20)$$

Here $[\mathcal{B}_n]_{\tau_n^-}^{\tau_n^+} = \mathcal{B}_n(\tau_n^+) - \mathcal{B}_n(\tau_n^-)$. Using (18), we find that Eq. (19a) yields

$$\frac{a_n(\tau)}{N_n(\tau)} \frac{d}{d\tau} \left(\frac{\dot{a}_n(\tau)}{N_n(\tau)} \right) + \frac{n-3}{2} \left[1 + \left(\frac{\dot{a}_n(\tau)}{N_n(\tau)} \right)^2 \right] = 0, \quad (21)$$

and Eq. (19b) yields

$$1 + \left(\frac{\dot{a}_n(\tau)}{N_n(\tau)} \right)^2 = \beta_n \left(\frac{a_n(\tau)}{R_M} \right)^{3-n}, \quad (22)$$

where β_n is a dimensionless constant given by

$$\beta_n \equiv \frac{6\pi^{(2-\frac{n}{2})} \Gamma(\frac{n}{2})}{(n-1)(n-2)} \left(\frac{\ell}{R_M} \right)^{n-4}. \quad (23)$$

Notice that $\beta_4 = 1$ and that Eq. (21) is redundant as it can be retrieved by differentiating (22) with respect to τ . Equations (21,22) are exactly the usual ones found by solving Einstein equations for the FRLW metric and a matter dominated universe. This concordance justifies the writing of the source term of the Action under the form given by Eq. (13). Indeed, due to the assumption of isotropy and homogeneity of the expanding universe, the global minimization of the total action S given by Eq. (14) performed here is equivalent to the usual local minimization leading to Einstein's equations.

Furthermore, the boundary terms in Eq. (20) simply give

$$[\mathcal{B}_n]_{\tau_n^-}^{\tau_n^+} = \left[2(n-1)\alpha_n a_n^{n-2} \delta \left(\frac{\dot{a}_n}{N_n} \right) \right]_{\tau_n^-}^{\tau_n^+}. \quad (24)$$

Using Eq. (22), which is also satisfied at the boundaries τ_n^\pm , we can relate $\delta(\dot{a}_n/N_n)$ to δa_n through

$$2R_M \left(\frac{\dot{a}_n}{N_n} \right) \delta \left(\frac{\dot{a}_n}{N_n} \right) = \beta_n (3-n) \left(\frac{a_n}{R_M} \right)^{2-n} \delta a_n. \quad (25)$$

Hence, substituting Eq. (25) into Eq. (24), we obtain

$$[\mathcal{B}_n]_{\tau_n^-}^{\tau_n^+} = R_M \frac{(n-3)}{(n-2)} \left[\left(\frac{\dot{a}_n}{N_n} \right)^{-1} \delta a_n \right]_{\tau_n^-}^{\tau_n^+}. \quad (26)$$

Let us define the cosmic time t using the transformation $dt = N_n(\tau)d\tau$. This allows us to rewrite the evolution equation (22) for $a_n(t)$ as

$$1 + \dot{a}_n(t)^2 = \beta_n \left(\frac{a_n(t)}{R_M} \right)^{3-n}, \quad (27)$$

with the boundary conditions at the transition between two successive states of the universe given by

$$[\mathcal{B}_n]_{t_n^-}^{t_n^+} = R_M \frac{(n-3)}{(n-2)} \left[\frac{\delta a_n(t)}{\dot{a}_n(t)} \right]_{t_n^-}^{t_n^+}. \quad (28)$$

Eq. (28) shows that $\mathcal{B}_3 \equiv 0$ which suggests that the cases $n = 3$ and $n \geq 4$ should be studied separately. In the following, we first focus on the latter and then analyse the former.

1. Case $n \geq 4$

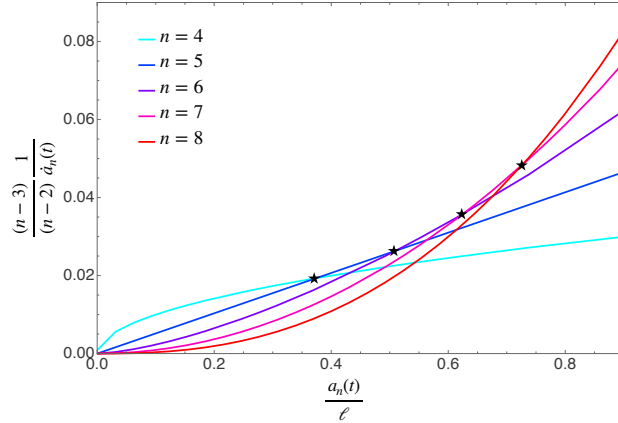


FIG. 1. Plot of the boundary term $\mathcal{B}_n(t)/(R_M \delta a_n(t))$ as function of $a_n(t)$ for different $n \geq 4$ and $\ell = 10^{-3} R_M$. Here, $\mathcal{B}_n(t)$ is given by Eq. (28) and the evolution equation (27) for $a_n(t)$ has been used. This result shows that for an expanding universe, the continuity conditions $\mathcal{B}_n(t) = \mathcal{B}_{n+1}(t)$, combined with the boundary condition (16), are satisfied if and only if the active dimension $n(t)$ is an *increasing* function of t . This result holds for any value of the parameter ℓ/R_M . Finally, star points indicate the transition points between two consecutive eras.

At the transition between two consecutive eras of different active dimensions, the minimisation of the total action involves boundary terms given by Eq. (28) that should satisfy continuity conditions.

If we assume that the scale factor $a_n(t)$ is continuous at the transition between two consecutive eras, Eq. (16), Fig. 1 shows that for $n \geq 4$, an incremental *increase* of the active dimension is the only possible solution for an expanding universe. Since the current state of the universe is characterised by an active dimension $n = 4$, higher dimensions $n > 4$ would be accessible in the future and lower ones $n < 4$ have been probed in the past. This result allows us to draw a possible scenario for an expanding universe with a dynamical active dimension. At time $t = 0$, the universe started to expand within a spacetime metric of active dimension $n(t) = 2$ and evolved so that $n(t)$ increased incrementally to $n(t) = 3$, then to $n(t) = 4$ which is the current era and may jump in the future to higher dimensions $n(t) = 5, 6$ and so on.

To proceed further in the proposed scenario, we introduce some useful notations. Continuity conditions on the cosmological time t and the scale factor $a_n(t)$ at the transition from n to $(n + 1)$ are written as

$$t_n^+ = t_{n+1}^- \equiv t_{n+1}^* > t_n^* , \quad (29)$$

$$a_n(t_n^+) = a_{n+1}(t_{n+1}^-) \equiv a_{n+1}^* > a_n^* , \quad (30)$$

for all $n \geq 2$. Also, we define the origin of time at $t_2^- \equiv t_2^* = 0$ and impose that universe starts its expansion from $a_2(t_2^*) \equiv a_2^* = 0$.

Returning to the case $n \geq 4$, we define the conformal time coordinate η using $dt = a_n(\eta)d\eta$. Then, the solution of Eq. (27) is parametrically given by

$$a_n(\eta) = R_M \left(\sqrt{\beta_n} \sin \left(\frac{(n-3)}{2} \eta \right) \right)^{\frac{2}{n-3}} , \quad (31)$$

$$t(\eta) = t_n^* + \int_{\eta_n^-}^{\eta} a_n(\tau) d\tau , \quad (32)$$

with $\eta_n^- \leq \eta \leq \eta_n^+$ and

$$t_{n+1}^* = t_n^* + \int_{\eta_n^-}^{\eta_n^+} a_n(\tau) d\tau . \quad (33)$$

The boundaries η_n^\pm of eras with an active dimension n are determined using continuity conditions which can be rewritten as

$$a_n(\eta_n^+) = a_{n+1}(\eta_{n+1}^-) , \quad (34)$$

$$\frac{(n-2)}{(n-3)} \dot{a}_n(\eta_n^+) = \frac{(n-1)}{(n-2)} \dot{a}_{n+1}(\eta_{n+1}^-) . \quad (35)$$

Equations (34,35) allow us to determine the η_n^\pm for all $n \geq 4$ except η_4^- which is involved in the boundary conditions at the transition from $n = 3$ to $n = 4$. The numerical computation of these

constants is straightforward, however it turns out in the following that we will need to compute only η_4^+ . It can be shown that for $\ell \ll R_M$ one finds

$$\eta_4^+ \simeq \frac{3\sqrt{6}}{8} \sqrt{\frac{\ell}{R_M}}. \quad (36)$$

Now, let us proceed further and study the special case $n = 3$.

2. Case $n = 3$

Eq. (27) applied to the case $n = 3$ yields a constant expansion speed given by

$$\dot{a}_3(t) = \sqrt{\frac{3\pi R_M}{2\ell} - 1}, \quad (37)$$

where we make the likely satisfied assumption $R_M > 2\ell/3\pi$. Then, the scale factor $a_3(t)$ is simply given by

$$a_3(t) = a_3^* + \sqrt{\frac{3\pi R_M}{2\ell} - 1} (t - t_3^*), \quad (38)$$

where t_3^* and a_3^* are unknown constants. As mentioned above, the boundary conditions at the transition between eras $n = 3$ and $n = 4$ are particular. Eq. (28) shows that $\mathcal{B}_3(t) = 0$ and thus the condition $\mathcal{B}_4(t_4^*) = 0$, or equivalently $\delta\dot{a}_4(t_4^*) = 0$, should be satisfied. This means that $\dot{a}_4(t_4^*)$ should be fixed a priori. The only available constant speed is provided by $\dot{a}_3(t)$ in the era $n = 3$. Therefore, one expects that both the scale factor and the expansion speed are continuous at the transition from $n = 3$ to $n = 4$. these considerations lead to boundary conditions given by

$$a_3(t_4^*) = a_4(t_4^*) \quad \text{and} \quad \dot{a}_3(t_4^*) = \dot{a}_4(t_4^*), \quad (39)$$

which allows us to find

$$a_4^* = \frac{2\ell}{3\pi}, \quad (40)$$

$$t_4^* = t_3^* + \frac{\left(\frac{2\ell}{3\pi} - a_3^*\right)}{\sqrt{\frac{3\pi R_M}{2\ell} - 1}}, \quad (41)$$

where t_3^* and a_3^* are still unknown constants. In addition, one has

$$\eta_4^- = 2 \sin^{-1} \sqrt{\frac{2\ell}{3\pi R_M}}. \quad (42)$$

Comparing Equations (36,42), one shows that $\eta_4^- > \eta_4^+$ for $\ell \ll R_M$ which would yield $t_5^* < t_4^*$. This result can be generalised for all ℓ . However, this contradicts the assumption of an expanding

universe. This result means that once the transition from $n(t) = 3$ to $n(t) = 4$ occurs it rules out the possibility of a transition from $n(t) = 4$ to $n(t) = 5$. Therefore within the classical FRLW cosmological model of an isotropic and homogeneous universe, our model predicts that the active dimension $n = 4$ is the upper limit for an expanding universe. It is noteworthy that this result is independent of the fact that the universe would have started its expansion from $n(t = 0) = 3$ or $n(t = 0) = 2$.

B. Case $n = 2$

Here, we explore the possibility that at earlier times, the universe was in a state with an active dimension $n(t) = 2$. Using the JT gravity model, the total action is explicitly given by

$$S_2 = \int_{\tau_2^-}^{\tau_2^+} d\tau \left(-\alpha_2 \left(2\Phi(\tau) \frac{d}{d\tau} \left(\frac{\dot{a}_2(\tau)}{N_2(\tau)} \right) + \lambda \frac{a_2(\tau)}{N_2(\tau)} \dot{\Phi}^2(\tau) \right) + R_M N_2(\tau) \right). \quad (43)$$

Here, we extended the hypothesis of isotropy and homogeneity to the dilaton field Φ . The minimisation of the action (43) with respect to variations of $\Phi(\tau)$, $a_2(\tau)$ and $N_2(\tau)$ yields three Euler-Lagrange equations and boundary terms at the extremities of the time domain of the era $n(t) = 2$. In terms of the cosmic time t , Euler-Lagrange equations read

$$\ddot{a}_2(t) - \lambda \frac{d}{dt} \left(a_2(t) \dot{\Phi}(t) \right) = 0, \quad (44)$$

$$2\ddot{\Phi}(t) + \lambda \dot{\Phi}(t)^2 = 0, \quad (45)$$

$$2\dot{a}_2(t) \dot{\Phi}(t) - \lambda a_2(t) \dot{\Phi}(t)^2 = \frac{R_M}{\alpha_2}, \quad (46)$$

and the boundary terms are given by

$$[\mathcal{B}_2]_{t_2^*}^{t_3^*} = -2\alpha_2 \left[\lambda a_2 \dot{\Phi} \delta\Phi + \Phi \delta \dot{a}_2 - \dot{\Phi} \delta a_2 \right]_{t_2^*}^{t_3^*}. \quad (47)$$

It is straightforward to show that the only solution of Equations (44-46) that satisfy the boundary conditions $\mathcal{B}_2(t_2^* = 0) = 0$ is given by

$$a_2(t) = \frac{3\pi\lambda}{2} \frac{R_M t^2}{\ell^2}, \quad (48)$$

$$\Phi(t) = \frac{2}{\lambda} \log \frac{t}{t_3^*} + \Phi_0, \quad (49)$$

where Φ_0 is a constant. The boundary conditions at the transition from $n = 2$ to $n = 3$ read

$$a_2(t_3^*) = a_3(t_3^*), \quad \dot{a}_2(t_3^*) = \dot{a}_3(t_3^*), \quad \delta\Phi(t_3^*) = 0. \quad (50)$$

The justification of the first two equations in (50) is the same as for the boundary conditions at the transition from $n = 3$ to $n = 4$. On the other hand, the boundary condition $\delta\Phi(t_3^*) = 0$ allows us to

fix the constant Φ_0 . However, whichever is the physical condition that determines Φ_0 , it does not influence the continuity conditions of the scale factor $a_n(t)$. Finally, using (50) it is straightforward to show that the constants a_3^* and t_3^* are given by

$$a_3^* = \frac{\ell}{4\lambda} \left(1 - \frac{2\ell}{3\pi R_M} \right), \quad (51)$$

$$t_3^* = \frac{\ell^2}{3\lambda\pi R_M} \sqrt{\frac{3\pi R_M}{2\ell} - 1}. \quad (52)$$

Notice that a transition from $n(t) = 2$ to $n(t) = 3$ within the assumption of an expanding universe is possible only if $a_3^* < a_4^*$. This condition imposes a lower bound on the value of the parameter λ given by

$$\lambda > \frac{3\pi}{8} \left(1 - \frac{2\ell}{3\pi R_M} \right). \quad (53)$$

In the following, we will assume that this condition is fulfilled.

IV. DISCUSSION

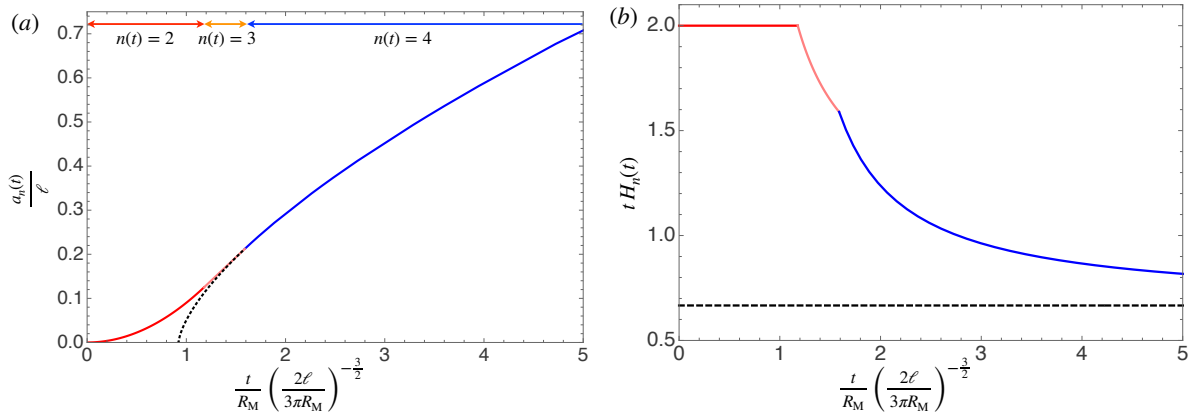


FIG. 2. Expansion of the universe within a cosmological model that allows for a dynamical spacetime dimension $n(t)$. Here, $\ell = 10^{-3}R_M$ and $\lambda = 2$. (a) Evolution of the scale factor $a_n(t)$ during the different eras $2 \leq n(t) \leq 4$. The dashed curve is the continuation of $a_4(t)$ that would arise from solving the classical cosmological model with $n = 4$. (b) The corresponding variation of $tH_n(t)$. The dashed curve is the classical solution with $tH_4(t) < 2/3$.

The results of the present cosmological model are more telling through a graphical representation of its solution. Fig. 2 shows the evolution of the scale factor $a_n(t)$ and the Hubble rate $H_n(t)$ from $t = 0$ and $n(t) = 2$ up to a given time t_0 in the era $n(t) = 4$ for which the universe is still expanding. Notice that when both t , $a_n(t)$ and $H_n(t)$ are scaled appropriately, the results depend

on the dimensionless parameters ℓ/R_M and λ . Nevertheless, the behavior of $a_n(t)/\ell$ and $tH_n(t)$ shown in Fig. 2 vary little with ℓ/R_M as long as $\ell \ll R_M$.

Fig. 2a shows that the dynamics of the scale factor $a_2(t)$ is characterised by a constant acceleration $\ddot{a}_2(t) = 3\pi\lambda R_M/\ell^2$. This behavior close to $t = 0$ within the era $n(t) = 2$ allows for a weak inflation scenario. Since the conditions $a_2(0) = \dot{a}_2(0) = 0$ are satisfied, one can extend the solution to negative times $t < 0$ and build up a periodic behavior of universe dynamics with a succession of bounces and crunches. Another important feature of the model concerns the dynamics of the scale factor within the era $n(t) = 4$. Except for fixing a_4^* and t_4^* , the evolution of $a_4(t)$ does not involve neither the “hidden” length scale ℓ nor the parameter λ . Indeed $a_4(t)$ is exactly the solution of the classical four-dimensional cosmological model in which the origin of time is shifted and the expansion started from a finite value $a_4^* \neq 0$.

Fig. 2(b) shows that the Hubble rate in the era $n(t) = 4$ satisfies the inequality $tH_4(t) < 2$ contrary to the classical case of a purely four-dimensional FRLW cosmological model for which one has $tH_4(t) \leq \frac{2}{3}$. This result is a direct consequence of the shift in the origin of time due to the existence of lower spacetime dimension eras. For $\ell \ll R_M$, one can show that the age of the universe t_0 is given by

$$\frac{t_0}{R_M} \simeq \left(1 + \frac{9\pi}{8\lambda}\right) \left(\frac{t_0 H_4(t_0)}{3t_0 H_4(t_0) - 2}\right) \left(\frac{2\ell}{3\pi R_M}\right)^{\frac{3}{2}}. \quad (54)$$

Assuming the standard estimates $t_0 H_4(t_0) \simeq 1$, $R_M \simeq 10^{29}$ m, $t_0 \simeq 10^{26}$ m and taking $\lambda = 2$, Eq. (54) yields $\ell \simeq 10^{26}$ m $\simeq 10^{-3} R_M$. Our model is then relevant if ℓ is a “cosmic” length scale which is in agreement with our purely “classical” approach. This result is in contrast with the usual “quantum” origin assumption invoked by compactification theories and yielding to the Planck scale $\ell_P \sim 10^{-35}$ m. Finally, an outcome of our model is that the universe has spent equivalent times in the two, three, and four dimensional eras so far.

V. CONCLUSION

The central assumption of this work is that the current four-dimensional spacetime is selected by a dynamical dimension of the metric associated with a homogeneous, isotropic expanding universe. To exploit this idea, we use the generalised action of a gravitational field to any integer spacetime dimension and give dimensional arguments for the existence of a physical length scale ℓ that characterises the spatial extension, or “thickness”, of “hidden” passive components of the metric. Then, we solve a generalised FRLW cosmological model in which the active dimension $n(t)$ is

allowed to jump incrementally. Within this approach and assuming the condition (16) that governs the jump condition between two eras we show the existence of a single possible scenario of an increasing spacetime dimension from $n(t) = 2$ to $n(t) = 3$ and then to $n(t) = 4$ which appears as an upper possible dimension. In the following, we enumerate the main assumptions, outcomes and drawbacks of our approach.

- The model mainly necessitates the introduction of a single new physical length scale ℓ . To be physically relevant, it turns out that ℓ should be a cosmic scale unrelated to the quantum-based Planck scale. Fortunately, this possibility is not forbidden by any cosmological observation.
- The weak point of our reasoning lies in the choice of the jump condition at the transition between two eras with different active dimensions. Even it is realistic the assumption (16) obviously needs more physical and mathematical foundations. However, there is no particular reason to not impose different boundary conditions induced by other types of constraints related to either metric components (eg. conservation of area) or source field (eg. equation of state). Such alternative conditions may modify the results and the conclusions of this study.
- Our model is consistent in the only case of a universe with elliptic geometry; hyperbolic or plane geometries being discarded. Moreover, it is likely compatible with the description of a purely matter dominated universe only. Indeed, we can show that purely radiation dominated universe is not allowed within our approach (see Appendix). However, we did not address the possibility of a universe constituted of both matter, radiation and dark energy.
- In the early stage of universe expansion, the possible existence of an era with a dimension $n(t) = 2$ yields a weak inflation characterised by a constant acceleration of the scale factor. Consequently, a scenario in which universe is periodically bouncing can built upon our results [16]. However, this behavior is closely related to the specific choice of JT action as a 2D gravity model. Nevertheless, since $\ddot{a}_3(t) = 0$ and $\ddot{a}_4(t) < 0$, one expects that any physically relevant 2D gravity model should be constrained to exhibit an inflation scenario.
- The existence of an intermediate era with $n(t) = 3$ imposes $n(t) = 4$ as an upper spacetime dimension of the universe, whereas if $n(t = 0) = 4$ higher dimensions $n > 4$ are possible. This strong result means that the current dimension of our physical world is stable, consistently with Ehrenfest paradigm, if its dimension evolved from $n(t = 0) = 2$ or $n(t = 0) = 3$.
- We have shown that the current Hubble rate $H_4(t_0)$ satisfies the inequality $t_0 H_4(t_0) < 2$. Consequently, one can achieve a universe of age $t_0 \gtrsim H_4^{-1}(t_0)$ contrary to the classical case of a purely four-dimensional FRLW cosmological model for which one has $t_0 \leq \frac{2}{3} H_4^{-1}(t_0)$.
- If we assume that light manifests only in four-dimensional spacetime, the dynamics of the universe

at lower dimensions $n(t) < 4$ would be opaque to conventional cosmological measurements. This would provide a lower bound on what can be “observed”. However, one could imagine that the jump in dimension at $t = t_4^*$ could be a trigger for fluctuations that may possibly reveal quantities associated with eras $n(t) < 4$ such as the “hidden” length scale ℓ .

Finally, the applicability of our model to a specific expanding universe (closed and matter dominated) would make it unsuitable for other variants. Nevertheless, we could advantageously argue that it is because it implicitly implements a selection mechanism of the cosmological model of the universe. However, some cosmological observations, such as the current acceleration of the universe, are still missing. Despite the possible shortcomings, we believe that the hypothesis of a dynamical spacetime dimension of the universe is an attractive proposition worth exploring. For example, the restriction that the dimension of spacetime only covers integer values, which is the only possibility within the framework of a “classical” cosmological approach, could be lifted in a more elaborate model.

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DATA AVAILABILITY STATEMENT

No Data associated in the manuscript

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- [1] J. M. Overduin and P. S. Wesson. Kaluza-Klein gravity. *Physics Reports*, 283(5-6):303–378, 1997.
 - [2] P. Halpern. How many dimensions does the universe really have? public broadcasting service, 2014.
 - [3] P. Ehrenfest. In what way does it become manifest in the fundamental laws of physics that space has three dimensions? In *Proceedings of the Royal Netherlands Academy of Arts and Sciences (KNAW)*, volume 20 I, pages 200–209, 1918.
 - [4] F. R. Tangherlini. Schwarzschild field in n dimensions and the dimensionality of space problem. *II Nuovo Cimento (1955-1965)*, 27(3):636–651, 1963.
 - [5] L. D. Landau and E. M. Lifshitz. *The Classical Theory of Fields*, volume 2. Butterworth-Heinemann, 1975.

- [6] K. Kang, H. Fried, and P. Frampton. Fifth workshop on Grand Unification, eds. World Scientific, Singapore, 1986.
- [7] M. J. Duff, B. E. W. Nilsson, and C. N. Pope. Kaluza-Klein supergravity. Physics Reports, 130(1-2):1–142, 1986.
- [8] D. Bailin and A. Love. Supersymmetric gauge field theory and string theory. Taylor & Francis, 1993.
- [9] C. Teitelboim. Gravitation and hamiltonian structure in two spacetime dimensions. Physics Letters B, 126(1-2):41–45, 1983.
- [10] R. Jackiw. Lower dimensional gravity. Nuclear Physics B, 252:343–356, 1985.
- [11] D. Grumiller, W. Kummer, and D. V. Vassilevich. Dilaton gravity in two dimensions. Physics Reports, 369(4):327–430, 2002.
- [12] E. Witten. Deformations of JT gravity and phase transitions. arXiv preprint arXiv:2006.03494, 2020.
- [13] A. A. Garcia and S. Carlip. n -dimensional generalizations of the Friedmann-Robertson-Walker cosmology. Phys. Lett. B, 645(2-3):101–107, 2007.
- [14] P. S. Letelier and J. P. M. Pitelli. n -dimensional FLRW quantum cosmology. Phys. Rev. D, 82(10):104046, 2010.
- [15] S. M. Carroll. The cosmological constant. Living Reviews in Relativity, 4(1):1–56, 2001.
- [16] M. Novello and S. E. P. Bergliaffa. Bouncing cosmologies. Physics Reports, 463(4):127–213, 2008.

Appendix A: Case of radiation dominated universe

If we assume that the source term in the total action originates from a radiation energy, the corresponding action S_r is given by

$$S_r = \sigma_r \sum_{n \geq 2} \int_{\tau_n^-}^{\tau_n^+} d\tau \frac{N_n(\tau)}{a_n(\tau)}, \quad (\text{A1})$$

where σ_r has physical dimensions of an action. The total action $S = S_g + S_r$ can be simplified into

$$S = \frac{3\pi c^3}{4G} \sum_{n \geq 2} \int_{\tau_n^-}^{\tau_n^+} d\tau \left(-\alpha_n N_n(\tau) a_n(\tau)^{n-1(n)} \mathcal{L}\{g, \Phi\} + R_r^2 \frac{N_n(\tau)}{a_n(\tau)} \right), \quad (\text{A2})$$

where

$$R_r = \sqrt{\frac{4G\sigma_r}{3\pi c^3}}, \quad (\text{A3})$$

is a length scale associated with radiation content of the universe. The minimisation of Eq. (A2) can be performed following the same steps as for matter dominated universe. For the present case we show that an incrementally increasing $n(t)$ is still the only possible solution for an expanding universe with a dynamical spacetime metric. However, contrarily to matter dominated case, we

find that $n(t)$ increases indefinitely without reaching an upper bound. In the following, we briefly present the main results.

For $n \geq 3$, the minimisation of S as given by Eq. (A2) yields

$$1 + \dot{a}_n(t)^2 = \beta_n^{(r)} \left(\frac{a_n(t)}{R_r} \right)^{2-n}, \quad (\text{A4})$$

where $\beta_n^{(r)}$ is given by Eq. (23) with R_M replaced by R_r . Eq. (A4) is exactly the one derived by solving Einstein equations for the FRLW metric for a radiation dominated universe. In addition, the jump conditions at the transition between two successive eras are given by

$$\dot{a}_n(t_n^*) = \dot{a}_{n+1}(t_n^*). \quad (\text{A5})$$

Notice that $\dot{a}_n(t)$ is continuous at the transition for all $n \geq 3$ contrarily to the matter dominated case. Using the conformal time η defined by $dt = a_n(\eta)d\eta$, the solution of Eq. (A4) yields

$$a_n(\eta) = \left[\sqrt{\beta_n} \sin \left(\frac{n-2}{2} \eta \right) \right]^{\frac{2}{n-2}}, \quad (\text{A6})$$

$$t_n(\eta) = t_n^* + \int_{\eta_n^-}^{\eta} a_n(\tau) d\tau, \quad (\text{A7})$$

with $\eta_n^- \leq \eta \leq \eta_n^+$. The boundary conditions at the transition read

$$a_n(\eta_n^+) = a_{n+1}(\eta_{n+1}^-), \quad (\text{A8})$$

$$\dot{a}_n(\eta_n^+) = \dot{a}_{n+1}(\eta_{n+1}^-). \quad (\text{A9})$$

These conditions allow us to determine η_n^\pm for all $n \geq 3$ except η_3^- which should be fixed by the boundary condition at the transition from $n = 2$ to $n = 3$.

For $n = 2$, we also follow the same steps as in the matter dominated case and find

$$\ddot{a}_2(t) - \lambda \frac{d}{dt} \left(a_2(t) \dot{\Phi}(t) \right) = 0, \quad (\text{A10})$$

$$2\ddot{\Phi}(t) + \lambda \dot{\Phi}(t)^2 = -\frac{R_r}{\alpha_2} \frac{1}{a_2(t)^2}, \quad (\text{A11})$$

$$2\dot{a}_2(t) \dot{\Phi}(t) - \lambda a_2(t) \dot{\Phi}(t)^2 = \frac{R_r}{\alpha_2} \frac{1}{a_2(t)}, \quad (\text{A12})$$

while the boundary conditions are still given by Eq. (47). The only possible solution of Eqs. (A10-A12) are given by $a_2(t) \sim t$ and $\dot{\Phi}(t) \sim 1/t$. However these behavior do not satisfy the boundary conditions $\mathcal{B}_2(t=0) = 0$.

On the other hand, if one assumes that universe started its expansion at $t = 0$ with a metric of dimension $n(t) = 3$, inflation scenario does not take place. Because either cases are not conclusive, we conclude that purely radiation dominated universe is unlikely possible in our approach.