A BIRATIONAL INVOLUTION

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Abstract. Given a general K3 surface S of degree 18, lattice theoretic considerations allow to predict the existence of an anti-symplectic birational involution of the Hilbert cube $S^{[3]}$. We describe this involution in terms of the Mukai model of S, with the help of the famous transitive action of the exceptional group $G_2(\mathbb{R})$ on the six-dimensional sphere. We make a connection with Homological Projective Duality by showing that the indeterminacy locus of the involution is birational to a $\mathbb{P}^2$-bundle over the dual K3 surface of degree two.

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1. Introduction

Consider a complex algebraic $K3$ surface S whose Picard group is generated by an ample line bundle H such that $H^2 = 2t$ (a very general element of the 19-dimensional moduli space of $2t$-polarized $K3$ surfaces). A classification of the group of biregular automorphisms of the punctual Hilbert scheme $S^{[n]} := Hilb^n(S)$ has been given by Boissière, An. Cattaneo, Nieper-Wißkirchen and Sarti [7] for $n = 2$, and by Al. Cattaneo for all $n \geq 3$ [8]. In particular $Aut(S^{[n]})$ is either trivial or generated by an involution which is non-symplectic. More recently, extended lattice theoretic considerations allowed Al. Cattaneo and the first author to decide whether $S^{[n]}$ admits non trivial birational endomorphisms [4]. When it does, there exists a unique such endomorphism, a birational involution that may be symplectic or non symplectic. The precise result is the following:

Theorem 1. *Suppose $t \geq 2$. There exists a non-trivial birational automorphism $\sigma \in Bir(S^{[n]})$, which is necessarily an involution, if and only if:*

1. *$d = t(n-1)$ is not a square,*
2. *the minimal non trivial solution (X, Y) of Pell's equation*
$$X^2 - dY^2 = 1 \quad \textit{with} \quad X = \pm 1 \; mod \; n-1$$
is such that Y is even and $(X, X) = (1, 1), (1, -1)$ or $(-1, -1)$ in $\mathbb{Z}/2(n-1)\mathbb{Z} \times \mathbb{Z}/2t\mathbb{Z}$.

Such a statement does not provide much insight on how to construct this birational involution geometrically, when it does exist. Some cases have been explicitely described in [4, Examples 6.1, 6.2], for $t = 2$ and $n = 6, 8, 18$. A few other cases had been known before:

- Suppose $t = n$, so that H embeds S into $\mathbb{P}^{n+1}$, as a surface of degree $2n$. If we choose n points $p_1, \ldots, p_n$ in S in general linear position, they generate a codimension two subspace of $\mathbb{P}^{n+1}$, which in general cuts S at $2n$ distinct points $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$. This yields the *Beauville involution*, which is non-symplectic, and biregular only for $n = 2$ (that is for quartics in $\mathbb{P}^3$).
- Suppose $n = 2$ and $t = 5$, so that S is embedded in the Grassmannian $G(2, 5)$ as the tranverse intersection of a quadric Q and three hyperplanes H_1, H_2, H_3. In particular the linear span of S intersects the Grassmannian along $F = G(2, 5) \cap H_1 \cap H_2 \cap H_3$, which is in general a smooth Fano threefold of index two. Now consider two

general points p_1, p_2 in S, defining two transverse planes P_1, P_2 in $\mathbb{C}^5$. The Grassmannian $G(2, P_1 \oplus P_2)$ is a four dimensional quadric in $G(2,5)$, its intersection with F is a conic, and its intersection with $S = F \cap Q$ therefore consists in general in four points p_1, p_2 and q_1, q_2. This yields the O'Grady involution [32, section 4.3].

- Suppose $n = 3$ and again $t = 5$, so that S can be described as before. Given three general points p_1, p_2, p_3 in S, defining three planes P_1, P_2, P_3 in $\mathbb{C}^5$, or equivalently three lines ℓ_1, ℓ_2, ℓ_3 in $\mathbb{P}^4$, it is a classical fact that there exists a unique line ℓ_0 meeting these three lines. Equivalently, if $\Sigma_\ell \subset G(2,5)$ is the codimension two Schubert cycle parametrizing lines in $\mathbb{P}^4$ that meet a given line ℓ, then $\Sigma_{\ell_1}.\Sigma_{\ell_2}.\Sigma_{\ell_3} = 1$. Since Σ_ℓ has degree three, the intersection $\Sigma_{\ell_0} \cap S$ consists in the three points p_1, p_2, p_3 (since $\ell_0 \in \Sigma_{\ell_i}$ is equivalent to $\ell_i \in \Sigma_{\ell_0}$), and three other points q_1, q_2, q_3 [10, Example 4.12].

In fact there are some easily identified infinite sequences of pairs (n, t), described in [4, Prop. 2.6 (i)], that satisfy the conditions of Theorem 1: fix any $k > 0$ and take $t = (n-1)k^2 + 1$, for any $n \geq 2$. Taking $k = 1$ we recover the Beauville involution. For $k = 2$ only the case $n = 2$ has been described, this is the O'Grady involution. In this paper we describe the next case, $n = 3$ and $t = 9$.

A general $K3$ surface of degree $2t = 18$, or equivalently of genus 10 admits a Mukai model: it can be described as a codimension three linear section of the adjoint variety of the exceptional Lie group G_2, the closed G_2-orbit inside the projectivized adjoint representation [30]. We denote this five dimensional homogeneous space by $X_{ad}(G_2) \subset \mathbb{P}(\mathfrak{g}_2)$, so that

$$S = X_{ad}(G_2) \cap L$$

for some codimension three linear subspace L of $\mathbb{P}(\mathfrak{g}_2)$.

Recall that G_2 can be described as a subgroup of $SO(7)$. This means in particular that G_2 has a natural representation V_7 of dimension 7, which is irreducible, and that it preserves some non degenerate quadratic form Q. Since then $\mathfrak{g}_2 \subset \mathfrak{so}_7 \simeq \wedge^2 V_7$, we deduce that the adjoint variety $X_{ad}(G_2)$, which is the minimal orbit in $\mathbb{P}(\mathfrak{g}_2)$, must be a subvariety of $G(2, V_7)$. In fact there is a diagram

$$\begin{array}{ccccc} X_{ad}(G_2) & \hookrightarrow & OG(2, V_7) & \hookrightarrow & G(2, V_7) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}(\mathfrak{g}_2) & \hookrightarrow & \mathbb{P}(\wedge^2 V_7) & = & \mathbb{P}(\wedge^2 V_7) \end{array}$$

where the vertical arrows are embeddings.

So let p_1, p_2, p_3 be generic points on S. They can be identified with three planes $P_1, P_2, P_3 \subset V_7$. In general, these three planes generate a hyperplane $V_6 \subset V_7$. Let ℓ denote the orthogonal line to H with respect to the quadratic form Q. We will prove that the stabilizer of ℓ in $\mathfrak{g}_2$ is a copy $\mathfrak{s}$ of $\mathfrak{sl}_3$. Hence a linear space $\mathbb{P}(\mathfrak{s}) \simeq \mathbb{P}^7$ inside $\mathbb{P}(\mathfrak{g}_2)$. We will see that $\mathbb{P}(\mathfrak{s})$ meets the adjoint variety $X_{ad}(G_2)$ along a copy F of the complete flag variety of $\mathfrak{sl}_3$. But F has dimension three and degree 6, so its intersection with L in general consist in six distinct points, p_1, p_2, p_3 plus three other points q_1, q_2, q_3. This is our main construction of a birational endomorphism φ of $S^{[3]}$:

Theorem 2. *φ is a non trivial birational involution of $S^{[3]}$.*

We also provide another interpretation of φ in terms of the extremal ray $H_3 - 2\delta$ of $S^{[3]}$, where H_3 is the line bundle induced by the polarization H of S and 2δ is, as usual, the class of the locus of non reduced subschemes. A technical deformation argument allows us to prove:

Theorem 3. *The linear system $|H_3 - 2\delta|$ is base point free.*

We show that the associated morphism $\phi_{|H_3-2\delta|}$ is generically finite of degree two, and that the birational involution φ is the corresponding deck transformation. Moreover the indeterminacy locus I of φ can be described in terms of subschemes of length three that do not span a hyperplane in V_7. Such subschemes are obtained from certain cubic scrolls in the adjoint variety, parametrized by what we call decomposing five planes: those codimension two subspaces of V_7 over which the invariant three-form is completely decomposed. We show:

Proposition 4. *The variety of decomposing five planes is projectively equivalent to the orthogonal Grassmannian $OG(2, 7)$.*

This statement is closely related to a recent paper by Guseva [15].

From our cubic scrolls we then get a rank five vector bundle $\mathcal{K}_5$ over $OG(2, 7)$, which is a subbundle of the trivial bundle with fiber $\mathfrak{g}_2$. This vector bundle already appears in Kuznetsov's description of Homological Projective Duality for the adjoint variety of G_2, that he calls the G_2-Grassmannian [23, section 9]. Remarkably, the restriction of this bundle to $X_{ad}(G_2) \subset OG(2, 7)$ is the affine version of the contact distribution on the adjoint variety, which is a notorious holomorphic contact manifold. This connects the construction to the Grothendieck-Springer resolution of G_2, a classical construction of great importance in geometric representation theory [9, 5].

We will recover, with a slightly different perspective, Kuznetsov's result that this resolution factorizes through a degree two (generically finite) cover of $\mathbb{P}(\mathfrak{g}_2)$, branched over the projective dual Δ_6 of the adjoint variety (which is, interestingly, one of the two components of the discriminant hypersurface, both of degree six). This is precisely this morphism which defines (outside some closed subset) the Homological Projective Dual to $X_{ad}(G_2)$ [23, Corollary 9.10]. In particular, to the degree 18 K3 surface $S = X_{ad}(G_2) \cap \mathbb{P}(L)$ is associated a degree 2 K3 surface Σ, derived-equivalent to S (up to the twist by a Brauer class), defined as the double cover of the projective plane $\mathbb{P}(L^\perp)$ branched over its intersection with the sextic Δ_6.

This finally allows us to describe our indeterminacy locus:

Theorem 5. *The indeterminacy locus I of φ is birational to a $\mathbb{P}^2$-bundle over the degree two $K3$ surface Σ.*

The techniques we use are a mixture of G_2-geometry with moduli interpretations and stability conditions. Basic facts about G_2 are recalled in section 2. The involution φ is defined in section 3 where we prove Theorem 2. The connections with the linear system $|H_3 - 2\delta|$ are discussed in section 4. Section 5 focuses on decomposing five-planes, and the proof of Proposition 4. The relations with the Grothendieck-Springer resolution are discussed is section 6. Theorem 5 is finally established in section 7. The proof of Theorem 3, which involves a specific deformation argument, was postponed to section 8.

All along we will find strong analogies, but also important differences, with the story of Gushel-Mukai varieties and (double) EPW sextics, a story which includes the O'Grady involution. Being able to upgrade the constructions we discuss, in order to construct a new locally complete family of polarized hyperKähler manifolds, is the main challenge that remains to be met.

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2. Some G_2 geometry

2.1. **Classical preliminaries.** There are two classical ways to understand G_2, which may be both useful according to the context. Here we work over the complex numbers, and most of what follows is translated from the real setting. In particular, each time in the sequel we deal with octonions and the Cayley algebra $\mathbb{O}$, we will be dealing with the *complexified* octonions and the *complexified* Cayley algebra. We will not use any special notation for this complexification, hoping this will not cause any confusion.

(1) (Cartan 1914) G_2 can be defined as the automorphism group of the Cayley algebra of octonions. Since the Cayley algebra $\mathbb{O}$ is normed, it admits a G_2-invariant quadratic form Q, and it splits othogonally as $\mathbb{O} = \mathbb{C}\mathbf{1} \oplus V_7$, where V_7 is the space of imaginary octonions.

(2) (Engel 1900) $G_2 \subset SL(V_7)$ can be defined as the stabilizer of a generic skew-symmetric three-form $\omega \in \wedge^3 V_7^\vee$. Then V_7 inherits an invariant quadratic form, defined up to scalar by

$$Q(x, y) = \iota_x\omega \wedge \iota_y\omega \wedge \omega \in \wedge^7 V_7^\vee \simeq \mathbb{C},$$

where ι_x denotes the contraction by the vector x. Conversely, on the space of imaginary octonions the skew-symmetric three-form is given by the formula $\omega(x, y, z) = \mathrm{Re}(x(yz - zy))$ (where Re must be interpreted as the projection on the invariant part in the decomposition $\mathbb{O} = \mathbb{C} \oplus V_7$. From this three-form one easily recovers the octonionic product.

The two fundamental representations of G_2 are V_7 and the adjoint representation on the Lie algebra $\mathfrak{g}_2$, whose dimension is 14. The two associated generalized Grassmannians are $G_2/P_1 = \mathbb{Q}^5 \subset \mathbb{P}(V_7)$, the invariant quadric, and the adjoint variety $X_{ad}(G_2) \subset \mathbb{P}(\mathfrak{g}_2)$. There is a fundamental correspondence

$$\begin{tikzcd} & G_2/B \arrow[dl] \arrow[dr] & \\ \mathbb{Q}^5 & & X_{ad}(G_2), \end{tikzcd}$$

where the two morphisms are $\mathbb{P}^1$-fibrations over Fano fivefolds, $\mathbb{Q}^5$ of index five and $X_{ad}(G_2)$ of index three; both of them dominated by the flag manifold G_2/B, where B is a Borel subgroup.

As we already mentionned, since $\mathfrak{g}_2 \subset \mathfrak{so}_7$, the adjoint variety $X_{ad}(G_2)$ is a subvariety of $G(2, V_7)$, hence parametrizes some special planes. According to our two viewpoints:

(1) $X_{ad}(G_2)$ parametrizes the family of *null-planes* in V_7, that is, those planes $P \subset V_7 = Im(\mathbb{O})$ on which the octonionic product vanishes identically [25, 28].
(2) Equivalently, $X_{ad}(G_2)$ parametrizes the planes $P = \langle x, y\rangle \subset V_7$ such that the double contraction $\iota_x\iota_y\omega = 0$. In other words, $\omega(x, y, \bullet) = 0$, showing that ω defines a global section of $Q^\vee(1)$, the twisted dual of the quotient bundle on $G(2, V_7)$, whose zero-locus is precisely the adjoint variety.

2.2. **Action of G_2 on the six-dimensional sphere.** One of the beautiful properties of the real compact form of G_2, which is the automorphism group of the real octonions, is that it acts transitively on the six-dimensional sphere, seen as the set of imaginary octonions of unit norm. This leads to the well-known identification of the six-dimensional sphere with the quotient $G_2(\mathbb{R})/SU_3$, or to the fact that $G_2(\mathbb{R})$ can be seen as the total space of a SU_3-principal bundle over S^6. Note that over the real numbers, the sphere is a double cover of the projective space of imaginary octonions.

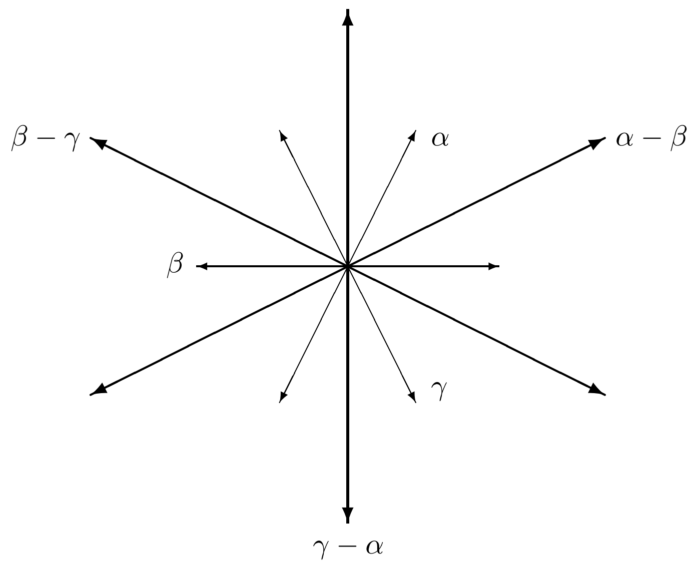
Over the complex numbers this is if course no longer true, since there exists octonions of norm zero. Instead we have the following statement.

Proposition 6. *The action of G_2 on $\mathbb{P}(V_7)$ admits only two orbits, the five dimensional quadric $\mathbb{Q}^5$ defined by Q, and its complement*

$$\mathbb{P}(V_7) - \mathbb{Q}^5 \simeq G_2/SL_3.$$

That $\mathfrak{sl}_3$ can be embedded inside $\mathfrak{g}_2$ is completely clear from the root system of $\mathfrak{g}_2$ (just forget the short roots!). Another reformulation of the previous result is that the possible embeddings are parametrized by $\mathbb{P}(V_7) - \mathbb{Q}^5$.

As in [28] we denote by α, β, γ three short roots summing to zero. Then $\alpha - \beta$, $\beta - \gamma$, $\gamma - \alpha$ are three long roots also summing to zero, and we get the twelve roots of $\mathfrak{g}_2$ by including the opposite of those six roots.



The root system of type G_2

In fact, once we have fixed a maximal torus in $\mathfrak{g}_2$ and obtained the previous decomposition, we also get a basis of V_7 consisting in weight vectors:

$$V_7 = \mathbb{C}e_0 \oplus \mathbb{C}e_\alpha \oplus \mathbb{C}e_{-\alpha} \oplus \mathbb{C}e_\beta \oplus \mathbb{C}e_{-\beta} \oplus \mathbb{C}e_\gamma \oplus \mathbb{C}e_{-\gamma},$$

and we can normalize so that the invariant three-form ω and the invariant quadratic form Q have the following expressions in the dual basis:

$$\omega = v_0 \wedge (v_\alpha \wedge v_{-\alpha} + v_\beta \wedge v_{-\beta} + v_\gamma \wedge v_{-\gamma}) + v_\alpha \wedge v_\beta \wedge v_\gamma + v_{-\alpha} \wedge v_{-\beta} \wedge v_{-\gamma},$$

$$Q = v_0^2 + v_\alpha v_{-\alpha} + v_\beta v_{-\beta} + v_\gamma v_{-\gamma}.$$

In particular e_0 has norm one, and it is clearly stabilized by the copy of $\mathfrak{sl}_3 \subset \mathfrak{g}_2$ generated by the long roots. The restriction of V_7 to this $\mathfrak{sl}_3$ decomposes into $V_7 = \mathbb{C}e_0 \oplus V_3 \oplus V_3^*$, where $V_3 = \langle e_\alpha, e_\beta, e_\gamma \rangle$ is a copy of the natural representation of $\mathfrak{sl}_3$, in duality with $V_3^* = \langle e_{-\alpha}, e_{-\beta}, e_{-\gamma} \rangle$ through the quadratic form.

Proposition 7. *The intersection of $X_{ad}(G_2) \subset \mathbb{P}(\mathfrak{g}_2)$ with $\mathbb{P}(\mathfrak{sl}_3)$ is a copy $Fl_3 \subset \mathbb{P}(\mathfrak{sl}_3)$, the variety of complete flags in V_3, which is the adjoint variety of $\mathfrak{sl}_3$.*

Proof. Denote by $\theta : \mathfrak{g}_2 \to End(V_7)$ the Lie algebra action. Consider the incidence variety

$$I := \{(p, \ell) \in X_{ad}(G_2) \times (\mathbb{P}(V_7) - \mathbb{Q}^5), \quad \theta(p)(\ell) = 0\}.$$

The fibers of the projection $I \to \mathbb{P}(V_7) - \mathbb{Q}^5$ are the intersections that we want to describe. Let us start with the fibers of the other projection $I \to X_{ad}(G_2)$. By homogeneity we may consider a point p in $Fl_3 \subset \mathbb{P}(\mathfrak{sl}_3)$, so that $p = [e^* \otimes f]$ for some non zero vector $f \in V_3$ and some non zero linear form $e^* \in V_3^*$ with $\langle e^*, f \rangle = 0$. As we already observed, the restriction of V_7 to $\mathfrak{sl}_3$ decomposes into $\mathbb{C} \oplus V_3 \oplus V_3^*$, and it readily follows that the kernel of $\theta(p)$ is $\mathbb{C} \oplus H_{e^*} \oplus H_f$, where $H_{e^*} \subset V_3$ and $H_f \subset V_3^*$ are the hyperplanes defined by e^* and f, respectively. This implies that the fiber of I over p is an open subset of $\mathbb{P}^4$. So the dimension of I is equal to nine, and its relative dimension over $\mathbb{P}(V_7) - \mathbb{Q}^5$ is three. This exactly means that for any $\mathfrak{sl}_3 \subset \mathfrak{g}_2$, stabilizing some non isotropic line $\ell \in V_7$, the intersection of $X_{ad}(G_2)$ with $\mathbb{P}(\mathfrak{sl}_3)$ is three dimensional. Therefore it has to coincide with Fl_3, the minimal orbit in $\mathbb{P}(\mathfrak{sl}_3)$ and the only one to be three-dimensional. $\square$

Given a point in $X_{ad}(G_2)$, it is easy to describe the kernel of its Lie algebra action on V_7.

Proposition 8. *Let p be a point in the adjoint variety of G_2, let π be a generator of the corresponding line in $\mathfrak{g}_2$. Then the action of π on V_7 kills a vector v if and only if v is Q-orthogonal to the null-plane P defined by p.*

Proof. By homogeneity, we may suppose that $p = [e^* \otimes f]$ is a point of the adjoint variety of some $\mathfrak{sl}_3 \subset \mathfrak{g}_2$. We have seen in the proof of Proposition 7 that the restriction of V_7 to such an $\mathfrak{sl}_3$ is of the form $\mathbb{C} \oplus H_{e^*} \oplus H_f$. By invariance, the quadratic form Q on $V_7 = \mathbb{C} \oplus V_3 \oplus V_3^*$ has to be of the form $Q(x + u + v^*) = ax^2 + b\langle v^*, u \rangle$ for some non zero scalars a, b. This implies that the Q-orthogonal to $\mathbb{C} \oplus H_{e^*} \oplus H_f$ is $\mathbb{C}f \oplus \mathbb{C}e^*$, hence exactly the plane defined by p. $\square$

3. The involution

In all the sequel we consider a very general $K3$ surface S of genus 10. By the work of Mukai [30], S can be described as a generic codimension three linear section of the adjoint variety of G_2,

$$S = X_{ad}(G_2) \cap L.$$

Let p_1, p_2, p_3 be generic points on S. They can be identified with three null planes $P_1, P_2, P_3 \subset V_7$. In general, these three planes generate a hyperplane $H \subset V_7$. Let ℓ denote the orthogonal line to H with respect to the quadratic form Q. In general ℓ is not isotropic and by Proposition 6, the stabilizer of ℓ in $\mathfrak{g}_2$ is a copy $\mathfrak{s}$ of $\mathfrak{sl}_3$. By Proposition 7, the projectivization of $\mathfrak{s}$ meets $X_{ad}(G_2)$ along a copy F of the complete flag variety Fl_3 of $\mathfrak{sl}_3$. By Proposition 8, the three points p_1, p_2, p_3 belong to $\mathbb{P}(\mathfrak{s})$, hence to F by Proposition 7.

Now, since F has dimension three and degree 6, its intersection with L consists, in general, in p_1, p_2, p_3 plus three other points q_1, q_2, q_3. This defines a rational endomorphism φ of $S^{[3]}$.

Theorem 9. *φ is a non trivial birational involution of $S^{[3]}$.*

In particular, φ has to coincide with the non natural, non symplectic birational involution whose existence was predicted in [4].

Proof. In general, with the previous notation, the intersection $F \cap L$ consists in 6 simple points p_1, p_2, p_3 and q_1, q_2, q_3 where $F = X_{ad}(\mathfrak{g}_2) \cap \mathbb{P}(\mathfrak{s})$ and $\mathfrak{s}$ is the stabilizer of a non isotropic line $\ell \subset V_7$. By Proposition 8, a point $p \in S$ belongs to F if and only if the corresponding null-plane P is orthogonal to ℓ, that is $P \subset \ell^\perp$. In particular, the three null-planes Q_1, Q_2, Q_3 corresponding to q_1, q_2, q_3 are contained in $H = \ell^\perp$, and since in general they must be transverse, we deduce that $Q_1 + Q_2 + Q_3 = H$. So starting from the three points q_1, q_2, q_3 of S rather than p_1, p_2, p_3 we would define the same copy $\mathfrak{s}$ of $\mathfrak{sl}_3$ and the same copy F of Fl_3. This implies that φ is a rational involution, and thus birational. $\square$

Remark. A more direct proof would simply consist in checking that, given a general hyperplane $H \subset V_7$, there are exactly 6 points in S corresponding to null planes contained in H. This follows from a straightforward Chern class computation. Indeed recall that $X_{ad}(G_2)$ is defined in $G(2, V_7)$ by a general section of $Q^\vee(1)$. Restricting to $G(2, H)$ we get $Q_H^\vee(1) \oplus \mathcal{O}(1)$, where Q_H denotes the rank four quotient bundle on $G(2, H)$, and the top Chern class of $Q_H^\vee(1)$ is $c_4(Q_H^\vee(1)) = \sigma_4 + \sigma_{31} + \sigma_{22}$, of degree $1+3+2 = 6$. Alternatively, it is classical that the zero-locus of a general section of $Q_H^\vee(1)$ on $G(2, H)$ is a copy of $\mathbb{P}^2 \times \mathbb{P}^2$, whose degree is equal to six.

One can wonder what does happen when $H^\perp \subset V_7$ is an isotropic line. Let us briefly answer this question. Up to the action of G_2 we may suppose that the isotropic line is generated by $e_{-\gamma}$, in which case the restriction of the invariant three-form to H is

$$\omega_H = v_0 \wedge v_\alpha \wedge v_{-\alpha} + v_0 \wedge v_\beta \wedge v_{-\beta} + v_\alpha \wedge v_\beta \wedge v_\gamma.$$

This is a general element of the affine tangent space to $G(3, H^\vee)$ at the point defined by $\langle v_0, v_\alpha, v_\beta \rangle$, and it distinguishes uniquely this space, as well as the orthogonal projective plane $\mathbb{P}(A)$, where $A = \langle e_{-\alpha}, e_{-\beta}, e_\gamma \rangle$. Moreover ω_H induces an isomorphism ι between A and $B = \langle e_\beta, e_\alpha, e_0 \rangle$. A straightforward computation shows that the section of $Q^\vee(1)$ on $G(2, H)$ defined by ω_H vanishes at a point defined by a plane $P \subset H$ if either $P \subset A$, or $P = \langle v, w + \iota(v) \rangle$ for some $v, w \in A$. This defines a four dimensional subvariety P of $G(2, H)$, which can be described as the image of a birational morphism from the total space of the vector bundle $C := Q \oplus \mathcal{O}(-1)$ over $\mathbb{P}(A) \simeq \mathbb{P}^2$:

$$\begin{tikzcd}
 & \mathbb{P}_{\mathbb{P}(A)}(C) \arrow[r] \arrow[dl] & P \arrow[r, hook] & G(2,H) \\
\mathbb{P}(A) & F \arrow[l] \arrow[r] \arrow[u, hook] & \mathbb{P}(A^\vee) \arrow[u, hook] &
\end{tikzcd}$$

The exceptional divisor $F = \mathbb{P}(Q) \simeq Fl_3$ is contracted to the dual plane $\mathbb{P}(A^\vee)$, which is the singular locus of P, with transverse singularities of type A_1. We thus get a singular, flat degeneration of $\mathbb{P}^2 \times \mathbb{P}^2$.

Now, the additional condition that $\omega(e_{-\gamma}, ., .) = 0$ defines a hyperplane section $\tilde{Fl}_3$ of P, which is transverse outside $\mathbb{P}(A^\vee)$, but $\tilde{Fl}_3$ is singular along a line in $\mathbb{P}(A^\vee)$. Its preimage in $\mathbb{P}(C)$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}(A)$ outside the point $[e_\gamma]$ over which we get the whole fiber. In particular the exceptional divisor is the blowup of $\mathbb{P}(A)$ at that point. We finally get a diagram

$$\begin{tikzcd} & I_6 \arrow[dl] \arrow[dr] & \\ X_{ad}(\mathfrak{g}_2) & & \mathbb{P}(V_7^\vee) \end{tikzcd}$$

where the incidence variety I_6 is a $\mathbb{P}^4$-bundle over $X_{ad}(\mathfrak{g}_2)$, while the fibers of its projection to $\mathbb{P}(V_7^\vee)$ are copies of Fl_3 outside $\mathbb{Q}^5$, and copies of its degeneration $\tilde{Fl}_3$ over $\mathbb{Q}^5$.

Another consequence of Proposition 8 is the following statement, which will be used in the next section:

Proposition 10. *The direct sum map induces a rational map $S^{[3]} \dashrightarrow \mathbb{P}(V_7^*)$ which is generically finite of degree* 20.

Proof. Suppose a general hyperplane H in V_7 defines six points $p_1, p_2, p_3, q_1, q_2, q_3$ in S as above. Then the corresponding null planes are in general linear position and the direct sum of any three of them is contained, hence equal to H. This gives 20 points in the fiber of the direct sum map. That there is no other follows from Proposition 8. $\square$

This direct sum map is very similar to the one that, given a smooth quartic surface $\Sigma \subset \mathbb{P}^3$, maps $\Sigma^{[2]}$ to $G(2,4)$ by sending a length two subscheme of S to the line it generates in $\mathbb{P}^3$. Obviously this is a generically finite cover of degree 6 (finite if Σ contains no line).

4. The linear system $|H_3 - 2\delta|$

Recall that the second cohomology group $H^2(S^{[n]}, \mathbb{Z}) \simeq H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta$, where 2δ is the class of the divisor of non reduced schemes. This decomposition is orthogonal with respect to the Beauville-Bogomolov form q_{BB}, which restricts to the intersection form on $H^2(S, \mathbb{Z})$. If we denote by $L \mapsto L_n$ the embedding of $H^2(S, \mathbb{Z})$ inside $H^2(S^{[n]}, \mathbb{Z})$, this means that $q_{BB}(L_n) = L^2$. On the other hand, $q_{BB}(\delta) = -2(n-1)$ (see e.g. [10, 3.2.1]).

The goal of this section is to show that the linear system that defines φ is $|H_3 - 2\delta|$.

4.1. **The movable and the nef cones.** We first describe the movable cone and the nef cone of $S^{[3]}$. The following result is [4, Proposition 4.1] ($d = 2$ for $t = 9$) and [4, Lemma 3.6] with $n = 3$ and $t = 4n - 3$, we write down a direct proof for the reader's convenience.

Proposition 11. *The movable cone of $S^{[3]}$ has two chambers, exchanged by the action of φ.*

Proof. The structure of the movable cone of $S^{[3]}$ is described in [3, Theorem 13.1] in terms of Pell's equations: it is the interior of the convex cone generated by H_3 and $4H_3 - 9 \times 17\delta$. The walls in the movable cone are spanned by vectors of the form $XH_3 - 18Y\delta$ for (X, Y) a positive solution of $X^2 - 72Y^2 = 8 + \alpha^2$, and $\alpha \in \{1, 2\}$; this is [3, Theorem 12.3], translated in term of generalized Pell's equations using [8, Lemma 2.5] (see also [8, Remark 2.8]). Actually, the vectors cutting chambers of the movable cone are those for which $\frac{2Y}{X} < \frac{4}{17}$.

One can immediately check that there are no integral solutions when $\alpha = 2$, so we are left to consider solutions of $X^2 - 72Y^2 = 9$, which in turn are in the form $(3\tilde{X}, Y)$ with $\tilde{X}^2 - 8Y^2 = 1$; the latter is a Pell equation with minimal positive solution $(3, 1)$. All the solutions of this equation can be found recursively by letting

$$(\tilde{X}_{k+1}, Y_{k+1}) = (3\tilde{X}_k + 8Y_k, \tilde{X}_k + 3Y_k).$$

Moreover $\frac{\tilde{X}_{k+1}}{Y_{k+1}} < \frac{\tilde{X}_k}{Y_k}$. The first two solutions are $(3, 1)$, $(17, 6)$. Note that for the second one we have $2\frac{Y_2}{3\tilde{X}_2} = \frac{4}{17}$, so this solution corresponds to a boundary of the movable cone, and there is therefore exactly one wall inside the movable cone. This also means that the ample cone is the interior of the cone generated by H_3 and $9H_3 - 18\delta$; this last vector is proportional to $H_3 - 2\delta$, which is the generator of the invariant lattice for the action of φ by [4, First case of Proposition 2.2]. Then the action of φ in cohomology is a reflection w.r.t. $H_3 - 2\delta$, exchanging the two rays of the movable cone, hence the two chambers. □

Corollary 12. $\mathrm{Nef}(S^{[3]}) = \langle H_3, H_3 - 2\delta \rangle$.

Moreover there is a biregular automorphism between $S^{[3]}$ and its birational model corresponding to the non ample chamber of the movable cone, and we get the following statement.

Corollary 13. *φ is the composition of a biregular morphism with the flop associated to the wall between the two chambers of the movable cone.*

Note that the extremal contraction asociated to the extremal ray $H_3 - 2\delta$ is defined by the linear system $|k(H_3 - 2\delta)|$ for $k >> 1$. In the sequel we focus on the linear system system $|H_3 - 2\delta|$ itself.

4.2. **First observations.** We start by computing the dimension of the linear system $|H_3 - 2\delta|$.

Proposition 14. $|H_3 - 2\delta| \simeq \mathbb{P}^9$.

Proof. We know that $H_3 - 2\delta$ is nef by Corollary 12. Moreover

$$q_{BB}(H_3 - 2\delta) = H^2 + 4\delta^2 = 18 + 4 \times (-4) = 2.$$

Since the Fujiki constant of $S^{[3]}$ is 15 [10, 3.2.1], this implies that $(H_3 - 2\delta)^6 = 15 q_{BB}(H_3 - 2\delta)^3 = 120$, in particular the class $H_3 - 2\delta$ is big as well. We can therefore invoke Kawamata-Viehweg and conclude that the number of sections is given by the Riemann-Roch polynomial. But on a hyperKähler fourfold X of K3-type we have (see e.g. [10, 3.3])

$$\chi(X, L) = \binom{\frac{1}{2} q_{BB}(L) + 4}{3},$$

hence the claim. $\square$

Proposition 15. *There is a natural identification*

$$|H_3 - 2\delta| \simeq |I_3(Sec(S))|.$$

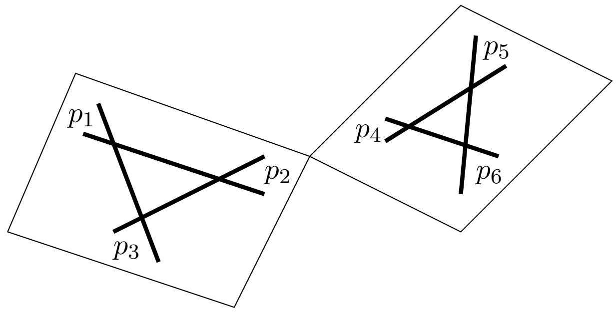
Moreover, given a length three subscheme Z of S, not in the base locus, its image in $|I_3(Sec(S))|^\vee$ is the hyperplane of cubics containing the projective plane spanned by Z.

Proof. Given three distinct points p_1, p_2, p_3 of S, corresponding to three lines $\ell_1, \ell_2, \ell_3 \subset \mathfrak{g}_2$, the fiber of H_3 at $Z = p_1 \cup p_2 \cup p_3$ is $\ell_1^\vee \otimes \ell_2^\vee \otimes \ell_3^\vee$. A cubic polynomial P on L defines a section of H_3 whose evaluation at Z is given by the polarisation $\tilde{P}$ of P, restricted to $\ell_1 \otimes \ell_2 \otimes \ell_3$. This section vanishes on the locus of non reduced schemes $E \simeq 2\delta$ if and only if $\tilde{P}(x_2, x_2, x_3) = 0$ for any $p_2 = [x_2], p_3 = [x_3]$ in S. This is equivalent to the condition that $P(sx_2 + tx_3) = 0$ for any $p_2 = [x_2], p_3 = [x_3]$ in S and any scalars s, t, hence to the condition that P vanishes on $Sec(S)$.

If again $Z = p_1 \cup p_2 \cup p_3$ is reduced, the secant variety of S contains the three lines $\overline{p_1p_2}$, $\overline{p_2p_3}$ and $\overline{p_3p_1}$. So the restriction to the plane $\langle Z \rangle$ of a cubic polynomial vanishing on $Sec(S)$ is completely determined up to scalar. If $p_1 = [x_1], p_2 = [x_2], p_3 = [x_3]$, it is clear that P vanishes on the whole plane if and only if $\tilde{P}(x_1, x_2, x_3) = 0$. This proves our last claim. $\square$

If $\varphi(Z) = p_4 \cup p_5 \cup p_6$, we know there exists a copy $\mathfrak{s}$ of $\mathfrak{sl}_3$ in $\mathfrak{g}_2$ such that the six points $p_1, \dots, p_6$ belong to $\mathbb{P}(\mathfrak{s})$, hence to $\mathbb{P}(\mathfrak{s}) \cap L$. But the latter is in general a $\mathbb{P}^4$, so the two planes $\langle Z \rangle$ and $\langle \varphi(Z) \rangle$ have to meet. Generically they will meet outside the lines of the two triangles defined by Z and $\varphi(Z)$, and then a cubic vanishing on one triangle plus the span of the other triangle has to vanish on both spans. This proves:

Corollary 16. *The map $\phi_{|H_3 - 2\delta|}$ factorises through the involution φ.*



Remark. Here again there is a strong analogy with conics on Gushel-Mukai fourfolds, which are all linear sections of copies of $G(2,4)$ inside $G(2,5)$. In particular the linear system of quadratic equations of the Gushel-Mukai, which contains the Pfaffian quadrics as a hyperplane, restricts to a pencil on the plane spanned by the conic. So exactly as before, containing this plane is just a codimension one condition on the linear system. See [18] for more details.

Theorem 17. *The linear system $|H_3 - 2\delta|$ is base point free.*

Proof. The proof relies on a technical deformation argument, and we postpone it to section 8. $\square$

4.3. **The secant variety and Pfaffian cubics.** Since obviously $Sec(S) \subset Sec(X_{ad}(\mathfrak{g}_2))$, let us describe the latter. By [21] and [20], the secant variety of $X_{ad}(\mathfrak{g}_2)$ has dimension 10, hence defect one, and the generic entry locus is a conic. (Recall that the entry locus of a general point p on the secant variety of some variety Z, is the set of points $z \in Z$ such that the line joining p to z is a bisecant to Z, see [35].)

The action of G_2 on $Sec(X_{ad}(\mathfrak{g}_2))$ has only finitely many orbits. This can be seen directly by using the fact that since the secant variety is defective, it is equal to the tangent variety. This reduces us to understanding the isotropy representation. Since we will not use it in the sequel we omit the proof of the next statement:

Proposition 18. *The secant variety $Sec(X_{ad}(G_2))$ has four G_2-orbits:*

1. *The adjoint variety $X_{ad}(G_2)$ itself.*
2. *The orbit of $[e_\alpha \wedge e_{-\beta} + e_{-\beta} \wedge e_\gamma + e_\alpha \wedge e_\gamma - e_0 \wedge e_{-\beta}]$, for which the entry-locus is a double line.*
3. *The codimension one orbit of $[e_\alpha \wedge e_{-\beta} + e_\beta \wedge e_{-\gamma}]$, for which the entry-locus is a degenerate, reduced conic.*
4. *The open orbit of $[e_\alpha \wedge e_{-\beta} + e_\beta \wedge e_{-\alpha}]$, for which the entry-locus is a smooth conic.*

For a variety like $X_{ad}(G_2)$, which is cut-out by quadrics, the general expectation is that the secant variety should be cut-out by cubics. Of course there are many exceptions, typically when the codimension is not large enough.

Proposition 19. *$Sec(X_{ad}(G_2))$ is not cut out by cubics, and in fact*

$$I_3(Sec(X_{ad}(G_2))) = V_7.$$

We call the cubics defined by vectors $v \in V_7$ the *Pfaffian cubics*, since they are given by

$$P_v(x) = v \wedge x \wedge x \wedge x \in \wedge^7 V_7 \simeq \mathbb{C}, \qquad x \in \mathfrak{g}_2 \subset \wedge^2 V_7.$$

Note that the Pfaffian cubics have actually little to do with $\mathfrak{g}_2$; in fact they cut out in $\mathbb{P}(\wedge^2 V_7)$ the secant variety of the whole Grassmannian $G(2, V_7)$; otherwise said, tensors of rank at most four (when considered as skew-symmetric forms on $V_7^\vee$).

Remark. Note the strong analogy with the *Pfaffian quadrics* that cut out $G(2, V_5)$ in $\mathbb{P}(\wedge^2 V_5)$, and are parametrized by V_5; these quadrics play a major rôle in the study of Gushel-Mukai varieties.

Proof. According to Lie, $S^3\mathfrak{g}_2^\vee = [0,3] \oplus [2,1] \oplus [3,0] \oplus [0,1] \oplus [1,0]$, where we denote by $[a,b]$ the irreducible $\mathfrak{g}_2$-module with highest weight $a\omega_1 + b\omega_2$. (In particular $[1,0] = V_7$ and $[0,1] = \mathfrak{g}_2$.) Since there is no multiplicity bigger than one, the submodule spanned by cubics vanishing on the adjoint variety must be the sum of some of these irreducible modules. It certainly does not contain $[0,3]$, which is the space of cubics on $X_{ad}(G_2)$, but it certainly contains $[1,0] = V_7$, which can be identified with the space of cubics vanishing on the whole secant variety of $G(2, V_7)$ (more precisely, on its intersection with $\mathbb{P}(\mathfrak{g}_2)$). In order to complete the discussion, recall that $I_2(X_{ad}(G_2)) = [2,0] \oplus [0,0]$, the invariant factor being defined by the Killing form K. This implies that the embedding of $[0,1]$ in $S^3\mathfrak{g}_2^\vee$ is given by $X \mapsto P_X$ with $P_X(Y) = K(X,Y)K(Y,Y)$. So the corresponding cubics are in fact reducible and do not vanish on the secant variety.

There remains to understand the embeddings of $[2,1]$ and $[3,0]$ inside $S^3\mathfrak{g}_2^\vee$. For this we can use the embedding of S^2V_7 in $S^2\mathfrak{g}_2^\vee$ sending uv to the polynomial $Q_{uv}(Y) = q(Yu, Yv)$. This induces an embedding of $[2,1]$ and $[3,0]$ inside $S^3\mathfrak{g}_2^\vee$ since they are both contained in $[2,0] \otimes [0,1] = S^{(2)}V_7 \otimes \mathfrak{g}_2$, where $S^{(2)}V_7$ denotes the hyperplane in S^2V_7 spanned by tensors v^2 with $Q(v) = 0$. In fact there is a unique line of vectors of weight $2\omega_1 + \omega_2 = 2\alpha + \psi$ in this tensor product, where ψ is the highest root, generated by $e_\alpha^2 \otimes X_\psi$. Its image is the reducible polynomial $Y \mapsto K(Y, X_\psi)q(Ye_\alpha)$, which does not vanish identically on the secant.

For the weight $3\omega_1 = 3\alpha$, the situation is different since there is a three-dimensional space of weight vectors, among which we claim that

$$e_\alpha^2 \otimes X_\alpha - e_\alpha e_{-\beta} \otimes X_{\alpha-\gamma} - e_\alpha e_{-\gamma} \otimes X_{\alpha-\beta}$$

is a highest weight vector. There remains to check that the corresponding polynomial $R(e_\alpha)$, sending $Y \in \mathfrak{g}_2$ to $R(e_\alpha)(Y)$ given by

$$q(Ye_\alpha)K(X_\alpha, Y) - q(Ye_\alpha, Ye_{-\beta})K(X_{\alpha-\gamma}, Y) - q(Ye_\alpha, Ye_{-\gamma})K(X_{\alpha-\beta}, Y)$$

does not vanish identically on $Sec(X_{ad}(G_2))$. For this we let $Y = e_{-\alpha} \wedge e_\gamma + e_\beta \wedge e_{-\gamma}$, that belongs to the line joining the two null-planes $\langle e_{-\alpha}, e_\gamma\rangle$ and $\langle e_\beta, e_{-\gamma}\rangle$. Then $Ye_\alpha = e_\gamma$, $Ye_{-\beta} = e_{-\gamma}$ and $Ye_{-\gamma} = -e_{-\alpha}$ hence $q(Ye_\alpha) = 0$, $q(Ye_\alpha, Ye_{-\beta}) = 1$ and $q(Ye_\alpha, Ye_{-\gamma}) = 0$. We get $R(e_\alpha)(Y) = -K(X_{\alpha-\gamma}, Y) \neq 0$, since Y is a linear combination of $X_{\gamma-\alpha}$ and $X_{\beta-\gamma}$ with non zero coefficients. $\square$

By restricting Pfaffian cubics we get a natural inclusion $V_7 \subset I_3(Sec(S))$, and this sublinear system defines a rational map $S^{[3]} \dashrightarrow \mathbb{P}(V_7^\vee)$. The next easy claim is that this coincides with the direct sum map.

Lemma 20. *Consider $S^{[3]} \dashrightarrow \mathbb{P}(V_7^\vee)$ as in Proposition 10. The diagram*

$$\begin{tikzcd}
 & S^{[3]} \arrow[dl, dotted] \arrow[dr, "\phi_{|H_3-2\delta|}"] & \\
\mathbb{P}(V_7^\vee) & & \mathbb{P}(I_3(Sec(S))^\vee) \arrow[ll, dotted, "p_{V_7^\vee}"']
\end{tikzcd}$$

commutes. In particular $\phi_{|H_3-2\delta|} : S^{[3]} \longrightarrow \mathbb{P}(I_3(Sec(S))^\vee)$ is generically finite of degree d with $d|20$.

Proof. The rational map $p_{V_7^\vee}$ sends a projective hyperplane of cubics H to the hyperplane in V_7 consisting in the Pfaffian cubics P_v that belong to H. Consider a general $x = p_1 + p_2 + p_3 \in S^{[3]}$. Choose a basis $e_1, \ldots, e_7$ of V_7 such that the plane associated to p_h is $\langle e_h, e_{h+3}\rangle$ for $h = 1, 2, 3$. Then $x \wedge x \wedge x = 6e_1 \wedge e_4 \wedge e_2 \wedge e_5 \wedge e_3 \wedge e_6$, so $P_v(x) = 0$ if and only if v belongs to $\langle e_1, \ldots, e_6\rangle$. This implies the claim. $\square$

Remark. The situation is in some sense close to that of Gushel-Mukai varieties, which are defined by Pfaffian quadrics plus one extra, general quadric. Here, the linear system of cubics containing $Sec(S)$ is made of the Pfaffian cubics, plus three extra cubics that remain mysterious. Equivalently, the projective plane $\mathbb{P}(V_7^\perp)$ which is the center of the projection $p_{V_7^\vee}$ remains elusive.

4.4. **Computation of the degree.** Let us determine the degree d of $\phi_{|H_3-2\delta|}$. We will use a monodromy argument, based on some simple combinatorics. First note that by Corollary 16, d is even, and by Lemma 20, d divides 20. So $d = 2, 4, 10$ or 20.

Proposition 21. $d = 2$.

Proof. Let us exclude the other possibilities. Recall that at the generic point $p_1 + p_2 + p_3$ of $S^{[3]}$ we have another point $p_4 + p_5 + p_6$ of $S^{[3]}$ in the same fiber, and that any other point in the fiber must be of the form $p_i + p_j + p_k$ and come with its complement $p_\ell + p_m + p_n$, where $\{i, j, k\} \cap \{\ell, m, n\} = \emptyset$. To simplify notations we will denote these pairs by $(123|456)$ and $(ijk|\ell mn)$, where we can permute the two triples and the three integers in each triple.

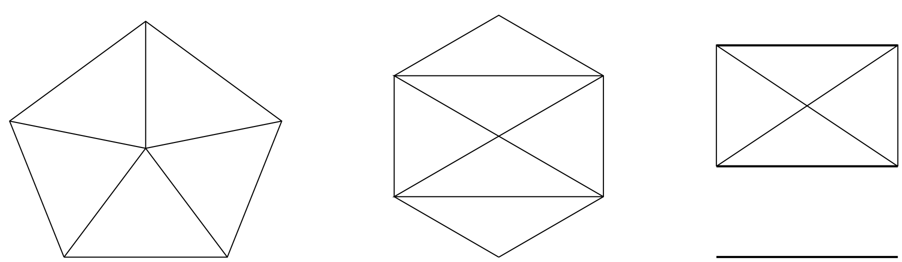
$d = 4$. Up to permutation of the indices, we can always suppose that the four points in the fiber are given by the triples $(123|456)$ and $(124|356)$. Then each triple contains a unique pair of points that are also contained in another triple. This means that we would be able to split the generic point of $S^{[3]}$ as the sum of a point of $S^{[2]}$ and a point of S, which is absurd!

$d = 10$. We will use the same idea as before, although this case is a bit more involved combinatorially. We first remark that up to permuting the indices, there are only three ways to choose 5 pairs of complementary triples among six indices. We leave the following lemma to the reader.

Lemma 22. *Up to permuting indices, any* 5*-tuple of complementary triples of indices between* 1 *and* 6 *is equivalent to one of the following:*

$(123\|456)$	$(123\|456)$	$(123\|456)$
$(134\|256)$	$(124\|356)$	$(124\|356)$
$(145\|236)$	$(125\|346)$	$(134\|256)$
$(156\|234)$	$(126\|345)$	$(135\|246)$
$(126\|345)$	$(156\|234)$	$(145\|236)$

A nice combinatorial gadget in order to distinguish these 5-tuples is to associate them a little graph by using the previous remark that each time we have two pairs of complementary triples, we can arrange them in the form $(ijk|\ell mn)$ and $(ij\ell|kmn)$. In particular the pair $(k\ell)$ is distinguished. So each 5-tuple yields ten pairs, which we can vizualise as the edges of a graph. We get the three following graphs:



In particular the six points do not have the same combinatorial properties, and in each case it easy to see that one of the points in the triples can be distinguished, which is absurd.

$d = 20$. This means that all the triples $p_i + p_j + p_k$ belong to the fiber. Denote by π_{ijk} the projective plane in L spanned by these three points, when they are in general position. By Proposition 14, we conclude that there is a hyperplane in $I_3(Sec(S))$ consisting in cubics that vanish on the 20-planes π_{ijk}. Recall that the six points $p_1, \dots, p_6$ span the linear space $\mathbb{P}(\mathfrak{s}) \cap L$, where $\mathfrak{s} \simeq \mathfrak{sl}_3$, which is a $\mathbb{P}^4$. It is easy to check that a cubic on this $\mathbb{P}^4$ vanishing on the 20 planes must vanish identically. We conclude that the linear system $|I_3(Sec(S))|$ reduces on $\mathbb{P}(\mathfrak{s}) \cap L$ to a unique cubic. But since these projective four spaces cover L, this would mean that the linear system itself reduces to a single cubic, which is absurd. The proof is complete. $\square$

Corollary 23. *φ is the covering involution associated to $\phi_{|H_3-2\delta|}$.*

Recall that $\phi_{|H_3-2\delta|}$ is only generically finite; we will denote by I its exceptional locus, defined as the union of its positive dimensional fibers.

Corollary 24. *The indeterminacy locus of φ coincides with the exceptional locus of $\phi_{|H_3-2\delta|}$.*

In particular, call N the contraction of the indeterminacy locus of φ, as given by the Stein factorization of $\phi_{|H_3-2\delta|}$. Then N is normal and φ descends to a regular involution $\bar{\varphi}$ of N. The quotient $N/\langle\bar{\varphi}\rangle$ is still normal, hence isomorphic to the normalisation of $\phi_{|H_3-2\delta|}(S^{[3]})$. This is summarized in the following diagram:

$$\begin{tikzcd}
S^{[3]} \arrow[loop left, "\varphi"] \arrow[rr, "\phi_{|H_3-2\delta|}"] \arrow[dr, "c"'] & & \phi_{|H_3-2\delta|}(S^{[3]}) \\
& N \arrow[loop left, "\bar{\varphi}"] \arrow[ur, "\nu"] \arrow[dr] & \\
& & N/\bar{\varphi} \arrow[uu, "\bar{\nu}"']
\end{tikzcd}$$

Remark. Note the analogy with the O'Grady involution, which yields a similar picture where the map $c : S^{[2]} \to N$ contracts a $\mathbb{P}^2$, N is a double EPW sextic singular in $c(\mathbb{P}^2)$ which is the inverse image of the Plücker point, and $N \to \phi_{|H_3-2\delta|}(S^{[2]})$ is the double cover $\tilde{Y}_A \to Y_A$ of a special EPW sextic [32]. This is a situation where $Y_A = \phi_{|H_3-2\delta|}(S^{[2]})$ is normal (although $Y_A^{[3]} \neq \emptyset$ does not have the expected dimension).

In our situation we do not know whether $\bar{\nu}$ is an isomorphism. But pursuing the analogy with EPW sextics, it would be tempting to imagine that the double cover $\nu : N \to \phi_{|H_3-2\delta|}(S^{[3]})$ can be deformed to a locally complete family of polarized hyperKähler sixfolds. This very natural question is also discussed in [22].

4.5. **Structure of the indeterminacy locus.** Now we focus on the indeterminacy locus I of the birational involution φ. By the previous corollary, I coincides with the exceptional locus of the generically finite morphism $\phi_{|H_3-2\delta|}$. By [27, Lemma 2.1.28], this is also the exceptional locus of $\phi_{|k(H_3-2\delta)|}$, which is for $k >> 1$ nothing else than the extremal contraction defined by the extremal ray $H_3 - 2\delta$.

Extremal contractions of holomorphic symplectic manifolds have been extensively studied (see for example [16, 1]). We will use the methods and results of [3] to describe the structure of our exceptional locus I. Our final result will be that this locus can be described in terms of the degree two $K3$ surface associated to S by Homological Projective Duality [23, section 8] (to be precise, the HPD to S is the degree-two $K3$ surface twisted by a Brauer class). We will first use stability conditions to show that I is birational to a $\mathbb{P}^2$-bundle over such a surface. In the next section, we will use the geometry of G_2 to show that the base of the fibration is exactly the expected surface.

Let us consider, see for example [3, Section 13] for a reference, $S^{[3]}$ as the moduli space $M_\sigma(1, 0, -2)$ of stable sheaves on S with Mukai vector $v = (1, 0, -2)$, for some generic stability condition σ with $\ell(\sigma)$ lying in the ample chamber of $S^{[3]}$, where $\ell : \mathrm{Stab}^\dagger(S) \to \mathrm{NS}(S^{[3]})$ is as in [3, Theorem 1.2]. As in the previous diagram we denote by $c : S^{[3]} \to N$ the contraction associated to the flopping wall generated by $H_3 - 2\delta$; here N is $M_{\sigma_0}(v)$ for a stability condition σ_0 with $\ell(\sigma_0) \in \mathbb{R}(H_3 - 2\delta)$. Once again, the indeterminacy locus of φ is also the exceptional locus of c.

Proposition 25. *The indeterminacy locus of φ is birational to a $\mathbb{P}^2$-bundle over a K3 surface Σ. In particular it is irreducible.*

Proof. In the following, we denote by $v(E)$ the Mukai vector of an object $E \in K(D^b(S))$. A numerical description of the wall $\mathbb{R}(H_3 - 2\delta)$ is given by [8, Lemma 2.5], and [4, Section 4]: the associate primitive, rank two, hyperbolic lattice in $H^*_{\mathrm{alg}}(S, \mathbb{Z})$ is $\mathcal{H} = v\mathbb{Z} + a\mathbb{Z}$ with $a = -(2, -H, 5)$; its Gram matrix with respect to the base (v, a) is

$$\begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}.$$

We take a look at the stability conditions in the potential walls associated to $\mathcal{H}$, see [3, Definition 5.2]. In particular, we focus on stability conditions in the form

$$\sigma_{\alpha,\beta} = (\mathbf{Coh}^\beta(S), Z_{\alpha,\beta}) \quad \text{for} \quad (\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0},$$

whose central charge are of the form $Z_{\alpha,\beta} = (e^{i\alpha H + \beta H}, -)$, so

$$Z_{\alpha,\beta}(w_0, w_1, w_2) = i\alpha H \cdot (w_1 - \beta w_0 H) - w_2 + \beta H \cdot w_1 + \frac{H^2}{2}(\alpha^2 - \beta^2)w_0.$$

In the (β, α) half-plane in $\mathrm{Stab}^\dagger(S)$, we obtain a numerical semicircular wall of center $(-\frac{1}{2}, 0)$ and radius $\frac{1}{6}$. The only spherical class in $\mathcal{H}$ is $a = -(2, -H, 5)$, so all the pairs (β, α) on the wall effectively correspond to stability conditions by [2, Theorem 3.1], except for $(\beta, \alpha) = (-\frac{1}{2}, \frac{1}{6})$; the latter does not give a stability condition, since in this case the evaluation of $Z_{\alpha,\beta}$ at a is zero. We pick (β_0, α_0) on the wall and just on the right side of the critical value $(-\frac{1}{2}, \frac{1}{6})$, so that the class a is effective with respect to the stability condition (see [3, Proposition 5.5] for a definition of effective classes in $\mathcal{H}$).

We fix then $\sigma_0 = \sigma_{\alpha_0,\beta_0}$; since $a^2 = -2$ and it is effective for σ_0, the moduli space $M_\sigma(a)$ is a single point and is equal to $M_{\sigma_0}(a)$.

By [3, Theorem 2.18], curves inside the exceptional locus are given by σ-stable objects which are S-equivalent with respect to the stability condition σ_0. Recall that S-equivalent objects with respect to a stability condition σ are objects whose Jordan-Hölder filtrations have the same stable quotients; the factors of the filtration have same phase with respect to σ_0, hence their Mukai vectors are in $\mathcal{H}$, see [3, Proposition 5.1 (d)]. So we need classes $w_1, w_2 \in \mathcal{H}$ such that $w_1 + w_2 = v$ and $w_h^2 \geq -2$ for $h = 1, 2$. More precisely, the decompositions really inducing contractions are already classified by Bayer-Macrì in [3, Theorem 5.7], or rather [3, Theorem 12.1] for the particular case of Hilbert schemes of points.
As we expected, a straightforward computation shows that the only possible decomposition in the form of the ones listed in [3] is $v = a + (v - a)$. Moreover, we cannot further decompose a and $v - a$ as above.

Now we follow the proof of [3, Proposition 9.1] as an algorithm to find the exceptional locus of c. We are in the case where a is effective, $a^2 = -2$ and $(a, v) = 1$, and the parallelogram with vertices $0, a, v - a, v$ does not contain any integer point other than the vertices. The description holds up to birationality since, for particular elements in the contracted locus, the filtration can be more complicated.

Call A the object with $v(A) = a$ and F an object with $v(F) = v - a$. By definition of the Mukai pairing, we have $-(v(A), v(F)) = \chi(A, F) = \sum_{i=1}^{3} \dim(\mathrm{Ext}^i(A, F))$, thus $\dim(\mathrm{Ext}^1(A, F)) \geq (a, v - a) = 3$. An explicit computation shows that $\phi(A) > \phi(F)$, for ϕ the phase associated to $\sigma = (\mathbf{Coh}^\beta(S), Z_{\alpha,\beta})$ such that $\ell(\sigma)$ lies in the ample cone of $S^{[3]}$. So, by [3, Lemma 9.3], any extension $F \hookrightarrow E \twoheadrightarrow A$ is σ-semistable. Clearly any such objects E, E' have $v(E) = v(E') = v$ and are S-equivalent, so the morphism c contracts the whole $\mathbb{P}(\mathrm{Ext}^1(A, F))$ to a point.

Moreover, F varies in a moduli space $M_\sigma(v - a)$, which is a K3 surface Σ. So we obtain a family of dimension $2 + \dim(\mathrm{Ext}^1(A, F)) - 1$. The exceptional locus of c is the indeterminacy locus of a flop between hyperKähler manifolds, so it has codimension at least two, hence we must have $\dim(\mathrm{Ext}^1(A, F)) = 3$. $\square$

As it is clear from the proof above, up to birationality the morphism $c : S^{[3]} \to N$ contracts the fibers of a $\mathbb{P}^2$-bundle to the surface Σ, which is a double cover of the projective plane $\mathbb{P}(V_7^\perp)$, center of the projection $p_{V_7^\vee}$.

Lemma 26. *The K3 surface Σ is a double cover of a projective plane, ramified over a sextic curve.*

Proof. We have seen that $\Sigma = M_\sigma(v - a)$, so

$$\mathrm{NS}(\Sigma) \cong \frac{(v-a)^\perp}{(v-a)\mathbb{Z}} \cong (1, 0, -1)\mathbb{Z}.$$

This implies the claim, the covering involution of Σ being the Mukai reflection associated to v. $\square$

5. More G_2-geometry

5.1. **Another perspective on the indeterminacy locus.** An obvious problem in the definition of φ occurs when we consider a scheme $Z \in S^{[3]}$ that does not generate a hyperplane $V_6 \subset V_7$, which means that if $\mathcal{U}$ denotes the restriction of the tautological rank two vector bundle on $G(2, V_7)$, the restriction map

$$V_7^\vee \longrightarrow H^0(Z, \mathcal{O}_Z \otimes \mathcal{U}^\vee)$$

has a two-dimensional kernel; this is exactly the indeterminacy locus of the rational map $S^{[3]} \dashrightarrow |V_7|^\vee$.

Outside this locus, Lemma 20 implies that any positive dimensional fiber of $\phi_{|H_3-2\delta|}$ must also be contracted by the direct sum map $S^{[3]} \dashrightarrow |V_7|^\vee$. If a length three subscheme Z of S generates a hyperplane $V_6 \subset V_7$, recall from the beginning of section 3 that $\varphi(Z)$ is defined as a subscheme of $F \cap L$, where F is the copy of Fl_3 defined by V_6. In particular, there can exist a positive dimensional fiber of the map $S^{[3]} \dashrightarrow |V_7|^\vee$ passing through Z only when the length six scheme $F \cap L$ contains infinitely many subschemes. (Note that $F \cap L$ can never be positive dimensional withouth violating the condition that $Pic(S) = \mathbb{Z}H$.) This requires that $F \cap L$ has multiplicity at least three at some point of its support, something that for L generic occurs only in codimension at least four in $S^{[3]}$, and would give rise to one-dimensional fibers. So we would get a component of I of dimension at most three, and by Proposition 25 this is not possible. We conclude:

Proposition 27. *I is contained in $J = \phi^{-1}_{|H_3-2\delta|}(\mathbb{P}(V_7^\perp))$, the locus parametrizing length-three subschemes Z of S that only generate a $V_5 \subset V_7$.*

On the other hand, we know that I is contracted in N to a two-dimensional variety, birationally equivalent to Σ, and that $N \xrightarrow{\nu} \mathbb{P}^9$ is finite on its image, since it is the second factor in the Stein decomposition of $\phi_{|H_3-2\delta|}$. So necessarily $\phi_{|H_3-2\delta|}(I) = \mathbb{P}(V_7^\perp)$.

Unfortunately, at this point we are not able to conclude that $I = J$. Note that $J \setminus I$ would be mapped to $\mathbb{P}(V_7^\perp)$ with finite fibers, each point of which would be a connected component of the corresponding fiber of $\phi_{|H_3-2\delta|}$. Such a fiber would then have more than two connected components, and since the degree is two this would mean that $\phi_{|H_3-2\delta|}(S^{[3]})$ cannot be normal, which we do not know (see the discussion after Corollary 24).

Proposition 27 urges the need to understand length-3 subschemes of the adjoint variety of G_2, that generate a $V_5 \subset V_7$. The main result of this section will be that these subschemes can be constructed from certain special codimension two subspaces of V_7. The proof that $I = J$ will only come in Proposition 43.

5.2. **Hyperplanes and the three-form.** Let us start by classifying hyperplanes. Taking their orthogonals with respect to the invariant quadratic form, this is equivalent to classifying lines, so there are only two types, that we will call (with some abuse of terminology) isotropic and non isotropic hyperplanes, respectively.

What about the restriction of the invariant-three form ω to these hyperplanes? Recall that non zero three-forms in six variables have only four orbits, with the following normal forms (for $e_1, \dots, e_6$ a basis of linear forms):

$$\begin{array}{ll} \text{Type } 0: & e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6, \\ \text{Type } 1: & e_1 \wedge e_2 \wedge e_4 + e_2 \wedge e_3 \wedge e_5 + e_3 \wedge e_1 \wedge e_6, \\ \text{Type } 2: & e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_4 \wedge e_5, \\ \text{Type } 3: & e_1 \wedge e_2 \wedge e_3. \end{array}$$

In type 0, the two planes $\langle e_1, e_2, e_3\rangle$ and $\langle e_4, e_5, e_6\rangle$ are uniquely defined by the three-form (this is the *one apparent double point property* of the Grassmannian $G(3,6)$). In type 1 only the plane $\langle e_1, e_2, e_3\rangle$ is canonically attached to the three-from, which defines a tangent vector to $G(3,6)$ at the corresponding point. By [24, Lemma 3.6], $\omega_{|V_6}$ has type 0 when $V_6 \subset V_7$ is a non isotropic hyperplane, and $\omega_{|V_6}$ has type 1 when V_6 is isotropic.

5.3. **Codimension two subspaces.** Slightly more difficult than the classification of hyperplanes, we proceed to the classification of codimension two subspaces (or equivalently, of dimension two) up to the action of G_2. Such a plane defines a line in $\mathbb{P}(V_7)$, which can be contained in $\mathbb{Q}^5$, or a tangent line, or a bisecant line. We split the classification in the two following statements.

Proposition 28. *Consider two points $x \neq y$ on the quadric $\mathbb{Q}^5 \subset \mathbb{P}(V_7)$. Up to the action of G_2, there are three possible relative positions.*

1. *They are joined by a special line in the quadric, which means that they span a null plane.*
2. *They are joined by a non special line in the quadric, which means that they span an isotropic plane which is not a null plane.*
3. *They are not joined by a line in the quadric, which means that they span a non degenerate plane.*

Representatives of the three cases are $(e_\alpha, e_{-\beta}), (e_\alpha, e_\beta), (e_\alpha, e_{-\alpha})$.

Proof. Suppose that the line $\overline{xy}$ is not contained in the quadric. This means that the plane $P \subset V_7$ generated by x and y is non degenerate. So it suffices to show that G_2 acts transitively on such planes. Let us prove the stronger statement that G_2 acts transitively on pairs (P, z), with P non degenerate and $z \subset P$ a non isotropic line. Recall that the stabilizer of z in G_2 is a copy of SL_3, whose action on V_7 decomposes as $z \oplus V_3 \oplus V_3^\vee$. So we are reduced to proving that SL_3 acts transitively on the invariant quadric in $\mathbb{P}(V_3 \oplus V_3^\vee)$, which is clear.

The remaining claims follow from the general fact that there are at most two G-orbits of lines in any generalized Grassmannian G/P, P being a maximal parabolic subgroup of a simple complex Lie group G [26, Theorem 4.3]. ∎

Proposition 29. *If $x \in \mathbb{Q}^5$ and $\overline{xy}$ is a tangent line not contained in the quadric, then up to the action of G_2 there are two possibilities.*

(1) *The linear form $\omega(x, y, \bullet)$ is a non zero multiple of $q(x, \bullet)$.*
(2) *These two linear forms are not proportional.*

Representatives of the two cases are (e_α, e_0) and $(e_\alpha, e_\beta + e_{-\beta})$. The first case is easier to express in the language of octonions: it means that the plane $\langle x, y\rangle$ contains a unique isotropic line, generated by x such that the octonionic product xy is a (non zero) multiple of x. We call such a plane a *semi-null* plane. Alternatively, we may observe that (e_α, e_0) generate a Lie subalgebra of $\mathfrak{g}_2$ (whose derived algebra is generated by e_α); this is the point of view chosen in [15, Appendix A].

In the second case xy is still non zero, but does not belong to $\langle x, y\rangle$.

Proof. We may suppose that x is the line generated by e_α, in which case the sub-Lie algebra of $\mathfrak{g}_2$ stabilizing x is

$$\mathfrak{p} = \langle \mathfrak{t}, X_{\beta-\gamma}, X_{\gamma-\beta}, X_\alpha, X_{-\beta}, X_{-\gamma}, X_{\alpha-\beta}, X_{\alpha-\gamma}\rangle.$$

The first three terms generate a copy of $\mathfrak{gl}_2$ which is the Levi part of this parabolic subalgebra. The remaining terms generate the nilpotent part. One checks that there is a minimal filtration of $e_\alpha^\perp/\mathbb{C}e_\alpha$ preserved by $\mathfrak{p}$, which is

$$\langle e_\beta, e_\gamma\rangle \longrightarrow \langle e_0\rangle \longrightarrow \langle e_{-\beta}, e_{-\gamma}\rangle.$$

Vectors in $\langle e_{-\beta}, e_{-\gamma}\rangle$ generate with e_α special lines contained in the quadric, so we can discard them. Consider a vector of the form $e_0 + ae_{-\beta} + be_{-\gamma}$. It belongs to the P-orbit of e_0, and we are in the first case of the Proposition. Consider then a vector of the form $u + ze_0 + v$. Up to the action of P we may suppose that $u = e_\beta$. Using the action of P we can let $z = 0$. Then we remain with the action of GL_2 on $V_2 \oplus V_2^*$, whose only covariant is the evaluation map. This means that we may suppose that v is orthogonal to e_β, in which case $u + v$ is isotropic and we are back to a previous case, or it is not, and we are in the second case of the Proposition. □

This finally yields a complete classification of the G_2-orbits in $G(2, V_7)$. Define a *rank one plane* to be a plane in V_7 on which the restriction of the invariant quadratic form Q has rank one.

Corollary 30. *Up to the action of G_2, a plane $V_2 \subset V_7$ can be:*

(1) *a null-plane,*
(2) *an isotropic plane which is not a null-plane,*
(3) *a semi-null plane,*
(4) *a rank one plane which is not semi-null,*
(5) *a non-degenerate plane.*

Explicit representatives are $\langle e_\alpha, e_{-\beta}\rangle$, $\langle e_\alpha, e_\beta\rangle$, $\langle e_0, e_\alpha\rangle$, $\langle e_0, e_\alpha + e_\beta\rangle$, $\langle e_\alpha, e_{-\alpha}\rangle$. Each case defines a unique orbit of the G_2-action on $G(2, V_7)$, and the incidence diagram is

$$
\begin{tikzcd}
 & & \mathcal{O}^7_{semi} \arrow[dr] \arrow[dd, dotted, leftrightarrow] & \\
\mathcal{O}^{10}_{gen} \arrow[r] & \mathcal{O}^9_{rank1} \arrow[ur] \arrow[dr] & & \mathcal{O}^5_{null} \\
 & & \mathcal{O}^7_{iso} \arrow[ur] &
\end{tikzcd}
$$

Here the exponents are the dimensions of the orbits. We will see in the next subsection that $\mathcal{O}_{semi}$ and $\mathcal{O}_{iso}$ are projectively equivalent. Considering the Zariski closures of these orbits we get the diagram

$$
\begin{tikzcd}
 & & LieGr(2,V_7) \arrow[dl, hook] \arrow[dd, dotted, leftrightarrow] & \\
G(2,V_7) & \mathcal{C}_3 \arrow[l, hook'] & & X_{ad}(\mathfrak{g}_2) \arrow[ul, hook] \arrow[dl, hook] \\
 & & OG(2,V_7) \arrow[ul, hook] &
\end{tikzcd}
$$

where $\mathcal{C}_3$ is a cubic hypersurface (indeed the restriction of the invariant quadratic forms to planes yields a section of $S^2\mathcal{U}^\vee$, and the condition that this restriction is degenerate gives a section of $\det(S^2\mathcal{U}^\vee) = \mathcal{O}_{G(2,V_7)}(3)$). The notation $LieGr(2, V_7)$ is taken from [15, Appendix A].

5.4. **Decomposing five-planes.** When we restrict the invariant three-form to a codimension two subspace $V_5 \subset V_7$, since a three-form in five variables is the same as a two-form in five variables, there are three possibilities: we could get zero, a form of rank two, or a form of rank four. The first case is actually impossible, because if $\omega_{|V_5} = 0$, then the restriction of ω to any hyperplane $V_6 \supset V_5$ must be of type 2 or 3, and we know that this cannot happen.

Definition. A subspace $V_5 \subset V_7$ is *decomposing* if $\omega_{|V_5}$ has rank two. Equivalently, there exists linear forms e_1, e_2, e_3 such that $\omega_{|V_5} = e_1 \wedge e_2 \wedge e_3$. So if $V_5 = \langle e_6, e_7\rangle^\perp$, there exists two-forms θ_6 and θ_7 such that

$$\omega = e_1 \wedge e_2 \wedge e_3 + e_6 \wedge \theta_6 + e_7 \wedge \theta_7.$$

Note in particular that the five linear forms $\langle e_1, e_2, e_3, e_6, e_7\rangle$ define a plane $A_2 \subset V_5$, that we call the *axis* of the decomposing plane.

It is straightforward to translate this property in terms of our previous classification of planes. Since being decomposing is a G_2-invariant property, it must hold when $V_2 = V_5^\perp$ belongs to a certain union of G_2-orbits in $G(2, V_7)$. An explicit check yields the following conclusion.

Proposition 31. *A five-plane V_5 is decomposing if and only if $V_2 = V_5^\perp$ is a null or a semi-null plane. Its axis A_2 is an isotropic plane, which coincides with V_2 if and only if it is a null-plane.*

From the axis A_2, it is actually easy to reconstruct V_5. The correspondence between A_2 and V_2 induces an isomorphism between $LieGr(2, V_7)$ and $OG(2, V_7)$, as noticed in [15, Proposition A.7]. This isomorphism is actually linear and can be described as follows. Recall the decomposition into irreducible components

$$\wedge^2 V_7 = \mathfrak{g}_2 \oplus V_7,$$

where $V_7 \simeq V_7^\vee$ embeds into $\wedge^2 V_7$ by contraction with the (dual) invariant three-form, and $\wedge^2 V_7$ maps to V_7 by a similar contraction, whose kernel is $\mathfrak{g}_2$. Denote by ι the involution of $\wedge^2 V_7$ acting by $+1$ on $\mathfrak{g}_2$ and by -1 on V_7. Starting from the Plücker representative $e_\alpha \wedge e_\beta$ of the general isotropic plane A_2 as above, we first compute its image in V_7 to be $\omega(e_\alpha, e_\beta) = e_{-\gamma}$. Then we identify this vector with the two-form $e_{-\gamma}.\omega = e_0 \wedge e_{-\gamma} - e_\alpha \wedge e_\beta$. As a consequence, if we apply the symmetry ι of $\wedge^2 V_7$ defined by its decomposition into $\mathfrak{g}_2$-modules, we see that

$$\iota(e_\alpha \wedge e_\beta) = e_0 \wedge e_{-\gamma}.$$

This is nothing else than the Plücker representative of $V_5^\perp$, where V_5 is the unique decomposing five plane containing A_2, that we computed above.

Corollary 32. $LieGr(2, V_7) = \iota(OG(2, V_7))$.

Now we turn to the property that makes decomposing five-planes relevant for our problems.

Proposition 33. *Let V_5 be a decomposing five-plane.*

(1) *The kernel of the map $\omega : \wedge^2 V_5 \to V_7^\vee$ is a five-plane K_5.*
(2) *If $V_5^\perp$ is not a null-plane, the intersection of $X_{ad}(G_2)$ with $\mathbb{P}(K_5)$ is a cubic scroll.*

Proof. For $\psi \in \wedge^2 V_5$, the contraction with ω yields

$$\iota_\psi(\omega) = \iota_\psi(e_1 \wedge e_2 \wedge e_3) + \iota_\psi(\theta_6)e_6 + \iota_\psi(\theta_7)e_7,$$

showing that the image of $\omega : \wedge^2 V_5 \to V_7^\vee$ is $V_2^\perp$. So the kernel K_5 is five dimensional.

Since K_5 is contained in $\mathfrak{g}_2$, the intersection of $X_{ad}(G_2)$ with $\mathbb{P}(K_5)$ is the set of planes $P = \langle p_1, p_2 \rangle$ such that $\psi = p_1 \wedge p_2$ belongs to K_5 . Concretely, this means that $\iota_\psi(\theta_6) = \iota_\psi(\theta_7) = 0$, and that $\iota_\psi(e_1 \wedge e_2 \wedge e_3)$ also vanishes. This latter condition precisely means that P meets V_2 non trivially. Given a line $\ell \subset V_2$, θ_6 and θ_7 cut the four-dimensional space $\ell \wedge V_5$ along a plane, that does not contain ψ (for V_5 generic at least). This implies that $\Sigma(V_5) := X_{ad}(G_2) \cap \mathbb{P}(K_5)$ is a surface scroll. In order to compute its degree, we cut Σ with two extra hyperplanes, whose equations can be expressed as the restriction to $K_5 \subset \wedge^2 V_5$ of two extra linear forms θ_8 and θ_9. We get the set of planes $\langle p,q\rangle$, where $p\in V_2$ and $\theta_i(p,q)=0$ for $i=6...9$. For a given line $\langle p\rangle$, q varies in $V_5/\langle p\rangle$ and we need the four linear forms $\theta_i(p,\bullet)$ on this space to be linearly independent. This is a cubic condition, and we are done. □

We can compute the intersection $\Sigma(V_5)=G(2,V_5)\cap\mathbb{P}(\mathfrak{g}_2)$ for each type of five planes. The result is presented in the table below.

$V_2=V_5^\perp$	Representative	$\Sigma(V_5)$
non degenerate	$\langle e_\alpha, e_{-\alpha}\rangle$	smooth conic
rank one	$\langle e_0, e_\alpha+e_\beta\rangle$	reduced singular conic
isotropic	$\langle e_\alpha, e_\beta\rangle$	double line
seminull	$\langle e_0, e_\alpha\rangle$	smooth cubic scroll
null	$\langle e_\alpha, e_{-\beta}\rangle$	cone over a rational cubic

An important consequence is the following. We say that a scheme $Z\subset G(2,V_7)$ generates the linear space $W\subset V_7$ if the linear span of Z is contained in $\mathbb{P}(\wedge^2 W)$, and W is minimal for this property.

Proposition 34. *Let $Z\subset S$ be a length three subscheme that does not generate a hyperplane. Then it generates a decomposing codimension two subspace of V_7.*

Proof. First observe that Z cannot generate a V_4, because its linear span would intersect $G(2,V_4)$ along a conic, that would be contained in S.

If Z generates a V_5, it must be contained inside $G(2,V_5)\cap X_{ad}(\mathfrak{g}_2)$. If V_5 is not decomposing, then $V_2=V_5^\perp$ is either non degenerate, rank one or isotropic, and by the previous table $G(2,V_5)\cap X_{ad}(\mathfrak{g}_2)$ is a conic, so its intersection with L is either a line, a conic or a finite scheme of length two. But since it contains Z, the latter case is impossible and S must contain a line or a conic, a contradiction. □

5.5. **Vector bundles interpretations.** A consequence of Corollary 32 is that $OG(2,V_7)$ has two distinct G_2-equivariant embeddings inside $G(2,V_7)$, the natural one and its twist by ι. To each point A_2 of $OG(2,V_7)$, we have associated a decomposing five-plane V_5 and its orthogonal $V_2=V_5^\perp$, and then the rank five kernel K_5 of the contraction $\wedge^2V_5\to V_7^\vee\simeq V_7$.

Definition. We denote by $\mathcal{E}_2$ and $\mathcal{F}_2$ the corresponding tautological vector bundles on $OG(2,V_7)$. The rank two bundles $\mathcal{E}_2$ and $\mathcal{F}_2$ are subbundles of the trivial bundle with fiber V_7, and they coincide on $X_{ad}(G_2)$. We also let $\mathcal{V}_5=\mathcal{F}_2^\perp$ which contains $\mathcal{E}_2$ as a subbundle. Note that the quadratic form on V_7 descends to a non degenerate quadratic form on the quotient $\mathcal{V}_5/\mathcal{F}_2$, which is in particular self-dual. Finally, the morphism $\wedge^2\mathcal{V}_5\to V_7^\vee\otimes\mathcal{O}_{OG(2,V_7)}$ has constant rank by Proposition 33, its kernel is thus a rank five vector bundle that we denote $\mathcal{K}_5$. All these bundles are G_2-equivariant.

By construction there is a commutative diagram

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & \wedge^2(\mathcal{V}_5/\mathcal{E}_2) \arrow[u] \arrow[r, equal] & \wedge^2(\mathcal{V}_5/\mathcal{E}_2) \arrow[u] & \\
0 \arrow[r] & \mathcal{K}_5 \arrow[r] & \wedge^2\mathcal{V}_5 \arrow[u] \arrow[r] & \mathcal{E}_2^\perp \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{K}_5 \arrow[u, equal] \arrow[r] & \mathcal{E}_2 \wedge \mathcal{V}_5 \arrow[u] \arrow[r] & \mathcal{F}_2 \arrow[u] \arrow[r] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

We deduce that the determinants of our bundles are

$$\det(\mathcal{E}_2) = \det(\mathcal{F}_2) = \det(\mathcal{V}_5) = \mathcal{O}_{OG(2,V_7)}(-1), \quad \det(\mathcal{K}_5) = \mathcal{O}_{OG(2,V_7)}(-3).$$

Recall that the adjoint variety $X_{ad}(G_2)$ is a *contact manifold*, which means that it admits a contact distribution, that is a corank one subbundle $\mathcal{H}$ of the tangent bundle which is maximally non integrable. Exactly as there is also an affine tangent bundle $\hat{\mathcal{T}}$ for the adjoint variety in its minimal embedding, there is an affine contact bundle $\hat{\mathcal{H}}$, fitting in the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{O}(-1) \arrow[d] \arrow[r, equal] & \mathcal{O}(-1) \arrow[d] & & \\
0 \arrow[r] & \hat{\mathcal{H}} \arrow[d] \arrow[r] & \hat{\mathcal{T}} \arrow[d] \arrow[r] & \mathcal{O} \arrow[d, equal] \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{H}(-1) \arrow[d] \arrow[r] & \mathcal{T}(-1) \arrow[d] \arrow[r] & \mathcal{O} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Proposition 35. *On the closed orbit $X_{ad}(G_2) \subset OG(2, V_7)$, the vector bundle $\mathcal{K}_5$ restricts to the affine contact bundle.*

Proof. Recall that $\mathcal{K}_5$ is a subbundle of $\mathcal{E}_2 \wedge \mathcal{V}_5$. Denote by $\mathcal{L}_5$ its restriction to $X_{ad}(G_2)$. By the definition of null-planes, $\mathcal{L}_5$ contains $\wedge^2\mathcal{E}_2$ as a subbundle of rank one. We get an exact sequence

$$0 \longrightarrow \mathcal{L}_5/\wedge^2\mathcal{E}_2 \longrightarrow \mathcal{E}_2 \otimes \mathcal{V}_5/\mathcal{E}_2 \longrightarrow \mathcal{E}_2 \longrightarrow 0.$$

We can complete this sequence into a big commutative diagram as follows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{L}_5/\wedge^2\mathcal{E}_2 \arrow[r] \arrow[d] & \mathcal{E}_2\otimes\mathcal{V}_5/\mathcal{E}_2 \arrow[r] \arrow[d] & \mathcal{E}_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{P} \arrow[r] \arrow[d] & \mathcal{E}_2\otimes V_7/\mathcal{E}_2 \arrow[r] \arrow[d] & \mathcal{V}_5 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d] & \mathcal{E}_2\otimes\mathcal{E}_2^\vee \arrow[r] \arrow[d] & \mathcal{V}_5/\mathcal{E}_2 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where we have used the fact that on the adjoint variety, $\mathcal{V}_5 = \mathcal{E}_2^\perp$ and therefore $V_7/\mathcal{V}_5$ is naturally identified with $\mathcal{E}_2^\vee$. For the same reason $\mathcal{V}_5$ is identified with $(V_7/\mathcal{E}_2)^\vee$, the dual of the quotient bundle $\mathcal{Q}$. But recall that inside $G(2,V_7)$, the adjoint variety can be defined as the zero locus of the (generic) section of $\mathcal{Q}^\vee(1)$ defined by the invariant three-form. We conclude that the middle horizontal exact sequence of the diagram above is the twist by $\mathcal{O}(-1)$ of the normal exact sequence of the embedding of $X_{ad}(G_2)$ inside $G(2,V_7)$. In particular, the bundle $\mathcal{P}$ is nothing else than the twisted tangent bundle $\mathcal{T}(-1)$. Thus $\mathcal{L}_5/\wedge^2\mathcal{E}_2 = \mathcal{H}(-1)$, and therefore $\mathcal{L}_5 = \hat{\mathcal{H}}$. $\square$

Remark. The previous observations already appear in [23, Lemma 9.4], although the point of view there is a bit different.

6. A story with two planes

Our next task will be to describe the locus $I \subset S^{[3]}$ parametrizing length three subschemes in S that do not span a hyperplane, which is the indeterminacy locus of the direct sum map $S^{[3]} \dashrightarrow \mathbb{P}(V_7^\vee)$. By Proposition 27 this is also the exceptional locus of $\phi_{|H_3-2\delta|}$. Our description will be closely related with Homological Projective Duality for the G_2-Grassmannian, as developped in [23].

6.1. **The two planes.** Note that $\mathcal{K}_5$ is by construction a subbundle of the trivial bundle with fiber $\mathfrak{g}_2$. In particular its dual is globally generated. Let us first determine its space of global sections.

Proposition 36. $H^0(OG(2,V_7),\mathcal{K}_5^\vee) \simeq \mathfrak{g}_2$.

Proof. We will use the exact sequence

$$0 \longrightarrow \mathcal{F}_2^\vee \longrightarrow (\mathcal{E}_2\wedge\mathcal{V}_5)^\vee \longrightarrow \mathcal{K}_5^\vee \longrightarrow 0,$$

whose dual appeared in section 6.4. As observed in [15, Remark A.8], $\mathcal{F}_2^\vee$ is the spinor bundle on $OG(2,V_7)$, an irreducible homogeneous vector bundle.

The Bott-Borel-Weil theorem implies that its higher cohomology vanishes, while its space of global sections is isomorphic with the spin representation Δ_8 of $\mathfrak{so}_7$. Note that as a $\mathfrak{g}_2$-module, $\Delta_8 \simeq \mathbb{C} \oplus V_7$.

Now we consider $(\mathcal{E}_2 \wedge \mathcal{V}_5)^\vee$, and the exact sequence

$$0 \longrightarrow \mathcal{E}_2^\vee \otimes (V_7/\mathcal{E}_2)^\vee \longrightarrow (\mathcal{E}_2 \wedge \mathcal{V}_5)^\vee \longrightarrow \wedge^2 \mathcal{E}_2^\vee \longrightarrow 0.$$

Note that $\wedge^2 \mathcal{E}_2^\vee$ is the Plücker line bundle; it has no higher cohomology on $OG(2, V_7)$ and its space of global sections is $\wedge^2 V_7^\vee$. Also $\mathcal{E}_2^\vee \otimes (V_7/\mathcal{E}_2)^\vee$ has a canonical section given by the morphism $V_7/\mathcal{E}_2 \rightarrow V_7/\mathcal{V}_5 \simeq \mathcal{E}_2^\vee$. Using the fact that this bundle is the restriction of an irreducible homogeneous vector bundle on the full Grassmannian $G(2, V_7)$, it is easy to check that this gives the full cohomology, namely $H^q(OG(2, V_7), \mathcal{E}_2^\vee \otimes (V_7/\mathcal{E}_2)^\vee) = \delta_{q,0}\mathbb{C}$. Putting everything together we get an exact sequence

$$0 \longrightarrow \Delta_8 \longrightarrow \mathbb{C} \oplus \wedge^2 V_7 \longrightarrow H^0(OG(2, V_7), \mathcal{K}_5^\vee) \longrightarrow 0,$$

and the result follows. □

Proposition 37. *The zero locus of a general section of $\mathcal{K}_5^\vee$ is the union of two anticanonically embedded projective planes.*

Proof. Since $\omega_{OG(2,V_7)} = \mathcal{O}_{OG(2,V_7)}(-4)$ and $\det(\mathcal{K}_5) = \mathcal{O}_{OG(2,V_7)}(-3)$, the zero locus of a general section of $\mathcal{K}_5^\vee$ must be a smooth anticanonically embedded del Pezzo surface Z.

We may suppose that our section is defined by a general element of $\mathfrak{g}_2$, which we may choose to be a general element h in the Cartan subalgebra $\mathfrak{t}$. Then we can compute explicitely the number of fixed points in $Z = Z_h$ of the corresponding maximal torus T, as follows. We are looking for isotropic planes A_2, fixed by the torus, such that the corresponding K_5 is orthogonal to h. There exists 12 isotropic planes fixed by the torus: 6 of type $\langle e_\alpha, e_{-\beta}\rangle$, which are null planes, and 6 of type $\langle e_\alpha, e_\beta\rangle$, which are not. The first plane $A_2 = \langle e_\alpha, e_{-\beta}\rangle$ is the axis of $V_5 = \langle e_\alpha, e_{-\beta}, e_0, e_\gamma, e_{-\gamma}\rangle$, in which case

$$K_5 = \langle e_\alpha \wedge e_{-\beta}, e_\alpha \wedge e_{-\gamma}, e_\gamma \wedge e_{-\beta}, e_0 \wedge e_\alpha - e_{-\beta} \wedge e_{-\gamma}, e_0 \wedge e_{-\beta}, e_\alpha \wedge e_\gamma\rangle$$

is orthogonal to the whole Cartan subalgebra, hence to h in particular. The second plane $A_2 = \langle e_\alpha, e_\beta\rangle$ is the axis of $V_5 = \langle e_\alpha, e_{-\alpha}, e_\beta, e_{-\beta}, e_{-\gamma}\rangle$, in which case $\mathcal{K}_5$ contains $e_\alpha \wedge e_{-\alpha} - e_\beta \wedge e_{-\beta}$ which is not orthogonal to h.

So finally we find exactly 6 fixed points, which are all null planes. As a consequence of the Byalinicki-Birula decomposition [6], this implies that the topological Euler characteristic $\chi(Z_h) = 6$. This leaves only two possibilities: either Z_h is a del Pezzo surface of degree 6, or it is the disjoint union of two planes. We exclude the first case by observing that the difference with the second case is the existence of lines.

Claim. For h generic, Z_h contains no line.

By contradiction, consider a line $d \subset OG(2, V_7)$. Recall that a line in $G(2, V_7)$ is the set of planes A such that $L_1 \subset A \subset L_3$ for some fixed line L_1

and three-plane L_3; it is contained in $OG(2,V_7)$ when L_3 is isotropic. As a consequence d moves in an eight-dimensional family.

To each point x of d corresponds an isotropic plane $A_2(x)$ which is the axis of a unique decomposing five-plane $V_5(x)$, to which we associate $K_5(x) \subset \mathfrak{g}_2$. Then x belongs to $p_2^{-1}([h])$ if and only if $K_5(x) \subset h$. So let $K(d)$ be the linear span of the five-planes $K_5(x)$ for $x \in d$. If its dimension is δ, then $[h]$ must belong to a $\mathbb{P}^{13-\delta}$. Since d moves in an eight-dimensional family, it would be enough to check that $\delta \geq 8$ for any line d. By semi-continuity, it is enough to check this for the most special lines.

What are these lines? The most special isotropic three-planes are those that contain null-planes, in which case they contain a pencil of null-planes, meeting along one line which is the most special line in the three-plane. In other words, we may suppose that $L_1 = \langle e_\alpha \rangle$ and $L_3 = \langle e_\alpha, e_{-\beta}, e_{-\gamma}\rangle$. The computation goes as follows. Suppose $A_2(x) = \langle e_\alpha, ue_{-\beta} + ve_{-\gamma}\rangle$. Since this is a null-plane, we know that the corresponding $V_5(x) = A_2(x)^\perp = \langle e_0, e_\alpha, e_{-\beta}, e_{-\gamma}, ve_\beta - ue_\gamma\rangle$. Then we get

$$\begin{aligned} K_5(x) = \langle e_{\alpha,-\beta}, e_{\alpha,-\gamma}, e_{0,\alpha} - e_{-\beta,-\gamma}, x(e_{\alpha,\gamma} - e_{0,-\beta}) - y(e_{\alpha,\beta} + e_{0,-\gamma}),\\ y^2 e_{\beta,-\gamma} - x^2 e_{\gamma,-\beta} + xy(e_{\beta,-\beta} - e_{\gamma,-\gamma})\rangle. \end{aligned}$$

So $K(d)$ has dimension eight. This concludes the proof. $\square$

6.2. **The Grothendieck-Springer simultaneous resolution.** Consider the incidence variety

$$\mathfrak{J}^{OG} := \Big\{(x,s) \in OG(2,V_7) \times \mathbb{P}H^0(\mathcal{K}_5^\vee),\ s(x) = 0\Big\},$$

a smooth $\mathbb{P}^8$-bundle over $OG(2,V_7)$. The Stein factorization of the second projection yields

$$\begin{tikzcd} \mathfrak{J}^{OG} \arrow[rr, "p_2"] \arrow[d, "p_1"] \arrow[dr, "\mathbb{P}^2"] & & \mathbb{P}H^0(\mathcal{K}_5^\vee) \simeq \mathbb{P}\mathfrak{g}_2 \\ OG(2,V_7) & \overline{\mathfrak{J}}^{OG} \arrow[ur, "2:1"'] & \end{tikzcd}$$

where it follows from Proposition 37 that $\mathfrak{J}^{OG} \to \overline{\mathfrak{J}}^{OG}$ is generically a $\mathbb{P}^2$-bundle, and $\overline{\mathfrak{J}}^{OG} \to \mathbb{P}(\mathfrak{g}_2)$ is a double cover.

In fact this double cover already appears in [23, Theorem 9.9], where it is shown that it provides (at least outside some closed subset, and up to a twist by a Brauer class) the Homological Projective Dual to the adjoint variety of G_2. Denote by $\mathfrak{J}^{G_2}$ the restriction of $\mathfrak{J}^{OG}$ to $X_{ad}(\mathfrak{g}_2) \subset OG(2,V_7)$.

Proposition 38. *$\mathfrak{J}^{G_2}$ is the projectivisation of the vector bundle $G_2 \times_{P_2} \mathfrak{p}_2$ over $G_2/P_2 = X_{ad}(G_2)$.*

Proof. By definition, $\mathfrak{J}^{OG} \to OG(2,V_7)$ is the projective bundle $\mathbb{P}(\mathcal{K}_5^\perp)$. When we restrict to $X_{ad}(G_2)$, we know by Proposition 35 that $\mathcal{K}_5$ coincides with the affine contact bundle, and we need to check that the orthogonal bundle (inside the trivial bundle with fiber $\mathfrak{g}_2$) can be identified with the homogeneous bundle defined by the adjoint representation of P_2 on its Lie algebra, which is an easy exercise. $\square$

A classical result in Lie theory asserts that for $B \subset G$ a Borel subgroup of a simple Lie group, with Lie algebras $\mathfrak{b} \subset \mathfrak{g}$, the adjoint action of B on $\mathfrak{b}$ defines a homogeneous vector bundle $\tilde{\mathfrak{g}} := G \times_B \mathfrak{b}$ on G/B, with a natural projection to $\mathfrak{g}$ known as the *Grothendieck-Springer simultaneous resolution* [9, Chapter 3]. Although it extends the classical Springer resolution of the nilpotent cone, this is not a resolution stricto sensu, but a generically finite cover. Note that it restricts to an unramified Galois cover, with the Weyl group W of $\mathfrak{g}$ for Galois group, over the open subset of regular semisimple elements. In particular the branch locus is contained in the discriminant hypersurface, the complement of the set of regular semisimple elements.

There also exists a variant of this construction for a parabolic subgroup $P \subset G$, with Lie algebra $\mathfrak{p} \subset \mathfrak{g}$: the adjoint action of P on $\mathfrak{p}$ defines a homogeneous vector bundle $\tilde{\mathfrak{g}}_P := G \times_P \mathfrak{p}$ on G/P, and the natural projection to $\mathfrak{g}$ is a generically finite cover of degree equal to the cardinality of W/W_P [5, 2.1.4]. Supposing that $P \supset B$ we get a factorization of the usual Grothendieck-Springer simultaneous resolution:

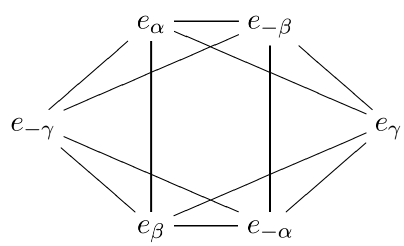
$$\begin{tikzcd}
G \times_B \mathfrak{b} \arrow[r] \arrow[d] & G \times_P \mathfrak{p} \arrow[r] \arrow[d] & \mathfrak{g} \\
G/B \arrow[r] & G/P &
\end{tikzcd}$$

Specializing these results to $P = P_2 \subset G = G_2$, and projectivizing, we deduce that $\mathfrak{J}^{G_2} \longrightarrow \mathbb{P}(\mathfrak{g}_2)$ is a generically $6:1$ cover that factorizes the Grothendieck-Springer simultaneous reolution. Let us make this statement more concrete.

Lemma 39. *Let $X \in \mathfrak{g}_2$ be regular semisimple.*

(1) *There exists exactly* 12 *isotropic planes in V_7 stabilized by X, among which* 6 *are null planes.*
(2) *The* 6 *null-planes define in $\mathbb{P}(\mathfrak{g}_2)$ the six sides of a hexagon. There are only two triples of these null planes that are linearly independent.*
(3) *The* 6 *other isotropic planes define in $\mathbb{P}(\mathfrak{g}_2)$ the sides of two triangles, inscribed in the hexagon.*

Proof. We may suppose that X belongs to our standard Cartan subalgebra $\mathfrak{t}$, and then a plane stabilized by X must be generated by two eigenvectors of the action of X on V_7. Among those, there are 6 null planes of type $\langle e_\alpha, e_{-\beta}\rangle$ (up to permutation of α, β, γ), and 6 other isotropic planes of type $\langle e_\alpha, e_\beta\rangle$. $\square$



Because G_2 is not simply laced, the discriminant hypersurface Δ is the union of two irreducible sextics Δ_6 and Δ_6' that can be characterized as follows: $X \in \mathfrak{t}$ belongs to Δ_6 if a long root vanishes on X, and to Δ_6' if a short root vanishes on X. (But beware that the general elements of Δ_6 and Δ_6' are not semisimple.) Moreover, according to [34], the sextic Δ_6 is the projective dual variety of $X_{ad}(\mathfrak{g}_2)$.

Lemma 40. *Let $X \in \mathfrak{g}_2$ belong to $\Delta_6' \backslash \Delta_6$.*

(1) *If X is not semisimple, it stabilizes exactly 4 null planes.*
(2) *If X is semisimple, the variety of null planes stabilized by X is the disjoint union of two points and two rational normal cubics.*

Proof. We start with the case where X is semisimple, so we may suppose that X belongs to our fixed Cartan subalgebra, and that no long root, but at least one short root vanishes on X. Up to the action of the Weyl group, we may suppose this short root is γ, so that α and $-\beta$ take the same value on X. Then the action of X on V_7 has three eigenspaces $A_0 = \langle e_0, e_\gamma, e_{-\gamma}\rangle$, $A_+ = \langle e_\alpha, e_{-\beta}\rangle$, $A_- = \langle e_{-\alpha}, e_\beta\rangle$. A null plane fixed by X must be generated by two eigenvectors, and a straightforward computation shows that apart from A_+ and A_-, we can get planes of the form $\langle se_\alpha + te_{-\beta}, ste_0 + s^2 e_\gamma - t^2 e_{-\gamma}\rangle$ or $\langle se_{-\alpha} + te_\beta, ste_0 + s^2 e_\gamma - t^2 e_{-\gamma}\rangle$, for $[s,t] \in \mathbb{P}^1$. Hence the two rational cubics.

If X is not semisimple, we may suppose that its semisimple part X_s is as before, and then its nilpotent part can be supposed to be X_γ. Since they commute, a null plane stabilized by X must be stabilized by both X_s and X_γ. One readily checks that among the planes parametrized by the two cubic rational normal curves, only $\langle e_{-\beta}, e_\gamma\rangle$ and $\langle e_{-\alpha}, e_\gamma\rangle$ are stabilized (in fact, annihilated) by X_γ. $\square$

Lemma 41. *Let $X \in \mathfrak{g}_2$ belong to $\Delta_6 \backslash \Delta_6'$.*

(1) *If X is not semisimple, it stabilizes exactly 3 null planes.*
(2) *If X is semisimple, the variety of null planes stabilized by X is the disjoint union of a conic and two lines.*

Proof. As for the previous lemma we start with the case where X is semisimple, so we may suppose that X belongs to our fixed Cartan subalgebra, and that no short root, but at least one long root vanishes on X. Up to the action of the Weyl group, we may suppose this short root is $\alpha - \beta$, so that

α and β take the same value on X, half that of $-\gamma$. Then the action of X on V_7 has five eigenspaces $A_0 = \langle e_0\rangle$, $B_+ = \langle e_\alpha, e_\beta\rangle$, $B_- = \langle e_{-\alpha}, e_{-\beta}\rangle$, $C_+ = \langle e_\gamma\rangle$ and $C_- = \langle e_{-\gamma}\rangle$. A straightforward computation shows that the null planes stabilized by X are parametrized by a conic in $\mathbb{P}(B_+)\times\mathbb{P}(B_-)$ and the two lines $\mathbb{P}(B_+)\times\mathbb{P}(C_-)$ and $\mathbb{P}(B_-)\times\mathbb{P}(C_+)$.

When X is not semisimple we may suppose that its semisimple part X_s is as before, while its nilpotent part is $X_{\alpha-\beta}$. Then there is one fixed point of $X_{\alpha-\beta}$ in the conic and each of the two lines, namely $\langle e_\alpha, e_{-\beta}\rangle$, $\langle e_\beta, e_{-\gamma}\rangle$ and $\langle e_\gamma, e_{-\beta}\rangle$. □

7. Back to the indeterminacy locus : conclusions

7.1. **The branch locus.** We first draw the conclusions of the previous discussion on the morphism $\mathfrak{J}^{OG}\to\mathbb{P}(\mathfrak{g}_2)$. Restricting its Stein factorization to $\mathfrak{J}^{G_2}$ we get a diagram

$$\begin{tikzcd}
\mathfrak{J}^{OG} \arrow[r, "p_2"] \arrow[dr] & \mathbb{P}\mathfrak{g}_2 \\
\mathfrak{J}^{G_2} \arrow[u] \arrow[r] \arrow[ur] & \overline{\mathfrak{J}}^{OG} \arrow[u, "\overline{p_2}"', "2:1"]
\end{tikzcd}$$

which shows that the degree two morphism $\mathfrak{J}^{OG}\to\mathbb{P}\mathfrak{g}_2$ factorizes the Grothendieck-Springer resolution. Since we know the branch locus of the latter, we can get precise information on the branch locus of the former (compare with [23, Corollary 9.10]):

Proposition 42. *The ramification locus of the map $\overline{p_2} : \overline{\mathfrak{J}}^{OG}\to\mathbb{P}(\mathfrak{g}_2)$ is the projective dual of the adjoint variety $X_{ad}(G_2)$.*

Proof. We know that the Grothendieck-Springer resolution is ramified over the discriminant hypersurface $\Delta = \Delta_6 \cup \Delta_6'$. The branch locus of $\overline{p_2}$ must therefore be (the projectivization of) one of these irreducible sextics, or their union. So we just need to exclude Δ_6' according to which the preimage of a general point consists in four points. If $\overline{p_2}$ was ramified at such a point, we would then get four points in a fiber of a morphism which is generically finite of degree three: this is absurd. □

7.2. **The dual K3 surface.** Consider a decomposing five-plane $V_5 \subset V_7$. When we cut out the cubic scroll

$$\Sigma(V_5) = X_{ad}(G_2)\cap\mathbb{P}(K_5) = G(2, V_5)\cap\mathbb{P}\mathfrak{g}_2$$

with the codimension three linear space $L = \mathbb{P}(V_{11}) \subset \mathbb{P}\mathfrak{g}_2$ that defines the K3 surface S, we get the empty set in general, just for dimensional reasons. But we get three points in general when we suppose $L = \mathbb{P}(V_{11})$ to be non transverse to $\mathbb{P}(K_5)$. This happens when the restriction morphism $V_{11}^\perp\to K_5^\vee$ is not injective, hence in codimension three. So we get a four-dimensional family $D(S)$ of decomposing planes, which essentially parametrizes J.

We will at last be able to deduce that J does coincide with the indeterminacy locus I of $\phi_{|H_3-2\delta|}$.

Proposition 43. *$J = I$.*

Proof. We know by Proposition 34 that a scheme Z from $J \subset S^{[3]}$ defines a decomposing five-plane $V_5 \subset V_7$. Then Z is contained in $\Sigma(V_5)$, which is by the Table before the same Proposition, either a smooth cubic scroll or a cone over a twisted rational cubic. The linear span of $\Sigma(V_5)$ is $\mathbb{P}(K_5) \simeq \mathbb{P}^4$, and it cannot intersect L properly: indeed, if it was the case Z would be contained in a line, and since S is cut-out by quadrics this line would be contained in S, which is impossible. So V_5 belongs to the degeneracy locus $D(S)$ parametrizing the decomposing five planes for which $\mathbb{P}(K_5)$ meets L in codimension two.

Since $\mathcal{K}_5^\vee$ is generated by global sections, for L generic $D(S)$ has pure codimension three in $OG(2, V_7)$, and its singular locus is the next degeneracy locus, of codimension eight; so $D(S)$ is in fact smooth, and everywhere of dimension four. For V_5 in $D(S)$, the intersection $\Sigma(V_5) \cap L$ must have dimension zero, since otherwise, $\Sigma(V_5)$ being cut out by three quadrics, we would get in S a curve of degree at most eight. So this intersection is a length three scheme. We thus get a flat family of finite schemes over $D(S)$, inducing a morphism $D(S) \longrightarrow S^{[3]}$. By construction the image of this morphism is J, and it is injective since V_5 can easily be reconstructed from $Z = \Sigma(V_5) \cap L$.

Although we do not know whether J is normal (and then isomorphic to $D(S)$, thus smooth in particular), we can at least deduce that J has pure codimension four everywhere. In particular $\phi_{|H_3-2\delta|}$ must contract it to $\mathbb{P}(V_7^\vee)$ with non trivial fibers at every point, which exactly means that $J \subset I$. Since we already know from Proposition 27 that $I \subset J$, this concludes the proof. $\square$

With the notation $L = \mathbb{P}(V_{11}) \subset \mathbb{P}\mathfrak{g}_2$, the fact that for $V_5 \in D(S)$, the projective four-space $\mathbb{P}(K_5)$ meets L in codimension two rather than three means that the natural morphism $V_{11}^\perp \to K_5^\vee$ has a one dimensional kernel. This yields a morphism $D(S) \to \mathbb{P}(V_{11}^\perp) \simeq \mathbb{P}^2$. What are the fibers? A point in $\mathbb{P}(V_{11}^\perp)$ is nothing else than a hyperplane H in $\mathfrak{g}_2$ containing V_{11}, and we look at the locus where $K_5 \subset H$. But this is exactly the zero locus of the section of $\mathcal{K}_5^\vee$ defined by an equation of H, hence in general the union of two projective planes, by Proposition 37. So we have recovered the birational structure of I obtained in Proposition 25, as a $\mathbb{P}^2$-bundle over a surface.

We can be more precise. The Stein factorization of $D(S) \longrightarrow \mathbb{P}(V_{11}^\perp)$ is

$$\begin{tikzcd} D(S) \arrow[rr] \arrow[dr, "p"'] & & \mathbb{P}(V_{11}^\perp) = \mathbb{P}^2 \\ & \Sigma \arrow[ur, "2:1"'] & \end{tikzcd}$$

where the general fiber of p is a projective plane and the degree two finite cover $\Sigma \longrightarrow \mathbb{P}(V_{11}^{\perp})$ is ramified along the smooth sextic curve cut out by the projective dual Δ_6 of the adjoint variety. As a consequence, Σ (which may a priori be singular) must be birational to the K3 surface of degree two which is homologically projectively equivalent, up to the twist by a Brauer class, to the initial K3 surface S. We have finally proved:

Theorem 44. *I is birational to a $\mathbb{P}^2$-bundle over the K3 surface of degree two which is dual to S.*

Remark. Note the following strange phenomenon. According to the previous discussion, the projective plane covered by the degree two $K3$ surface Σ can be naturally identified with $L^{\perp} = \mathbb{P}(V_{11}^{\perp})$. But it can also be identified with $\mathbb{P}(V_7^{\perp})$, when we see $\mathbb{P}(V_7)$ as the linear system of Pfaffian cubics inside $|I_3(Sec(S))|$, see the comment after the proof of Proposition 25. The identification

$$L^{\perp} \simeq \mathbb{P}(V_7^{\perp})$$

is certainly an important clue for understanding the linear system $|I_3(Sec(S))|$.

7.3. **More about isotropic three-planes.** In this section we suggest another construction of the double cover $\overline{p_2} : \overline{\mathfrak{J}}^{OG} \rightarrow \mathbb{P}(\mathfrak{g}_2)$ from the action of G_2 on the isotropic subspaces of V_7 of dimension two and three. Recall that the orthogonal Grassmannian $OG(3,7)$ of isotropic 3-planes in V_7 is isomorphic to any of the two components of $OG(4,8)$, which by triality are isomorphic to $\mathbb{Q}^6$.

Proposition 45. *The action of G_2 on $OG(2,7)$ has two orbits, the closed orbit of null planes, and its complement.*

The action of G_2 on $OG(3,7) \simeq \mathbb{Q}^6$ has also two orbits; the closed orbit is isomorphic with $\mathbb{Q}^5$ and parametrizes the isotropic three-planes that contain a pencil of null planes; those in the open orbit do no contain any null-plane.

Proof. The first assertion follows from Corollary 30. For the second one, consider for example the three-plane $\langle e_\alpha, e_\beta, e_\gamma \rangle$. A direct computation shows that its stabilizer in $\mathfrak{g}_2$ is copy of $\mathfrak{sl}_3$, so its orbit has dimension $\dim \mathfrak{g}_2 - \dim \mathfrak{sl}_3 = 14 - 8 = 6$. We conclude that $OG(3,7)$ contains an open orbit of G_2, and since any non trivial projective homogeneous space under G_2 has dimension five, the complement of this open orbit must be a union of closed orbits. We can locate these closed orbits by looking at the decomposition of $\wedge^3 V_7$ into irreducible $\mathfrak{g}_2$-components, which is

$$\wedge^3 V_7 = \mathbb{C} \oplus V_{\omega_1} \oplus V_{2\omega_1}.$$

This implies that there are three closed G_2-orbits inside $\mathbb{P}(\wedge^3 V_7)$: a point and two quadrics, one being embedded by a quadratic Veronese morphism. The point simply corresponds to the (dual) invariant three-form, so it does not belong to $G(3, V_7)$. The first quadric $\mathbb{Q}_5$ is embedded inside $\mathbb{P}(V_7^{\vee}) = \mathbb{P}(V_{\omega_1})$, which is itself embedded in $\mathbb{P}(\wedge^3 V_7)$ by wedging vectors with the invariant three-form ω and considering the result inside $\wedge^4 V_7^\vee \simeq \wedge^3 V_7$; again we never get a decomposed four-form and therefore, this copy of $\mathbb{Q}^5$ is not contained in $G(3, V_7)$, a fortiori not either in $OG(3, V_7)$. We conclude that $OG(3, 7)$ contains a unique closed orbit, which is a copy of $\mathbb{Q}^5$, and our claims follow. $\square$

Now consider the incidence variety

$$\mathcal{I} = \{(P, X) \in OG(3, V_7) \times \mathbb{P}(\mathfrak{g}_2),\ X(P) \subset P\},$$

with its two projections p_1 and p_2.

For $h \in \mathfrak{g}_2$ generic, say regular semisimple, the action of h on V_7 is diagonalizable with distinct eigenvalues, so the three-planes P stabilized by h are generated by three eigenvectors. Among these 35 three-planes, exactly 8 are isotropic, among which only two do not contain null-planes. In particular p_2 is generically finite of degree 8.

Now consider $P \in OG(3, 7)$ and the fiber $p_1^{-1}(P)$. If P belongs to the open orbit we have already used in the proof of Proposition 45 that the fiber is a copy of $\mathfrak{sl}_3$. If P belongs to the closed orbit, we can suppose that $P = \langle e_\alpha, e_{-\beta}, e_{-\gamma}\rangle$. Then a direct computation shows that

$$stab(P) = \mathfrak{t} \oplus \mathfrak{g}_{\beta-\gamma} \oplus \mathfrak{g}_{\gamma-\beta} \oplus \mathfrak{g}_{\alpha-\beta} \oplus \mathfrak{g}_{\alpha-\gamma} \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\beta} \oplus \mathfrak{g}_{-\gamma}$$

has dimension nine. We deduce:

Proposition 46. *The incidence variety $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$ has two irreducible components: $\mathcal{I}_1$ is a $\mathbb{P}^8$-bundle over $\mathbb{Q}^5$, while $\mathcal{I}_0$ projects surjectively to $OG(3, V_7)$, with eight-dimensional fibers. The projection to $\mathbb{P}\mathfrak{g}_2$ is generically finite of degree 2.*

In words, the degree two morphism $\mathcal{I}_0 \longrightarrow \mathbb{P}\mathfrak{g}_2$ can be described as follows: if $h \in \mathfrak{g}_2$ is generic, its centralizer is a maximal torus and defines a root space decomposition of $\mathfrak{g}_2$; keeping only the long roots we get a copy of $\mathfrak{sl}_3$, whose action on V_7 decomposes as

$$V_7 = \mathbb{C} \oplus U_3 \oplus U_3'.$$

The two isotropic three-planes U_3 and U_3' represent the two points in the fiber over $[h]$.

8. Exceptional locus: the deformation argument

In this section we provide the proof of Theorem 3 or 17. The deformation argument that we use relies on the properties of the period map [11], and is also suggested in [22].

8.1. **The Heegner divisor $\mathcal{D}_{18}$.** For the notation and the general ideas about period maps, we refer to [11]. We fix $\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \mathbb{Z}\ell$, with $\ell^2 = -4$; this is the model lattice for the cohomology group of a manifold of $K3^{[3]}$-type, with its Beauville-Bogomolov form. We will denote by $\{e, f\}$ a standard base for U, a copy of the hyperbolic plane. For any lattice R we denote the discriminant group by A_R. For any $r \in R$, we write $div_R(r)$ for the divisibility of r in R. The class $r_* = [r/div_R(r)] \in A_R$ has order $div_R(r)$.

In order to use the period map in a proper way, in particular its surjectivity, we fix a connected component of the moduli space of marked manifolds of $K3^{[3]}$-type and any manifold of this kind considered in the sequel will be taken from this component.

We denote by $\mathcal{P}$ the period domain associated to 2-polarized manifolds of $K3^{[3]}$-type - and divisibility 1, the only possibility for those degree and dimension. Recall the construction of this period domain. We first fix $h \in \Lambda$ with the prescribed length and divisibility. In our case the divisibility is 1, so by [14, Example 3.8] the construction does not depend on the choice of h, and we take $h = e + f$. Then $\mathcal{P}$ is obtained by quotienting one of the two connected components of

$$\Omega = \Big\{x \in \mathbb{P}(h^\perp) \text{ such that } x^2 = 0, (x, \bar{x}) > 0\Big\}$$

by the restriction to $h^\perp$ of the group of monodromies fixing h (see [29, Definition 7.2] for a definition of the group of monodromies of the abstract lattice Λ). For us it is actually more convenient to see the period domain as the quotient of the whole Ω by $\widehat{O}(\Lambda, h) = \{\theta \in O(\Lambda)$ such that the action of θ on $A_{h^\perp}$ is $\pm id\}$, which is a double extension of the group of monodromies, see [29, Lemma 9.2].

Definition. The *Heegner divisor* $\mathcal{D}_{2d}$ is the image, in $\mathcal{P}$, of the union of $K^\perp \cap \Omega$ for all the primitive, hyperbolic, rank two lattices $K \subset \Lambda$ such that $h \in K$ and $K^\perp$ has discriminant $-2d$.

Inside $\mathcal{P}$, we restrict to the Heegner divisor $\mathcal{D}_{18}$. Consider a manifold of $K3^{[3]}$-type X, together with a line bundle L of length 2 and a marking sending L to h: the associated period point lies in $\mathcal{D}_{18}$ if and only if $\mathrm{NS}(X)$ contains a primitive, rank-2 lattice K such that $L \in K$ and $K^{\perp_{H^2(X,\mathbb{Z})}}$ has discriminant -18.

For (X, D) a manifold of $K3^{[3]}$-type with a line bundle whose class has degree two, we call *period point of* (X, D), the point of $\mathcal{P}$ obtained by fixing a marking ψ on X such that $\psi(D) = h$: there is a unique $O(\Lambda)$-orbit of vectors of degree 2 and divisibility 1, see [12, Corollary 3.7], so for any D of degree 2 there exists such a marking. By the surjectivity of the period map, any point of $\mathcal{P}$ is obtained that way.

Definition. For any hyperKähler manifold X, we denote by W_{exc} the subgroup of $Mon^2_{Hdg}(X)$ generated by reflections with respect to classes of stably prime-exceptional divisors, see [29, Definition 6.8] for details. An important feature of W_{exc} is that it acts simply transitively on the set of exceptional chambers of X [29, Theorem 6.18].

By the action of $W_{exc} \subseteq Mon^2_{Hdg}(X)$, there is a line bundle $\tilde{D}$ of the same degree as D inside the movable cone of X [29, Theorem 6.18 and Lemma 6.22]. This means that there exists a birational morphism $\phi : X' \dashrightarrow X$ such that $(X', \phi^*\tilde{D})$ is a manifold of $K3^{[3]}$-type with a big and nef line bundle (and even ample if the period point falls outside a finite number of Heegner divisors). As a consequence, the periods of manifolds with a degree two big and nef line bundle cover the whole of $\mathcal{P}$.

Lemma 47. *The Heegner divisor $\mathcal{D}_{18}$ is irreducible.*

Proof. The Heegner divisor can be seen as the image, in $\mathcal{P}$, of the union of the hyperplanes $\kappa^\perp \subset h^\perp$ such that $\kappa \in h^\perp$, $\kappa^2 < 0$ and the saturation of $\mathbb{Z}h \oplus \mathbb{Z}\kappa$ in Λ has discriminant -18.

The restriction of $\widehat{O}(\Lambda, h)$ to $h^\perp$ clearly contains the stable orthogonal group $\widetilde{O}(h^\perp) = \{\theta \in h^\perp$ acting trivially on the discriminant of $h^\perp\}$, by [13, Lemma 7.1]. Since $\widehat{O}(\Lambda, h)$ acts on a projective space, we also quotient out by the action of $-id_{h^\perp}$.

So, to prove the irreducibility of $\mathcal{D}_{18}$ it is sufficient to ask that, *up to a sign*, there is a unique orbit, by the action of the polarized monodromy, of vectors $\kappa \in h^\perp$ such that the orthogonal complement of $\mathbb{Z}h \oplus \mathbb{Z}\kappa \subset \Lambda$ has discriminant -18. The main tool we use is Eichler's criterion [14, Lemma 3.3]: the $\widetilde{O}(h^\perp)$-orbit of a non-isotropic vector κ is determined by κ^2 and $\kappa_* \in A_{h^\perp} = (e-f)_* \times \ell_* \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{4\mathbb{Z}}$.

Decompose $\kappa = a(e-f) + b\ell + cm$, where $\gcd(a, b, c) = 1$ and $m \in U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}$ a primitive vector. For simplicity, we denote by s the divisibility of κ in $h^\perp$, $s = \gcd(2a, 4b, c)$. Clearly $s \in \{1, 2, 4\}$. In $\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{4\mathbb{Z}}$, we have $\kappa_* = (\frac{2a}{s}, \frac{4b}{s})$.

Using [13, Lemma 7.2], we check that $\kappa^2 = -\frac{18 \cdot s^2}{8}$; since κ^2 has to be an even integer, we deduce that $s = 4$ and $\kappa^2 = -36$. Moreover a is even and $c \equiv 0 \pmod 4$. Since κ is primitive, b has to be odd. Hence $\kappa_* = (\frac{a}{2}, b) = (\frac{a}{2}, \pm 1)$. We also know that $36 = 2a^2 + 4b^2 - c^2m^2$ (with m^2 even and negative), so

$$9 = 2(a/2)^2 + b^2 - \frac{c^2m^2}{4},$$

which reduces to $0 \equiv (a/2)^2 \pmod 4$, and finally to $\kappa_* = (0, \pm 1)$. □

Denote by $\mathcal{K}_{2t}$ the moduli space of $2t$-quasi-polarized K3 surfaces. There is a map $\psi : \mathcal{K}_{18} \dashrightarrow \mathcal{P}$ sending (S, H) to the period point of $(S^{[3]}, H_3 - 2\delta)$.

Lemma 48. *The map ψ is generically injective, and its image is a dense subset of the Heegner divisor $\mathcal{D}_{18}$.*

Proof. First observe that the image of ψ is contained in $\mathcal{D}_{18}$: indeed the lattice $K = \mathbb{Z}H_3 \oplus \mathbb{Z}\delta \subset H^2(S^{[3]}, \mathbb{Z})$ has the required properties to ensure that $(S^{[3]}, H_3 - 2\delta) \in \mathcal{D}_{18}$.

To prove the generic injectivity of ψ, note that at a point (S, H) such that H generates $\mathrm{NS}(S)$, injectivity is equivalent to asking that, if there is $(S', H') \in \mathcal{K}_{18}$ such that $(S^{[3]}, H_3 - 2\delta) \cong ((S')^{[3]}, H'_3 - 2\delta')$, then $(S, H) \cong (S', H')$. This can be checked by a straightforward computation, using [4, Theorem 5.2]. Finally, the moduli space and the Heegner divisor have the same dimension, and $\mathcal{D}_{18}$ is irreducible, so the image of ψ must be dense in the latter. $\square$

8.2. **On degree two $K3$ surfaces.** We turn for a moment to 2-polarized K3 surfaces.

Proposition 49. *Let T be a K3 surface, D a big and nef line bundle of degree 2 on it. If $|D|$ is base point free, then $|D_3|$ on $T^{[3]}$ is base point free as well.*

Proof. Denote by $\mu : S \to \mathbb{P}(H^0(T, D)^\vee) \simeq \mathbb{P}^2$ the morphism defined by $|D|$. Then $|H^0(T^{[3]}, D_3)|$ is the linear system of cubics on $\mathbb{P}(H^0(T, D)^\vee) = \mathbb{P}^2$. By definition $\phi_{|D_3|}$ factors through the Hilbert-Chow morphism $T^{[3]} \to T^{(3)}$, so that the image by $\phi_{|D_3|}$ of a scheme $Z \in T^{[3]}$ only depend on its support, actually the image of its support by μ.

More precisely, for $Z \in T^{[3]}$ such that the support of $\mu(Z)$ is $\{p_1, p_2, p_3\}$, then $\phi_{|D_3|}(Z)$ is the hyperplane of cubics C on $\mathbb{P}^2$ whose polarization is such that $C(p_1, p_2, p_3) = 0$. For any triple this is clearly a non trivial condition on C, hence the claim. $\square$

Now we consider a K3 surface T whose Néron-Severi group is generated by $\{D, \Gamma\}$, with associated Gram matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

Lemma 50. *Up to a change of sign, the class D is big and nef and $|D|$ is base-point-free. Moreover the period point of $(T^{[3]}, D_3)$ lies in $\mathcal{D}_{18}$.*

Proof. The only (-2)-classes of T are $\pm\Gamma$, so the positive cone of T has two chambers exchanged by $\Gamma \mapsto -\Gamma$ [17, VIII, Section 2]; one of the two chambers is the ample cone. Since $D.\Gamma = 0$, D cannot be ample since, by Riemann-Roch Theorem, Γ or $-\Gamma$ must be effective. Nevertheless, no matter which chamber of $NS(T)$ is the actual ample cone, D or $-D$ belongs to the closure of the ample cone. So up to a change of sign, D is nef and big. Since $div_{\mathrm{NS}(T)}(w)$ is even for any $w \in \mathrm{NS}(T)$, any nef and big divisor has empty base locus by [17, II.3.15 (ii)].

Consider inside the lattice $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ a copy U_1 of the hyperbolic plane, with a standard basis $\{e_1, f_1\}$ orthogonal to $\{e, f\}$ defined at the beginning of the section. By [17, Corollary XIV.1.9], there exists an isometry

$\psi' : H^2(T, \mathbb{Z}) \to U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ such that $\psi'(D) = h$ and $\psi'(\Gamma) = e_1 - f_1$, Thus we can fix a marking ψ, on $T^{[3]}$, restricting to ψ' on $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ and such that $\psi(\delta) = \ell$. Finally, set $K = \mathbb{Z}h \oplus \mathbb{Z}(4(e_1 - f_1) + \ell)$: this is a primitive, hyperbolic lattice and $K^\perp = \mathbb{Z}(e - f) \oplus N \oplus U \oplus E_8(-1)^{\oplus 2}$, where N is the orthogonal complement of $4(e_1 - f_1) + \ell$ in $U_1 \oplus \mathbb{Z}\ell$; by [13, Lemma 7.2], $K^\perp$ has discriminant $-2 \cdot 9 = -18$. $\square$

8.3. **End of the proof.** We will use the fact that for a family of polarized manifolds (X, D) such that $h^0(X, D)$ remains constant, base point freeness is an open condition on the base of the family.

We consider the period point of $(T^{[3]}, D_3)$ and a small neighborhood $\mathcal{U}$ of this point inside the Heegner divisor $\mathcal{D}_{18}$. By Lemma 48, the image of ψ contains an open and dense subset of $\mathcal{U}$. The corresponding points in $\mathcal{K}_{18}$ are birationally equivalent to $S^{[3]}$ for some 18-polarized $K3$ surface (S, H); this actually implies that, very generally, they are in fact isomorphic to $S^{[3]}$, since there is no non-trivial smooth birational model of $S^{[3]}$ when its Picard rank is two (recall that any birational model has a birational morphism to $S^{[3]}$ which by Proposition 11 is in fact biregular, up to composing it with φ).

On this $S^{[3]}$ we also get a degree two class Θ by deformation of D_3. We do not know, a priori, if Θ is big and nef, but for sure it is base point free (up to shrinking again $\mathcal{U}$), hence movable. But any degree two class on $S^{[3]}$, lying in the interior of the movable cone of $S^{[3]}$, produces a birational involution on it, since minus the reflection in the class is a Hodge monodromy of $H^2(S^{[3]}, \mathbb{Z})$ [4, proof of Thm. 1.1]. Since when S has Picard rank one, the only birational involution on $S^{[3]}$ is the one fixing $H_3 - 2\delta$ (recall Theorem 1), we conclude that Θ has to coincide with $H_3 - 2\delta$. In particular, the latter is base point free. $\square$

Remark. While $(T^{[3]}, D_3)$ deforms to $(S^{[3]}, H_3 - 2\delta)$, the natural involution i on $(T^{[3]}, D_3)$ does not deform to the involution φ, since the fixed lattice of the natural involution has rank two. Indeed this must remain true for any deformation, while the fixed lattice of φ has only rank one; the theory of deformation of involutions has been studied by Joumaah [19].

8.4. **Non separated points in the moduli space.** We conclude this section with some heuristic observations. Indeed the 2-polarized K3 surface T seems to come out of the blue, but a posteriori it is not so out of place.

First of all it has a nice geometric description, as it is the very general 2-polarized K3 surface obtained by desingularizing the 2-to-1 cover of $\mathbb{P}^2$ ramified over a sextic curve with a single node. But the most interesting thing is that there is a natural dominant map $M : \mathcal{K}_{18} \dashrightarrow \mathcal{K}_2$, sending (S, H) to the HPD (up to a twist by a Brauer class) 2-polarized K3 surface described in the previous sections. Equivalently, at least for a general element of $\mathcal{K}_{18}$, the image of this map is the K3 surface which is the base locus of the contracted locus of the flop on $S^{[3]}$ i.e. $M_\sigma(3, -H, 3)$, where

σ is a general stability condition on S, see Corollary 13. (The attentive reader will have observed that $(3, -H, 3)$ is precisely the Mukai vector $v - a$ from Lemma 26.) So, it is natural to wonder if there exists (S, H) such that $M(S) = M_\sigma(3, -H, 3)$ has Néron-Severi lattice $R = \langle 2\rangle \oplus \langle -2\rangle$.

Consider a K3 surface S' whose Néron-Severi lattice is again R, with basis $\{D, \Gamma\}$ as above (that is $D^2 = 2$, $D.\Gamma = 0$, $\Gamma^2 = -2$). In the basis given by $H' = 5D + 4\Gamma$ and $E = D + \Gamma$, the associated Gram matrix is

$$\begin{bmatrix} 18 & 2 \\ 2 & 0 \end{bmatrix}.$$

Since H' does not lie in a wall and has positive intersection with D, we can suppose, up to a finite number of reflections in $H^2(S', \mathbb{Z})$, that (S', H') is an 18-polarized K3 surface.

Lemma 51. *The K3 surface $M(S')$ is isomorphic to S', in particular its Néron-Severi lattice is again R.*

Proof. The Néron-Severi lattice of $M(S')$ can be computed as in Lemma 26 and turns out to be again R: in the present case $(3, -H', 3)^\perp \subset H^*_{alg}(S', \mathbb{Z})$ is generated by $\{(3, -H', 3), (1, 0, -1), (1, -3E, 1)\}$, so

$$\frac{(3, -H', 3)^\perp}{(3, -H', 3)\mathbb{Z}} \cong \mathbb{Z}(1, 0, -1) \oplus \mathbb{Z}(1, -3E, 1) \cong R.$$

Moreover, $M(S')$ is a Fourier-Mukai partner of S': indeed it is sufficient, in order to prove this claim, to check the existence of a universal family on $M(S') \times S'$, see [33, Theorem 1.4]. For this, by [17, X.2.2., item (i)] we just need to find $w \in H^*_{alg}(S', \mathbb{Z})$ such that $(3, -H', 3) \cdot w = 1$. We can choose $w = (-1, E, 0)$.

Finally, there remains to check that S' is in general its only Fourier-Mukai partner with the correct Néron-Severi group. Recall that the number m of isomorphism classes of Fourier-Mukai partners of S', whose Néron-Severi group is isomorphic to R, is bounded by the number m' of non-isomorphic embeddings of the abstract lattice $NS(S') \oplus NS(S')^{\perp_{H^2(S',\mathbb{Z})}}$ in the unique even unimodular lattice of signature $(3, 19)$, such that $NS(S')$ and $NS(S')^{\perp_{H^2(S',\mathbb{Z})}}$ are primitively embedded (see [33, Section 3]). In turn, as explained in [31, Section 5], the number of such embeddings is bounded by the number m'' of isotropic subgroups of maximal cardinality of $A_{NS(S')} \oplus A_{NS(S')^{\perp_{H^2(S',\mathbb{Z})}}}$ such that the projection to both discriminant groups is injective.

As $NS(S') \cong R$, its discriminant group is isomorphic to $\langle \frac{1}{2}\rangle \oplus \langle -\frac{1}{2}\rangle$ as an abelian group endowed with a finite form q_R with values in $\mathbb{Q}/2\mathbb{Z}$. The discriminant group of the orthogonal complement of R is isomorphic to A_R, since $H^2(S', \mathbb{Z})$ is unimodular; the corresponding finite form is again $\langle \frac{1}{2}\rangle \oplus \langle -\frac{1}{2}\rangle$, since $q_{R^\perp} \cong -q_R$ by unimodularity and $-q_R \cong q_R$. So the finite form $q := q_R \oplus q_{R^\perp} : (\mathbb{Z}/2\mathbb{Z})^4 \to \mathbb{Q}/2\mathbb{Z}$ has values $q(1,0,0,0) = q(0,0,1,0) = \frac{1}{2}$ and $q(0,1,0,0) = q(0,0,0,1) = -\frac{1}{2}$.

An explicit computation shows that there are only two isotropic maximal subgroups, namely $\{(0,0,0,0),(1,0,1,0),(0,1,0,1),(1,1,1,1)\}$ and $\{(0,0,0,0),(1,1,0,0),(0,0,1,1),(1,1,1,1)\}$. For the latter the projections to the two discriminant groups are not injective; in particular it corresponds to an embedding $R \to H^2(S',\mathbb{Z})$ with respect to which the saturation of R is a copy of the hyperbolic plane. We conclude that $m''=1$, hence $m'=1$ and finally $m=1$: any Fourier-Mukai partner of S', whose Néron-Severi group is isomorphic to R, is in fact isomorphic to S'. In particular this holds for $M(S')$, and we are done. $\square$

In other words, there is an irreducible codimension one family in $\mathcal{K}_{18}$ made of $K3$ surfaces which are mapped by $M : \mathcal{K}_{18} \dashrightarrow \mathcal{K}_2$ to the same $K3$ surface, but with a polarization of degree two. (Note that $M(S)$ and S are not even Fourier-Mukai partners in general.) For Hilbert schemes of three points, this translates into the following phenomenon:

Proposition 52. *For $H' = 5D + 4\Gamma$ the pairs $((S')^{[3]}, H'_3 - 2\delta)$ and $((S')^{[3]}, D_3)$ have the same period point.*

Proof. We consider, on $H^2((S')^{[3]},\mathbb{Z})$, the rational isometry given by the reflection with respect to the class $\nu = -2E_3 + \delta$, denoted by ρ. Because E_3 lies in the unimodular lattice $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$, it has divisibility 1, so ν has divisibility 2 and, since $\nu^2 = -4$, ρ is actually integral and a monodromy operator [29, Proposition 9.12]. More precisely, $\rho \in W_{exc}$, since we are in the fifth case of [29, Theorem 9.17], as one can see directly from the table in [29, Proposition 9.16 (i)]. Hence the action of ρ does not change the period point. It is then sufficient, in order to conclude the proof, to observe that $\rho(H'_3 - 2\delta) = D_3$. $\square$

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