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A BIRATIONAL INVOLUTION

PIETRO BERI, LAURENT MANIVEL

ABSTRACT. Given a very general K3 surface S of degree 18, lattice theoretic considerations allow to predict the existence of an anti-symplectic birational involution φ of the Hilbert cube $S^{[3]}$. We describe this involution in terms of the Mukai model of S , with the help of the geometry of the exceptional Lie group G_2 . We make a connection with Homological Projective Duality by showing that the indeterminacy locus of the involution is a \mathbb{P}^2 -fibration over the dual K3 surface of degree two. We deduce that φ is an instance of a Mukai flop.

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1. INTRODUCTION

Consider a complex algebraic $K3$ surface S whose Picard group is generated by an ample line bundle \mathcal{H} such that $\mathcal{H}^2 = 2t$ (a very general element of the 19-dimensional moduli space of $2t$ -polarized $K3$ surfaces). A classification of the group of biregular automorphisms of the punctual Hilbert scheme $S^{[n]} := \text{Hilb}^n(S)$ has been given by Boissière, An. Cattaneo, Nieper-Wißkirchen and Sarti [8] for $n = 2$, and by Al. Cattaneo for all $n \geq 3$ [12]. In particular, $\text{Aut}(S^{[n]})$ is either trivial or generated by an involution which is non-symplectic for $t \geq 2$. More recently, extended lattice theoretic considerations allowed Al. Cattaneo and the first author to decide whether $S^{[n]}$ admits non-trivial birational automorphisms [6, Theorem 1.1], symplectic or not. When it does, there exists a unique such automorphism, a birational involution that may be symplectic or non symplectic. The precise result is the following:

Theorem 1. *Suppose $t \geq 2$. Then $\text{Bir}(S^{[n]})$ is either trivial or generated by a birational involution φ . The latter holds if and only if:*

- (1) $d = t(n - 1)$ is not a square,
- (2) the minimal non-trivial solution (X, Y) of Pell's equation

$$X^2 - dY^2 = 1 \quad \text{with} \quad X \equiv \pm 1 \pmod{n - 1}$$

is such that Y is even and $(X, X) = (1, 1), (1, -1)$ or $(-1, -1)$ in $\mathbb{Z}/2(n - 1)\mathbb{Z} \times \mathbb{Z}/2t\mathbb{Z}$.

When $(X, X) = (1, 1)$, the involution is symplectic and never biregular. When $(X, X) = (\pm 1, -1)$, it is always non-symplectic, and biregular if and only if it fixes an ample class of $S^{[n]}$; for that case the classification can be found in [8, 12], the precise condition for regularity being condition (iii) of [12, Theorem 1.3].

Such a statement does not provide much insight on how to construct this birational involution geometrically, when it does exist. Some cases have been explicitly described in [6, Examples 6.1, 6.2], for $t = 2$ and $n = 6, 8, 18$. A few other cases had been known before:

- Suppose $t = n$, so that \mathcal{H} embeds S into \mathbb{P}^{n+1} , as a surface of degree $2n$. If we choose n points p_1, \dots, p_n in S in general linear position, they generate a codimension two subspace of \mathbb{P}^{n+1} , which in general cuts S at $2n$ distinct points p_1, \dots, p_n and q_1, \dots, q_n . This yields the *Beauville involution*, which is non-symplectic, and biregular only for $n = 2$ (that is for quartics in \mathbb{P}^3).
- Suppose $n = 2$ and $t = 5$, so that S is embedded in the Grassmannian $G(2, 5)$ as the transverse intersection of a quadric Q and three hyperplanes H_1, H_2, H_3 . In particular, the linear span of S intersects the Grassmannian along $F = G(2, 5) \cap H_1 \cap H_2 \cap H_3$, which is a smooth Fano threefold of index two. Now consider two general points p_1, p_2 in S , defining two transverse planes P_1, P_2 in \mathbb{C}^5 .

The Grassmannian $G(2, P_1 \oplus P_2)$ is a four dimensional quadric in $G(2, 5)$, its intersection with F is a conic, and its intersection with $S = F \cap Q$ therefore consists in general of four points p_1, p_2 and q_1, q_2 . This yields the O'Grady involution [41, section 4.3].

- Suppose $n = 3$ and again $t = 5$, so that S can be described as before. Given three general points p_1, p_2, p_3 in S , defining three planes P_1, P_2, P_3 in \mathbb{C}^5 , or equivalently three lines ℓ_1, ℓ_2, ℓ_3 in \mathbb{P}^4 , it is a classical fact that there exists in general a unique line ℓ_0 meeting these three lines. Equivalently, if $\Sigma_\ell \subset G(2, 5)$ is the codimension two Schubert cycle parametrizing lines in \mathbb{P}^4 that meet a given line ℓ , then $\Sigma_{\ell_1} \cdot \Sigma_{\ell_2} \cdot \Sigma_{\ell_3} = 1$. Since Σ_ℓ has degree three, the intersection $\Sigma_{\ell_0} \cap S$ consists of the three points p_1, p_2, p_3 (since $\ell_0 \in \Sigma_{\ell_i}$ is equivalent to $\ell_i \in \Sigma_{\ell_0}$), and three other points q_1, q_2, q_3 [13, Example 4.12].

In fact, there are some easily identified infinite sequences of pairs (n, t) , described in [6, Prop. 2.6 (i)], that satisfy the conditions of Theorem 1: fix any $k > 0$ and take $t = (n - 1)k^2 + 1$, for any $n \geq 2$. For $k = 1$ we recover the Beauville involution. For $k = 2$ only the case $n = 2$ has been described, this is the O'Grady involution. In this paper, we describe the next case, $n = 3$ and $t = 9$.

A general $K3$ surface of degree $2t = 18$, or equivalently of genus 10, admits a Mukai model: it can be described as a codimension three linear section of the adjoint variety of the exceptional Lie group G_2 , the closed G_2 -orbit inside the projectivized adjoint representation [38]. We denote this five dimensional homogeneous space by $X_{ad}(G_2) \subset \mathbb{P}(\mathfrak{g}_2)$, so that

$$S = X_{ad}(G_2) \cap L$$

for some codimension three linear subspace L of $\mathbb{P}(\mathfrak{g}_2)$.

Recall that G_2 can be described as a subgroup of $SO(7)$ [1]. This means in particular that G_2 has a natural representation V_7 of dimension 7, which is irreducible, and that it preserves some non-degenerate quadratic form Q . Since then $\mathfrak{g}_2 \subset \mathfrak{so}_7 \simeq \wedge^2 V_7$, we deduce that the adjoint variety $X_{ad}(G_2)$, which is the minimal orbit in $\mathbb{P}(\mathfrak{g}_2)$, must be a subvariety of $G(2, V_7)$. There is a diagram

$$\begin{array}{ccccc} X_{ad}(G_2) & \hookrightarrow & OG(2, V_7) & \hookrightarrow & G(2, V_7) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}(\mathfrak{g}_2) & \hookrightarrow & \mathbb{P}(\wedge^2 V_7) & = & \mathbb{P}(\wedge^2 V_7) \end{array}$$

where the vertical arrows are embeddings. In fact, the Mukai model for S is constructed from the unique stable vector bundle \mathcal{U}_2^\vee on S with Mukai vector $(2, \mathcal{H}, 5)$, where $\mathcal{H}^2 = 18$: this bundle is generated by its seven global sections, which embed S into $G(2, V_7)$.

So let p_1, p_2, p_3 be generic points on S . They can be identified with three planes $P_1, P_2, P_3 \subset V_7$. In general, these three planes generate a hyperplane $V_6 \subset V_7$, and $F\ell(V_6) := X_{ad}(G_2) \cap G(2, V_6)$ is a copy of the

complete flag variety $F\ell_3$ of \mathfrak{sl}_3 . Since $F\ell_3$ has dimension three and degree 6, its intersection with L consists in general of six distinct points, p_1, p_2, p_3 , plus three other points q_1, q_2, q_3 . We define

$$(1) \quad \varphi : S^{[3]} \dashrightarrow S^{[3]}, \quad p_1 + p_2 + p_3 \mapsto q_1 + q_2 + q_3.$$

Note that, at this stage, we can consider any K3 surface S of genus ten whose projective model is a three dimensional section of $X_{ad}(G_2)$.

Theorem 2. *(Theorem 18) For any K3 surface $S = X_{ad}(G_2) \cap L$, φ is a non-trivial birational involution of $S^{[3]}$.*

We also provide another interpretation of φ in terms of the extremal ray $\mathcal{H}_3 - 2\delta$ of $S^{[3]}$, where \mathcal{H}_3 is the line bundle induced by the polarization \mathcal{H} of S and 2δ is, as usual, the class of the locus of non-reduced subschemes.

Theorem 3. *(Theorem 45, Proposition 50, Corollary 52) Let S be a very general K3 surface of genus ten. The linear system $|\mathcal{H}_3 - 2\delta|$ on the punctual Hilbert scheme $S^{[3]}$ is base point free. The associated morphism $\phi_{|\mathcal{H}_3 - 2\delta|}$ is generically finite of degree two, and the birational involution φ is the corresponding deck transformation.*

Moreover the indeterminacy locus $I \subseteq S^{[3]}$ of φ can be described in terms of subschemes of length three that do not span a hyperplane in V_7 . Such subschemes are obtained from certain cubic scrolls in the adjoint variety, parametrized by what we call decomposing five-planes: those codimension two subspaces of V_7 over which the G_2 -invariant three-form is completely decomposed. Part of our arguments rely on a good understanding of these codimension two subspaces. We show:

Proposition 4. *(Corollary 13) If $\omega \in \bigwedge^3 V_7^\vee$ is a G_2 -invariant 3-form, then the subvariety of $G(5, V_7)$ parameterizing ω -decomposing five-planes is projectively equivalent to the orthogonal Grassmannian $OG(2, 7)$.*

This statement is closely related to a recent paper by Guseva [16].

Cubic scrolls in $X_{ad}(G_2)$ and its hyperplane sections were studied by M. Kapustka and Ranestad [23]; they proved the remarkable fact that a smooth hyperplane section $F = X_{ad}(G_2) \cap H$ is isomorphic to a linear section of the Grassmannian $G(3, U_6)$, and deduced that cubic scrolls in F are parametrized by the disjoint union of two Veronese surfaces. We integrate their description with results from [26], thus extending it to the case where F is a nodal hyperplane section of $X_{ad}(G_2)$.

When H contains L , the section F contains the K3 surface S and we can pull-back the tautological and quotient bundles on $G(3, U_6)$ to get two vector bundles on S ; but only one when we pull-back from $LG(3, U_6)$, where the tautological and quotient bundles gets identified. This realizes the moduli space $M_{\mathcal{H}}(3, \mathcal{H}, 3)$ of \mathcal{H} -stable vector bundles on S with the corresponding invariants, as a smooth double cover of $L^\perp \simeq \mathbb{P}^2$, branched over its intersection with the dual hypersurface to $X_{ad}(G_2)$, which is a sextic hypersurface.

This double cover, described in [23], is precisely the degree two K3 surface Σ , derived equivalent to S up to the twist by a Brauer class, that appears in Kuznetsov's Homological Projective Duality for $X_{ad}(G_2)$ [26, Corollary 9.10]. Note that although the moduli space $\Sigma = M_{\mathcal{H}}(3, \mathcal{H}, 3)$ is not fine, there exists a twisted universal bundle \mathcal{F} on $S \times \Sigma$. With this in mind, we can describe the indeterminacy locus:

Theorem 5. *(Proposition 33) Let S be a very general K3 surface of genus ten. The indeterminacy locus I of φ on the punctual Hilbert scheme $S^{[3]}$ is a smooth fourfold with an étale locally trivial \mathbb{P}^2 -fibration over the smooth degree two K3 surface Σ , given either by*

- (1) *the locus of length-three subschemes of S contained in $G(2, V_5)$ for some decomposing five-plane $V_5 \subset V_7$;*
- (2) *the family of cubic scrolls on hyperplane sections of $X_{ad}(G_2)$ containing S ;*
- (3) *the projectivization $\mathbb{P}_{\Sigma}(\mathcal{G})$ of the twisted vector bundle $\mathcal{G} = p_{2*}\mathcal{H}om(\mathcal{F}, p_1^*\mathcal{U}_2)$ over Σ , where p_1 and p_2 are the projections from $S \times \Sigma$ to S and Σ , respectively.*

Finally, since I is a smooth codimension two subvariety of $S^{[3]}$, fibered in \mathbb{P}^2 's, we can perform a Mukai flop to obtain another smooth, simply connected sixfold M with a non-degenerate two-form; but a priori not necessarily hyperKähler. It turns out that M is in fact isomorphic with $S^{[3]}$ and that, up to identifying the two models, the Mukai flop is nothing else than our involution φ !

Theorem 6. *(Corollary 36) For $S^{[3]}$ the punctual Hilbert scheme over a very general K3 surface of genus ten, φ is a Mukai flop.*

The techniques we use are a mixture of G_2 -geometry with moduli interpretations and stability conditions. Basic facts about G_2 are recalled in section 2, where we also discuss decomposing five-planes and cubic scrolls. The involution φ is defined in section 3, where we prove Theorem 2 and we put into the picture the extremal contraction induced by a multiple of $\mathcal{H}_3 - 2\delta$, which ultimately leads to Theorem 6. These two results can be seen as different descriptions of φ and the key point to identify them is Theorem 6. The connections with the linear system $|\mathcal{H}_3 - 2\delta|$ itself are discussed in section 4: we first prove base point freeness, then that the associated morphism is generically finite of degree two.

All along we will find strong analogies, but also important differences, with the story of Gushel-Mukai varieties and (double) EPW sextics, a story which includes the O'Grady involution. Some of those observations are gathered in section 4.5, where we also briefly discuss the indirect and intriguing connections, also present in [26], of some of our constructions with the Grothendieck-Springer resolution. Being able to upgrade the present work in order to construct a new locally complete family of polarized hyperKähler manifolds, is one of the challenges that remain to be met.

One can also wonder to which extent our constructions can be generalized. The definition of our birational involution still makes sense for certain punctual Hilbert schemes of K3 surfaces with different polarizations, and the geometric study presented in this paper can be partially extended, although we of course lose all the interpretations we gave in terms of G_2 -geometry [7]. Parallel constructions, presented in the more flexible setting of derived categories of moduli spaces of sheaves on K3's, were very recently presented in [14]. But the case of K3 surfaces of genus ten, with its so many facets, will certainly remain a unique gem.

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2. SOME G_2 GEOMETRY

2.1. Classical preliminaries. A useful, wide-ranging reference for the exceptional Lie group G_2 is [1]. There are two classical ways to understand G_2 , which may be both useful according to the context. Here we work over the complex numbers, and most of what follows is translated from the real setting. In particular, each time in the sequel we deal with octonions and the Cayley algebra \mathbb{O} , we will in fact be dealing with the *complexified* octonions and the *complexified* Cayley algebra. We will not use any special notation for this complexification, hoping this will not cause any confusion.

- (1) (Cartan 1914) G_2 can be defined as the automorphism group of the Cayley algebra of octonions. Since the Cayley algebra \mathbb{O} is normed, it admits a G_2 -invariant quadratic form Q , and it splits orthogonally as $\mathbb{O} = \mathbb{C}\mathbf{1} \oplus V_7$, where V_7 is the space of imaginary octonions.
- (2) (Engel 1900) $G_2 \subset SL(V_7)$ can be defined as the stabilizer of a generic skew-symmetric three-form $\omega \in \wedge^3 V_7^\vee$. Then V_7 inherits an invariant quadratic form, defined up to scalar by

$$Q(x, y) = \iota_x \omega \wedge \iota_y \omega \wedge \omega \in \wedge^7 V_7^\vee \simeq \mathbb{C},$$

where ι_x denotes the contraction by the vector x . Conversely, on the space of imaginary octonions the skew-symmetric three-form is given by the formula $\omega(x, y, z) = \operatorname{Re}(x(yz))$ (where Re must be interpreted as the projection on the invariant part in the decomposition $\mathbb{O} =$

$\mathbb{C} \oplus V_7$). From this three-form one easily recovers the octonionic product.

The two fundamental representations of G_2 are V_7 and the adjoint representation on the Lie algebra \mathfrak{g}_2 , whose dimension is 14. The two associated generalized Grassmannians (orbits of highest weight vectors) are $G_2/P_1 = \mathbb{Q}^5 \subset \mathbb{P}(V_7)$, the invariant quadric, and the adjoint variety $X_{ad}(G_2) = G_2/P_2 \subset \mathbb{P}(\mathfrak{g}_2)$. There is a fundamental correspondence

$$(2) \quad \begin{array}{ccc} & G_2/B & \\ p_{\mathbb{Q}} \swarrow & & \searrow p_X \\ \mathbb{Q}^5 & & X_{ad}(G_2), \end{array}$$

where the two projections are \mathbb{P}^1 -fibrations over Fano fivefolds, \mathbb{Q}^5 of index five and $X_{ad}(G_2)$ of index three; both of them dominated by the flag manifold G_2/B , where B is a Borel subgroup containing the maximal parabolic subgroups P_1 and P_2 .

Since $\mathfrak{g}_2 \subset \mathfrak{so}_7 \simeq \wedge^2 V_7$, the adjoint variety $X_{ad}(G_2) \subset \mathbb{P}(\mathfrak{g}_2)$ is a subvariety of $G(2, V_7) \subset \mathbb{P}(\wedge^2 V_7)$, hence parametrizes some special planes. According to our two viewpoints:

- (1) $X_{ad}(G_2)$ parametrizes the family of *null-planes* in V_7 , that is, those planes $P \subset V_7 = \text{Im}(\mathbb{O})$ on which the octonionic product vanishes identically [29, 34].
- (2) Equivalently, $X_{ad}(G_2)$ parametrizes the planes $P = \langle x, y \rangle \subset V_7$ such that $\iota_x \iota_y \omega = \omega(x, y, \bullet) = 0$. Let \mathcal{U}_2 and \mathcal{Q}_5 denote the tautological and quotient vector bundles over $G(2, V_7)$; this shows that ω defines a global section of the bundle $\wedge^2 \mathcal{U}_2^\vee \otimes \mathcal{Q}_5^\vee = \mathcal{Q}_5^\vee(1)$, whose zero-locus is precisely the adjoint variety [26, Section 8].

For our purposes, we will mostly adopt this second point of view, but it may be useful to keep the first one in mind as well.

Note that in diagram (2), G_2/B is the projectivized bundle $\mathbb{P}(\mathcal{U}_2)$ over $X_{ad}(G_2)$. Similarly, it is the projectivization of a G_2 -equivariant rank two bundle \mathcal{C} on \mathbb{Q}^5 called the *Cayley bundle* [43], and such that

$$H^0(\mathbb{Q}^5, \mathcal{C}^\vee) = \mathfrak{g}_2^\vee \simeq \mathfrak{g}_2.$$

As in [34] we fix a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}_2$, which yields a root space decomposition, and we denote by α, β, γ three short roots summing to zero. Then $\alpha - \beta, \beta - \gamma, \gamma - \alpha$ are three long roots also summing to zero, and we get the twelve roots of \mathfrak{g}_2 by including the opposites of those six roots.

In fact, once we have fixed a Cartan subalgebra in \mathfrak{g}_2 and obtained the previous decomposition, we also get a basis of V_7 consisting of weight vectors:

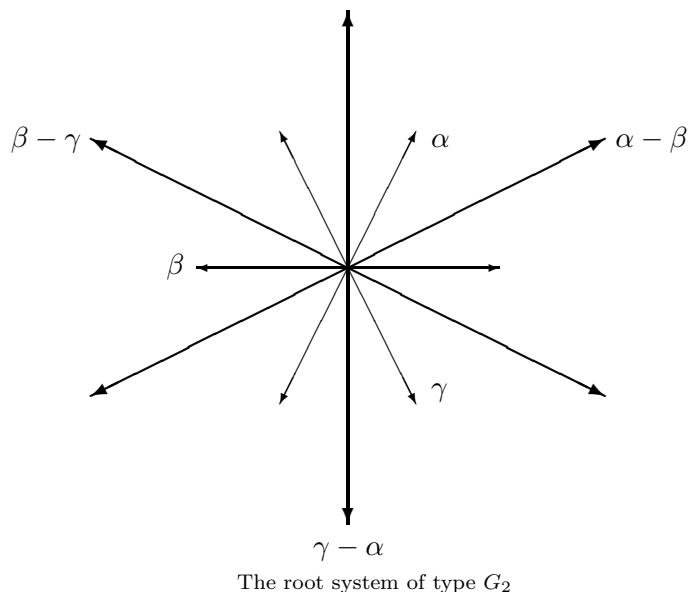
$$V_7 = \mathbb{C}e_0 \oplus \mathbb{C}e_\alpha \oplus \mathbb{C}e_{-\alpha} \oplus \mathbb{C}e_\beta \oplus \mathbb{C}e_{-\beta} \oplus \mathbb{C}e_\gamma \oplus \mathbb{C}e_{-\gamma}$$

(where e_α has weight α , etc.), and we can normalize so that the invariant three-form ω and the invariant quadratic form Q have the following expressions in the dual basis:

$$\omega = v_0 \wedge (v_\alpha \wedge v_{-\alpha} + v_\beta \wedge v_{-\beta} + v_\gamma \wedge v_{-\gamma}) + v_\alpha \wedge v_\beta \wedge v_\gamma + v_{-\alpha} \wedge v_{-\beta} \wedge v_{-\gamma},$$

$$Q = -v_0^2 + v_\alpha v_{-\alpha} + v_\beta v_{-\beta} + v_\gamma v_{-\gamma}.$$

Note in particular that all our basis vectors, except e_0 , are isotropic with respect to Q . Note also that Q being non-degenerate, it allows to identify V_7 and V_7^\vee . In particular we will use the same symbol \perp for the orthogonal in V_7 or V_7^\vee , with respect to Q or to the canonical duality.



2.2. The action of G_2 on imaginary octonions. The action of G_2 on V_7 provides a classical example of a prehomogeneous space. The following result is essentially [44, Proposition 42].

Proposition 7. *The action of G_2 on $\mathbb{P}(V_7)$ has two orbits, the five dimensional quadric \mathbb{Q}^5 defined by Q , and its complement*

$$\mathbb{P}(V_7) - \mathbb{Q}^5 \simeq G_2/H,$$

where H is locally isomorphic to SL_3 .

(Here locally isomorphic means that the two groups have the same Lie algebra.) For example, the Lie algebra of the stabilizer of $[e_0]$ is the subalgebra of \mathfrak{g}_2 generated by the long root spaces: this is clearly a subalgebra, since adding two long roots never gives a short root, of type A_2 , and it certainly kills e_0 since no weight of V_7 is a long root.

Remark. One of the beautiful properties of the real compact form of G_2 , which is the automorphism group of the usual octonions with real coefficients, is that it acts transitively on the six-dimensional sphere, seen as the set of imaginary octonions of unit norm. This leads to the well-known identification of the six-dimensional sphere with the quotient space $G_2(\mathbb{R})/SU_3$, and to the fact that $G_2(\mathbb{R})$ can be seen as the total space of a SU_3 -principal bundle over S^6 . Note that over the real numbers, the sphere is a double cover of the projective space of imaginary octonions. Over the complex numbers, the corresponding statement is that the generic stabilizer H is isomorphic to $Aut(SL_3)$, which is a degree two extension of SL_3 . We will not use this fact in the sequel.

Taking orthogonals with respect to the quadratic form Q , Proposition 7 also means that there are only two types of hyperplanes $V_6 \subset V_7$ up to the action of G_2 : they can either be transverse or tangent to the quadric. Another convenient way to distinguish them is by the restriction of the invariant form ω .

If $V_6 = v_0^\perp$, then

$$\omega|_{V_6} = v_\alpha \wedge v_\beta \wedge v_\gamma + v_{-\alpha} \wedge v_{-\beta} \wedge v_{-\gamma}$$

belongs to the open $GL(V_6)$ -orbits in $\mathbb{P}(\wedge^3 V_6^\vee)$. The two planes $\mathbb{P}\langle e_\alpha, e_\beta, e_\gamma \rangle$ and $\mathbb{P}\langle e_{-\alpha}, e_{-\beta}, e_{-\gamma} \rangle$ are uniquely determined (see e.g. [27, 3.1]): this is the *one apparent double point property* of the Grassmannian $G(3, V_6)$.

If $V_6 = v_{-\gamma}^\perp$, then

$$\omega|_{V_6} = v_0 \wedge v_\alpha \wedge v_{-\alpha} + v_0 \wedge v_\beta \wedge v_{-\beta} + v_\alpha \wedge v_\beta \wedge v_\gamma$$

belongs to the tangent space to the Grassmannian $G(3, V_6^\vee)$ at the uniquely defined plane $\mathbb{P}\langle v_0, v_\alpha, v_\beta \rangle$. (Unicity follows from the facts that, $G(3, V_6^\vee)$ being Legendrian [11], its tangent and dual varieties are identified, and that a generic singular hyperplane section is singular at a unique point.)

Denote by $\mathfrak{sl}_3 \subset \mathfrak{g}_2$ the annihilator of $e_0 \in V_7$. This is also the Lie algebra of the group H from Proposition 7.

Proposition 8. *The intersection of $X_{ad}(G_2) \subset \mathbb{P}(\mathfrak{g}_2)$ with $\mathbb{P}(\mathfrak{sl}_3)$ is a copy of $F\ell_3 \subset \mathbb{P}(\mathfrak{sl}_3)$, the variety of complete flags in a three-dimensional space, which is also the adjoint variety of \mathfrak{sl}_3 .*

Proof. Let $V_6 := v_0^\perp$. Recall that $X_{ad}(G_2)$ is defined in $G(2, V_7)$ as the zero-locus of the section s_ω of $\mathcal{Q}_5^\vee(1)$ defined by ω . Since an element of the cone over $G(2, V_7)$, considered as an element of $\mathfrak{so}_7 \simeq \wedge^2 V_7$, kills e_0 if and only if it is contained in $G(2, V_6)$, we have

$$(3) \quad X_{ad}(G_2) \cap \mathbb{P}(\mathfrak{sl}_3) = X_{ad}(G_2) \cap G(2, V_6).$$

Then the conclusion follows from [23, Lemma 4]. \square

Remark. Recall that $F\ell_3$, being a hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$, has degree six. Also, the intersection $X_{ad}(G_2) \cap G(2, V_6)$ for V_6^\perp isotropic has been

described in [23, Lemma 4] as a specific hyperplane section of the image of $P = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus T_{\mathbb{P}^2}(-1))$ in \mathbb{P}^8 by the linear system $|\mathcal{O}_P(1)|$. This P has also degree 6 and is a union of projective planes parametrized by \mathbb{P}^2 .

2.3. Action of G_2 on planes in V_7 . The action of $GL_2 \times G_2$ on $\mathbb{C}^2 \otimes V_7$ is prehomogeneous [44, Proposition 43]; this implies that the action of G_2 on the Grassmannian $G(2, V_7)$ has an open orbit. In this section we show that there are in fact finitely many orbits, that we completely classify. Of course, one obvious invariant for a line in $\mathbb{P}(V_7)$ is its position with respect to the invariant quadric \mathbb{Q}^5 . So we split the classification into the following two statements.

Proposition 9. *[Bisecant lines] Consider two points $x \neq y$ on the quadric $\mathbb{Q}^5 \subset \mathbb{P}(V_7)$. Up to the action of G_2 , there are only three possibilities.*

- (1) *The line \overline{xy} is a special line contained in the quadric, which means that x and y span a null plane.*
- (2) *The line \overline{xy} is a non-special line contained in the quadric, which means that x and y span an isotropic plane which is not a null plane.*
- (3) *The line \overline{xy} is not contained in the quadric, which means that x and y span a non-degenerate plane.*

Proof. It follows from [44, Proposition 43] that the complement of the open orbit of G_2 in $G(2, V_7)$ must be irreducible. Since it certainly contains the hypersurface of lines that are tangent to \mathbb{Q}^5 , there must be equality, which means that G_2 acts transitively on the lines that are true bisecants to the quadric. The remaining claims follow from the general fact that there are at most two G -orbits of lines in any generalized Grassmannian G/P , where P is a maximal parabolic subgroup of a simple complex Lie group G [30, Theorem 4.3]. \square

Proposition 10 (Tangent lines). *If $x \in \mathbb{Q}^5$ and \overline{xy} is a tangent line to \mathbb{Q}^5 , not contained in the quadric, then up to the action of G_2 there are only two possibilities.*

- (1) *The linear form $\omega(x, y, \bullet)$ is a non zero multiple of $Q(x, \bullet)$.*
- (2) *These two linear forms are not proportional.*

For simplicity, we slightly abused notations in (1) by using the same letter x to denote any generator of the line (idem for y). Continuing with the same abuse, we note that the first case is easier to express in the language of octonions: it means that the plane $\langle x, y \rangle$ contains a unique isotropic line (generated by x) such that the octonionic product xy is a (non-zero) multiple of x . We call such a plane a *semi-null* plane. Alternatively, we may observe that $\langle x, y \rangle$ is closed under the bracket on V_7 defined by ω ; this is the point of view chosen in [16, Appendix A], see in particular Lemma A.5.

In the second case xy is still non-zero, but does not belong to $\langle x, y \rangle$.

Proof. We may suppose that x is the line generated by e_α , in which case the stabilizer P of x in G_2 has for Lie algebra

$$\mathfrak{p} = \langle \mathfrak{t}, X_{\beta-\gamma}, X_{\gamma-\beta}, X_{-\beta}, X_{-\gamma}, X_\alpha, X_{\alpha-\beta}, X_{\alpha-\gamma} \rangle \subset \mathfrak{g}_2,$$

where \mathfrak{t} is our Cartan subalgebra and, as usual, for any root τ we denote by X_τ a generator of the root space $\mathfrak{g}_\tau \subset \mathfrak{g}_2$. The first three terms generate a copy of \mathfrak{gl}_2 which is the Levi part of this parabolic subalgebra. The remaining terms generate the nilpotent part.

The Zariski tangent space to the quadric at x identifies with $e_\alpha^\perp/\mathbb{C}e_\alpha$, in which there is a maximal filtration preserved by \mathfrak{p} :

$$\langle e_{-\beta}, e_{-\gamma} \rangle \subset \langle e_{-\beta}, e_{-\gamma}, e_0 \rangle \subset \langle e_{-\beta}, e_{-\gamma}, e_0, e_\beta, e_\gamma \rangle.$$

Represent an element in $\mathbb{P}(e_\alpha^\perp/\mathbb{C}e_\alpha)$ by a vector of the form $t = v + ze_0 + u$, with $u \in \langle e_\beta, e_\gamma \rangle$ and $v \in \langle e_{-\beta}, e_{-\gamma} \rangle$. We claim that up to the action of P there are only three orbits defined by the conditions that $u \neq 0$, or $u = 0, z \neq 0$, or $u = 0, z = 0, v \neq 0$.

If $u \neq 0$, using the action of the Levi part of P we can suppose that $u = e_\beta$. Then it is easy to check that $[t]$ belongs to the P -orbit of $[e_\beta]$, by simply using the action of the nilpotent subgroup N generated by $X_{-\beta}, X_{-\gamma}$, and of the maximal torus. We are then in case (2) of the Proposition.

If $u = 0$ and $z \neq 0$, the action of N is enough to check that $[t]$ belongs to the P -orbit of $[e_0]$. We are then in case (1) of the Proposition.

Finally, if $u = 0$ and $z = 0$, t spans a special line contained in the quadric, a situation which is excluded by our hypothesis. \square

This finally yields a complete classification of the G_2 -orbits in $G(2, V_7)$. Recall that in terms of the equivariant map $\wedge^2 V_7 \rightarrow V_7^\vee \rightarrow V_7$ deduced from the invariant three-form and quadratic form, a *null-plane* is characterized by the fact that a Plücker representative maps to zero, while for a *semi-null plane* it maps to a nonzero vector in the plane. Also, a plane is *isotropic*, *rank one* or *non-degenerate* when the rank of the restriction of the invariant quadratic Q is 0, 1, or 2, respectively.

Corollary 11. *Up to the action of G_2 , a plane $V_2 \subset V_7$ can be:*

- (1) *a null-plane,*
- (2) *an isotropic plane which is not a null-plane,*
- (3) *a semi-null plane,*
- (4) *a rank one plane which is not semi-null,*
- (5) *a non-degenerate plane.*

Explicit representatives are, respectively:

$$(4) \quad \langle e_\alpha, e_{-\beta} \rangle, \quad \langle e_\alpha, e_\beta \rangle, \quad \langle e_0, e_\alpha \rangle, \quad \langle e_0, e_\alpha + e_\beta \rangle, \quad \langle e_\alpha, e_{-\alpha} \rangle.$$

Each case defines a unique orbit of the G_2 -action on $G(2, V_7)$, and the incidence diagram is

$$(5) \quad \begin{array}{ccccc} & & & \mathcal{O}_{semi}^7 & \\ & & & \nearrow & \searrow \\ \mathcal{O}_{gen}^{10} & \longrightarrow & \mathcal{O}_{rank1}^9 & & \mathcal{O}_{null}^5 = X_{ad}(G_2) \\ & & \searrow & \nearrow & \\ & & \mathcal{O}_{iso}^7 & & \end{array}$$

Here the superscripts are the dimensions of the orbits and an arrow from \mathcal{O} to \mathcal{O}' means that \mathcal{O}' is contained in the Zariski closure of \mathcal{O} . We will see in Corollary 13 that \mathcal{O}_{semi} and \mathcal{O}_{iso} are projectively equivalent. By definition, the closure of \mathcal{O}_{iso}^7 is $OG(2, V_7)$. Following [16, Appendix A], we denote by $LieGr(2, V_7)$ the closure of \mathcal{O}_{semi}^7 .

2.4. Decomposing five-planes. When we restrict the invariant three-form to a codimension two subspace $V_5 \subset V_7$, since a three-form on V_5 is the same as a two-form on V_5^\vee , there are three possibilities a priori: we could get zero, or a form of rank two, or a form of rank four. The first case is actually impossible: we know that the most degenerate situation is when V_5 is orthogonal to a null-plane, say $\langle e_\alpha, e_{-\beta} \rangle$, and the restriction of ω to $\langle e_\alpha, e_{-\beta} \rangle^\perp = \langle e_\alpha, e_{-\beta}, e_0, e_\gamma, e_{-\gamma} \rangle$ is $v_0 \wedge v_\gamma \wedge v_{-\gamma} \neq 0$.

Definition. A subspace $V_5 \subset V_7$ is *decomposing* if $\omega|_{V_5}$ is decomposable, meaning that there exist linear forms v_1, v_2, v_3 such that $\omega|_{V_5} = v_1 \wedge v_2 \wedge v_3$. The kernel of $\omega|_{V_5}$ is a plane $A_2 \subset V_5$, that we call the *axis* of the decomposing plane.

Proposition 12. *A five-plane V_5 is decomposing if and only if its Q -orthogonal $V_2 = V_5^\perp$ is a null or a semi-null plane in V_7 . Its axis A_2 is an isotropic plane, which coincides with V_2 if and only if it is a null-plane.*

Proof. Since a skew-symmetric form has rank at most two if and only if its square is zero, the locus of decomposing five-planes is the zero-locus, in $G(5, V_7)$, of the global section t_ω of $\mathcal{U}_5^\vee(1)$ defined by ω (here \mathcal{U}_5 denotes the rank five tautological bundle on $G(5, V_7)$). If we identify $G(5, V_7)$ with $G(2, V_7)$ by orthogonality with respect to the invariant quadratic form, the bundle \mathcal{U}_5^\vee is identified with the quotient bundle \mathcal{Q}_5 on $G(2, V_7)$, and t_ω becomes a section of $\mathcal{Q}_5(1) = \wedge^2 \mathcal{U}_2^\vee \otimes \mathcal{Q}_5$. By the Borel-Weil theorem,

$$H^0(G(2, V_7), \mathcal{Q}_5(1)) = S_{21111} V_7 = Ker(V_7 \otimes \wedge^5 V_7 \longrightarrow \wedge^6 V_7),$$

with the usual notation for Schur functors. As a G_2 -module, $\wedge^5 V_7 = \wedge^2 V_7 = \mathfrak{g}_2 \oplus V_7$, so $V_7 \otimes \wedge^5 V_7$ has a unique invariant line, while $\wedge^6 V_7 = V_7$ has none. So $H^0(G(2, V_7), \mathcal{Q}_5(1))$ contains a unique invariant line; this implies that t_ω

coincides, up to scalar, with the invariant section t'_ω of $\mathcal{Q}_5(1)$ obtained as the composition

$$\wedge^2 \mathcal{U}_2 \hookrightarrow \wedge^2 V_7 \otimes \mathcal{O}_{G(2, V_7)} \xrightarrow{\omega} V_7 \otimes \mathcal{O}_{G(2, V_7)} \longrightarrow \mathcal{Q}_5,$$

(where we use once again the isomorphism $V_7^\vee \simeq V_7$ defined by the invariant quadratic form). By [16, Lemma A.5], the zero-locus of t'_ω is exactly $LieGr(2, V_7)$. The second part of the statement can be checked explicitly, using the explicit representatives of orbits given after Corollary 11. \square

From the axis A_2 , it is actually easy to reconstruct V_5 . Indeed, the correspondence between A_2 and V_2 induces an isomorphism between $LieGr(2, V_7)$ and $OG(2, V_7)$, as noticed in [16, Proposition A.7]. This isomorphism is actually linear and can be described as follows. Recall the decomposition into irreducible components

$$(6) \quad \wedge^2 V_7 = \mathfrak{g}_2 \oplus V_7,$$

where $V_7 \simeq V_7^\vee$ embeds into $\wedge^2 V_7$ by contraction with the (dual) invariant three-form, and $\wedge^2 V_7$ maps to V_7 by a similar contraction, whose kernel is \mathfrak{g}_2 . Denote by ι the involution of $\wedge^2 V_7$ acting by $+1$ on \mathfrak{g}_2 and by -1 on V_7 . It obviously fixes (the cone over) $\mathcal{O}_{null}^5 = X_{ad}(G_2)$, but a priori not (the cone over) $G(2, V_7)$.

Proposition 13. \mathcal{O}_{iso}^7 and \mathcal{O}_{semi}^7 are projectively equivalent via ι . Taking closures, we get

$$LieGr(2, V_7) = \iota(OG(2, V_7)).$$

Proof. Starting from the Plücker representative $e_\alpha \wedge e_\beta$ of the general isotropic plane A_2 as above, we first compute its image in V_7 to be $\omega(e_\alpha, e_\beta) = e_{-\gamma}$. Then we identify this vector with the two-form $e_{-\gamma} \cdot \omega = e_0 \wedge e_{-\gamma} - e_\alpha \wedge e_\beta$. As a consequence, if we apply the symmetry ι of $\wedge^2 V_7$ defined by its decomposition into \mathfrak{g}_2 -modules, we see that

$$\iota(e_\alpha \wedge e_\beta) = e_0 \wedge e_{-\gamma}.$$

This is nothing else than the Plücker representative of V_5^\perp , where V_5 is the unique decomposing five-plane containing A_2 , that we computed above. \square

Consider again (5). Taking closures, we get the diagram

$$\begin{array}{ccccc}
 & & LieGr(2, V_7) & & \\
 & \swarrow & \uparrow & \searrow & \\
 G(2, V_7) & \longleftarrow & \mathcal{C}_3 & \longrightarrow & X_{ad}(G_2) \\
 & \swarrow & \downarrow \iota & \searrow & \\
 & & OG(2, V_7) & &
 \end{array}$$

where \mathcal{C}_3 is a cubic hypersurface (indeed the restriction of the invariant quadratic form to planes yields a section of $S^2 \mathcal{U}_2^\vee$, and the condition that

this restriction is degenerate gives a section of $\det(S^2\mathcal{U}_2^\vee) = \mathcal{O}_{G(2,V_7)}(3)$. The dashed vertical arrow indicates the symmetry induced by ι .

Now we turn to the property that makes decomposing five-planes relevant for our problems.

Proposition 14. *Let V_5 be a decomposing five-plane.*

- (1) *The kernel of the map $\omega : \wedge^2 V_5 \rightarrow V_7^\vee$ is a five-plane K_5 .*
- (2) *The intersection $X_{ad}(G_2) \cap \mathbb{P}(K_5) = G(2, V_5) \cap \mathbb{P}(K_5)$ is a (possibly degenerate) cubic scroll.*

Proof. Consider the diagram

$$\begin{array}{ccccccc}
 & & & \wedge^2 V_5 & & & \\
 & & & \downarrow & \searrow^{\omega|_{V_5}} & & \\
 0 & \longrightarrow & V_5^\perp & \longrightarrow & V_7^\vee & \longrightarrow & V_5^\vee \longrightarrow 0
 \end{array}$$

The kernel of the diagonal arrow is $A_2 \wedge V_5$, where $A_2 \subset V_5$ denotes the axis; its dimension is seven. The kernel K_5 of the vertical arrow is equal to the kernel of the induced map $A_2 \wedge V_5 = \text{Ker}(\omega|_{V_5}) \rightarrow V_5^\perp$, in other words, we have a commutative diagram

$$(7) \quad \begin{array}{ccccccc}
 0 & \longrightarrow & K_5 & \longrightarrow & \wedge^2 V_5 & \longrightarrow & V_7^\vee \\
 & & \parallel & & \uparrow & & \uparrow \\
 0 & \longrightarrow & K_5 & \longrightarrow & A_2 \wedge V_5 & \longrightarrow & V_5^\perp \longrightarrow 0
 \end{array}$$

and K_5 has dimension at least five. If this dimension was bigger, the intersection $X_{ad}(G_2) \cap G(2, V_5)$ would contain a hyperplane section of $\mathbb{P}(A_2 \wedge V_5) \cap G(2, V_5)$. The latter intersection is the set of planes in V_5 that meet A_2 non trivially; geometrically, this is a cone over a cubic scroll isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$. But a hyperplane section of such a cone always contains a plane, and we would conclude that $X_{ad}(G_2)$ contains a plane, which is not the case [23, Lemma 3]. This proves (1), and then $X_{ad}(G_2) \cap \mathbb{P}(K_5) = G(2, V_5) \cap \mathbb{P}(K_5)$ is a codimension two linear section of a cone over $\mathbb{P}^1 \times \mathbb{P}^2$, hence a cubic scroll by [23, Lemma 5]. More precisely, one easily checks that the scroll is smooth when the section does not contain the vertex, which means that A_2 is not a null-plane; otherwise it is a cone over a twisted cubic. This proves (2). \square

Remark. Consider a decomposing five-plane V_5 , so that $\omega|_{V_5} = e_1 \wedge e_2 \wedge e_3$. If $V_5 = \langle e_6, e_7 \rangle^\perp$, there exists two-forms θ_6 and θ_7 such that

$$\omega = e_1 \wedge e_2 \wedge e_3 + e_6 \wedge \theta_6 + e_7 \wedge \theta_7,$$

and the axis is $A_2 = \langle e_1, e_2, e_3, e_6, e_7 \rangle^\perp$. For $\psi \in \wedge^2 V_5$, the contraction with ω yields

$$\iota_\psi(\omega) = \iota_\psi(e_1 \wedge e_2 \wedge e_3) + \iota_\psi(\theta_6)e_6 + \iota_\psi(\theta_7)e_7,$$

showing that the image of $\omega : \wedge^2 V_5 \rightarrow V_7^\vee$ is contained in A_2^\perp , hence it is equal to it.

The fact that an intersection $X_{ad}(G_2) \cap G(2, V_5)$ is always a conic or a cubic scroll, possibly singular, was already observed in [23, Lemma 5]. We can be more precise and compute this intersection for the five types of five-planes orthogonal to the five types of planes given in Corollary 11; using the explicit representatives of (4), this is a straightforward computation whose result is presented in the table below.

TABLE 1

$V_2 = V_5^\perp$	Representative	$X_{ad}(G_2) \cap G(2, V_5)$
<i>non-degenerate</i>	$\langle e_\alpha, e_{-\alpha} \rangle$	smooth conic
<i>rank one</i>	$\langle e_0, e_\alpha + e_\beta \rangle$	reduced singular conic
<i>isotropic</i>	$\langle e_\alpha, e_\beta \rangle$	double line
<i>semi-null</i>	$\langle e_0, e_\alpha \rangle$	smooth cubic scroll
<i>null</i>	$\langle e_\alpha, e_{-\beta} \rangle$	cone over a rational cubic

Let us briefly treat the isotropic and semi-null cases; the other ones are obtained with similar explicit computations and will be left to the reader.

If $V_2 = \langle e_\alpha, e_\beta \rangle$, then $V_5 = V_2^\perp = \langle e_\alpha, e_\beta, e_0, e_\gamma, e_{-\gamma} \rangle$. We deduce that

$$\wedge^2 V_5 \cap \mathfrak{g}_2 = \langle e_\alpha \wedge e_{-\gamma}, e_\beta \wedge e_{-\gamma}, e_\alpha \wedge e_\beta - e_0 \wedge e_{-\gamma} \rangle.$$

For a form $\theta = xe_\alpha \wedge e_{-\gamma} + ye_\beta \wedge e_{-\gamma} + z(e_\alpha \wedge e_\beta - e_0 \wedge e_{-\gamma})$ in this three-plane we have $\theta \wedge \theta = -2z^2 e_\alpha \wedge e_\beta \wedge e_0 \wedge e_{-\gamma}$, hence the double line.

If $V_2 = \langle e_0, e_\alpha \rangle$ then $V_5 = V_2^\perp = \langle e_\alpha, e_\beta, e_{-\beta}, e_\gamma, e_{-\gamma} \rangle$ and

$$\wedge^2 V_5 \cap \mathfrak{g}_2 = \langle e_\alpha \wedge e_{-\beta}, e_\alpha \wedge e_{-\gamma}, e_\beta \wedge e_{-\gamma}, e_\gamma \wedge e_{-\beta}, e_\beta \wedge e_{-\beta} - e_\gamma \wedge e_{-\gamma} \rangle.$$

For $\theta = pe_\alpha \wedge e_{-\beta} + qe_\alpha \wedge e_{-\gamma} + re_\beta \wedge e_{-\gamma} + se_{-\beta} \wedge e_\gamma + t(e_\beta \wedge e_{-\beta} - e_\gamma \wedge e_{-\gamma})$, we compute that $\frac{1}{2}\theta \wedge \theta$ is equal to

$$(pr-qt)e_\alpha \wedge e_{-\beta} \wedge e_\beta \wedge e_{-\gamma} + (qs-pt)e_\alpha \wedge e_{-\gamma} \wedge e_{-\beta} \wedge e_\gamma + (rs-t^2)e_\beta \wedge e_{-\gamma} \wedge e_{-\beta} \wedge e_\gamma.$$

So $\theta \wedge \theta = 0$ if and only if the rank of the matrix $\begin{pmatrix} p & s & t \\ q & t & r \end{pmatrix}$ is at most one, hence the smooth cubic scroll.

2.5. Some vector bundles. By Proposition 13, $LieGr(2, V_7)$ has two G_2 -equivariant embeddings in $G(2, V_7)$, the natural one and its twist by ι .

Definition. To keep the exposition readable, we use the same letter for the tautological bundle \mathcal{U}_2 of $G(2, V_7)$ and its restriction to $LieGr(2, V_7)$. We denote by $\mathcal{V}_5 = \mathcal{U}_2^\perp$ the bundle of decomposing five-planes (orthogonality being taken with respect to the invariant quadratic form Q). The rank two subbundle $\mathcal{A}_2 \subset \mathcal{V}_5$ defined by their axis (recall Proposition 12) is the tautological rank two bundle on $LieGr(2, V_7)$ corresponding to the twisted embedding in $G(2, V_7)$; in the terminology of [16, Remark A.8], \mathcal{A}_2 is the dual spinor bundle.

By definition of decomposing five-planes and their axis, ω descends to a global section of $\wedge^3(\mathcal{V}_5/\mathcal{A}_2)^\vee$, everywhere nonzero. Also recall that the morphism $\wedge^2\mathcal{V}_5 \rightarrow V_7^\vee \otimes \mathcal{O}_{LieGr(2, V_7)}$ has constant rank by Proposition 14.

Definition. We denote by \mathcal{K}_5 the kernel of this morphism, a rank five vector bundle on $LieGr(2, V_7)$.

This bundle admits a natural inclusion

$$(8) \quad \mathcal{K}_5 \hookrightarrow \mathfrak{g}_2 \otimes \mathcal{O}_{LieGr(2, V_7)}.$$

All these bundles are G_2 -equivariant and, taking into account the remark after Proposition 14, the relative version of diagram (7) can be completed into the following commutative diagram of vector bundles:

$$(9) \quad \begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \uparrow & & \uparrow & \\ & & & \wedge^2(\mathcal{V}_5/\mathcal{A}_2) & \xlongequal{\quad} & (\mathcal{V}_5/\mathcal{A}_2)^\vee & \\ & & & \uparrow & & \uparrow & \\ 0 & \longrightarrow & \mathcal{K}_5 & \longrightarrow & \wedge^2\mathcal{V}_5 & \longrightarrow & \mathcal{A}_2^\perp & \longrightarrow & 0 \\ & & \parallel & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{K}_5 & \longrightarrow & \mathcal{A}_2 \wedge \mathcal{V}_5 & \longrightarrow & \mathcal{U}_2 & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Since $\wedge^3(\mathcal{V}_5/\mathcal{A}_2)$ is trivial, \mathcal{V}_5 and \mathcal{A}_2 have the same determinant, and their orthogonal as well. We deduce that

$$(10) \quad \begin{aligned} \det(\mathcal{U}_2) &= \det(\mathcal{A}_2) = \det(\mathcal{V}_5) &= \mathcal{O}_{LieGr(2, V_7)}(-1), \\ \det(\mathcal{K}_5) &= \mathcal{O}_{LieGr(2, V_7)}(-3). \end{aligned}$$

Recall from the proof of Proposition 14 that the cubic scroll associated to a decomposing five-plane V_5 is a codimension two linear section, by $\mathbb{P}(K_5)$, of the Schubert cycle of planes in V_5 that meet its axis A_2 . This Schubert

cycle is contained in $\mathbb{P}(A_2 \wedge V_5)$, and can be identified with a cone over $\mathbb{P}(A_2) \times \mathbb{P}(V_5/A_2)$ with vertex $\mathbb{P}(\wedge^2 A_2)$. The linear section is a cone over a rational cubic when it contains the vertex (meaning that A_2 is a null plane), otherwise it is a smooth cubic scroll.

For future use (more precisely, for section 4.3), let us denote by \mathcal{R} and \mathcal{C} the corresponding families of cubic scrolls and Schubert cycles. They fit in the following diagram:

$$(11) \quad \begin{array}{ccc} & \mathbb{P}_{LieGr(2,V_7)}(\mathcal{A}_2 \wedge \mathcal{V}_5) & \longrightarrow \mathbb{P}(\wedge^2 V_7) \\ \mathcal{C} \nearrow & \uparrow & \nearrow \\ \mathcal{C} & \longrightarrow G(2, V_7) & \\ \mathcal{R} \nearrow & \downarrow & \downarrow \\ & \mathbb{P}_{LieGr(2,V_7)}(\mathcal{K}_5) & \longrightarrow \mathbb{P}(\mathfrak{g}_2) \\ \mathcal{R} \nearrow & \uparrow & \nearrow \\ \mathcal{R} & \longrightarrow X_{ad}(G_2) & \end{array}$$

The whole lefthand part of this diagram is fibered over $LieGr(2, V_7)$.

Moreover, \mathcal{R} is defined in \mathcal{C} as the intersection with $\mathbb{P}_{LieGr(2,V_7)}(\mathcal{K}_5)$. This will be used in the proof of Theorem 45.

2.6. Cubic scrolls in hyperplane sections of $X_{ad}(G_2)$. Let us come back to diagram (2). We denote by $\Upsilon(X_{ad}(G_2))$ the Hilbert scheme of cubic scrolls and cones in $X_{ad}(G_2)$.

Lemma 15. *For any line $L \subset \mathbb{Q}^5$, the scheme $R_L := p_X(p_{\mathbb{Q}}^{-1}L)$ is a cubic scroll or cone in $X_{ad}(G_2)$. Moreover the map*

$$OG(2, V_7) \rightarrow \text{Hilb}(X_{ad}(G_2)), \quad L \mapsto R_L,$$

is an isomorphism onto $\Upsilon(X_{ad}(G_2))$.

Proof. The correspondence (2) allows to identify \mathbb{Q}^5 with the Hilbert scheme $F_1(X_{ad}(G_2))$ of lines in $X_{ad}(G_2)$. Recall from Proposition 9 that there are two types of lines in \mathbb{Q}^5 . If L is special, it defines a point x in $X_{ad}(G_2)$, and R_L is the union of lines in $X_{ad}(G_2)$ passing through that point - hence a cubic cone. If L is not special, then R_L is still a surface of degree three. If R is a cubic scroll or cone, the pencil of lines in the ruling of R defines a curve $C \in \mathbb{Q}^5$, which for degree reasons must be a line. \square

Let $\mathcal{X} \subset X_{ad}(G_2) \times \mathbb{P}(\mathfrak{g}_2^\vee)$ be the universal hyperplane section of $X_{ad}(G_2)$, and $\Upsilon(\mathcal{X}/\mathbb{P}(\mathfrak{g}_2^\vee))$ be the relative Hilbert scheme of cubic scrolls and cones.

There is a diagram

$$(12) \quad \begin{array}{ccc} & \Upsilon(\mathcal{X}/\mathbb{P}(\mathfrak{g}_2^\vee)) & \\ p_\Upsilon \swarrow & & \searrow p_{\mathfrak{g}} \\ \Upsilon(X_{ad}(G_2)) & & \mathbb{P}(\mathfrak{g}_2^\vee), \end{array}$$

and $\Upsilon(X_{ad}(G_2) \cap H) = p_{\mathfrak{g}}^{-1}([H])$ for any hyperplane $H \subset \mathfrak{g}_2$. The following information can be extracted from [26].

- (1) p_Υ is a \mathbb{P}^8 -bundle by [26, Lemma 8.5] (the scheme Σ' defined there is the universal cubic scroll or cone by Lemma 15).
- (2) $\Upsilon(\mathcal{X}/\mathbb{P}(\mathfrak{g}_2^\vee)) \simeq \tilde{Y}$ in the notation of [26], and the map $p_{\mathfrak{g}}$ coincides with \tilde{g} therein.
- (3) The Stein factorization of this map takes the form $p_{\mathfrak{g}} = \tilde{g} = \bar{g}_3 \circ \phi$ by [26, Lemma 8.8].
- (4) The dual of the decomposition (6) yields a projection $\mathbb{P}(\wedge^2 V_7^\vee) \dashrightarrow \mathbb{P}(\mathfrak{g}_2^\vee)$. The image of $G(2, V_7^\vee)$ via the projection is Z_1 in Kuznetsov's notation, see [26, Lemma 8.2 and Corollary 8.11]. Over the complement of Z_1 , the map ϕ is an étale-locally trivial \mathbb{P}^2 -bundle by [26, Proposition 8.9], and the map $\bar{g}_3 : \tilde{Y}_3 \rightarrow \mathbb{P}(\mathfrak{g}_2^\vee)$ is a double cover branched along the dual variety of $X_{ad}(G_2)$, i.e. the discriminant sextic hypersurface $\Delta \subset \mathbb{P}(\mathfrak{g}_2^\vee)$.

Since Z_1 is irreducible, a union of G_2 -orbits and 10-dimensional, it has to be the closure of O_{10} in [22, Lemma 3.1]. The intersection $X_{ad}(G_2) \cap H$ is smooth for $[H] \notin \Delta$, and nodal with a single node for $[H] \in \Delta \setminus Z_1$ ([22, Proof of Theorem 2.3]), while Z_1 parametrizes highly singular hyperplane sections.

Putting everything together, we deduce the following statement:

Proposition 16. *The Hilbert scheme $\Upsilon(X_{ad}(G_2) \cap H)$ is isomorphic to*

- (1) $\mathbb{P}^2 \sqcup \mathbb{P}^2$ if $[H] \notin \Delta$, i.e. $X_{ad}(G_2) \cap H$ is smooth;
- (2) \mathbb{P}^2 if $[H] \in \Delta \setminus Z_1$, i.e. $X_{ad}(G_2) \cap H$ is nodal.

The first part of the statement can also almost be deduced directly from Lemma 15, as follows. An equation of H defines a section of the dual Cayley bundle \mathcal{C}^\vee , and the zero locus Z of this section is the locus of points in \mathbb{Q}^5 defining lines in $X_{ad}(G_2)$ that are contained in H . So cubic scrolls and cones contained in $X_{ad}(G_2) \cap H$ are in correspondence with lines in Z . But by [43, Theorem 3.7], when Z is smooth it is isomorphic to the flag manifold Fl_3 , hence lines in Z are parametrized by two copies of \mathbb{P}^2 .

Cubic surface scrolls in smooth hyperplane sections $F = X_{ad}(G_2) \cap H$ were also studied in [23] by reducing to surface scrolls in $G(3, U_6)$. Indeed, fix a vector space U_6 of dimension six. By [23, Theorem 1], there exists a unique linear embedding $\psi : H \hookrightarrow \mathbb{P}(\wedge^3 U_6)$ sending F isomorphically to a four dimensional linear section of $G(3, U_6)$. Now, inside $G(3, U_6)$ there are two

different types of cubic scrolls, yielding two families in F both parametrized by \mathbb{P}^2 .

When F is nodal with a single node f , the projection ψ from the node sends F birationally to a codimension two linear section $\tilde{F} = LG(3, U_6) \cap P$ of the Lagrangian Grassmannian $LG(3, U_6)$, see [22]. The cubic scrolls and cones in F are sent to maximal quadrics in \tilde{F} , and one can check that such quadrics are parametrized by a projective plane.

Remark. Not surprisingly, the cubic scrolls in $X_{ad}(G_2)$ defined by decomposing five-planes in V_7 also yield twisted cubic curves in Fano fourfolds and threefolds of genus ten. As observed by an anonymous referee, Proposition 16 is closely related to Theorem 9.11 of [28], which asserts that the Hilbert scheme of rational cubic curves inside a smooth prime Fano threefold of genus ten is a \mathbb{P}^2 -bundle over a genus two curve.

3. THE INVOLUTION

In all the sequel we consider a general K3 surface S of genus 10. (Beware that S will be very general starting from section 3.2.) By the work of Mukai [38], S can be described as a generic codimension three linear section of the adjoint variety of G_2 ,

$$(13) \quad S = X_{ad}(G_2) \cap L \subset \mathbb{P}(\mathfrak{g}_2),$$

where $L = \mathbb{P}(V_{11})$ for $V_{11} \subset \mathfrak{g}_2$ a codimension three subspace.

We will first present our geometric construction of the birational involution φ of $S^{[3]}$ whose existence is known abstractly from [6]. This will immediately stress the rôle of the variety J parametrizing schemes in $S^{[3]}$ that do not generate a hyperplane. These subschemes of S are closely related to the cubic scrolls we studied in the previous section. In order to get a better understanding of the indeterminacy locus I of φ , we will then need to dig a little deeper in Bayer-Macri's description of the Mori cone and of wall-crossing for $S^{[3]}$. We will deduce a description of I in terms of vector bundles, implying that $I = J$ and that it admits a structure of \mathbb{P}^2 -fibration. This is exactly what allows to define a Mukai flop, and we will finally prove that this Mukai flop is nothing else than φ .

Hypothesis. In the next section 3.1, S will be any smooth polarized K3 surface of genus ten admitting a Mukai model (13). According to Mukai, this is equivalent to asking that the polarized K3 surface is Brill-Noether general [40, Theorem 3.10 and Theorem 4.7] (but see also [3, Section 1.2-1.3]). Polarized K3 surfaces which are Brill-Noether general fill a Zariski open subset of the 19-dimensional moduli space of K3 surfaces of genus ten, whose complement has been explicitly computed [15, Lemma 2.8].

Starting from section 3.2 we will always suppose that S is very general, in the sense that $Pic(S)$ is generated by the hyperplane divisor (the restriction of the hyperplane divisor from $\mathbb{P}(\mathfrak{g}_2)$). We will use more than once that this implies that S contains no line and no conic (more generally, no curve of

degree smaller than 18). For statements like Lemma 21, we will also need L to be generic enough for certain degeneracy schemes to be well-behaved. Some specific genericity assumptions are also required in order to be able to apply the results of [23]. In the end we will not attempt to make our genericity conditions precise.

3.1. Construction of the involution. Let S be a smooth K3 surface obtained as a linear section (13) of $X_{ad}(G_2)$.

Lemma 17. *Let p_1, p_2, p_3 be generic points on S . They correspond to three null planes P_1, P_2, P_3 spanning a hyperplane of V_7 .*

Proof. By contradiction, suppose that P_1, P_2, P_3 are contained in a five-plane V_5 . Since S is a linear section of $X_{ad}(G_2)$, Table 1 shows that V_5 must be decomposing, since otherwise we would conclude that there is a conic or a line in S passing through the generic point, which is not the case.

Now, fix P_1 and P_2 , spanning a four-plane V_4 . Recall that by the generalized Bruhat decomposition (see e.g. [9, Proposition 3.16]), the relative positions of two points in a rational homogeneous space are in bijection with double cosets of the Weyl group. Applying this to $X_{ad}(G_2)$, we conclude that, up to the action of G_2 , there are only four possible relative positions for two points p_1 and p_2 of the adjoint variety. These relative positions are easy to identify: either p_1 and p_2 coincide, or they are joined by a line in $X_{ad}(G_2)$, or they are orthogonal with respect to the Killing form, or they are in general position.

A dimension count shows that in the latter case, there is only a one-dimensional family of decomposing five-planes V_5 containing V_4 . If V_5 is constant when we move the third point P_3 , the surface S must be contained in $G(2, V_5)$, which is clearly absurd by Table 1 again. If V_5 moves in a one-dimensional family, then S is covered by its intersections with the corresponding $G(2, V_5)$'s. In particular these intersections must be curves, and linear sections of the cubic scrolls or cones $X_{ad}(G_2) \cap G(2, V_5)$; so they are contained in hyperplane sections of these scrolls, hence they must be rational curves or union of rational curves. We would thus conclude again that S is covered by rational curves, which is not the case.

We conclude that p_1 and p_2 cannot be in general position in $X_{ad}(G_2)$, which means that they are orthogonal with respect to the Killing form. Since this must be true for any two general points in S , this implies that the span L of S is isotropic; a contradiction since the Killing form is non-degenerate, and L has dimension bigger than 6. \square

For p_1, p_2, p_3 generic points on S , let V_6 denote the span of the corresponding null planes in V_7 . By Proposition 8 and (3),

$$\begin{aligned} S \cap G(2, V_6) &= (X_{ad}(G_2) \cap L) \cap G(2, V_6) = \\ &= (X_{ad}(G_2) \cap G(2, V_6)) \cap L = Fl(V_6) \cap L \end{aligned}$$

is a codimension three section of the flag variety, hence in general consists in six reduced points: p_1, p_2, p_3 plus three other points q_1, q_2, q_3 . Let

$$\varphi(p_1 + p_2 + p_3) = q_1 + q_2 + q_3.$$

This gives a rational map from $S^{[3]}$ to itself, more formally described in the proof of the next statement.

Theorem 18. *Let $S = X_{ad}(G_2) \cap L$ be a K3 surface of genus 10. Then there is a unique birational involution $\varphi : S^{[3]} \dashrightarrow S^{[3]}$ such that for a subscheme $Z \subset S$ defined by a general point of $S^{[3]}$, there is a unique hyperplane $V_6 \subset V_7$ such that*

$$S \cap G(2, V_6) = Z \cup \varphi(Z).$$

In particular, for S very general φ has to coincide with the non symplectic birational involution whose existence was predicted in [6].

Proof. Let \mathcal{Q}_5 denote the quotient bundle on the Grassmannian $G(2, V_7)$, and consider the natural morphism $\mathbb{P}_S(\mathcal{Q}_5^\vee) \rightarrow \mathbb{P}(V_7^\vee)$. Since the pre-image of a point defined by a hyperplane V_6 is the set of pairs $(P \subset V_6)$ where the plane P defines a point of S , this morphism is generically 6 to 1. Therefore, the relative Hilbert scheme $Hilb^3(\mathbb{P}_S(\mathcal{Q}_5^\vee)/\mathbb{P}(V_7^\vee))$ admits an obvious non-trivial birational involution. Finally, the projection of $\mathbb{P}_S(\mathcal{Q}_5^\vee)$ to S induces a birational morphism from $Hilb^3(\mathbb{P}_S(\mathcal{Q}_5^\vee)/\mathbb{P}(V_7^\vee))$ to $S^{[3]}$, whose inverse associates to a triple of general points of S the hyperplane generated by the three corresponding planes. It is clear that the induced birational involution of $S^{[3]}$ coincides with φ . \square

3.2. Schemes generating a non-maximal linear space. From now on, S is a very general linear section of $X_{ad}(G_2)$. The definition of φ stresses the condition, for three points in S , that the corresponding planes in V_7 generate a hyperplane. In order to understand better this condition, consider the universal scheme \mathcal{Z} of length three over S , with the two projections

$$(14) \quad \begin{array}{ccc} & \mathcal{Z} & \\ p \swarrow & & \searrow q \\ S^{[3]} & & S. \end{array}$$

Denote again by \mathcal{U}_2 the restriction to S of the rank two tautological bundle on $G(2, V_7)$. The tautological epimorphism $V_7^\vee \otimes \mathcal{O}_S \rightarrow \mathcal{U}_2^\vee$ induces a morphism

$$(15) \quad \phi : V_7^\vee \otimes \mathcal{O}_{S^{[3]}} \rightarrow p_*(q^*\mathcal{U}_2^\vee).$$

Since p is flat, $p_*(q^*\mathcal{U}_2^\vee)$ is a vector bundle of rank six, sometimes denoted $(\mathcal{U}_2^\vee)^{[3]}$. We say that a scheme $Z \subset G(2, V_7)$ generates the linear space $W \subset V_7$ if

$$W^\perp = Ker(V_7^\vee \rightarrow H^0(Z, \mathcal{U}_2^\vee|_Z)).$$

Then of course $Z \subset G(2, W)$, since this sub-Grassmannian is cut out in $G(2, V_7)$ by the linear forms from $W^\perp \wedge V_7^\vee \subset \wedge^2 V_7^\vee$.

Definition. Let $J \subset S^{[3]}$ denote the degeneracy locus of ϕ , with the scheme structure defined by the corresponding Fitting ideal. This is the locus of length three subschemes of S that fail to generate a hyperplane of V_7 .

When Z is a reduced length three subscheme of S , outside J , it consists in three points corresponding to three planes in general position in V_7 , whose linear span is a hyperplane. By Lemma 17, J is a proper subset of $S^{[3]}$.

The locus $J \subset S^{[3]}$ is closely related to decomposing five-planes, as our next result shows.

Proposition 19. *Let $Z \subset S$ be a length three subscheme that does not generate a hyperplane. Then it generates a decomposing five-plane in V_7 .*

Proof. First observe that Z cannot generate a subspace of V_7 of dimension smaller than five. Indeed, for any $V_4 \subset V_7$ the intersection $X_{ad}(G_2) \cap G(2, V_4) = \mathbb{P}(\mathfrak{g}_2) \cap G(2, V_4)$ is a conic, or a line, or a finite scheme of length at most two. Since S contains no conic and no line, a fortiori $S \cap G(2, V_4)$ must be a finite scheme of length at most two, and cannot contain Z .

If Z generates a V_5 , it must be contained inside $G(2, V_5) \cap X_{ad}(G_2)$. But recall from Table 2.4 that this is a conic when V_5 is not decomposing, and by the same argument as before $S \cap G(2, V_5)$ cannot contain Z . \square

The previous result and Proposition 12 imply that there is a map

$$\alpha : J \longrightarrow \text{LieGr}(2, V_7)$$

associating to a subscheme Z the decomposing five-plane V_5 it generates. The map is injective, since we recover Z from V_5 by intersecting with $L = \mathbb{P}(V_{11})$ the cubic surface scroll $X_{ad}(G_2) \cap G(2, V_5)$. Since $L \subset \mathbb{P}(\mathfrak{g}_2)$ has codimension three, the intersection cannot be transverse: we will detect this defect of transversality through the morphism

$$(16) \quad \psi : \mathcal{K}_5 \longrightarrow (\mathfrak{g}_2/V_{11}) \otimes \mathcal{O}_{\text{LieGr}(2, V_7)}$$

deduced from (8). We denote by $\mathcal{D}(\psi)$ its first degeneration scheme.

Lemma 20. $\mathcal{D}(\psi) = \alpha(J)$.

Proof. Let V_5 be a decomposing five-plane, and suppose that the corresponding morphism $K_5 \rightarrow \mathfrak{g}_2/V_{11}$ is surjective, so that its kernel is two-dimensional. Then by Proposition 14 and Table 2.4 the subscheme

$$S \cap G(2, V_5) = \mathbb{P}(V_{11}) \cap X_{ad}(G_2) \cap G(2, V_5)$$

is the intersection of a cubic scroll (or cone) with a line, hence it has length at most two since S contains no line. So $\alpha(J)$ is contained in $\mathcal{D}(\psi)$.

Conversely, if V_5 belongs to $\mathcal{D}(\psi)$, $L = \mathbb{P}(V_{11})$ meets $\mathbb{P}(K_5)$ along a codimension two subspace, that cuts the (possibly degenerate) cubic scroll $X_{ad}(G_2) \cap G(2, V_5)$ along a length three subscheme of S (once again, this

intersection cannot be positive dimensional, since it is defined by quadrics and S contains no line or conic.) \square

Lemma 21. *ψ has constant corank one over $\alpha(J)$ and α is an isomorphism onto its image. In particular, J is smooth.*

Proof. If $\mathbb{P}(K_5) \cap L$ has dimension three, L cuts the corresponding cubic scroll along a degree 3 curve which is contained inside S . But there is no such curve in S . Since $\alpha(J)$ is a degeneracy locus by Lemma 20, for L generic it is singular where the rank of ψ drops by more than one, which never happens. So $\alpha(J)$ is smooth, and we conclude by injectivity of α . \square

In words, we conclude that J parametrizes pairs (V_5, H) , where $V_5 \subset V_7$ is a decomposing five-plane and $H \subset \mathbb{P}(\mathfrak{g}_2)$ is a hyperplane through S that also contains the cubic scroll $X_{ad}(G_2) \cap G(2, V_5)$; moreover this hyperplane is unique, hence there is a morphism

$$(17) \quad \pi : J \longrightarrow L^\perp \simeq \mathbb{P}^2.$$

We are now ready to describe the structure of J , using results from Section 2.6. Because of [23, Lemma 7], the fiber of π over the hyperplane H can be identified with the Hilbert scheme of cubic surface scrolls in $X_{ad}(G_2) \cap H$, so

$$(18) \quad J = \bigcup_{[H] \in L^\perp} \Upsilon(X_{ad}(G_2) \cap H).$$

Let C be the intersection of $L^\perp \subset \mathbb{P}(\mathfrak{g}_2^\vee)$ with the dual variety Δ of $X_{ad}(G_2)$; since L is generic, this is a smooth plane sextic curve. Denote by $g : \Sigma \longrightarrow L^\perp \simeq \mathbb{P}^2$ the double cover branched over the sextic C , so that Σ is a smooth K3 surface.

Corollary 22. *J is irreducible. The Stein factorization of $\pi : J \longrightarrow L^\perp$ is*

$$(19) \quad J \xrightarrow{f} \Sigma \xrightarrow{g} L^\perp,$$

where f is an étale locally trivial \mathbb{P}^2 -fibration.

Proof. Via (18), the map (17) is nothing but the restriction of p_g in (12) over $L^\perp \subset \mathbb{P}(\mathfrak{g}_2^\vee)$. Generically $L^\perp \cap Z_1 = \emptyset$, then the Stein factorization follows from Proposition 16. \square

The surface Σ is the K3 surface of degree two which is associated to S (together with a Brauer class), in the context of Homological Projective Duality for the adjoint variety of G_2 [26, section 6.4].

Moreover, in [23] the authors give an explicit description of Σ in terms of stable vector bundles on S . Indeed, denote by \mathcal{E}^σ and \mathcal{E}^τ , respectively, the dual of the tautological bundle and the tautological quotient bundle over $G(3, U_6)$, both of rank three; over the Lagrangian Grassmannian $LG(3, U_6)$ the two bundles are isomorphic, we denote them by $\tilde{\mathcal{E}}$.

For $[H] \in L^\perp$, so that $F = X_{ad}(G_2) \cap H$ contains S , recall from Section 2.6 that there is a regular map $\psi_H : F \rightarrow G(3, U_6)$ when F is smooth, and a rational map $\psi_H : F \dashrightarrow LG(3, U_6)$ when F is nodal. Their restrictions ψ_S to S are always regular. [23, Theorem 2] and its proof yield the following statement.

Theorem 23. Σ is isomorphic with the moduli space $M_{\mathcal{H}}(3, \mathcal{H}, 3)$ of \mathcal{H} -stable vector bundles with Mukai vector $(3, \mathcal{H}, 3)$. The fiber over $[H]$ of the double cover $g : \Sigma \rightarrow L^\perp$ is

- the classes of the two bundles $\mathcal{E}_H^\sigma = \psi_S^* \mathcal{E}^\sigma$ and $\mathcal{E}_H^\tau = \psi_S^* \mathcal{E}^\tau$ when $[H] \notin C$,
- the class of the bundle $\psi_S^* \tilde{\mathcal{E}}$ when $[H] \in C$.

Definition. We say that a cubic surface scroll in $G(3, U_6)$ is of σ -type if it is contained in the Schubert variety $Fl(U_1, 3, U_6)$ of spaces containing some line $U_1 \subset U_6$. It is of τ -type if it is contained in the Grassmannian $G(3, U_5)$ for some $U_5 \subset U_6$. (Note that $Fl(U_1, 3, U_6)$ and $G(3, U_5)$ are both isomorphic with $G(2, 5)$.)

When $[H] \notin C$, both \mathcal{E}_H^σ and \mathcal{E}_H^τ have a six-dimensional space of global sections: in the projectivizations of those spaces lie the two copies of \mathbb{P}^2 (two Veronese surfaces, see [23, Proposition 4]) parametrizing the fiber of $f : J \rightarrow \Sigma$ over $g^{-1}([H])$. More precisely, and taking into account also the singular case, one gets:

Proposition 24. For $\mathcal{E} \in \Sigma \simeq M_{\mathcal{H}}(3, \mathcal{H}, 3)$, a finite scheme $Z \in S^{[3]}$ belongs to $f^{-1}(\mathcal{E})$ if and only if it is the zero locus of some section of \mathcal{E} .

For $\{\mathcal{E}_H^\sigma, \mathcal{E}_H^\tau\}$ the fiber over $[H] \notin C$, all subschemes in $f^{-1}(\mathcal{E}_H^\sigma)$, resp. $f^{-1}(\mathcal{E}_H^\tau)$, are a linear section of the pullback via ψ_H of a cubic surface scroll of σ -type (resp. τ -type). This property characterizes the two fibers.

Proof. See [23, proof of Lemma 8] for the smooth case. The nodal case is completely analogous, once we have observed that the maximal quadrics $Q_L \subset LG(3, U_6)$, in the singular case of Section 2.6, are the zero loci of the non-zero sections of the dual tautological vector bundle. \square

The moduli space $M_{\mathcal{H}}(3, \mathcal{H}, 3)$ will play a crucial rôle in the sequel. In the next subsection we will take a different perspective on the involution φ , but the moduli space will soon return to stage.

3.3. The nef and movable cones. The goal of this section is to prove the existence of a (unique) flop over $S^{[3]}$. We will also gather some information about its extremal contractions, that will be made more precise in the next subsections.

Recall that the second cohomology group

$$(20) \quad H^2(S^{[n]}, \mathbb{Z}) \simeq H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta,$$

where 2δ is the class of the divisor of non-reduced schemes. The embedding of $H^2(S, \mathbb{Z})$ is given by $\xi \mapsto \xi_n := HC^* \xi^{(n)}$, where $HC : S^{[n]} \rightarrow S^{(n)}$ is

the Hilbert-Chow morphism and $\xi^{(n)}$ is the class in the second cohomology group of the symmetric product, whose pullback to S^n is the sum of the pullbacks of ξ itself via all the projections $p_i : S^n \rightarrow S$, i.e. $\xi^{(n)} = \sum_{i=1}^n p_i^* \xi$.

We first describe the movable cone and the nef cone of $S^{[3]}$. The following result is [6, Proposition 4.1] ($d = 2$ for $t = 9$) and [6, Lemma 3.6] with $n = 3$ and $t = 4n - 3$. We write down a direct proof for the reader's convenience.

Proposition 25. *The nef cone of $S^{[3]}$ is*

$$\text{Nef}(S^{[3]}) = \langle \mathcal{H}_3, \mathcal{H}_3 - 2\delta \rangle.$$

The movable cone has two chambers, exchanged by the action of φ :

$$\text{Mov}(S^{[3]}) = \text{Nef}(S^{[3]}) \cup \varphi^*(\text{Nef}(S^{[3]})).$$

Proof. The structure of the movable cone of $S^{[3]}$ is described in [4, Proposition 13.1] in terms of Pell's equations: it is the interior of the convex cone generated by \mathcal{H}_3 and $17\mathcal{H}_3 - 9 \cdot 4\delta$, the minimal solution of $X^2 - 18Y^2 = 1$ being $(17, 4)$. The walls in the movable cone are spanned by vectors of the form $X\mathcal{H}_3 - 18Y\delta$ for (X, Y) a positive solution of $X^2 - 72Y^2 = 8 + \alpha^2$, and $\alpha \in \{1, 2\}$; this is [4, Theorem 12.1 and Theorem 12.3], translated in terms of generalized Pell's equations using [12, Lemma 2.5] (see also [12, Remark 2.8]). Actually, the vectors cutting chambers of the movable cone are those for which $\frac{2Y}{X} < \frac{4}{17}$. (These Pell's equations will appear again in the proof of Lemma 29, where their relation with the birational geometry of $S^{[3]}$ will become apparent.)

One can immediately check that there are no integral solutions when $\alpha = 2$. So we are left to consider solutions of $X^2 - 72Y^2 = 9$, which in turn are in the form $(3Z, Y)$ with $Z^2 - 8Y^2 = 1$; the latter is a Pell's equation with minimal positive solution $(3, 1)$. All the solutions of this equation can be found recursively by letting

$$(Z_{k+1}, Y_{k+1}) = (3Z_k + 8Y_k, Z_k + 3Y_k).$$

Moreover $\frac{Z_{k+1}}{Y_{k+1}} < \frac{Z_k}{Y_k}$. The first two solutions are $(3, 1)$, $(17, 6)$. Note that for the second one we have $2\frac{Y_2}{3Z_2} = \frac{4}{17}$, so this solution corresponds to a boundary of the movable cone, and there is therefore exactly one wall inside the movable cone. This also means that the ample cone is the interior of the cone generated by \mathcal{H}_3 and $9\mathcal{H}_3 - 18\delta$; this last vector is proportional to $\mathcal{H}_3 - 2\delta$, which is the generator of the invariant lattice for the action of φ by [6, first case of Proposition 2.2]. Then the action of φ in $\text{NS}(S^{[3]})$ is $-R_{\mathcal{H}_3 - 2\delta}$, the opposite of the reflection with respect to $\mathcal{H}_3 - 2\delta$ [6, Proposition 2.1], exchanging the two rays of the movable cone, hence the two chambers. \square

Remark. For future use, let us record that the solution $(X, Y) = (9, 1)$ giving the second wall corresponds, going back to [4, Theorem 12.1] via [12, Lemma 2.5], to the class $a = -(2, -\mathcal{H}, 5) \in H_{\text{alg}}^*(S, \mathbb{Z})$, with $a^2 = -2$. Also,

$(a, v) = 1$ for $v = (1, 0, -2)$. (Here and in all the sequel, since $\text{Pic}(S) = \mathbb{Z}\mathcal{H}$ we identify $H_{alg}^*(S, \mathbb{Z})$ with $\mathbb{Z} \oplus \mathbb{Z}\mathcal{H} \oplus \mathbb{Z}$.)

The wall-and-chamber decomposition of the movable cone carries a lot of information about the birational geometry of $S^{[3]}$: each chamber corresponds to a pair (M, f) , where $f : S^{[3]} \dashrightarrow M$ is a birational, not biregular map between hyperKähler manifolds, unique up to automorphisms of the two manifolds [35, Theorem 1.5 and Lemma 6.22]. Walls in the *interior* of the movable cone correspond to flopping contractions and crossing them, in the sense of [4], gives rise to flops to hyperKähler birational models.

Corollary 26. *Any hyperKähler birational model of $S^{[3]}$ is isomorphic to $S^{[3]}$.*

Proof. Let $f : S^{[3]} \dashrightarrow M$ be a birational map, with M hyperKähler. By Proposition 25, the pullback of the ample cone of M via f is the interior of either $\text{Nef}(S^{[3]})$ or $\varphi^*\text{Nef}(S^{[3]})$. Hence either f or $f \circ \varphi$ sends Kähler classes on $S^{[3]}$ to Kähler classes on M , and then it is an isomorphism by the Torelli Theorem for hyperKähler manifolds. \square

When we fix such an identification $M = S^{[3]}$, φ corresponds to the flop associated to the wall and the chamber $\varphi^*(\text{Nef}(S^{[3]}))$ is associated to the pair $(S^{[3]}, \varphi)$. By Bayer-Macri's study of the Mori cone of $S^{[3]}$ (as a cone in $H^2(S^{[3]}, \mathbb{R})$, see [4, Section 12]), we know that there is an extremal contraction associated to φ . We call this flopping contraction $c : S^{[3]} \rightarrow N$, with N a normal irreducible projective variety.

The diagram associated to the flop is

$$(21) \quad \begin{array}{ccc} S^{[3]} & \xrightarrow{\varphi} & S^{[3]} \\ & \searrow c & \swarrow d \\ & N & \end{array}$$

As a corollary of Proposition 25, φ is the only flop on $S^{[3]}$! More precisely, it is the only flop for which the space obtained by the transformation is a hyperKähler manifold (contrary to what can happen for certain Mukai flops). This observation will play a key rôle in Section 3.6.

3.4. Interpretations in terms of wall-crossing. In this section we take a closer look at the description of the flop in terms of moduli spaces of objects in the derived category of S . This approach is mainly due to Bayer-Macri; [5, 4, 2] will be our main references for this section. The main idea is that the moduli space changes when the stability condition does; more precisely, when one crosses certain *walls* in the manifold $\text{Stab}(S)$ parametrizing stability conditions. In our setting, the flop φ can be realized through such a wall-crossing, and we will deduce precise information about it.

Actually, the manifold $\text{Stab}(S)$ is not completely described, but we will use as a substitute an explicit submanifold $\text{Stab}^\dagger(S)$ constructed in [10].

This submanifold parametrizes stability conditions of the form

$$\sigma_{\alpha,\beta} = (\mathbf{Coh}^\beta(S), Z_{\alpha,\beta}) \quad \text{for } (\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0},$$

where $\mathbf{Coh}^\beta(S)$ is a certain abelian subcategory of $D^b(S)$, and the central charge is the homomorphism $Z_{\alpha,\beta} = (e^{i\alpha\mathcal{H} + \beta\mathcal{H}}, -) : H_{alg}^*(S, \mathbb{Z}) \rightarrow \mathbb{C}$. In the sequel we stick to these stability conditions and use them to analyze our wall-crossing.

Any non-zero Mukai vector u induces a wall-chamber decomposition of $\text{Stab}^\dagger(S)$ [2, Corollary 3.5], the complement of the walls being the set of stability conditions σ for which any σ -semistable object with Mukai vector u is also σ -stable.

The stability condition σ is *generic with respect to u* if it does not lie on a wall. In that case, there exists a moduli space $M_\sigma(u)$ of σ -stable objects with Mukai vector u in the derived category of S . For example, for a well-chosen σ_+ , we can get the following moduli spaces:

- For $a = -(2, -\mathcal{H}, 5)$, $M_{\sigma_+}(-a)$ is a single point [5, Lemma 7.1], that we can identify with the bundle \mathcal{U}_2 on S : by [39, Theorem 3], \mathcal{U}_2^\vee is the unique stable bundle on S whose Mukai vector is $a' = (2, \mathcal{H}, 5)$ (and the change of sign in the Mukai vector amounts to taking duals [18, Definition 6.1.2]). As a consequence, $M_{\sigma_+}(a)$ is also a single point, corresponding to the shifted bundle $\mathcal{U}_2[1]$.
- For $w = (3, -\mathcal{H}, 3)$, $M_{\sigma_+}(w)$ is isomorphic to $\Sigma = M_{\mathcal{H}}(3, \mathcal{H}, 3)$ (see Theorem 23, with a change of sign for taking duals). Indeed, $M_{\sigma_+}(w)$ is a K3 surface by [5, Lemma 7.2], moreover, there exists a stability condition τ such that $M_\tau(w) \simeq \Sigma$ by [2, Thm 4.4]. Since $M_{\sigma_+}(w)$ is obtained from $M_\tau(w)$ by some wall-crossings, they must be birational, hence isomorphic.
- For $v = (1, 0, -2)$, $M_{\sigma_+}(v)$ is isomorphic to $S^{[3]}$, its closed points corresponding to ideal sheaves of length three subschemes of S , see [2, Theorem 4.4] and [45, Example p.71]. We will be more precise in Proposition 27.

Starting from the latter interpretation of $S^{[3]}$ as $M_{\sigma_+}(v)$, we will change the stability condition and use wall-crossing to describe the flop φ . We first locate the relevant wall.

The numerical condition that for some object T , its Mukai vector t has the same phase as v , that is, $\arg(Z_{\alpha,\beta}(t)) = \arg(Z_{\alpha,\beta}(v))$, defines either a semicircular or a vertical real codimension one submanifold of $\text{Stab}^\dagger(S)$, see [32, 1.3]. A *potential wall* associated to the primitive lattice containing $\langle v, t \rangle$ is any connected component of such a submanifold [4, Definition 5.2]. Being a potential wall is a necessary condition for being a wall.

Potential walls in $\text{Stab}^\dagger(S)$ that are walls inducing non-trivial birational transformations are exactly those for which either $t = a$ as in [4, Theorems 12.1], or $t = s$ as defined in [4, Theorems 12.3]. Moreover they correspond via wall-crossing to an extremal contraction of $S^{[3]}$ if and only if

$\theta_v(t^\perp \cap v^\perp)$ contains the corresponding wall of the nef cone of $S^{[3]}$, where $\theta_v : v^\perp \rightarrow \text{NS}(S^{[3]})$ is the Mukai morphism [4, Remark 2.17]. Since S has Picard rank one, the latter nef cone has only two walls, giving respectively the Hilbert-Chow contraction for $t = s = (0, 0, 1)$, and the flopping contraction c for $t = a = -(2, -\mathcal{H}, 5)$ (for the latter, see the Remark after Proposition 25).

Definition. Let Λ be the rank two, primitive sublattice

$$\Lambda = \langle v, a \rangle \subset H_{\text{alg}}^*(S, \mathbb{Z}), \quad M = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix},$$

with M the associated Gram matrix.

Definition. Let $W \subset \text{Stab}^\dagger(S)$ be the quarter circle parametrized by

$$\left(\frac{1}{6} \cos(t) - \frac{1}{2}, \frac{1}{6} \sin(t) \right), \quad 0 < t < \frac{\pi}{2}.$$

Let $W' \supset W$ be the semicircle parametrized by $0 < t < \pi$.

By [2, Theorem 3.1] all the pairs $(\beta, \alpha) \in W'$ correspond to stability conditions, except for $(\beta, \alpha) = (-\frac{1}{2}, \frac{1}{6})$, for which $\Re(Z_{\alpha, \beta}(-a)) = 0$. An explicit computation shows that $Z_{\alpha, \beta}(v)$ and $Z_{\alpha, \beta}(a)$ have the same phase for any $(\beta, \alpha) \in W'$, hence W is a potential wall for Λ . By the discussion above, W is in fact a wall, giving via wall-crossing the flopping contraction c . A similar computation shows that the walls associated to $\langle s, v \rangle$, and giving the Hilbert-Chow contraction, lie in the α -axis of the (β, α) -plane.

From now on, we fix a stability condition $\sigma_0 = \sigma_{\alpha_0, \beta_0}$ on the wall W ; an associated slope function ν_0 can be defined on $\mathbf{Coh}^{\beta_0}(S)$, see [2, (4)]. The full subcategory of $\mathbf{Coh}^{\beta_0}(S)$ of objects with the same phase $\phi = \frac{1}{\pi} \arg(Z_{\alpha_0, \beta_0}(v))$ as v and semistable with respect to ν_0 will be denoted by $\mathcal{P}(\phi)$. An object E in the derived category is σ_0 -semistable of phase ϕ if some shift $E[k]$ lies in $\mathcal{P}(\phi)$, and σ_0 -stable if additionally it is simple in $\mathcal{P}(\phi)$, i.e. it has no non-trivial subobject. Since $\mathcal{P}(\phi)$ is abelian of finite length, every object admits a finite Jordan-Hölder filtration, which is trivial if and only if the object is σ_0 -stable. Objects with the same Jordan-Hölder factors are called S-equivalent with respect to σ_0 .

We also fix two stability conditions on the two sides of the wall, namely $\sigma_\pm = \sigma_{\alpha_0 \pm \epsilon, \beta_0}$, which can be chosen to be generic with respect to v, w and a since the set of walls is locally finite, see [33, Section 6.4] (for convenience we keep β_0 fixed). The following result is due to Bayer and Macrì.

Proposition 27. *There is a natural isomorphism $S^{[3]} \simeq M_{\sigma_+}(v)$ sending Z to \mathcal{I}_Z . Via this isomorphism, the wall W induces via wall-crossing the flopping contraction $c : S^{[3]} \rightarrow N$, more precisely:*

- (1) *c contracts a curve C if and only if, for $p_1, p_2 \in C$ general, the associated objects in $D^b(S)$ are S-equivalent with respect to σ_0 .*

(2) For any p lying in a contracted curve, the Mukai vector of every Jordan-Hölder factor with respect to σ_0 lies in Λ .

Moreover (1), (2) also hold for the contraction $d_{\sigma_0} : M_{\sigma_-}(v) \rightarrow N$ on the other side of the wall.

Proof. Stability conditions in $\text{Stab}^\dagger(S)$ corresponding to the natural isomorphism described in the statement correspond to (β, α) 's with $\alpha \gg 0$ and $\beta < 0$, see for example [2, Theorem 4.4]. The analysis of the potential walls carried before the proof shows that those conditions lie in the same chamber as σ_+ . Moreover the wall W induces the contraction c since $\mathbb{R}_{>0}(\mathcal{H}_3 - 2\delta) \subset \text{NS}(S^{[3]})$ is a connected component of $\theta_v(v^\perp \cap a^\perp)$.

Now, item (1) is [4, Theorem 2.18 (b)]. For (2), let Q be the last Jordan-Hölder factor of an object E . By definition of the filtration, their central charges with respect to $\sigma_0 \in W$ are aligned, so their Mukai vectors lie in Λ by definition of potential wall associated to Λ . Then we can proceed by induction. Finally, the proof is identical for both sides of the wall. \square

Different types of walls are defined in [4, Definition 2.20]. We exclude an unpleasant case.

Lemma 28. *The wall W is not totally semistable.*

Proof. By [4, Theorem 5.7], a wall is totally semistable if there is either an effective spherical class with negative product with v , or an isotropic class $u \in \Lambda$ satisfying $(v, u) = 1$.

An isotropic class u with $(v, u) = 1$ would generate with v a unimodular, even, rank two sublattice of Λ , which is absurd since Λ has discriminant -9 . For the first case, we note by solving $(xv + ya)^2 = -2$ for $x, y \in \mathbb{Z}$, that the only spherical classes in Λ are a and $-a$. By [4, Proposition 5.5], only a is effective, and we are done since $(a, v) = 1 > 0$. \square

Since W is not totally semistable, the space $M_{\sigma_0}^s(v)$ of σ_0 -stable objects is non-empty. Then openness of stability [4, Theorem 4.2] ensures that $M_{\sigma_0}^s(v)$ admits an open embedding in both $M_{\sigma_+}(v)$ and $M_{\sigma_-}(v)$. The flop

$$\varphi_{\sigma_0} : S^{[3]} \simeq M_{\sigma_+}(v) \dashrightarrow M_{\sigma_-}(v) \simeq S^{[3]}$$

is induced by this identification, see [5, proof of Theorem 1.4 (b)]; recall that $M_{\sigma_-}(v) \simeq S^{[3]}$ by Corollary 26. This yields an identification of φ_{σ_0} with φ and of d_{σ_0} with the flopping contractions $d : S^{[3]} \rightarrow N$, so that we recover (21).

3.5. The indeterminacy locus. The main advantage of describing the flop in terms of wall-crossing is that we now have a precise description of the exceptional loci of c and $d = d_{\sigma_0}$.

Definition. The indeterminacy locus of φ will be denoted by I . We call *base of the contraction* the image of the exceptional locus of c in N .

In this subsection we deduce a precise description of I , and ultimately show that it identifies with J .

Lemma 29. *The indeterminacy locus I of φ coincides with the exceptional locus of c . The base $c(I)$ of the contraction c is an irreducible surface in N , birational to Σ .*

Proof. According to Proposition 27, to prove the second claim we need to analyse the possible Jordan-Hölder factors of the strictly σ_0 -semistable objects in $D^b(S)$ with Mukai vector v . Suppose that, up to a shift, such an object E fits into an exact sequence in $\mathcal{P}(\phi)$, as an extension of two σ_0 -semistable objects F_1, F_2 with Mukai vectors $a_1, a_2 = v - a_1 \in \Lambda$. For E to be contained in a curve contracted by c , we need $\text{ext}^1(F_2, F_1) > 1$: by [4, Theorem 5.7], the objects with the same image as E are exactly those that are also extensions of F_1 by F_2 .

We claim that the only possible Mukai vectors are $a_1 = a, a_2 = w$. Indeed, since W is not totally semistable by Lemma 28, any effective class in Λ has non-negative intersection with respect to the Mukai pairing with v so, by [4, Proposition 5.5], (a_1, a_2) must be of the form $(a, v - a)$ for a listed in [12, Remark 2.8]. This can be translated via [12, Lemma 2.5] in terms of solutions to certain Pell's equations, which coincide with the equations that already appeared in the proof of Proposition 25 – namely the equations $X^2 - 72Y^2 = 8 + \alpha^2$, such that $X\mathcal{H}_3 - 18Y\delta$ is a multiple of $\mathcal{H}_3 - 2\delta$. We have shown in the proof of Proposition 25 that the only solution is $(9, 1)$, which corresponds to $a = -(2, -\mathcal{H}, 5)$, and we conclude that the base of c is birational to $M_{\sigma_+}(a) \times M_{\sigma_+}(w) \simeq \{pt\} \times \Sigma$, as claimed.

The same analysis also applies to $d = d_{\sigma_0}$. In fact, [4, Proposition 9.1] allows to produce, for a generic element of the base, a positive dimensional fiber for both c and d . In other words, the bases of the contractions c and d coincide. This implies our first claim. \square

Now we take a closer look to the subcategory $\mathbf{Coh}^{\beta_0}(S)$ associated to σ_0 and σ_+ . According to [2, Section 2], it can be constructed from two subcategories T^{β_0} and F^{β_0} of $\text{Coh}(S)$, as the extension-closure $\langle T^{\beta_0}, F^{\beta_0}[1] \rangle$ in $D^b(S)$. One gets an abelian category in which short exact sequences are the exact triangles of the derived category whose objects lie in $\mathbf{Coh}^{\beta_0}(S)$.

An easy computation shows that $\mathcal{I}_Z, \mathcal{E}^\vee \in T^{\beta_0}$ for any Z in $S^{[3]}$ and any vector bundle \mathcal{E} corresponding to a point of $\Sigma = M_{\mathcal{H}}(3, \mathcal{H}, 3)$. Moreover $\mathcal{U}_2 \in F^{\beta_0}$.

Lemma 30. *A length three subscheme Z of S defines a point of I if and only if the sheaf \mathcal{I}_Z lies in a short exact sequence*

$$(22) \quad 0 \longrightarrow \mathcal{U}_2 \xrightarrow{\nu^t} \mathcal{E}^\vee \longrightarrow \mathcal{I}_Z \longrightarrow 0$$

for some vector bundle \mathcal{E} from $\Sigma = M_{\mathcal{H}}(3, \mathcal{H}, 3)$. Moreover Z and Z' lie in the same fiber of c if and only if the two associated exact sequences (22) involve the same bundle \mathcal{E} .

Proof. By Lemma 29, the only possible Mukai vectors for two objects in $\mathcal{P}(\phi)$ of which \mathcal{I}_Z is an extension are a and w . An explicit computation of the phase ϕ_+ of a, v, w with respect to σ_+ shows that $\phi_+(w) < \phi_+(v) < \phi_+(a)$. Therefore the first subobject A in the Jordan-Hölder filtration of an object \mathcal{I}_Z in $\mathcal{P}(\phi)$ has necessarily Mukai vector w [4, def 2.1 (b)]. This yields a short exact sequence in $\mathcal{P}(\phi)$:

$$(23) \quad 0 \longrightarrow A \longrightarrow \mathcal{I}_Z \longrightarrow U \longrightarrow 0,$$

with $v(A) = w$, $v(U) = a$. Now consider (23) as a sequence in $\mathbf{Coh}^{\beta_0}(S)$. Objects of $M_{\sigma_+}(v)$ are concentrated in degree 0 by Proposition 27, hence by [2, Proposition 2.4], A is a semistable sheaf with Mukai vector w , so $A = \mathcal{E}^\vee$ for some vector bundle \mathcal{E} corresponding to a point of Σ . Moreover, again by [2, Proposition 2.4], (23) is equivalent to a long exact sequence of sheaves

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E}^\vee \xrightarrow{\nu^t} \mathcal{I}_Z \longrightarrow \mathcal{Q} \longrightarrow 0$$

with $\mathcal{K} \in F^{\beta_0}$, of rank two (we cannot have $\nu^t = 0$, since $\mathcal{K} \in F^{\beta_0}$ and $\mathcal{E}^\vee \in T^{\beta_0}$). We claim that ν^t is necessarily surjective. By contradiction, suppose that $\mathcal{Q} \neq 0$: then \mathcal{Q} is torsion. Since \mathcal{E}^\vee is semistable, \mathcal{K} has slope $\mu_{\mathcal{H}}(\mathcal{K}) = \frac{c_1(\mathcal{K}) \cdot \mathcal{H}}{2} \leq -\frac{\mathcal{H}^2}{3}$, so $c_1(\mathcal{K}) = -\lambda \mathcal{H}$ with $\lambda > 0$. The Mukai vectors of \mathcal{K} and \mathcal{Q} are respectively $v_1 = (2, -\lambda \mathcal{H}, r)$ and $v_2 = (0, -(\lambda - 1)\mathcal{H}, r - 5)$. For v_2 to be the Mukai vector of a coherent sheaf we need $-(\lambda - 1) \geq 0$, so $\lambda = 1$. Hence $v_2 = (0, 0, r - 5)$ with $r > 5$, which is absurd since then $v_1^2 = 18 - 4 \cdot r < -2$. So $\mathcal{Q} = 0$, hence \mathcal{K} has Mukai vector $-a$ and thus $\mathcal{K} = \mathcal{U}_2$: we have obtained (22). The last assertion is clear. \square

Lemma 31. *For any vector bundle \mathcal{E} from $M_{\mathcal{H}}(3, \mathcal{H}, 3)$, we have*

$$\mathrm{hom}(\mathcal{E}, \mathcal{U}_2^\vee) = 3, \quad \mathrm{ext}^1(\mathcal{E}, \mathcal{U}_2^\vee) = \mathrm{ext}^2(\mathcal{E}, \mathcal{U}_2^\vee) = \mathrm{hom}(\mathcal{U}_2^\vee, \mathcal{E}) = 0.$$

Proof. Riemann-Roch yields $\chi(\mathrm{Hom}(\mathcal{E}, \mathcal{U}_2^\vee)) = 3$. Since \mathcal{E} is semistable, $\mathrm{hom}(\mathcal{U}_2^\vee, \mathcal{E}) = 0$. Suppose $\mathrm{ext}^1(\mathcal{E}, \mathcal{U}_2^\vee) \neq 0$, hence, there is a nontrivial extension $0 \rightarrow \mathcal{U}_2^\vee \rightarrow \mathcal{K} \rightarrow \mathcal{E} \rightarrow 0$, with $\mu_{\mathcal{H}}(\mathcal{K}) = 2\mathcal{H}^2/5$.

We claim that \mathcal{K} must be stable. By contradiction, suppose it contains a destabilizing subsheaf \mathcal{K}' , with the induced extension $0 \rightarrow \mathcal{A}' \rightarrow \mathcal{K}' \rightarrow \mathcal{E}' \rightarrow 0$. If \mathcal{A}' and \mathcal{E}' have rank smaller than \mathcal{U}_2^\vee and \mathcal{E} respectively, then by semistability $c_1(\mathcal{A}') \leq 0$ and $c_1(\mathcal{E}') \leq 0$, hence also $c_1(\mathcal{K}') \leq 0$, a contradiction. So \mathcal{A}' or \mathcal{E}' must be zero (otherwise both must have full rank, hence also \mathcal{K}' , which is excluded). If $\mathcal{E}' = 0$, \mathcal{K}' is a subsheaf of \mathcal{U}_2^\vee and cannot destabilize \mathcal{E} , so neither \mathcal{K} . If $\mathcal{A}' = 0$, then we need $c_1(\mathcal{E}') = \mathcal{H}$, which implies that the inclusion of \mathcal{E}' in \mathcal{E} is an isomorphism in codimension one. But then the embedding of \mathcal{E}' into \mathcal{K} can be lifted to \mathcal{E} , which is impossible since this would trivialize the extension that defines \mathcal{K} .

Now, the bundle \mathcal{K} being semistable, it defines a point of a moduli space of dimension $v_{\mathcal{K}}^2 + 2$, the Mukai vector of \mathcal{K} being $v_{\mathcal{K}} = (5, 2\mathcal{H}, 8)$. But this gives $v_{\mathcal{K}}^2 + 2 = -6$, so the moduli space is empty – a contradiction! \square

Lemma 32. *Any nonzero $\nu \in \text{Hom}(\mathcal{E}, \mathcal{U}_2^\vee)$ yields an extension*

$$(24) \quad 0 \longrightarrow \mathcal{U}_2 \xrightarrow{\nu^t} \mathcal{E}^\vee \longrightarrow \mathcal{I}_Z \longrightarrow 0$$

for some length 3 subscheme $Z \in S^{[3]}$.

Proof. Let \mathcal{K} be the kernel of ν^t and \mathcal{Q} its image. The rank of \mathcal{K} is at most one since ϕ is nonzero. If \mathcal{K} (hence also \mathcal{Q}) has rank one, the stability of \mathcal{U}_2 and \mathcal{E} implies that $c_1(\mathcal{K}) \leq -\mathcal{H}$ and $c_1(\mathcal{Q}) \leq -\mathcal{H}$, hence $c_1(\mathcal{U}_2) \leq -2\mathcal{H}$, a contradiction. So $\mathcal{K} = 0$ since \mathcal{U}_2 is torsion free.

Note that ν induces a morphism from $\wedge^2 \mathcal{U}_2 = \mathcal{O}_S(-1)$ to $\wedge^2 \mathcal{E}^\vee = \mathcal{E}(-1)$, hence a non-trivial section s of \mathcal{E} . Since \mathcal{E} is stable, s vanishes in codimension two and the associated Koszul complex is exact, hence the sequence $\mathcal{U}_2 \rightarrow \mathcal{E}^\vee \rightarrow \mathcal{O}_S$ is also exact. So the cokernel of ν^t is a rank one subsheaf of \mathcal{O}_S with $c_1 = 0$, hence the ideal sheaf of some finite scheme Z . Since $ch(\mathcal{E}) = 3 + \mathcal{H}$ and $ch(\mathcal{U}_2) = 2 + \mathcal{H} + 3[pt]$, we get

$$ch(\mathcal{O}_Z) = 1 - (3 - \mathcal{H}) + (2 - \mathcal{H} + 3[pt]) = 3[pt],$$

which means that $\ell(Z) = 3$. \square

Let \mathcal{F} be the *twisted* universal rank three vector bundle on $S \times \Sigma$ (see [17, X.2.2 (ii)] for the definition and existence of \mathcal{F}). Let p_1 and p_2 be the projections from $S \times \Sigma$ to the two factors. Since \mathcal{F} is twisted with respect to a Brauer class that is pulled-back from Σ , the push-forward

$$\mathcal{G} = p_{2*} \text{Hom}(\mathcal{F}, p_1^* \mathcal{U}_2^\vee)$$

makes sense and defines a twisted sheaf on Σ . By Lemma 31, \mathcal{G} is actually a rank three twisted vector bundle on Σ , and its projectivization is well-defined. Moreover, Lemmas 30 and 32 yield a surjective morphism

$$\theta : \mathbb{P}_\Sigma(\mathcal{G}) \longrightarrow I \subset S^{[3]}.$$

Proposition 33. *θ maps $\mathbb{P}_\Sigma(\mathcal{G})$ isomorphically to $I = J \subset S^{[3]}$ and we get a commutative diagram*

$$(25) \quad \begin{array}{ccc} \mathbb{P}_\Sigma(\mathcal{G}) & \xrightarrow{\theta} & J \\ & \searrow & \swarrow f \\ & \Sigma & \end{array}$$

Moreover the fibers of f coincide with those of the restriction of c to J .

Proof. We consider any \mathcal{I}_Z lying in an exact sequence as in (24) and we prove that $Z \in J$. To do that, we tensor (24) by \mathcal{U}_2^\vee and take cohomology. Since \mathcal{U}_2 is stable on S (hence simple) and rigid, we get an exact sequence

$$0 \longrightarrow \text{End}(\mathcal{U}_2) = \mathbb{C} \longrightarrow \text{Hom}(\mathcal{E}, \mathcal{U}_2^\vee) \longrightarrow H^0(\mathcal{I}_Z \otimes \mathcal{U}_2^\vee) \longrightarrow 0.$$

Since $\text{Hom}(\mathcal{E}, \mathcal{U}_2^\vee)$ is three-dimensional by Lemma 31, we deduce that $H^0(\mathcal{I}_Z \otimes \mathcal{U}_2^\vee)$ is two-dimensional, which means that Z only spans a codimension two subspace of V_7 , that is, $Z \in J$. Moreover, the long exact

sequence continues and yields

$$\text{Ext}^1(\mathcal{E}, \mathcal{U}_2^\vee) \longrightarrow H^1(\mathcal{I}_Z \otimes \mathcal{U}_2^\vee) \longrightarrow \text{Ext}^2(\mathcal{U}_2^\vee, \mathcal{U}_2^\vee) \longrightarrow \text{Ext}^2(\mathcal{E}, \mathcal{U}_2^\vee).$$

By Serre duality $\text{ext}^2(\mathcal{E}, \mathcal{U}_2^\vee) = 0$ and $\text{ext}^2(\mathcal{U}_2^\vee, \mathcal{U}_2^\vee) = 1$, while we have $\text{ext}^1(\mathcal{E}, \mathcal{U}_2^\vee) = 0$ by Lemma 31. So $h^1(\mathcal{I}_Z \otimes \mathcal{U}_2^\vee) = 1$, meaning that the extension (24) is unique.

Hence θ is injective, with image $I = J$; and since J is smooth by Lemma 21, θ must be an isomorphism. Finally, the fibers of f parametrize subschemes of $S^{[3]}$ that are S-equivalent, since the S-equivalence class is determined by the isomorphism class of the vector bundle \mathcal{E} in (24); so they coincide with the non-trivial fibers of c . \square

3.6. The Mukai flop. Since $\mathbb{P}_\Sigma(\mathcal{G}) \simeq J$ embeds in $S^{[3]}$ as a codimension two subvariety and is a \mathbb{P}^2 -fibration, one can perform a Mukai flop

$$(26) \quad \begin{array}{ccc} & B & \\ b \swarrow & & \searrow \tilde{b} \\ S^{[3]} & \xrightarrow{\tilde{\varphi}} & M \end{array}$$

where M is again simply connected and admits an everywhere non-degenerate two-form [37, Theorem 0.7]. Recall that b is just the blow-up of $S^{[3]}$ along J . The exceptional divisor E is a $F\ell_3$ -fibration over Σ ; the contractions to $S^{[3]}$ and to M contract each copy of $F\ell_3 \subset \mathbb{P}^2 \times \mathbb{P}^2$ on one of the two factors.

It is of course tempting to imagine that $\tilde{\varphi}$ should coincide with φ , since as a consequence of Proposition 25 the latter is the unique flop of $S^{[3]}$, and the two transformations have the same indeterminacy locus $I = J$. But the unicity statement only holds in the algebraic setting, while the result of a Mukai flop needs not always be Kähler; a counter-example was given by Yoshioka [48, Section 4.4]. So we need to provide an extra argument. This will be based on an explicit extension of φ to B .

Proposition 34. *The birational involution φ lifts to a biregular involution τ of B , preserving the exceptional divisor E .*

Proof. First recall that J was defined as the degeneracy locus of the morphism ϕ in (15). By Proposition 19, the rank of ϕ never drops to four, so the restriction of ϕ to J yields a long exact sequence

$$0 \longrightarrow \mathcal{Q}^\vee \longrightarrow V_7^\vee \otimes \mathcal{O}_J \longrightarrow p_*(q^*\mathcal{U}_2^\vee)|_J \longrightarrow \mathcal{L} \longrightarrow 0,$$

for some line bundle \mathcal{L} , and a vector bundle \mathcal{Q}^\vee whose fiber at $[V_5]$ is the space of linear forms on V_7 that vanish on V_5 . The normal bundle of J in $S^{[3]}$ can thus be described as $\text{Hom}(\mathcal{Q}^\vee, \mathcal{L}) = \mathcal{Q} \otimes \mathcal{L}$; its projectivization, whose total space is the exceptional divisor E , can therefore be identified with $\mathbb{P}(\mathcal{Q})$. This means that if $Z \in S^{[3]}$ spans $V_5 \subset V_7$, a point in E above Z can be identified with a hyperplane $V_6 \supset V_5$.

Recall that V_5 is a decomposing five-plane by Proposition 19, hence Z is contained in a unique hyperplane section $F = X_{ad}(G_2) \cap H$ containing S , and $R = F \cap G(2, V_5)$ is a cubic scroll. By Proposition 8 and the remark that follows, $X_{ad}(G_2) \cap G(2, V_6) \cap H$ is a surface T of degree 6. Thus, $T = R \cup R'$ for another surface R' of degree three. The intersection of R' with L is then a length three subscheme Z' of S , and mapping (Z, V_6) to Z' defines a regular extension $\bar{\varphi}$ of φ from $S^{[3]}$ to B , coinciding with φ outside E .

In general V_6^\perp is not isotropic, and $F\ell(V_6) = X_{ad}(G_2) \cap G(2, V_6)$ is a copy of $F\ell_3$ by Proposition 8. The surface T is therefore cut out in $\mathbb{P}^2 \times \mathbb{P}^2$ by a pencil of hyperplanes, defined by 3×3 matrices. Since it contains the cubic scroll $R = \mathbb{P}^1 \times \mathbb{P}^2 \cap H$ (see the proof of Proposition 14), one of the hyperplanes in the pencil has to contain the linear span of $\mathbb{P}^1 \times \mathbb{P}^2$, which implies that it is defined by a matrix of rank one. But then the corresponding hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$ is the union of $\mathbb{P}^1 \times \mathbb{P}^2$ with a copy of $\mathbb{P}^2 \times \mathbb{P}^1$, and R' is a linear section of the latter. So R' is another cubic scroll, necessarily contained in $G(2, V_5')$ for some $V_5' \subset V_6$, by [23, Lemma 7]. This proves that $\bar{\varphi}$ sends E to J , and by the universal property of blowups, $\bar{\varphi}$ lifts to an involution of B , sending E to itself (and (Z, V_6) to (Z', V_6)). \square

The restriction of τ to E is an automorphism over L^\perp via $\pi \circ b|_E : E \rightarrow L^\perp$ (see (17)) i.e. $\pi \circ b|_E = \pi \circ b|_E \circ \tau$. Consider the Stein factorization (19) and let $\{\mathcal{E}_H^\sigma, \mathcal{E}_H^\tau\}$ be the fiber over a generic $[H] \in L^\perp$ (recall Theorem 23 and the Definition that follows). A general element Z of $f^{-1}(\mathcal{E}_H^\sigma)$ lies in the pullback via ψ_H of a cubic scroll of σ -type R ; for any $(Z, V_6) \in b^{-1}(Z)$, the scroll R' such that $\tau(Z, V_6) = (R' \cap L, V_6)$ is the pullback of a scroll of σ or τ -type. By Corollary 24, this does not depend on the choice of V_6 and by continuity it also does not depend on the choice of H . Thus τ induces a birational, hence biregular, involution on Σ , which is the identity or the covering involution according to whether R' is still of σ -type or not. We denote this involution by γ .

Lemma 35. *The involution γ is the covering involution of $g : \Sigma \rightarrow L^\perp$.*

Proof. Recall that according to [23] our scrolls come from $G(3, U_6)$, in which copies of $\mathbb{P}^2 \times \mathbb{P}^2$ are obtained by decomposing $U_6 = A \oplus B$ into the sum of two three-dimensional subspaces, and considering the family of subspaces of the form $a \oplus b^\perp$, for a, b one dimensional in A and B^\vee , respectively. When a hyperplane section decomposes into two cubic scrolls, we have seen that it is defined by a rank-one pairing $\nu \in A^\vee \otimes B = \text{Hom}(A, B)$, the condition being that $\langle \nu(a), b \rangle = 0$. Then the two components are the families of spaces of the form $a \oplus b^\perp$ for $a \subset \text{Ker}(\nu)$, which contain the fixed hyperplane $\text{Ker}(\nu) \oplus B$, or $b \subset \text{Ker}(\nu^t)$, which contain the fixed line $\text{Ker}(\nu^t)^\perp \subset B$. This shows that the two scrolls R and R' have different types. \square

Corollary 36. *$\varphi = \tilde{\varphi}$ is a Mukai flop.*

Proof. Since τ coincides with φ outside E , we have $\varphi = b \circ \tau \circ b^{-1}$, so the birational maps φ and $\tilde{\varphi}$ are obtained by composing b^{-1} with two a priori different projections from B , respectively $b \circ \tau$ and \check{b} .

Consider $p \in \Sigma$. We denote

$$\mathbb{P}_p^2 = c^{-1}(\gamma(p)) \quad E_p = b^{-1}(\mathbb{P}_p^2) \quad \check{\mathbb{P}}_p^2 = \check{b}(E_p)$$

and by b_p, τ_p, \check{b}_p the restrictions of the corresponding morphisms. Note that $E_p = \{(x, [H]) \mid x \in H\} \subset \mathbb{P}_p^2 \times \check{\mathbb{P}}_p^2$ is the flag variety of \mathbb{P}_p^2 , and that b_p, \check{b}_p are respectively the first and second projection. Also $b_p \circ \tau_p : E_p \rightarrow \check{\mathbb{P}}_p^2$ is a projection contracting a family of \mathbb{P}^1 's, so it is either b_p or \check{b}_p . Suppose that $b_p \circ \tau_p = b_p$ for one p . Then φ could be extended over \mathbb{P}_p^2 by imposing, for any $Z \in \mathbb{P}_p^2$, $\varphi(Z) = b\tau(\zeta)$ for any $\zeta \in b^{-1}(Z)$, which is absurd by Proposition 33. So $b_p \circ \tau_p$ coincide with \check{b}_p for any p , hence $(b \circ \tau)|_E : E \rightarrow J$ can be identified with the projection $\check{b}|_E$. Hence, $\check{b} \circ (b\tau)^{-1} : S^{[3]} \dashrightarrow M$ extends to a biregular map. \square

The following diagram summarizes the situation; $\tilde{\varphi}$ is the genuine Mukai flop; it is identified with φ through the isomorphism between M and $S^{[3]}$, and can be lifted to the regular involution τ of B :

$$(27) \quad \begin{array}{ccccc} & B & \xrightarrow{\tau} & B & \\ & \swarrow b & & \searrow b & \\ S^{[3]} & & & & S^{[3]} \\ & \searrow \check{b} & & \swarrow b & \\ & M & \xrightarrow{\cong} & S^{[3]} & \\ & \swarrow \tilde{\varphi} & & & \end{array}$$

As an immediate consequence of Corollary 36 together with (21), the Mukai flop is a flop also in the sense of [25, Definition 2.1], with associated contraction c . This implies the smoothness of the base of the contraction, yielding an identification of $f : J \rightarrow \Sigma$ with the restriction of c to J , see Proposition 33.

Putting together all the results of the last two Sections, we obtain:

Theorem 37. *The indeterminacy locus of φ is the locus J of schemes generating a codimension two linear space in V_7 . It is a \mathbb{P}^2 -fibration over the $K3$ surface Σ and φ is the Mukai flop associated to J .*

4. THE LINEAR SYSTEM $|\mathcal{H}_3 - 2\delta|$

The extremal, flopping contraction $c : S^{[3]} \rightarrow N$ coincides with the morphism $\phi_{|k(\mathcal{H}_3 - 2\delta)}$ for $k \gg 0$ (see the proof of [36, Theorem 8.1.3]). In this section we conduct an in-depth study of the linear system $|\mathcal{H}_3 - 2\delta|$ itself, and show how it is related to the birational involution φ . We also study the direct sum map $\sigma : S^{[3]} \dashrightarrow \mathbb{P}(V_7^\vee)$, showing that it is defined by a linear subsystem of $|\mathcal{H}_3 - 2\delta|$.

We start by computing the dimension of the linear system $|\mathcal{H}_3 - 2\delta|$. The decomposition (20) is orthogonal with respect to the Beauville-Bogomolov form q_{BB} , which restricts to the intersection form on $H^2(S, \mathbb{Z})$, hence $q_{BB}(\xi_n) = \xi^2$. On the other hand, $q_{BB}(\delta) = -2(n-1)$ [13, 3.2.1].

Proposition 38. $|\mathcal{H}_3 - 2\delta| \simeq \mathbb{P}^9$.

Proof. We know that $\mathcal{H}_3 - 2\delta$ is nef by Proposition 25. Moreover

$$q_{BB}(\mathcal{H}_3 - 2\delta) = \mathcal{H}^2 + 4\delta^2 = 18 + 4 \cdot (-4) = 2.$$

Since the Fujiki constant of $S^{[3]}$ is 15 according to [13, 3.2.1], this implies that $(\mathcal{H}_3 - 2\delta)^6 = 15q_{BB}(\mathcal{H}_3 - 2\delta)^3 = 120$. In particular, the class $\mathcal{H}_3 - 2\delta$ is big as well. We can therefore invoke Kawamata-Viehweg's theorem and conclude that the number of sections is given by the Riemann-Roch polynomial. But on a hyperKähler sixfold X of K3-type we have

$$\chi(X, \mathcal{L}) = \binom{\frac{1}{2}q_{BB}(\mathcal{L}) + 4}{3}$$

by [13, 3.3], hence the claim. \square

4.1. The secant variety and Pfaffian cubics. Recall that S is cut in $X_{ad}(G_2)$ by a codimension three linear space $\mathbb{P}(V_{11}) \subset \mathbb{P}(\mathfrak{g}_2)$. Let $|I_3(\text{Sec}(S))|$ denotes the linear system of cubics of $\mathbb{P}(V_{11})$ containing the secant variety $\text{Sec}(S)$.

Proposition 39. *There is a natural identification*

$$|\mathcal{H}_3 - 2\delta| \simeq |I_3(\text{Sec}(S))|.$$

Moreover, given a length three subscheme Z of S , not in the base locus of the linear system, its image in $|I_3(\text{Sec}(S))|^\vee$ is the hyperplane of cubics containing the projective plane spanned by Z .

Remark. Since \mathcal{H} is 3-very ample [24, Theorem 1.1], any length three subscheme Z of S spans a plane.

Proof. First of all, since \mathcal{H}_3 is the pull-back, by the Hilbert-Chow morphism, of the line bundle $\mathcal{H}^{(3)}$ on $S^{(3)}$, one has

$$H^0(S^{[3]}, \mathcal{H}_3) \simeq H^0(S^{(3)}, \mathcal{H}^{(3)}) \simeq \text{Sym}^3 H^0(S, \mathcal{H}).$$

Explicitly, given a length three scheme Z supported at $p_1 + p_2 + p_3$, the sum of three points of S , corresponding to three lines $\ell_1, \ell_2, \ell_3 \subset \mathfrak{g}_2$, the fiber of $\mathcal{H}^{(3)}$ at $p_1 + p_2 + p_3$ is $\ell_1^\vee \otimes \ell_2^\vee \otimes \ell_3^\vee$. Then the cubic polynomial P on L defines a section of \mathcal{H}_3 whose evaluation at Z is given by the polarization \tilde{P} of P , restricted to $\ell_1 \otimes \ell_2 \otimes \ell_3$.

Since 2δ is the class of the divisor parametrizing non-reduced schemes, we need to understand when this section vanishes on the locus of non-reduced

schemes. But since the section is pulled-back from $S^{(3)}$, this simply means that $\tilde{P}(x_2, x_2, x_3) = 0$ for any $p_2 = [x_2], p_3 = [x_3]$ in S . This implies that

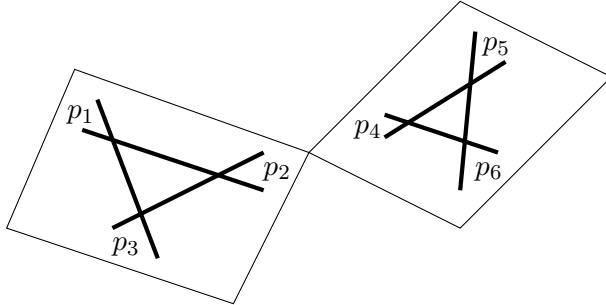
$$P(sx_2 + tx_3) = s^3\tilde{P}(x_2, x_2, x_2) + 3s^2t\tilde{P}(x_2, x_2, x_3) + 3st^2\tilde{P}(x_2, x_3, x_3) + t^3\tilde{P}(x_3, x_3, x_3) = 0$$

for any $p_2 = [x_2], p_3 = [x_3]$ in S and any scalars s, t , and the converse is also clearly true. This shows that $|\mathcal{H}_3 - 2\delta|$ identifies with the linear system of cubic polynomials P vanishing on $\text{Sec}(S)$.

To prove the second claim, we can suppose that $Z = p_1 \cup p_2 \cup p_3$ is reduced. The secant variety of S contains the three lines $\overline{p_1 p_2}, \overline{p_2 p_3}$ and $\overline{p_3 p_1}$, so the restriction to the plane $\langle Z \rangle$ of a cubic polynomial vanishing on $\text{Sec}(S)$ is completely determined up to scalar. If $p_1 = [x_1], p_2 = [x_2], p_3 = [x_3]$, it is clear that P vanishes on the whole plane if and only if $\tilde{P}(x_1, x_2, x_3) = 0$. This concludes the proof. \square

Corollary 40. *The map $\phi_{|\mathcal{H}_3 - 2\delta|}$ associated to the linear system $|\mathcal{H}_3 - 2\delta|$ factorizes through the involution φ .*

Proof. Let $Z = p_1 \cup p_2 \cup p_3$ be reduced as well as $\varphi(Z) = p_4 \cup p_5 \cup p_6$. By definition of φ , for Z general there exists a copy \mathfrak{s} of \mathfrak{sl}_3 in \mathfrak{g}_2 such that the six points p_1, \dots, p_6 belong to $\mathbb{P}(\mathfrak{s})$, hence to $\mathbb{P}(\mathfrak{s}) \cap L$. But the latter is in general a \mathbb{P}^4 , so the two planes $\langle Z \rangle$ and $\langle \varphi(Z) \rangle$ have to meet. Generically, we claim they will meet outside the lines of the two triangles defined by Z and $\varphi(Z)$. Indeed, since the planes defined by p_1, p_2, p_3 are in general position, those are the only rank two points in $\langle Z \rangle$ (where the rank refers to the usual rank as skew-symmetric tensors), and the other points on the three sides of the triangle are the only rank four points. Obviously the two planes cannot meet at a rank two point since otherwise two of the points p_1, \dots, p_6 would coincide. They cannot either meet at a rank four point, since then two sides of the two triangles, say $\overline{p_2 p_3}$ and $\overline{p_4 p_5}$ would meet; but then p_2, p_3, p_4, p_5 would be coplanar, contradicting the fact that p_2, p_3, p_4 are the only rank two points inside the plane they generate.



We conclude that a cubic polynomial vanishing on one triangle plus the span of the other triangle has to vanish on both spans. This proves the claim. \square

Remark. Here again there is a strong analogy with the conics on Gushel-Mukai fourfolds obtained as linear sections of copies of $G(2, 4)$ inside $G(2, 5)$. The linear system of quadratic equations of the Gushel-Mukai, which contains the Plücker quadrics as a hyperplane, restricts to a pencil on the plane spanned by such a conic. So, exactly as before, containing this plane is just a codimension one condition on the linear system. See [19] for more details.

Since obviously $\text{Sec}(S) \subset \text{Sec}(X_{ad}(G_2))$, let us describe the latter. The secant variety is irreducible and reduced, and has dimension 10 according to [20].

Lemma 41. $\text{Sec}(X_{ad}(G_2)) = \text{Sec}(G(2, V_7)) \cap \mathbb{P}(\mathfrak{g}_2)$.

Proof. The inclusion $\text{Sec}(X_{ad}(G_2)) \subset \text{Sec}(G(2, V_7)) \cap \mathbb{P}(\mathfrak{g}_2)$ is obvious, and we only need to prove the reverse inclusion. Note that

$$\text{Sec}(G(2, V_7)) = \bigcup_{V_4 \subset V_7} \mathbb{P}(\wedge^2 V_4),$$

so it is enough to check that $\mathbb{P}(\wedge^2 V_4 \cap \mathfrak{g}_2) \subset \text{Sec}(X_{ad}(G_2))$ for each $V_4 \subset V_7$. Recall that \mathfrak{g}_2 is the kernel of the contraction $\wedge^2 V_7 \rightarrow V_7^\vee$ by the invariant three-form ω . There are two cases.

If $\omega|_{V_4} = 0$, the composition $\wedge^2 V_4 \rightarrow \wedge^2 V_7 \rightarrow V_7^\vee$ factors through V_4^\perp , hence its kernel has dimension at least three and its intersection with $X_{ad}(G_2)$, being also the intersection with the Grassmannian, is a quadric hypersurface in $\mathbb{P}(\wedge^2 V_4 \cap \mathfrak{g}_2)$, whose secants span the whole space.

If $\omega|_{V_4} \neq 0$, it is a nonzero decomposable three-form on V_4 , hence corresponds to a line $V_1 \subset V_4$. Then the kernel of the composition $\wedge^2 V_4 \rightarrow \wedge^2 V_7 \rightarrow V_7^\vee \rightarrow V_4^\vee$ is $V_1 \wedge V_4$. Since $\mathbb{P}(V_1 \wedge V_4) \subset G(2, V_4)$, we conclude that $\mathbb{P}(\wedge^2 V_4 \cap \mathfrak{g}_2)$ is contained in $X_{ad}(G_2)$, and a fortiori also in its secant variety. \square

For a variety like $X_{ad}(G_2)$, which is cut-out by quadrics, the general expectation is that the secant variety should be cut out by cubics. Of course, there are many exceptions, typically when the codimension is not large enough. But this is the case for $G(2, V_7)$, whose secant variety consists in tensors of rank at most four (when considered as skew-symmetric forms on V_7^\vee), and is cut out by the 6×6 Pfaffians. Restricting to \mathfrak{g}_2 , we call *Pfaffian cubics* those cubics defined, for $v \in V_7$, by

$$P_v(x) = v \wedge x \wedge x \wedge x \in \wedge^7 V_7 \simeq \mathbb{C}, \quad x \in \mathfrak{g}_2 \subset \wedge^2 V_7.$$

Proposition 42. *There is an exact sequence*

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{\mathbb{P}(\mathfrak{g}_2)}(-7) \rightarrow V_7^\vee \otimes \mathcal{O}_{\mathbb{P}(\mathfrak{g}_2)}(-4) \rightarrow \\ \rightarrow V_7 \otimes \mathcal{O}_{\mathbb{P}(\mathfrak{g}_2)}(-3) \rightarrow \mathcal{O}_{\mathbb{P}(\mathfrak{g}_2)} \rightarrow \mathcal{O}_{\text{Sec}(X_{ad}(G_2))} \rightarrow 0. \end{aligned}$$

As a consequence, $I_3(\text{Sec}(X_{ad}(G_2))) = V_7$.

Proof. Recall that $\text{Sec}(G(2, V_7)) \subset \mathbb{P}(\wedge^2 V_7)$ is Cohen-Macaulay of codimension three [47, Theorem 6.4.1]. Lemma 41 shows that $\text{Sec}(G(2, V_7)) \cap \mathbb{P}(\mathfrak{g}_2)$ still has codimension three in $\mathbb{P}(\mathfrak{g}_2)$, and therefore it is also Cohen-Macaulay. It is easy to check that the intersection of $\text{Sec}(G(2, V_7))$ with $\mathbb{P}(\mathfrak{g}_2)$ is transverse at the general point of $\text{Sec}(X_{ad}(G_2))$, and since the latter is irreducible and $\text{Sec}(G(2, V_7)) \cap \mathbb{P}(\mathfrak{g}_2)$ is Cohen-Macaulay, this intersection must be reduced everywhere. Hence $\text{Sec}(X_{ad}(G_2)) = \text{Sec}(G(2, V_7)) \cap \mathbb{P}(\mathfrak{g}_2)$ schematically. Moreover the restriction of the standard locally free resolution

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbb{P}(\wedge^2 V_7)}(-7) \longrightarrow V_7^\vee \otimes \mathcal{O}_{\mathbb{P}(\wedge^2 V_7)}(-4) \longrightarrow \\ \longrightarrow V_7 \otimes \mathcal{O}_{\mathbb{P}(\wedge^2 V_7)}(-3) \longrightarrow \mathcal{O}_{\mathbb{P}(\wedge^2 V_7)} \longrightarrow \mathcal{O}_{\text{Sec}(G(2, V_7))} \longrightarrow 0 \end{aligned}$$

to $\mathbb{P}(\mathfrak{g}_2)$ gives a resolution of the structure sheaf of $\text{Sec}(G(2, V_7))$, hence the first claim. We can then twist the resolution by $\mathcal{O}_{\mathbb{P}(\mathfrak{g}_2)}(3)$: the first two terms are clearly acyclic, so that taking cohomology we simply get the exact sequence

$$0 \longrightarrow V_7 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}(\mathfrak{g}_2)}(3)) \longrightarrow H^0(\mathcal{O}_{\text{Sec}(G(2, V_7))}(3)) \longrightarrow 0.$$

This implies the second claim. \square

4.2. The direct sum map. Let Z be a length three subscheme of S outside J , see Section 3.2. If Z is reduced, it corresponds to three planes in general position in V_7 . The hyperplane they generate is simply their direct sum, and we call the corresponding rational map from $S^{[3]}$ to $\mathbb{P}(V_7^\vee)$ the direct sum map.

Proposition 43. *The direct sum map is a rational map*

$$(28) \quad \sigma : S^{[3]} \dashrightarrow \mathbb{P}(V_7^\vee),$$

which is regular outside J , and generically finite of degree 20.

Proof. For a general hyperplane V_6 of V_7 , $S \cap G(2, V_6)$ is the union W of six reduced points. These points define six planes in V_7 such that any three of them generate V_6 by Proposition 19, since a dimension count shows that V_6 cannot contain any decomposing five-plane. So any scheme Z in the preimage of $[V_6]$ by σ must be the union of three reduced points from W , and we conclude that σ is generically finite of degree $\binom{6}{3} = 20$. \square

Remark. The direct sum map σ is very similar to the one that, given a smooth quartic surface $T \subset \mathbb{P}^3$, maps $T^{[2]}$ to $G(2, 4)$ by sending a length two subscheme of S to the line it generates in \mathbb{P}^3 . Obviously, this is a generically finite cover of degree 6 (finite if T contains no line).

Restricting Pfaffian cubics yields a natural inclusion $j : V_7 \hookrightarrow I_3(\text{Sec}(S))$, see Proposition 42, hence a linear projection

$$j^t : \mathbb{P}(I_3(\text{Sec}(S))^\vee) \dashrightarrow \mathbb{P}(V_7^\vee).$$

This linear subsystem induces a rational map $S^{[3]} \dashrightarrow \mathbb{P}(V_7^\vee)$. The next easy claim is that this coincides with the direct sum map.

Lemma 44. *The following diagram commutes:*

$$(29) \quad \begin{array}{ccc} & S^{[3]} & \\ \sigma \swarrow & & \searrow \phi_{|H_3-2\delta|} \\ \mathbb{P}(V_7^\vee) & \xleftarrow{j^t} & \mathbb{P}(I_3(\text{Sec}(S))^\vee). \end{array}$$

In particular, $\phi_{|H_3-2\delta|} : S^{[3]} \dashrightarrow \mathbb{P}(I_3(\text{Sec}(S))^\vee)$ is generically finite of degree d such that $d|20$.

Proof. The rational map j^t sends a projective hyperplane H of cubics to the hyperplane in V_7 consisting of the Pfaffian cubics P_v that belong to H . Consider a general $x = p_1 + p_2 + p_3 \in S^{[3]}$. Choose a basis e_1, \dots, e_7 of V_7 such that the plane associated to p_h is $\langle e_h, e_{h+3} \rangle$ for $h = 1, 2, 3$. Then $x \wedge x \wedge x = 6e_1 \wedge e_4 \wedge e_2 \wedge e_5 \wedge e_3 \wedge e_6$, so $P_v(x) = 0$ if and only if v belongs to $\langle e_1, \dots, e_6 \rangle$. This implies the first claim. The second claim follows, since σ is generically finite of degree 20. \square

4.3. Base point freeness.

Theorem 45. *The linear system $|\mathcal{H}_3 - 2\delta|$ is base point free.*

The proof of Theorem 45 that we are going to present, after a few technical preliminaries, is entirely due to an anonymous referee. It will use results from sections 2.5 and 3.2.

Our initial proof relied on a completely different, and more involved, deformation argument. We took a well-chosen 2-polarized K3 surface (T, \mathcal{D}) such that a general deformation, along a divisor in the moduli space, of the pair $(T^{[3]}, \mathcal{D}_3)$ is of the form $(S^{[3]}, \mathcal{H}_3 - 2\delta)$ for (S, \mathcal{H}) one of our 10-polarized K3 surfaces. Base point freeness is clear for $|\mathcal{D}_3|$ on $T^{[3]}$, and we deduced base point freeness for $|\mathcal{H}_3 - 2\delta|$ on $S^{[3]}$, since this is an open condition in a family of polarized manifolds (X, \mathcal{D}) such that $h^0(X, \mathcal{D})$ remains constant.

Let us denote by $i : J \hookrightarrow S^{[3]}$ the inclusion morphism. Recall that J was defined in Section 3.2 as the degeneracy locus of the morphism

$$\phi : V_7^\vee \otimes \mathcal{O}_{S^{[3]}} \longrightarrow p_*(q^*\mathcal{U}_2^\vee).$$

Proposition 46. *There is an exact sequence*

$$0 \longrightarrow p_*(q^*\mathcal{U}_2^\vee)^\vee \xrightarrow{\phi^\vee} V_7 \otimes \mathcal{O}_{S^{[3]}} \longrightarrow \mathcal{O}_{S^{[3]}}(\mathcal{H}_3 - 2\delta) \longrightarrow i_*i^*\mathcal{O}_{S^{[3]}}(\mathcal{H}_3 - 2\delta) \longrightarrow 0.$$

Proof. Outside J , the morphism ϕ is surjective. So the kernel of ϕ is reflexive of rank one, hence a line bundle. Since J has codimension two by Corollary 22, we can compute this line bundle by taking determinants. We get

$$(30) \quad \text{Ker}(\phi) \simeq \det p_*(q^*\mathcal{U}_2^\vee)^\vee \simeq \mathcal{O}_{S^{[3]}}(2\delta - \mathcal{H}_3)$$

(see [46, Lemma 1.5] for the latter isomorphism).

On the other hand, the cokernel of ϕ is supported on J , and the corank is never bigger than one by Proposition 21. So $\text{Coker}(\phi) \simeq i_*\mathcal{L}$ for some line bundle \mathcal{L} on J , and we get an exact sequence

$$0 \rightarrow \mathcal{O}_{S^{[3]}}(2\delta - \mathcal{H}_3) \rightarrow V_7^\vee \otimes \mathcal{O}_{S^{[3]}} \rightarrow p_*(q^*\mathcal{U}_2^\vee) \rightarrow i_*\mathcal{L} \rightarrow 0.$$

Since J has codimension two in $S^{[3]}$, $\mathcal{H}om(i_*\mathcal{L}, \mathcal{O}) = \mathcal{E}xt^1(i_*\mathcal{L}, \mathcal{O}) = 0$. Dualizing the sequence above then yields the exact sequence

$$0 \rightarrow p_*(q^*\mathcal{U}_2^\vee)^\vee \rightarrow V_7 \otimes \mathcal{O}_{S^{[3]}} \rightarrow \mathcal{O}_{S^{[3]}}(\mathcal{H}_3 - 2\delta) \rightarrow \mathcal{E}xt^2(i_*\mathcal{L}, \mathcal{O}) \rightarrow 0.$$

Finally, the last term is a line bundle on J , and therefore the surjectivity of the previous arrow implies that this line bundle is isomorphic to the restriction of $\mathcal{O}_{S^{[3]}}(\mathcal{H}_3 - 2\delta)$. \square

Now, recall from diagram (11) that the family \mathcal{R} parametrizing cubic scrolls and cones in $X_{ad}(G_2)$ is an incidence correspondence

$$(31) \quad \begin{array}{ccc} & \mathcal{R} & \\ \tilde{p} \swarrow & & \searrow \tilde{q} \\ LieGr(2, V_7) & & X_{ad}(G_2). \end{array}$$

Moreover, \mathcal{R} is contained in the family \mathcal{C} of Schubert cycles over $LieGr(2, V_7)$, parametrizing planes in the family \mathcal{V}_5 of decomposing five-planes, that meet their axis from the family \mathcal{A}_2 . Inside the incidence correspondence

$$(32) \quad \begin{array}{ccc} & Gr_{LieGr(2, V_7)}(2, \mathcal{V}_5) & \\ \hat{p} \swarrow & & \searrow \hat{q} \\ LieGr(2, V_7) & & G(2, V_7), \end{array}$$

\mathcal{C} is the degeneration scheme of the natural morphism

$$\hat{q}^*\mathcal{U}_2 \hookrightarrow \hat{p}^*\mathcal{V}_5 \rightarrow \hat{p}^*(\mathcal{V}_5/\mathcal{A}_2).$$

Hence we have the following Eagon-Northcott resolution where \mathcal{H} is the pullback of the hyperplane divisor by \hat{q} :

$$(33) \quad 0 \rightarrow \hat{q}^*\mathcal{U}_2(-\mathcal{H}) \rightarrow \hat{p}^*(\mathcal{V}_5/\mathcal{U}_2) \otimes \mathcal{O}(-\mathcal{H}) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{\mathcal{C}} \rightarrow 0.$$

Next, recall from the end of subsection 2.5 that \mathcal{R} is obtained by intersecting \mathcal{C} with $\mathbb{P}_{LieGr(2, V_7)}(\mathcal{K}_5)$ inside $\mathbb{P}_{LieGr(2, V_7)}(\mathcal{A}_2 \wedge \mathcal{V}_5)$. Equivalently, it is the zero locus of the tautological section of $\mathcal{H}om(\mathcal{O}(-\mathcal{H}), \hat{p}^*((\mathcal{A}_2 \wedge \mathcal{V}_5)/\mathcal{K}_5))$ on \mathcal{C} . Diagram (9) shows that the bundle $(\mathcal{A}_2 \wedge \mathcal{V}_5)/\mathcal{K}_5$ is isomorphic to \mathcal{U}_2 , so \mathcal{R} is defined in \mathcal{C} as the zero locus of a global section of $\hat{p}^*\mathcal{U}_2 \otimes \mathcal{O}(\mathcal{H})$; hence the Koszul resolution

$$(34) \quad 0 \rightarrow \det(\hat{p}^*\mathcal{U}_2^\vee) \otimes \mathcal{O}(-2\mathcal{H}) \rightarrow \hat{p}^*\mathcal{U}_2^\vee \otimes \mathcal{O}(-\mathcal{H}) \rightarrow \mathcal{O}_{\mathcal{C}} \rightarrow \mathcal{O}_{\mathcal{R}} \rightarrow 0.$$

Lemma 47.

$$\mathbf{R}\tilde{p}_*(\tilde{q}^*\mathcal{U}_2^\vee(-t\mathcal{H})|_{\mathcal{R}}) \simeq \begin{cases} \mathcal{V}_5^\vee & \text{if } t = 0, \\ 0 & \text{if } t = 1, \\ \det(\mathcal{V}_2)^\vee \otimes \det(\mathcal{V}_5)^{\otimes 2}[-2] & \text{if } t = 2. \end{cases}$$

As usual the shift notation $[-2]$ means that the corresponding term appears in cohomological degree two, while one gets zero in the other degrees.

Proof. First twist (33) by $\tilde{q}^*\mathcal{U}_2^\vee(-t\mathcal{H})$ and use the relative Borel-Weil-Bott theorem to deduce that

$$\mathbf{R}\hat{p}_*(\hat{q}^*\mathcal{U}_2^\vee(-t\mathcal{H})|_{\mathcal{R}}) \simeq \begin{cases} \mathcal{V}_5^\vee & \text{if } t = 0, \\ 0 & \text{if } 1 \leq t \leq 3, \\ \det(\mathcal{V}_5)^{\otimes 2}[-4] & \text{if } t = 4. \end{cases}$$

Then use (34) to complete the computation. \square

Recall that J comes with a natural map $\pi : J \rightarrow \mathbb{P}^2$, defined in (17).

Lemma 48. *One has $i^*\mathcal{O}_{S^{[3]}}(\mathcal{H}_3 - 2\delta) \simeq \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$.*

Proof. Since the morphism $p : \mathcal{Z} \rightarrow S^{[3]}$ in (14) is flat, we can use base change to get, from (30),

$$i^*\mathcal{O}_{S^{[3]}}(\mathcal{H}_3 - 2\delta) \simeq i^*\det p_*(q^*\mathcal{U}_2^\vee) \simeq \det p_*(q^*\mathcal{U}_2^\vee|_{\mathcal{Z}_J}),$$

where $\mathcal{Z}_J = p^{-1}(J)$. We will use the identification $J \simeq \mathcal{D}(\psi)$ from Lemma 20 to compute the right hand side. Indeed, note that the family \mathcal{R} of cubic scrolls and cones is contained in $\mathbb{P}_{LieGr(2, V_7)}(\mathcal{K}_5)$ and that

$$\mathcal{Z}_J \simeq \mathcal{R} \times_{\mathbb{P}_{LieGr(2, V_7)}(\mathcal{K}_5)} \mathbb{P}_J(\text{Ker}(\psi_J)).$$

In words, the subscheme $Z \subset S$ of length three corresponding to a point of J is equal to the intersection of the corresponding cubic scroll or cone $\mathbb{P}(\mathcal{K}_5) \cap X_{ad}(G_2)$ with the plane $\mathbb{P}(\text{Ker}(\mathcal{K}_5 \rightarrow \mathfrak{g}_2/V_{11})) \subset \mathbb{P}(\mathcal{K}_5)$. But by definition of ψ , we have

$$\text{Coker}(\psi|_J) = \pi^*\mathcal{O}_{\mathbb{P}^2}(1) \quad \text{and} \quad \text{Ker}(\psi_J) = \text{Ker}(\mathcal{K}_5 \rightarrow \pi^*(\Omega(1))).$$

Hence the Koszul resolution

$$0 \rightarrow \mathcal{O}_{\mathcal{R}_J}(-2\mathcal{H}) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(1) \rightarrow \mathcal{O}_{\mathcal{R}_J}(-\mathcal{H}) \otimes \pi^*(\mathcal{T}(-1)) \rightarrow \mathcal{O}_{\mathcal{R}_J} \rightarrow \mathcal{O}_{\mathcal{Z}_J} \rightarrow 0,$$

where $\mathcal{R}_J = \mathcal{R} \times_{LieGr(2, V_7)} J$ and \mathcal{H} is again the pullback of the hyperplane class of $X_{ad}(G_2)$. Now we can tensorize this exact sequence by $\tilde{q}^*\mathcal{U}_2^\vee$ and push it forward to J . Using Lemma 47, we get the exact sequence

$$0 \rightarrow \mathcal{V}_{5|J}^\vee \rightarrow p_*(q^*\mathcal{U}_2^\vee|_{\mathcal{Z}_J}) \rightarrow (\det(\mathcal{V}_2)^\vee \otimes \det(\mathcal{V}_5)^{\otimes 2})|_J \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(1) \rightarrow 0.$$

Taking determinants, and using (10), we deduce the Lemma. \square

Proof of Theorem 45. Combining Proposition 46 and Lemma 48, we get the exact sequence

$$(35) \quad 0 \rightarrow p_*(q^*\mathcal{U}_2^\vee)^\vee \rightarrow V_7 \otimes \mathcal{O}_{S^{[3]}} \rightarrow \mathcal{O}_{S^{[3]}}(\mathcal{H}_3 - 2\delta) \rightarrow i_*\pi^*\mathcal{O}_{\mathbb{P}^2}(1) \rightarrow 0.$$

We claim that $|\pi^*\mathcal{O}_{\mathbb{P}^2}(1)| \simeq |\mathcal{O}_{\mathbb{P}^2}(1)| = \mathbb{P}^2$. Indeed, the Stein factorization of π described in Corollary 22 yields

$$\pi_*\mathcal{O}_J = g_*\mathcal{O}_\Sigma = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3),$$

since g is a degree two cover branched over a sextic curve. We thus get an exact sequence

$$(36) \quad 0 \longrightarrow V_7 \longrightarrow H^0(\mathcal{O}_{S^{[3]}}(\mathcal{H}_3 - 2\delta)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \longrightarrow 0,$$

where a dimension count provides the surjectivity of the rightmost map, see Proposition 38. This means that every global section of $\pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ can be lifted to a global section of $\mathcal{O}_{S^{[3]}}(\mathcal{H}_3 - 2\delta)$, and since the former is base point free, we can conclude from (35) that the latter is also base point free. \square

Recall from Lemma 29 that the indeterminacy locus J of φ is also the exceptional locus of the extremal contraction $c : S^{[3]} \rightarrow N$, induced by a multiple of $\mathcal{H}_3 - 2\delta$. Since $|\mathcal{H}_3 - 2\delta|$ itself is base point free, by [31, Lemma 2.1.28], c is also the first map of the Stein factorization of $\phi_{|\mathcal{H}_3 - 2\delta|}$,

$$(37) \quad \begin{array}{ccc} S^{[3]} & \xrightarrow{\phi_{|\mathcal{H}_3 - 2\delta|}} & |\mathcal{H}_3 - 2\delta|^\vee \\ & \searrow c & \nearrow \nu \\ & N & \end{array}$$

In the next section we will compute the degree of $\phi_{|\mathcal{H}_3 - 2\delta|}$, hence of ν .

In $|\mathcal{H}_3 - 2\delta|^\vee$ there is a distinguished plane $\mathbb{P}(V_7^\perp)$, of codimension one linear subsystems containing the Pfaffian cubics. It is the center of the projection j^t from Lemma 44.

Proposition 49. *The image of J via $\phi_{|\mathcal{H}_3 - 2\delta|}$ is $\mathbb{P}(V_7^\perp)$.*

Proof. A length three scheme Z corresponding to a point in J is contained in $G(2, V_5)$ for some $V_5 \subset V_7$. Since a form in five variables has rank at most four, any Pfaffian cubic vanishes on $\langle Z \rangle \subset \mathbb{P}(\wedge^2 V_5)$. Hence the claim by Proposition 39. \square

Remark. Here again, the situation is in some sense close to that of Gushel-Mukai varieties in $G(2, V_5)$. The latter Grassmannian is cut out in $\mathbb{P}(\wedge^2 V_5)$ by the Plücker quadrics, parametrized by V_5 itself. A Gushel-Mukai variety is cut out by the Plücker quadrics plus one extra, general quadric. Here, the linear system of cubics containing $\text{Sec}(S)$ is made of the Pfaffian cubics, plus three extra cubics that remain mysterious. Equivalently, the projective plane $\mathbb{P}(V_7^\perp)$ which is the center of the projection $p_{V_7^\vee}$ remains elusive.

4.4. Computation of the degree. Let us finally determine the degree d of $\phi_{|\mathcal{H}_3 - 2\delta|}$. By Corollary 40, d is even, and by Lemma 44, d divides 20. So $d = 2, 4, 10$ or 20 . We will exclude the three last cases thanks to some simple combinatorics.

Proposition 50. $d = 2$.

Proof. Let us exclude the other possibilities. Recall that for the generic fiber of σ there are six distinct points $\{p_i\}_{i=1,\dots,6}$ such that the fiber is $\{p_i + p_j + p_k\}_{i < j < k}$. Thus, at the generic point $p_1 + p_2 + p_3$ of $S^{[3]}$ we have another point $p_4 + p_5 + p_6$ of $S^{[3]}$ in the same fiber of $\phi_{|H_3-2\delta|}$, and any other point in that fiber must be of the form $p_i + p_j + p_k$ and come with its complement $p_\ell + p_m + p_n$, where $\{i, j, k\} \cap \{\ell, m, n\} = \emptyset$. To simplify notations, we will denote these pairs by $(123|456)$ and $(ijk|\ell mn)$, where we can permute the two triples and the three integers in each triple.

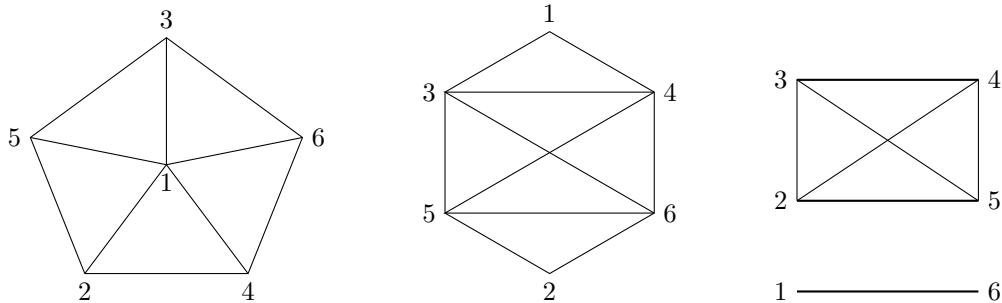
$d = 4$. Up to permutation of the indices, we can always suppose that the four points in the fiber are given by the triples $(123|456)$ and $(124|356)$. Then each triple contains a unique pair of points (either (12) or (56)) that is also contained in another triple. This means that we would be able to split the generic point of $S^{[3]}$ as the sum of a point of $S^{[2]}$ and a point of S . But then the universal family $\mathcal{Z} \rightarrow S^{[3]}$ would admit a rational section, which is impossible since \mathcal{Z} is irreducible.

$d = 10$. We will use the same idea as before, although this case is a bit more involved combinatorially. We first remark that up to permuting the indices, there are only three ways to choose 5 pairs of complementary triples among six indices. We leave the following lemma to the reader.

Lemma 51. *Up to permuting indices, any 5-tuple of complementary triples of indices between 1 and 6 is equivalent to one of the following:*

$$\begin{array}{ccc}
 (123|456) & (123|456) & (123|456) \\
 (134|256) & (124|356) & (124|356) \\
 (145|236) & (125|346) & (134|256) \\
 (156|234) & (126|345) & (135|246) \\
 (126|345) & (156|234) & (145|236)
 \end{array}$$

A nice combinatorial gadget in order to distinguish these 5-tuples is to associate to them a little graph by using the previous remark that each time we have two pairs of complementary triples, we can arrange them in the form $(ijk|\ell mn)$ and $(ij\ell|kmn)$. In particular, the pair $(k\ell)$ is distinguished. So each 5-tuple yields ten such pairs, which we can visualize as the edges of a graph. We get the three following graphs:



(Beware that on the rightmost graph, the three horizontal edges are double and represented by a thick line). In particular the six points do not have the same combinatorial properties, and in each case it is easy to see that in each triple, one point can be distinguished. (In the first graph the vertex 1 is clearly special, which gives one distinguished point in five of the ten triples; in the remaining ones like (456), only one vertex, here 5, is not connected to the other ones and can be distinguished. In the second graph, 1 and 2 are special, allowing to distinguish one vertex in each triple that contains one of them or both; in the remaining triples, two vertices are connected to the same special vertex, and the other one is distinguished. In the third graph, 1 and 6 are special and give a distinguished point in each triple.) Exactly as in the previous case, we would thus be able to construct a rational section of the universal family $\mathcal{Z} \rightarrow S^{[3]}$, which is impossible.

$d = 20$. This means that all the triples $p_i + p_j + p_k$ belong to the fiber. We denote by π_{ijk} the projective plane in L spanned by p_i, p_j, p_k . By Proposition 39, we conclude that there is a hyperplane in $I_3(\text{Sec}(S))$ consisting of cubics that vanish on the 20 planes π_{ijk} . Recall that the six points p_1, \dots, p_6 span the linear space $\mathbb{P}(\mathfrak{s}) \cap L \subset \mathbb{P}(\mathfrak{g}_2)$, where $\mathfrak{s} \simeq \mathfrak{sl}_3$, which is a \mathbb{P}^4 , see the proof of Corollary 40. If the corresponding planes in V_7 span the hyperplane V_6 , this is also $\mathbb{P}(\wedge^2 V_6) \cap L \subset \mathbb{P}(\wedge^2 V_7)$. An explicit computation in coordinates shows that a cubic on this \mathbb{P}^4 vanishing on the 20 planes must vanish identically. We conclude that the linear system $|I_3(\text{Sec}(S))|$ reduces on $\mathbb{P}(\mathfrak{s}) \cap L$ to a unique cubic. But since the projective four spaces $\mathbb{P}(\wedge^2 V_6) \cap L$ cover an open subset of L (because any element of $\wedge^2 V_7$ has rank at most six), this would mean that the linear system itself reduces to a single cubic, which is absurd. The proof is complete. \square

Corollary 52. φ is the covering involution associated to $\phi_{|\mathcal{H}_3 - 2\delta|}$.

4.5. Odds and ends. We conclude the paper with a few observations.

Deformations. The variety N is normal and φ descends to an involution $\bar{\varphi}$ of N which is regular, since it fixes the ample class whose pullback via c is a multiple of $\mathcal{H}_3 - 2\delta$ [31, Theorem 2.1.27]. The quotient $N/\langle \bar{\varphi} \rangle$ is still normal, hence isomorphic to the normalization of $\phi_{|\mathcal{H}_3 - 2\delta|}(S^{[3]})$. This is summarized in the following diagram:

$$(38) \quad \begin{array}{ccc} S^{[3]} & \xrightarrow{\phi_{|\mathcal{H}_3 - 2\delta|}} & \phi_{|\mathcal{H}_3 - 2\delta|}(S^{[3]}) \\ \swarrow c & & \uparrow \nu \\ N & & \\ \downarrow \bar{\varphi} & & \downarrow \bar{\nu} \\ N/\bar{\varphi} & & \end{array}$$

Remark. Note the analogy with the O’Grady involution, which yields a similar picture where the map $c : S^{[2]} \rightarrow N$ contracts a \mathbb{P}^2 , N is a double EPW sextic singular in $c(\mathbb{P}^2)$ which is the inverse image of the Plücker point, and $N \rightarrow \phi_{|\mathcal{H}_3-2\delta|}(S^{[2]})$ is the double cover $\tilde{Y}_A \rightarrow Y_A$ of a special EPW sextic [42]. This is a situation where $Y_A = \phi_{|\mathcal{H}_3-2\delta|}(S^{[2]})$ is normal (although $Y_A^{[3]} \neq \emptyset$ does not have the expected dimension).

In our situation, we do not know whether $\bar{\nu}$ is an isomorphism. But pursuing the analogy with EPW sextics, it would be tempting to imagine that the double cover $\nu : N \rightarrow \phi_{|\mathcal{H}_3-2\delta|}(S^{[3]})$ can be deformed to a locally complete family of polarized hyperKähler sixfolds. This very natural question is also discussed in [21].

On the contact structure of $X_{ad}(G_2)$. Since $X_{ad}(G_2)$ is an adjoint variety, it is well known that it admits an algebraic contact structure (and a famous conjecture by Lebrun and Salamon asserts that all contact structures on Fano manifolds of Picard number one arise from adjoint varieties). Although we did not insist on this, there is an interesting connection between the contact structure on $X_{ad}(G_2)$ and our constructions, which roughly goes as follows.

Recall from Corollary 12 that $LieGr(2, V_7) \simeq OG(2, V_7)$ parametrizes decomposing five planes $V_5 \subset V_7$. In section 2.5 we have constructed a rank five subbundle \mathcal{K}_5 of the trivial bundle with fiber \mathfrak{g}_2 over $LieGr(2, V_7)$. By item (2) of Proposition 14, its fibers are the affine spans of the corresponding cubic scrolls. One can prove that this bundle \mathcal{K}_5 restricted to $X_{ad}(G_2)$ is nothing else than the affine version of the contact vector bundle (sitting inside the affine tangent bundle exactly as the contact hyperplane bundle sits inside the tangent bundle). Moreover, the space of global sections of \mathcal{K}_5^\vee is isomorphic with \mathfrak{g}_2 , and a generic section vanishes along two anticanonically embedded Veronese surfaces. So we recover the two Veronese surfaces of Proposition 16 in a completely different way.

Considering the relative zero locus over the projectivized space of sections, and its Stein factorization, we get a double cover of $\mathbb{P}(\mathfrak{g}_2)$, branched over the sextic hypersurface which is projectively dual to $X_{ad}(G_2)$. This double cover plays a crucial rôle in Homological Projective Duality for the G_2 -Grassmannian, as developed in [26]. It was already observed in loc. cit. that this double cover factorizes the so-called Grothendieck-Springer simultaneous resolution. Ultimately, it is related to the classical fact that a del Pezzo surface of degree six can be represented as the blowup of the projective plane at three points, in exactly two different ways.

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