HAL open science

Unipotent Subgroups of Stabilizers

Philippe Gille, Robert M. Guralnick

To cite this version:

Philippe Gille, Robert M. Guralnick. Unipotent Subgroups of Stabilizers. Transformation Groups, In press, 28 (3), pp.1149-1164. 10.1007/s00031-023-09802-7 . hal-03864542v2

HAL Id: hal-03864542

https://cnrs.hal.science/hal-03864542v2

Submitted on 16 Jan 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

UNIPOTENT SUBGROUPS OF STABILIZERS

PHILIPPE GILLE AND ROBERT GURALNICK

For James Humphreys

Abstract. We consider semi-continuity of certain dimensions on group schemes.

1. Introduction

Let G be an algebraic group over a field k. Let $d_u(G)$ the maximal dimension of a (smooth) connected unipotent subgroup of $G_{\overline{k}}$. Using techniques à la Demazure-Grothendieck-Raynaud, we show the following result.

Theorem 1.1. *Let S be a scheme and let G be an S-group scheme of finite presentation. Then the function d_u on S is upper semi-continuous.*

Upper semi-continuous means informally that the function jumps along closed sets. In particular, the function is locally constant at the points of the minimal value locus. If S is irreducible, this implies that the minimal value is reached at the generic point of U. This gives the following useful corollary.

Corollary 1.2. *Let S be an irreducible scheme and let G be an S-group scheme of finite presentation. If for the generic point $\xi \in S$, $G_{\overline{\kappa(\xi)}}$ contains a d-dimensional smooth unipotent subgroup, then the same is true for $G_{\overline{\kappa(s)}}$ for all $s \in S$.*

We observe that the proof of Theorem 1.1 goes by reduction to the case of $S = \operatorname{Spec}(A)$ where A is a DVR so that the corollary implies actually 1.1.

A case of special interest is the group scheme of stabilizers called also the stabilizer of the diagonal [SGA3, V.10.2]. More precisely we assume that the scheme S is noetherian and that G is a separated S–group scheme of finite presentation acting on a separated S–scheme X of finite presentation, we consider the fiber product

$$\begin{tikzcd} F \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ G \times_S X \arrow[r] & X \times_S X \end{tikzcd} \qquad (g,x) \longmapsto (x, g.x).$$

It defines an X-group scheme F which is a closed X-subgroup scheme of $G \times_S X$ (here separability is used) of finite presentation (by noetherianity) such that for each $x \in X$ of

Date: January 10, 2023.
2020 *Mathematics Subject Classification.* Primary 14L15; secondary 14L30, 20B15, 20G15.
The second author was partially supported by the NSF grant DMS-1901595 and a Simons Foundation Fellowship 609771.

image $s \in S$, $F_{\kappa(x)} \subset G \times_{\kappa(s)} \kappa(x)$ is the stabilizer of the point x for the action of $G \times_{\kappa(s)} \kappa(x)$ on $X \times_{\kappa(s)} \kappa(x)$. In case $\kappa(s) = \kappa(x)$, the usual notation for $F_{\kappa(x)}$ is G_x.

If X is furthermore irreducible with generic point η, the result shows that if $F_{\overline{\kappa(\eta)}}$ contains a d-dimensional smooth unipotent subgroup, then the same is true for $F_{\overline{\kappa(x)}}$ for all $x \in X$.

A typical example is that of a linear representation $G \to \mathrm{GL}_{n,S}$, that is, a linear action on the S–affine space $X = \mathbb{A}^n_S$ (which is irreducible [GW, ex. 3.22]). If the geometric generic stabilizer $G_\eta \times_{\kappa(\eta)} \overline{\kappa(\eta)}$ contains a d-dimensional smooth unipotent subgroup, then the same is true for $G_x \times_{\kappa(x)} \overline{\kappa(x)}$ for all $x \in X$. Another important example is for G an algebraic group and X a direct product of homogeneous spaces for G.

One motivation for this question was related to the base size of finite groups acting primitively on a set and the existence of regular orbits for nontransitive actions. An important case is that of a finite simple group of Lie type over a finite field where the action comes from the algebraic group. See [BGS]. Another interesting case is when G is a reductive algebraic group acting linearly on X (or acting on the Grassmanian of a rational module). See [GG, GL, PV] for more on this. We give more details on this in Section 5.

Note that by the Lang-Steinberg theorem, an algebraic group defined over $\mathbb{F}_q$ has a Borel subgroup defined over $\mathbb{F}_q$. We also know that if U is a d-dimensional unipotent subgroup defined over $\mathbb{F}_q$, then $|U(\mathbb{F}_q)| = q^d$ [B, GG]. We can apply Corollary 1.2 to the stabilizer scheme to obtain the following result.

Corollary 1.3. *Let G be an algebraic group acting faithfully on an irreducible variety X and assume that the G, X and the action are defined over a finite field $\mathbb{F}_q$. Assume that there is a nonempty open subset X_0 of X such that the stabilizer G_x of $x \in X_0$ has a d-dimensional unipotent subgroup. Then for all $x \in X(\mathbb{F}_q)$, $G_x(\mathbb{F}_q)$ contains a subgroup of size q^d.*

One can ask more generally what other functions are upper semi-continuous. Of course, dimension is [SGA3, VI_B.4.3]. In fact, we will show that other such functions are also upper semi-continuous. On the other hand, if we define $d^0(G)$ to be the dimension of the derived subgroup of the connected component of G, it is not true that d^0 need be upper semi-continuous. We study this in Section 6 and particularly for the smooth case. Smoothness rarely holds in the case of the stabilizer scheme and we give an example to show the failure of upper semi-continuity for $d^0(G)$ in stabilizer schemes.

We use mostly the terminology of Borel's book [B] and time to time the more general setting Demazure-Gabriel's book [D-G] and the SGA3 seminar [SGA3] where in particular an algebraic group is not supposed smooth. All definitions are coherent.

In the next sections, we prove some preliminary results. We prove Theorem 1.1 and other upper semi-continuity results in Section 4. In Section 6, we consider the dimension of the derived subgroup. In the final section, we present an alternate proof of Theorem 1.1 due to Brian Conrad.

Acknowledgments: We thank Jean-Pierre Serre, Brian Conrad and Matthieu Romagny for valuable comments on a preliminary version of the paper. We also thank Brian Conrad for allowing us to include his alternate proof of Theorem 1.1, Skip Garibaldi for the applications to the abelianization (Cor. 6.4) and Laurent Moret-Bailly for the improvement of Proposition 3.6.

2. Definition of the rank functions

Let k be a field. In this paper a k–variety is a separated k-scheme of finite type which is geometrically integral. We remind the reader that a linear algebraic group G (i.e. smooth affine algebraic k-group) is unipotent if for each (or any) faithful $\overline{k}$–representation $\rho : G_{\overline{k}} \to \mathrm{GL}_{n,\overline{k}}$, $\rho(G(\overline{k}))$ consists of unipotent elements. This agrees with the general definition given in [D-G, 4.2.2.1], see 4.3.26.b of this reference. In practice we use the equivalent definition that G admits a closed k-embedding in a k–group of strictly upper triangular matrices (*ibid*, 4.2.2.5).

Similarly a closed smooth k–subgroup G of GL_n is trigonalizable if there exists $h \in \mathrm{GL}_n(k)$ such that $hGh^{-1} \subset B_n$ where B_n is the k–Borel subgroup of GL_n consisting of upper triangular matrices; the same holds for any linear representation $G \to \mathrm{GL}_r$ [B, 15.5]. That definition holds more generally for an arbitrary affine algebraic k–group since it is equivalent to the Demazure-Gabriel's definition [D-G, 4.2.3.4].

2.1. **Relative version.** We define the following invariants:

(i) $D_u(k,G)$ = Maximal dimension of an unipotent k-subgroup of G;

(ii) $D_{cu}(k,G)$ = Maximal dimension of a commutative unipotent k-subgroup of G;

(iii) $D_t(k,G)$ = Maximal dimension of a trigonalizable k-subgroup of G;

(iv) $D_n(k,G)$ = Maximal dimension of an affine nilpotent k-subgroup of G;

(v) $D_{nt}(k,G)$ = Maximal dimension of a nilpotent trigonalizable k-subgroup of G;

(vi) $D_s(k,G)$ = Maximal dimension of an affine solvable k-subgroup of G;

(vii) $D_c(k,G)$ = Maximal dimension of an affine commutative k-subgroup of G;

(viii) $D'_c(k,G)$ = Maximal dimension of a commutative k-subgroup of G;

(ix) $D'_n(k,G)$ = Maximal dimension of a nilpotent k-subgroup of G;

(x) $D'_s(k,G)$ = Maximal dimension of a solvable k-subgroup of G.

We have $D_c(k,G) \leq D'_c(k,G)$, $D_n(k,G) \leq D'_n(k,G)$ with equalities if G is affine. We have the obvious inequalities $D'_c(k,G) \leq D'_n(k,G) \leq D'_s(k,G)$ and $D_{cu}(k,G) \leq \mathrm{Min}(D_c(k,G), D_u(k,G))$ and $D_u(k,G) \leq D_n(k,G)$, $D_{nt}(k,G) \leq \mathrm{Max}\big(D_n(G), D_t(k,G)\big) \leq D_s(k,G)$.

All these functions are increasing by change of fields.

Lemma 2.1. *If k is algebraically closed, then $D(k,G) = D(F,G)$ for any field extension F/k and for each function D as above.*

Proof. We do it for d_u, the other cases being similar. We can assume that F is algebraically closed so that G_F admits a smooth unipotent F–subgroup U of dimension d. There exists a finitely generated k–subextension E of F such that U is defined over E. The field E is function field of a smooth k-variety X. Up to shrinking X, U extends to a closed subgroup scheme $\mathfrak{U}$ of $G \times_k X$ which is smooth in view of [SGA3, VI_B.10]. Up to shrinking one more time, $\mathfrak{U}$ is a closed subgroup scheme of the X-group scheme of strictly upper triangular matrices. Since k is algebraically closed, we have $X(k) \neq \emptyset$. The fiber at $x \in X(k)$ provides a smooth unipotent k–subgroup $\mathfrak{U}_x \subset G$ of dimension d. Thus $D(k,G) \geq d$. $\square$

2.2. **Absolute version.** We define now

(i) $d_u(G) = D_u(\overline{k}, G)$, i.e. the maximal dimension of an unipotent $\overline{k}$-subgroup of $G_{\overline{k}}$;
(ii) $d_{cu}(G) = D_{cu}(\overline{k}, G)$;
(iii) $d_t(G) = D_t(\overline{k}, G)$;
(iv) $d_n(G) = D_n(\overline{k}, G)$;
(v) $d_{nt}(G) = D_{nt}(\overline{k}, G)$;
(vi) $d_s(G) = D_s(\overline{k}, G)$;
(vii) $d_c(G) = D_c(\overline{k}, G)$;
(viii) $d'_c(G) = D'_c(\overline{k}, G)$;
(ix) $d'_n(G) = D'_n(\overline{k}, G)$;
(x) $d'_s(G) = D'_s(\overline{k}, G)$.

Clearly it does not depend of the choice of $\overline{k}$. All these functions are insensitive to change of fields.

Lemma 2.2. *We have $d(k, G) = d(F, G)$ for any field extension F/k and for each function d as above.*

Proof. Let $\overline{F}$ be an algebraic closure of F containing $\overline{k}$. Lemma 2.1 shows that $D(\overline{k}, G) = D(\overline{F}, G)$ whence $d(k, G) = d(F, G)$. □

Let $G = G_{\overline{k}}$ be an algebraic goup. Denote the maximal smooth subgroup and maximal smooth connected subgroup of $G_{\overline{k}}$ by $G_{\overline{k},red}$ and $G^0_{\overline{k},red}$ respectively. Both of these subgroups have the same dimension as G.

From that observation it follows that

$$d_u(G) = \text{Maximal dimension of a smooth connected unipotent } \overline{k}\text{-subgroup of } G_{\overline{k}};$$

and similarly for the other absolute rank functions. Since affine smooth connected solvable $\overline{k}$–subgroups are trigonalizable by the Lie-Kolchin's theorem [B, 10.5] it follows that $d_s(k, G) = d_t(k, G)$ and similarly we have $d_n(k, G) = d_{nt}(k, G)$.

We have $d_c(G) \leq d'_c(k, G)$, $d_n(G) \leq d'_n(G)$ and $d_c(G) \leq d'_c(G)$ with equalities if G is affine. We have the obvious inequalities $d'_c(G) \leq d'_n(G) \leq d'_s(G)$ and $d_{cu}(G) \leq \mathrm{Min}(d_c(G), d_u(G))$ and $d_u(G) \leq d_n(G) \leq d_s(G)$. We investigate the invariance under an isogeny.

Lemma 2.3. *Let $f : G \to H$ be an isogeny of algebraic k–groups.*

(1) We have $d'_\bullet(k, G) = d'_\bullet(k, H)$ for $\bullet = c, n, s$.

(2) We have $d_\bullet(k, G) = d_\bullet(k, H)$ for $\bullet = c, u, cu, n, s$.

Proof. We can assume that k is algebraically closed.

(1) For $\bullet = c, n, s$ we have $d'_\bullet(k, G) \leq d'_\bullet(k, H)$. Conversely we assume that H contains a smooth connected k–group H' which is commutative (resp. nilpotent, solvable) of dimension $d = d'_\bullet(k, H)$ for the relevant case. We put $G' = f^{-1}(H')^0_{red}$ and the map $G' \to H'$ is an isogeny between smooth connected groups. The k–group G' has dimension d so that $d'_\bullet(k, G) \geq d$.

If H' is commutative we have that $f([G',G']) \subset \ker(f)$ so that $[G',G']=1$ and G' is commutative. In this case we have $d'_c(G) \geq d = d'_c(H)$.

The nilpotent and solvable cases are similar and based on the characterization by sequences of subgroups [SGA3, prop. $\mathrm{VI}_B.8.2$]. Let us consider the solvable case. The sequence $H'_0 = H', H'_1 = [H'_0, H'_0], H'_2 = [H'_1, H'_1], \dots$ vanishes at some $n \geq 1$. Using the same notation, it follows that $f(G'_n) = 1$ so that $G'_n = 1$ since G'_n is smooth connected. Thus G'_n is solvable. We conclude that $d'_s(G) \geq d = d'_s(H)$.

(2) For $\bullet = c, u, cu, n, s$ we have $d_\bullet(k,G) \leq d_\bullet(k,H)$. Conversely we assume that H contains a smooth connected k–group H' which is affine commutative (resp. unipotent, commutative unipotent, affine nilpotent, affine solvable) of dimension $d_\bullet(k,H)$ for the relevant case. We put $G' = f^{-1}(H')^0_{red}$ and the map $G' \to H'$ is an isogeny between smooth connected groups. The k–group G' has dimension d and is affine according to [D-G, III.3.2.6]. Lemma 7.1.(2) shows that G' is commutative (resp. unipotent, commutative unipotent, nilpotent, solvable). We conclude that $d_\bullet(G) \geq d = d_\bullet(k,H)$ for $\bullet = c, u, cu, n, s$. □

2.3. **Connections with the literature.** (a) In [SGA3, XII.1], Grothendieck defines related rank functions but which are different. For example the Grothendieck unipotent rank $\rho_u(G)$ of a smooth connected group G over an algebraically closed field is $d_u(C)$ where C is a Cartan subgroup of G^0. This function ρ_u is upper semi-continuous if G is smooth affine over a base [SGA3, XII.2.7.(i)]. Our result does not require smoothness (nor flatness).

(b) If G is a smooth connected affine algebraic group over an algebraically closed field k, the Borel subgroups are the maximal smooth connected solvable subgroups, they are all conjugate and so $d_s(G)$ is nothing but the dimension of a Borel subgroup. The unipotent radicals are then the maximal smooth connected unipotent subgroups, they are all conjugate and so $d_u(G)$ is the dimension of a the unipotent radical of a Borel subgroup of G. For $d_n(G)$ the situation is more complicated since maximal smooth connected nilpotent subgroups of G do not consist of a single conjugacy class. However there are finitely many conjugacy classes (Platonov, [P, thm 2.13]).

3. Specialization over a regular local ring

3.1. **Group schemes over a DVR.** Let A be a discrete valuation ring with fraction field K and residue field k. If G is an A–group scheme of finite presentation, we would like to list properties on the generic fiber G_K which are inherited by the closed fiber G_k. For example, if G is flat, then G_K and G_k share the same dimension [SGA3, $\mathrm{VI}_B.4.3$]. Also if G is separated and flat and G_K is affine, then G is affine (Raynaud, [PY, prop 3.1]) so that G_k is affine (the separability assumption can be removed, see Lemma 3.3.(5) below). Flatness is then an important property, we recall that an A–scheme $\mathfrak{X}$ is flat if and only if it is torsion free, this second condition being equivalent to the density of the generic fiber $\mathfrak{X}_K$ in $\mathfrak{X}$ [GW, §14.3]. For the study of the function d_u, the next statement is the key step.

Lemma 3.1. *We assume that G is flat and affine of finite presentation. If G_K is trigonalizable (resp. unipotent) so is G_k.*

Proof. We assume that G_K is trigonalizable, that is, G_K is an extension of a diagonalizable K-group by an unipotent K-group. According to [BT, 1.4.5], there exists a closed monomorphism $\rho : G \to \mathrm{GL}_N$. Since G_K is trigonalizable, its stabilizes a flag of K^N [D-G, prop. 4.2.3.4, $(i) \Longrightarrow (iv)$]. In other words ρ factorizes through a Borel subgroup B_K of GL_N. Since the A–scheme of Borel subgroups of GL_N is projective [SGA3, XXII.5.8.3], B_K extends uniquely to a Borel A–subgroup scheme B of GL_N [a concrete way is to use a filtration $V_0 = 0 \subsetneq V_1 \subsetneq V_2 \cdots \subsetneq V_N = K^N$ and put $\widetilde{V}_i = A^n \cap V_i$ for each i. It defines an A-flag of lattices of A^n which is stabilized by G]. Its reduction to k provides an embedding of G_k to a Borel k–subgroup of $\mathrm{GL}_{N,k}$. The last quoted result, $(v) \Longrightarrow (i)$, enables us to conclude that G_k is trigonalizable.

We assume furthermore that G_K is unipotent. We have an exact sequence of A–group schemes $1 \to U \to B \xrightarrow{q} \mathbb{G}_m^N \to 1$. Since G_K is unipotent $q_K \circ \rho_K = G_K \to (\mathbb{G}_m^N)_K$ is trivial according to [D-G, IV.2.2.4]. Since G_K is dense in G it follows that $q \circ \rho = G \to \mathbb{G}_m^N$ is trivial so that ρ factorizes through the unipotent radical U of B. A fortiori G_k admits a representation in a strictly upper triangular k-group so is unipotent according to [D-G, IV.2.2.5, $(vi) \Longrightarrow (i)$]. □

Remark 3.2. (a) If G_K is split K–unipotent, a result of Veĭsfeĭler-Dolgachev on unipotent group schemes [VD, thm. 1.1] shows also that G_k is unipotent. This is a very different proof.

(b) If G has smooth fibers and G_K is unipotent, the fact that G_k is unipotent follows from [SGA3, VI_B.8.4.(ii)]. This is a quite different proof.

(c) One simpler proof of (b) occurs in the alternate proof below of Theorem 1.1 using the smoothness of the scheme of maximal tori of G, see §7.

For dealing with the other rank functions, we need more facts.

Lemma 3.3. *We assume that the A–group scheme G is flat of finite presentation. Let H be a K–subgroup of G_K and let $\mathfrak{H}$ be the schematic closure of H in G.*

(1) $\mathfrak{H}$ is a closed A–subgroup scheme of G which is flat of finite presentation. If H is central, then $\mathfrak{H}$ is central.

(2) The fppf quotient $G/\mathfrak{H}$ is representable by a separated A-scheme of finite presentation which is flat.

(3) If H is normal in G_K, then $\mathfrak{H}$ is a normal A–subgroup scheme of G and $G/\mathfrak{H}$ carries a natural structure of A-group scheme.

(4) Let G_0 be the schematic closure of $\mathrm{Spec}(K) \to G$. *Then G_0 is étale over A and is a normal closed A–subgroup scheme of G. Furthermore G/G_0 is representable by a separated A-scheme of finite presentation and is flat.*

(5) If G is flat and G_K is affine, then G_k is affine.

Proof. (1) According to [SGA3, VIII.7.1], $\mathfrak{H}$ is a closed A–subgroup scheme of G which is flat. It of finite presentation since G is a noetherian scheme [St, Tag 04ZL].

Assume that H is central, that is the commutator map $G_K \times_K H \to G_K$ is trivial. Since $G_K \times_K H$ is dense in $G \times_A \mathfrak{H}$, it follows that the commutator map $G \times_A \mathfrak{H} \to G$ is trivial, so that $\mathfrak{H}$ is central in G.

(2) The representability is a result of Anantharaman [A, IV, th. 4.C] so that $G/\mathfrak{H}$ is separated and of finite presentation [SGA3, VI_B.9.2.(x) and (xiii)]. Finally $G/\mathfrak{H}$ is flat according to [SGA3, VI_B.9.2.(xi)].

(3) Assume that H is normal in G_K, that is, the commutator map $G_K \times_K H \to G_K/H$ is trivial. Since $G_K \times_K H$ is dense in $G \times_A \mathfrak{H}$, it follows that the commutator map $G \times_A \mathfrak{H} \to G/\mathfrak{H}$ is trivial so that $\mathfrak{H}$ is a normal A–subgroup scheme of $\mathfrak{G}$. According to [SGA3, VI_B.9.2.(iv)], it follows that $G/\mathfrak{H}$ carries a natural structure of A-group scheme.

(4) We follow the argument of [SGA3, VI_B.12.10.(3)]. Since G_0 is flat over A and étale over K, the fiberwise criterion of étaleness [EGA4, 17.8.2] boils down to establish that $G_{0,k}$ is étale over k. Let π be an uniformizing parameter of A. Let $x \in G_{0,k}$ and let U_x be an affine open neighborhood of x in G_0. Since G_0 is flat over A, it follows that $U_x \cap G_{0,K}$ is non empty so is $G_{0,K} = \mathrm{Spec}(K)$. We consider the flat A–algebra $B = H^0(U_x, \mathcal{O}_{U_x})$. We have $B \otimes_A K = K$ and $\pi^{-1} \not\in B$ (since π belong to the maximal ideal of B_x). It follows that $A = B$, so that the projection $U_x \to \mathrm{Spec}(A)$ is an isomorphism. We have proven that N is étale over A. Next G_0 is a normal A–subgroup of G by (3) and (2) shows that G/G_0 is representable by a separated A-group scheme of finite presentation. Finally G/G_0 is flat by (2).

(5) If $G_K = (G/G_0)_K$ is affine so is G/G_0 according to Raynaud's criterion (quoted above). Since $G_{0,k}$ is étale, it is affine so that the morphism $G_k \to G_k/G_{0,k} = (G/G_0)_k$ is affine [D-G, III.3.2.6]. Thus G_k is affine. □

Remark 3.4. Lemma 3.3 holds more generally in the case of algebraic spaces (Raynaud [R, prop. 3.3.5]). We have chosen to write a proof for schemes and not to deduce it from this stronger statement.

Lemma 3.5. *We assume that the A–group scheme G is flat, and of finite presentation. If G_K is commutative (resp. nilpotent, solvable). Then G_k is commutative (resp. nilpotent, solvable).*

Proof. If G_K is commutative, so is G_k according to Lemma 3.3.(5).

We assume now that G_K is nilpotent, that is, admits a central composition series $H_0 = 1 \subset H_1 \subset H_2 \subset \cdots \subset H_{n-1} \subset H_n = G_K$ where the H_i's are normal K–subgroups of G_K and such that each H_{i+1}/H_i is central in G_K/H_i. Let G_i be the schematic closure of H_i in G, this is a flat A-group scheme and all H_i's are normal A–subgroups of G according to Lemma 3.3.(3). Furthermore each quotient G_{i+1}/G_i is central in G/H_i. By extending the scalars to k we get then a central composition series for G_k.

The argument is similar for the solvable case. □

3.2. **The regular local ring case.** Let A be a regular local ring with fraction field K and residue field k.

Proposition 3.6. *Let G be a flat A-group scheme of finite presentation.*

(1) Assume that G_K contains an algebraic subgroup (resp. normal subgroup) of dimension d. Then G_k contains an algebraic subgroup (resp. normal subgroup) of dimension d.

(2) Assume that G_K contains an algebraic subgroup which is commutative (resp. nilpotent, solvable) of dimension d. Then G_k contains an algebraic subgroup which is commutative (resp. nilpotent, solvable) of dimension d.

(3) Assume that G is separated and that G_K contains an algebraic subgroup which is affine commutative (resp. unipotent, commutative unipotent, affine nilpotent, nilpotent trigonalizable, trigonalizable, affine solvable) of dimension d. Then G_k contains an algebraic subgroup which is affine commutative (resp. unipotent, commutative unipotent, affine nilpotent, nilpotent trigonalizable, trigonalizable, affine solvable) of dimension d.

Proof. The proof goes by induction on the dimension n of A. If A is of dimension zero, it is a field and the three assertions are obvious.

We follow a blowing-up argument arising from [EGA4, lemma 15.1.1.6]. We denote by X the blow-up of $\mathrm{Spec}(A)$ at his closed point, this is a regular scheme [L, §8.1, th. 1.19] and the exceptional divisor $E \subset X$ is a Cartier divisor isomomorphic to $\mathbb{P}^{n-1}_k$. We denote by $B = \mathcal{O}_{X,\eta}$ the local ring at the generic point η of E. The ring B is a DVR of fraction field K and of residue field $l = k(t_1, \dots, t_{n-1})$.

(1) Our assumption is that G_K contains an algebraic subgroup (resp. normal subgroup) H which is of dimension d. We consider the schematic closure of H_K in G_B (which is flat), this defines a flat B–group scheme (resp. normal B–subgroup scheme) according to Lemma 3.3.(3)) $\mathfrak{H}$ of closed fiber $\mathfrak{H}_l$ which is a subgroup of $(G_k)_l$. Applying the induction assumption to the regular local ring $R = k[t_1, \dots, t_{n-1}]_{t_1 \dots t_{n-1}}$ of fraction field $k(t_1, \dots, t_{n-1})$ and residue field k and to $(G_k) \times_k R$ we obtain that G_k contains an algebraic subgroup (resp. normal subgroup) of dimension d.

(2) The proof is similar by using Lemma 3.5.

(3) Our assumption is that G_K contains an algebraic subgroup H which is affine commutative (resp. unipotent, commutative unipotent, affine nilpotent, nilpotent trigonalizable, trigonalizable, affine solvable). We consider the schematic closure of H_K in G_B, this defines a flat B–group scheme $\mathfrak{H}$ of closed fiber $\mathfrak{H}_l$ which is a subgroup of $(G_k)_l$. Since G_B is separated so is $\mathfrak{H}$. Raynaud's affiness criterion [PY, prop 3.1] ensures that $\mathfrak{H}$ is affine over B. The same induction argument involing this times Lemmas 3.1 and 3.5 permits to conclude that that G_k carries a k–subgroup which is affine commutative (resp. unipotent, commutative unipotent, affine nilpotent, nilpotent trigonalizable, trigonalizable, affine solvable) of dimension d. □

Remark 3.7. Moret-Bailly suggested the following alternative proof of Proposition 3.6 by proceeding also by induction on the dimension. We observe first that it holds in dimension 1. Let t be a regular parameter of A and consider the local ring $B = A_t$ which is a DVR of residual field $K_1 = \mathrm{Frac}(A/tA)$. The one dimensional case provides the wished K_1–subgroup of $G \times_A K_1$ and by induction a similar k–subgroup of G_k since A/tA is a regular local ring of dimension $\dim(A) - 1$.

3.3. **More permanence properties.**

Lemma 3.8. *Let G be an algebraic group defined over a field k.*

(1) For each function D as above, we have $D(k, G) = D(k(t), G) = D(k((t)), G)$.

(2) If X is a connected smooth k–variety such that $X(k) \neq \emptyset$, we have $D(k, G) = D(k(X), G)$.

Proof. (1) We have the obvious inequalities $D(k, G) \leq D(k(t), G) \leq D(k((t)), G)$. The fact that $D(k((t)), G) \leq D(k, G)$ follows from Proposition 3.6 applied to $A = k[[t]]$ and the group scheme $G \times_k k[[t]]$.

(2) We have $D(k,G) \leq D(k(X),G)$. For proving the converse inequality, we pick a point $x \in X(k)$. Let $(t_1,\dots,t_d)$ be a system of parameters of the regular local ring $R = \mathcal{O}_{X,x}$. Then its completion is k–isomorphic to $k[[t_1,\dots,t_d]]$ which embeds in the field of iterated Laurent series $k((t_1))\dots((t_d))$. It follows that $k(X)$ embeds in $k((t_1))\dots((t_d))$ so that $D(k(X),G) \leq D(k((t_1))\dots((t_d)),G)$. By induction on d, (1) provides $D(k,G) = D(k((t_1))\dots((t_d)),G)$ so that $D(k(X),G) \leq D(k,G)$.

□

4. Upper semi-continuity

Theorem 4.1. *Let S be a scheme and let G be an S-group scheme of finite presentation. For each function $d_\bullet$ as defined in §2.2 we define $d_\bullet(s) = d_\bullet(G_{\kappa(s)})$ for each $s \in S$. The functions d'_c, d'_n, d'_s, $d_c, d_u, d_{cu}, d_n, d_s$ on S are upper semi-continuous.*

Proof. The problem is local so we can assume that $S = \mathrm{Spec}(A)$ for A an integral local ring of fraction field K and residue field k. We have to show that $d_\bullet(G_k) \geq d_\bullet(G_K) = d$ for one of the functions $d_\bullet$. By using the standard yoga of noetherian reduction [SGA3, VI_B.10.2], we can assume that A is furthermore noetherian. If A is a field, we have $K = k$ and this is obvious. We assume that A is not a field. According to [EGA2, prop. 7.1.7], there exists an extension L of K (of finite type) and equipped with a discrete valuation such that its valuation ring B dominates A, that is, $A \subset B$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$. We denote by l the residue field of B. Since $d_\bullet(G_k) = d_\bullet(G_l)$ and $d_\bullet(G_K) = d_\bullet(G_L)$, it is enough to show that $d_\bullet(G_l) \geq d_\bullet(G_L) = d$.

We can assume then that A is a DVR and the insensitivity of $d_\bullet$ by change of fields (Lemma 2.2) allows us to assume that A is complete and that its residue field k is algebraically closed.

Our assumption is that $G_{\overline{K}}$ contains a closed subgroup of dimension d which is commutative (resp. nilpotent, solvable, affine commutative, unipotent, commutative unipotent, affine nilpotent, affine solvable). Then there exists a finite K–subextension K' of $\overline{K}$ such that the same holds for $G_{K'}$. Let v' be an extension of the valuation v_K to K' and denote by A' its valuation ring and by k' its residue field. Once again we have $d_\bullet(G_k) = d_\bullet(G_{k'})$ and $d = d_\bullet(G_K) = d_\bullet(G_{K'})$. Up to replacing A by A' we can assume that G_K contains a closed subgroup H_K of dimension d commutative (resp. nilpotent, solvable, affine commutative, unipotent, commutative unipotent, affine nilpotent, affine solvable). Furthermore up to replacing G by the schematic closure of G_K we can assume that G is flat.

In the commutative (resp. nilpotent, solvable) case, Proposition 3.6.(1) and (2) shows that G_k contains a closed subgroup of dimension d which is commutative (resp. nilpotent, solvable). For $\bullet = c, n, s$, we get that $d'_\bullet(G_k) \geq d$.

For the commutative (resp. unipotent, commutative unipotent, affine nilpotent, affine solvable) case, we assume first that G is separated. Then the same argument as above with Proposition 3.6.(3) shows that $d_\bullet(G_k) \geq d$ for $\bullet = c, u, cu, n, s$.

It remains to deal with the non separated case. According to Lemma 3.3.(4) we have an exact sequence of flat A–group schemes

$$1 \to G_0 \to G \to H \to 1$$

where G_0 is étale, H is separated and $G_K \cong H_K$. The separated case shows that $d_\bullet(H_k) \geq d$ for $\bullet = c, u, cu, n, s$. Since $d_\bullet(H_k) = d_\bullet(G_k)$ according to Lemma 2.3, we conclude that $d_\bullet(H_k) \geq d$ for $\bullet = c, u, cu, n, s$. $\square$

5. Finite Groups

One motivation for considering this question comes from a problem about finite groups.

Let G be an (abstract) group acting on a set X. A base of G acting on X is a subset Y of X such any element of $g \in G$ which fixed Y pointwise acts trivially on X. The base size $b(G, X)$ is the minimal cardinality of a base. In the case of finite groups, this has been classical object of study for more than 150 years. This has had many applications (e.g. in computational group theory). One is also interested in this from a probabilistic point of view; what is the proportion of the subsets of size b which are a base.

Now suppose that K is an algebraically closed field. In [BGS], base size was considered for $G = G(K)$ a simple algebraic group over K acting on the homogeneous space G/M with M a maximal closed subgroup and in almost cases $b(G, M)$ was determined exactly (in a few cases, there was a small range of possible values). In this case one can consider two other quantities. We define $b^0(G, X) = c$ to be the smallest positive integer c so that there is subset Y of X of size c so that the pointwise stabilizer of Y is finite and $b^1(G, X) = e$ where e is the smallest positive integer such that the pointwise stabilizer of e generic points is trivial. It is easy to show that $b^0 \leq b \leq b^1 \leq b^0 + 1$.

Note that this can be rephrased in terms of G acting on $(G/M)^e$ and asking if there is a regular orbit or an orbit of $\dim G$ or if the generic orbit is regular.

More generally let G be an algebraic group acting on a quasiprojective irreducible variety X and assume that the action is defined over the finite field $\mathbb{F}_q$. We are interested the stabilizers in $G(\mathbb{F}_q)$ of a point $x \in X(\mathbb{F}_q)$ (and more generally we consider Steinberg-Lang endomorphisms with finite group of fixed points). Note that $X(\mathbb{F}_q)$ may not be a single orbit for $G(\mathbb{F}_q)$ if the stabilizer G_x is not connected.

As noted the stabilizer scheme $\{(g, x) | g \in G, x \in X, gx = x\}$ satisfies our hypotheses and so our results apply in this case. In particular, if for generic x, G_x contains a d-dimensional connected unipotent subgroup, then G_y does for all $y \in X$. Then Corollary 1.3 implies that if $y \in X(\mathbb{F}_q)$ and G_y has a smooth d-dimensional connected unipotent subgroup, then $G_y(\mathbb{F}_q)$ contains a subgroup of order q^d.

In particular, if the stabilizer of G for a generic point of $X = (G/M)^e$ contains a d-dimensional unipotent subgroup, it follows that for every point $x \in X(\mathbb{F}_q)$, $G_x(\mathbb{F}_q)$ has a subgroup of order q^d and so $b(G(\mathbb{F}_q), X(\mathbb{F}_q)) > e$. There are connections between the base size of the algebraic group and finite group of Lie type – see [BGS, GG] for some examples.

6. Derived Subgroups

6.1. For an algebraic group G defined over a field k, we define $d(G)$ (resp. $d^0(G)$) the dimension of the derived group of the smooth $\overline{k}$–group $G_{\overline{k},red}$ (resp. the smooth connected $\overline{k}$–group $G^0_{\overline{k},red}$). If G is smooth, we have $d(G) = \dim_k(DG)$ and $d^0(G) = \dim_k(D(G^0))$.

It is convenient to introduce a third dimension function $d^+(G)$ which is the supremum of the dimensions of the C_n's where C_n stands for the schematic image of the commutator map $c_n : G^{2n} \to G$, $c_n(x_1, y_1, \dots, x_n, y_n) = [x_1, y_1] \dots [x_n, y_n]$.

Since the formation of the schematic image commutes with flat base change, $d^+(G)$ is insensitive to an arbitrary field extension. We have $d^0(G) \leq d(G) \leq d^+(G)$ and $d(G) = d^+(G)$ for G smooth.

Proposition 6.1. *Let S be a scheme and let G be a flat S–group scheme of finite presentation. Then the function $s \mapsto d^+(G_{\kappa(s)})$ is lower semi-continuous.*

Proof. Using the same kind of argument as in the proof of Theorem 4.1 it is enough to deal with the case $S = \mathrm{Spec}(A)$ where A is a DVR with fraction field K and of residue field k. Let $n \geq 1$ be an integer and consider the commutator map $c_n : G^{2n} \to G$. Let $\mathfrak{C}_n \subset G$ be the schematic closure of $C_{n,K}$; it is flat over A so equidimensional of dimension d according to [EGA4, 12.1.1.5]. Since G_K^{2n} is dense in G^{2n}, it follows that c_n factorizes through $\mathfrak{C}_n$ so that $c_{n,k}$ factorizes through $(\mathfrak{C}_n)_k$. Therefore $C_{n,k} \subset (\mathfrak{C}_n)_k$ whence $d^+(G_k) \leq d^+(G_K)$. □

Corollary 6.2. *Let S be a scheme and let G be an S–group scheme of finite presentation.*

(1) Assume that G is smooth. Then the functions $s \mapsto d(G_{\kappa(s)})$ and $s \mapsto d^0(G_{\kappa(s)})$ are lower semi-continuous.

(2) Assume that S is irreducible with generic point ξ such that $G_{\kappa(\xi)}$ is smooth. Then $d(G_{\kappa(\xi)}) \geq d(G_{\kappa(s)})$ for each $s \in S$.

Proof. (1) In this case $d^+(G_{\kappa(s)}) = d(G_{\kappa(s)})$ so that Proposition 6.1 implies that d is lower semi-continuous. For the other function we consider the (smooth) S–group scheme G^0 defined in [SGA3, VI_B.3.10]. We have $d^0(G_{\kappa(s)}) = d(G^0_{\kappa(s)})$ so d^0 is upper semi-continuous.

(2) We have $d(G_{\kappa(\xi)}) = d^+(G_{\kappa(\xi)}) \geq d^+(G_{\kappa(s)}) \geq d(G_{\kappa(s)})$.

□

This function d may fail to be upper semi-continuous even in the case of a smooth affine group scheme over a DVR with connected fibers; in that case it would be locally constant according to Corollary 6.2.(1).

Lemma 6.3. *Let k be a an algebraically closed field and G be a split semisimple simply connected $k((t))$-group assumed almost simple of rank r. Let $\mathfrak{B}$ be Bruhat-Tits $k[[t]]$–group scheme attached to an Iwahori subgroup of $G(k((t)))$. Then we have*

$$d(\mathfrak{B}_k) \leq d(G) - r < d(G) = \dim_{k((t))}(\mathfrak{B}_{k((t))})).$$

Such a $\mathfrak{B}$ is smooth and has connected fibers according to [BT, prop. 4.6.32].

Proof. In this case $G_{k[[t]]}$ is a Bruhat-Tits group scheme attached to the maximal parahoric subgroup $G(k[[t]])$ of $G(k((t)))$. The Bruhat-Tits correspondence [BT, th. 4.6.35] is a bijection between the k–parabolic subgroups of G_k and the parahoric subgroups of $G(k((t)))$ included in $G(k[[t]])$. By taking a Borel subgroup B_k of G_k, we then get an Iwahori subgroup $\mathcal{B}$ of $G(k((t)))$ and a Bruhat-Tits group scheme $\mathfrak{B}$ such that B_k occurs as quotient of $\mathfrak{B}_k$. In particular $\mathfrak{B}_k$ maps onto a Borel k–subgroup of G_k so admits a commutative quotient of dimension r. It follows that $d(\mathfrak{B}_k) \leq \dim(\mathfrak{B}_k) - r = \dim(G) - r = d(G) - r < d(G)$. We have proven that for one specific Iwahori subgroup but this is enough by conjugacy. □

We conclude by giving an example of a stabilizer scheme where the generic stabilizer is simple of dimension 3 but some stabilizer is abelian (also of dimension 3).

Let $G = \mathrm{Sp}_4(k) = \mathrm{Sp}(V)$ for k any algebraically closed field. Let $X = V \oplus V$. If $x = (v_1, v_2)$ is a generic point, then the stabilizer of x is the subgroup acting trivially on the nondegenerate 2-space spanned by v_1 and v_2 and so $G_x \cong \mathrm{Sp}_2(k)$. If $y = (w_1, w_2)$ with w_1 and w_2 spanning a totally singular 2-space, then G_y is the unipotent radical of the parabolic subgroup stabilizing the space spanned by w_1 and w_2. In particular, for a generic point x, G_x is nonsolvable while G_y is abelian.

6.2. **Abelianization.** A variant is the following. For an algebraic group G defined over a field k, we define $d_{ab}(G)$ (resp. $d^0_{ab}(G)$) the dimension of the abelianization of the smooth $\overline{k}$–group $G_{\overline{k},red}$ (resp. the smooth connected $\overline{k}$–group $G^0_{\overline{k},red}$). We have $d(G) + d_{ab}(G) = \dim_k(G)$ and similarly $d^0_{ab}(G) + d^0_{ab}(G) = \dim_k(G)$.

Corollary 6.4. *Let S be a scheme and let G be an S–group scheme of finite presentation.*

(1) Assume that G is smooth. Then the functions $s \mapsto d_{ab}(G_{\kappa(s)})$ and $s \mapsto d^0_{ab}(G_{\kappa(s)})$ are upper semi-continuous.

(2) Assume that S is irreducible with generic point ξ such that $G_{\kappa(\xi)}$ is smooth. Then $d_{ab}(G_{\kappa(\xi)}) \leq d_{ab}(G_{\kappa(s)})$ for each $s \in S$.

Proof. (1) Since the dimension is a locally constant function, the statement follows from Corollary 6.2.(1).

(2) Corollary 6.2.(2) states that $d(G_{\kappa(\xi)}) \geq d(G_{\kappa(s)})$ so that $-d(G_{\kappa(\xi)}) \leq -d(G_{\kappa(s)})$. On the other hand we have $\dim(G_{\kappa(\xi)}) \leq \dim(G_{\kappa(s)})$ according to [SGA3, VI_B.4.1]. By summing the inequalities we obtain $d_{ab}(G_{\kappa(\xi)}) \leq d_{ab}(G_{\kappa(s)})$ as desired. □

7. Appendix: Alternate proof of Theorem 1.1

We present an alternate proof of the upper semi-continuity of d_u. It involves the following preliminary fact.

Lemma 7.1. *Let F be a field. Let M be an affine smooth connected F–group.*

(1) The following assertions are equivalent:

(i) M is unipotent;

(i') $M_{\overline{F}}$ is unipotent;

(ii) All F–tori of M are trivial;

(ii') All $\overline{F}$–tori of M are trivial;

(2) Let $f : G \to H$ be an isogeny between affine smooth connected F–groups. Then G is unipotent (resp. solvable, nilpotent, commutative) if and only if H is unipotent (resp. solvable, nilpotent, commutative).

Proof. (1) Let $\rho : M \to \mathrm{GL}_n$ be faithful linear representation. The equivalence $(i) \Longleftrightarrow (i')$ is by definition since in both cases it says that $\rho(G(\overline{F}))$ consists of unipotent elements.

$(i') \Longrightarrow (ii')$. Let $\mathbb{G}^r_{m,\overline{F}} \hookrightarrow M_{\overline{F}}$ be a $\overline{F}$–subtorus. If $r \geq 1$, we pick an element $t \neq 1 \in (\overline{F})^r$. We have that $\rho(t) \in GL_n(\overline{k})$ is unipotent and semisimple so is 1 contradicting $t \neq 1$. We conclude that $r = 1$.

$(ii') \Longrightarrow (ii)$. This is obvious.

$(ii) \Longrightarrow (ii')$. According to the conjugacy theorem [B, 11.3], all maximal $\overline{F}$–subtori of $G_{\overline{F}}$ are conjugate. According to a result of Grothendieck [B, 18.2(i)], there is a maximal $\overline{F}$-torus T in $M_{\overline{F}}$ that is defined over F. Our assumption implies that $T = 1$. According to the conjugacy theorem for maximal $\overline{F}$-subtori of $G_{\overline{F}}$, [B, 11.3] we conclude that all $\overline{F}$–tori of M are trivial.

(2) The direct implication is obvious. Conversely we assume that H is unipotent (resp. solvable, nilpotent, commutative). We can assume F is algebraically closed (this is (1) in the unipotent case and holds by definition for the other cases).

Unipotent case. It suffices to show that some maximal torus is trivial. For a maximal torus T in G its image $f(T)$ is a maximal torus in H by [B, Prop. 11.14.(1)]. Since $\ker(f)$ is finite we have $f(T) = 1$ if and only if $T = 1$. The proof is completed.

Solvable case. Let B a Borel subgroup of G. The quoted citation shows that $f(B)$ is a Borel subgroup of H, so $H = f(B)$. It follows that B has same dimension than G. Thus $G = B$ is solvable.

Nilpotent case. By the preceding case, G is solvable. Let T be a maximal torus of G. We have seen that $f(T)$ is a maximal torus of H. Since H is nilpotent, $f(T)$ is central in H [B, prop. 12.5] so that $f([T, G]) = 1$ whence $[T, G] \subset \ker(f)$. Since $[T, G]$ is connected, we conclude that $[T, G] = 1$. The torus T is central in G so that G is a Cartan subgroup of G. According [B, §12.6, theorem], G is nilpotent.

Commutative case. The argument is similar and simpler. We have $f([G, G]) = 1$ so that $[G, G] = 1$.

□

We come now to the proof of Theorem 1.1. The beginning is same as that of the proof of Theorem 4.1, that is, a reduction to the case of the spectrum of a DVR A with fraction field K and residue field k. We are given an A–group scheme G of finite presentation and want to show the inequality $d_u(G_k) \geq d_u(G_K)$. The reduction to the separated case is the same that as the end of the proof of Theorem 4.1 so we can assume that G is separated. We put $d = d_u(G_K)$. Using the insensitivity of d_u by change of fields (Lemma 2.2) we can assume that A is complete.

Also we are allowed to make arbitrary finite extensions of K so that we can assume that G_K admits a smooth connected unipotent K–group U of dimension d. Letting $\mathfrak{H}$ be the schematic closure of U in G, it is separated flat over A and of finite presentation. Raynaud's affiness criterion (see §3.1) shows that $\mathfrak{H}$ is affine. Replacing G by $\mathfrak{H}$ we can then assume that the A–group scheme G is affine flat, and that G_K is smooth unipotent connected.

We consider the case when G is smooth. This enables to deal with the neutral component G^0 of G [SGA3, VI$_B$.3.10] which is a smooth open A–subgroup G such that G^0_K (resp. G^0_k) is the neutral component of the algebraic group G_K (resp. G_k). Also the A–group scheme G^0 is affine according to Raynaud's criterion.

Lemma 7.1.(1) states that G_k^0 is unipotent if and only if all tori of G_k^0 are trivial. Let T_0 be a maximal torus of G_k^0. Since A is henselian, T_0 lifts to a subtorus T of G^0 by using Grothendieck's representability theorem [SGA3, XI.4.1]. Since G_K is unipotent, T_K is trivial so that $T = 1$ and $T_0 = 1$. It follows that G_k^0 is unipotent so that $d_u(G) = d$.

We consider now the general case. According to [PY, prop. 3.4] (based on [A, app. II]), there exists a local extension A' of A of DVR's such that the normalization $\widetilde{G}'$ of $G' = G \times_A A'$ is smooth over A' and such that the fraction field K' is finite over K. We denote by k' the residue field of A'. According to [PY, thm. A.6] the normalization morphism $h : \widetilde{G}' \to G'$ is finite. In particular the morphism of smooth affine connected K'–groups $h^0_{K'} : \bigl((\widetilde{G}')_{K'}\bigr)^0 \to G_{K'}$ is an isogeny between smooth affine connected K'–groups. Since $G_{K'}$ is unipotent, Lemma 7.1.(2) shows that $(\widetilde{G}')^0_{K'}$ is unipotent.

The smooth case applied to $(\widetilde{G}')^0$ over A' shows that $(\widetilde{G}')^0_{k'}$ is unipotent of dimension d. Since the homomorphism $h^0_k : (\widetilde{G}')^0_{k'} \to (G'_{k'})^0 \cong (G_k)^0 \times_k k'$ is finite, Lemma 7.1.(2) shows that the k'–subgroup $(G'_{k'})^0/\ker(h^0_{k'})$ of $G_{k'}$ is unipotent of dimension d. Thus $d_u(G_k) \geq d$.

References

[A] S. Anantharaman, *Schémas en groupes, espaces homogènes et espaces algébriques sur une base de dimension 1*, Bull. Soc. Math. France, Mém. **3**3 (1973). 3.1, 7

[B] A. Borel, *Linear Algebraic Groups*, Second edition. Graduate Texts in Mathematics, 126. Springer–Verlag, New York, 1991. 1, 1, 2, 2.2, 7

[BT] F. Bruhat et J. Tits, *Groupes réductifs sur un corps local II*, Publ. Math. IHES **6**0 (1984). 3.1, 6.1

[BGS] T. Burness, R. Guralnick and J. Saxl, *On base sizes for algebraic groups*, J. Eur. Math. Soc. (JEMS) **1**9 (2017), no. 8, 2269–2341. 1, 5

[D-G] M. Demazure, P. Gabriel, *Groupes algébriques*, North-Holland (1970). 1, 2, 2.2, 3.1, 3.1

[GG] S. Garibaldi and R. Guralnick, *Generic stabilizers for simple algebraic groups*, Michigan Math. J. **7**2 (2022), 343-387. 1, 5

[EGA2] A. Grothendieck (avec la collaboration de J. Dieudonné), *Eléments de Géométrie Algébrique II*, Publications mathématiques de l'I.H.É.S. no 8 (1961). 4

[EGA4] A. Grothendieck (avec la collaboration de J. Dieudonné), *Eléments de Géométrie Algébrique IV*, Publications mathématiques de l'I.H.É.S. no 20, 24, 28 and 32 (1964 - 1967). 3.1, 3.2, 6.1

[GL] R. Guralnick, R. Lawther, Generic stabilizers in actions of simple algebraic groups, Mem. Amer. Math. Soc., to appear. 1

[GW] U. Görtz and T. Wedhorn, Algebraic Geometry I, second edition, Springer Fachmedien Wiesbaden, 2020. 1, 3.1

[L] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics **6** (2002), Oxford University Press, Oxford. 3.2

[P] V.P. Platonov, *Theory of algebraic linear groups and periodic groups* (in Russian), Izv. Akad. Nauk SSSR Ser. Mat. **3**0 (1966), 573-620; English translation, AMS Translations **6**9 (1968), 61-110. 2.3

[PV] V. L. Popov and E. B. Vinberg, Invariant theory, Encyclopedia of Mathematical Sciences, vol. 55, pp. 123–284, Springer-Verlag, 1994. 1

[PY] G. Prasad, J.-Y. Yu, *On quasi-reductive group schemes*, with an appendix by Brian Conrad. J. Algebraic Geom. **1**5 (2006), 507-549. 3.1, 3.2, 7

[R] M. Raynaud, *Spécialisation du foncteur de Picard*, Inst. Hautes Études Sci. Publ. Math. **3**8 (1970), 27-76. 3.4

[SGA3] *Séminaire de Géométrie algébrique de l'I. H. E. S., 1963-1964, schémas en groupes, dirigé par M. Demazure et A. Grothendieck*, Lecture Notes in Math. 151-153. Springer (1970). 1, 1, 2.1, 2.2, 2.3, 3.1, 3.1, 3.2, 3.1, 4, 6.1, 6.2, 7

[St] Stacks project, https://stacks.math.columbia.edu 3.1

[VD] B. Ju Veĭsfeĭler and I. V. Dolgachev, *Unipotent group schemes over integral group rings*, Izv. Akad. Nauk SSSR Ser. Mat. **3**8 (1974), 757-799. 3.2

UMR 5208 Institut Camille Jordan - Université Claude Bernard Lyon 1 43 boulevard du 11 novembre 1918 69622 Villeurbanne cedex - France

Email address: gille@math.univ-lyon1.fr

Department of Mathematics, University of Southern California, Los Angeles, CA 90089-2532, USA

Email address: guralnic@usc.edu