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Yves Achdou, Claude Le Bris

HOMOGENIZATION OF SOME PERIODIC HAMILTON-JACOBI EQUATIONS WITH DEFECTS

YVES ACHDOU* AND CLAUDE LE BRIS†

Abstract. We study homogenization for a class of stationnary Hamilton-Jacobi equations in which the Hamiltonian is obtained by perturbing near the origin an otherwise periodic Hamiltonian. We prove that the limiting problem consists of a Hamilton-Jacobi equation outside the origin, with the same effective Hamiltonian as in periodic homogenization, supplemented at the origin with an effective Dirichlet condition that keeps track of the perturbation. Various comments and extensions are discussed.

1. Introduction. This work discusses the homogenization limit for a first order stationary Hamilton-Jacobi equation of the form

$$\alpha\, u_\varepsilon + H\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) = 0 \quad \text{in } \mathbb{R}^d,$$

and some related equations, where the Hamiltonian H, besides satisfying some classical assumptions, is obtained by a *local*, compactly supported, perturbation of a periodic Hamiltonian H_{per}. We henceforth refer to such a perturbation as a local *defect*.

Homogenization theory in the presence of local defects within an otherwise periodic environment was first introduced in [10], in the first of a series of works by the second author, in collaboration with X. Blanc and P-L. Lions. It was further developed in [11, 12] and other subsequent works by various authors, considering different classes of defects such as, in particular, interfaces between two different periodic media. In those works, the typical setting is that of a linear non-degenerate elliptic equation, first in divergence form and next in more general form. Only recently, some quasilinear equation was considered in [30].

On the other hand, extensions to the context of nonlinear Hamilton-Jacobi equations arising from optimal control theory were first addressed in a series of lectures [24] at Collège de France by P-L. Lions, where some results obtained in collaboration with P. Souganidis, [26], were described. Since the present contribution is very much related to [24, 26], we will devote a special paragraph below to commenting our results in view of the latter references and to stressing the similarities and differences. In [15], P. Cardaliaguet, P. Souganidis and the second author considered the extension to a fully nonlinear equation of a specific *probabilistic* setting originally introduced, in the linear elliptic setting and from a computational perspective, in [5]. In that setting, the defect is assumed to occur only with a small probability, and the homogenized limit is identified at the leading order in probability. More generally, an introductory account of the entire mathematical endeavor together with a large variety of possible applications and extensions, has appeared in [23].

In a distinct line of works, the first author, in collaboration with N. Tchou in particular, studied the asymptotic behaviour of the solutions to some classes of Hamilton-Jacobi equations depending on a vanishing parameter, for which the limit is characterized by boundary value problems with transmission conditions involving effective

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Hamiltonians. In particular, [3] dealt with dimension reduction for Hamilton-Jacobi equations posed in thin domains converging to networks. When the thickness of the domain vanishes, one finds at the limit a Hamilton-Jacobi equation posed on the network with a special transmission condition at the vertices. The latter involves a so-called flux limiter which keeps track of the microscopic geometry of the thin domains. Subsequently, the works [2] and [4] were concerned with Hamilton-Jacobi equations in an environment consisting of two different homogeneous media separated by an oscillatory interface. The oscillations of the interface have small period and amplitude, and as both the latter parameters vanish, the interface tends to an hyperplane. At the limit, one finds on the flat interface an effective nonlinear transmission condition keeping memory of the previously mentioned microscopic oscillations. Similarly, G. Galise, C. Imbert and R. Monneau studied in [19] a family of time dependent Hamilton-Jacobi equations on the simplest possible network composed of two half-lines with a perturbation of the Hamiltonian localized in a small region close to the junction. Related homogenization problems with applications to traffic flows were discussed in [18, 16] by N. Forcadel and his coauthors. Key arguments in [3, 2, 4, 19] are the construction of families of correctors that account for the localized pertubations of the environment and are defined in unbounded domains. We will see that the construction of such correctors also plays a key role in the present work. Note also that another common feature between the references [3, 2, 4, 19] is that at the limit, the new effective transmission conditions are all posed on manifolds of codimension one. The study of such homogenization problems was indeed part of the significant research effort that took place in the last decade on the analysis of Hamilton-Jacobi equations posed on heterogeneous structures such as networks [20, 1, 27], booklet-like geometries or multidimensional junctions [8, 21, 28]. These problems all involve Hamilton-Jacobi equations on manifolds with nonlinear transmission conditions on submanifolds with codimension one. In sharp contrast, in the present work, the problem arising at the limit involves a boundary condition posed at the origin, which is generally not a manifold of codimension one. This causes technical difficulties and a priori prevents one from relying on the theories contained in [20, 1, 27]. On the contrary, the problem arising at the limit falls into the class of *stratified* problems (and is a particularly simple example of such problems) introduced by A. Bressan and coauthors, see [13] and later studied by G. Barles and E. Chasseigne, see [9]. By and large, these problems involve Hamilton-Jacobi equations with discontinuities in a more general geometric setting, namely *stratifications* of $\mathbb{R}^d$. The latter references contain in particular comparison principles for viscosity solutions of stratified problems, but, up to our knowledge, there is no literature yet on homogenization or singular perturbations leading to stratified problems, and in particular no proofs of convergence.

Finally, in connection with [2, 4], the present contribution can be seen as a first step toward homogenization theory for Hamilton-Jacobi equations with two different possibly periodic media separated by an interface presenting localized defects of periodicity. We plan to address this aspect in a forecoming work.

As already mentioned, the present study (essentially) considers a single localized defect, supported say in a neighborhood of the origin, within a periodic medium. It identifies the homogenized limit in the case of a *first order* Hamilton-Jacobi equation arising from optimal control, thus with a convex Hamiltonian; we will assume that the Hamiltonian has agreeable mathematical properties, which will be made precise in the next section. It is natural to expect that at the limit, except possibly at the

origin, the Hamilton-Jacobi equation involving the effective Hamiltonian obtained from periodic homogenization is satisfied. We will see that if the defect has the effect of attracting the optimal trajectories starting not too far from the origin, (which occurs when the running cost or potential of the optimal control problem displays a prominent *downward* bump within the periodic environment), then the homogeneity of the environment is broken in the homogenized limit. We will indeed prove that, in this case, the limit problem involves a Dirichlet boundary condition at the origin. Besides, it is well known that, in singular domains such as $\mathbb{R}^d \backslash \{0\}$, uniqueness does not hold for the now classical formulation of Dirichlet problems due to H. Ishii, see e.g. [6, chapter V, section 4]. By contrast, uniqueness holds for the formulation of the Dirichlet problem that we find here from homogenization. We stress again that because of the effective Dirichlet condition, the limit is *generally not* an Hamilton-Jacobi equation on the whole ambient space.

We wish to emphasize that the results exposed here are to be considered in the light of those obtained earlier by P-L. Lions and P. Souganidis and summarized in [24] (and, likewise, of [15] for randomized defects). In the control theoretic interpretation of that work, the typical local perturbation of the running cost is a bump oriented *upwards* so that the neighborhood of the origin becomes repulsive. In [24], the presence of such a defect indeed does not affect the homogenized limit, but only possibly *"the next order correction"*, that is the definition of the corrector function itself. In sharp contrast, if the defect makes the origin attractive, then it affects the homogenized limit itself. Note that it is not unexpected that signs, in the wide acceptation of that term, play a critical role in the context of nonlinear equations.

In the authors' view, the present work, together with the results of [24], lay the groundwork for a complete theory for Hamilton-Jacobi equations in periodic environments with local defects. Many challenging questions, some of them presumably quite difficult, however remain unsolved.

Our article is more precisely organized as follows. In the rest of this introduction, we first make precise, in Section 1.1, the mathematical setting and assumptions of the problem we study. The Hamilton-Jacobi equation under consideration reads as (1.7) below. We next state and comment in Section 1.2 our main result, namely Theorem 1.1 which establishes for (1.7) the homogenized limit (1.11) through (1.14) made precise therein. Section 2 then contains the detailed proof of Theorem 1.1. The proof essentially falls in four steps, respectively contained in Sections 2.1 through 2.4. The final Section 3 contains several comments upon and extensions of the problem considered. First, in Section 3.1, we illustrate in the one-dimensional setting the general results of Section 1.2. We in particular consider, in Section 3.1.4, a probabilized variant of our problem inspired by some of the earlier works recalled above. Section 3.2 discusses the extension to higher dimensions of the considerations and results given in Section 3.1.

1.1. Setting and assumptions. Let us define the problem and give the assumptions that will be used in the whole paper.

We consider Hamilton-Jacobi equations linked to infinite horizon optimal control problems in $\mathbb{R}^d$. The Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is of the form

$$(1.1) \qquad H(x, p) = \max_{a \in A} \Big(-p \cdot f(x, a) - \ell(x, a) \Big).$$

Here, A is a compact metric space, $f : \mathbb{R}^d \times A \to \mathbb{R}^d$ is a bounded and continuous function, Lipschitz continuous with respect to its first variable uniformly with respect to its second variable, i.e. for any $x, y \in \mathbb{R}^d$ and $a \in A$,

$$|f(x, a) - f(y, a)| \leqslant L_f |x - y|,$$

for some positive constant L_f. We also suppose that there exists some radius $r_f > 0$ such that for any $x \in \mathbb{R}^d$, $\{f(x, a), a \in A\}$ contains the ball $B_{r_f}(0)$, which implies that the trajectories are locally strongly controllable.

We suppose that the function $\ell : \mathbb{R}^d \times A \to \mathbb{R}$ is bounded and continuous and that here is a modulus of continuity ω_ℓ such that for any $x, y \in \mathbb{R}^d$ and $a \in A$,

$$|\ell(x, a) - \ell(y, a)| \leqslant \omega_\ell(|x - y|).$$

Define $M_f = \sup_{x \in \mathbb{R}^d, a \in A} |f(x, a)|$ and $M_\ell = \sup_{x \in \mathbb{R}^d, a \in A} |\ell(x, a)|$. It is easy to check that the Hamiltonian H defined in (1.1) has the following properties: H is convex with respect to its second argument, and for any $x, y, p, q \in \mathbb{R}^d$,

$$(1.2) \qquad\qquad H(x, p) \geqslant r_f |p| - M_\ell,$$
$$(1.3) \qquad |H(x, p) - H(y, p)| \leqslant L_f |p| |x - y| + \omega_\ell(|x - y|),$$
$$(1.4) \qquad |H(x, p) - H(x, q)| \leqslant M_f |p - q|.$$

Property (1.2) implies the coercivity of H w.r.t. its second variable uniformly in its first variable, i.e. $\lim_{|p| \to \infty} \inf_{x \in \mathbb{R}^d} H(x, p) = +\infty$.

We next assume that, except in a neighborhood of the origin, the dynamics f and cost ℓ coincide with periodic functions. Let $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ denote the torus of $\mathbb{R}^d$. We assume that there exist $R_0 > 0$, $f_{\mathrm{per}} : \mathbb{T}^d \times A \to \mathbb{R}^d$ and $\ell_{\mathrm{per}} : \mathbb{T}^d \times A \to \mathbb{R}$ such that, if $|x| > R_0$, then $f(x, a) = f_{\mathrm{per}}(x, a)$ and $\ell(x, a) = \ell_{\mathrm{per}}(x, a)$. Hence, the suprema M_f and M_ℓ introduced above are indeed maxima. Let H_{per} stand for the related periodic Hamiltonian:

$$(1.5) \qquad H_{\mathrm{per}}(x, p) = \max_{a \in A} \Big(-p \cdot f_{\mathrm{per}}(x, a) - \ell_{\mathrm{per}}(x, a) \Big).$$

Let α be a positive scalar and ε be a small positive parameter that will eventually vanish. It is well known, see e.g. [6], that the value function u_ε of the following optimal control problem:

(1.6)

$$u_\varepsilon(x) = \inf \int_0^\infty e^{-\alpha t} \ell \left(\frac{y(t)}{\varepsilon}, a(t) \right) dt \quad \text{subject to} \quad \begin{cases} y(t) = x + \int_0^t f\left(\frac{y(\tau)}{\varepsilon}, a(\tau) \right) d\tau, \\ a \in L^\infty(\mathbb{R}_+) \\ a(t) \in A, \text{ a.e.} \end{cases}$$

is the unique viscosity solution in $\mathrm{BUC}(\mathbb{R}^d)$ of

$$(1.7) \qquad \alpha\, u_\varepsilon + H\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) = 0 \quad \text{in } \mathbb{R}^d.$$

Our goal is to study the asymptotic behaviour of u_ε as $\varepsilon \to 0$.

Homogenization of the periodic Hamilton-Jacobi equation involving H_{per} instead of H is well understood since the pioneering work [25]. In the periodic case, the homogenized equation is

$$(1.8) \qquad \alpha\, u + \overline{H}(Du) = 0 \quad \text{in } \mathbb{R}^d,$$

where the effective Hamiltonian is characterized as follows: for any $p \in \mathbb{R}^d$, $\overline{H}(p)$ is the unique real number such that there exists a periodic corrector $\chi_{\mathrm{per},p}$, i.e. a viscosity solution $\chi_{\mathrm{per},p} \in C(\mathbb{T}^d)$ of

$$(1.9) \qquad H_{\mathrm{per}}(y, p + D\chi_{\mathrm{per},p}) = \overline{H}(p) \quad \text{in } \mathbb{R}^d.$$

In general, the latter periodic corrector is not unique even up to the addition of a scalar constant. It is well known that, under the assumptions made above, $\overline{H}$ is convex on $\mathbb{R}^d$, Lipschitz continuous and coercive. Therefore, there exists a vector $p_0 \in \mathbb{R}^d$, possibly non unique, such that

$$(1.10) \qquad \overline{H}(p_0) = \min_{q \in \mathbb{R}^d} \overline{H}(q).$$

We choose such a vector p_0 and fix it for all what follows.

1.2. The main result. Our main result is the following:

THEOREM 1.1. *We consider the solution u_ε of (1.7). As $\varepsilon \to 0$, the family u_ε converges locally uniformly to the unique function $u \in \mathrm{BUC}(\mathbb{R}^d)$ defined by the following properties:*

1. *u is a viscosity solution of*

$$(1.11) \qquad \alpha\, u + \overline{H}(Du) = 0 \quad \text{in } \mathbb{R}^d \backslash \{0\},$$

   with the effective Hamiltonian $\overline{H}$ defined above by periodic homogenization, see (1.9)

2. *The condition*

$$(1.12) \qquad \alpha\, u(0) + E \leqslant 0$$

   holds, where E is the effective Dirichlet datum defined in paragraph 2.1 below. In addition, if $\phi \in C^1(\mathbb{R}^d)$ is such that $u - \phi$ has a local maximum at the origin, then

$$(1.13) \qquad \alpha\, u(0) + \overline{H}(D\phi(0)) \leqslant 0.$$

3. *If $\phi \in C^1(\mathbb{R}^d)$ is such that $u - \phi$ has a local minimum at the origin, then*

$$(1.14) \qquad \alpha\, u(0) + \max\left(E, \overline{H}(D\phi(0))\right) \geqslant 0.$$

REMARK 1.1. *The conditions (1.11) through (1.14) may be seen as a weak formulation of a Dirichlet boundary value problem comprising the Hamilton-Jacobi equation (1.11) in the singular open set $\mathbb{R}^d \backslash \{0\}$ and the Dirichlet boundary condition $u(0) =$*

$-E/\alpha$. *Note that this formulation is stronger than that due to H. Ishii, which expresses that at the origin, u is both a viscosity subsolution of $\alpha\,u(0) + \min(\overline{H}(Du(0)), E) \leqslant 0$ and a viscosity supersolution of $\alpha\,u(0) + \max(\overline{H}(Du(0)), E) \geqslant 0$, see for example [6, Chapter V, Section 4], [7] and [22]. Indeed, while the supersolution condition* (1.14) *coincides with that of Ishii, the subsolution condition* (1.12)-(1.13) *is stronger.*

A consequence of the difference between the two formulations regards uniqueness. Reference [6, Chapter V, Section 4] contains a counterexample for the uniqueness of viscosity solutions in the sense of Ishii of Dirichlet problems posed in $\mathbb{R}^d\backslash\{0\}$. Note also that an interior cone condition is sufficient for uniqueness, see (4.30) in the latter reference, but this condition is not satisfied by $\mathbb{R}^d\backslash\{0\}$. By contrast, and as we will see in the proof of Theorem 1.1 *in Section* 2.4 *below, uniqueness holds for* (1.11), (1.12), (1.13), (1.14).

REMARK 1.2. *The conditions* (1.11) *through* (1.14) *fall into the general notion of stratified solutions of Hamilton-Jacobi equations, see [13, 9], and correspond to the partition of $\mathbb{R}^d$ into the sets $\mathbb{R}^d\backslash\{0\}$ and $\{0\}$. The final step in the proof of Theorem* 1.1 *needs a comparison result which can be found in [9] in a much more general setting. In order to keep the paper self-contained, we will give a short proof of the comparison principle, see Section* 2.4*, because it is quite simple in the particular case under consideration.*

REMARK 1.3. *With the same assumptions on H, Theorem* 1.1 *may be easily generalized to the homogenization of Hamilton-Jacobi equations of the form*

$$\alpha\,u_\varepsilon + H\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) = b(x) \quad in\ \mathbb{R}^d,$$

where $b \in \mathrm{BUC}(\mathbb{R}^d)$. The effective Dirichlet datum E would then depend on b only through its value at the origin.

Similarly, the result can be easily extended to problems of the form

$$\alpha\,u_\varepsilon + H_\varepsilon(x, Du_\varepsilon) = 0 \quad in\ \mathbb{R}^d,$$

where $H_\varepsilon(x, p) = \max_{a\in A}\left(-p \cdot f_\varepsilon(x, a) - \ell_\varepsilon(x, a)\right)$ and

$$\ell_\varepsilon(x, a) = \ell_1(x, \frac{x}{\varepsilon}, a) + \ell_0(\frac{x}{\varepsilon}, a),$$
$$f_\varepsilon(x, a) = f_1(x, \frac{x}{\varepsilon}, a) + f_0(\frac{x}{\varepsilon}, a),$$

the functions ℓ_0, f_0 are as above (they model the perturbation located near the origin) and ℓ_1 and f_1 are smooth with respect to their first argument and periodic with respect to the second one.

Other generalizations to different settings may be considered. While those to evolutive problems seem fairly easy, extensions to viscous Hamilton-Jacobi equations seem more challenging.

REMARK 1.4. *As mentioned in the introduction, the homogenization of periodic Hamilton-Jacobi equations in the presence of a defect has been first addressed by P-L. Lions and P. Souganidis, see the results announced in [24, 26]. The assumptions made therein imply that $E \leqslant \min_{p\in\mathbb{R}^d} \overline{H}(p)$ and that the defect does not show up in the homogenized limit, i.e. u is a viscosity solution of $\alpha\,u + \overline{H}(Du) = 0$ in $\mathbb{R}^d$. We therefore recover as a particular case of our setting the homogenized limit obtained in [24, 26], under the particular assumptions made there.*

For example, in the case when $p_0 = 0$ in (1.10), we find that u is then identically equal to $-\frac{1}{\alpha}\overline{H}(0) = -\frac{1}{\alpha}\min_p \overline{H}(p)$.

On the contrary, if

$$(1.15) \qquad\qquad E > \overline{H}(0) = \min_{p \in \mathbb{R}^d} \overline{H}(p),$$

then the defect is visible in the homogenized limit. Indeed, it can first be proven (see § 3.2.1 below) that

$$(1.16) \qquad\qquad \lim_{|x| \to \infty} u(x) = -\frac{1}{\alpha}\overline{H}(0).$$

Combining (1.12), (1.15) and (1.16), we then see that u is not a constant function, so it differs from the homogenization limit in the absence of defect. We also claim that in this particular case, the global infimum of u is reached at the origin and only at the origin. Indeed, (1.12), (1.15) and (1.16) altogether imply that there exists a minimizer x_0 of u. If $x_0 \neq 0$, then (1.11) yields that $\alpha\, u(x_0) \geqslant -\overline{H}(0)$, thus $\alpha\, u(x_0) > -E \geqslant \alpha\, u(0)$, and we reach a contradiction.

Note also that, in [24], thanks to the assumption made, the authors are able to construct correctors χ_p in the whole space $\mathbb{R}^d$ for all vectors $p \in \mathbb{R}^d$ (p stands for the gradient of a smooth test-function at the origin). Their construction uses arguments from the theory of optimal control. These correctors, which differ from those used in periodic homogenization, can be used for proving the convergence to the homogenized problem by means of the now classical method of perturbed test-functions, see [17].

In contrast, when $E > \min_{p \in \mathbb{R}^d} \overline{H}(p)$, we will have to adjust the strategy proposed in [24] in order to keep relying on the theory of optimal control : using arguments somewhat reminiscent of those proposed in [3], we will construct correctors associated to suitable piecewise affine functions rather than smooth or linear functions.

2. Homogenization of (1.7): the proof of Theorem 1.1. Recall that our goal is to understand the asymptotic behavior of u_ε as ε tends to 0. First, using either comparison principles, see for example [6, Chapter II, Theorem 3.5] or arguments from the theory of optimal control, we see that

$$-\max_{y \in \mathbb{R}^d} H(y, 0) \leqslant \alpha\, u_\varepsilon(x) \leqslant -\min_{y \in \mathbb{R}^d} H(y, 0).$$

From this estimate and (1.7), we infer from the coercivity of the Hamiltonian that u_ε is Lipschitz continuous in $\mathbb{R}^d$ with a Lipschitz constant independent of ε.

In order to study the asymptotic behaviour of u_ε, we consider

$$(2.1) \qquad\qquad \overline{u}(x) = \limsup_{\varepsilon \to 0} u_\varepsilon(x),$$

$$(2.2) \qquad\qquad \underline{u}(x) = \liminf_{\varepsilon \to 0} u_\varepsilon(x).$$

Note that, from the observation above on the regularity of u_ε, $\overline{u}$ and $\underline{u}$ coincide respectively with the half-relaxed semi-limits $\limsup_{x' \to x, \varepsilon \to 0} u_\varepsilon(x')$ and $\liminf_{x' \to x, \varepsilon \to 0} u_\varepsilon(x')$, that are classically used in the homogenization of Hamilton-Jacobi equations. It is clear that the functions $\overline{u}$ and $\underline{u}$ are bounded and Lipschitz continuous.

We now observe that, in $\mathbb{R}^d \backslash \{0\}$, the homogenized equation does not keep track of the defect of periodicity and is identical to (1.8). This is not surprising, since the support of the defect shrinks to $\{0\} \times A$ as $\varepsilon \to 0$, while the related Hamiltonian does not become singular.

PROPOSITION 2.1. *The functions $\overline{u}$ and $\underline{u}$ are respectively a bounded subsolution and a bounded supersolution of (1.11) in $\mathbb{R}^d \backslash \{0\}$.*

Proof. The proof is classical and relies on perturbed test-functions techniques, see [17]. □

The strategy of proof for Theorem 1.1. Below, the remainder of the proof of Theorem 1.1 is done in four different steps. Accordingly, Section 2 is cut into four parts. The four steps, which will be rapidly summarized below, mostly rely on the theory of viscosity solutions to first order Hamilton-Jacobi equations, except the third one, which also contains arguments from the theory of optimal control.

1. The first step consists of constructing the ergodic constant E associated to the defect and a related corrector w. It will be proved that $E \geqslant \overline{H}(p_0)$. An important difficulty is that the corrector w must be a function defined in the whole space $\mathbb{R}^d$, which makes it necessary to impose some condition at infinity. We will see that the latter amounts to the fact that w is the locally uniform limit as $R \to +\infty$ of a family $(w^R)_{R>0}$ of solutions of problems with state constraints posed in the balls $B_R(0)$. From the optimal control theory viewpoint, these problems, refered to as *truncated cell problems*, account for trajectories that remain close to the defect at the microscopic scale. Proposition 2.2 below contains information on the growth of w at infinity.

2. In the second step, we prove that the upper-limit $\overline{u}$ satisfies conditions (1.12) and (1.13). While (1.13) is a rather easy consequence of Proposition 2.1, the proof of (1.12) relies on Evans' method of perturbed test-functions, see [17]. The construction of the perturbed test-function involves the above mentioned solution w^R to the truncated cell problem in the ball $B_R(0)$.

3. The third step, the most involved one, consists of proving that the lower-limit $\underline{u}$ satisfies condition (1.14). Most of the work concerns the situation in which $E > \overline{H}(p_0)$. Here again, one needs to construct a suitable perturbation of a test-function by a corrector. The crucial theoretical novelty is the choice of the test-function. Instead of using smooth or affine test-functions as it is classically done, we consider suitable piecewise affine test-functions, in the spirit of [3]. Their behaviour at infinity makes it possible to construct associated correctors in the whole space $\mathbb{R}^d$. The key intermediate result is Proposition 2.6 which states the existence of such correctors with a strictly sublinear growth at infinity. Its proof uses as an important ingredient the function w introduced in the first step and Proposition 2.2. Note also that the proof of Proposition 2.6 relies on control theoretical arguments in the same spirit as in [24]. The choice of the above mentioned piecewise affine test-functions is precisely made so that the latter arguments can be applied when $E > \overline{H}(p_0)$.

4. The fourth step of the proof mostly consists of deducing from the previously obtained results that $\underline{u} = \overline{u}$, by means of a comparison principle. Once this is done, the results announced in Theorem 1.1 follow.

2.1. The ergodic constant associated to the defect. We next introduce ingredients which play a key role in the effective boundary condition at the origin.

2.1.1. Ergodic constants for state-constrained problems in truncated domains. In order to understand the effect of the defect on the asymptotics of u_ε, we start by solving *truncated cell problems* in balls centered at the origin; these are associated to state constrained boundary conditions. From the optimal control theory viewpoint, these problems account for trajectories that remain close to the defect at

the microscopic scale.

For $\lambda > 0$, $R > 0$, we know from e.g. [29, 14] that there exists a unique function $w^{\lambda,R} \in C(\overline{B_R(0)})$ such that

$$(2.3) \qquad \lambda w^{\lambda,R} + H(y, Dw^{\lambda,R}) \leqslant 0 \quad \text{in } B_R(0),$$

$$(2.4) \qquad \lambda w^{\lambda,R} + H(y, Dw^{\lambda,R}) \geqslant 0 \quad \text{in } \overline{B_R(0)},$$

the above inequalities being understood in the sense of viscosity. An equivalent way to write (2.3)-(2.4) is the following:

$$(2.5) \qquad \lambda w^{\lambda,R} + H(y, Dw^{\lambda,R}) = 0 \quad \text{in } B_R(0),$$

$$(2.6) \qquad \lambda w^{\lambda,R} + H^{\uparrow}(y, Dw^{\lambda,R}) = 0 \quad \text{on } \partial B_R(0),$$

in the sense of viscosity, where, for $y \in \partial B_R(0)$,

$$(2.7) \qquad H^{\uparrow}(y, p) = \max_{a \in A, \; f(y,a) \cdot y \leqslant 0} \Big(-p \cdot f(y, a) - \ell(y, a) \Big).$$

is the Hamiltonian associated to the admissible trajectories that do not exit the ball $\overline{B(0, R)}$ through y. The function $w^{\lambda,R}$ is the value function of the following infinite horizon state constrained optimal control problem in $\overline{B_R(0)}$,

(2.8)

$$w^{\lambda,R}(z) = \inf \int_0^\infty e^{-\lambda t} \ell\left(\frac{y(t)}{\varepsilon}, a(t)\right) dt \quad \text{subj. to} \quad \begin{cases} y(t) = z + \int_0^t f\left(\frac{y(\tau)}{\varepsilon}, a(\tau)\right) d\tau \\ y(t) \in \overline{B_R(0)}, \\ a \in L^\infty(\mathbb{R}_+) \\ a(t) \in A, \text{ a.e.} \end{cases}$$

which the reader may compare to (1.6). Since ℓ is bounded on $\mathbb{R}^d \times A$ (because, in particular, A is compact and ℓ coincides with ℓ_{per} out of $B_{R_0}(0)$), $\lambda \|w^{\lambda,R}\|_{L^\infty(B_R(0))}$ is bounded uniformly in λ and R. More precisely, $\min_{(y,a) \in \mathbb{R}^d \times A} \ell(y, a) \leqslant \lambda w^{\lambda,R} \leqslant \max_{(y,a) \in \mathbb{R}^d \times A} \ell(y, a)$. This and the uniform coercivity of H imply with (2.5) that $\|Dw^{\lambda,R}\|_{L^\infty(B_R(0))}$ is bounded uniformly in λ and R.

Using Ascoli-Arzelà theorem, we may suppose that up to the extraction of a sequence, as $\lambda \to 0$, $\lambda w^{\lambda,R}$ tends uniformly on $\overline{B_R(0)}$ to some *ergodic* constant $-E^R$ which is bounded from above and below uniformly in R, and that $w^{\lambda,R} - w^{\lambda,R}(0)$ tends uniformly on $\overline{B_R(0)}$ to some function w^R such that $w^R(0) = 0$ and which is Lipschitz continuous with a Lipschitz constant independent of R. By classical results on the stability of viscosity solutions of state constrained problems, w^R is a viscosity solution of

$$(2.9) \qquad H(y, Dw^R) \leqslant E^R \quad \text{in } B_R(0),$$

$$(2.10) \qquad H(y, Dw^R) \geqslant E^R \quad \text{in } \overline{B_R(0)}.$$

The comparison principle for state constrained problems, see [29, 14], yields the uniqueness of E^R such that (2.9)-(2.10) has a solution. Thus, $\lim_{\lambda \to 0} \|\lambda w^{\lambda,R} + E^R\|_{C(\overline{B}_R(0))} = 0$ (uniform convergence and not only for a subsequence).

2.1.2. The ergodic constant and the cell problem. We deduce for example from (2.8) that

$$R_1 \geqslant R_2 \quad \Rightarrow \quad \lambda w^{\lambda,R_1} \leqslant \lambda w^{\lambda,R_2},$$

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and passing to the limit as $\lambda \to 0$, we obtain the monotonicity property of the ergodic constants E^R:

$$(2.11) \qquad\qquad R_1 \geqslant R_2 \quad \Rightarrow \quad E^{R_1} \geqslant E^{R_2}.$$

Since E^R is bounded from above independently of R (2.11) implies that

$$(2.12) \qquad\qquad E = \lim_{R \to \infty} E^R$$

exists in $\mathbb{R}$.

Similarly, since $w^R(0) = 0$, w^R is Lipschitz continuous on $\overline{B_R(0)}$ with a Lipschitz constant independent of R, we may construct by Ascoli-Arzelà theorem and a diagonal extraction argument a sequence $(R_n)_{n \in \mathbb{N}}$, $R_n \to +\infty$ as $n \to \infty$, such that w^{R_n} tends to some function w locally uniformly in $\mathbb{R}^d$; we then see that $w(0) = 0$ and w is a Lipschitz continuous viscosity solution of

$$(2.13) \qquad\qquad H(y, Dw) = E \quad \text{in } \mathbb{R}^d.$$

Let us now zoom out and pass to the macroscopic scale by considering the function $w_\varepsilon : x \mapsto \varepsilon w(\frac{x}{\varepsilon})$; it is clearly a viscosity solution of $H(\frac{x}{\varepsilon}, D_x w_\varepsilon) = E$, and it is Lipschitz continuous with the same constant as w. Hence, after the extraction of a sequence, we may assume that w_ε converges locally uniformly to some Lipschitz function W on $\mathbb{R}^d$. As for Proposition 2.1, a standard argument yields that W is a viscosity solution of $\overline{H}(DW) = E$. This implies the important inequality

$$(2.14) \qquad\qquad E \geqslant \min_{p \in \mathbb{R}^d} \overline{H}(p) = \overline{H}(p_0),$$

the right hand side equality holding because of (1.10).

Proposition (2.2) below gives some information on the behaviour of w at infinity. It will be useful for proving that $\underline{u}$ satisfies (1.14).

PROPOSITION 2.2. *If $E > \overline{H}(p_0) = \min_{p \in \mathbb{R}^d} \overline{H}(p)$, then*

$$(2.15) \qquad\qquad \lim_{|y| \to +\infty} w(y) - p_0 \cdot y = +\infty,$$

where w is the corrector constructed in (2.13).

If $E = \overline{H}(p_0) = \min_{p \in \mathbb{R}^d} \overline{H}(p)$, then $w(y) - p_0 \cdot y$ is bounded from below.

Proof.

Step 1. Let us start by proving the desired results in the case when $p_0 = 0$. With R_0 defined in Section 1.1, we already know that, for all $R > R_0$, the corrector w^R solution to (2.9)-(2.10), is Lipschitz continuous with a Lipschitz constant $C > 0$ independent of R.

For $R_1 > R_0$, let Q_{R_1} be the cube

$$Q_{R_1} = \{y \in \mathbb{R}^d, \ \|y\|_\infty \leqslant R_1\}.$$

It contains $\overline{B_{R_0}(0)}$. It is clear that, for all $R > R_0$ and $y \in \partial Q_{R_1}$,

$$(2.16) \qquad\qquad |w^R(y)| \leqslant C\sqrt{d}R_1,$$

using the facts that $w^R(0) = 0$ and that C is a Lipschitz constant for w^R. We distinguish two cases:

1. $E > \overline{H}(0)$.

   Let $(e_i)_{i=1,\ldots,d}$ be the canonical basis of $\mathbb{R}^d$ associated with the system of coordinates $(y_i)_{i=1,\ldots,d}$. Since $E > \overline{H}(0)$, we may choose δ, $0 < \delta < E - \overline{H}(0)$. Now, because of the convergence of E^R to E, we have for R large enough (that we can always suppose larger than $\sqrt{d}R_1$, so that the ball $B_R(0)$ contains $\overline{Q_{R_1}}$),

   (2.17) $$E \geqslant E^R > E - \delta.$$

   Since $\overline{H}(0) = \min_p \overline{H}(p)$, the continuity and the coercivity of $\overline{H}$ together with $E - \delta > \overline{H}(0)$ imply that for all $i \in \{1,\ldots,d\}$, there exists $(\underline{p}_{i,\delta}, \overline{p}_{i,\delta}) \in \mathbb{R}^2$, $\underline{p}_{i,\delta} < 0 < \overline{p}_{i,\delta}$ such that

   (2.18) $$\overline{H}(\underline{p}_{i,\delta}e_i) = \overline{H}(\overline{p}_{i,\delta}e_i) = E - \delta.$$

   Consider now the functions

   (2.19) $$\overline{w}_{i,\delta}(y) = \overline{c}_{i,\delta} + \overline{p}_{i,\delta}y_i + \chi_{\mathrm{per},\overline{p}_{i,\delta}e_i}(y),$$
   (2.20) $$\underline{w}_{i,\delta}(y) = \underline{c}_{i,\delta} + \underline{p}_{i,\delta}y_i + \chi_{\mathrm{per},\underline{p}_{i,\delta}e_i}(y).$$

   where $\overline{c}_{i,\delta}$ and $\underline{c}_{i,\delta}$ are scalars chosen such that, for any $y \in \partial Q_{R_1}$,

   (2.21) $$\overline{w}_{i,\delta}(y) < -C\sqrt{d}R_1 \leqslant w^R(y) \quad \text{and} \quad \underline{w}_{i,\delta}(y) < -C\sqrt{d}R_1 \leqslant w^R(y),$$

   recalling that $\overline{Q_{R_1}} \subset B_R(0)$. Moreover, since H coincides with H_{per} in $(\mathbb{R}^d \backslash \overline{Q_{R_1}}) \times \mathbb{R}^d$, we see from (1.9) that

   (2.22) $$H(y, D\overline{w}_{i,\delta}(y)) = E - \delta,$$
   (2.23) $$H(y, D\underline{w}_{i,\delta}(y)) = E - \delta,$$

   in the sense of viscosity in $\mathbb{R}^d \backslash \overline{Q_{R_1}}$.

   Since w^R is a viscosity supersolution of (2.10), we may use a comparison principle in $\overline{B_R(0)} \backslash Q_{R_1}$ and deduce from (2.17) and (2.21)-(2.22)-(2.23) that for R large enough and for all $y \in \overline{B_R(0)} \backslash Q_{R_1}$,

   $$\overline{w}_{i,\delta}(y) \leqslant w^R(y), \quad \text{and} \quad \underline{w}_{i,\delta}(y) \leqslant w^R(y).$$

   To summarize, we have proved that, for all $y \in \overline{B}_R(0) \backslash Q_{R_1}$:

   (2.24) $$\max\{\overline{w}_{i,\delta}(y), \underline{w}_{i,\delta}(y), \; i = 1,\ldots,d\} \leqslant w^R(y).$$

   By passing to the limit in R (possibly after the extraction of a sequence), we deduce that for any $y \in \mathbb{R}^d \backslash Q_{R_1}$,

   (2.25) $$\max\{\overline{w}_{i,\delta}(y), \underline{w}_{i,\delta}(y), \; i = 1,\ldots,d\} \leqslant w(y).$$

   Since the correctors $\chi_{\mathrm{per},\overline{p}_{i,\delta}e_i}$ are bounded functions, it is clear from (2.19)-(2.20) that there exists a constant c such that for all $y \in \mathbb{R}^d \backslash Q_{R_1}$,

   $$\max\{\overline{w}_{i,\delta}(y), \underline{w}_{i,\delta}(y), \; i = 1,\ldots,d\}$$
   $$\geqslant \min\{|\overline{p}_{i,\delta}|, |\underline{p}_{i,\delta}|, \; i = 1,\ldots,d\}\|y\|_\infty - c.$$

   This yields the desired result, namely (2.15) in the case when $p_0 = 0$ and $E > \overline{H}(0)$.

2. $E = \overline{H}(0)$. For any c, the function $\underline{w}(y) = \chi_{\mathrm{per},0}(y) + c$ thus satisfies

$$H(y, D\underline{w}) = E, \quad \text{in } B_R(0)\backslash\overline{Q}_{R_1},$$

in the sense of viscosity. Moreover, it is possible to choose c such that $\underline{w}(y) = \chi_{\mathrm{per},0}(y) + c < -C\sqrt{d}R_1$ for all $y \in \partial Q_{R_1}$. By a comparison principle, it follows that $\underline{w} \leqslant w^R$ in $\overline{B_R(0)}\backslash Q_{R_1}$. By passing to the limit in R (possibly after the extraction of a sequence), we deduce that w is bounded from below, which concludes the proof of Proposition 2.2 when $p_0 = 0$.

Step 2. We need to prove the result for a general p_0. The idea consists of suitably shifting the Hamiltonians. More explicitly, we consider the new running costs and Hamiltonians:

$$(2.26) \qquad \widetilde{\ell}(x, a) = \ell(x, a) + p_0 \cdot f(x, a),$$

$$(2.27) \qquad \widetilde{\ell_{\mathrm{per}}}(x, a) = \ell_{\mathrm{per}}(x, a) + p_0 \cdot f_{\mathrm{per}}(x, a),$$

$$(2.28) \qquad \widetilde{H}(x, p) = H(x, p + p_0) = \sup_{a \in A} \left(-p \cdot f(x, a) - \widetilde{\ell}(x, a) \right),$$

$$(2.29) \qquad \widetilde{H_{\mathrm{per}}}(x, p) = H_{\mathrm{per}}(x, p + p_0) = \sup_{a \in A} \left(-p \cdot f_{\mathrm{per}}(x, a) - \widetilde{\ell_{\mathrm{per}}}(x, a) \right),$$

which satisfy all the stuctural assumptions made on ℓ, ℓ_{per}, H and H_{per} in Section 1.1. It is clear that the new effective Hamiltonian $\widetilde{\overline{H}}(p) = \overline{H}(p + p_0)$ is obtained from $\widetilde{H_{\mathrm{per}}}$ by solving the shifted cell problem:

$$(2.30) \qquad H_{\mathrm{per}}(y, p_0 + p + D\widetilde{\chi_{\mathrm{per},p}}) = \widetilde{\overline{H}}(p) \quad \text{in } \mathbb{R}^d,$$

the solutions of which are of the form $\widetilde{\chi_{\mathrm{per},p}} = \chi_{\mathrm{per},p+p_0}$, where $\chi_{\mathrm{per},p+p_0}$ is a solution of the original periodic cell problem (1.9) associated to $p+p_0$. Note that $0 \in \arg\min \widetilde{\overline{H}}$. It is straightforward to realize that $\widetilde{w}^R(y) = w^R(y) - p_0 \cdot y$ is a viscosity solution of the state constrained problem:

$$(2.31) \qquad\qquad \widetilde{H}(y, D\widetilde{w}^R) \leqslant E^R \quad \text{in } B_R(0),$$

$$(2.32) \qquad\qquad \widetilde{H}(y, D\widetilde{w}^R) \geqslant E^R \quad \text{in } \overline{B_R(0)}.$$

and that $\widetilde{w}^R$ tends to $\widetilde{w}(y) = w(y) - p_0 \cdot y$ locally uniformly. We can apply the results proven above to $\widetilde{H}$ and $\widetilde{w}$ because $\widetilde{\overline{H}}$ reaches its minimal value at 0. This yields the desired result in the general case. □

2.2. The function $\overline{u}$ is a subsolution of (1.11) and satisfies (1.12)-(1.13).

PROPOSITION 2.3. *The upper limit $\overline{u}$ satisfies (1.12).*

Proof. Let us proceed by contradiction and assume that

$$(2.33) \qquad\qquad\qquad \alpha\overline{u}(0) + E = \theta > 0.$$

Using w^R defined in paragraph 2.1.2 (recall $w^R(0) = 0$), we define

$$\phi^{\varepsilon,R} = \overline{u}(0) + \varepsilon w^R(\frac{x}{\varepsilon}).$$

We deduce from (2.10) that $\phi^{\varepsilon,R}$ is a viscosity supersolution of

$$\alpha\phi^{\varepsilon,R}(x) + H\left(\frac{x}{\varepsilon}, D\phi^{\varepsilon,R}\right) \geqslant \alpha\varepsilon w^R\left(\frac{x}{\varepsilon}\right) + E^R - E + \theta \quad \text{in } \overline{B_{\varepsilon R}(0)}.$$

There exists $r > 0$ such that $E^R - E \geqslant -\frac{\theta}{4}$ for any $R \geqslant r$. Let us fix such a value of R.

Having fixed R, we see that for ε_0 sufficiently small and any ε such that $0 < \varepsilon < \varepsilon_0$,

$$\alpha\varepsilon w^R\left(y\right) \geqslant -\frac{\theta}{4} \quad \text{for any } y \in \overline{B_R(0)}.$$

We deduce that, for any $\varepsilon < \varepsilon_0$,

$$\alpha\phi^{\varepsilon,R} + H\left(\frac{x}{\varepsilon}, D\phi^{\varepsilon,R}\right) \geqslant \frac{\theta}{2} \quad \text{in } \overline{B_{\varepsilon R}(0)}.$$

Next, using (2.1), consider a vanishing sequence $0 < \varepsilon_n < \varepsilon_0$ such that $u_{\varepsilon_n}(0)$ tends to $\overline{u}(0)$. We know that u_{ε_n} satisfies in the sense of viscosity

$$\alpha\, u_{\varepsilon_n} + H\left(\frac{x}{\varepsilon_n}, Du_{\varepsilon_n}\right) \leqslant 0 \quad \text{in } B_{\varepsilon_n R}(0).$$

The comparison principle for Hamilton-Jacobi equations with state constraints, see [29, 14] and [6, Th. 5.8, Chapter IV, page 278], then implies that

$$\phi^{\varepsilon_n,R} - \frac{\theta}{2\alpha} \geqslant u_{\varepsilon_n} \quad \text{in } B_{\varepsilon_n R}(0),$$

or, put differently,

$$\overline{u}(0) + \varepsilon_n w^R(\frac{x}{\varepsilon_n}) - \frac{\theta}{2\alpha} \geqslant u_{\varepsilon_n}(x) \quad \text{for any } x \in B_{\varepsilon_n R}(0).$$

Taking $x = 0$ and letting n tend to $+\infty$ yields $\overline{u}(0) - \frac{\theta}{2\alpha} \geqslant \overline{u}(0)$, the desired contradiction. $\qquad\square$

The next proposition states that $\overline{u}$ is a viscosity subsolution of (1.11) (which is already known from Proposition 2.1) and satisfies condition (1.13).

PROPOSITION 2.4. *The function $\overline{u}$ is a viscosity subsolution of $\alpha v + \overline{H}(Dv) \leqslant 0$ in the whole space $\mathbb{R}^d$.*

Proof. We know from Proposition 2.1 that $\overline{u}$ is Lipschitz continous in $\mathbb{R}^d$. Hence, since $\overline{u}$ is a viscosity subsolution of $\alpha v + \overline{H}(Dv) \leqslant 0$ in $\mathbb{R}^d \backslash \{0\}$, it satisfies $\alpha\overline{u}(x) + \overline{H}(D\overline{u}(x)) \leqslant 0$ at almost every $x \in \mathbb{R}^d$, see [6, Prop. 1.9, Chapter I, page 31] and its proof. But $\overline{H}$ is convex. Therefore, from [6, Prop. 5.1, Chapter II, page 77], $\overline{u}$ is a viscosity subsolution of $\alpha v + \overline{H}(Dv) \leqslant 0$ in the whole space $\mathbb{R}^d$. $\qquad\square$

2.3. The function $\underline{u}$ is a supersolution of (1.11) and satisfies (1.14). We already know from Proposition 2.1 that $\underline{u}$ is a bounded supersolution of (1.11).

Let $\phi \in C^1(\mathbb{R}^d)$ be such that $\underline{u} - \phi$ has a local minimum at the origin. We wish to prove that $\alpha\underline{u}(0) + \max\left(E, \overline{H}(D\phi(0))\right) \geqslant 0$. The proof differs depending whether $\overline{H}(D\phi(0)) > E$ or $\overline{H}(D\phi(0)) \leqslant E$. It is based on control theoretic arguments and partly inspired from the ideas proposed by P-L Lions and P. Souganidis when they dealt with the case $E = \overline{H}(p_0)$, see [24]. However, various new arguments, reminiscent of those used in [3], will be needed to address the more difficult case $E > \overline{H}(p_0)$.

Let us start by considering $p \in \mathbb{R}^d$ such that $\overline{H}(p) > E$. It is obvious that $p \neq p_0$ because $E \geqslant \overline{H}(p_0) = \min_q \overline{H}(q)$. From the convexity and the coercivity of $\overline{H}$, we see that there exists a unique unit vector e colinear to $p - p_0$ such that $\mathbb{R} \ni t \mapsto \overline{H}(p + te)$ is strictly decreasing in a neighborhood of $t = 0$, and a unique vector $\tilde{p}$ such that

1. $\tilde{p} - p_0$ is colinear to $p - p_0$
2. $\overline{H}(\tilde{p}) = \overline{H}(p)$
3. $t \mapsto \overline{H}(\tilde{p} + te)$ is strictly increasing in a neighborhood of $t = 0$.

We easily check that $(p - p_0) \cdot e < 0$ and that $(\tilde{p} - p_0) \cdot e > 0$.

The following observation will be useful for proving that $\underline{u}$ satisfies (1.14).

REMARK 2.1. *Note that if the function* $x \mapsto \underline{u}(x) - \underline{u}(0) - p \cdot x$ *has a local minimum at* $x = 0$, *then the function* $x \mapsto \underline{u}(x) - \underline{u}(0) - \min(p \cdot x, \tilde{p} \cdot x)$ *also has a local minimum at* $x = 0$.

We now assume that, for any $\lambda > 0$ and $y \in \mathbb{R}^d$, there exists an optimal trajectory (z_λ, a_λ) of the optimal control problem

$$\chi^\lambda_{\mathrm{per},p}(y) = \inf_{z,a} \left\{ \int_0^\infty e^{-\lambda t} \Big(\ell_{\mathrm{per}}(z(t), a(t)) + p \cdot f_{\mathrm{per}}(z(t), a(t)) \Big) dt \right\},$$

subject to

(2.34) $\qquad z'(t) = f_{\mathrm{per}}(z(t), a(t)), \quad a(t) \in A \ \text{ for almost } t > 0,$

(2.35) $\qquad z(0) = y.$

This assumption on the existence of such an optimal trajectory is made for simplicity. In the full generality, we may find trajectories as close to optimal as needed and what follows remains true with such trajectories.

The value function $\chi^\lambda_{\mathrm{per},p}$ is a continuous function defined on $\mathbb{R}^d/\mathbb{Z}^d$; let $\langle \chi^\lambda_{\mathrm{per},p} \rangle$ stand for its mean value. It is well known, see [25], that $-\lambda \langle \chi^\lambda_{\mathrm{per},p} \rangle$ tends to $\overline{H}(p)$ as $\lambda \to 0$ and that after the extraction of a sequence, $y \mapsto \chi^\lambda_{\mathrm{per},p}(y) - \langle \chi^\lambda_{\mathrm{per},p} \rangle$ tends to $\chi_{\mathrm{per},p}$ uniformly, where $\chi_{\mathrm{per},p}$ is a corrector associated with p for the periodic homogenization problem. We recall that in periodic homogenization, the corrector may not be unique, even up to the addition of a constant.

Similarly, let us assume that there exists $(\tilde{z}_\lambda, \tilde{a}_\lambda)$, an optimal trajectory for the optimal control problem

$$\chi^\lambda_{\mathrm{per},\tilde{p}}(y) = \inf_{z,a} \left\{ \int_0^\infty e^{-\lambda t} \left(\ell_{\mathrm{per}}(z(t), a(t)) + \tilde{p} \cdot f_{\mathrm{per}}(z(t), a(t)) \right) dt \right\},$$

subject to (2.34)-(2.35).

The next lemma gives a uniform upper bound on the time spent by the optimal trajectories z_{λ_n} in the half-space $\{x \in \mathbb{R}^d : e \cdot x \leqslant r\}$ for $r > 0$, as λ_n converges to 0, and a symmetric estimate concerning $\tilde{z}_{\lambda_n}$. This lemma will be used in the proof of Proposition 2.6 below.

LEMMA 2.5. *If* $\overline{H}(p) > E$, *let* $(\lambda_n)_{n \in \mathbb{N}}$ *and* $(\tilde{\lambda}_n)_{n \in \mathbb{N}}$ *be sequences converging to* 0 *such that* $y \mapsto \chi^{\lambda_n}_{\mathrm{per},p}(y) - \langle \chi^{\lambda_n}_{\mathrm{per},p} \rangle$ *converges to* $\chi_{\mathrm{per},p}$ *uniformly and* $y \mapsto \chi^{\tilde{\lambda}_n}_{\mathrm{per},\tilde{p}}(y) -$ $\langle \chi^{\tilde{\lambda}_n}_{\mathrm{per},\tilde{p}} \rangle$ *converges to* $\chi_{\mathrm{per},\tilde{p}}$ *uniformly. There exists a positive constant* c *that only depends on* $\overline{H}(p)$, *such that for any* $T > 0$, *there exist subsequences still denoted by* $(\lambda_n)_{n \in \mathbb{N}}$ *and* $(\tilde{\lambda}_n)_{n \in \mathbb{N}}$ *satisfying the following properties: for the unit vector* e *introduced in the beginning of Section 2.3,*

$$e \cdot (z_{\lambda_n}(t) - y) \geqslant c(t - 1) \quad and \quad e \cdot (\tilde{z}_{\lambda_n}(t) - y) \leqslant -c(t - 1),$$

for all n *and* $t \in [0, T]$.

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Proof. We will omit the index n, except at the end of the proof. We focus on the first assertion (concerning z_λ) since the proof of the second assertion (on $\tilde{z}_\lambda$) is similar. Since $\overline{H}(p_0) \leqslant E < \overline{H}(p)$, there exists $q = p_0 + \pi_0(p - p_0)$, with $0 < \pi_0 < 1$ such that $E < \overline{H}(q) < \overline{H}(p)$. The trajectory (z_λ, a_λ) is then strictly suboptimal for the optimal control problem :

$$\chi^\lambda_{\mathrm{per},q}(y) = \inf_{z,a}\left\{ \int_0^\infty e^{-\lambda t}\Big(\ell_{\mathrm{per}}(z(t), a(t)) + q \cdot f_{\mathrm{per}}(z(t), a(t))\Big)dt\right\},$$

subject to (2.34)-(2.35). We obtain that

$$(2.36) \qquad \chi^\lambda_{\mathrm{per},p}(y) = \int_0^t e^{-\lambda s}\Big(\ell_{\mathrm{per}}(z_\lambda(s), a_\lambda(s)) + p \cdot f_{\mathrm{per}}(z_\lambda(s), a_\lambda(s))\Big)ds$$
$$+ e^{-\lambda t}\chi^\lambda_{\mathrm{per},p}(z_\lambda(t)),$$

$$(2.37) \qquad \chi^\lambda_{\mathrm{per},q}(y) \leqslant \int_0^t e^{-\lambda s}\Big(\ell_{\mathrm{per}}(z_\lambda(s), a_\lambda(s)) + q \cdot f_{\mathrm{per}}(z_\lambda(s), a_\lambda(s))\Big)ds$$
$$+ e^{-\lambda t}\chi^\lambda_{\mathrm{per},q}(z_\lambda(t)).$$

Using a Taylor expansion of $s \mapsto e^{-\lambda s}$ as λ vanishes, we see that there exists a positive constant C such that

$$\left| \chi^\lambda_{\mathrm{per},p}(y) - \chi^\lambda_{\mathrm{per},p}(z_\lambda(t)) - \int_0^t \ell_{\mathrm{per}}(z_\lambda(s), a_\lambda(s))ds - p \cdot (z_\lambda(t) - y) + \lambda t \chi^\lambda_{\mathrm{per},p}(z_\lambda(t)) \right|$$
$$\leqslant C\lambda t^2$$

and

$$\chi^\lambda_{\mathrm{per},q}(y) - \chi^\lambda_{\mathrm{per},q}(z_\lambda(t)) - \int_0^t \ell_{\mathrm{per}}(z_\lambda(s), a_\lambda(s))ds - q \cdot (z_\lambda(t) - y) + \lambda t \chi^\lambda_{\mathrm{per},q}(z_\lambda(t))$$
$$\leqslant C\lambda t^2$$

We deduce from the latter two inequalities that

$$(2.38) \qquad -\left[\chi^\lambda_{\mathrm{per},q}(z_\lambda) - \langle \chi^\lambda_{\mathrm{per},q} \rangle - \chi^\lambda_{\mathrm{per},p}(z_\lambda) + \langle \chi^\lambda_{\mathrm{per},p} \rangle \right]_0^t$$
$$\leqslant (q - p) \cdot (z_\lambda(t) - y) - \lambda t(\chi^\lambda_{\mathrm{per},q}(z_\lambda(t)) - \chi^\lambda_{\mathrm{per},p}(z_\lambda(t))) + 2C\lambda t^2.$$

We know that for two correctors $\chi_{\mathrm{per},p}$ and $\chi_{\mathrm{per},q}$ of the periodic problems (1.9) respectively associated with p and q, it is possible to find some subsequence λ such that
- $\chi^\lambda_{\mathrm{per},q} - \langle \chi^\lambda_{\mathrm{per},q} \rangle$ converges uniformly to $\chi_{\mathrm{per},q}$
- $\chi^\lambda_{\mathrm{per},p} - \langle \chi^\lambda_{\mathrm{per},p} \rangle$ converges uniformly to $\chi_{\mathrm{per},p}$
- $\lambda(\chi^\lambda_{\mathrm{per},q} - \chi^\lambda_{\mathrm{per},p})$ converges uniformly to $-\overline{H}(q) + \overline{H}(p)$.

Note that $0 < \overline{H}(p) - \overline{H}(q) < \overline{H}(p)$ and that $\chi_{\mathrm{per},q}$ and $\chi_{\mathrm{per},p}$ are uniformly bounded by a constant which depends on $\overline{H}(p)$. Hence the left hand side of (2.38) is bounded from below by $-c_0$, for a suitable constant c_0 depending on $\overline{H}(p)$ only.

From (2.38) and the observations above, we deduce that there exists a constant c_1 depending on $\overline{H}(p)$ only, such that for all $T > 0$, we may select a subsequence (λ_n) such that
$$(p - q) \cdot (z_{\lambda_n}(t) - y) \leqslant (\overline{H}(q) - \overline{H}(p))t + c_1$$

for all $t \in [0, T]$ and all n. The desired result follows. $\qquad\square$

The next proposition deals with the existence of a corrector associated to the piecewise linear function $y \mapsto \min(p \cdot y, \tilde{p} \cdot y)$, in the whole space $\mathbb{R}^d$. In view of Remark 2.1, this corrector will be useful for proving that $\underline{u}$ satisfies (1.14). As already mentioned, this strategy differs from that used in [24] in which the corrector is associated to the linear function $y \mapsto p \cdot y$, under stronger assumptions.

PROPOSITION 2.6. *For any $p \in \mathbb{R}^d$ such that $\overline{H}(p) > E$, let e and $\tilde{p}$ be defined as in the beginning of Section 2.3. Then there exists $\chi_{p,\tilde{p}} \in C(\mathbb{R}^d)$ such that*

$$\text{(2.39)} \qquad \lim_{|y| \to \infty} \frac{|\chi_{p,\tilde{p}}(y) - \min(p \cdot y, \tilde{p} \cdot y)|}{|y|} = 0,$$

$$\text{(2.40)} \qquad H\left(y, D\chi_{p,\tilde{p}}\right) = \overline{H}(p) \quad in \ \mathbb{R}^d,$$

where (2.40) is understood in the sense of viscosity.

Proof. Fix $\chi_{\mathrm{per},p}$ and $\chi_{\mathrm{per},\tilde{p}}$ as in Lemma 2.5. For a radius $R > R_0 + 1$ that will eventually converge to $+\infty$, consider the Dirichlet problem

$$\text{(2.41)} \qquad H\left(y, D\chi_{p,\tilde{p}}^R\right) = \overline{H}(p) \quad in \ B_R(0),$$

$$\text{(2.42)} \qquad \chi_{p,\tilde{p}}^R(y) = \min\left(p \cdot y + \chi_{\mathrm{per},p}(y), \tilde{p} \cdot y + \chi_{\mathrm{per},\tilde{p}}(y)\right) \quad \text{on } \partial B_R(0).$$

As a first step, let us construct a subsolution to (2.41)-(2.42). For a constant $c > 0$ that will be chosen below, we set

$$\text{(2.43)} \qquad \sigma(y) = \min\left(w(y) - c, p \cdot y + \chi_{\mathrm{per},p}(y), \tilde{p} \cdot y + \chi_{\mathrm{per},\tilde{p}}(y)\right),$$

where w is the viscosity solution of (2.13) constructed in Section 2.1.2, which is Lipschitz continuous in $\mathbb{R}^d$ and thus satisfies (2.13) almost everywhere in $\mathbb{R}^d$ from [6, Prop. 1.9, Chapter I, page 31].

We next choose c such that, for any $y \in B_{R_0+1}(0)$,

$$w(y) - c < \min\left(p \cdot y + \chi_{\mathrm{per},p}(y), \tilde{p} \cdot y + \chi_{\mathrm{per},\tilde{p}}(y)\right),$$

thus

$$\text{(2.44)} \qquad \sigma(y) = w(y) - c, \quad \text{for any } y \in B_{R_0+1}(0).$$

On the other hand, since $E < \overline{H}(p)$, w, $y \mapsto p \cdot y + \chi_{\mathrm{per},p}(y)$ and $y \mapsto \tilde{p} \cdot y + \chi_{\mathrm{per},\tilde{p}}(y)$ are three Lipschitz continuous viscosity subsolutions of $H_{\mathrm{per}}(y, Dv) \leqslant \overline{H}(p)$ in $B_R(0) \backslash \overline{B_{R_0+1}(0)}$. We deduce from [6, Prop. 1.9, Chapter I, page 31] that

$$\text{(2.45)} \qquad H\left(y, D\sigma(y)\right) \leqslant \overline{H}(p) \quad \text{for almost all } y \in B_R(0) \backslash \overline{B_{R_0+1}(0)}.$$

Combining this with (2.44), σ actually satisfies (2.45) for almost all $y \in B_R(0)$. Then, since H is convex with respect to its second argument, [6, Prop. 5.1, Chapter II, page 77] can be applied and yields that σ is a viscosity subsolution of

$$\text{(2.46)} \qquad H\left(y, D\sigma\right) \leqslant \overline{H}(p) \quad in \ B_R(0).$$

On the other hand, it is clear that

$$\sigma(y) \leqslant \min\left(p \cdot y + \chi_{\mathrm{per},p}(y), \tilde{p} \cdot y + \chi_{\mathrm{per},\tilde{p}}(y)\right) \quad \text{for any } y \in \partial B_R(0).$$

Hence, σ is a subsolution of the Dirichlet problem (2.41)-(2.42).

Our second step is to establish the existence of a constant $C > 0$ independent of $R > R_0 + 1$ such that, for all $y \in B_R(0)$,

$$(2.47) \qquad \sigma(y) - \min\Big(p \cdot y + \chi_{\mathrm{per},p}(y), \tilde{p} \cdot y + \chi_{\mathrm{per},\tilde{p}}(y)\Big) \geq -C.$$

For this purpose, we distinguish two cases:

Case 1: $E > \overline{H}(p_0)$. We first observe that $y \mapsto \min\Big((p-p_0)\cdot y, (\tilde{p}-p_0)\cdot y\Big)$ is bounded from above. Therefore, $y \mapsto \min\Big((p-p_0)\cdot y + \chi_{\mathrm{per},p}(y), (\tilde{p}-p_0)\cdot y + \chi_{\mathrm{per},\tilde{p}}(y)\Big)$ is also bounded from above. Since $\overline{H}(p) > E > \overline{H}(p_0)$, we deduce from this and (2.15) in Proposition 2.2 that for $|y|$ large enough, $w(y) - p_0 \cdot y - c$ is larger that $\min\Big((p - p_0) \cdot y + \chi_{\mathrm{per},p}(y), (\tilde{p} - p_0) \cdot y + \chi_{\mathrm{per},\tilde{p}}(y)\Big)$, hence

$$\sigma(y) = \min\Big(p \cdot y + \chi_{\mathrm{per},p}(y), \tilde{p} \cdot y + \chi_{\mathrm{per},\tilde{p}}(y)\Big).$$

This yields (2.47).

Case 2: $E = \overline{H}(p_0)$. From the second conclusion of Proposition 2.2, we know that there exists a a constant $C > 0$ independent of R such that, for all $y \in \mathbb{R}^d$,

$$(2.48) \qquad w(y) - p_0 \cdot y > -C.$$

Hence, (2.44), which also reads

$$(2.49) \quad w(y) - p_0 \cdot y - c < \min\Big((p-p_0) \cdot y + \chi_{\mathrm{per},p}(y), (\tilde{p} - p_0) \cdot y + \chi_{\mathrm{per},\tilde{p}}(y)\Big)$$

then yields that $|e \cdot y|$ is bounded by a constant uniform in R.

In turn, this implies that

$$\min\Big((p - p_0) \cdot y + \chi_{\mathrm{per},p}(y), (\tilde{p} - p_0) \cdot y + \chi_{\mathrm{per},\tilde{p}}(y)\Big)$$

is bounded above uniformly in R. Combining this with (2.48), we deduce that for a possibly different value of the constant C,

$$w(y) - c - \min\Big(p \cdot y + \chi_{\mathrm{per},p}(y), \tilde{p} \cdot y + \chi_{\mathrm{per},\tilde{p}}(y)\Big) \geq -C.$$

Inserting this minoration into (2.43) yields (2.47).

Our third step consists of finding a supersolution to (2.41)-(2.42). This is easy, because for $q \in \mathbb{R}^d$, $|q|$ large enough and $y_0 \in \mathbb{R}^d$ such that $-q \cdot y_0$ is large enough, $y \mapsto q \cdot (y - y_0)$ is indeed a supersolution of (2.41)-(2.42).

The existence of a solution $\chi^R_{p,\tilde{p}}$ to (2.41)-(2.42) is then obtained by Perron's method. It can be proved in a classical way that this solution is unique. Besides, we have the following representation formula:

$$(2.50) \quad \chi^R_{p,\tilde{p}}(y) = \inf_{z,a,\tau} \left\{ \begin{array}{l} \displaystyle\int_0^\tau \Big(\ell(z(t), a(t)) + \overline{H}(p)\Big) dt \\ + \min\Big(p \cdot z(\tau) + \chi_{\mathrm{per},p}(z(\tau)), \tilde{p} \cdot z(\tau) + \chi_{\mathrm{per},\tilde{p}}(z(\tau))\Big) \end{array} \right\},$$

subject to

$$z'(t) = f(z(t), a(t)), \quad a(t) \in A \text{ for almost } t > 0,$$
$$z(0) = y,$$
$$\tau = \inf\{t > 0 : |z(t)| = R\}.$$

Now, our goal is to obtain $\chi_{p,\tilde{p}}$ as a locally uniform limit of $\chi_{p,\tilde{p}}^{R_n}$ for some sequence R_n tending to $+\infty$. For that purpose, we will use (2.47) which readily gives a bound from below on $\chi_{p,\tilde{p}}^R$, but we also need an accurate bound from above. Our fourth step therefore consists of obtaining this bound by using the representation formula (2.50) and Lemma 2.5.

With c given in Lemma 2.5, which depends only on $\overline{H}(p)$, let $T > 0$ be such that $c(T - 1) > 2R$. We denote by λ any term in the sequence appearing in Lemma 2.5. With (z_λ, a_λ) introduced above, let τ_λ be the first time at which z_λ hits $\partial B_R(0)$, which is smaller than T thanks to Lemma 2.5. From (2.50), we deduce that

$$\chi_{p,\tilde{p}}^R(y) \leq \left(\begin{array}{l} \displaystyle\int_0^{\tau_\lambda} \big(\ell(z_\lambda(t), a_\lambda(t)) + \overline{H}(p)\big)\, dt \\[2mm] + \min\Big(p \cdot z_\lambda(\tau_\lambda) + \chi_{\mathrm{per},p}(z_\lambda(\tau_\lambda)), \tilde{p} \cdot z_\lambda(\tau_\lambda) + \chi_{\mathrm{per},\tilde{p}}(z_\lambda(\tau_\lambda))\Big) \end{array} \right)$$
$$\leq \int_0^{\tau_\lambda} \big(\ell(z_\lambda(t), a_\lambda(t)) + \overline{H}(p)\big)\, dt + p \cdot z_\lambda(\tau_\lambda) + \chi_{\mathrm{per},p}(z_\lambda(\tau_\lambda))$$
$$= A + p \cdot y$$

where, for brevity, we have set

$$A = \int_0^{\tau_\lambda} \Big(\ell(z_\lambda(t), a_\lambda(t)) + p \cdot f_{\mathrm{per}}(z_\lambda(t), a_\lambda(t)) + \overline{H}(p)\Big) dt + \chi_{\mathrm{per},p}(z_\lambda(\tau_\lambda)).$$

Observing that, from Lemma 2.5, the measure of $\{t : z^\lambda(t) \in B_{R_0}(0)\}$ is bounded, and that ℓ coincides with ℓ_{per} outside $B_{R_0}(0)$, we first obtain that, for a positive constant c_1 depending only on $\overline{H}(p)$,

$$A \leq \int_0^{\tau_\lambda} \Big(\ell_{\mathrm{per}}(z_\lambda(t), a_\lambda(t)) + p \cdot f_{\mathrm{per}}(z_\lambda(t), a_\lambda(t)) + \overline{H}(p)\Big) dt + \chi_{\mathrm{per},p}(z_\lambda(\tau_\lambda)) + c_1$$
$$= \int_0^{\tau_\lambda} e^{-\lambda t} \Big(\ell_{\mathrm{per}}(z_\lambda(t), a_\lambda(t)) + p \cdot f_{\mathrm{per}}(z_\lambda(t), a_\lambda(t))\Big) dt + e^{-\lambda \tau_\lambda} \chi_{\mathrm{per},p}(z_\lambda(\tau_\lambda))$$
$$+ \tau_\lambda \overline{H}(p) + c_1 + B,$$

where

$$B = (1 - e^{-\lambda \tau_\lambda})\chi_{\mathrm{per},p}(z_\lambda(\tau_\lambda))$$
$$+ \int_0^{\tau_\lambda} (1 - e^{-\lambda t})\Big(\ell_{\mathrm{per}}(z_\lambda(t), a_\lambda(t)) + p \cdot f_{\mathrm{per}}(z_\lambda(t), a_\lambda(t))\Big) dt.$$

But τ_λ is bounded uniformly in λ (it does depend on R), so $\lim_{\lambda \to 0}(1 - e^{-\lambda \tau_\lambda}) = 0$, and we see that for λ smaller than a constant depending on R, $|B| \leq c_1$, hence

$$A \leq \int_0^{\tau_\lambda} e^{-\lambda t} \Big(\ell_{\mathrm{per}}(z_\lambda(t), a_\lambda(t)) + p \cdot f_{\mathrm{per}}(z_\lambda(t), a_\lambda(t))\Big) dt + e^{-\lambda \tau_\lambda} \chi_{\mathrm{per},p}(z_\lambda(\tau_\lambda))$$
$$+ \tau_\lambda \overline{H}(p) + 2c_1$$

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Using (2.36), the right hand side in the latter inequality can be written

$$\chi_{\mathrm{per},p}^{\lambda}(y) - e^{-\lambda\tau_\lambda}\chi_{\mathrm{per},p}^{\lambda}(z_\lambda(\tau_\lambda)) + e^{-\lambda\tau_\lambda}\chi_{\mathrm{per},p}(z_\lambda(\tau_\lambda)) + \tau_\lambda\overline{H}(p) + 2c_1.$$

Next, since $\lim_{\lambda\to 0}\lambda\langle\chi_{\mathrm{per},p}^{\lambda}\rangle = -\overline{H}(p)$, we get that for λ small enough,

$$
\begin{aligned}
A &\leqslant \chi_{\mathrm{per},p}^{\lambda}(y) - e^{-\lambda\tau_\lambda}\chi_{\mathrm{per},p}^{\lambda}(z_\lambda(\tau_\lambda)) + e^{-\lambda\tau_\lambda}\chi_{\mathrm{per},p}(z_\lambda(\tau_\lambda)) - \lambda\tau_\lambda\langle\chi_{\mathrm{per},p}^{\lambda}\rangle + 3c_1 \\
&= \chi_{\mathrm{per},p}^{\lambda}(y) - \langle\chi_{\mathrm{per},p}^{\lambda}\rangle + \left(-e^{-\lambda\tau_\lambda} + 1 - \lambda\tau_\lambda\right)\langle\chi_{\mathrm{per},p}^{\lambda}\rangle \\
&\quad - e^{-\lambda\tau_\lambda}\left(\chi_{\mathrm{per},p}^{\lambda}(z_\lambda(\tau_\lambda)) - \langle\chi_{\mathrm{per},p}^{\lambda}\rangle - \chi_{\mathrm{per},p}(z_\lambda(\tau_\lambda))\right) + 3c_1
\end{aligned}
$$

The uniform convergence of $\chi_{\mathrm{per},p}^{\lambda} - \langle\chi_{\mathrm{per},p}^{\lambda}\rangle$ to $\chi_{\mathrm{per},p}$ then implies that for λ small enough

$$A \leqslant \chi_{\mathrm{per},p}(y) + 4c_1.$$

We have proven that for all R,

$$(2.51) \qquad \chi_{p,\tilde{p}}^{R}(y) \leqslant \chi_{\mathrm{per},p}(y) + p\cdot y + 4c_1.$$

Similarly, using Lemma 2.5, we see that there exists a positive constant $\tilde{c}_1$ such that

$$(2.52) \qquad \chi_{p,\tilde{p}}^{R}(y) \leqslant \chi_{\mathrm{per},\tilde{p}}(y) + \tilde{p}\cdot y + 4\tilde{c}_1.$$

This concludes our fourth step. We deduce from (2.51) and (2.52) that there exists a constant c_2 independent of R, (which depends only on $\overline{H}(p)$) such that for all $y \in \overline{B}_R(0)$,

$$(2.53) \qquad \sigma(y) \leqslant \chi_{p,\tilde{p}}^{R}(y) \leqslant \min\left(\chi_{\mathrm{per},p}(y) + p\cdot y, \chi_{\mathrm{per},\tilde{p}}(y) + \tilde{p}\cdot y\right) + c_2.$$

From (2.53) and the fact that $\chi_{p,\tilde{p}}^{R}$ is locally Lipschitz with a Lipschitz constant independent of R, we can find a sequence $(R_n)_n$ which tends to $+\infty$ such that $\chi_{p,\tilde{p}}^{R_n}$ tends to some $\chi_{p,\tilde{p}}$ locally uniformly, and by passing to the limit in (2.41), (2.53) (using (2.47)), we obtain that $\chi_{p,\tilde{p}}$ satisfies (2.39)-(2.40). □

PROPOSITION 2.7. *If there exists $\phi \in C^1(\mathbb{R}^d)$ such that 0 is a local minimizer of $\underline{u} - \phi$ and $\overline{H}(D\phi(0)) > E$, then*

$$(2.54) \qquad \alpha\underline{u}(0) + \overline{H}(D\phi(0)) \geqslant 0.$$

Proof. We can always assume that $\phi(0) = \underline{u}(0)$ and that $\underline{u} - \phi$ has a strict local minimum at the origin. For brevity, let us set $p = D\phi(0)$. Because $\overline{H}(p) > E$, we know that $p \neq p_0$ and we can apply Proposition 2.6, with e and $\tilde{p}$ defined as above. Suppose by contradiction that

$$(2.55) \qquad \alpha\underline{u}(0) + \overline{H}(p) = -\theta < 0,$$

and consider the perturbed test-function

$$\phi_\varepsilon(x) = \phi(0) + \varepsilon\chi_{p,\tilde{p}}(\frac{x}{\varepsilon}),$$

where $\chi_{p,\tilde{p}}$ is the function appearing in Proposition 2.6.

The definition of ϕ_ε, (2.40) and (2.55) imply that $\alpha\phi_\varepsilon + H(\frac{x}{\varepsilon}, D\phi_\varepsilon) \leqslant \alpha\varepsilon\chi_{p,\tilde{p}}(\frac{x}{\varepsilon}) - \theta$ in the sense of viscosity in $\mathbb{R}^d$.

From this and (2.39), we deduce that there exists $r_0 > 0$ and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and $0 < r < r_0$, ϕ_ε is a viscosity subsolution of

$$(2.56) \qquad \alpha\phi_\varepsilon + H(\frac{x}{\varepsilon}, D\phi_\varepsilon) \leqslant -\frac{\theta}{2} \quad \text{in } B_r(0).$$

On the other hand, since 0 is a strict local minimizer of $\underline{u} - \phi$, there exists $r_1 > 0$ and a function $k : (0, r_1] \to (0, 1]$, such that $\lim_{r \to 0} k(r) = 0$ and for any $r \in (0, r_1]$,

$$\phi(x) \leqslant \underline{u}(x) - k(r) \quad \text{on } \partial B_r(0).$$

From (2.39), we know that for $x \neq 0$, $\varepsilon\chi_{p,\tilde{p}}(\frac{x}{\varepsilon}) = \min(p \cdot x, \tilde{p} \cdot x) + |x|o_{\varepsilon \to 0}(1)$.

Using (2.2), this implies that first fixing $r > 0$ small enough, we have for ε small enough,

$$(2.57) \qquad \phi_\varepsilon(x) < u_\varepsilon(x) - \frac{k(r)}{2} \quad \text{on } \partial B_r(0).$$

From (2.56) and (2.57) and since u_ε is a viscosity solution of (1.7), the comparison principle yields that it is possible to choose $r > 0$ such that, for ε small enough, $\phi_\varepsilon(x) \leqslant u_\varepsilon(x) - \frac{k(r)}{2}$ in $\overline{B_r(0)}$. By choosing a sequence ε_n such that $u_{\varepsilon_n}(0)$ tends to $\underline{u}(0)$, we deduce that $\phi(0) + \frac{k(r)}{2} \leqslant \underline{u}(0)$, the desired contradiction. $\qquad\square$

PROPOSITION 2.8. *If there exists $\phi \in C^1(\mathbb{R}^d)$ such that 0 is a local minimizer of $\underline{u} - \phi$ and $\overline{H}(D\phi(0)) \leqslant E$, then*

$$(2.58) \qquad \alpha\underline{u}(0) + E \geqslant 0.$$

Proof. As above we may assume without loss of generality that $\phi(0) = \underline{u}(0)$ and that 0 is a strict local minimizer of $\underline{u} - \phi$. Let us again set $p = D\phi(0)$.

If $\overline{H}(p) > \overline{H}(p_0) = \min_q \overline{H}(q)$, we set e and $\tilde{p}$ as above. For $\eta, \tilde{\eta} > 0$, the function $x \mapsto \underline{u}(x) - \underline{u}(0) - \min((p_0 + (1 + \eta)(p - p_0)) \cdot x, (p_0 + (1 + \tilde{\eta})(\tilde{p} - p_0)) \cdot x)$ has a local minimum at at $x = 0$. Given $\varepsilon > 0$, we may choose η and $\tilde{\eta}$ such that $\overline{H}(p_0 + (1 + \eta)(p - p_0)) = \overline{H}(p_0 + (1 + \tilde{\eta})(\tilde{p} - p_0)) = E + \varepsilon$. The same argument as in the proof of Proposition 2.7 with $\chi_{p_0 + (1+\eta)(p-p_0), p_0 + (1+\tilde{\eta})(\tilde{p}-p_0)}$, the corrected version of the piecewise affine function $y \mapsto \min((p_0 + (1 + \eta)(p - p_0)) \cdot y, (p_0 + (1 + \tilde{\eta})(\tilde{p} - p_0)) \cdot y)$, yields $\alpha\underline{u}(0) + \overline{H}(p_0 + (1 + \tilde{\eta})(\tilde{p} - p_0)) \geqslant 0$, that is

$$\alpha\underline{u}(0) + E + \varepsilon \geqslant 0.$$

Letting ε tend to 0, we obtain (2.58).

On the other hand, if $\overline{H}(p) = \min_q \overline{H}(q) = \overline{H}(p_0)$, we then consider two cases:

1. If $p \neq p_0$, then from the convexity of $= \overline{H}$, the set $\{t \in \mathbb{R}, \overline{H}(p_0 + t(p - p_0)) = \overline{H}(p_0)\}$ is an interval $[\beta, \gamma]$, with $\beta \leqslant 0$ and $\gamma \geqslant 1$. Let e be the unique unit vector aligned with $p - p_0$ such that $t \mapsto \overline{H}(p_0 + \gamma(p - p_0) + te)$ is non increasing near $t = 0$. For $\eta, \tilde{\eta} > 0$, the function $x \mapsto \underline{u}(0) + \min((p_0 + (\gamma + \eta)(p - p_0)) \cdot x, (p_0 + (\beta - \tilde{\eta}))(p - p_0) \cdot x)$ touches $\underline{u}$ from below at $x = 0$. Given $\varepsilon > 0$, we may choose η and $\tilde{\eta}$ such that $\overline{H}(p_0 + (\gamma + \eta)(p - p_0)) = \overline{H}(p_0 + (\beta - \tilde{\eta})(p - p_0)) = E + \varepsilon$. The same argument as in the proof of Proposition 2.7 with

   $\chi_{p_0 + (\gamma+\eta)(p-p_0), p_0 + (\beta-\tilde{\eta})(p-p_0)}$, the corrected version of the piecewise affine function $y \mapsto \min((p_0 + (\gamma + \eta)(p - p_0)) \cdot y, (p_0 + (\beta - \tilde{\eta}))(p - p_0) \cdot y)$, yields that $\alpha\underline{u}(0) + E + \varepsilon \geqslant 0$. Then (2.58) is obtained by letting ε tend to 0.

2. If $p = p_0$, we may choose any non zero vector q_ε such that $\overline{H}(q_\varepsilon) = E + \varepsilon$: let e be the unique unit vector aligned with $q_\varepsilon - p_0$ such that $t \mapsto \overline{H}(q_\varepsilon + te)$ is non increasing near $t = 0$ and let $\tilde{q}_\varepsilon$ be the unique vector different from q_ε such that $\tilde{q}_\varepsilon - p_0$ aligned with $q_\varepsilon - p_0$, and $\overline{H}(\tilde{q}_\varepsilon) = E + \varepsilon$. The function $x \mapsto \underline{u}(0) + \min(q_\varepsilon \cdot x, \tilde{q}_\varepsilon \cdot x)$ touches $\underline{u}$ from below at $x = 0$, which yields $\alpha \underline{u}(0) + E + \varepsilon \geqslant 0$ as in the proof of Proposition 2.7, using the corrector $\chi_{q_\varepsilon, \tilde{q}_\varepsilon}$. Letting ε tend to 0, we obtain (2.58). □

We have proved that $\underline{u}$ satisfies (1.14). In order to prove Theorem 1.1, there only remains to establish that $\underline{u} = \overline{u}$.

2.4. End of the proof of Theorem 1.1. We consider two cases:

Case 1. Suppose first that there does not exist any function $\phi \in C^1(\mathbb{R}^d)$ such that $\underline{u} - \phi$ has a local minimum at the origin and that $\overline{H}(D\phi(0)) \leqslant E$. Hence, from Proposition 2.7, for any function $\phi \in C^1(\mathbb{R}^d)$ such that $\underline{u} - \phi$ has a local minimum at the origin,

$$\alpha \underline{u}(0) + \overline{H}(D\phi(0)) \geqslant 0.$$

We deduce from this and Lemma 2.1 that $\underline{u}$ is a bounded supersolution of (1.8) in $\mathbb{R}^d$. On the other hand, from Proposition 2.4, $\overline{u}$ is a bounded subsolution of (1.8) in $\mathbb{R}^d$. Hence, from the comparison principle, [6, Th. 2.12, Chapter III, page 107], we deduce that for any $x \in \mathbb{R}^d$, $\overline{u}(x) \leqslant \underline{u}(x)$. From the definitions (2.1)-(2.2) of $\overline{u}$ and $\underline{u}$, we deduce that $\underline{u} = \overline{u}$.

Case 2. If, on the contrary, there exists $\phi \in C^1(\mathbb{R}^d)$ such that $\underline{u} - \phi$ has a local minimum at the origin and that $\overline{H}(D\phi(0)) \leqslant E$, Proposition 2.8 yields that $\alpha \underline{u}(0) + E \geqslant 0$. On the other hand, from Proposition 2.3, we know that $\alpha \overline{u}(0) + E \leqslant 0$. From the definitions (2.1)-(2.2) of $\overline{u}$ and $\underline{u}$, we deduce that $\underline{u}(0) = \overline{u}(0)$. This allows us to apply the comparison principle in $\mathbb{R}^d \backslash \{0\}$, [6, Remark 2.14, Chapter III, page 109], because $\underline{u}$ and $\overline{u}$ are respectively a bounded supersolution and a bounded subsolution of (1.11) in $\mathbb{R}^d \backslash \{0\}$. Therefore, for any $x \in \mathbb{R}^d$, $\overline{u}(x) \leqslant \underline{u}(x)$, and finally, $\overline{u} = \underline{u}$.

In both cases, we deduce that the whole family u_ε converges locally uniformly to a solution of (1.11) through (1.14), which concludes the proof of Theorem 1.1.

3. Miscellaneous comments and extensions. This section contains various comments about, and illustrations of the general result proven above.

3.1. A simple and explicit one-dimensional example. We begin by providing in this Section 3.1 some illustrations of the results in a simple one-dimensional setting.

Upon considering this simple situation, we expect to illustrate as sharply as possible, the homogenization process described by Theorem 1.1. We also intend to (a) make explicit in those simple settings the key quantities involved in Theorem 1.1, in particular the constant E and the Dirichlet condition (1.12)-(1.13)-(1.14) and (b) provide some additional light on some qualitative aspects of the problem that are best exposed using simple examples and that will be generalized in higher dimensions later on.

We will exploit the peculiarity of the considered one-dimensional setting. In the course of our arguments, we will make several simplifying assumptions about the type of Hamiltonian H and the shape of the defect(s) that we consider: see for instance (3.2), (3.7), (3.31) below, etc. Even if we expect that several of our arguments may carry over when some of these simplifying assumptions are relaxed, it might be the case that not everything can be generalized. We do not claim that the

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techniques we are employing below, which are specific to the particular setting, carry over to higher dimensions and other Hamiltonians. We only intend to illustrate, on some simple enough cases, some *phenomena* that we find interesting, with no sake of generality whatsoever.

We consider in this section what is, to some extent, the *simplest possible setting* where the homogenization process we have established in the general setting occurs. We pose the Hamilton-Jacobi equation on the real line $\mathbb{R}$. We pick a *separate* Hamiltonian H (by this term, we mean, throughout this section, an Hamiltonian that depends *separately* of x and p), with a potential part that reads as the sum

$$(3.1) \qquad \qquad \ell = \ell_{\mathrm{per}} + \ell_0,$$

of a periodic function ℓ_{per} and a local defect ℓ_0. Both ℓ_{per} and ℓ_0 are assumed smooth. We also assume that ℓ_0 is compactly supported, with, say,

$$(3.2) \qquad \qquad \operatorname{Supp} \ell_0 \subset [-\frac{1}{2}, \frac{1}{2}].$$

In order to observe an actual perturbation by ℓ_0 of the homogenized equation, we further assume

$$(3.3) \qquad \qquad \inf_{\mathbb{R}}(\ell_{\mathrm{per}} + \ell_0) < \inf_{\mathbb{R}} \ell_{\mathrm{per}} .$$

The role of this condition will become clear below but it is already intuitive that its spirit is to make sure that the defect indeed locally *lower* the periodic environment (or put differently in the optimal control interpretation, diminishes the periodic cost) and therefore indeed shows up in the homogenized limit. Condition (3.3) is our only *actual* assumption, all the other assumptions above and below (smoothness of the data, compact support of ℓ_0, and others to come) being only used to simplify the algebraic expressions manipulated and to spare the reader unnecessary technicalities. As for the kinetic part, we choose $|p|$, that is a simple, commonly used convex nonlinearity. In our notation of Section 1.1, our choices of course correspond to $A = [-1, 1]$ and $f(x, a) = f_{\mathrm{per}}(x, a) = a$. Put differently, there is neither oscillation nor defect in this kinetic part. We thus manipulate

$$(3.4) \qquad \qquad H(x, p) = |p| - \ell_{\mathrm{per}}(x) - \ell_0(x)$$

throughout this Section 3.1.

Because this positive constant is irrelevant in our arguments and can be easily reinstated in our results, we choose $\alpha = 1$ in (1.7) and keep this value throughout the section.

3.1.1. A downward defect inserted in a flat environment. To start with, let us temporarily further simplify the above model. We take $\ell_{\mathrm{per}} = 0$ and recall the trivial example already introduced in [11, Section 6]. The Hamilton-Jacobi equation considered then reads as

$$(3.5) \qquad \qquad u_\varepsilon(x) + \left|(u_\varepsilon)'(x)\right| = \ell_0\left(\frac{x}{\varepsilon}\right),$$

and should be thought as the perturbation, by the local potential ℓ_0, of the periodic equation

$$(3.6) \qquad \qquad u(x) + \left|u'(x)\right| = 0.$$

As is well known, the only bounded C^1 solution of (3.6) is $u = 0$. We may likewise make explicit the bounded solution u_ε to (3.5). For this purpose, and just to keep algebraic manipulations and expressions simple, we assume, in addition to (3.2)-(3.3), that ℓ_0 satisfies

(3.7) $\ell_0(0) = \inf_{x\in\mathbb{R}} \ell_0(x)$ and $\begin{cases} (\ell_0)'(x) < 0, & \forall x \in\,]-\frac{1}{2}, 0[\, \cap \operatorname{Supp}\ell_0, \\[2mm] (\ell_0)'(x) > 0, & \forall x \in\,]0, \frac{1}{2}[\, \cap \operatorname{Supp}\ell_0. \end{cases}$

Precisely under such conditions, it is easy to verify that the unique C^1 solution to (3.5) reads as

(3.8) $u_\varepsilon(x) = \begin{cases} e^x \left(\ell_0(0) + \displaystyle\int_x^0 e^{-t} \ell_0\left(\frac{t}{\varepsilon}\right) dt \right) & \text{for } x < 0, \\[4mm] e^{-x} \left(\ell_0(0) + \displaystyle\int_0^x e^{t} \ell_0\left(\frac{t}{\varepsilon}\right) dt \right) & \text{for } x > 0, \end{cases}$

Indeed, a simple computation allows to check that, for instance if $x > 0$, the function

$$g(x) = e^x\, b(x) - b(0) - \int_0^x e^t\, b(t)\, dt$$

is such that $g(0) = 0$ and $g'(x) = e^x\, b'(x)$. Hence, if b reaches its global minimum at $x = 0$, is non-increasing for $x < 0$ and non-decreasing for $x > 0$, then the function g satisfies $g(0) = 0$, is non-increasing for $x < 0$ and non-decreasing for $x > 0$. This clearly implies that $g \geqslant 0$ on $\mathbb{R}$. Applying this to $b = \ell_0\left(\frac{\cdot}{\varepsilon}\right)$ we deduce that

$$e^x \ell_0\left(\frac{x}{\varepsilon}\right) - \ell_0(0) - \int_0^x e^t \ell_0\left(\frac{t}{\varepsilon}\right) dt \geqslant 0.$$

This means that, u_ε being defined by (3.8), $(u_\varepsilon)'(x)$ is non-negative for $x > 0$, hence $(u_\varepsilon)'(x) = |(u_\varepsilon)'(x)|$. In addition, since $u_\varepsilon(x) + (u_\varepsilon)'(x) = \ell_0\left(\frac{x}{\varepsilon}\right)$, we obtain equation (3.5) for $x > 0$. A similar argument holds for $x < 0$.

Using the explicit expression (3.8) of the solution to (3.5), we now easily identify its limit u as $\varepsilon \to 0$. Since the function ℓ_0 has compact support, the sequence of functions $\ell_0\left(\frac{\cdot}{\varepsilon}\right)$ converges strongly to zero in L^p for any $p < \infty$. Hence, in any such space, the locally uniform limit u of u_ε, is

(3.9) $u(x) = \ell_0(0)\, e^{-|x|}.$

Put differently, the homogenized limit u is a bounded (in fact, vanishing at infinity) solution to

(3.10) $\begin{cases} u(x) + |u'(x)| = 0 & \text{for } x \neq 0, \\[2mm] u(0) = \ell_0(0). \end{cases}$

It is enlightening to reconcile this particular result with the general result stated in Theorem 1.1. Since the periodic Hamiltonian identically vanishes here, its homogenized limit is evidently $\overline{H} = 0$. The first line of (3.10) thus agrees with (1.11) (for the choice of constant $\alpha = 1$ of course). Far more interesting is the Dirichlet

condition $u(0) = \ell_0(0)$ posed at the origin. In the particular setting of the present section, (1.12) and (1.14) respectively read as $u(0) \leqslant -E$ and $u(0) + \max(E, 0) \geqslant 0$. The effective Dirichlet datum E is itself obtained, in whole generality, as the limit, first as $\lambda \to 0$ and next as $R \to +\infty$, of $-\lambda w^{\lambda,R}$, where $w^{\lambda,R}$ is the solution to (2.3)-(2.4). In our setting again, $w^{\lambda,R}$ is actually indeed independent of R and is easily obtained from the explicit expression (3.8). Using a rescaling argument (setting $w^{\lambda,R}(x) = u_{\varepsilon=1}(\lambda x)$ for the rescaled defect $\lambda^{-1}\ell_0(\lambda^{-1}x)$), we realize that

$$(3.11) \qquad w^{\lambda,R}(x) = \begin{cases} \lambda^{-1} e^{\lambda x} \left(\ell_0(0) + \displaystyle\int_{\lambda x}^{0} e^{-t}\, \ell_0\left(\lambda^{-1}t\right)\, dt \right) & \text{for } x < 0, \\[4mm] \lambda^{-1} e^{-\lambda x} \left(\ell_0(0) + \displaystyle\int_{0}^{\lambda x} e^{t}\, \ell_0\left(\lambda^{-1}t\right)\, dt \right) & \text{for } x > 0, \end{cases}$$

(actually independent of R) is the bounded C^1 solution to the equation

$$(3.12) \qquad \lambda\, w^{\lambda,R} + \left|(w^{\lambda,R})'\right| = \ell_0 \quad \text{in } \mathbb{R}.$$

Taking the limit $\lambda \to 0$ of $-\lambda\, w^{\lambda,R}$ readily yields the effective Dirichlet condition $E = -\ell_0(0)$. But, since we have assumed that $\ell_0(0) < 0$, E is positive. Thus $u(0) \leqslant -E$ and $u(0) + \max(E, 0) \geqslant 0$ combine with one another into

$$(3.13) \qquad u(0) = -E = \ell_0(0).$$

We have thus recovered in (3.10) the general limit (1.11)-(1.12)-(1.13)-(1.14) stated in Theorem 1.1.

In addition, taking the limit, as $\lambda \to 0$ and say for $x > 0$, of

$$w^{\lambda,R}(x) - w^{\lambda,R}(0) = \lambda^{-1}\left((e^{-\lambda x} - 1)\, \ell_0(0) + e^{-\lambda x} \int_0^{\lambda x} e^t\, \ell_0\left(\lambda^{-1}t\right)\, dt \right),$$

yields

$$w^{\lambda,R}(x) - w^{\lambda,R}(0) \to -\ell_0(0)\, x + \int_0^x \ell_0(t)\, dt,$$

and we thus find that, as predicted by (2.15) in Proposition 2.2, $\lim_{x\to+\infty} w(x) = +\infty$. A similar argument using the first line of (3.11) confirms the same limit in $-\infty$.

REMARK 3.1. *We note in passing that, had we assumed, say, that the local defect ℓ_0 is everywhere nonnegative, satisfies*

$$(3.14) \qquad \ell_0(0) = \sup_{x\in\mathbb{R}} \ell_0(x) \quad and \quad \begin{cases} (\ell_0)'(x) > 0, & \forall x \in\,]-1, 0[\, \cap\, Supp\, \ell_0, \\[3mm] (\ell_0)'(x) < 0, & \forall x \in\,]0, 1[\, \cap\, Supp\, \ell_0, \end{cases}$$

instead of (3.7) (and assumed also, again a technicality, that ℓ_0 is even), then an easy adaptation of the above algebraic expressions shows that, mutatis mutandis,

$$(3.15) \qquad u_\varepsilon(x) = \begin{cases} e^{-x} \displaystyle\int_{-\infty}^{x} e^t\, \ell_0\left(\frac{t}{\varepsilon}\right)\, dt & for\ x < 0, \\[5mm] e^{x} \displaystyle\int_{x}^{+\infty} e^{-t}\, \ell_0\left(\frac{t}{\varepsilon}\right)\, dt & for\ x > 0. \end{cases}$$

This explicit calculation thus shows that, instead of solving (3.10), *the homogenized limit u vanishes over the real line, just like it is the case for the periodic, null, Hamiltonian. As briefly mentioned in the introduction, the presence of this "upward" defect $\ell_0 \geqslant 0$ only affects the next order term. This relates to the "other regime" considered in the works* [26, 15].

REMARK 3.2. *We also store for future use (see Remark 3.3 and Section 3.2.1) the observation that the homogenized limit u given by* (3.9) *does not have compact support, and therefore only agrees at infinity $x = \pm\infty$ with the (here, trivial) homogenized limit of the periodic case, solution to* (3.6).

3.1.2. Periodic unperturbed environment: the homogenized limit. We now reinstate a non trivial (that is, non constant) periodic potential ℓ_{per} in (3.1), which, without loss of generality, we assume of period 1. We also assume that there exists only one point per unit interval where ℓ_{per} reaches its minimum : $\ell_{\mathrm{per}}(x_0) = \inf_{\mathbb{R}} \ell_{\mathrm{per}}$. This is again just for simplicity. We recall that we additionally assume (3.2)-(3.3) and the separated form (3.4) of the Hamiltonian H. Before we insert any defect ℓ_0 in the periodic environment described by ℓ_{per}, we need to lay some groundwork and make explicit the homogenized limit in the absence of defect.

Homogenized equation. For convenience, we now recall some basic facts.

We consider the one-dimensional equation

$$(3.16) \qquad u_{\mathrm{per},\varepsilon}(x) + \left|(u_{\mathrm{per},\varepsilon})'(x)\right| = \ell_{\mathrm{per}}\left(\frac{x}{\varepsilon}\right),$$

(obviously a particular case of (1.8)) where the subscript per in $u_{\mathrm{per},\varepsilon}$ refers to the fact that the right-hand side is only the *periodic* cost ℓ_{per}. Actually, using elementary facts from the theory of viscosity solutions, $u_{\mathrm{per},\varepsilon}$ is indeed periodic (of period ε); we will use this property below, but at this time this is irrelevant.

The homogenized Hamiltonian $\overline{H}$ arising from (3.16) is characterized, for any $p \in \mathbb{R}$, as the unique real number $\overline{H}(p)$ such that there exists a periodic corrector $\chi_{\mathrm{per},p}$, viscosity solution to

$$(3.17) \qquad \left|p + (\chi_{\mathrm{per},p})'(y)\right| = \overline{H}(p) + \ell_{\mathrm{per}}(y) \quad \text{in } \mathbb{R},$$

which is a particular case of (1.9). Mimicking the explicit calculations performed in [25] that address the case of the quadratic Hamiltonian $|p|^2 - \ell_{\mathrm{per}}(y)$ instead of $|p| - \ell_{\mathrm{per}}(y)$, it is easy to identify the periodic solution $\chi_{\mathrm{per},p}$ and the homogenized Hamiltonian

$$(3.18) \qquad \overline{H}(p) = \begin{cases} -\inf_{\mathbb{R}} \ell_{\mathrm{per}} & \text{for} \quad |p| \leqslant \langle \ell_{\mathrm{per}} \rangle - \inf_{\mathbb{R}} \ell_{\mathrm{per}}\,, \\[2ex] |p| - \langle \ell_{\mathrm{per}} \rangle & \text{for} \quad |p| \geqslant \langle \ell_{\mathrm{per}} \rangle - \inf_{\mathbb{R}} \ell_{\mathrm{per}}. \end{cases}$$

It follows that the (periodic) homogenized equation, defined by (1.8) in whole generality, here reads as

$$u + \overline{H}(u') = 0\,,$$

for the particular Hamiltonian (3.18). We immediately notice that its unique BUC viscosity solution is the constant function

$$u = \inf_{\mathbb{R}} \ell_{\mathrm{per}}\,,$$

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which we henceforth denote by u_{per} to distinguish it from the homogenized limit u in the presence of defect, which we are going to manipulate shortly.

An alternative "direct" proof. Our next step is to remark that in the present setting, besides being a consequence of the general periodic homogenization theory, the limit $u_{\mathrm{per},\varepsilon} \to \inf_{\mathbb{R}} \ell_{\mathrm{per}}$ may alternatively and independently be obtained from the original equation (3.16), as soon as we have some elementary information on the solution $u_{\mathrm{per},\varepsilon}$. The interest of following this alternative route is to better understand the phenomena at play and prepare the ground for the homogenization limit in the presence of a defect that will be addressed in Section 3.1.3 below.

It is immediate to deduce from (3.16) that its solution $u_{\mathrm{per},\varepsilon}$ consists of the combination of two different branches, respectively solving

$$(3.19) \qquad u_{\mathrm{per},\varepsilon}(x) + (u_{\mathrm{per},\varepsilon})'(x) = \ell_{\mathrm{per}}\left(\frac{x}{\varepsilon}\right) \qquad \text{when } (u_{\mathrm{per},\varepsilon})'(x) \geqslant 0$$

and

$$(3.20) \qquad u_{\mathrm{per},\varepsilon}(x) - (u_{\mathrm{per},\varepsilon})'(x) = \ell_{\mathrm{per}}\left(\frac{x}{\varepsilon}\right) \qquad \text{when } (u_{\mathrm{per},\varepsilon})'(x) \leqslant 0 .$$

We now introduce

$$(3.21) \qquad g_{\mathrm{per},\varepsilon}(x) = e^{-x} \int_{-\infty}^{x} e^{t}\, \ell_{\mathrm{per}}\left(\frac{t}{\varepsilon}\right) dt,$$

which is not an unexpected function given the calculations in Section 3.1.1 above. It is evidently a periodic function, which satisfies $g_{\mathrm{per},\varepsilon} + (g_{\mathrm{per},\varepsilon})' = \ell_{\mathrm{per}}\left(\frac{\cdot}{\varepsilon}\right)$ and thus $\langle g_{\mathrm{per},\varepsilon} \rangle = \langle \ell_{\mathrm{per}} \rangle$. We deduce that in the branch described by (3.19), $u_{\mathrm{per},\varepsilon}(x) = g_{\mathrm{per},\varepsilon}(x) + \alpha\, e^{-x}$ for some constant α (depending on ε but this is irrelevant).

Should the branch (3.19) extend to $+\infty$, we would therefore obtain, using periodicity and shifting to $+\infty$, that $u_{\mathrm{per},\varepsilon}(x) = g_{\mathrm{per},\varepsilon}(x)$ for all x in that branch. Since $\langle g_{\mathrm{per},\varepsilon} \rangle = \langle \ell_{\mathrm{per}} \rangle$ while, in view of (3.16), $u_{\mathrm{per},\varepsilon} \leqslant \ell_{\mathrm{per}}\left(\frac{\cdot}{\varepsilon}\right)$ everywhere, this implies $g_{\mathrm{per},\varepsilon} = \ell_{\mathrm{per}}\left(\frac{\cdot}{\varepsilon}\right)$. Given the differential equation satisfied by $g_{\mathrm{per},\varepsilon}$, this may only hold when ℓ_{per} is constant. We reach a contradiction with the assumption made on ℓ_{per}.

Similarly to (3.21), we introduce the function $h_{\mathrm{per},\varepsilon}(x) = e^{x} \int_{x}^{\infty} e^{-t} \ell_{\mathrm{per}}\left(\frac{t}{\varepsilon}\right) dt$, and get that in the branch (3.20), $u_{\mathrm{per},\varepsilon}(x) = h_{\mathrm{per},\varepsilon}(x) + \beta\, e^{x}$. It immediately follows that (3.20) cannot extend to $+\infty$ either, since the asymptotic blow-up implied by the term e^{x} would contradict the boundedness of the solution.

A symmetric argument allows to exclude that the branches extend to $-\infty$. We conclude that the solution $u_{\mathrm{per},\varepsilon}$ consists of an effective, infinite alternation of the two branches (3.19) and (3.20). In our particular periodic setting, we *could* have obtained this fact simply using the uniqueness, thus the periodicity, of the viscosity solution to (3.16). But the specific argument we have developed will be useful later, in the presence of a defect.

We next remark that, because of the generic properties of viscosity solutions, a transition, at some $\overline{x}$, from the branch (3.20) at the left of $\overline{x}$ to the branch (3.19) at its right, may only occur when $(u_{\mathrm{per},\varepsilon})'$ is continuous at $\overline{x}$, nonpositive before and

nonnegative after $\overline{x}$. Thus $(u_{\mathrm{per},\varepsilon})'(\overline{x}) = 0$ and $\overline{x}$ is a local minimizer of $u_{\mathrm{per},\varepsilon}$. Using the equation, $u_{\mathrm{per},\varepsilon}(\overline{x}) = \ell_{\mathrm{per}}(\overline{x})$. Since we know that $u_{\mathrm{per},\varepsilon} \leqslant \ell_{\mathrm{per}}$ everywhere, the three facts altogether imply that ℓ_{per} reaches its minimum at $\overline{x}$. For simplicity, we have assumed that such a minimizer is unique per period. So we exactly know that, per period, there is one transition (3.20)-(3.19) (at the minimizer $\overline{x} = \arg\min \ell_{\mathrm{per}}$) and, thus also, one transition (3.19)-(3.20) (thus necessarily *above* the level $\min_{\mathbb{R}} \ell_{\mathrm{per}}$).

An interesting corollary of the previous observation is the following. Since $u_{\mathrm{per},\varepsilon} \geqslant \inf_{\mathbb{R}} \ell_{\mathrm{per}}$ everywhere, $|(u_{\mathrm{per},\varepsilon})'| = \ell_{\mathrm{per}} - u_{\mathrm{per},\varepsilon}$ is thus bounded independently of ε. It follows from these two facts together with the periodicity of $u_{\mathrm{per},\varepsilon}$ (with period ε) that, necessarily, $u_{\mathrm{per},\varepsilon} \to \inf_{\mathbb{R}} \ell_{\mathrm{per}}$ as ε vanishes. In dimension 1, this provides us with an independent and alternative proof of the homogenization limit, as announced above.

3.1.3. A downward defect in this periodic environment. We now insert the defect ℓ_0. The equation under consideration is

$$(3.22) \qquad u_\varepsilon(x) + |(u_\varepsilon)'(x)| = \ell_{\mathrm{per}}\left(\frac{x}{\varepsilon}\right) + \ell_0\left(\frac{x}{\varepsilon}\right).$$

Using the exact same argument as above in (3.19)-(3.20), the two auxiliary functions $g_{\mathrm{per},\varepsilon}$ and $h_{\mathrm{per},\varepsilon}$, along with the functions $g_{0,\varepsilon}(x) = e^{-x} \int_{-\infty}^{x} e^t \ell_0\left(\frac{t}{\varepsilon}\right) dt$, $h_{0,\varepsilon}(x) = e^x \int_{x}^{\infty} e^{-t} \ell_0\left(\frac{t}{\varepsilon}\right) dt$, we may similarly conclude that no branch can extend to $\pm\infty$. Recall indeed that ℓ_0 vanishes at infinity so does not modify the argument performed in the periodic case above.

We can thus claim that u_ε is an infinite alternation of the two branches. Again similarly as above, we know that the junction (3.20)-(3.19) (in this order from left to right) may only occur at local minimizers of $(\ell_{\mathrm{per}} + \ell_0)\left(\frac{\cdot}{\varepsilon}\right)$, and that at such points, $u_\varepsilon = (\ell_{\mathrm{per}} + \ell_0)\left(\frac{\cdot}{\varepsilon}\right)$. Outside $\mathrm{Supp}\, \ell_0\left(\frac{\cdot}{\varepsilon}\right)$, this thus occurs at some minimizers of $\ell_{\mathrm{per}}\left(\frac{\cdot}{\varepsilon}\right)$, at which u_ε coincides with $\ell_{\mathrm{per}}\left(\frac{\cdot}{\varepsilon}\right)$.

We next recall that, precisely at all minimizers of $\ell_{\mathrm{per}}\left(\frac{\cdot}{\varepsilon}\right)$, we also have the equality $u_{\mathrm{per},\varepsilon} = \ell_{\mathrm{per}}\left(\frac{\cdot}{\varepsilon}\right)$, where $u_{\mathrm{per},\varepsilon}$ denotes the solution in the absence of defects. Thus, at any minimizer of $\ell_{\mathrm{per}}\left(\frac{\cdot}{\varepsilon}\right)$ outside $\mathrm{Supp}\, \ell_0\left(\frac{\cdot}{\varepsilon}\right)$ where u_ε coincides with $\ell_{\mathrm{per}}\left(\frac{\cdot}{\varepsilon}\right)$, we *also* have $u_\varepsilon = u_{\mathrm{per},\varepsilon}$. Between two such minimizers, using a comparison principle for the equation, we deduce that $u_\varepsilon \equiv u_{\mathrm{per},\varepsilon}$. So $u_\varepsilon \equiv u_{\mathrm{per},\varepsilon}$ at least everywhere outside a bounded interval containing the origin and presumably slightly exceeding $\mathrm{Supp}\, \ell_0\left(\frac{\cdot}{\varepsilon}\right)$.

Our next task is thus to fully identify the homogenized limit of u_ε. Since we do not have as much information on u_ε as we used to have on $u_{\mathrm{per},\varepsilon}$ in the previous section, we need to make a small detour (in fact by defining the homogenized problem itself and next backpedalling) .

We introduce the function

$$(3.23) \qquad u(x) = \begin{cases} \langle \ell_{\mathrm{per}} \rangle - e^{-|x|}\left(\langle \ell_{\mathrm{per}} \rangle - \inf_{\mathbb{R}}(\ell_{\mathrm{per}} + \ell_0)\right) & \text{for } |x| \leqslant \mu, \\[2ex] \inf_{\mathbb{R}} \ell_{\mathrm{per}} & \text{for } |x| \geqslant \mu, \end{cases}$$

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for μ defined by

$$(3.24) \qquad e^{-\mu} = \frac{\langle \ell_{\mathrm{per}} \rangle - \inf_{\mathbb{R}} \ell_{\mathrm{per}}}{\langle \ell_{\mathrm{per}} \rangle - \inf_{\mathbb{R}} (\ell_{\mathrm{per}} + \ell_0)}.$$

This function u is indeed the homogenized limit of u_ε, as we will see below. For the time being, we just remark that

(i) when $\ell_0 \equiv 0$ and more generally exactly when condition (3.3) is not satisfied, then $\mu = 0$ and $u = \inf_{\mathbb{R}} \ell_{\mathrm{per}}$ is indeed the homogenized limit we found for $u_{\mathrm{per}, \varepsilon}$.

(ii) when $\ell_{\mathrm{per}} \equiv 0$, then $\mu = +\infty$, $u(x) = (\inf_{\mathbb{R}} \ell_0) e^{-|x|}$ and we recover the homogenized limit (3.9) if we additionally assume (3.7) as we did in Section 3.1.1.

We also remark that u solves

$$\begin{cases} u(x) + \overline{H}\left(u'(x)\right) = 0 \quad \text{for } x \neq 0, \\[2mm] u(0) = \inf_{\mathbb{R}}(\ell_{\mathrm{per}} + \ell_0), \end{cases}$$

where $\overline{H}$ is defined in (3.18). In this specific setting, we again recover, as in Section 3.1.1, the general limit (1.11)-(1.12)-(1.13)-(1.14) stated in Theorem 1.1, this time with the specific value

$$(3.25) \qquad E = -\inf_{\mathbb{R}}(\ell_{\mathrm{per}} + \ell_0)$$

of the effective Dirichlet condition. Expression (3.25) is obviously a generalization of (3.13). And (1.12) and (1.14) again combine with one another to yield the above Dirichlet condition, precisely because of the condition (3.3). We omit here the detailed verification that the explicit value of E given in (3.25) may indeed be independently obtained using the sequence of approximate correctors, as in the general theory using (2.3)-(2.4), similarly to what we did above in Section 3.1.1 in the particular setting where ℓ_{per} identically vanishes.

In order to prove that u defined in (3.23) is indeed the limit of u_ε, let us consider

$$v_\varepsilon(x) = e^{-x} \int_0^x e^t \, \ell\left(\frac{t}{\varepsilon}\right) dt + e^{-x} \inf_{\mathbb{R}} \ell.$$

As ε vanishes, this function pointwise (and in fact locally uniformly) converges to

$$\langle \ell_{\mathrm{per}} \rangle \left(1 - e^{-x}\right) + e^{-x} \inf_{\mathbb{R}} \ell = u(x),$$

as defined by (3.23), for all $0 < x < \mu$. Since evidently $\ell_{\mathrm{per}}\left(\dfrac{x}{\varepsilon}\right) \geqslant \inf_{\mathbb{R}} \ell_{\mathrm{per}}$,

$$\ell_{\mathrm{per}}\left(\frac{x}{\varepsilon}\right) - v_\varepsilon(x) \geqslant \inf_{\mathbb{R}} \ell_{\mathrm{per}} - e^{-x} \int_0^x e^t \, \ell\left(\frac{t}{\varepsilon}\right) dt - e^{-x} \inf_{\mathbb{R}} \ell,$$

where, as ε vanishes, the right-hand side likewise converges to

$$\inf_{\mathbb{R}} \ell_{\mathrm{per}} - \langle \ell_{\mathrm{per}} \rangle \left(1 - e^{-x}\right) - e^{-x} \inf_{\mathbb{R}} \ell = \left(\inf_{\mathbb{R}} \ell_{\mathrm{per}} - \inf_{\mathbb{R}} \ell\right) - \left(\langle \ell_{\mathrm{per}} \rangle - \inf_{\mathbb{R}} \ell\right)\left(1 - e^{-x}\right).$$

Note that this quantity is nonnegative when $x \leqslant \mu$, since

$$\left(\inf_{\mathbb{R}} \ell_{\mathrm{per}} - \inf_{\mathbb{R}} \ell\right) - \left(\langle \ell_{\mathrm{per}} \rangle - \inf_{\mathbb{R}} \ell\right)\left(1 - e^{-x}\right)$$

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$$\geqslant (\inf_{\mathbb{R}} \ell_{\text{per}} - \inf_{\mathbb{R}} \ell) - (\langle \ell_{\text{per}} \rangle - \inf_{\mathbb{R}} \ell)(1 - e^{-\mu})$$
$$= 0,$$

and only vanishes at $x = \mu$.

If we now take $0 < x < \mu$, outside $\text{Supp}\,\ell_0\left(\frac{\cdot}{\varepsilon}\right)$, and slightly bounded away from μ (by some irrelevant constant, say of order ε), we thus have, for ε sufficiently small,

$$\ell\left(\frac{x}{\varepsilon}\right) - v_\varepsilon(x) \geqslant (\text{actually} =)\,\ell_{\text{per}}\left(\frac{x}{\varepsilon}\right) - v_\varepsilon(x) > 0.$$

If follows that, in that region, $(v_\varepsilon)'(x) = \ell\left(\frac{x}{\varepsilon}\right) - v_\varepsilon > 0$ and thus v_ε is solution to (3.22). We may proceed similarly for $-\mu < x < 0$ and the other branch, that of type (3.20). The function v_ε that we have constructed may then be continued close to and beyond $\pm\mu$. Eventually, it yields the global solution u_ε.

REMARK 3.3. *In echo to Remark 3.2, we notice that, here, the two homogenized limits obtained respectively for ℓ_{per} and $\ell_{\text{per}} + \ell_0$ exactly agree outside a bounded interval surrounding the origin, unless ℓ_{per} is constant. The latter case is indeed the only case where the homogenized Hamiltonian $\overline{H}$ is not flat around $p = 0$ (it is $|p|$). The agreement at infinity of the two homogenized solutions, respectively in the absence and in the presence of defects, reflects this property. We will further investigate this question in higher dimensions in Section 3.2.1 below.*

3.1.4. A randomized variant. As briefly mentioned in Section 1, a randomized variant of the theory of local defects has been introduced by A. Anantharaman and the second author in [5]. In that work, the equation under consideration was a linear elliptic equation in conservation form. The adaptation of the setting considered therein to our equation (3.22) is as follows.

We consider as a perturbation of the underlying periodic environment encoded in ℓ_{per}, the *random* running cost

$$\ell_S(x, \omega) = \sum_{k \in \mathbb{Z}} X_k^\eta(\omega)\,\ell_0(x - k),$$

where $\{X_k^\eta(\omega)\}_{k \in \mathbb{Z}}$ is a sequence of i.i.d. random variables that all follow a Bernoulli law of parameter η, that is, $X_k^\eta(\omega) = 0$ or $= 1$ with probability $1 - \eta$ and η respectively. The parameter $\eta > 0$ is to be thought of as a small parameter, which will eventually be sent to zero. This models that the periodic environment encoded in ℓ_{per} is only *slightly* perturbed.

After rescaling in ε of the potential ℓ_S, the equation for which we study homogenization thus reads as

$$u_\varepsilon(x, \omega) + |(u_\varepsilon)'(x, \omega)| = \ell_{\text{per}}\left(\frac{x}{\varepsilon}\right) + \ell_S\left(\frac{x}{\varepsilon}, \omega\right) = \ell_{\text{per}}\left(\frac{x}{\varepsilon}\right) + \sum_{k \in \mathbb{Z}} X_k^\eta(\omega)\,\ell_0\left(\frac{x - k\varepsilon}{\varepsilon}\right).$$

We intend to identify the homogenized limit $\varepsilon \to 0$ for this equation *in the regime when the parameter η vanishes*. Here, to keep things simple and allow for analytic calculations, we only consider the case $\ell_{\text{per}} \equiv 0$ of a flat unperturbed environment, that is

(3.26) $$u_\varepsilon(x, \omega) + |(u_\varepsilon)'(x, \omega)| = \sum_{k \in \mathbb{Z}} X_k^\eta(\omega)\,\ell_0\left(\frac{x - k\varepsilon}{\varepsilon}\right),$$

using a "shape function" ℓ_0 for the defects that satisfies (3.7).

REMARK 3.4. *Let us emphasize that, even though the specific algebraic manipulations below depend on the technical assumption that $\ell_{\mathrm{per}} = 0$, the general result that we are going to obtain carries over to other cases.*

Extensions to other ambient dimensions will be addressed in Section 3.2.

In order to understand the homogenized limit, we need a preliminary step, namely the case of *two* localized (downward) defects, instead of a single one at the origin as in Section 3.1.3. Intuitively, $\ell_S(x, \omega)$ is the random superposition of an infinite number of such defects. Since *"He who can do more can do less"*, we need to first understand the case of two defects, that is:

$$(3.27) \qquad z_\varepsilon(x) + \left|(z_\varepsilon)'(x)\right| = \ell_0\left(\frac{x}{\varepsilon}\right) + \ell_0\left(\frac{x - \varepsilon}{\varepsilon}\right).$$

It is then easy to realize that the solution reads as

$$(3.28) \qquad z_\varepsilon(x) = \inf\left(u_\varepsilon(x), u_\varepsilon(x - \varepsilon)\right),$$

where u_ε is the solution for one single defect at the origin, which we made explicit in (3.8). In order to realize that (3.28) holds, we just need to establish that, in $\operatorname{Supp}\ell_0\left(\frac{\cdot}{\varepsilon}\right)$, we have $u_\varepsilon(x) \leqslant u_\varepsilon(x - \varepsilon)$. A similar argument will prove a symmetric result on $\operatorname{Supp}\ell_0\left(\frac{x - \varepsilon}{\varepsilon}\right)$, while, outside the two supports, both $u_\varepsilon(\cdot)$ and $u_\varepsilon(\cdot - \varepsilon)$ solve (3.27). Let us recall that, because of (3.2), the two supports $\operatorname{Supp}\ell_0$ and $\operatorname{Supp}\ell_0(\cdot - 1)$ are disjoint. Given the explicit expression (3.8) of u_ε and its monotonicity on $\mathbb{R}^+$, we then have that

$$(3.29) \qquad u_\varepsilon(x) \leqslant e^{-\varepsilon a}\left(\ell_0(0) + \int_0^{\varepsilon a} e^t \ell_0\left(\frac{t}{\varepsilon}\right) dt\right),$$

for $a \leqslant \frac{1}{2}$ and all $0 < x < \varepsilon a$. On the other hand, again for all $0 < x < \varepsilon a$, we have, by the same argument but this time looking at the explicit expression of $u_\varepsilon(x - \varepsilon)$ that ,

$$(3.30) \qquad u_\varepsilon(x - \varepsilon) \geqslant e^{\varepsilon a - \varepsilon}\left(\ell_0(0) + \int_{\varepsilon - \varepsilon a}^{\varepsilon} e^{-(t - \varepsilon)} \ell_0\left(\frac{t - \varepsilon}{\varepsilon}\right) dt\right).$$

Since $a \leqslant \frac{1}{2}$, we have $e^{\varepsilon a - \varepsilon} \leqslant e^{-\varepsilon a}$. On the other hand, for simplicity let us further assume that

$$(3.31) \qquad \ell_0 \quad \text{is an even function},$$

so that the two integrals appearing in the right-hand sides of (3.29) and (3.30) are identical. We deduce that

$$u_\varepsilon(x) \leqslant u_\varepsilon(x - \varepsilon) \quad \text{for all } 0 < x < \varepsilon a,$$

that region being the part of $\operatorname{Supp}\ell_0\left(\frac{\cdot}{\varepsilon}\right)$ at the right of the origin. A similar argument allows to conclude for all the other regions. Note that a simple estimation of the terms in (3.29) and (3.30) shows that, had we not assumed (3.31) for simplicity, the argument above would also hold true provided the two defects are separated from a

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distance $q\varepsilon$ for an integer q chosen sufficiently large. We will see a similar argument in Section 3.2.2 below.

Equation (3.28) readily implies that, returning to the random setting (3.26),

$$(3.32) \qquad u_\varepsilon(x,\omega) \;=\; \inf_{k\in\mathbb{Z}}\left\{X_k^\eta(\omega)\,u_\varepsilon(x-k\varepsilon)\right\},$$

since any $k \in \mathbb{Z}$ for which $X_k^\eta(\omega) = 1$ contributes to lowering the solution, while others k do not. The "min-formula" (3.32) is the main ingredient in the rest of our argument and will be generalized in Section 3.2.

REMARK 3.5. *Note that we work here with the solution u_ε of the original equation (3.26), prove it explicitly reads as (3.32) and proceed from there to find its homogenized limit. In fact, we could, using a similar string of arguments, identify the solution to the discounted problem*

$$(3.33) \qquad \delta\,w_\delta(y,\omega) + \left|(w_\delta)'(y,\omega)\right| \;=\; \sum_{k\in\mathbb{Z}} X_k^\eta(\omega)\,\ell_0\,(y-k),$$

for $\delta > 0$, that replaces the corrector problem in the context of stochastic homogenization. The similarity between (3.26) and (3.33) is evident. Using a simple rescaling (exactly as we did in the specific deterministic case (3.11)-(3.12) above), any explicit expression of the solution u_ε of the former equation yields an expression of the solution w_δ to the latter equation. Taking the limit $\lim_{\delta\to 0}\delta\,w_\delta$, we may then identify the value $\overline{H}_\eta(0)$ of the homogenized Hamiltonian in this stochastic setting. Note that the same remark applies to the multidimensional context we will address in Section 3.2.2 below, but we will not repeat it there.

Since we already know that, in the homogenized limit, u_ε given by (3.8) converges to $u(x) = \ell_0(0)\,e^{-|x|}$ in (3.9) (later generalized in (3.23)), we *temporarily* replace (3.32) by

$$(3.34) \qquad u_\varepsilon(x) \;=\; \ell_0(0)\,\sup_{k\in\mathbb{Z}}\left\{X_k^\eta(\omega)\,e^{-|x-k\varepsilon|}\right\},$$

where we have used that $\ell_0(0) < 0$. Take now $x = 0$. The key quantity appearing on the right-hand side of (3.34) is in fact the random variable

$$(3.35) \qquad \sup_{k\in\mathbb{Z}}\left\{X_k^\eta(\omega)\,e^{-|k|\varepsilon}\right\},$$

and to get a grasp on this quantity, let us only consider therein the positive indices, so that we focus our attention on

$$Z_\varepsilon^\eta(\omega) \;=\; \sup_{k\in\mathbb{Z},\,k\geqslant 0}\left\{X_k^\eta(\omega)\,e^{-k\varepsilon}\right\}.$$

Since the function $k \to e^{-k\varepsilon}$ is decreasing with respect to $k > 0$, it is evident that $Z_\varepsilon^\eta = e^{-k\varepsilon}$ for the first index k such that $X_k^\eta = 1$. But now, since the X_k^η are i.i.d. and all follow a Bernoulli law, it is well known that the law of this first index is geometric. More precisely, we have

$$(3.36) \qquad Z_\varepsilon^\eta(\omega) \;=\; e^{-k\varepsilon} \quad \text{with probability } (1-\eta)^k\eta.$$

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If we now let $\varepsilon \to 0$, we obtain $Z_\varepsilon^\eta(\omega) \to 1$ almost surely. It is easy to see that a similar argument applies to (3.35), since only $|k|$ matters there and $\max(X_{-k}^\eta, X_k^\eta)$ are i.i.d. Bernoulli variables of parameter $1 - (1 - \eta)^2$, which scales as η, when η is small. Thus, eventually we may guess that

$$u_\varepsilon(0, \omega) \to \ell_0(0)$$

almost surely in the limit $\varepsilon \to 0$. The actual proof of this fact necessitates to realize that we have replaced u_ε given by (3.8) by its homogenized limit $u(x) = \ell_0(0) e^{-|x|}$. The error committed in this approximation may easily be controlled (at least in this case where everything is explicitly known by the formulae (3.8) and (3.9)) and proven to be irrelevant in the limit process above. In addition, the asymptotics that we have found for $x = 0$ in (3.34) readily carry over to any $x \in \mathbb{R}$, since all what matters is the decay of the function $k \to e^{-|x - k\varepsilon|}$ away from x. We have thus obtained

$$(3.37) \qquad \qquad \text{a.s.} \lim_{\varepsilon \to 0} u_\varepsilon(x, \omega) = \ell_0(0).$$

The convergence (3.37) is both *disappointing* and *intuitive*. It is disappointing because, the homogenized limit being flat, there is not much interesting mathematical phenomenon to discuss. The convergence (3.37) is, on the other hand, intuitive, because as soon as a defect occurs at a location $k\varepsilon$ for some index k such that $k\varepsilon \to 0$ in the homogenized limit, this defect is brought to the origin by the rescaling and everything happens *as if* we had a deterministic localized defect there. For instance, even the probability of having *at least one* defect in the $\dfrac{1}{\sqrt{\varepsilon}}$ first indices (say at the right of the origin) is

$$(3.38) \qquad \qquad \sum_{0 \leqslant k \leqslant \frac{1}{\sqrt{\varepsilon}}} (1 - \eta)^k \, \eta \approx 1 - (1 - \eta)^{1/\sqrt{\varepsilon}} \xrightarrow{\varepsilon \to 0} 1,$$

for η fixed, so all this happens with probability one. A similar argument again applies if we consider both sides of the origin, and likewise when we consider any point x since all points play identical roles. In the end, the limit (3.37) is flat. It is also independent of η.

This comes in sharp contrast to the case of a similar random coefficient inserted in an elliptic equation, which is the setting studied in the work [5]. This striking difference may be intuitively explained by the fact that, at the macroscopic scale, an elliptic equation only *sees* suitable *averages* of the oscillatory coefficient, while an Hamilton-Jacobi type equation sensitively *sees extremal values* of that coefficient. Put in an even more simplified language, what we observe is nothing but the difference between

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N X_k^\eta(\omega) \quad \text{and} \quad \lim_{N \to \infty} \sup_{1 \leqslant k \leqslant N} \{X_k^\eta(\omega)\},$$

for the i.i.d. Bernoulli random variables X_k^η of parameter η. The leftmost limit scales as η while the rightmost limit is independent of η .

The net conclusion of the above discussion is that, in the model we have considered so far, the probability of having defects is so large that, in the limit, there are almost surely defects everywhere. In order to get a much more interesting regime, we need to

relate the small parameter ε, which measures the distances at the microscopic scale, with the parameter η, which measures the *amount of random perturbation present in the environment* (recall that, for $\eta = 0$, we recover the unperturbed environment). A possible way to obtain this property is to ensure that, for instance, there is, asymptotically, a fixed, small, proportion of defects per unit *macroscopic* length (or, in higher dimensional settings, volume). More precisely, we now take

$$(3.39) \qquad\qquad \eta = \overline{\eta}\,\varepsilon, \quad \text{for} \quad \overline{\eta} \quad \text{fixed.}$$

The scaling law (3.39) is precisely adjusted so that, in contrast to what we observed above in (3.38) for η fixed, the probability of seeing at least one defect over a unit macroscopic length around the origin scales as

$$(3.40) \qquad \sum_{0 \leqslant k \leqslant \frac{1}{\varepsilon}} (1 - \overline{\eta}\varepsilon)^k \,\overline{\eta}\varepsilon \;\approx\; 1 - (1 - \overline{\eta}\varepsilon)^{1/\varepsilon} \stackrel{\varepsilon \to 0}{\longrightarrow} 1 - e^{-\overline{\eta}} \stackrel{\overline{\eta} \to 0}{\approx} \overline{\eta} \ll 1.$$

If we revisit our calculations above with the particular value (3.39), we may mimic the argument step by step. We realize that the key quantity from (3.36) is now

$$(3.41) \qquad\qquad Z_\varepsilon^\eta(\omega) = e^{-k\varepsilon} \quad \text{with probability } (1 - \overline{\eta}\varepsilon)^k\,\overline{\eta}\,\varepsilon.$$

A simple calculation shows that, for all $\mu \geqslant 0$,

$$\mathbb{P}\left(Z_\varepsilon^\eta \leqslant e^{-\mu}\right) = (1 - \overline{\eta}\varepsilon)^{\mu/\varepsilon} \approx e^{-\overline{\eta}\mu}, \quad \text{as} \quad \varepsilon \to 0,$$

that is, for all $0 \leqslant t \leqslant 1$,

$$(3.42) \qquad\qquad \mathbb{P}\left(Z_\varepsilon^\eta \leqslant t\right) \approx t^{\overline{\eta}}, \quad \text{as} \quad \varepsilon \to 0.$$

Upon differentiating with respect to t, we obtain that, in the limit $\varepsilon \to 0$, the law is $\overline{\eta}\,t^{\overline{\eta}-1}$. This suffices to establish that the limit is not deterministic. Put differently, (i) when $\eta > 0$ is fixed (or $\eta \gg \varepsilon$) the limit u_ε is deterministic and flat at the level $\ell_0(0)$, (ii) when $\eta = 0$ (or $\eta \ll \varepsilon$) the limit u_ε is deterministic and flat at the level 0 (that is the unperturbed solution), and (iii) when exactly η scales as $\overline{\eta}\varepsilon$ in (3.39), then the limit is flat, and its value is a random variable in the interval $[\ell_0(0), 0]$.

3.2. Generalization to higher dimensions. Motivated by the phenomena we have just observed in the one-dimensional setting of Section 3.1, we consider two specific questions in a higher dimensional situation:

(i) the behavior when $|x| \to +\infty$ of the homogenized solution in presence of a defect at the origin, as compared to that of the homogenized solution in the unperturbed periodic setting

(ii) the random superposition of point defects.

The two issues are respectively studied in Sections 3.2.1 and 3.2.2 below.

3.2.1. Influence of the defect at infinity. We return to the general setting of Section 1.1 with, in ambient dimension $d \geqslant 1$, a general Hamiltonian H (defined in (1.1)) satisfying the classical properties made precise in that section, in particular (1.2) through (1.4) and that is the perturbation of a periodic Hamiltonian H_{per} (defined in (1.5)) in the following sense: $f = f_{\mathrm{per}}$ and $\ell = \ell_{\mathrm{per}}$ outside a neighborhood of the origin.

We denote by u the homogenized solution provided by Theorem 1.1. In order to avoid any confusion, we denote by u_{per} the homogenized solution associated with the *periodic* Hamiltonian H_{per} in equation (1.8). We now intend to prove that

(3.43) $$|u(x) - u_{\text{per}}(x)| \to 0 \quad \text{as} \quad |x| \to +\infty.$$

This property of course mathematically encodes that the local defect (that is, the difference between (f, ℓ) and $(f_{\text{per}}, \ell_{\text{per}})$ assumed compactly supported near the origin) only affects the homogenized solution *locally*. In Section 3.1, we have seen that it is true in the one-dimensional setting and for the separate Hamiltonian considered, and even that, if ℓ_{per} is not constant, then $u(x) = u_{\text{per}}(x)$ outside a neighborhood of the origin.

The proof of (3.43) turns out to be rather simple. Let us consider, say, the first canonical vector $\mathbf{e}_1$ in $\mathbb{R}^d$, and the function $u_n(x) = u(x - n\,\mathbf{e}_1)$ obviously solution to the same problem as (1.11) through (1.14), but with an effective Dirichlet condition set at the point $x_n = n\,\mathbf{e}_1$ instead of the origin. The sequence of functions u_n inherits, uniformly in $n \in \mathbb{N}$, of the properties of u itself. We therefore have uniform almost everywhere pointwise bounds on u_n and Du_n. It follows that, up to an extraction (and a diagonal argument) which we henceforth omit to explicitly denote, u_n locally uniformly converges to some function v. Fix now a unit ball $B_1(y_0)$ around an arbitrary point $y_0 \in \mathbb{R}^d$. For any sufficiently large $n \in \mathbb{N}$, the point x_n lies outside that ball, and the function u_n thus solves (in the viscosity sense) the periodic homogenized equation (1.11), that is

$$\alpha u_n + \overline{H}(Du_n) = 0$$

on that ball. Given the above convergence and using the result of stability of viscosity solutions for that equation, so does the limit v. By uniqueness of the bounded uniformly continuous viscosity solution to (1.11) posed on the entire space $\mathbb{R}^d$, we thus know that $v = u_{\text{per}}$. Since the limit is unique, we conclude that the sequence u_n itself converges to u_{per} uniformly locally. The convergence (3.43) follows.

Two remarks are in order.

First, we note that we have only used the homogenized form of the equation provided by Theorem (1.1) and *not* the assumption *itself* that $f = f_{\text{per}}$ and $\ell = \ell_{\text{per}}$ outside a neighborhood of the origin. So, if Theorem 1.1 turns out to hold true for some defect ℓ_0 that does not necessarily have compact support (or alternatively and more generally, in some specific settings, for more general (f, ℓ) that converge to $(f_{\text{per}}, \ell_{\text{per}})$ in a milder sense), the convergence (3.43) still holds true. On the other hand and as briefly mentioned above, unless peculiar conditions are met, there is no reason for u and u_{per} to exactly agree outside a bounded domain.

Second, both above facts may be intuitively understood when resorting to the optimal control interpretation of the equations and solutions at play (in the spirit of our arguments of Section 2.3). When starting farther and farther away from the defect located at the origin, the optimal trajectory has lesser reason to visit the neighborhood of the origin. As $|x| \to +\infty$, the value function therefore becomes decreasingly affected by the presence of the defect. Thus the convergence (3.43). Additionally and depending upon the specific landscape, such an excursion toward the origin may come, or not, at a price. If for points x located far enough from the origin, this price becomes too high, then the optimal paths leaving x do not

visit the support of the defect. In this case, the exact equality $u = u_{\mathrm{per}}$ may hold outside a bounded domain. This is what happens for a nontrivial one-dimensional setting as that of Section 3.1.2, but not for that of Section 3.1.1 or for a higher dimensional setting where some specific paths escaping from the wells of ℓ_{per} can be cost-advantageous. In the latter situations, only (3.43) holds.

3.2.2. Randomized variant. We devote this section to randomized lattices of points defects in $\mathbb{R}^d$, $d > 1$. We focus our attention to the case when the background environment is constant. The extension to random perturbations of periodic Hamiltonians is left for future work, because it is not obvious that a counterpart of formula (3.32) holds in this situation.

Radially symmetric situations. We consider the Hamiltonians

$$(3.44) \qquad H\,(y, p) = \overline{H}(p) - \ell_0\,(y)\,,$$

and for each $\mathbf{k} \in \mathbb{Z}^d$,

$$(3.45) \qquad H_{\mathbf{k}}^{\varepsilon}(x\,,\,p\,,\,\omega) = \overline{H}(p) - \ell_0\left(\frac{x - \varepsilon\,q\,\mathbf{k}}{\varepsilon}\right) X_{\mathbf{k}}^{\eta}(\omega),$$

where $\{X_{\mathbf{k}}^{\eta}(\omega)\}_{\mathbf{k} \in \mathbb{Z}^d}$ is a set of i.i.d. real valued, Bernoulli random variables of parameter η, that is, a multi-dimensional analogue of the sequence introduced in Section 3.1.4. We consider the case when

- $\overline{H}$ is a C^2, convex, globally Lipschitz and radially symmetric function of p (it therefore reaches its minimum at $p = 0$)
- ℓ_0 is smooth, non positive and radially symmetric function, supported in the unit ball centered at the origin, and such that $\arg\min \ell_0 = \{0\}$ and $\ell_0(0) < 0$
- the ergodic constant E defined in Section 2.1 satisfies

$$(3.46) \qquad E > \overline{H}(0).$$

We will see later that (3.46) is in fact a consequence of the assumptions made on ℓ_0.

The Hamiltonian $H_{\mathbf{k}}^{\varepsilon}$ corresponds to a rescaled Hamiltonian similar to $H\left(\dfrac{x}{\varepsilon}, p\right)$, when the local defect (originally set at the origin) is shifted at each of the positions $\varepsilon\,q\,\mathbf{k}$, and only occurs there with the probability η, encoded in the random variable $X_{\mathbf{k}}(\omega)$. The parameter q is a positive integer that will be adjusted in the course of the proof, see (3.66) below. We then form the random Hamiltonian

$$(3.47) \qquad H_S^{\varepsilon}(x\,,\,p\,,\,\omega) = \sup_{\mathbf{k} \in \mathbb{Z}^d} H_{\mathbf{k}}^{\varepsilon}(x\,,\,p\,,\,\omega),$$

which also reads as

$$(3.48) \qquad H_S^{\varepsilon}(x\,,\,p\,,\,\omega) = \overline{H}(p) - \sum_{\mathbf{k} \in \mathbb{Z}^d} X_{\mathbf{k}}^{\eta}(\omega)\,\ell_0\left(\frac{x - \varepsilon\,q\,\mathbf{k}}{\varepsilon}\right).$$

This is the d-dimensional counterpart of the one-dimensional Hamiltonian considered in Section 3.1.4. The expression (3.48) proceeds from the definition (3.45)-(3.47) along with the observation that the supports of the two functions $\ell_0\left(\dfrac{x - \varepsilon\,q\,\mathbf{k}}{\varepsilon}\right)$

and $\ell_0 \left(\dfrac{x - \varepsilon \, q \, \mathbf{k}'}{\varepsilon} \right)$ for different indices $\mathbf{k} \neq \mathbf{k}'$ do not overlap and the latter two functions are nonpositive, so, for all $x \in \mathbb{R}^d$ and $\varepsilon > 0$, hence

$$\ell_0 \left(\frac{x - \varepsilon \, q \, \mathbf{k}}{\varepsilon} \right) \, + \, \ell_0 \left(\frac{x - \varepsilon \, q \, \mathbf{k}'}{\varepsilon} \right) \; = \; \min \left[\ell_0 \left(\frac{x - \varepsilon \, q \, \mathbf{k}}{\varepsilon} \right) \, , \, \ell_0 \left(\frac{x - \varepsilon \, q \, \mathbf{k}'}{\varepsilon} \right) \right].$$

We now consider, for almost all ω, the equation

(3.49) $$\alpha \, U_\varepsilon(x, \omega) \, + \, H_S^\varepsilon(x \, , \, DU_\varepsilon \, , \, \omega) \; = \; 0.$$

We intend to first prove that this equation has a unique, bounded uniformly continuous solution $U_\varepsilon(x, \omega)$ in the viscosity sense, and next study the limit of this solution as ε vanishes. For this purpose, under a suitable assumption on q, we will in fact explicitly characterize the solution U_ε as

(3.50) $$U_\varepsilon(x, \omega) \; = \; \inf_{\mathbf{k} \in \mathbb{Z}^d} \{ X_{\mathbf{k}}^\eta(\omega) \, u_\varepsilon(x - \varepsilon \, q \, \mathbf{k}) \},$$

where $u_\varepsilon(x)$ is the unique viscosity solution in $\mathrm{BUC}(\mathbb{R}^d)$ to (1.7), that is

$$\alpha u_\varepsilon + H \left(\frac{x}{\varepsilon}, Du_\varepsilon \right) = 0 \quad \text{in } \mathbb{R}^d.$$

The expression (3.50) evidently generalizes (3.32) to the present multi-dimensional context.

In order to establish (3.50), it is obviously sufficient to understand the setting where only two defects are present and are respectively localized, say, at the origin $x_0 = 0$ and at $x_{\mathbf{k}} = \varepsilon \, q \, \mathbf{k}$, for some $\mathbf{k} \in \mathbb{Z}^d \backslash \{0\}$. We thus need to manipulate the two Hamiltonians $H_{\mathbf{k}}^\varepsilon$ and H_0^ε (the latter is obtained by setting $\mathbf{k} = 0$ in (3.45)). In this simplified setting, we claim that the solution to

(3.51) $$\alpha \, V_\varepsilon(x, \omega) \, + \, \max \{ H_0^\varepsilon \, , \, H_{\mathbf{k}}^\varepsilon \} \, (x, DV_\varepsilon, \omega) \; = \; 0$$

actually reads as
(3.52)
$$V_\varepsilon(x, \omega) \; = \; \min \left\{ X_0^\eta(\omega) \, u_\varepsilon(x) + (1 - X_0^\eta(\omega)) \, \overline{u} \, , \, X_{\mathbf{k}}^\eta(\omega) \, u_\varepsilon(x - \varepsilon \, q \, \mathbf{k}) + (1 - X_{\mathbf{k}}^\eta(\omega)) \, \overline{u} \right\},$$

where u_ε solves (1.7) in the sense of viscosity, and the constant function $\overline{u} = -\overline{H}(0)/\alpha$ corresponds to the situation when there is no defect.

If we temporarily admit that (3.52) holds true, then the generalization to infinitely many randomized defects is immediate and yields (3.50), where we note that all the terms in $(1 - X_{\mathbf{k}}^\eta(\omega)) \overline{u}$ originally present in (3.52) may be discarded because, with full probability, one at least of the $X_{\mathbf{k}}^\eta(\omega)$, $\mathbf{k} \in \mathbb{Z}^d$, has value one and $u_\varepsilon(\cdot - \varepsilon \, q \, \mathbf{k}) \leqslant \overline{u}$ by the comparison principle (using the non-positiveness of ℓ_0).

In order to establish (3.52), we proceed as follows. First, when $X_0^\eta(\omega) = X_{\mathbf{k}}^\eta(\omega) = 0$, $H_0^\varepsilon(\cdot, p, \omega) = H_{\mathbf{k}}^\varepsilon(\cdot, p, \omega) = \overline{H}(p)$, the constant function $\overline{u}$ is the solution to (3.51) and (3.52) holds. Second, if $X_0^\eta(\omega) = 1$ while $X_{\mathbf{k}}^\eta(\omega) = 0$, then $H_0^\varepsilon(\cdot, p, \omega) = H \left(\dfrac{\cdot}{\varepsilon}, p \right)$ and $H_{\mathbf{k}}^\varepsilon(\cdot, p, \omega) = \overline{H}(p)$, thus, given the non-positiveness of ℓ_0,

$$\max \{ H_0^\varepsilon(\cdot, p, \omega) \, , \, H_{\mathbf{k}}^\varepsilon(\cdot, p, \omega) \} \; = \; H \left(\frac{\cdot}{\varepsilon}, p \right)$$

and the solution is indeed u_ε. We conclude "symmetrically" when $X_0^\eta(\omega) = 0$ and $X_{\mathbf{k}}^\eta(\omega) = 1$. The only interesting case is the third one, when $X_0^\eta(\omega) = X_{\mathbf{k}}^\eta(\omega) = 1$, and we wish to prove that

$$(3.53) \qquad V_\varepsilon(x, \omega) = \min\Big\{ u_\varepsilon(x), \, u_\varepsilon(x - \varepsilon\, q\, \mathbf{k}) \Big\}.$$

To address the latter case, we partition $\mathbb{R}^d$ into three non-overlapping regions, namely $\mathcal{A}_0^\varepsilon = \text{Supp}\,\ell_0\left(\frac{\cdot}{\varepsilon}\right)$, $\mathcal{A}_{\mathbf{k}}^\varepsilon = \text{Supp}\,\ell_0\left(\frac{\cdot}{\varepsilon} - q\,\mathbf{k}\right) = \varepsilon\, q\, \mathbf{k} + \text{Supp}\,\ell_0\left(\frac{\cdot}{\varepsilon}\right)$, and the remaining part $(\mathcal{A}_0^\varepsilon \cup \mathcal{A}_{\mathbf{k}}^\varepsilon)^c$ of $\mathbb{R}^d$.

From the non-positiveness of ℓ_0, we notice that

$$(3.54) \qquad \max\{H_0^\varepsilon, H_{\mathbf{k}}^\varepsilon\} = \begin{cases} H_0^\varepsilon & \text{in } \mathcal{A}_0^\varepsilon, \\ H_{\mathbf{k}}^\varepsilon & \text{in } \mathcal{A}_{\mathbf{k}}^\varepsilon, \\ H_0^\varepsilon = H_{\mathbf{k}}^\varepsilon = \overline{H} & \text{in } (\mathcal{A}_0^\varepsilon \cup \mathcal{A}_{\mathbf{k}}^\varepsilon)^c. \end{cases}$$

We are going to prove that for ε sufficiently small,

$$(3.55) \qquad \begin{cases} u_\varepsilon(x) \leqslant u_\varepsilon(x - \varepsilon\, q\, \mathbf{k}) & \text{in } \mathcal{A}_0^\varepsilon, \\ u_\varepsilon(x - \varepsilon\, q\, \mathbf{k}) \leqslant u_\varepsilon(x) & \text{in } \mathcal{A}_{\mathbf{k}}^\varepsilon. \end{cases}$$

Assume temporarily that (3.55) is true. Since u_ε and $u_\varepsilon(\cdot - \varepsilon\, q\, \mathbf{k})$ are respectively viscosity solutions to (1.7) and that same equation translated from $-\varepsilon\, q\, \mathbf{k}$, and since they are both locally Lipschitz continuous, we know from [6, Prop. 1.9, Chapter I] that they are solutions almost everywhere of those equations, respectively. It follows from (3.54) and (3.55) together, that $\min\big\{ u_\varepsilon(x), \, u_\varepsilon(x - \varepsilon\, q\, \mathbf{k}) \big\}$ is solution almost everywhere to (3.51). We also know that this minimum is a viscosity supersolution of the equation. Finally, since the minimum of two Lipschitz continuous functions is also Lipschitz continuous and since the Hamiltonian is convex, we know from [6, Prop. 5.1, Chapter II] that this almost everywhere, Lipschitz continuous, solution is also a viscosity subsolution. We thus conclude that it is a viscosity solution and that (3.53). Consequently, if (3.55) holds for all $\mathbf{k} \in \mathbb{Z}^d \backslash \{0\}$, then (3.50) holds.

The inequations in (3.55) in fact only rely upon an interplay between the properties of the solution u_ε for the problem with *one* defect (essentially that $u_\varepsilon(x)$ grows as x departs from the defect) and the distance between the *two contiguous* defects considered. At this point, we may assume that $\alpha = 1$, just for alleviating notation.

To start with, we observe that the solution u to the Dirichlet problem (1.11)-(1.14) in the sense of Theorem 1.1, (a) is radially symmetric and (b) is a strictly increasing function of the radius $|x|$. The radial symmetry comes from the uniqueness of that solution stated in Theorem 1.1. The increasing character comes from our assumption (3.46). Indeed, because of this assumption and (1.12), we have $u(0) \leqslant -E < -\overline{H}(0)$. By continuity, $u(x) < -\overline{H}(0)$ on a neighborhood $B_R(0)$ for some radius $R > 0$. The continuity of $\overline{H}$ then implies that $|Du(x)| \geqslant \delta > 0$ for some $\delta > 0$ and almost all $x \in B_R(0)$. By radial symmetry, this amounts to $\left|\dfrac{\partial u}{\partial r}\right| \geqslant \delta > 0$ for almost all $r \leqslant R$. But the notion of viscosity solution then implies that, on $B_R(0)$, $\dfrac{\partial u}{\partial r}$, which is strictly positive around the origin (given that we have already established that u reaches its global infimum there) can change sign and become negative at most once. If this is the case, then u decreases on the region where $\dfrac{\partial u}{\partial r} \leqslant -\delta$, so the

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inequality $u(x) < -\overline{H}(0)$ is all the more true. We may thus enlarge the ball $B_R(0)$ and since $|Du|$ cannot be arbitrarily small on this region, the viscosity solution u cannot become increasing again. Therefore, u remains decreasing as $|x|$ grows to infinity. On the other hand, we also already know from (3.43) that $|u(x) - \overline{u}| \to 0$ at infinity and from the non-positiveness of ℓ_0 that $u \leqslant \overline{u}$, so we reach a contradiction. In conclusion, $\dfrac{\partial u}{\partial r} \geqslant \delta > 0$ everywhere outside the origin and u is indeed a strictly increasing, radially symmetric function.

We now return to u_ε, for which we will apply a similar argument, using the uniformity of properties with respect to ε. To start with, we notice that, because ℓ_0 is assumed radially symmetric and $\overline{H}$ is isotropic, u_ε is also radially symmetric. Next, since ℓ_0 is nonpositive, comparison yields as above that $u_\varepsilon \leqslant \overline{u} = -\overline{H}(0)$ in $\mathbb{R}^d$. On the other hand, by Theorem 1.1, $u_\varepsilon(0) \to u(0) \leqslant -E < -\overline{H}(0)$.

Let us fix an arbitrary positive constant β such that $0 < 2\beta < E - \overline{H}(0)$. There exists a positive radius R_0 such that $u(x) < -\overline{H}(0) - 2\beta$ on $\overline{B_{R_0}(0)}$. From the uniform convergence of u_ε to u on $\overline{B_{R_0}(0)}$, we know that there exists $\varepsilon_0 > 0$ such that for any ε such that $0 < \varepsilon \leqslant \varepsilon_0$,

$$(3.56) \qquad u_\varepsilon(x) < -\overline{H}(0) - \beta, \qquad \text{on } \overline{B_{R_0}(0)}.$$

Step 1. Let us start by studying u_ϵ in $B_{R_0}(0) \backslash B_\varepsilon(0)$. Given that H_ε coincides with $\overline{H}$ outside $\mathrm{Supp}\,\ell_0\left(\dfrac{\cdot}{\varepsilon}\right)$, (3.56) implies that there exists some $\delta > 0$, independent of $\varepsilon \leqslant \varepsilon_0$, such that

$$(3.57) \qquad |Du_\varepsilon(x)| \geqslant \delta > 0, \quad \text{for almost all } \varepsilon \leqslant |x| \leqslant R_0.$$

Therefore, for almost all $\varepsilon \leqslant |x| \leqslant R_0$,

$$(3.58) \qquad \left| \frac{\partial u_\varepsilon}{\partial r} \right| \geqslant \delta.$$

Since we know, from the notion of viscosity solution and from (3.57), that Du_ε cannot be arbitrarily small on $B_{R_0}(0) \backslash B_\varepsilon(0)$, $\dfrac{\partial u_\varepsilon}{\partial r}$ may change sign at most once in this ring, and if it is the case, then the jump is from positive to negative as $|x|$ grows.

Hence, if $\lim_{r \to \varepsilon_-} \dfrac{\partial u_\varepsilon}{\partial r}(r) < 0$, then $\dfrac{\partial u_\varepsilon}{\partial r} \leqslant -\delta$ in $B_{R_0}(0) \backslash B_\varepsilon(0)$. This implies that we may find a larger R_0 still satisfying $u_\varepsilon(x) \leqslant -\overline{H}(0) - \beta$ for $x \in B_{R_0}(0) \backslash B_\varepsilon(0)$. Repeating the argument, we obtain that we may indeed choose $R_0 = +\infty$ and that $\lim_{|x| \to \infty} u_\varepsilon(x) = -\infty$, which contradicts the convergence of u_ε to $-\overline{H}(0)$ at infinity.

Therefore, $\lim_{r \to \varepsilon_-} \dfrac{\partial u_\varepsilon}{\partial r}(r) > 0$ and $\dfrac{\partial u_\varepsilon}{\partial r}$ may change sign at most once on $B_{R_0}(0) \backslash B_\varepsilon(0)$ to become negative, say at some radius $R_1 < R_0$, where (this is therefore the only possible case) it jumps:

$$(3.59) \qquad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial r} \geqslant \delta & \text{for } \varepsilon \leqslant |x| < R_1, \\ \dfrac{\partial u_\varepsilon}{\partial r} \leqslant -\delta & \text{for } R_1 < |x| \leqslant R_0. \end{cases}$$

We claim that this situation cannot occur. Indeed, if it was the case, then $u_\varepsilon(x) < -\overline{H}(0) - \beta$ for $|x| = R_0$ and we would be able to enlarge R_0 while keeping the

inequality true. Then u_ε would stay decreasing in the whole region $|x| \geq R_1$, which again would contradict the convergence of u_ε to $-\overline{H}(0)$ at infinity.

To summarize, we have proven that

$$(3.60) \qquad \frac{\partial u_\varepsilon}{\partial r} \geq \delta, \quad \text{in the ring } \{x : \varepsilon \leq |x| \leq R_0\}.$$

Step 2. Let us now study u_ε in $B_\varepsilon(0)$.

Assume first that u_ε has a local maximum at 0. Then, $u_\varepsilon(0) \leq -\overline{H}(0) + \ell_0(0)$, and the function u_ε cannot have a minimum at some $\underline{x}$, $0 < |\underline{x}| < \varepsilon$. Indeed, if it was the case, $u_\varepsilon(\underline{x}) \geq -\overline{H}(0) + \ell_0(\underline{x}) > -\overline{H}(0) + \ell_0(0) \geq u_\varepsilon(0)$, which is contradictory. Therefore, the minimum of u_ε in $\overline{B_\varepsilon(0)}$ is achieved at $|x| = \varepsilon$, which is impossible because $\lim_{r \to \varepsilon_-} \frac{\partial u_\varepsilon}{\partial r}(r) > 0$. We have proved by contradiction that u_ε does not have a local maximum at 0.

On the other hand, the semi-concavity and the radial symmetry of u_ε imply that $\frac{\partial u_\varepsilon}{\partial r}$ has a limit at 0, which is nonpositive.

Combining the latter two points yields that u_ε has a derivative at 0 and that $\frac{\partial u_\varepsilon}{\partial r}(0) = 0$. This implies that

$$(3.61) \qquad u_\varepsilon(0) = -\overline{H}(0) + \ell_0(0),$$

and from the convergence of $u_\varepsilon(0)$ to $u(0)$, that

$$(3.62) \qquad E = \overline{H}(0) - \ell_0(0).$$

We have also proved that $-E$ is the minimal value of u_ε on $B_\varepsilon(0)$. Note that (3.61) and (3.62) may be obtained in an easier way by using arguments from the theory of optimal control.

Step 3. Our next step consists of proving that it possible to choose a positive integer $\underline{q}$ such that

$$(3.63) \qquad \min_{|x| \geq \underline{q}\varepsilon} u_\varepsilon(x) \geq \max_{|x| \leq \varepsilon} u_\varepsilon(x).$$

First, we know that there exists $M > 0$ independent of ε such that $\|Du_\varepsilon\|_\infty \leq M$. This and (3.61)-(3.62) imply that

$$(3.64) \qquad \max_{|x| \leq \varepsilon} u_\varepsilon(x) \leq -E + M\varepsilon.$$

On the other hand, (3.60) implies

$$(3.65) \qquad \begin{aligned} \min_{\underline{q}\varepsilon \leq |x| \leq R_0} u_\varepsilon(x) &\geq u_\varepsilon(|y| = \varepsilon) + \delta(\underline{q} - 1)\epsilon \\ &\geq -E + \delta(\underline{q} - 1)\epsilon. \end{aligned}$$

We deduce from (3.64) and (3.65) that if $\underline{q} > 1 + M/\delta$, then $\min_{\underline{q}\varepsilon \leq |x| \leq R_0} u_\varepsilon(x) \geq \max_{|x| \leq \varepsilon} u_\varepsilon(x)$.

Moreover, u_ε cannot reach a value smaller than $\max_{|x| \leq \varepsilon} u_\varepsilon(x)$ at some y such that $|y| > R_0$, because, if it was the case, then the infimum of $u_\varepsilon|_{\{|x| \geq R_0\}}$ would be achieved by some $\underline{x}$, $R_0 < |\underline{x}|$, and

$$\max_{|x| \leq \varepsilon} u_\varepsilon(x) \geq u_\epsilon(\underline{x}) \geq -\bar{H}(0),$$

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which is a contradiction for $\varepsilon \leqslant \varepsilon_0$. Hence (3.63) holds for $\underline{q} > 1 + M/\delta$ and $\varepsilon \leqslant \varepsilon_0$.

It is then easy to obtain (3.55) for any $\varepsilon \leqslant \varepsilon_0$, with

$$(3.66) \qquad\qquad q = \underline{q} + 2, \quad \underline{q} > 1 + M/\delta.$$

Finally, the explicit expression (3.50) of the solution to (3.49) is established, it is a straightforward adaptation of our results of Section 3.1.4 to indeed identify the limit as ε vanishes. All our arguments and conclusions hold *mutatis mutandis* with minor modifications.

Generalization. The results obtained above can be generalized to a class of non-radially symmetric situations under the following assumptions:

- The Hamiltonian $\overline{H}$ defined by $\overline{H}(p) = \max_{a \in A} -\overline{f}(a) \cdot p - \overline{\ell}(a)$ is a C^2, convex, globally Lipschitz function which reaches its minimum at $p = 0$. This implies that $\overline{H}(0) = -\min_{a \in A} \overline{\ell}(a) = -\min_{a \in A : \overline{f}(a)=0} \overline{\ell}(a)$
- The function $\ell_0 : \mathbb{R}^d \times A \to \mathbb{R}_-$ is smooth, and for all $a \in A$, $\ell_0(\cdot, a)$ is supported in the unit ball $B_1(0) \subset \mathbb{R}^d$. The perturbed Hamiltonian is $H(y, p) = \max_{a \in A} \overline{f}(a) \cdot p - \overline{\ell}(a) - \ell_0(y, a)$
- The ergodic constant E defined in Section 2.1 satisfies (3.46)
- There exists two Hamiltonians $\overline{H}_1$ and $\overline{H}_2$ which are radially symmetric, C^2, convex, globally Lipschitz fonctions defined on $\mathbb{R}^d$ and two smooth, radially symmetric functions $\ell_{0,1}$ and $\ell_{0,2}$, defined on $\mathbb{R}^d$ with values in $\mathbb{R}_-$, supported in $B_1(0)$, such that the following conditions are satisfied:
  - the pairs $(\overline{H}_1, \ell_{0,1})$ and $(\overline{H}_2, \ell_{0,2})$ satisfy all the assumptions made in the radially symmetric setting. In particular, as we have seen above, the related ergodic constants E_1 and E_2 are given by $E_1 = \overline{H}_1(0) - \ell_{0,1}(0)$ and $E_2 = \overline{H}_2(0) - \ell_{0,2}(0)$
  - there holds $\overline{H}_1(p) \leqslant \overline{H}(p) \leqslant \overline{H}_2(p)$ for all $p \in \mathbb{R}^d$, $\ell_{0,1}(x) \geqslant \ell_0(x, a) \geqslant \ell_{0,2}(x)$ for all $x \in \mathbb{R}^d$, $a \in A$. Moreover, $\overline{H}_1(0) = \overline{H}(0) = \overline{H}_2(0)$ and $\ell_{0,1}(0) = \ell_{0,2}(0)$.

The corresponding perturbed Hamiltonians are $H_1(y, p) = \overline{H}_1(p) - \ell_{0,1}(y)$ and $H_2(y, p) = \overline{H}_2(p) - \ell_{0,2}(y)$. Obviously,

$$(3.67) \qquad\qquad H_1(y, p) \leqslant H(y, p) \leqslant H_2(y, p),$$

and it can be checked that the assumptions imply that $E_1 = E = E_2$.

Using (3.67) and comparison principles, it is possible to prove that (3.52) still holds in this situation, provided that q is chosen large enough, and all the results proved in the radially symmetric setting remain valid. Note that the assumptions made are rather strong: in particular they require that $a \mapsto \ell_0(0, a)$ be constant.

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