

How to extract a spectrum from hydrodynamic equations

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A paper dedicated to the memory of Charles Doering (1956–2021)

Abstract

Practical results gained from statistical theories of turbulence usually appear in the form of an inertial range energy spectrum $\mathcal{E}(k) \sim k^{-q}$ and a cut-off wavenumber k_c . For example, the values $q = 5/3$ and $\ell k_c \sim \text{Re}^{3/4}$ are intimately associated with Kolmogorov’s 1941 theory. To extract such spectral information from the Navier–Stokes equations, Doering and Gibbon [2002] introduced the idea of forming a set of dynamic wave-numbers $\kappa_n(t)$ from ratios of norms of solutions. The time averages of the $\kappa_n(t)$ can be interpreted as the $2n$ th-moments of the energy spectrum. They found that $1 < q \leq 8/3$, thereby confirming the earlier work of Sulem and Frisch [1975] who showed that when spatial intermittency is included, no inertial range can exist in the limit of vanishing viscosity unless $q \leq 8/3$. Since the $\kappa_n(t)$ are based on Navier–Stokes weak solutions, this approach connects empirical predictions of the energy spectrum with the mathematical analysis of the Navier–Stokes equations. This method is developed to show how it can be applied to many hydrodynamic models such as the two dimensional Navier–Stokes equations (in both the direct- and inverse-cascade regimes), the forced Burgers equation and shell models.

1 Introduction

The energy spectrum of the velocity field plays an important role in fluid dynamics, since it describes how kinetic energy distributes across scales. In turbulent flows, the energy spectrum generally behaves as a power law in the range between the forcing and dissipation characteristic wavenumbers, with a slope that depends critically on the space dimension. In view of their highly fluctuating nature, turbulent flows have been studied with statistical tools, and the form of the energy spectrum has been predicted by using dimensional analysis, renormalization-group techniques, and stochastic or closure

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models. For a recent review of this topic, the reader is referred to Alexakis and Biferale [2018] and Verma [2019].

Establishing a rigorous connection between the statistical theory of turbulence and the mathematical analysis of the Navier–Stokes equations is a difficult problem [Doering and Gibbon, 1995; Foias et al., 2001; Constantin, 2006; Doering, 2009; Kuksin and Shirikyan, 2012; Bardos and Titi, 2013]. Let us first summarize how empirical estimates for length scales in the statistical theory of homogeneous and isotropic turbulence have traditionally been obtained in terms of the energy spectrum. In a d -dimensional space, this is defined as

$$\mathcal{E}(k) = c_d k^{d-1} \text{Tr} \mathbb{F}(k), \quad (1)$$

where c_d is a positive constant which depends on the spatial dimension and

$$\mathbb{F}(k) = \int_{\mathbb{R}^d} e^{-i\mathbf{k}\cdot\mathbf{r}} \overline{\mathbf{u}(\mathbf{x} + \mathbf{r}, t) \cdot \mathbf{u}(\mathbf{x}, t)} dV_{\mathbf{r}} \quad (2)$$

is the Fourier transform of the velocity spatial correlation function [Monin and Yaglom, 1975]. The overline denotes an ensemble average over the realizations of the velocity field in the statistically steady state. For a statistically stationary, homogeneous, and isotropic field, the spatial correlation does not depend on time and the position \mathbf{x} , but only on the separation \mathbf{r} . For $d = 3$ assume that $\mathcal{E}(k)$ has an inertial range between the forcing wavenumber ℓ^{-1} and a cut-off wavenumber k_c of the form

$$\mathcal{E}(k) \sim \epsilon^{2/3} \ell^{q-5/3} k^{-q} \quad (1 < q < 3), \quad (3)$$

where

$$\epsilon = \nu \int_0^\infty k^2 \mathcal{E}(k) dk \quad (4)$$

is the mean energy dissipation rate. By using (3) and ignoring the energy content in the range $k > k_c$, the mean energy dissipation rate can be estimated as $\epsilon^{1/3} \sim \nu \ell^{5/3-q} k_c^{3-q}$. This, together with the empirical prediction $\epsilon \sim U^3/\ell$ yields

$$\ell k_c \sim \text{Re}^{\frac{1}{3-q}}, \quad (5)$$

which can be found in Frisch [1995]. Here U is the root-mean square velocity and $\text{Re} = U\ell/\nu$ is the Reynolds number. The $2n$ -th moment of the energy spectrum, i.e.

$$K_n^{2n} = \frac{\int_0^\infty k^{2n} \mathcal{E}(k) dk}{\int_0^\infty \mathcal{E}(k) dk} \quad (n \geq 1), \quad (6)$$

is then estimated as

$$\ell K_n \sim (\ell k_c)^{1-\frac{q-1}{2n}} \sim \text{Re}^{\frac{1}{3-q} - \frac{1}{2n} \left(\frac{q-1}{3-q}\right)}. \quad (7)$$

Kolmogorov's 1941 theory sets q to $5/3$, which gives

$$\ell k_c \sim \text{Re}^{3/4} \quad \text{and} \quad \ell K_n \sim \text{Re}^{3/4-1/4n}. \quad (8)$$

Table 1: Estimates for the time average of $\mathcal{L}\langle\kappa_n\rangle_T$ and corresponding predictions for the inertial-range energy spectrum. \mathcal{L} is the box size L for $d = 2$ and the forcing length scale (ℓ) in all the other cases. Unless otherwise specified, $q > 1$.

System	Upper bounds on $\mathcal{L}\langle\kappa_n\rangle_T$	$\mathcal{E}(k) \sim k^{-q}$
3D Navier–Stokes	$a_\ell^{3-\frac{7}{2n}} \text{Re}^{3-\frac{5}{2n}+\frac{\delta}{n}}$	$q \leq \frac{8}{3}$
3D Navier–Stokes with suppressed fluctuations	$a_\ell^{\frac{3(n-1)(p-2)}{n(p+6)}-\frac{1}{2n}} \text{Re}^{\frac{6np-5p+6}{2n(p+6)}}$	$q \leq \frac{8}{3} - \frac{2}{p}$
2D Navier–Stokes (direct cascade)	$a_\ell^{\frac{3}{2}(1-\frac{1}{n})} \text{Re}^{\frac{3}{4}-\frac{1}{2n}}$	$q \leq \frac{11}{3}$
2D Navier–Stokes (direct cascade) with monochromatic or constant ϵ -forcing	$a_\ell^{\frac{3}{2}(1-\frac{1}{n})} \text{Re}^{\frac{1}{2}}$	$q \leq 3$
2D Navier–Stokes (inverse cascade)	$a_\ell^{n/2} \text{Re}^{\frac{1}{2}}$	$\frac{5}{3} \leq q$
Burgers	$a_\ell^{\frac{1}{3}-\frac{5}{6n}} \text{Re}^{1-\frac{1}{2n}}$	$q \leq 2$
Shell model	$a_\ell^{-\frac{1}{2n}} \text{Re}^{\frac{3}{4}-\frac{1}{4n}}$	$q \leq \frac{5}{3}$

How can a result like (7) be achieved for the incompressible Navier–Stokes equations? More specifically, how can the value of q be determined from the analysis? Rigorous results for partial differential equations are conventionally expressed as estimates of time-averages of spatial norms and not in terms of spectra. Indeed, in the language of Sobolev norms the idea of a spectrum associated with an inertial range, as in (3), has no meaning. How to circumvent this difficulty and extract results corresponding to (7) for weak solutions of the three-dimensional Navier–Stokes equations was first addressed by Doering and Gibbon [2002] twenty years ago. Moreover, in a separate but parallel paper, Doering and Foias [2002] also addressed how length scales in the forcing can be used to achieve estimates in terms of the Reynolds number Re instead of the Grashof number Gr , which is less common in the statistical theory of turbulence : see (23) in §2 for definitions of these dimensionless quantities.

A summary of these ideas is the following: first write down the Navier–Stokes equations on a periodic d -dimensional domain $\mathcal{V} = [0, L]^d$, where $d = 2, 3$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} + \mathbf{f}(\mathbf{x}), \quad \nabla \cdot \mathbf{u} = 0. \quad (9)$$

Here $\mathbf{u}(\mathbf{x}, t)$ is the velocity field, p is pressure, ν is the kinematic viscosity, and $\mathbf{f}(\mathbf{x})$ is a time-independent, mean-zero, and divergence-free body forcing. For simplicity, we follow Doering and Foias [2002] in assuming that the forcing takes the form $\mathbf{f}(\mathbf{x}) = f\Phi(\ell^{-1}\mathbf{x})$, where ℓ^{-1} is the smallest wavenumber in the forcing, f is the magnitude of the forcing, and $\Phi(\mathbf{y})$ its shape. Therefore,

$$\|\mathbf{f}\|_2 = c_n \ell^n \|\nabla^n \mathbf{f}\|_2, \quad (10)$$

where $\|\cdot\|_2^2 = \int_V |\cdot|^2 dV$ and c_n only depends on the shape of the forcing (see Doering and Foias [2002]). The aspect ratio of the box size to the forcing scale is denoted as

$$a_\ell = \frac{L}{\ell}. \quad (11)$$

As a consequence of Poincaré's inequality, $a_\ell \geq 2\pi$. The initial velocity field is taken mean-zero, so that $\mathbf{u}(\mathbf{x}, t)$ remains mean-zero at all times. In Doering and Gibbon [2002] the following sequence of squared L^2 -norms was introduced ($n = 1, 2, \dots$):

$$F_n(t) = H_n(t) + \tau_n^2 \|\nabla^n \mathbf{f}\|_2^2 \quad (12)$$

with

$$H_n(t) = \|\nabla^n \mathbf{u}(\cdot, t)\|_2^2. \quad (13)$$

The forcing term is included for the technical reason that the analysis involves a division of a differential inequality by F_n and thus H_n may be small on certain time intervals. The time scales¹ τ_n are chosen in such a way that the contribution of the forcing does not dominate the time average of H_n in the turbulent regime so the Re-scaling of the time averages of F_n and H_n remains the same. These technical issues are addressed in §2. Then the following family of time-dependent ratios was introduced

$$\kappa_{n,r}(t) = \left(\frac{F_n}{F_r} \right)^{\frac{1}{2(n-r)}} \quad (0 \leq r < n). \quad (14)$$

The $\kappa_{n,r}$ have the dimension of a wavenumber and are ordered according to $\kappa_{n,r} \leq \kappa_{n+1,r}$ and $\kappa_{n,r} \leq \kappa_{n,r+1}$, which follow from the inequality (see Lemmas 6.2 and 6.3 in Doering and Gibbon [1995])

$$F_m \leq F_{m-j}^{\frac{i}{i+j}} F_{m+i}^{\frac{j}{i+j}}, \quad 1 \leq j \leq m, i \geq 1. \quad (15)$$

The quantities $\kappa_n \equiv \kappa_{n,0}$ play a special role because of their physical meaning. Indeed, Parseval's equality yields

$$H_n(t) = L^d \sum_{\mathbf{k}} k^{2n} |\hat{\mathbf{u}}(\mathbf{k}, t)|^2 \quad (16)$$

with $\hat{\mathbf{u}}(\mathbf{k}, t)$ as the inverse spatial Fourier transform of $\mathbf{u}(\mathbf{x}, t)$. Hence

$$\kappa_n^{2n}(t) = \frac{\sum_{\mathbf{k}} k^{2n} \left(|\hat{\mathbf{u}}(\mathbf{k}, t)|^2 + \tau_n^2 |\hat{\mathbf{f}}(\mathbf{k})|^2 \right)}{\sum_{\mathbf{k}} \left(|\hat{\mathbf{u}}(\mathbf{k}, t)|^2 + \tau_n^2 |\hat{\mathbf{f}}(\mathbf{k})|^2 \right)}. \quad (17)$$

At large Reynolds numbers, $\kappa_n^{2n}(t)$ can therefore be regarded as the $2n$ -th moment of the (instantaneous) energy spectrum. The strategy in Doering and Gibbon [2002], which

¹In Doering and Gibbon [2002] the τ_n had no n -dependence. However, the development of the method to other cases sometimes requires this dependence so it has been introduced at this point.

also adopted some ideas on the forcing from Doering and Foias [2002], was to find a class of estimates of the type

$$\langle \ell \kappa_n \rangle_T \leq c_n \text{Re}^{\xi_n}, \quad (18)$$

for the set of time averages $\langle \ell \kappa_n \rangle_T$, where the brackets

$$\langle \cdot \rangle_T = \frac{1}{T} \int_0^T \cdot dt \quad (19)$$

denote a long-time average. The specific form of ξ_n found in Doering and Gibbon [2002] is given in Theorem 1 and is also displayed in the first line of Table 1. The estimate in (18) in terms of Re then allowed them to make the final step which was to compare the exponent ξ_n with that in (7) that comes from Frisch [1995]

$$\frac{1}{3-q} - \frac{1}{2n} \left(\frac{q-1}{3-q} \right) \leq \xi_n. \quad (20)$$

The direction of the inequality in (20) reflects that in (18). In reality, results from statistically stationary, homogeneous, isotropic turbulence theory are being compared with estimates of the long-time averages of ratios of Navier-Stokes spatial norms. The value of ξ_n from Theorem 1 gives the range of q and this turns out to be precisely

$$1 < q \leq 8/3 \quad (21)$$

as in Sulem and Frisch [1975], where a bound on the energy spectrum was obtained by considering a ‘shell decomposition’ of the velocity field and examining the energy flux across wavenumbers.

Here we will endeavour to show that this method has much greater scope and can be applied in other circumstances, such as the 2D Navier–Stokes equations (in both the direct- and inverse-cascade regimes), Burgers equation, and shell models. Table 1 summarises the range of q for each of these cases with the details provided in the rest of the paper. Although the spectral slopes for these systems are known, our study shows that they can be obtained in a systematic way within the same mathematical framework and thus confirms the wide applicability of these methods to the analysis of hydrodynamic equations.

2 The Navier–Stokes equations in three and two dimensions

In the following, we consider the Navier–Stokes equations in both $d = 3$ and $d = 2$ dimensions. For weak solutions with initial data in $L^2(\mathcal{V})$, the root-mean square velocity

$$U = L^{-d/2} \sqrt{\langle \|\mathbf{u}\|_2^2 \rangle_T} < \infty. \quad (22)$$

Suitable definitions of the Grashof and Reynolds numbers are

$$\text{Gr} = \frac{f \ell^3}{\nu^2}, \quad \text{Re} = \frac{U \ell}{\nu}. \quad (23)$$

The former is a dimensionless measure of the magnitude of the forcing, whereas the latter is the system response. The reason for using the forcing scale instead of the domain size in (23) is that the estimates based on these definitions of Gr and Re remain valid in the infinite volume limit $a_\ell \rightarrow \infty$ [Doering and Foias, 2002].

Gr and Re satisfy the bound [Doering and Foias, 2002]

$$\text{Gr} \leq c(\text{Re} + \text{Re}^2), \quad (24)$$

which shows that the turbulent regime is achieved for $\text{Gr} \gg 1$. In Doering and Foias [2002], the bound in (24) is also rewritten in terms of the mean energy dissipation rate,²

$$\epsilon = \nu L^{-d} \langle H_1 \rangle_T, \quad (25)$$

as

$$\epsilon \leq c\nu^3 \ell^{-4} (\text{Re}^2 + \text{Re}^3). \quad (26)$$

In addition, Doering and Foias [2002] proved the inequality

$$c_1 f \leq c_2 \nu^{1/2} \ell^{-1} \epsilon^{1/2} + c_3 \nu^{-1/2} U \epsilon^{1/2}, \quad (27)$$

which, in turn, gives the lower bound

$$\epsilon \geq c\nu^3 \ell^{-4} \frac{\text{Gr}^2}{(1 + \text{Re})^2}. \quad (28)$$

The L^2 -norms F_n include a contribution from the forcing which must not dominate $\langle H_n \rangle_T$ as $\text{Gr} \rightarrow \infty$. This is achieved by suitably choosing the time scales τ_n . Using Poincaré's inequality, (10), and (28) yields

$$\frac{\tau_n^2 \|\nabla^n \mathbf{f}\|_2^2}{\langle H_n \rangle_T} \leq c_n L^{2(n-1)} \frac{\tau_n^2 \|\nabla^n \mathbf{f}\|_2^2}{\langle H_1 \rangle_T} = c_n \nu \epsilon^{-1} L^{2(n-1)} \ell^{-2n} \tau_n^2 f^2 \quad (29)$$

$$= c_n \nu^5 \epsilon^{-1} L^{2(n-1)} \ell^{-2(n+3)} \tau_n^2 \text{Gr}^2 \leq c_n \nu^2 \ell^{-4} a_\ell^{2(n-1)} \tau_n^2 (1 + \text{Re})^2. \quad (30)$$

Therefore, a suitable definition of τ_n is

$$\tau_n = \frac{\nu^{-1} \ell^2}{a_\ell^{(n-1)} (1 + \text{Re})^{1+2\delta}} \quad (31)$$

with

$$0 < \delta < \frac{1}{6} \quad \text{for } d = 3 \quad \text{and} \quad \delta = 0 \quad \text{for } d = 2, \quad (32)$$

so that

$$\tau_n^2 \|\nabla^n \mathbf{f}\|_2^2 \leq c_n \text{Re}^{-4\delta} \langle H_n \rangle_T \quad \text{as} \quad \text{Gr} \rightarrow \infty. \quad (33)$$

The non-zero δ -correction is required when $d = 3$ because, for technical reasons, the forcing contribution to $\langle F_n \rangle_T$ needs to become negligible as $\text{Gr} \rightarrow \infty$ [Doering and

²Estimates of time averaged quantities all have a correction term $O(T^{-1})$ on their right hand side. From now on this will be ignored.

Gibbon, 2002]. When $d = 2$, the contribution of the forcing simply must not grow faster than $\langle H_n \rangle_T$ [Gibbon and Pavliotis, 2007]. We shall see that the use of definition (31) systematically improves the power of a_ℓ in the estimates of $\langle \kappa_n \rangle_T$. Although this is of little importance in most cases because generally $a_\ell = O(1)$, it becomes essential in the study of the inverse-cascade regime of the 2D Navier–Stokes equations, which is characterized by large values of a_ℓ .

Finally, when $d = 2$ and $d = 3$ the F_n satisfy the following ‘ladder’ of differential inequalities as $\text{Gr} \rightarrow \infty$ (see qualifications in Appendix A):

$$\frac{1}{2} \dot{F}_n \leq -\nu F_{n+1} + c_n (\|\nabla \mathbf{u}\|_\infty + \tau_n^{-1}) F_n. \quad (34)$$

Now we shall see that estimates for $\langle \kappa_n \rangle_T$ differ, leading to different ranges of q . Throughout this paper c and c_n denote dimensionless, generic constants.

2.1 Three examples involving the 3D Navier–Stokes equations

The main result of Doering and Gibbon [2002] is an estimate for the time average of κ_n for weak solutions of the 3D Navier–Stokes equations.³

Theorem 1 (Doering and Gibbon [2002]). *For $n \geq 2$ and $0 < \delta < \frac{1}{6}$,*

$$\ell \langle \kappa_n \rangle_T \leq c_n a_\ell^{3 - \frac{7}{2n}} \text{Re}^{3 - \frac{5}{2n} + \frac{\delta}{n}} \quad \text{as} \quad \text{Gr} \rightarrow \infty. \quad (35)$$

Remark 1. *Comparing the exponents of Re in (35) and (7) gives*

$$\frac{1 - \frac{q-1}{2n}}{3 - q} \leq 3 - \frac{5}{2n} + \frac{\delta}{n}, \quad (36)$$

whence

$$q \leq \frac{8}{3} + \frac{2\delta}{3(n-1)}. \quad (37)$$

Thus, for every value of n , we find that $1 < q \leq \frac{8}{3}$ as advertised in Table 1.

Doering and Gibbon [2002] also investigated how the energy spectrum is modified when the spatial fluctuations of the velocity gradients are suppressed through the assumption

$$\|\nabla \mathbf{u}\|_\infty \approx c L^{-3/p} \|\nabla \mathbf{u}\|_p, \quad 2 \leq p \leq \infty. \quad (38)$$

For $p = 2$, this means that as Re increases, the maximum velocity scales as the root-mean square velocity. Higher values of p correspond to a milder suppression of fluctuations, and (35) is recovered for $p = \infty$. It can also be shown that decreasing p in (38) corresponds to a stronger and stronger suppression of intermittency in the multifractal description of turbulence [Dubrulle and Gibbon, 2022].

With approximation (38), Theorem 1 is modified as follows⁴

³The difference in the power of a_ℓ compared to the original version of the theorem is due to the use of definition (31), which gives an extra factor a_ℓ^{-1} in the estimate of $\langle \kappa_1^2 \rangle_T$: $\ell^2 \langle \kappa_1^2 \rangle_T \leq c a_\ell^{-1} \text{Re}^{1+2\delta}$ as $\text{Gr} \rightarrow \infty$. The rest of the proof is unchanged.

⁴The exponent δ can be set to zero when assumption (38) is used.

Theorem 2 (Doering and Gibbon [2002]). *Under assumption (38) and for $n \geq 2$*

$$\ell \langle \kappa_n \rangle_T \leq c_n a_\ell^{\frac{3(n-1)(p-2)}{n(p+6)} - \frac{1}{2n}} \text{Re}^{\frac{6np-5p+6}{2n(p+6)}} \quad \text{as} \quad \text{Gr} \rightarrow \infty. \quad (39)$$

If the energy spectrum is as in (3), an argument analogous to that used for $p = \infty$ shows that the scaling in Theorem 2 is consistent with

$$1 < q \leq \frac{8}{3} - \frac{2}{p}. \quad (40)$$

In particular, $p = 2$ yields the Kolmogorov spectrum $q = 5/3$. More generally, by altering the value of p in the range $2 \leq p \leq \infty$ we find that the upper bound of q , designated as q_{ub} , lies in the range

$$\frac{5}{3} \leq q_{ub} \leq \frac{8}{3}. \quad (41)$$

These methods have also been applied to magnetohydrodynamic turbulence to show that the Iroshnikov–Kraichnan total-energy spectrum can be excluded when there is no cross-correlation between the velocity and magnetic fields [Gibbon et al., 2016].

The bounds on $\langle \kappa_n \rangle_T$ come out from scaling arguments and, for $n = 1$, yield the estimate $L \text{Re}^{-3/4}$ for the Kolmogorov scale, which is universally acknowledged to be sharp [Gibbon, 2019]. This suggests, although does not prove, that the all hierarchy is sharp.

2.2 The 2D Navier–Stokes equations

Consider the Navier–Stokes equations on the periodic square $\mathcal{V} = [0, L]^2$. The definitions introduced in §1 extend unchanged to two dimensions ($d = 2$). However, the absence of vortex stretching leads to a different estimate for the time average of κ_n .

Two-dimensional turbulence is characterized by a dual cascade consisting of a direct cascade of enstrophy (defined as $\|\boldsymbol{\omega}\|_2^2$ with $\boldsymbol{\omega} = \nabla \times \mathbf{u}$) from ℓ^{-1} to high wavenumbers and an inverse cascade of energy from ℓ^{-1} to low wavenumbers [Kraichnan and Montgomery, 1980; Kellay and Goldburg, 2002; Tabeling, 2002; Boffetta and Ecke, 2012]. The enstrophy cascade ends at a cutoff wavenumber k_c , beyond which enstrophy is dissipated by viscosity. In an unbounded domain or in a bounded domain before statistical equilibrium is established, the energy cascade continues to extend to ever smaller wavenumbers, and a quasi-steady spectrum forms at wavenumbers between the inverse integral scale and ℓ^{-1} .

We study the spectra of the two cascades separately by considering first the case $\ell \sim L/2\pi$ (direct cascade) and then $\ell \ll L$ (inverse cascade). Bounds on the energy spectrum for the dual cascade in 2D turbulence have been provided by Tran and Bowman [2003, 2004].

2.2.1 Direct cascade of enstrophy

The following theorem⁵ describes the behaviour of $\langle \kappa_n^2 \rangle_T$ as $\text{Gr} \rightarrow \infty$ while $a_\ell = O(1)$.

Theorem 3 (Gibbon and Pavliotis [2007]). *For $n \geq 2$*

$$L^2 \langle \kappa_n^2 \rangle_T \leq c_n a_\ell^{3(1-\frac{1}{n})} \text{Re}^{\frac{3}{2}-\frac{1}{n}} [\ln(a_\ell^2 \text{Re})]^{\frac{1}{2}-\frac{1}{n}} \quad \text{as} \quad \text{Gr} \rightarrow \infty. \quad (42)$$

It follows that

$$L \langle \kappa_n \rangle_T \leq c_n a_\ell^{\frac{3}{2}(1-\frac{1}{n})} \text{Re}^{\frac{3}{4}-\frac{1}{2n}} [\ln(a_\ell^2 \text{Re})]^{\frac{1}{4}-\frac{1}{2n}}. \quad (43)$$

We want to compare this bound with a practical estimate for LK_n (see (6) for the definition) under the assumption that $\ell \sim L/2\pi$. Consider the mean enstrophy dissipation rate

$$\eta_\nu = \nu L^{-2} \langle H_2 \rangle_T. \quad (44)$$

This is bounded [Alexakis and Doering, 2006, Gibbon and Pavliotis, 2007] and, at large Re , can be estimated as

$$\eta_\nu \sim \frac{U^3}{\ell^3}. \quad (45)$$

The flow is assumed to be isotropic and to have an energy spectrum of the form

$$\mathcal{E}(k) \sim \eta_\nu^{2/3} \ell^{3-q} k^{-q} \quad (\ell^{-1} \leq k \leq k_c) \quad (46)$$

with $1 < q < 5$. The mean enstrophy dissipation rate can be obtained from the energy spectrum via the relation [Monin and Yaglom, 1975]:

$$\eta_\nu \approx \nu \int_{\ell^{-1}}^{k_c} k^4 \mathcal{E}(k) dk \sim \nu \eta_\nu^{2/3} \ell^{3-q} k_c^{5-q}, \quad (47)$$

where the contributions coming from wavenumbers $k > k_c$ have been ignored. Combining (45), (46), and (47) yields

$$\ell k_c \sim \text{Re}^{\frac{1}{5-q}}. \quad (48)$$

By plugging (46) into (6) and using (48), we find:

$$LK_n \sim \ell K_n \sim (\ell k_c)^{1-\frac{q-1}{2n}} \sim \text{Re}^{\frac{1}{5-q}-\frac{1}{2n}(\frac{q-1}{5-q})}. \quad (49)$$

We now compare (49) with (43) and conclude that the Reynolds-number scaling of K_n is consistent with that of $\langle \kappa_n \rangle_T$ provided that

$$q \leq \frac{11n - 12}{3n - 4}. \quad (50)$$

⁵There is a small difference in the powers of a_ℓ and $\ln \text{Re}$ between (42) and the original version of the theorem. This is due to the choice of τ_n , which modifies the estimates of $\langle \kappa_1^2 \rangle_T$ and $\langle \kappa_{2,1}^2 \rangle_T$. With definition (31), as $\text{Gr} \rightarrow \infty$

$$L^2 \langle \kappa_1^2 \rangle_T \leq c a_\ell \text{Re} \quad \text{and} \quad L^2 \langle \kappa_{2,1}^2 \rangle_T \leq c a_\ell^2 \text{Re}.$$

Since this must hold for all $n \geq 2$ and the right-hand side of (50) is a decreasing function of n , we find that

$$1 < q \leq \frac{11}{3}. \quad (51)$$

Remark 2. *The bound in (50) can also be derived by comparing the high-Re scaling of the $2(n-1)$ -th moment of the enstrophy spectrum with the bound for $\langle \kappa_{n,1} \rangle_T^{2(n-1)}$ obtained in Gibbon and Pavliotis [2007].*

Remark 3. *The bound in (50) agrees with a practical estimate of Sulem and Frisch [1975] and a rigorous result of Eyink [1996]. The exponent $-11/3$ also describes the energy spectrum of spiral structures in two-dimensional turbulence [Gilbert, 1988].*

In numerical simulations of isotropic turbulence, the following two types of forcing are commonly used: (i) strictly monochromatic forcings with a single wavenumber ℓ^{-1} and (ii) forcings that maintain a constant energy injection rate ϵ , i.e.

$$\mathbf{f} = \epsilon L^2 \frac{\mathcal{P}\mathbf{u}}{\|\mathcal{P}\mathbf{u}\|_2}, \quad (52)$$

where the operator \mathcal{P} projects the velocity field on a finite set of spatial modes. For these forcings, it is possible to derive a more stringent bound on q . Indeed, as $\text{Gr} \rightarrow \infty$ the general estimate for the mean enstrophy dissipation rate [Alexakis and Doering, 2006, Gibbon and Pavliotis, 2007]

$$\langle H_2 \rangle_T \leq c \nu^2 \ell^{-4} a_\ell^2 \text{Re}^3 \quad (53)$$

is replaced with [Alexakis and Doering, 2006]

$$\langle H_2 \rangle_T \leq c \nu^2 \ell^{-4} a_\ell^2 \text{Re}^2. \quad (54)$$

Using (54) in the proof of Theorem 3 yields the following result.

Theorem 4. *For $n \geq 2$ and a monochromatic or a constant-energy-input forcing*

$$L^2 \langle \kappa_n^2 \rangle_T \leq c_n a_\ell^{3-3/n} \text{Re} [\ln(a_\ell^2 \text{Re})]^{1/2-1/n} \quad \text{as } \text{Gr} \rightarrow \infty. \quad (55)$$

By comparing (55) with (49), we find that for these types of forcing

$$q \leq 3 + \frac{2}{n-1}, \quad (56)$$

which yields for every n

$$1 < q \leq 3. \quad (57)$$

Note that, up to logarithmic corrections, Kraichnan's prediction for the energy spectrum in the enstrophy-cascade range is $\mathcal{E}(k) \sim k^{-3}$ [Kraichnan, 1967, 1971]. Alternative proofs of Kraichnan's scaling for the direct-cascade regime were provided by Eyink [2001], Foias et al. [2002], Tran and Dritschel [2006].

2.2.2 The inverse cascade of energy

To investigate the regime of the inverse cascade, we study the behaviour of $\langle \kappa_n \rangle_T$ in the limit in which $a_\ell \rightarrow \infty$ while the Reynolds number based on the characteristic velocity at the forcing scale is $O(1)$. More precisely, consider $u_f^2 = L^{-2} \langle \| \mathbf{u}_f \|_2^2 \rangle_T$, where

$$\mathbf{u}_f(\mathbf{x}, t) = \sum_{|\mathbf{k}| > \ell^{-1}} e^{i\mathbf{k} \cdot \mathbf{x}} \hat{\mathbf{u}}(\mathbf{k}, t), \quad (58)$$

and define

$$\text{Re}_f = \frac{u_f \ell}{\nu}. \quad (59)$$

The direct cascade of enstrophy is negligible when the forcing wavenumber is comparable to the viscous dissipation wavenumber and therefore $\text{Re}_f \sim 1$ [see Alexakis and Biferale, 2018]. Furthermore, if the energy spectrum $\mathcal{E}(k) \sim k^{-q}$ for $L^{-1} \leq k \leq \ell^{-1}$ and is negligible otherwise, it can be shown that [Smith and Yakhot, 1994, Tran, 2007]

$$U^2 = a_\ell^{q-1} u_f^2 \quad (60)$$

and hence

$$\text{Re} = a_\ell^{(q-1)/2} \text{Re}_f. \quad (61)$$

Therefore, in the regime considered here, $\text{Re} \sim a_\ell^{(q-1)/2}$. We can now prove a bound on $\langle \kappa_n \rangle_T$ which is relevant to the energy cascading range.

Theorem 5. *If $\mathcal{E}(k)$ is steeper than k^{-5} and $\text{Re}_f = O(1)$, then for $n \geq 1$ and as $a_\ell \rightarrow \infty$*

$$L^2 \langle \kappa_n^2 \rangle_T \leq c a_\ell^n \text{Re}. \quad (62)$$

Proof. Recall that from (33)

$$\langle F_2 \rangle_T \leq c \langle H_2 \rangle_T \leq c \nu^2 \ell^{-4} a_\ell^2 (\text{Re}^2 + \text{Re}^3), \quad (63)$$

where the bound on $\langle H_2 \rangle_T$ can be found in Alexakis and Doering [2006] and Gibbon and Pavliotis [2007]. Using the tighter bound for monochromatic or constant-energy-input forcings would not change the result in this case.

In addition, the following form of the Brezis–Gallouët inequality holds [Doering and Gibbon, 1995]

$$\| \nabla \mathbf{u} \|_\infty \leq c F_2^{1/2} [1 + \ln(L \kappa_{3,2})]^{1/2}. \quad (64)$$

Now note that, as $a_\ell \rightarrow \infty$, the F_n satisfy the same ladder as in (34) (see Appendix A). Thus, following Gibbon and Pavliotis [2007], we divide through the ladder by F_n and time average. We then use (64) together with $\kappa_{n,r} \leq \kappa_{n+1,r} \leq \kappa_{n+1,n}$ for $2 \leq r < n$ and Jensen's inequality on the logarithm to find

$$L^2 \langle \kappa_{n,r}^2 \rangle_T \leq c L^2 \nu^{-1} \langle F_2 \rangle_T^{1/2} [1 + \ln(L^2 \langle \kappa_{n,r}^2 \rangle_T)]^{1/2} + c a_\ell^{n+1} (1 + \text{Re}). \quad (65)$$

By using (61) and (63), we can see that the first term on the right-hand side behaves as $a_\ell^{3+3(q-1)/4} \text{Re}_f^{3/2}$, whereas the second behaves as $a_\ell^{n+1+(q-1)/2} \text{Re}_f$. Since $\text{Re}_f = O(1)$, $n \geq 3$, and $q < 5$, the second term dominates over the first. For $n > r \geq 2$ and in the limit $a_\ell \rightarrow \infty$ while $\text{Re}_f = O(1)$, we thus find

$$L^2 \langle \kappa_{n,r}^2 \rangle_T \leq c a_\ell^{n+1} (1 + \text{Re}). \quad (66)$$

By adapting the proofs of Gibbon and Pavliotis [2007] in the manner described in Appendix A, it is also possible to show that as $a_\ell \rightarrow \infty$

$$\frac{1}{2} \dot{F}_0 \leq -\nu F_1 + \frac{F_0}{2\tau_0} \quad \text{and} \quad \frac{1}{2} \dot{F}_1 \leq -\nu F_2 + \frac{F_1}{2\tau_1}, \quad (67)$$

which imply

$$L^2 \langle \kappa_1^2 \rangle_T \leq c a_\ell (1 + \text{Re}) \quad \text{and} \quad L^2 \langle \kappa_{2,1}^2 \rangle_T \leq c a_\ell^2 (1 + \text{Re}). \quad (68)$$

The advertised result follows from using (66) and (68) in

$$\langle \kappa_n^2 \rangle_T = \left\langle \left(\frac{F_n}{F_1} \right)^{1/n} \left(\frac{F_1}{F_0} \right)^{1/n} \right\rangle_T \leq \langle \kappa_{n,1}^2 \rangle_T^{(n-1)/n} \langle \kappa_1^2 \rangle_T^{1/n}. \quad (69)$$

□

We now move to the practical estimate for the moments of the spectrum. We remind the reader that we are assuming that $\ell \ll L$ and $\ell^{-1} \sim k_c$, so that the contribution from the spectrum at wavenumbers in the enstrophy cascading range is negligible. Assuming that $\mathcal{E}(k) \sim k^{-q}$ with $1 < q < 5$ in the range $L^{-1} \leq k \leq \ell^{-1}$, we find

$$L^{2n} K_n^{2n} \approx \frac{\int_{L^{-1}}^{\ell^{-1}} k^{2n} \mathcal{E}(k) dk}{\int_{L^{-1}}^{\ell^{-1}} \mathcal{E}(k) dk} \sim \left(\frac{L}{\ell} \right)^{2n+1-q}, \quad (70)$$

or

$$LK_n \sim a_\ell^{1-(q-1)/2n}. \quad (71)$$

In order to compare (71) with the mathematical bound for $L \langle \kappa_n \rangle_T$, we recall (61) and the assumption $\text{Re}_f = O(1)$. Thus, (71) can be recast as

$$LK_n \sim a_\ell^{1-(q-1)/2n-(q-1)/4} \text{Re}^{1/2} \quad (72)$$

and the practical estimate for LK_n is consistent with the bound for $L \langle \kappa_n \rangle_T$ if

$$q \geq \frac{2 + 5n - 2n^2}{n + 2}. \quad (73)$$

Since the right-hand side is a decreasing function of n , this means that the constraint on q is fixed by the $n = 1$ case, i.e.

$$q \geq \frac{5}{3}. \quad (74)$$

This lower bound agrees with an earlier result of Tran [2007] and with Kraichnan's prediction for the energy cascading range [Kraichnan, 1967, 1971]. That the bound is obtained for $n = 1$ rather than considering the large- n limit is consistent with the fact that the inverse energy cascade is a large-scale phenomenon.

3 Burgers equation

All the quantities introduced in §2 can be defined analogously for the Burgers equation by taking $d = 1$ on the periodic interval $\mathcal{V} = [0, L]$

$$\partial_t u + u \partial_x u = \nu \partial_x^2 u + f. \quad (75)$$

In particular, we can again set $\delta = 0$ in the definition of τ_n .

The following two lemmas can be proved by adapting the proofs for the 3D Navier–Stokes equations [see Doering and Gibbon, 2002] to the Burgers equation :

Lemma 1. *For $n \geq 1$ and as $\text{Gr} \rightarrow \infty$, the F_n satisfy the ladder in (34).*

Lemma 2. *There exists a positive constant c such that, as $\text{Gr} \rightarrow \infty$,*

$$\ell^2 \langle \kappa_1^2 \rangle_T \leq c a_\ell^{-1} \text{Re}. \quad (76)$$

We now prove the analogue of Theorems 1 and 3 for the Burgers equation.

Theorem 6. *For $n \geq 2$, as $\text{Gr} \rightarrow \infty$*

$$\ell^2 \langle \kappa_n^2 \rangle_T \leq c_n a_\ell^{\frac{2}{3} - \frac{5}{3n}} \text{Re}^{2 - \frac{1}{n}}. \quad (77)$$

Proof. The inequality

$$\|\partial_x u\|_\infty \leq c \|\partial_x u\|_2^{1/2} \|\partial_x^2 u\|_2^{1/2} \leq c F_1^{1/4} F_2^{1/4} \quad (78)$$

turns the ladder in (34) into

$$\frac{1}{2} \dot{F}_n \leq -\nu F_{n+1} + c_n \left(F_1^{1/4} F_2^{1/4} + \tau_n^{-1} \right) F_n. \quad (79)$$

By dividing through by F_n , time averaging, noting that the forcing term is subdominant and can therefore be ignored, and using the Cauchy–Schwarz inequality, we find

$$\langle \kappa_{n+1,n}^2 \rangle_T \leq \frac{c_n}{\nu} \langle F_1^{1/4} F_2^{1/4} \rangle_T = \frac{c_n}{\nu} \left\langle \left(\frac{F_2}{F_1} \right)^{1/4} F_1^{1/2} \right\rangle_T = \frac{c_n}{\nu} \langle \kappa_{2,1}^{1/2} F_1^{1/2} \rangle_T \leq \frac{c_n}{\nu} \langle \kappa_{2,1} \rangle_T^{1/2} \langle F_1 \rangle_T^{1/2} \quad (80)$$

and hence, by using Jensen's inequality,

$$\langle \kappa_{n+1,n}^2 \rangle_T \leq c_n \nu^{-1} \langle \kappa_{2,1}^2 \rangle_T^{1/4} \langle F_1 \rangle_T^{1/2}. \quad (81)$$

The estimate of Doering and Foias [2002] for the mean energy dissipation rate, and consequently the corresponding estimate for $\langle F_1 \rangle_T$, also hold for the Burgers equation. As $\text{Gr} \rightarrow \infty$, we thus have $\langle F_1 \rangle_T \leq c \langle H_1 \rangle_T \leq c \nu^2 \ell^{-3} a_\ell \text{Re}^3$. Inserting this estimate into (81) with $n = 1$ yields as $\text{Gr} \rightarrow \infty$

$$\langle \kappa_{2,1}^2 \rangle_T \leq c \ell^{-2} a_\ell^{2/3} \text{Re}^2. \quad (82)$$

Together (81) and (82) give

$$\langle \kappa_{n,1}^2 \rangle_T \leq \langle \kappa_{n+1,n}^2 \rangle_T \leq c_n \ell^{-2} a_\ell^{2/3} \text{Re}^2. \quad (83)$$

We also have

$$\langle \kappa_n^2 \rangle_T = \left\langle \left(\frac{F_n}{F_0} \right)^{\frac{1}{n}} \right\rangle_T = \left\langle \left(\frac{F_n}{F_1} \right)^{\frac{1}{n}} \left(\frac{F_1}{F_0} \right)^{\frac{1}{n}} \right\rangle_T = \left\langle \kappa_{n,1}^{\frac{2(n-1)}{n}} \kappa_1^{\frac{2}{n}} \right\rangle_T \leq \langle \kappa_{n,1}^2 \rangle_T^{\frac{n-1}{n}} \langle \kappa_1^2 \rangle_T^{\frac{1}{n}}. \quad (84)$$

Therefore, by using (83) and Lemma 2, we find

$$\langle \kappa_n^2 \rangle_T \leq \langle \kappa_{n,1}^2 \rangle_T^{\frac{n-1}{n}} \langle \kappa_1^2 \rangle_T^{\frac{1}{n}} \leq c \ell^{-2} a_\ell^{\frac{2}{3} - \frac{5}{3n}} \text{Re}^{2 - \frac{1}{n}}. \quad (85)$$

□

We assume again that the flow is isotropic, the forcing is large-scale with $\ell \sim L/2\pi$, and the energy spectrum is as in (3) with $1 < q < 3$. By proceeding as for $d = 3$ (see § 1), we find

$$\ell k_c \sim \text{Re}^{\frac{1}{3-q}} \quad (86)$$

and

$$\ell K_n \sim (\ell k_c)^{1 - \frac{q-1}{2n}} \sim \text{Re}^{\frac{1}{3-q} - \frac{1}{2n} \left(\frac{q-1}{3-q} \right)}. \quad (87)$$

Therefore, after comparing (87) with Theorem 6, we conclude that the scaling of ℓK_n is consistent with that of $\ell \langle \kappa_n \rangle_T$ provided that

$$1 < q \leq 2. \quad (88)$$

Remark 4. *The energy spectrum of the Burgers equation for a large-scale forcing is known to behave as k^{-2} [e.g. Frisch et al., 2013, Boritchev, 2014]. A k^{-2} energy spectrum is also found in the decaying Burgers equation [Tran and Dritschel, 2010].*

4 Shell models

In the ‘Sabra’ shell model [L’vov et al., 1998], the velocity variables u_j are complex and satisfy the system of ordinary differential equations⁶

$$\dot{u}_j = i(a_1 k_{j+1} u_{j+1}^* u_{j+2} + a_2 k_j u_{j+1} u_{j-1}^* - a_3 k_{j-1} u_{j-1} u_{j-2}) - \nu k_j^2 u_j + f_j, \quad j = 1, 2, 3, \dots, \quad (89)$$

where u_j^* is the complex conjugate of u_j , ν is the kinematic viscosity, f_j are the forcing variables, and $k_j = k_0 \lambda^j$ with $k_0 > 0$ and $\lambda > 1$. The ‘boundary conditions’ are $u_0 = u_{-1} = 0$, while the coefficients a_1, a_2, a_3 are real and such that $a_1 + a_2 + a_3 = 0$. This ensures that the kinetic energy

$$E = \sum_{j=1}^{\infty} |u_j|^2 \quad (90)$$

is conserved when $\nu = 0$ and $f_j = 0$ for all j . Moreover, the time-averaged energy dissipation rate is

$$\epsilon = \nu \left\langle \sum_{j=1}^{\infty} k_j^2 |u_j|^2 \right\rangle_T. \quad (91)$$

The forcing is assumed to be of the form $f_j = \mathcal{F} \phi_{j-j_f}$, where \mathcal{F} is a complex constant, ϕ_p is time independent,

$$\sum_{p=0}^{\infty} \lambda^{-2p} |\phi_p|^2 = 1, \quad (92)$$

and $\phi_p = 0$ for $p < 0$ and $p > j_{\max} - j_f$. Therefore, $k_f = k_0 \lambda^{j_f}$ and $k_{\max} = k_0 \lambda^{j_{\max}}$ are the minimal and maximum wavenumbers of the forcing, respectively. Under these assumptions and if the initial energy is finite, the shell model has globally regular solutions [Constantin et al., 2006]. Finally, Gr and Re are defined as in §1 with $U = \sqrt{\langle E \rangle_T}$, $\ell = k_f^{-1}$, $a_\ell = k_f/k_1$, and $f = |\mathcal{F}|$.

The shell-model analogues of H_n and F_n are

$$H_n = \sum_{j=1}^{\infty} k_j^{2n} |u_j|^2, \quad F_n = H_n + \tau_n^2 \sum_{j=1}^{\infty} k_j^{2n} |f_j|^2, \quad (93)$$

and τ_n is as in (31) with $\delta = 0$. As in the case of the Navier–Stokes equations, the definition of τ_n ensures that $\langle F_n \rangle_T$ and $\langle H_n \rangle_T$ scale in the same way as $\text{Gr} \rightarrow \infty$. Indeed, ϵ satisfies an inequality analogous to (27) (see (B.8) in Appendix B) which gives the same lower bound as in (28). Using a shell-model version of Poincaré’s inequality

$$\epsilon \leq \nu k_1^{-2(n-1)} \langle H_n \rangle_T. \quad (94)$$

⁶The results would be the same for the Gledzer–Ohkitani–Yamada (GOY) shell model [Gledzer, 1973, Yamada and Ohkitani, 1987].

we find

$$\begin{aligned}
 \tau_n^2 \sum_{j=1}^{\infty} k_j^{2n} |f_j|^2 &= b_n \nu^{-2} k_f^{2n-4} a_\ell^{-2(n-1)} (1 + \text{Re})^{-2} f^2 \\
 &= b_n \nu^2 k_f^{2(n+1)} a_\ell^{-2(n-1)} (1 + \text{Re})^{-2} \text{Gr}^2 \\
 &\leq c b_n \nu^{-1} k_f^{2(n-1)} a_\ell^{-2(n-1)} \epsilon \leq c b_n \langle H_n \rangle_T,
 \end{aligned} \tag{95}$$

where

$$b_n = \sum_{p=0}^{j_{\max} - j_f} \lambda^{2np} |\phi_p|^2 \tag{96}$$

and the constant c depends on ϕ_j but not on the magnitude of the forcing $|\mathcal{F}|$ (see Appendix B). It was proved in Vincenzi and Gibbon [2021] that, analogously to the Navier–Stokes equations,

$$\text{Gr} \leq c(\text{Re} + \text{Re}^2) \tag{97}$$

and as $\text{Gr} \rightarrow \infty$

$$\langle H_1 \rangle_T \leq c \nu^2 \ell^{-4} \text{Re}^3. \tag{98}$$

In addition, as $\text{Gr} \rightarrow \infty$ the F_n satisfy the same ladder of differential inequalities as in (34) with $\|\nabla \mathbf{u}\|_\infty$ replaced with $\sup_{1 \leq j \leq \infty} k_j |u_j|$:⁷

$$\frac{1}{2} \dot{F}_n \leq -\nu F_{n+1} + c_n \left(\sup_{1 \leq j \leq \infty} k_j |u_j| + \tau_n^{-1} \right) F_n. \tag{99}$$

In shell models, the energy spectrum is defined as $\mathcal{E}(k_j) = k_j^{-1} \langle |u_j(t)|^2 \rangle_T$ [Yamada and Ohkitani, 1987]. Therefore, in the limit $\text{Gr} \rightarrow \infty$, the quantity $\kappa_n^{2n} = F_n/F_0$ behaves as the ratio of the $(2n+1)$ -th to the first moment of the *instantaneous* energy spectrum. To obtain the Re-scaling of $\langle \kappa_n \rangle_T$, we first need the shell-model analogue of Lemma 2.

Lemma 3. As $\text{Gr} \rightarrow \infty$,

$$k_f^{-2} \langle \kappa_1^2 \rangle_T \leq c a_\ell^{-1} \text{Re}. \tag{100}$$

Proof. The energy evolution equation for the shell model is

$$\frac{dE}{dt} = -2\nu H_1 + \sum_{j=1}^{\infty} (f_j u_j^* + f_j^* u_j). \tag{101}$$

We add and subtract $b_1 \nu \tau_1^2 k_f^2 f^2$ to the right-hand side and apply the Cauchy–Schwarz inequality to the forcing term to obtain

$$\frac{1}{2} \dot{F}_0 \leq -\nu F_1 + b_1 \nu \tau_1^2 k_f^2 f^2 + b_0^{1/2} f H_0^{1/2} \tag{102}$$

⁷Vincenzi and Gibbon [2021] proved the ladder for shell models with a single time scale τ . The proof can be easily modified to include an n -dependent time scale by following the same approach as in Appendix A.

An application of Young's inequality with parameter $g\tau_0^2$ yields

$$\frac{1}{2}\dot{F}_0 \leq -\nu F_1 + \frac{H_0}{2g\tau_0^2} + \left(\frac{g}{2} + \frac{b_1\nu k_f^2}{b_0 a_\ell^2}\right) b_0 \tau_0^2 f^2, \quad (103)$$

where we have used $\tau_1 = a_\ell^{-1}\tau_0$ and g is such that the coefficients of H_0 and $b_0\tau^2 f^2$ are the same:

$$\begin{aligned} g &= -\frac{b_1\nu k_f^2}{b_0 a_\ell^2} + \left(\frac{b_1^2\nu^2 k_f^4}{b_0^2 a_\ell^4} + \frac{1}{\tau_0^2}\right)^{1/2} \\ &= \frac{1}{\tau_0} \left\{ -\frac{b_1}{b_0 a_\ell (1 + \text{Re})} + \left[\frac{b_1^2}{b_0^2 a_\ell^2 (1 + \text{Re})^2} + 1 \right]^{1/2} \right\}. \end{aligned} \quad (104)$$

Therefore, as $\text{Gr} \rightarrow \infty$ we find $g \sim \tau_0^{-1}$ and (103) becomes

$$\frac{1}{2}\dot{F}_0 \leq -\nu F_1 + c\tau_0^{-1}F_0. \quad (105)$$

By dividing by F_0 and time averaging, we finally get

$$\langle \kappa_1^2 \rangle_T \leq c\nu^{-1}\tau_0^{-1}. \quad (106)$$

The lemma is proved by replacing the definition of τ_0 . \square

Lemma 4. For $n \geq 1$, as $\text{Gr} \rightarrow \infty$

$$k_f^{-2} \langle \kappa_{n,1}^2 \rangle_T \leq c_n \text{Re}^{3/2}. \quad (107)$$

Proof. Dividing through (99) by F_n , time averaging, and ignoring the subdominant forcing term yields

$$\langle \kappa_{n+1,n}^2 \rangle_T \leq c_n \nu^{-1} \left\langle \sup_{1 \leq j} k_j |u_j| \right\rangle_T \leq c_n \nu^{-1} \langle F_1^{1/2} \rangle_T \leq c_n \nu^{-1} \langle F_1 \rangle_T^{1/2}. \quad (108)$$

The advertised result is obtained by using (98), $\langle F_1 \rangle_T \leq c \langle H_1 \rangle_T$, and $\langle \kappa_{n,1}^2 \rangle_T \leq \langle \kappa_{n+1,n}^2 \rangle_T$. \square

The estimates in the above lemmas can be used to prove the following theorem:

Theorem 7. For $n \geq 1$, as $\text{Gr} \rightarrow \infty$

$$k_f^{-1} \langle \kappa_n \rangle_T \leq c_n a_\ell^{-1/2n} \text{Re}^{3/4-1/4n}. \quad (109)$$

Proof. The proof is analogous to that of Theorem 1 in Doering and Gibbon [2002]. To achieve this, first note that

$$\langle \kappa_n \rangle_T \leq \left\langle \kappa_n^{\frac{2n}{2n-1}} \right\rangle_T^{\frac{2n-1}{2n}} = \left\langle \kappa_{n,1}^{\frac{2(n-1)}{2n-1}} (\kappa_1^2)^{\frac{1}{2n-1}} \right\rangle_T^{\frac{2n-1}{2n}} \leq \langle \kappa_{n,1} \rangle_T^{\frac{n-1}{n}} \langle \kappa_1^2 \rangle_T^{\frac{1}{2n}}, \quad (110)$$

and then use the estimates from Lemmas 3 and 4. \square

Remark 5. The scaling of $\langle \kappa_n \rangle_T$ is the same as in Theorem 2 for $p = 2$. This strengthens the parallel which was drawn in Vincenzi and Gibbon [2021] between shell models and the Navier–Stokes equations with suppressed velocity gradient fluctuations ($p = 2$).

By analogy with (6), we now define K_n^{2n} as

$$K_n^{2n} = \frac{\sum_{j=1}^{\infty} k_j^{2n} \langle |u_j|^2 \rangle_T}{\sum_{j=1}^{\infty} \langle |u_j|^2 \rangle_T} = \frac{\sum_{j=1}^{\infty} k_j^{2n+1} \mathcal{E}(k_j)}{\sum_{j=1}^{\infty} k_j \mathcal{E}(k_j)}. \quad (111)$$

We also assume $k_f = k_1$ and that there exists $k_c = k_0 \lambda^{j_c}$ such that $\mathcal{E}(k_j)$ decays rapidly for $k_j > k_c$, while

$$\mathcal{E}(k_j) \sim A k_j^{-q}, \quad 1 \leq j \leq j_c \quad (112)$$

with $1 < q < 3$ and $A \sim \epsilon^{2/3} k_f^{q-5/3} \sim U^2 k_f^{q+1/3}$ at large Re. In addition

$$\epsilon = \nu \sum_{j=1}^{\infty} k_j^3 \mathcal{E}(k_j) \sim \nu A k_c^{3-q}, \quad (113)$$

whence $k_c/k_f \sim \text{Re}^{\frac{1}{3-q}}$. We thus find

$$K_n^{2n} \sim k_c^{2n+1-q} k_f^{q-1} \quad (114)$$

and hence

$$K_n/k_f \sim \text{Re}^{\frac{1}{3-q} - \frac{q-1}{2n(3-q)}}. \quad (115)$$

We now follow the approach used for the Navier–Stokes and Burgers equations and compare the Re-scaling of K_n and $\langle \kappa_n \rangle_T$. The two scalings are consistent if

$$1 < q \leq \frac{5}{3}. \quad (116)$$

Remark 6. Numerical simulations show that, in the turbulent regime, the GOY and Sabra shell models display a $k^{-5/3}$ inertial-range spectrum [Yamada and Ohkitani, 1987, L’vov et al., 1998]. Moreover, in the inviscid unforced case, they possess fixed-point solutions with an energy spectrum scaling as $k^{-5/3}$ [Bohr et al., 1998]. Finally, Cheskidov et al. [2007] proved the $k^{-5/3}$ scaling of the energy spectrum for a dyadic model of the 3D incompressible Euler equations.

5 Summary and conclusion

This paper has developed the method of Doering and Gibbon [2002] in which a sequence of time-dependent wavenumbers, or inverse length scales $\kappa_n(t)$, was originally used to extract a spectrum from the 3D Navier–Stokes equations. These wavenumbers are ratios of volume integrals of velocity derivatives. For the 3D Navier–Stokes equations, and a version of them where large fluctuations of the velocity gradient are suppressed, Doering and Gibbon [2002] provided rigorous bounds for the time average of $\kappa_n(t)$ in

terms of Re . The wavenumbers $\kappa_n(t)$ are interpreted as moments of the energy spectrum and the bounds on the time average of these were then used to infer the slope of the energy spectrum in the inertial range of a turbulent velocity field. Since the $\kappa_n(t)$ are based on Navier–Stokes weak solutions, this approach connects empirical predictions of the energy spectrum with the mathematical analysis of the Navier–Stokes equations.

We have extended these methods to other hydrodynamic equations that display a turbulent regime at high Re , namely the 2D Navier–Stokes equations, the Burgers equation, and shell models. The results are summarized in Table 1. Previous predictions for the energy spectrum are recovered within the same mathematical framework, which confirms the appropriateness of $\langle \kappa_n \rangle_T$ as quantities suitable for the rigorous study of the energy spectrum of hydrodynamic partial differential equations.

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A Proof of the F_n ladder

Consider the ladder of inequalities for $n \geq 1$ [Doering and Gibbon, 1995]:

$$\frac{1}{2} \dot{H}_n \leq -\nu H_{n+1} + c_n \|\nabla \mathbf{u}\|_\infty H_n + H_n^{1/2} \|\nabla^n \mathbf{f}\|_2. \quad (\text{A.1})$$

In the case $d = 2$ the time differentiation of the higher order H_n is legal because the Navier–Stokes equations are regular. In the case $d = 3$ the result is formally true if one assumes there is a solution with sufficiently long interval of regularity. We proceed on this basis noting, however, that the estimates for the time-averages achieved in this paper can be shown to be true for weak solutions [Gibbon, 2019]. In turn, these are based on the work of Foias et al. [1981].

Add and subtract $\nu \tau_{n+1}^2 \|\nabla^{n+1} \mathbf{f}\|_2^2$ to obtain

$$\frac{1}{2} \dot{F}_n \leq -\nu F_{n+1} + c_n \|\nabla \mathbf{u}\|_\infty F_n + H_n^{1/2} \|\nabla^n \mathbf{f}\|_2 + \nu \tau_{n+1}^2 \|\nabla^{n+1} \mathbf{f}\|_2^2. \quad (\text{A.2})$$

Now apply Young’s inequality with parameter $g\tau_n^2$ to the last two terms of the right-hand side and use $\tau_{n+1} = a_\ell^{-1} \tau_n$ and $\|\nabla^{n+1} \mathbf{f}\|_2 = c \ell^{-2} \|\nabla^n \mathbf{f}\|_2$:

$$H_n^{1/2} \|\nabla^n \mathbf{f}\|_2 + \nu \tau_{n+1}^2 \|\nabla^{n+1} \mathbf{f}\|_2^2 \leq \frac{1}{2g\tau_n^2} H_n + \frac{g\tau_n^2}{2} \|\nabla^n \mathbf{f}\|_2 + \nu \tau_{n+1}^2 \|\nabla^{n+1} \mathbf{f}\|_2^2 \quad (\text{A.3})$$

$$= \frac{1}{2g\tau_n^2} H_n + \tau_n^2 \left(\frac{g}{2} + \frac{c\nu}{a_\ell^2 \ell^2} \right) \|\nabla^n \mathbf{f}\|_2^2. \quad (\text{A.4})$$

In order to have the same coefficients for H_n and $\tau_n^2 \|\nabla^n \mathbf{f}\|_2^2$ and thus form F_n , we must take

$$\begin{aligned} g &= -\frac{c\nu}{a_\ell^2 \ell^2} + \left(\frac{c^2 \nu^2}{a_\ell^4 \ell^4} + \frac{1}{\tau_n^2} \right)^{1/2} \\ &= \frac{1}{\tau_n} \left\{ -\frac{c}{a_\ell^{n+1} (1 + \text{Re})^{1+2\delta}} + \left[\frac{c^2}{a_\ell^{2(n+1)} (1 + \text{Re})^{2(1+2\delta)}} + 1 \right]^{1/2} \right\}. \end{aligned} \quad (\text{A.5})$$

As $\text{Gr} \rightarrow \infty$ or $a_\ell \rightarrow \infty$, we find $g \sim \tau_n^{-1}$ for all $n \geq 1$.

B Proof of an inequality for ϵ in shell models

The proof of the analogue of (27) for shell models follows the strategy used by Doering and Foias [2002] for the 3D Navier–Stokes equations. First define the constants

$$\begin{aligned} B_\lambda &= [(|a_1| + |a_2|)\lambda^{-1} + |a_1 + a_2|], \\ C_M &= \sum_{m=0}^{\infty} \lambda^{2mM} |\phi_m|^2, \\ D_M &= \sup_{m \geq 0} \lambda^{-m(2M-1)} |\phi_m|, \end{aligned} \quad (\text{B.1})$$

where M is any real number such that C_M and D_M are bounded. In particular, the following equality [Vincenzi and Gibbon, 2021] will be useful later :

$$\sum_{j=1}^{\infty} k_j^{2M} |f_j|^2 = C_M f^2 k_j^{2M}. \quad (\text{B.2})$$

Now multiply Eq. (89) by $k_j^{-2M} f_j^*$, sum over j , and average over time:

$$\begin{aligned} \sum_{j=1}^{\infty} k_j^{-2M} |f_j|^2 &= \left\langle \nu \sum_{j=1}^{\infty} k_j^{2-2M} u_j f_j^* \right\rangle_T \\ &\quad - \left\langle i \sum_{j=1}^{\infty} k_j^{-2M} f_j^* (a_1 k_{j+1} u_{j+1}^* u_{j+2} + a_2 k_j u_{j+1} u_{j-1}^* - a_3 k_{j-1} u_{j-1} u_{j-2}) \right\rangle_T. \end{aligned} \quad (\text{B.3})$$

Rearranging the terms in the first time average on the right-hand side and using the Cauchy–Schwartz inequality and (B.2) yields

$$\left| \left\langle \nu \sum_{j=1}^{\infty} k_j^{2-2M} u_j f_j^* \right\rangle_T \right| = \nu f \left| \left\langle \sum_{j=1}^{\infty} (k_j u_j) (k_j^{1-2M} \phi_{j-j_f}^*) \right\rangle_T \right|$$

$$\leq \nu^{1/2} \epsilon^{1/2} \sqrt{C_{1-2M}} f k_f^{1-2M}. \quad (\text{B.4})$$

The second time average can again be estimated by using the Cauchy–Schwartz inequality. Consider for instance the term with coefficient a_1 :

$$\begin{aligned} \left| \left\langle ia_1 \sum_{j=1}^{\infty} k_j^{-2M} f_j^* k_{j+1} u_{j+1}^* u_{j+2} \right\rangle_T \right| &= \frac{|a_1| f}{\lambda} \left| \left\langle \sum_{j=1}^{\infty} u_{j+1}^* (k_{j+2} u_{j+2}) (k_j^{-2M} \phi_{j-j_f}^*) \right\rangle_T \right| \\ &\leq \frac{|a_1|}{\lambda} \left(\frac{\epsilon}{\nu} \right)^{1/2} D_{M+\frac{1}{2}} U f k_f^{-2M}. \end{aligned} \quad (\text{B.5})$$

Likewise we have

$$\left| \left\langle ia_2 \sum_{j=1}^{\infty} k_j^{-2M} f_j^* k_j u_{j+1} u_{j-1}^* \right\rangle_T \right| \leq \frac{|a_2|}{\lambda} \left(\frac{\epsilon}{\nu} \right)^{1/2} D_{M+\frac{1}{2}} U f k_f^{-2M} \quad (\text{B.6})$$

and

$$\left| \left\langle ia_3 \sum_{j=1}^{\infty} k_j^{-2M} f_j^* k_{j-1} u_{j-1} u_{j-2} \right\rangle_T \right| \leq |a_1 + a_2| \left(\frac{\epsilon}{\nu} \right)^{1/2} D_{M+\frac{1}{2}} U f k_f^{-2M}. \quad (\text{B.7})$$

By combining (B.2) and the bounds in (B.4) to (B.7), we find

$$C_{-M} f \leq C_{1-2M}^{1/2} \nu^{1/2} \epsilon^{1/2} k_f + B_\lambda D_{M+\frac{1}{2}} \left(\frac{\epsilon}{\nu} \right)^{1/2} U. \quad (\text{B.8})$$

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