Acoustic and gravity waves in the ocean: a new derivation of a linear model from the compressible Euler equation

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Abstract

In this paper we construct an accurate linear model describing the propagation of both acoustic and gravity waves in water. This original model is obtained by the linearization of the compressible Navier-Stokes equations with free surface written in Lagrangian coordinates accounting for vertical variations of the background temperature and density. The models from the literature can be obtained from our model through two asymptotic analysis, one for the incompressible regime and one for the acoustic regime. We also propose a method to write the model in Eulerian coordinates. Our model includes many physical properties, such as the existence of internal gravity waves or the variation of the sound speed with depth.

1 Introduction

Several authors have proposed to use the propagation of acoustic waves in the ocean to detect tsunamis, as the sound travels in water at approximately $1500\,\mathrm{m\ s^{-1}}$ and the velocity of tsunami wave is around $300\,\mathrm{m\ s^{-1}}$ [4]. The existence of hydro-acoustic signals generated by tsunami sources such as earthquakes or landslides was shown in [27]. This motivates the mathematical modelling of the propagation of both surface waves – the tsunami – and underwater acoustic waves, also called hydroacoustic waves, in a compressible formulation.

The idea of using acoustic-gravity wave models for tsunami early-warning systems dates back from 1950 [5]. In [26], the analytical and numerical analysis of a two-dimensional linear model indicates that the pressure variations induced by the tsunami are significant enough to be used for the improvement of the tsunami early-warning systems.

For the description of the propagation of sound in water, the most common model is a linear wave equation for the fluid potential [13]. When both surface and acoustic waves are considered, different types of models were proposed. In

[26] the acoustic equation for the fluid potential is coupled with a free-boundary condition. The three-dimensional acoustic equation is analyzed in [22] and a depth-integrated version is proposed in [24] in order to reduce the computational costs. This approach was further developped in a serie of papers ([3], [1], [10]). Another approach was proposed in [20]. The equation is still on the fluid potential, but it includes a gravity term. This model was the starting point of an extensive work to describe the nonlinear interactions between acoustic and gravity waves [14]. In other works, such as [25] and [2], the flow is not assumed irrotational, so that the equations are written for the fluid velocity. They include gravity terms and a vertical stratification for the background density, temperature and salinity. This generalisation allows to study the internal waves caused by the stratification of the fluid, and dispersion relations for the three types of waves (acoustic, internal, surface) are obtained.

The above cited works share some common assumptions: irrotational flow, homogeneous background density or barotropic fluid, and a constant speed of sound. These modelling choices have a strong influence on the structure of the equations, resulting in a variety of tools for their analysis an their numerical approximation. For example, the irrotationality assumption allows to reduce the number of unknowns, but the validity of this assumption in the compressible case is not clear. The choice of a constant sound speed is also interesting to question: in the ocean, the variation of the sound speed creates the SOFAR channel, a horizontal strip in which the acoustic waves propagate with very little energy loss. It can be interesting to write a model in which this phenomenon appears naturally as it is the case when considering a vertically variable background density. Finally, because of the free-surface equation, the derivation of an accurate linear wave model is difficult, and existing justification are not completely convincing.

The aim of this work is to adress these modelling choices by deriving an accurate linear model as rigorously as possible with only very few assumptions. Salinity, thermal dissipation and viscosity are neglected, and in order to linearize the equations we assume that the ocean is at equilibrium and at rest before the earthquake, and that the source of the tsunami induces only a small displacement of the water. The obtained model describes the propagation of acoustic, internal and surface waves. In this model, the speed of sound results from the imposed background temperature profile, so that the effects of the SOFAR channel on the propagation of the hydroacoustic waves are naturally present.

Another advantage of having a model with few assumptions is that a cascade of simplified systems can be obtained from it. We indeed show that with some simplifying assumptions, our model reduces to the models proposed in the litterature. The analysis of these simplifications helps to understand the mathematical and physical choices made in these models. In this paper, this idea is applied for two common simplifications: the decoupling of acoustic and gravity waves, and the irrotationnality hypothesis.

The paper is organized as follows. In Sec. 2 the compressible Euler equations for a free-surface flow are written, then the system is transformed in Lagrangian

coordinates in order to keep an exact description of the free surface. After linearization, a wave-like equation for the fluid velocity is obtained, and we show that a particular formula for the energy is preserved locally and on the whole domain. Finally the barotropic case is studied. In Sec. 3 the incompressible limit and the acoustic limit of the wave equation are written in order to compare the obtained model with linear models from the litterature. In Sec. 4, we present a method allowing to write the model in Eulerian coordinates. The obtained system can be linearized at the cost of an additional approximation, namely that the equations have to be restricted to a fixed domain, and we show how to obtain a linear free-surface condition. Finally, in Sec 5 we obtain a dispersion relation which includes all of the physical effects mentioned above. In particular, it is a generalization of the dispersion relation studied in [2] to the case of a varying sound speed.

2 Linearization of compressible Euler equations in Lagrangian coordinates

We consider a portion of the ocean away from the coast and at equilibrium: there is no mean current and the temperature varies only vertically. In this work, we do not take the presence of salinity into account, hence the ocean is assimilated to pure water. The bottom and the surface of the domain are asssumed to be parametrized as graphs, respectively the topography $z_b(x,y) \geq 0$ and the free-surface elevation $\eta(x,y,t)$. The ground displacement is assumed to take place away from the coast, so that the domain is considered infinite in the (x,y) plane, see Figure 1. Mathematically speaking, the domain is assumed to have the following structure, for all time t of observation,

$$\Omega(t) = \{(x,y,z) \in \mathbb{R}^3 \mid z_b(x,y) < z < \eta(x,y,t)\}.$$

The boundary of the domain are then defined by

$$\Gamma_s(t) = \{(x,y,z) \in \mathbb{R}^3 \mid z = \eta(x,y,t)\},$$

and

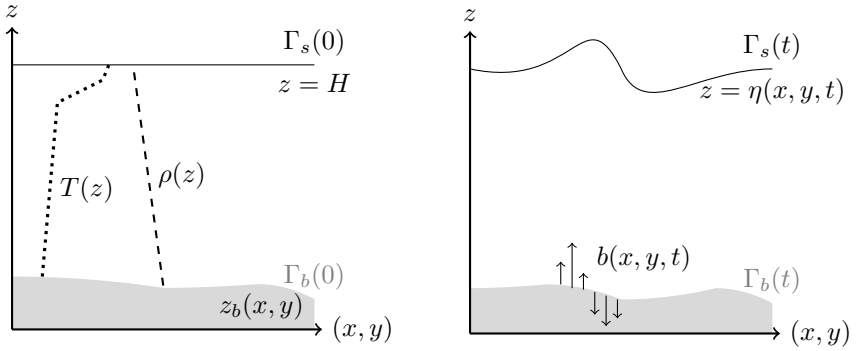
$$\Gamma_b(t) = \{(x,y,z) \in \mathbb{R}^3 \mid z = z_b(x,y) - b(x,y,t)\}.$$

The function b accounts for the only source term in our problem, namely the ground displacement at the sea bottom. It is assumed that this displacement starts at a time $t_0 > 0$, so that $b(x,y,0) = 0$.

2.1 Euler equation in Eulerian coordinate

2.1.1 Equations in the volume

The unknowns are the fluid velocity $\mathbf{U}$, its density ρ, its pressure p, its temperature T, its internal energy e and its entropy s. The gradient over the horizontal



(a) The domain $\Omega(0)$ at rest

(b) The domain $\Omega(t)$ for $t > 0$.

Figure 1: The domain $\Omega(t)$, (1a) for time $t = 0$, (1b) for time $t > 0$. In (1a) typical profiles for the temperature and the density at rest are drawn.

components will be denoted $\nabla_{x,y}$.

For future reference the equations are written for a viscous fluid with thermal dissipation. The stress tensor of a Newtonian fluid $\mathcal{T}$ has the form

$$\mathcal{T} = (-p + \lambda \nabla \cdot \mathbf{U})\mathbb{I} + 2\mu \mathbb{D}(\mathbf{U})$$

Where $\mathbb{D}(\mathbf{U})$ is defined by $\mathbb{D}(\mathbf{U}) = (\frac{1}{2}(\partial_i \mathbf{U}^j + \partial_j \mathbf{U}^i))_{i,j=x,y,z}$ and $\mathbb{I}$ is the identity matrix in $\mathbb{R}^3$. The heat flux is denoted by $\mathbf{q}$ and is a function of ρ and T.

The conservation of mass, momentum and energy of a Newtonian fluid read, in the domain $\Omega(t)$,

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0, & (1) \\ \dfrac{\partial}{\partial t}(\rho \mathbf{U}) + \nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p = \rho \mathbf{g} + \nabla(\lambda \nabla \cdot \mathbf{U}) + \nabla \cdot (2\mu \mathbb{D}(\mathbf{U})), & (2) \\ \dfrac{\partial}{\partial t}\left(\rho \dfrac{|\mathbf{U}|^2}{2} + \rho e\right) + \nabla \cdot \left((\rho \dfrac{|\mathbf{U}|^2}{2} + \rho e + p)\mathbf{U}\right) \\ \qquad = \rho \mathbf{g} \cdot \mathbf{U} + \nabla \cdot (\lambda \mathbf{U} \nabla \cdot \mathbf{U}) + \nabla \cdot (2\mu \mathbb{D}(\mathbf{U}) \cdot \mathbf{U}) - \nabla \cdot \mathbf{q}\ . & (3) \end{cases}$$

The acceleration of gravity is $\mathbf{g} = -g\,\mathbf{e}_3$ with $g > 0$ and $\mathbf{e}_3$ is the unit vector in the vertical direction, oriented upwards.

Among (ρ, e, T, p, s) only two variables are independent because of the Gibbs law and of the equation of state [7]. When considering ρ and s as independent, it is natural to introduce the scalar functions f_e, f_p and f_T satisfying

$$e = f_e(\rho, s), \quad p = f_p(\rho, s), \quad T = f_T(\rho, s),$$

and from physical considerations the function f_p must satisfy $\partial_\rho f_p(\rho, s) \geq 0$. When considering p and T as the independent variables, we introduce

$$s = f_s(p, T) \quad \text{and} \quad \rho = f_\rho(p, T).$$

Note that in this case the internal energy is given by

$$e = f_e(f_\rho(p, T), f_s(p, T)).$$

The Gibbs law, commonly written

$$\mathrm{d}e = \frac{p}{\rho^2}\mathrm{d}\rho + T\mathrm{d}s,$$

corresponds to a relation between the functions,

$$\frac{\partial f_e}{\partial \rho} = \frac{f_p}{\rho^2}, \quad \frac{\partial f_e}{\partial s} = f_T. \tag{4}$$

Our objective is to derive an equation for the pressure. Using (3) $- \mathbf{U} \cdot$ (2) and (4), one obtains as an intermediate step the evolution equation of the entropy,

$$\rho T(\frac{\partial s}{\partial t} + \mathbf{U} \cdot \nabla s) = D(\mathbf{U}) - \nabla \cdot \mathbf{q}. \tag{5}$$

The quantity $D(\mathbf{U})$ is the dissipation, defined by

$$D(\mathbf{U}) = \lambda(\nabla \cdot \mathbf{U})^2 + 2\mu\mathbb{D}(\mathbf{U}) : \mathbb{D}(\mathbf{U})$$

Now, since $p = f_p(\rho, s)$ we have

$$\frac{\partial p}{\partial t} + \mathbf{U} \cdot \nabla p = \frac{\partial f_p}{\partial \rho}\left(\frac{\partial \rho}{\partial t} + \mathbf{U} \cdot \nabla \rho\right) + \frac{1}{T\rho}\frac{\partial f_p}{\partial s}\left(\frac{\partial s}{\partial t} + \mathbf{U} \cdot \nabla s\right),$$

hence using (1) and (5) one obtains

$$\frac{\partial p}{\partial t} + \mathbf{U} \cdot \nabla p = -\frac{\partial f_p}{\partial \rho}\left(\rho \nabla \cdot \mathbf{U}\right) + \frac{1}{T\rho}\frac{\partial f_p}{\partial s}\left(D(\mathbf{U}) - \nabla \cdot \mathbf{q}\right). \tag{6}$$

At this point we use the common assumption that the viscous term and the thermal dissipation can be neglected compared to the advection term (see the discussion in [17], Chap. 1). Moreover, since $\partial_\rho f_p(\rho, s) \geq 0$, we can introduce the speed of sound c defined by

$$c^2 = \frac{\partial f_p}{\partial \rho}(\rho, s).$$

The equation (6) then reads

$$\frac{\partial p}{\partial t} + \mathbf{U} \cdot \nabla p + \rho c^2 \nabla \cdot \mathbf{U} = 0. \tag{7}$$

Eq. (7) is used for the study of a compressible fluid, in the isentropic case and when the viscosity and the thermal diffusion are neglected, see [7], Chap. 4. Note that the speed of sound c can also be viewed as a function of p and T, in that case we have

$$c^2(p,T) = \frac{\partial f_p}{\partial \rho}\left(f_\rho(p,T), f_s(p,T)\right).$$

In practice we choose to work directly with the expression $c = c(p,T)$ tabulated in [6]. Note that here the temperature intervenes as a side variable, because it is necessary to compute the speed of sound. However, we will see later that having the temperature profile of the state at rest is needed.

2.1.2 Boundary conditions

The following boundary conditions hold,

$$\begin{cases} \mathbf{U}\cdot\mathbf{n}_b = u_b = \partial_t b & \text{on } \Gamma_b, \quad (8) \\ p = p^a & \text{on } \Gamma_s. \quad (9) \end{cases}$$

The bottom boundary condition (8) is a non-penetration condition with a source term. It models the tsunami source as a displacement of the ocean bottom with velocity u_b. The second condition (9) is a dynamic condition, where we assume that the surface pressure is at equilibrium with a constant atmospheric pressure p^a. Also remark that one can show that the elevation η is solution of the following kinematic equation

$$\frac{\partial \eta}{\partial t} + \mathbf{U}\cdot\begin{pmatrix}\partial_x\eta \\ \partial_y\eta \\ -1\end{pmatrix} = 0 \quad \text{on} \quad \Gamma_s(t). \tag{10}$$

2.1.3 Initial conditions and equilibrium state

It is assumed that the initial state corresponds to the rest state, meaning that $\eta(x,y,0) = H$ with the elevation at rest H being independent of space and $H > z_b(x,y)$, therefore

$$\Omega(0) = \{(x,y,z)\in\mathbb{R}^3 \mid z_b(x,y) < z < H\}.$$

We choose the following initial conditions for the velocity, the temperature, the density and the pressure,

$$\mathbf{U}(x,y,z,0) = 0, \tag{11}$$

$$T(x,y,z,0) = T_0(z), \quad \rho(x,y,z,0) = \rho_0(z), \quad p(x,y,z,0) = p_0(z). \tag{12}$$

Where T_0, ρ_0, p_0 are functions defined on $(0,H)$ but because of the topography $z_b(x,y)$, the functions T, ρ, p need not to be defined from $z = 0$ for all (x,y).

When the source term B vanishes, we have an equilibrium state around $\mathbf{U} \equiv 0$ if the functions T_0, ρ_0, p_0 satisfy

$$\nabla p_0 = \rho_0 \mathbf{g}, \quad \rho_0 = f_\rho(p_0, T_0), \quad p_0(H) = p^a. \tag{13}$$

Hence, if $T_0(z)$ is given then the equation

$$\begin{cases} \dfrac{dp_0}{dz} = -g f_\rho(p_0, T_0), & z \in (0, H), \quad (14) \\ p_0 = p^a, & z = H. \quad (15) \end{cases}$$

can be solved to compute p_0, and then ρ_0 is computed with $\rho_0 = f_\rho(p_0, T_0)$. Note that, in the forthcoming sections the system (1), (2), (7) with boundary conditions (8), (9) and initial conditions (11), (12) will be linearized around the previously defined equilibrium state.

Remark: From the equations on p_0 in Eq. (13) one sees that at equilibrium the gradient of the pressure has only a component on the vertical coordinate, hence the pressure can vary only vertically. From the equation of state in Eq. (13) this implies that both the temperature and density depend also on the vertical coordinate only. Indeed if they had horizontal variations the pressure would also have vertical variations. The form of T_0, ρ_0, p_0 in Eq. (12) is a direct consequence of Eq. (11) when looking for an equilibrium state.

2.2 Lagrangian description

Although most of the works on free-surface flows are done in Eulerian coordinates, the Lagrangian formalism is sometimes preferred, see for example [23] and the references therein, or [9] for a precise derivation of linear models. Here we choose the Lagrangian description in order to avoid any approximation on the shape of the domain when we linearize the equations. The usual approximation made on the surface for the linear models in Eulerian coordinates consists in evaluating the surface condition on pressure at a fixed height, rather than at the actual, time-dependant free surface. The kinematic boundary condition is also replaced by its linear approximation. For the derivation and justification of the approximation, see [19], Chap. 3.

Let $\hat{\Omega}$ be the domain of the ocean at a reference time, with its surface boundary $\hat{\Gamma}_s$ and bottom boundary $\hat{\Gamma}_b$. The reference time is chosen before the tsunami generation, so that the surface of the domain is horizontal. In fact the following natural choice is made

$$\hat{\Omega} = \Omega(0), \quad \hat{\Gamma}_s = \Gamma_s(0), \quad \hat{\Gamma}_b = \Gamma_b(0).$$

The position at the reference time of a fluid particle is denoted

$$\boldsymbol{\xi} = (\xi^1, \xi^2, \xi^3) \in \hat{\Omega}.$$

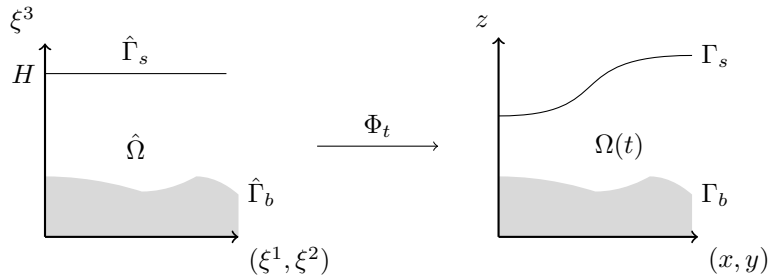


Figure 2: The mapping ϕ_t between the reference domain $\hat{\Omega}$ and the domain $\Omega(t)$.

At time t, the fluid has moved, the domain is $\Omega(t)$ and the new position of a fluid particle is $\mathbf{x} = (x(\boldsymbol{\xi}, t), y(\boldsymbol{\xi}, t), z(\boldsymbol{\xi}, t)) \in \Omega(t)$. We denote by $\boldsymbol{\phi}$ the transformation from $\hat{\Omega}$ to $\Omega(t)$ that maps each particle from its reference position $\boldsymbol{\xi}$ to its position $\mathbf{x}$ at time t (see Figure 2).

$$\boldsymbol{\phi} : \begin{cases} \hat{\Omega} & \to \Omega(t) \\ \boldsymbol{\xi} & \mapsto \mathbf{x}(\boldsymbol{\xi}, t) \end{cases}$$

Hence one has $\mathbf{x} = \boldsymbol{\phi}(\boldsymbol{\xi}, t)$. The transformation is assumed invertible, in particular we do not consider the case of wave breaking. We also define the displacement,

$$\mathbf{d}(\boldsymbol{\xi}, t) = \boldsymbol{\phi}(\boldsymbol{\xi}, t) - \boldsymbol{\xi}.$$

The gradient of $\boldsymbol{\phi}$ with respect to $\boldsymbol{\xi}$ is denoted F,

$$F = \nabla_\xi \boldsymbol{\phi},$$

and its determinant is denoted J. Both F and J can be expressed as functions of the displacement,

$$F = \mathbb{I} + \nabla_\xi \mathbf{d}, \quad J = \det F,$$

where ∇_ξ is the gradient with respect to the coordinate system $\boldsymbol{\xi}$. For a function $X(\mathbf{x}, t)$ defined on the domain $\Omega(t)$, we introduce $\hat{X}(\boldsymbol{\xi}, t)$ defined on $\hat{\Omega}$ by

$$\hat{X}(\boldsymbol{\xi}, t) = X(\boldsymbol{\phi}(\boldsymbol{\xi}, t), t).$$

Finally, note that the velocity $\hat{\mathbf{U}}(\boldsymbol{\xi}, t) = \mathbf{U}(\boldsymbol{\phi}(\boldsymbol{\xi}, t), t)$ is the time derivative of the displacement $\mathbf{d}$.

$$\hat{\mathbf{U}} = \frac{\partial \mathbf{d}}{\partial t}.$$

With this change of coordinates, the system (1), (2), (7) is now defined in the

time-independent reference domain $\hat{\Omega}$ and it reads

$$\begin{cases} \dfrac{\partial \hat{\rho}}{\partial t} + \dfrac{\hat{\rho}}{|J|} \nabla_\xi \cdot (|J| F^{-1} \hat{\mathbf{U}}) = 0, & (16) \\ \hat{\rho} \dfrac{\partial \hat{\mathbf{U}}}{\partial t} + F^{-T} \nabla_\xi \hat{p} = \hat{\rho} \mathbf{g}, & (17) \\ \dfrac{\partial \hat{p}}{\partial t} + \dfrac{\hat{\rho} \hat{c}^2}{|J|} \nabla_\xi \cdot (|J| F^{-1} \hat{\mathbf{U}}) = 0. & (18) \end{cases}$$

The boundary conditions become

$$\begin{cases} \hat{\mathbf{U}} \cdot \hat{\mathbf{n}_b} = \hat{u}_b & \text{on } \hat{\Gamma}_b, \\ \hat{p} = p^a & \text{on } \hat{\Gamma}_s, \end{cases}$$

where $\hat{\mathbf{n}_b}$ is a unit vector normal to $\hat{\Gamma}_b$ and pointing toward the exterior of the domain. The variables $\hat{\rho}, \hat{p}, \hat{T}$ satisfy the same equation of state

$$\hat{p} = f_p(\hat{\rho}, \hat{T}),$$

and the speed of sound is a function of the new variables, $\hat{c} = c(\hat{\rho}, \hat{s})$.

Remark: in Lagrangian coordinates the continuity equation is traditionally written (see e.g [16], Chap. 1)

$$\hat{\rho}(\boldsymbol{\xi}, 0) = \hat{\rho}(\boldsymbol{\xi}, t)\ |J(\boldsymbol{\xi}, t)|,$$

which is equivalent to our equation (16). Indeed, from the calculus of functions of several variables one can show that

$$\frac{\partial J}{\partial t} = \nabla_\xi \cdot (J F^{-1} \hat{\mathbf{U}}). \tag{19}$$

Since the transformation $\boldsymbol{\phi}$ is assumed invertible and close to the identity, its Jacobian is strictly positive, so that $J = |J|$. Using the equality (19) in Eq. (16) and multiplying by $|J|$ yields

$$|J| \frac{\partial \hat{\rho}}{\partial t} + \hat{\rho} \frac{\partial |J|}{\partial t} = 0,$$

which means that $\partial_t(\hat{\rho}|J|) = 0$. Hence we have $\hat{\rho}(\boldsymbol{\xi}, t)|J|(\boldsymbol{\xi}, t) = \hat{\rho}(\boldsymbol{\xi}, 0)|J|(\boldsymbol{\xi}, 0)$. At initial time the transformation $\boldsymbol{\phi}$ is the identity, then $|J(\boldsymbol{\xi}, 0)| = 1$.

2.3 Linearization and wave equation

We assume that the source of the tsunami is a small displacement – compared to the water height H – at the seafloor occuring in an ocean at rest as described in Sec. 2.1.3. In particular for this rest state there is no mean current and the temperature, pressure and density vary only vertically. The ratio of the bottom

displacement amplitude to the water height is denoted $\varepsilon \ll 1$, and the source term can be expressed as

$$\hat{u}_b = \varepsilon \hat{u}_{b,1} + \mathcal{O}(\varepsilon^2).$$

The linearization of the equations (16)-(18) around the rest state corresponds to the following asymptotic expansion,

$$\begin{aligned}
\mathbf{d}(\boldsymbol{\xi}, t) &= \varepsilon \mathbf{d}_1(\boldsymbol{\xi}, t) + \mathcal{O}(\varepsilon^2), \\
\hat{\rho}(\boldsymbol{\xi}, t) &= \hat{\rho}_0(\boldsymbol{\xi}) + \varepsilon \hat{\rho}_1(\boldsymbol{\xi}, t) + \mathcal{O}(\varepsilon^2), \\
\hat{p}(\boldsymbol{\xi}, t) &= \hat{p}_0(\boldsymbol{\xi}) + \varepsilon \hat{p}_1(\boldsymbol{\xi}, t) + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Note that the displacement has no zero order term, because the reference configuration used to define the Lagrangian description is the state given by the initial conditions. It holds then $\mathbf{d}_0 = 0$, $\hat{\mathbf{U}}_0 = 0$ and $\hat{\Omega} = \Omega(0)$. From the expansion one deduces the following Taylor expansions for the other functions,

$$\begin{aligned}
\hat{\mathbf{U}} &= \varepsilon \hat{\mathbf{U}}_1 + \mathcal{O}(\varepsilon^2), \\
F &= \mathbb{I} + \varepsilon \nabla_\xi \mathbf{d}_1 + \mathcal{O}(\varepsilon^2), \\
(F)^{-1} &= \mathbb{I} - \varepsilon \nabla_\xi \mathbf{d}_1 + \mathcal{O}(\varepsilon^2), \\
J &= 1 + \varepsilon \nabla_\xi \cdot \mathbf{d}_1 + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Injecting these expressions in Eq.(16)-(18) yields the system

$$\begin{cases}
\dfrac{\partial}{\partial t}\left(\hat{\rho}_0 + \varepsilon \hat{\rho}_1\right) + \varepsilon \hat{\rho}_0 \nabla_\xi \cdot \hat{\mathbf{U}}_1 = \mathcal{O}(\varepsilon^2), \\
\varepsilon \hat{\rho}_0 \dfrac{\partial \hat{\mathbf{U}}_1}{\partial t} + (I - \varepsilon \nabla_\xi \mathbf{d}_1)^T \nabla_\xi \hat{p}_0 + \varepsilon \nabla_\xi \hat{p}_1 = (\hat{\rho}_0 + \varepsilon \hat{\rho}_1)\mathbf{g} + \mathcal{O}(\varepsilon^2), \\
\dfrac{\partial}{\partial t}\left(\hat{p}_0 + \varepsilon \hat{p}_1\right) + \varepsilon \hat{\rho}_0 c^2(\hat{p}_0, \hat{T}_0) \nabla_\xi \cdot \hat{\mathbf{U}}_1 = \mathcal{O}(\varepsilon^2).
\end{cases}$$

By separating the powers of ε we obtain two systems: a limit system when $\varepsilon \to 0$ and a system for the first order corrections. Since the limit system correspond to the initial conditions described in Sec. 2.1.3, it remains to study the first order correction.

First-order correction: a wave-like equation for the velocity

The system for the correction terms reads in $\hat{\Omega}$,

$$\begin{cases}
\hat{\rho}_0 \dfrac{\partial \hat{\mathbf{U}}_1}{\partial t} + \nabla_\xi \hat{p}_1 - (\nabla_\xi \mathbf{d}_1)^T \ \nabla_\xi \hat{p}_0 = \hat{\rho}_1 \mathbf{g}, & (20) \\
\dfrac{\partial \hat{\rho}_1}{\partial t} + \hat{\rho}_0 \nabla_\xi \cdot \hat{\mathbf{U}}_1 = 0, & (21) \\
\dfrac{\partial \hat{p}_1}{\partial t} + \hat{\rho}_0 \hat{c}_0^2 \ \nabla_\xi \cdot \hat{\mathbf{U}}_1 = 0, & (22)
\end{cases}$$

with the boundary conditions

$$\begin{cases} \hat{\mathbf{U}}_1 \cdot \hat{\mathbf{n}}_b = \hat{u}_{b,1} & \text{on } \hat{\Gamma}_b, \quad (23) \\ \hat{p}_1 = 0 & \text{on } \hat{\Gamma}_s. \quad (24) \end{cases}$$

In this system the speed of sound is evaluated at the limit – or background – pressure and temperature, $\hat{c}_0 = c(\hat{p}_0, \hat{T}_0)$. In particular, $\hat{c}_0$ can be written as a function of depth. With an adapted temperature profile it is then possible to recover the typical speed of sound profile creating the SOFAR channel.

The correction pressure $\hat{p}_1$ and density $\hat{\rho}_1$ can be eliminated in (20) thanks to the other equations: differentiating in time (20) and replacing $\hat{\rho}_1$ and $\hat{p}_1$ with (21), (22) we obtain a second order equation for $\hat{\mathbf{U}}_1$,

$$\hat{\rho}_0 \frac{\partial^2 \hat{\mathbf{U}}_1}{\partial t^2} - \nabla_\xi \left(\hat{\rho}_0 \hat{c}_0^2 \nabla_\xi \cdot \hat{\mathbf{U}}_1 \right) - (\nabla_\xi \hat{\mathbf{U}}_1)^T \ \hat{\rho}_0 \mathbf{g} + \hat{\rho}_0 \nabla_\xi \cdot \hat{\mathbf{U}}_1 \ \mathbf{g} = 0 \ \text{ in } \hat{\Omega}. \quad (25)$$

Using (22), the surface boundary condition (24) is formulated for $\hat{\mathbf{U}}_1$, hence the two boundary conditions for the wave-like equation (25) are

$$\begin{cases} \hat{\mathbf{U}}_1 \cdot \hat{\mathbf{n}}_b = \hat{u}_{b,1} & \text{on } \hat{\Gamma}_b, \quad (26) \\ \nabla_\xi \cdot \hat{\mathbf{U}}_1 = 0 & \text{on } \hat{\Gamma}_s. \quad (27) \end{cases}$$

The wave-like equation (25) is completed with vanishing initial condition for $\hat{\mathbf{U}}_1(0)$ and $\partial_t \hat{\mathbf{U}}_1(0)$. System (25) includes both gravity and acoustic terms. This equation, which describes the velocity of a compressible, non-viscous fluid, in Lagrangian description, is called the Galbrun equation. It is used in helioseismology and in aeroacoustics ([18], [12]). However, the study of this equation with the boundary conditions (26) - (27) and a nonhomogeneous density is, to our knowledge, new.

An energy equation for the equation (25) can be obtained by taking the scalar product of (25) with $\partial_t \hat{\mathbf{U}}_1$ and integrating over the domain. After some computations (see Appendix, Sec. 7.1), we have

$$\frac{d}{dt}\mathcal{E} = \int_{\hat{\Gamma}_b} \hat{\rho}_0 \left(c_0^2 \nabla_\xi \cdot \hat{\mathbf{U}}_1 - \hat{\rho}_0 g \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \right) \frac{\partial \hat{u}_{b,1}}{\partial t} \, \mathrm{d}\sigma, \quad (28)$$

with the energy being the quadratic functional given by

$$\mathcal{E} = \int_{\hat{\Omega}} \rho_0 \frac{1}{2} \left| \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \right|^2 \mathrm{d}\boldsymbol{\xi} + \frac{1}{2} \int_{\hat{\Omega}} \hat{\rho}_0 \left(\hat{c}_0 \nabla_\xi \cdot \hat{\mathbf{U}}_1 - \frac{g}{\hat{c}_0} \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \right)^2 \mathrm{d}\boldsymbol{\xi} + \frac{1}{2} \int_{\hat{\Omega}} \hat{\rho}_0 N_b (\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3)^2 \, \mathrm{d}\boldsymbol{\xi} + \frac{1}{2} \int_{\hat{\Gamma}_s} \hat{\rho}_0 g (\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3)^2 \, \mathrm{d}\sigma.$$

In the definition of the energy the scalar N_b is given by

$$N_b(\xi^3) = -\left(\frac{g^2}{\hat{c}_0(\xi^3)^2} + g\frac{\rho_0'(\xi^3)}{\rho_0(\xi^3)}\right),$$

In order to have only positive terms in the energy equation (28), N_b should be positive. In this case we can define its square root, denoted N. The function N is called the Brunt-Väisälä frequency, or buoyancy frequency. It is closely related to the internal waves that appear in a stratified medium, see for example [19], Chap. 4 and [7], Chap. 6. Numerical values for the buoyancy frequency are available in [15]. In the ocean, the usual values of N^2 are around $10^{-8}\,\mathrm{rad}^2\mathrm{s}^{-2}$. The sign of N_b depends on the choice of the state at equilibrimum: $\rho_0' = d\rho_0/dz$ has to be negative and satisfy

$$\frac{|\hat{\rho}_0'|}{\hat{\rho}_0} > \frac{g}{\hat{c}_0^2}.$$

With the term in $g^2/\hat{c}_0^2$, we see that the compressibility tends to take the fluid away from its equilibrium. The stratification of the fluid must be strong enough to counter this effect and keep the system stable (see the discussion in [7], Chap.3). As a consequence, it is impossible to assume a homogeneous background density, to take the compressibility into account and to preserve an a priori positive energy of the system at the same time. In the following, we assume that the fluid has a stable stratification namely the function N_b is assumed always positive and we will use N^2 in the equations.

Remark: According to the equation of state (when the salinity is neglected) $\rho = f_\rho(p, T)$, the stratification has two factors: it comes from a variation in temperature and in pressure. The temperature profile can be chosen homogeneous, but the effect of gravity – see Eq. (15) – prevents the pressure to be independant of depth. Hence in a model with gravity, the fluid is always stratified with density increasing with depth.

Remark: One can notice that the condition on ρ_0' is not explicit in Eq. (25) and appears only when one is interested in the energy preservation, that is, in the well-posedness of the equation.

The barotropic case

Finally we consider the barotropic case, which is a very common assumption for the study of hydro-acoustic waves (see for example [2], [20], [26]). For a barotropic fluid the pressure is a function of the density only,

$$f_p(\rho, s) = f_p(\rho) = p.$$

Then, using (13) and the definition of the speed of sound,

$$\hat{p}_0' = \hat{\rho}_0'\frac{df_p}{d\rho}(\hat{\rho}_0) \quad \Rightarrow \quad -\hat{\rho}_0 g = \hat{\rho}_0'\hat{c}_0^2, \tag{29}$$

meaning that $N^2 = 0$. This corresponds to the limit case where the density is stratified because of the variation of pressure only. In order to use this equality, we divide Eq. (25) by $\hat{\rho}_0$,

$$\frac{\partial^2 \hat{\mathbf{U}}_1}{\partial t^2} - \nabla_\xi \left(\hat{c}_0^2 \nabla_\xi \cdot \hat{\mathbf{U}}_1 \right) - \left(\frac{\hat{\rho}_0'}{\rho_0} \hat{c}_0^2 + g \right) \nabla_\xi \cdot \hat{\mathbf{U}}_1 \mathbf{e}_3 + \nabla_\xi (\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3) g = 0, \quad (30)$$

and when Eq. (29) holds, the equation (30) can be simplified and reads

$$\frac{\partial^2 \hat{\mathbf{U}}_1}{\partial t^2} - \nabla_\xi \left(\hat{c}_0^2 \nabla_\xi \cdot \hat{\mathbf{U}}_1 \right) + \nabla_\xi (\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3) g = 0. \quad (31)$$

Taking the curl of Eq. (31) yields

$$\frac{\partial^2 \nabla_\xi \times \hat{\mathbf{U}}_1}{\partial t^2} = 0 \text{ in } \hat{\Omega}.$$

With the vanishing initial conditions we obtain that the velocity of a barotropic fluid is irrotational. This is a well-known result, since the fluid is also iviscid and subject to a potential force only ([11], Chap. 7). By the Helmholtz decomposition theorem [8] the fluid velocity is written as the gradient of a potential ψ defined up to a constant. The expression $\hat{\mathbf{U}}_1 = \nabla_\xi \psi$ is used in Eq. (31), to obtain

$$\nabla_\xi \left(\frac{\partial^2 \psi}{\partial t^2} - \hat{c}_0^2 \Delta_\xi \psi + g \frac{\partial \psi}{\partial \xi^3} \right) = 0. \quad (32)$$

The potential ψ being defined up to a constant. it can always be sought as the solution of

$$\frac{\partial^2 \psi}{\partial t^2} - \hat{c}_0^2 \Delta_\xi \psi + g \frac{\partial \psi}{\partial \xi^3} = 0, \quad (33)$$

The equation (33) is multiplied by $\hat{\rho}_0 / \hat{c}_0^2$, and we use $g/\hat{c}_0^2 = -\hat{\rho}_0'/\hat{\rho}_0$,

$$\frac{\hat{\rho}_0}{\hat{c}_0^2} \frac{\partial^2 \psi}{\partial t^2} - \hat{\rho}_0 \Delta_\xi \psi - \hat{\rho}_0' \frac{\partial \psi}{\partial \xi^3} = 0.$$

And since ρ_0 depends only on ξ^3, the two last terms can be rewritten,

$$\frac{\hat{\rho}_0}{\hat{c}_0^2} \frac{\partial^2 \psi}{\partial t^2} - \nabla_\xi \cdot (\hat{\rho}_0 \nabla_\xi \psi) = 0. \quad (34)$$

Hence ψ satisfies a wave equation. The boundary conditions are then deduced from Eq. (26) and Eq. (27),

$$\begin{cases} \nabla_\xi \psi \cdot \hat{\mathbf{n}}_b = \hat{u}_{b,1} \text{ on } \hat{\Gamma}_b, & (35) \\ \hat{c}_0^2 \Delta_\xi \psi = \dfrac{\partial^2 \psi}{\partial t^2} + g \dfrac{\partial \psi}{\partial \xi^3} = 0 \text{ on } \hat{\Gamma}_s. & (36) \end{cases}$$

The system (33),(35),(36) is the first-order system obtained in [20], where the irrotationality assumption is made independantly from the fact that the fluid is

barotropic and the boundary conditions are obtained with the linearized surface condition. This shows that the linearization made in [20] gives exactly the same result as the linearization strategy we have presented.

An energy equation can be then obtained for the system (34),(35),(36). The equation (34) is multiplied by $\partial_t \psi$ and integrated by parts,

$$\int_{\hat{\Omega}} \frac{\hat{\rho}_0}{\hat{c}_0^2} \frac{\partial \psi}{\partial t} \frac{\partial^2 \psi}{\partial t^2} \, \mathrm{d}\boldsymbol{\xi} + \int_{\hat{\Omega}} \hat{\rho}_0 \nabla \left(\frac{\partial \psi}{\partial t} \right) \cdot \nabla \psi \, \mathrm{d}\boldsymbol{\xi} \\ - \int_{\hat{\Gamma}_s} \hat{\rho}_0 \frac{\partial \psi}{\partial t} \nabla \psi \cdot \mathbf{e}_3 \, \mathrm{d}\sigma + \int_{\hat{\Gamma}_b} \hat{\rho}_0 \frac{\partial \psi}{\partial t} \nabla \psi \cdot \mathbf{n}_b \, \mathrm{d}\sigma = 0.$$

With the boundary conditions (35) - (36) and after simplifications it holds

$$\frac{d}{dt} \mathcal{E}_{bar} = - \int_{\hat{\Gamma}_b} \hat{\rho}_0 \frac{\partial \psi}{\partial t} \hat{u}_{b,1} \, \mathrm{d}\sigma,$$

where the energy $\mathcal{E}_{bar}$ is defined by

$$\mathcal{E}_{bar} = \frac{1}{2} \int_{\hat{\Omega}} \frac{\hat{\rho}_0}{\hat{c}_0^2} \left(\frac{\partial \psi}{\partial t} \right)^2 \mathrm{d}\boldsymbol{\xi} + \frac{1}{2} \int_{\hat{\Omega}} \hat{\rho}_0 |\nabla \psi|^2 \, \mathrm{d}\boldsymbol{\xi} + \frac{1}{2} \int_{\hat{\Gamma}_s} \frac{\hat{\rho}_0}{g} \left(\frac{\partial \psi}{\partial t} \right)^2 \mathrm{d}\boldsymbol{\xi}.$$

In order to write the barotropic system as a wave equation, and then obtain the energy equation, it is necessary to use the background density $\hat{\rho}_0$ even if it does not appear in Eq. (33). The correct manipulation was found by comparison with the general case described by Eq. (25).

In this section we have derived a linear model around a state at rest for the compressible Euler equation with free surface. The equations are valid for a generic equation of state and for a temperature varying vertically. An energy equation can be obtained for this system, and it gives a condition on the fluid stratification. We have shown that in the barotropic case the system (25) is equivalent to the first-order scalar equation of [20]. The goal of the next section is to continue the comparison of Eq. (25) with other models, by writing its approximation for two asymptotic regimes.

3 Two limits of the system

There are two main types of models for the inviscid free-surface flows: the incompressible models, when the acoustic waves are neglected, and the acoustic models, when the effect of gravity is neglected. In this section we write two simplified versions of the system (25)-(27) corresponding to each case. The wave equation (25) is written in non-dimensional form, and we show that it depends on a small nondimensional parameter. A simplified model is then obtained by passing formally to the limit when the small parameter vanishes. By making the appropriate choice for the time scale we obtain first an incompressible approximation, then an acoustic approximation.

3.1 Non-dimensional equation

We introduce the following characteristic scales for the system: a time τ, a horizontal scale L, a vertical scale H, a density $\bar{\rho}$, and a fluid velocity U. Since the speed of sound is not assumed constant, we denote by C its characteristic magnitude. Finally, the surface waves velocity is of the order of $\sqrt{gH}$ [4]. We focus on a non-shallow water formulation, hence we take $L = H$. For a shallow water version of the equation one would choose $H \ll L$.

Two dimensionless numbers are introduced: the Froude number and the Mach number, respectively defined by

$$\mathrm{Fr} = \frac{U}{\sqrt{gH}}, \qquad \mathrm{Ma} = \frac{U}{C}.$$

To fix the idea, we choose the following numerical values respectively for the speed of sound, the fluid velocity and the surface waves velocity: $C \sim 1480$ m s^{-1}, $U \sim 1$ m s^{-1} and $\sqrt{gH} \sim 100$ m s^{-1}. The dimensionless numbers are then

$$\mathrm{Fr} = 0.01, \quad \mathrm{Ma} = 6.10^{-4}.$$

The characteristic scale for time will be fixed later, as it will depend on the regime we want to study. The variables are put in non-dimensional form and the dimensionless variables are denoted with a $\tilde{}$, except for the space and time variable for the sake of conciseness. The adimensionned domain is denoted by $\tilde{\Omega}$ and its surface and bottom boundary are respectively $\tilde{\Gamma}_s$ and $\tilde{\Gamma}_b$. The non-dimensional system reads, after simplification by the factor $\bar{\rho}U$,

$$\frac{\tilde{\rho}_0}{\tau^2}\frac{\partial^2 \tilde{\mathbf{U}}_1}{\partial t^2} - \frac{C^2}{L^2}\nabla_\xi \left(\tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{\mathbf{U}}_1\right) + \frac{g}{L}\tilde{\rho}_0 \left(\nabla_\xi \left(\tilde{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3\right) - \nabla_\xi \cdot \tilde{\mathbf{U}}_{1,0}\ \mathbf{e}_3\right) = 0, \quad (37)$$

with the boundary conditions

$$\begin{cases} \tilde{\mathbf{U}}_1 \cdot \tilde{\mathbf{n}}_b = \tilde{u}_{b,1} \quad \text{on } \tilde{\Gamma}_b, & (38) \\ \nabla_\xi \cdot \tilde{\mathbf{U}}_1 = 0 \quad \text{on } \tilde{\Gamma}_s, & (39) \end{cases}$$

where $\tilde{u}_{b,1}$ is a dimensionless source term.

3.2 Incompressible limit

In order to study the incompressible limit, the characterisic time τ is chosen to follow the surface waves, which are much slower than the acoustic waves. We take $L/\tau = \sqrt{gH}$. The equation (37) becomes

$$\tilde{\rho}_0 \frac{\partial^2 \tilde{\mathbf{U}}_1}{\partial t^2} - \frac{\mathrm{Fr}}{\mathrm{Ma}}\nabla_\xi \left(\tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{\mathbf{U}}_1\right) + \tilde{\rho}_0 \left(\nabla_\xi \left(\tilde{\mathbf{U}}_1 \cdot \mathbf{e}_3\right) - \nabla_\xi \cdot \tilde{\mathbf{U}}_1\ \mathbf{e}_3\right) = 0.$$

The small parameter $\delta = \mathrm{Ma}/\mathrm{Fr} \sim 6.10^{-2}$ is introduced in the equation,

$$\tilde{\rho}_0 \frac{\partial^2 \tilde{\mathbf{U}}_1}{\partial t^2} - \frac{1}{\delta^2} \nabla_\xi \left(\tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{\mathbf{U}}_1 \right) + \tilde{\rho}_0 \left(\nabla_\xi \left(\tilde{\mathbf{U}}_1 \cdot \mathbf{e}_3 \right) - \nabla_\xi \cdot \tilde{\mathbf{U}}_1 \ \mathbf{e}_3 \right) = 0, \quad (40)$$

and the goal is now to study the behaviour of the problem (40) when δ goes to zero. We make the following ansatz for $\tilde{\mathbf{U}}_1$,

$$\tilde{\mathbf{U}}_1 = \tilde{\mathbf{U}}_{1,0} + \delta^2 \tilde{\mathbf{U}}_{1,2} + o(\delta^2),$$

where $\tilde{\mathbf{U}}_{1,0}$, $\tilde{\mathbf{U}}_{1,1}$ and $\tilde{\mathbf{U}}_{1,2}$ are independant of δ. Since Eq. (40) has only even powers of δ, the term $\tilde{\mathbf{U}}_{1,1}$ is equal to zero. Replacing $\tilde{\mathbf{U}}_1$ by its ansatz in the wave equation (40) and separating the powers of δ yields an equation for each term of the asymptotic developement of $\tilde{\mathbf{U}}_1$. The equation obtained with the terms in δ^{-2} reads

$$\nabla_\xi \left(\tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{\mathbf{U}}_{1,0} \right) = 0, \quad (41)$$

and the equation obtained with the terms δ^0 reads

$$\tilde{\rho}_0 \frac{\partial^2 \tilde{\mathbf{U}}_{1,0}}{\partial t^2} - \nabla_\xi \left(\tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{\mathbf{U}}_{1,2} \right) + \tilde{\rho}_0 \left(\nabla_\xi \left(\tilde{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3 \right) - \nabla_\xi \cdot \tilde{\mathbf{U}}_{1,0} \ \mathbf{e}_3 \right) = 0. \quad (42)$$

With the terms in δ^0 of the boundary conditions we have

$$\begin{cases} \nabla_\xi \cdot \tilde{\mathbf{U}}_{1,0} = 0 & \text{on } \tilde{\Gamma}_s, \quad (43) \\ \tilde{\mathbf{U}}_{1,0} \cdot \tilde{\mathbf{n}}_b = \tilde{u}_{b,1} & \text{on } \tilde{\Gamma}_b. \quad (44) \end{cases}$$

And the terms in δ^2 of the boundary conditions read

$$\begin{cases} \nabla_\xi \cdot \tilde{\mathbf{U}}_{1,2} = 0 & \text{on } \tilde{\Gamma}_s, \quad (45) \\ \tilde{\mathbf{U}}_{1,2} \cdot \tilde{\mathbf{n}}_b = 0 & \text{on } \tilde{\Gamma}_b. \quad (46) \end{cases}$$

Property of the limit solution

We focus first on Eq. (41) with the boundary conditions (43), (44). The Helmoltz decomposition of $\tilde{\mathbf{U}}_{1,0}$ reads

$$\tilde{\mathbf{U}}_{1,0} = \nabla_\xi \varphi_{1,0} + \nabla_\xi \times \boldsymbol{\psi}_{1,0},$$

where $\varphi_{1,0}$ vanishes on $\tilde{\Gamma}_s$ and $\tilde{\Gamma}_b$. Injecting the decomposition of $\tilde{\mathbf{U}}_{1,0}$ in Eq. (41) yields

$$\nabla_\xi \left(\tilde{\rho}_0 \tilde{c}_0^2 \ \Delta_\xi \varphi_{1,0} \right) = 0,$$

hence the term inside the gradient is constant in space. Since the velocity $\tilde{\mathbf{U}}_{1,0}$ is equal to zero at infinity, we obtain that $\Delta_\xi \varphi_{1,0} = 0$ in $\tilde{\Omega}$ (the quantity $\tilde{\rho}_0 \tilde{c}_0$ being always strictly positive). With the vanishing boundary conditions for $\varphi_{1,0}$, we obtain that $\varphi_{1,0}$ is equal to zero everywhere in $\tilde{\Omega}$. Then, taking the divergence of $\tilde{\mathbf{U}}_{1,0}$ yields

$$\nabla_\xi \cdot \tilde{\mathbf{U}}_{1,0} = \nabla_\xi \cdot (\nabla_\xi \times \boldsymbol{\psi}_{1,0}) = 0,$$

hence $\tilde{\mathbf{U}}_{1,0}$ is divergence-free.

Limit equation

We consider now the equation (42). Using the fact that the divergence of $\tilde{\mathbf{U}}_{1,0}$ vanishes and rearranging some terms we obtain

$$\tilde{\rho}_0 \frac{\partial^2 \tilde{\mathbf{U}}_{1,0}}{\partial t^2} - \nabla_\xi \left(\tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{\mathbf{U}}_{1,2} \right) + \nabla_\xi \left(\tilde{\rho}_0 \tilde{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3 \right) - \tilde{\rho}_0' (\tilde{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3)\, \mathbf{e}_3 = 0. \quad (47)$$

Taking the curl of this equation yields

$$\nabla_\xi \times \left(\tilde{\rho}_0 \frac{\partial^2 \tilde{\mathbf{U}}_{1,0}}{\partial t^2} - \tilde{\rho}_0' (\tilde{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3)\, \mathbf{e}_3 \right) = 0, \quad (48)$$

This means that these terms can be expressed as the gradient of a potential function defined up to a constant and denoted $-\tilde{\varphi}_0$,

$$\tilde{\rho}_0 \frac{\partial^2 \tilde{\mathbf{U}}_{1,0}}{\partial t^2} - \tilde{\rho}_0' (\tilde{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3)\, \mathbf{e}_3 = -\nabla_\xi \tilde{\varphi}_0. \quad (49)$$

The new function $\tilde{\varphi}_0$ can be understood as the Lagrange multiplier for the incompressibility constraint. However, one must be cautious that $\tilde{\varphi}_0$ is not similar to a pressure in this case, and rather plays the role of a velocity potential, as we will see later in the case of homogeneous density. The function $\tilde{\varphi}_0$ can be expressed differently, indeed by using its definition (49) in the equation (47) we have

$$\nabla_\xi \left(-\tilde{\varphi}_0 - \tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{\mathbf{U}}_{1,2} + \tilde{\rho}_0 \tilde{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3 \right) = 0,$$

and since the potential $\tilde{\varphi}_0$ is defined up to a constant, it can be chosen such that, in $\hat{\Omega}$, we have

$$\tilde{\varphi}_0 = -\tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{\mathbf{U}}_{1,2} + \tilde{\rho}_0 \tilde{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3. \quad (50)$$

We deduce from this equality and (45) the boundary condition

$$\tilde{\varphi}_0 = \tilde{\rho}_0 \tilde{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3 \quad \text{on } \tilde{\Gamma}_s. \quad (51)$$

To recover a dimensional system, the terms are multiplied by their corresponding characteristic scales, and $\hat{\varphi}_0 = \bar{\rho} U \tilde{\varphi}_0$ is defined. The limit solution $\hat{\mathbf{U}}_{1,0} = U \tilde{\mathbf{U}}_{1,0}$ satisfies

$$\begin{cases} \hat{\rho}_0 \dfrac{\partial^2 \hat{\mathbf{U}}_{1,0}}{\partial t^2} - g \hat{\rho}_0' (\hat{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3)\, \mathbf{e}_3 + g \nabla_\xi \hat{\varphi}_0 = 0 \quad \text{in } \hat{\Omega}, & (52) \\ \nabla_\xi \cdot \hat{\mathbf{U}}_{1,0} = 0 \quad \text{in } \hat{\Omega}, & (53) \end{cases}$$

with the boundary conditions

$$\begin{cases} \hat{\mathbf{U}}_{1,0} \cdot \hat{\mathbf{n}}_b = \hat{u}_{b,1} \quad \text{on } \hat{\Gamma}_b, & (54) \\ \nabla_\xi \cdot \hat{\mathbf{U}}_{1,0} = 0 \quad \text{on } \hat{\Gamma}_s, & (55) \\ \hat{\varphi}_0 = g \hat{\rho}_0 \hat{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3 \quad \text{on } \hat{\Gamma}_s, & (56) \end{cases}$$

We derive an energy equation for the system (52)-(56) with the boundary conditions (54), (55), (56). Taking the scalar product of (52) with $\partial_t \tilde{\mathbf{U}}_{1,0}$ and integrating over $\hat{\Omega}$ yields

$$\frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}} \hat{\rho}_0 \left|\frac{\partial \hat{\mathbf{U}}_{1,0}}{\partial t}\right|^2 \mathrm{d}\boldsymbol{\xi} - \int_{\hat{\Omega}} \hat{\rho}_0'(\hat{\mathbf{U}}_{1,0}\cdot\mathbf{e}_3)\,\mathbf{e}_3 \cdot \frac{\partial \hat{\mathbf{U}}_{1,0}}{\partial t}\,\mathrm{d}\boldsymbol{\xi} + \int_{\hat{\Omega}} g\frac{\partial \hat{\mathbf{U}}_{1,0}}{\partial t}\cdot\nabla_\xi \hat{\varphi}_0\,\mathrm{d}\boldsymbol{\xi} = 0. \tag{57}$$

The last term of Eq. (57) is integrated by parts. With the vanishing divergence of $\hat{\mathbf{U}}_{1,0}$ and the bottom condition (54) it holds

$$\int_{\hat{\Omega}} g\frac{\partial \hat{\mathbf{U}}_{1,0}}{\partial t}\cdot\nabla_\xi \hat{\varphi}_0\,\mathrm{d}\boldsymbol{\xi} = \int_{\hat{\Gamma}_s} g\hat{\varphi}_0\frac{\partial \hat{\mathbf{U}}_{1,0}}{\partial t}\cdot\mathbf{e}_3\,\mathrm{d}\sigma - \int_{\hat{\Gamma}_b} g\hat{\varphi}_0\frac{\partial \hat{u}_{b,1}}{\partial t}\,\mathrm{d}\sigma,$$

then $\hat{\varphi}_0$ is replaced in the surface integral using Eq. (56),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}} \hat{\rho}_0 \left|\frac{\partial \hat{\mathbf{U}}_{1,0}}{\partial t}\right|^2 \mathrm{d}\boldsymbol{\xi} - \int_{\hat{\Omega}} \hat{\rho}_0'(\hat{\mathbf{U}}_{1,0}\cdot\mathbf{e}_3)\,\mathbf{e}_3 \cdot \frac{\partial \hat{\mathbf{U}}_{1,0}}{\partial t}\,\mathrm{d}\boldsymbol{\xi} \\ + \int_{\hat{\Gamma}_s} g\hat{\rho}_0\hat{\mathbf{U}}_{1,0}\cdot\mathbf{e}_3\frac{\partial \hat{\mathbf{U}}_{1,0}}{\partial t}\cdot\mathbf{e}_3\,\mathrm{d}\sigma = \int_{\hat{\Gamma}_b} g\hat{\varphi}_0\frac{\partial \hat{u}_{b,1}}{\partial t}\,\mathrm{d}\sigma.\end{aligned} \tag{58}$$

By defining the energy

$$\mathcal{E}_{incomp} = \frac{1}{2}\int_{\hat{\Omega}} \hat{\rho}_0 \left|\frac{\partial \hat{\mathbf{U}}_{1,0}}{\partial t}\right|^2 \mathrm{d}\boldsymbol{\xi} - \frac{1}{2}\int_{\hat{\Omega}} \hat{\rho}_0'|\hat{\mathbf{U}}_{1,0}\cdot\mathbf{e}_3|^2 + \frac{1}{2}\int_{\hat{\Gamma}_s} \hat{\rho}_0|\hat{\mathbf{U}}_{1,0}\cdot\mathbf{e}_3|^2,$$

the equation (57) can be formulated in the following way,

$$\frac{d}{dt}\mathcal{E}_{incomp} = \int_{\hat{\Gamma}_b} g\varphi_0\frac{\partial \hat{u}_{b,1}}{\partial t}.$$

Recall that $\hat{\rho}_0' < 0$, so $\mathcal{E}_{incomp}$ is a positive quadratic functional.

Finally, note that the boudary conditions can be expressed differently, by using the equations (52) and (56). The bottom boundary condition is obtained by taking the scalar product of Eq. (52) with $\tilde{\mathbf{n}_b}$, and replacing the first term with Eq. (54) differentiated twice in time,

$$-\hat{\rho}_0\frac{\partial^2 \hat{u}_{b,1}}{\partial t^2} - g\hat{\rho}_0'(\hat{\mathbf{U}}_{1,0}\cdot\mathbf{e}_3)\,\mathbf{e}_3\cdot\hat{\mathbf{n}}_b + g\nabla_\xi\hat{\varphi}_0\cdot\hat{\mathbf{n}}_b = 0. \tag{59}$$

For the surface condition, the equation (56) is differentiated twice in time and the term in $\partial_{tt}\hat{\mathbf{U}}_{1,0}$ is replaced with (52),

$$\frac{\partial^2 \hat{\varphi}_0}{\partial t^2} - \hat{\rho}_0'(\hat{\mathbf{U}}_{1,0}\cdot\mathbf{e}_3) + \frac{\partial \hat{\varphi}_0}{\partial \xi^3} = 0 \quad \text{ on } \tilde{\Gamma}_s. \tag{60}$$

The boundary conditions (59), (60) will be more convenient for the homogeneous case.

The system (53)-(52) represents an incompressible fluid. However, this system is different from the classical Poisson equation found in the litterature [19] because of the assumption of a nonhomogeneous background density.

Remark: The condition $|\hat{\rho}_0'|/\hat{\rho}_0 > g/\hat{c}_0^2$ is no longer required because the destabilizing effects in the energy equation (28) come from the compressibility, and here it is neglected. This can be seen by formally assuming that the sound velocity is infinite, then $N^2 = -\hat{\rho}_0'/\hat{\rho}_0$. Nevertheless, density must still decrease with depth, but can be homogeneous.

Case of homogeneous density

For the sake of comparison with other models, let us assume now that the ocean at rest has a homogeneous density, $\tilde{\rho}_0' = 0$. Taking the divergence of (52) yields

$$\Delta_\xi \hat{\varphi}_0 = 0. \tag{61}$$

The equation is completed with the boundary conditions (59), (60)

$$\begin{cases} \nabla_\xi \hat{\varphi}_0 \cdot \hat{\mathbf{n}}_b = -\hat{\rho}_0 g\, \hat{u}_{b,1} \quad \text{on } \hat{\Gamma}_b, & (62) \\ \dfrac{\partial^2 \hat{\varphi}_0}{\partial t^2} + g \dfrac{\partial \hat{\varphi}_0}{\partial \xi^3} = 0 \quad \text{on } \hat{\Gamma}_s. & (63) \end{cases}$$

Eq. (61) - (62) is the Poisson equation, with boundary conditions, satisfied by the velocity flow in an incompressible homogeneous free-surface fluid ([19], Chap. 3.1.). Note that it was required that $\tilde{\rho}_0' \neq 0$ in the system (25) to obtain an a priori positive energy. Here this assumption is dropped, however a rather simple energy identity can be derived: multiplying Eq. (61) by $\partial_t \hat{\varphi}_0$, integrating by parts and using Eq. (62)- (63), we obtain

$$\begin{aligned} \int_{\hat{\Omega}} \Delta_\xi \hat{\varphi}_0 \frac{\partial \hat{\varphi}_0}{\partial t}\, \mathrm{d}\boldsymbol{\xi} = &- \int_{\hat{\Omega}} \nabla_\xi \varphi \cdot \nabla_\xi \left(\frac{\partial \hat{\varphi}_0}{\partial t} \right) \mathrm{d}\boldsymbol{\xi} \\ &- \int_{\hat{\Gamma}_s} \frac{1}{g} \frac{\partial \hat{\varphi}_0}{\partial t} \frac{\partial^2 \hat{\varphi}_0}{\partial t^2}\, \mathrm{d}\sigma + \int_{\hat{\Gamma}_b} \hat{\rho}_0 g \frac{\partial \hat{\varphi}_0}{\partial t} \hat{u}_{b,1}\, \mathrm{d}\sigma. \end{aligned} \tag{64}$$

We define the energy

$$\mathcal{E}_{Poisson} = \frac{1}{2} \left(\int_{\hat{\Omega}} |\nabla_\xi \hat{\varphi}|^2\, \mathrm{d}\boldsymbol{\xi} + \int_{\hat{\Gamma}_s} \frac{1}{g} (\frac{\partial \hat{\varphi}_0}{\partial t})^2\, \mathrm{d}\sigma \right),$$

Then it holds

$$\frac{d}{dt} \mathcal{E}_{Poisson} = - \int_{\hat{\Gamma}_b} \hat{\rho}_0 g \frac{\partial \hat{\varphi}_0}{\partial t} \hat{u}_{b,1}\, \mathrm{d}\sigma.$$

3.3 Acoustic limit

Another possible simplification of the system (25) - (27) is to keep only the acoustic terms. This choice is justified for short time scale, because the propagation speed of the acoustic waves and the gravity waves have different orders of magnitude ([20]).

With the timescale $L/\tau = C$, corresponding to the acoustic wave, and with the same small parameter $\delta = \mathrm{Ma}/\mathrm{Fr}$ as before, the system (37) becomes

$$\tilde{\rho}_0 \frac{\partial^2 \tilde{\mathbf{U}}_1}{\partial t^2} - \nabla_\xi \left(\tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{\mathbf{U}}_1 \right) + \delta^2 \tilde{\rho}_0 \left(\nabla_\xi \left(\tilde{\mathbf{U}}_1 \cdot \mathbf{e}_3 \right) - \nabla_\xi \cdot \tilde{\mathbf{U}}_1 \, \mathbf{e}_3 \right) = 0 \quad \text{in } \tilde{\Omega}, \quad (65)$$

With the boundary conditions

$$\begin{cases} \tilde{\mathbf{U}}_1 \cdot \tilde{\mathbf{n}}_b = \tilde{u}_{b,1} & \text{on } \tilde{\Gamma}_b, \\ \nabla_\xi \cdot \tilde{\mathbf{U}}_1 = 0 & \text{on } \tilde{\Gamma}_s. \end{cases}$$

As before, we make the following ansatz for $\tilde{\mathbf{U}}_1$,

$$\tilde{\mathbf{U}}_1 = \tilde{\mathbf{U}}_{1,0} + \delta^2 \tilde{\mathbf{U}}_{1,2} + o(\delta^2).$$

One can see that the limit term $\delta \to 0$ for the volumic equation (65) is

$$\tilde{\rho}_0 \frac{\partial^2 \tilde{\mathbf{U}}_{1,0}}{\partial t^2} - \nabla_\xi \left(\tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{\mathbf{U}}_{1,0} \right) = 0. \quad (66)$$

Taking the curl of this equation yields

$$\frac{\partial^2}{\partial t^2} \left(\nabla_\xi \times (\tilde{\rho}_0 \tilde{\mathbf{U}}_{1,0}) \right) = 0,$$

hence the curl of $\tilde{\rho}_0 \tilde{\mathbf{U}}_{1,0}$ is constant in time due to the vanishing initial conditions. By the Helmoltz decomposition theorem, it can be expressed as the gradient of some function $\tilde{\psi}_0$ defined up to a constant,

$$\tilde{\rho}_0 \tilde{\mathbf{U}}_{1,0} = \nabla_\xi \tilde{\psi}_0.$$

By subsituting in equation (66) we have

$$\nabla_\xi \left(\frac{\partial^2 \tilde{\psi}_0}{\partial t^2} - \tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \left(\tilde{\rho}_0^{-1} \nabla_\xi \tilde{\psi}_0 \right) \right) = 0,$$

then it holds

$$\frac{\partial^2 \tilde{\psi}_0}{\partial t^2} - \tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \left(\tilde{\rho}_0^{-1} \nabla_\xi \tilde{\psi}_0 \right) = 0,$$

since $\tilde{\psi}_0$ is defined up to a constant. We need the boundary conditions to conclude. Evaluating (66) at the surface yields

$$\tilde{\rho}_0 \frac{\partial^2 \tilde{\mathbf{U}}_{1,0}}{\partial t^2} - \frac{\partial}{\partial \xi^3}\left(\tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{\mathbf{U}}_{1,0}\right)\mathbf{e}_3 = 0 \qquad \text{on } \tilde{\Gamma}_s. \tag{67}$$

We take the cross product of (67) with the normal to the surface of the domain, which is $\mathbf{e}_3$, and replace the velocity by the scalar potential $\tilde{\psi}_0$.

$$\nabla\left(\frac{\partial^2 \tilde{\psi}_0}{\partial t^2}\right) \times \mathbf{e}_3 = 0 \qquad \text{on } \tilde{\Gamma}_s. \tag{68}$$

This implies that the tangential derivative of $\tilde{\psi}_0$ vanishes on the surface. Since $\tilde{\psi}_0$ is defined up to a constant, it holds $\partial^2_{tt}\tilde{\psi}_0 = 0$ on $\tilde{\Gamma}_s$, and with the vanishing initial conditions this implies that $\tilde{\psi}_0 = 0$ on $\tilde{\Gamma}_s$. To recover a dimensional system, the terms are multiplied by their corresponding characteristic scales, and $\hat{\psi}_0 = \bar{\rho} U L \tilde{\psi}_0$ is defined. The system reads then

$$\frac{\partial^2 \hat{\psi}_0}{\partial t^2} - \hat{\rho}_0 \hat{c}_0^2 \nabla_\xi \cdot \left(\hat{\rho}_0^{-1} \nabla_\xi \hat{\psi}_0\right) = 0 \qquad \text{in } \hat{\Omega}, \tag{69}$$

with the boundary conditions

$$\begin{cases} \nabla_\xi \hat{\psi}_0 \cdot \hat{\mathbf{n}}_b = \hat{u}_{b,1} & \text{on } \hat{\Gamma}_b, \quad (70) \\ \hat{\psi}_0 = 0 & \text{on } \hat{\Gamma}_s. \quad (71) \end{cases}$$

An energy equation can be obtained by multiplying Eq. (73) by $\partial_t \psi/(\rho_0 \hat{c}_0^2)$ and integrating over the domain. The result reads after an integration by parts

$$\frac{d}{dt}\mathcal{E}_{acoustic} = -\int_{\hat{\Gamma}_b} \frac{1}{\hat{\rho}_0} \frac{\partial \hat{\psi}_0}{\partial t} \hat{u}_{b,1} \, \mathrm{d}\sigma, \tag{72}$$

where the acoustic energy is

$$\mathcal{E}_{acoustic} = \frac{1}{2}\int_{\hat{\Omega}} \frac{1}{\hat{\rho}_0 \hat{c}_0^2} \frac{\partial \hat{\psi}_0}{\partial t} \, \mathrm{d}\boldsymbol{\xi} + \frac{1}{2}\int_{\hat{\Omega}} \frac{1}{\hat{\rho}_0} |\nabla \hat{\psi}_0|^2 \, \mathrm{d}\boldsymbol{\xi}.$$

Homogeneous case

When the density $\hat{\rho}_0$ is constant, the equation (69) becomes

$$\frac{\partial^2 \hat{\psi}_0}{\partial t^2} - \hat{c}_0^2 \Delta \psi_0 = 0 \quad \text{in } \hat{\Omega}. \tag{73}$$

Eq. (73) is completed with the boundary conditions

$$\begin{cases} \nabla_\xi \hat{\psi}_0 \cdot \hat{\mathbf{n}}_b = \hat{u}_{b,1} & \text{on } \hat{\Gamma}_b, \quad (74) \\ \hat{\psi}_0 = 0 & \text{on } \hat{\Gamma}_s. \quad (75) \end{cases}$$

We recover the well-known linear acoustic equation ([16], Chap. 10.).

In this section we obtained two limit systems for the equation (25), respectively in the incompressible regime and in the acoustic regime. We also showed that classical models, such as the Poisson equation for an incompressible fluid, or the acoustic wave equation, can be obtained as the formal limits of the equation (25) with the additional assumption of an homogeneous density. The equations with their boundary conditions and the associated energy for the general model and its different simplifications are summarized in Appendix, Table 1.

Remark: In the litterature, the boundary condition used for the propagation of acoustic waves in a free-surface, irrotational fluid, for a homogeneous density at equilibrium are the same as (63), see for example [28], [26], [24]. In those models, the acoustic and the gravity waves are decoupled, with the acoustic waves propagating inside the domain and the gravity waves propagating at the surface. One can also notice that the boundary conditions (75) and (74) obtained from the acoustic approximation coincide with [20].

Remark: In the sections 3.2 and 3.3, the equations (61) - (63) and (73)-(74) use the Lagrangian description whereas the equations from the litterature use the Eulerian description. In the general case, the use of different coordinate systems would cause two problems. First, when doing the change of coordinates, new terms should appear from the space or time differentiation. Second, the description of the domain is different, and this implies that the boundary conditions are not evaluated at the same location. In the next section we will show that the first problem does not exist in our case, due to the lack of a background velocity. As for the second problem, the linear eulerian models are obtained by evaluating the boundary conditions at a fixed water height. To this regard they use the same boundary as if they were in a Lagrangian description of the domain, so that the comparison remains valid.

4 The model in Eulerian coordinates

The equations we have been working on are defined on the reference domain $\hat{\Omega}$. However, the linear equations for the acoustic-gravity waves are generally written in Eulerian coordinates. In order to compare our model with those from the litterature, the equations must be formulated on the moving domain $\Omega(t)$. In this section we present a method to write the system in Eulerian coordinate.

4.1 General method

The aim is to write the equation on a moving domain $\Omega(t)$, hence a transformation $\boldsymbol{\phi} : \hat{\Omega} \to \Omega(t)$ is needed. We start by using a first order approximation

of the real transformation $\boldsymbol{\phi}$. The transformation $\boldsymbol{\phi}$ is developed for small displacements,

$$\boldsymbol{\phi}(\xi, t) = \mathbb{I} + \varepsilon \boldsymbol{\phi}_1(\xi, t) + \mathcal{O}(\varepsilon^2).$$

Let $\boldsymbol{\phi}_\varepsilon(\boldsymbol{\xi}, t) = \mathbb{I} + \varepsilon \boldsymbol{\phi}_1(\boldsymbol{\xi}, t)$ be its first order approximation. $\boldsymbol{\phi}_\varepsilon$ is used to define the equivalent domain and its boundary,

$$\Omega_\varepsilon(t) = \boldsymbol{\phi}_\varepsilon(\hat{\Omega}), \quad \Gamma_{s,eq} = (\boldsymbol{\phi}_\epsilon(\hat{\Gamma}_s)), \quad \Gamma_{b,eq} = (\boldsymbol{\phi}_\epsilon(\hat{\Gamma}_b)). \tag{76}$$

The coordinates on the equivalent domain are written $\mathbf{x} = (x, y, z)$. For any generic function $\hat{X}(\boldsymbol{\xi}, t)$ defined in $\hat{\Omega}$, a function $X(\mathbf{x}, t)$ is defined in Ω_ε by the following change of variables

$$X(\mathbf{x}, t) = \hat{X}(\boldsymbol{\phi}_\varepsilon^{-1}(\mathbf{x}, t), t),$$

which is equivalent to

$$\hat{X}(\boldsymbol{\xi}, t) = X(\boldsymbol{\phi}_\varepsilon(\boldsymbol{\xi}, t), t), \tag{77}$$

as long as $\boldsymbol{\phi}_\varepsilon$ is invertible. Then, if the function $\hat{X}$ has a first-order approximation $\hat{X} = \hat{X}_0 + \varepsilon \hat{X}_1 + \mathcal{O}(\varepsilon^2)$, then the function X also has a first-order approximation $X = X_0 + \varepsilon X_1 + \mathcal{O}(\varepsilon^2)$ and it holds (see Appendix, Sec. 7.2)

$$\nabla_\xi \hat{X}_0 = \nabla X_0, \tag{78}$$

$$\frac{\partial \hat{X}_0}{\partial t} = \frac{\partial X_0}{\partial t}, \tag{79}$$

$$\nabla_\xi \hat{X}_1 = (\nabla_\xi \mathbf{d}_1)^T \nabla X_0 + \nabla X_1, \tag{80}$$

$$\frac{\partial \hat{X}_1}{\partial t} = \frac{\partial X_1}{\partial t} + \mathbf{U}_1 \cdot \nabla X_0. \tag{81}$$

In the following, when writing the equations satisfied by the free surface of Ω_ϵ, we will also use

$$\frac{\partial \phi_\epsilon}{\partial t} = \varepsilon \hat{\mathbf{U}}_1. \tag{82}$$

4.2 The model in Eulerian coordinates

Using the change of variable (77) in the system (20)-(22) and with the equalities (78)-(81) we obtain the following system for $\mathbf{U}_1, p_1, \rho_1$ defined in Ω_ε,

$$\begin{cases} \rho_0 \dfrac{\partial \mathbf{U}_1}{\partial t} + \nabla p_1 = \rho_1 \mathbf{g}, & (83) \\ \dfrac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{U}_1) = 0, & (84) \\ \dfrac{\partial p_1}{\partial t} + \nabla p_0 \cdot \mathbf{U}_1 + \rho_0 c_0^2 \, \nabla \cdot \mathbf{U}_1 = 0. & (85) \end{cases}$$

And p_0, ρ_0 satisfy the limit equations

$$\begin{cases} \dfrac{\partial \rho_0}{\partial t} = 0, & (86) \\ \nabla p_0 = \rho_0 \mathbf{g}. & (87) \end{cases}$$

We assume in the following that the equations (83)-(87) are defined in $\hat{\Omega}$. It would be true if $\hat{\Omega} \subset \Omega_\varepsilon$, but the inclusion is in general not verified. The assumption is therefore not entirely satisfactory and has partly motivated the present work. Assuming that (83)-(87) are defined in $\hat{\Omega}$, it remains to write the boundary conditions on the boundary of $\hat{\Omega}$.

Boundary conditions and free surface description

Following the approach of [23], we show that a description for the free surface can be obtained. In the following, the components of the fluid velocity are denoted $\mathbf{U} = (U_1^1,\ U_1^2,\ U_1^3)^T$. The surface is defined by $\Gamma_{s,eq} = \boldsymbol{\phi}_\varepsilon(\hat{\Gamma}_s)$, and we assume that at each time t it can be parametrized as the graph η_ε. The elevation η_ε is a function of x, y, and t and can be decomposed in the following way,

$$\eta_\varepsilon(x, y, t) = H + \varepsilon\eta_1(x, y, t). \tag{88}$$

From the correspondence between the free surface and the particle displacement, it holds

$$\phi_\varepsilon^3(\xi^1, \xi^2, H, t) = \eta_\varepsilon\big(x(\xi^1, \xi^2, H, t),\ y(\xi^1, \xi^2, H, t), t\big). \tag{89}$$

Differentiating (89) in time and using the equation (82) yields

$$\varepsilon\hat{U}_1^3(\xi^1, \xi^2, H, t) = \frac{\partial\eta_\varepsilon}{\partial t} + \varepsilon\hat{U}_1^1(\xi^1, \xi^2, H, t)\,\frac{\partial\eta_\varepsilon}{\partial x} + \varepsilon\hat{U}_1^2(\xi^1, \xi^2, H, t)\,\frac{\partial\eta_\varepsilon}{\partial y}.$$

We use the change of variables $\phi_\varepsilon(\boldsymbol{\xi}, t) = Id + \varepsilon\phi_1(\boldsymbol{\xi}, t)$,

$$\varepsilon U_1^3(\phi_\varepsilon(\xi^1, \xi^2, H, t), t) = \frac{\partial\eta_\varepsilon}{\partial t} + \varepsilon U_1^1(\phi_\varepsilon(\xi^1, \xi^2, H, t), t)\,\frac{\partial\eta_\varepsilon}{\partial x} + \varepsilon U_1^2(\phi_\varepsilon(\xi^1, \xi^2, H, t), t)\,\frac{\partial\eta_\varepsilon}{\partial y}.$$

After a Taylor developement and keeping only the terms in ε, it holds

$$U_1^3(x, y, H, t) = \frac{\partial\eta_1}{\partial t}, \tag{90}$$

this is the linearized equation for the free surface.

Then the dynamic boundary conditions are linearized. With the change of variables, the boundary conditions (15), (23) and (24) become

$$\mathbf{U}_1 \cdot \mathbf{n}_b = u_{b,1} \quad \text{on } \Gamma_{b,eq}., \tag{91}$$

$$p_0 = p^a \quad \text{on } \Gamma_{s,eq}, \tag{92}$$

$$p_1 = 0 \quad \text{on } \Gamma_{s,eq}. \tag{93}$$

If we linearize (93) only we would miss the first-order term coming from (92). From (92) and (93) we deduce the boundary condition for the pressure

$$p_0 + \varepsilon p_1 = p^a \quad \text{on } \Gamma_{s,eq}. \tag{94}$$

A Taylor developement of p_0 and p_1 around $z = H$ on $\Gamma_{s,eq}$ yields

$$p_0(H) + \varepsilon(p_1(x, y, H, t) + p_0'(H)\eta_1) + \mathcal{O}(\varepsilon^2) = p^a.$$

After an identification of the powers of ε it holds

$$p_0(H) = p^a, \quad p_1(x, y, H, t) = \rho_0(x, y, H, t)g\,\eta_1(x, y, t). \tag{95}$$

In a similar way, the linearization of Eq. (91) reads

$$U_1^3(x, y, z_b) - U_1^1(x, y, z_b)\,\partial_x z_b - U_1^2(x, y, z_b)\,\partial_y z_b = u_{b,1}(x, y, t). \tag{96}$$

Hence the equations for $\mathbf{U}_1, \rho_1, p_1$ can be fully defined on the domain $\hat{\Omega}$, with an error in $\mathcal{O}(\varepsilon^2)$. Finally, note that for the system (83)-(85) with the boundary conditions (95),(96) and the kinematic condition (90), an energy is available, locally as well as over a whole water column (see [19], [21]).

In this section we have derived the linear equation in Eulerian coordinates, even though an approximation on the domain in which the equations are defined was necessary. The computations of Sec. 4.1 also justify that in the absence of mean flow and with the evaluation of the boundary conditions at a fixed height, the linear system in Eulerian coordinates is similar to the one in Lagrangian coordinate, up to terms in $\mathcal{O}(\varepsilon^2)$. At the same time, the linearization in the Lagrangian coordinates is better defined. For this reason the system in Lagrangian coordinates is preferred for the rest of this work. We conclude this paper by the study of the dispersion relation obtained from Eq. (25).

5 Dispersion relation

In this section we derive the dispersion relation from Eq. (25) and solve it numerically. First note that if one defines the equivalent pressure p_ε, the equivalent density ρ_ε and the equivalent velocity $\mathbf{U}_\varepsilon$ by

$$p_\varepsilon = p_0 + \varepsilon p_1, \quad \rho_\varepsilon = \rho_0 + \varepsilon\rho_1, \quad \mathbf{U}_\varepsilon = \varepsilon\mathbf{U}_1,$$

then a combination of the equations (83) - (87) yields the following system for p_ε, ρ_ε and $\mathbf{U}_\varepsilon$,

$$\rho_0 \frac{\partial \mathbf{U}_\varepsilon}{\partial t} + \nabla p_\varepsilon = \rho_\varepsilon \mathbf{g} + \mathcal{O}(\varepsilon^2), \tag{97}$$

$$\frac{\partial \rho_\varepsilon}{\partial t} + \nabla \cdot (\rho_0 \mathbf{U}_\varepsilon) = \mathcal{O}(\varepsilon^2), \tag{98}$$

$$\frac{\partial p_\varepsilon}{\partial t} + \nabla p_0 \cdot \mathbf{U}_\varepsilon + \rho_0 c_0^2\ \nabla \cdot \mathbf{U}_\varepsilon = \mathcal{O}(\varepsilon^2). \tag{99}$$

This system is comparable – up to the terms in $\mathcal{O}(\epsilon^2)$ – to the system studied in [2]. From this system of equation, the authors give a thorough analysis of the

dispersion relation for a stratified compressible fluid. This motivates the study of the dispersion relation for the wave-like equation (25).

In order to make the computations clearer, the problem is restricted to a 2-dimensional configuration in ξ^1 and ξ^3. Following the approach of [2], the wave angular frequency ω and the horizontal wave-number frequency k_x are defined, and we seek a solution of the form

$$\hat{\rho}_0 \hat{\mathbf{U}}_1(\xi^1, \xi^3, t) = \begin{pmatrix} \widetilde{U}^1(\xi^3) \\ \widetilde{U}^3(\xi^3) \end{pmatrix} e^{i(k_x \xi^1 - \omega t)}. \tag{100}$$

First Eq. (25) is written differently to make the unknown $\hat{\rho}_0 \hat{\mathbf{U}}_1$ appear.

$$\frac{\partial^2 \hat{\rho}_0 \hat{\mathbf{U}}_1}{\partial t^2} - \nabla_\xi (\hat{c}_0^2 \nabla_\xi \cdot (\hat{\rho}_0 \hat{\mathbf{U}}_1)) - \nabla_\xi \left(\frac{\hat{c}_0^2 N_0^2}{g} \rho_0 \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \right) - g \nabla_\xi \cdot (\hat{\rho}_0 \hat{\mathbf{U}}_1)\, \mathbf{e}_3 = 0 \tag{101}$$

Injecting the ansatz (100) in Eq. (101) yields

$$\omega^2 \widetilde{U}^1 + i k_x \left(\hat{c}_0^2 (i k_x \widetilde{U}^1 + (\widetilde{U}^3)') + \frac{\hat{c}_0^2 N^2}{g} \widetilde{U}^3 \right) = 0, \tag{102}$$

$$\omega^2 \widetilde{U}^3 + \partial_3 \left(\hat{c}_0^2 (i k_x \widetilde{U}^1 + (\widetilde{U}^3)') \right) + \partial_3 \left(\frac{\hat{c}_0^2 N^2}{g} \widetilde{U}^3 \right) + g (i k_x \widetilde{U}^1 + (\widetilde{U}^3)') = 0. \tag{103}$$

Using the equation (102), the horizontal component $\widetilde{U}^1$ is expressed as a function of the vertical component,

$$\widetilde{U}^1 = -i k_x \frac{\hat{c}_0^2 D (\widetilde{U}^3)' + (\hat{c}_0^2 - gD) \widetilde{U}^3}{D(\omega^2 - c_0^2 k_x^2)},$$

where D is a depth scale, defined by

$$\frac{1}{D} = \frac{N^2}{g} + \frac{g}{\hat{c}_0^2} = \frac{\rho_0'}{\rho_0}.$$

We also define the quantity

$$S = 2 \frac{\hat{c}_0'}{\hat{c}_0}.$$

Replacing $\widetilde{U}^1$ in the equation (103) yields, after some computations,

$$(\widetilde{U}^3)'' + \left(\frac{1}{D} + \omega^2 S^2 \right) (\widetilde{U}^3)' + \left(\frac{\omega^2}{c_0^2} + k_x^2 \frac{N^2 - \omega^2}{\omega^2} - \frac{D'}{D^2} + S \left(\frac{g}{\hat{c}_0^2} + \frac{N^2}{g} \frac{\omega^2}{\omega^2 - c_0^2 k_x^2} \right) \right) \widetilde{U}^3 = 0.$$

In order to write an harmonic equation the following change of variable is made,

$$\widetilde{U}^3(z) = \widetilde{U}^3(H)F(z)\exp\left(\int_z^H \frac{\alpha}{2}dz'\right), \quad \alpha = \frac{1}{D} + \omega^2 S.$$

Then $F(0) = 0$, $F(H) = 1$ and F satisfies the equation

$$F'' + k_z^2 F = 0 \tag{104}$$

where the vertical wave-number k_z is defined by

$$k_z^2 + k_x^2\frac{N^2-\omega^2}{\omega^2} + \frac{\omega^2}{\hat{c}_0^2} - \frac{1+2D'}{4D^2} - \frac{1}{2}\omega^2 S' + S\left(\frac{g}{\hat{c}_0^2} + \frac{N^2}{g}\frac{\omega^2}{\omega^2 - \hat{c}_0^2 k_x^2} - \frac{\omega^2}{2D} - \frac{1}{4}\omega^4 S\right) = 0, \tag{105}$$

Eq. (105) is the dispersion relation for the two wave-numbers k_x, k_z and the frequency ω. If the speed of sound is assumed constant, then $S \equiv 0$ and one recovers exactly the same inner dispersion relation as [2].

Remark: When k_z depends on z, it is not clear whether the solution to (104), and then the profile $\widetilde{U}^3$, can be written explicitely. When k_z is constant in z, as in [2], the expression of the profile $\widetilde{U}^3$ is used with the boundary conditions to obtain a boundary dispersion relation. In our case k_z is not a constant, and the boundary dispersion relation is not easily deduced.

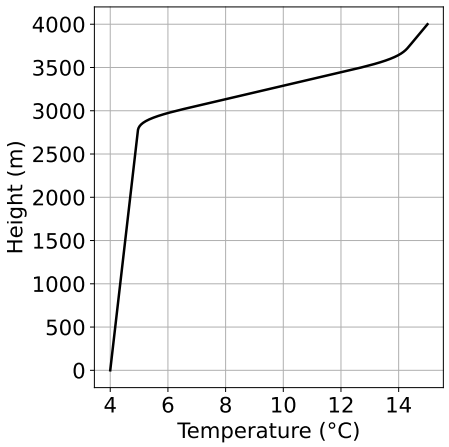
Numerical approximation of the dispersion relation

An evaluation of the equation (105) is possible once the limit state for the pressure and the density is computed. The differential equation for the pressure (13) is numerically solved for the temperature profile shown in Fig. 3a. Then the density and the speed of sound are computed from the tabulations given in [6]. Fig. 3b, 3c show the obtained density and speed of sound. With these profiles the dispersion relation (105) is computed. Fig 4 shows the contours of the vertical wave-number as a function of the horizontal wave-number and the angular frequency, at different depths. For the sake of comparison, the plotted variables are the adimensionned variables $\delta_x = k_x H$, $\delta_z = k_z H$ and $\log_{10}(\delta_\omega)$, where $\delta_\omega = \omega\sqrt{H/g}$.

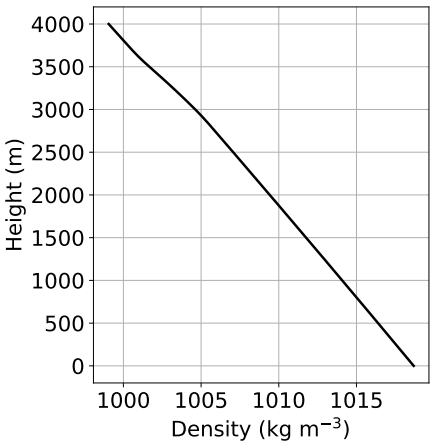
Although Fig 4 is close to the one in [2], one can notice the influence of the depth on the contour. The equation (105) is not further studied in this work, but this first result suggests that the variation of the parameters $\hat{c}_0, N, D$ with depth plays a non-negligible role in the waves dispersion.

6 Conclusion and future work

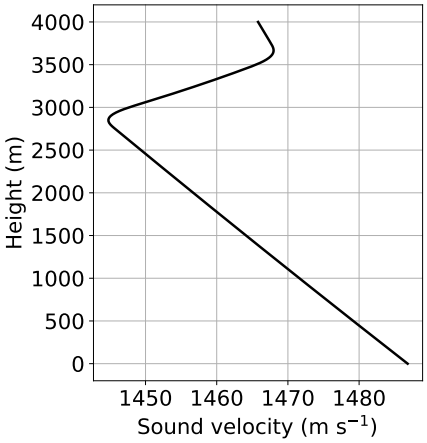
In this work we have presented an original system describing the propagation of the acoustic-gravity waves in an inviscid free-surface flow. Through a rigorous



(a) Temperature profile.

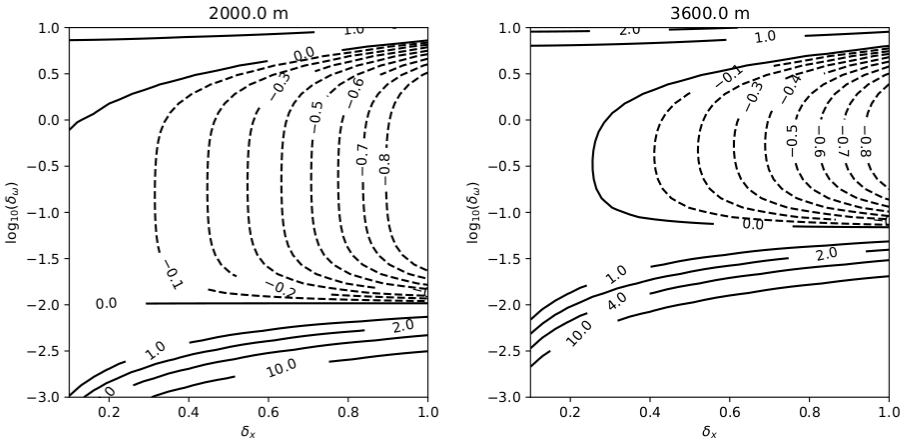


(b) Density.



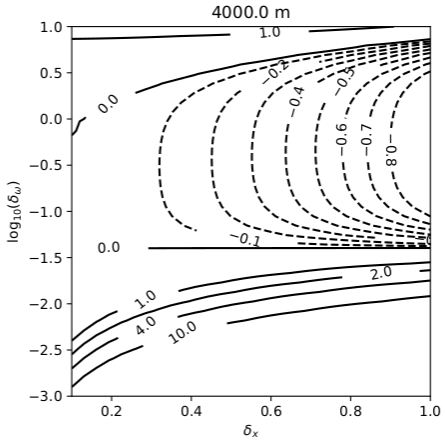
(c) Sound velocity

Figure 3: Temperature, density and sound velocity profiles used for the computation of the dispersion relation where $\xi^3 = 0$ is the seafloor and $\xi^3 = 4000$ m is the ocean surface.



(a) $\xi^3 = 2000$ m

(b) $\xi^3 = 3600$ m.



(c) $\xi^3 = 4000$ m.

Figure 4: Contour of the vertical wave-number as a function of the horizontal wave-number and the angular frequency, at different depths. 4a: $\xi^3 = 2000$ m, 4b: $\xi^3 = 3600$ m, 4c: $\xi^3 = 4000$ m.

linearization of the compressible Euler equation we have obtained a model able to represent many physical phenomenons, such as the SOFAR channel or the propagation of internal waves. The variety of these phenomenons is well represented in the dispersion relation.

In the derivation only few assumptions are made and some common simplifying hypotheses were avoided. In particular, the fluid is rotational and a generic equation of state can be used. Note also that in the present work the source term is a displacement of the seabed, but this is not restrictive and other source terms could be used (a change in the surface pressure for example). With additional assumptions compatible with the derivation of the system, such as considering a barotropic fluid, or restricting the model to the incompressible regime or to the acoustic regime, we are able to recover simpler models widely studied in the litterature. Hence the mathematical study of the more complete model can help gain insight on the other ones. The linear model in Lagrangian coordinates can also be used to recover the linearized Euler equations in Eulerian coordinates. This brings a clear understanding of the usual – nevertheless non satisfactory – assumption that is used to derive the aformentionned models in Eulerian coordinates.

The wave-like formulation of the model makes it a good candidate for a numerical approximation by the finite elements method. The fact that it preserves an energy suggests that the problem is well-posed, which motivates a more thorough study of the mathematical problem. These two aspects will be investigated in a future work.

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7 Appendix

7.1 The energy equation

In this section an energy equation for the system (25) is obtained. Recall that the system (25) reads in $\hat{\Omega}$,

$$\hat{\rho}_0 \frac{\partial^2 \hat{\mathbf{U}}_1}{\partial t^2} - \nabla_\xi \left(\hat{\rho}_0 \hat{c}_0^2 \nabla_\xi \cdot \hat{\mathbf{U}}_1 \right) - (\nabla_\xi \hat{\mathbf{U}}_1)^T \ \hat{\rho}_0 \mathbf{g} + \hat{\rho}_0 \nabla_\xi \cdot \hat{\mathbf{U}}_1 \ \mathbf{g} = 0,$$

with the boundary conditions

$$\begin{cases} \hat{\mathbf{U}}_1 \cdot \mathbf{n}_b = \hat{u}_{b,1} & \text{on } \hat{\Gamma}_b, \\ \nabla_\xi \cdot \hat{\mathbf{U}}_1 = 0 & \text{on } \hat{\Gamma}_s. \end{cases}$$

By taking the scalar product of (25) with $\partial_t \hat{\mathbf{U}}_1$ and integrating over the domain we have

$$\begin{aligned} &\int_{\hat{\Omega}} \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \cdot \left(\hat{\rho}_0 \frac{\partial^2 \hat{\mathbf{U}}_1}{\partial t^2} \right) \mathrm{d}\boldsymbol{\xi} - \int_{\hat{\Omega}} \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \cdot \left(\nabla_\xi \left(\hat{\rho}_0 \hat{c}_0^2 \nabla_\xi \cdot \hat{\mathbf{U}}_1 \right) \right) \mathrm{d}\boldsymbol{\xi} \\ &\quad + \int_{\hat{\Omega}} \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \cdot \left(\nabla_\xi (\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3)\ \hat{\rho}_0 g \right) \mathrm{d}\boldsymbol{\xi} - \int_{\hat{\Omega}} \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \cdot \left(\hat{\rho}_0 \nabla_\xi \cdot \hat{\mathbf{U}}_1\ g\, \mathbf{e}_3 \right) \mathrm{d}\boldsymbol{\xi} = 0. \end{aligned} \tag{106}$$

For the first integral of (106) it holds

$$\int_{\hat{\Omega}} \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \cdot \left(\hat{\rho}_0 \frac{\partial^2 \hat{\mathbf{U}}_1}{\partial t^2} \right) \mathrm{d}\boldsymbol{\xi} = \frac{d}{dt} \int_{\hat{\Omega}} \rho_0 \frac{1}{2} \left| \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \right|^2 \mathrm{d}\boldsymbol{\xi}. \tag{107}$$

The second term of (106) is integrated by parts, using $\nabla_\xi \cdot \hat{\mathbf{U}}_1 = 0$ on the surface and $\hat{\mathbf{U}}_1 \cdot \hat{\mathbf{n}}_b = \hat{b}_1$ at the bottom (hence $\partial_t (\hat{\mathbf{U}}_1 \cdot \hat{\mathbf{n}}_b) = \partial_t \hat{b}_1$),

$$\begin{aligned} -\int_{\hat{\Omega}} \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \cdot \nabla_\xi \left(\rho_0 c_0^2 \nabla_\xi \cdot \hat{\mathbf{U}}_1 \right) \mathrm{d}\boldsymbol{\xi} &= \frac{1}{2} \frac{d}{dt} \int_{\hat{\Omega}} \hat{\rho}_0 \hat{c}_0^2 |\nabla_\xi \cdot \hat{\mathbf{U}}_1|^2 \, \mathrm{d}\boldsymbol{\xi} \\ &\quad - \int_{\hat{\Gamma}_b} \rho_0 c_0^2 \nabla_\xi \cdot \hat{\mathbf{U}}_1 \frac{\partial \hat{b}_1}{\partial t} \, \mathrm{d}\sigma. \end{aligned} \tag{108}$$

For the computation of the two last integral of (106), we define

$$(I) = \int_{\hat{\Omega}} \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \cdot \left(\nabla_\xi (\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3)\ \hat{\rho}_0 g \right) \mathrm{d}\boldsymbol{\xi} - \int_{\hat{\Omega}} \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \cdot \left(\hat{\rho}_0 \nabla_\xi \cdot \hat{\mathbf{U}}_1\ g\, \mathbf{e}_3 \right) \mathrm{d}\boldsymbol{\xi},$$

an we denote by $\hat{\mathbf{n}}$ be the vector normal to the boundary $\partial\Omega$. (I) is integrated by parts and reads

$$\begin{aligned} (I) = \int_{\partial\Omega} \hat{\rho}_0 g \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3\ \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \cdot \hat{\mathbf{n}} \, \mathrm{d}\sigma - \int_{\hat{\Omega}} g \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \nabla_\xi \cdot (\hat{\rho}_0 \frac{\partial \hat{\mathbf{U}}_1}{\partial t}) \, \mathrm{d}\boldsymbol{\xi} \\ - \int_{\hat{\Omega}} \hat{\rho}_0 g \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \cdot \mathbf{e}_3 \nabla_\xi \cdot \hat{\mathbf{U}}_1 \, \mathrm{d}\boldsymbol{\xi}. \end{aligned}$$

The boundary term is simplifed using $\partial_t (\hat{\mathbf{U}}_1 \cdot \hat{\mathbf{n}}_b) = \partial_t \hat{b}_1$ at the bottom. On the boundary $\hat{\Gamma}_s$, the surface is horizontal hence the normal vector is the unit

vector $\mathbf{e}_3$, so it holds

$$(I) = \int_{\hat{\Gamma}_b} \hat{\rho}_0 g \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \frac{\partial \hat{b}_1}{\partial t} \,\mathrm{d}\sigma + \int_{\hat{\Gamma}_s} \hat{\rho}_0 g \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \cdot \mathbf{e}_3 \,\mathrm{d}\sigma$$
$$- \int_{\hat{\Omega}} g \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \nabla_\xi \cdot (\hat{\rho}_0 \frac{\partial \hat{\mathbf{U}}_1}{\partial t}) \,\mathrm{d}\boldsymbol{\xi} - \int_{\hat{\Omega}} \hat{\rho}_0 g \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \cdot \mathbf{e}_3 \nabla_\xi \cdot \hat{\mathbf{U}}_1 \,\mathrm{d}\boldsymbol{\xi}. \quad (109)$$

Next we develop the gradient in the third integral of Eq. (109). Note that $\hat{\rho}_0$ depends only on the vertical coordinate, then we have

$$- \int_{\hat{\Omega}} g \, \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \cdot \nabla_\xi \hat{\rho}_0 = - \int_{\hat{\Omega}} g \, \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \frac{\partial \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3}{\partial t} \frac{d\hat{\rho}_0}{d\xi^3} = -\frac{1}{2} \frac{d}{dt} \int_{\hat{\Omega}} \hat{\rho}_0' g \, |\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3|^2,$$

hence we obtain

$$(I) = \int_{\hat{\Gamma}_b} \hat{\rho}_0 g \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \, \frac{\partial \hat{b}_1}{\partial t} \,\mathrm{d}\sigma + \frac{1}{2} \frac{d}{dt} \int_{\hat{\Gamma}_s} \hat{\rho}_0 g |\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3|^2 \,\mathrm{d}\sigma$$
$$- \frac{1}{2} \frac{d}{dt} \int_{\hat{\Omega}} g \hat{\rho}_0' |\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3|^2 \,\mathrm{d}\boldsymbol{\xi} - \int_{\hat{\Omega}} \hat{\rho}_0 g \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \frac{\partial}{\partial t} (\nabla_\xi \cdot \hat{\mathbf{U}}_1) \,\mathrm{d}\boldsymbol{\xi}$$
$$- \int_{\hat{\Omega}} \hat{\rho}_0 g \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \cdot \mathbf{e}_3 \nabla_\xi \cdot \hat{\mathbf{U}}_1 \,\mathrm{d}\boldsymbol{\xi}. \quad (110)$$

The two last terms of (110) are put together,

$$(I) = \int_{\hat{\Gamma}_b} \hat{\rho}_0 g \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \, \frac{\partial \hat{b}_1}{\partial t} \,\mathrm{d}\sigma + \frac{1}{2} \frac{d}{dt} \int_{\hat{\Gamma}_s} \hat{\rho}_0 g |\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3|^2 \,\mathrm{d}\sigma$$
$$- \frac{1}{2} \frac{d}{dt} \int_{\hat{\Omega}} g \hat{\rho}_0' |\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3|^2 \,\mathrm{d}\boldsymbol{\xi} - \frac{d}{dt} \int_{\hat{\Omega}} \hat{\rho}_0 g \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \nabla_\xi \cdot \hat{\mathbf{U}}_1 \,\mathrm{d}\boldsymbol{\xi}. \quad (111)$$

Summing the terms (107), (108) and (111) yields

$$\frac{d}{dt} \int_{\hat{\Omega}} \rho_0 \frac{1}{2} \left| \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \right|^2 \,\mathrm{d}\boldsymbol{\xi} + \frac{1}{2} \frac{d}{dt} \int_{\hat{\Omega}} \hat{\rho}_0 \left(\hat{c}_0 \nabla_\xi \cdot \hat{\mathbf{U}}_1 - \frac{g}{\hat{c}_0} \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \right)^2 \,\mathrm{d}\boldsymbol{\xi}$$
$$- \frac{1}{2} \frac{d}{dt} \int_{\hat{\Omega}} \hat{\rho}_0 (\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3)^2 \left(\frac{g^2}{\hat{c}_0^2} + \frac{g \hat{\rho}_0'}{\hat{\rho}_0} \right) \,\mathrm{d}\boldsymbol{\xi} + \frac{1}{2} \frac{d}{dt} \int_{\Gamma_s} \hat{\rho}_0 g (\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3)^2 \,\mathrm{d}\sigma$$
$$= \int_{\hat{\Gamma}_b} \rho_0 \left(c_0^2 \nabla_\xi \cdot \hat{\mathbf{U}}_1 - \hat{\rho}_0 g \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \right) \frac{\partial \hat{u}_{b,1}}{\partial t} \,\mathrm{d}\sigma,$$

and by defining

$$N_b = - \left(\frac{g^2}{\hat{c}_0^2} + \frac{g \hat{\rho}_0'}{\hat{\rho}_0} \right),$$

we obtain the energy equation (28).

7.2 Space and time derivatives in Ω_ϵ

In this section we derive the relations between the zero- and first-order approximation in Eulerian and in Lagrangian coordinates, when differentiating with respect to time or space. First note that ϕ_0 and ϕ_1 can be expressed in terms of the displacement $\mathbf{d}$. From the assumption of small displacements it holds $\mathbf{d} = \varepsilon \mathbf{d}_1 + \mathcal{O}(\varepsilon^2)$, then identifying the powers of ε and summing yields

$$\boldsymbol{\phi}_\varepsilon(\boldsymbol{\xi}, t) = \boldsymbol{\xi} + \varepsilon \mathbf{d}_1(\boldsymbol{\xi}, t).$$

From the change of coordinate we have

$$\nabla_\xi \hat{X} = (\nabla_\xi \phi_\varepsilon)^T \nabla X = (Id + \varepsilon \nabla_\xi \mathbf{d}_1)^T \nabla X,$$

and using this identity for $\hat{X} = \hat{X}_0 + \varepsilon \hat{X}_1$ yields

$$\nabla_\xi (\hat{X}_0 + \varepsilon \hat{X}_1) = \nabla X_0 + \varepsilon \left((\nabla_\xi \mathbf{d}_1)^T \nabla X_0 + \nabla X_1 \right) + \mathcal{O}(\varepsilon^2).$$

By identifying the powers of ε it holds

$$\nabla_\xi \hat{X}_0 = \nabla X_0, \qquad \nabla_\xi \hat{X}_1 = (\nabla_\xi \mathbf{d}_1)^T \nabla X_0 + \nabla X_1.$$

The same method is used for the time derivative. Starting with

$$\frac{\partial \hat{X}}{\partial t}(\boldsymbol{\xi}, t) = \frac{\partial X}{\partial t}(\boldsymbol{\phi}_\varepsilon(\boldsymbol{\xi}), t) + \frac{\partial \boldsymbol{\phi}_\varepsilon}{\partial t}(\boldsymbol{\xi}) \cdot \nabla X(\boldsymbol{\phi}_\varepsilon(\boldsymbol{\xi}), t),$$

we obtain after replacing X and $\hat{X}$ by their first order approximation,

$$\frac{\partial \hat{X}_0}{\partial t} + \varepsilon \frac{\partial \hat{X}_1}{\partial t} = \frac{\partial X_0}{\partial t} + \varepsilon \left(\frac{\partial X_1}{\partial t} + \frac{\partial \mathbf{d}_1}{\partial t} \cdot \nabla X_0 \right) + \mathcal{O}(\varepsilon^2).$$

With $\partial_t \mathbf{d}_1(\boldsymbol{\xi}, t) = \hat{\mathbf{U}}_1(\boldsymbol{\xi}, t) = \mathbf{U}_1(\mathbf{x}, t)$ it holds

$$\frac{\partial X_0}{\partial t} + \varepsilon \left(\frac{\partial X_1}{\partial t} + \mathbf{U}_1 \cdot \nabla X_0 \right) + \mathcal{O}(\varepsilon^2).$$

We identify the powers of ε,

$$\frac{\partial \hat{X}_0}{\partial t} = \frac{\partial X_0}{\partial t}, \qquad \frac{\partial \hat{X}_1}{\partial t} = \frac{\partial X_1}{\partial t} + \mathbf{U}_1 \cdot \nabla X_0.$$

	Equation in $\hat{\Omega}$	Boundary conditions	Energy
General model (Sec. 2.3)	$\hat{\rho}_0 \dfrac{\partial^2 \hat{\mathbf{U}}_1}{\partial t^2} - \nabla_\xi \left(\hat{\rho}_0 \hat{c}_0^2 \nabla_\xi \cdot \hat{\mathbf{U}}_1 \right)$ $- \rho_0 g \left((\nabla_\xi (\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3) + \nabla_\xi \cdot \hat{\mathbf{U}}_1 \mathbf{e}_3 \right) = 0$	$\hat{\mathbf{U}}_1 \cdot \mathbf{n}_b = \hat{u}_{b,1}, \quad \hat{\Gamma}_b$ $\nabla_\xi \cdot \hat{\mathbf{U}}_1 = 0, \quad \hat{\Gamma}_s$	$\mathcal{E} = \dfrac{1}{2} \displaystyle\int_{\hat{\Omega}} \hat{\rho}_0 \left(\left| \dfrac{\partial \hat{\mathbf{U}}_1}{\partial t} \right|^2 + \left(\hat{c}_0 \nabla_\xi \cdot \hat{\mathbf{U}}_1 - \dfrac{g}{\hat{c}_0} \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \right)^2 \right)$ $+ \dfrac{1}{2} \displaystyle\int_{\hat{\Omega}} \hat{\rho} N^2 (\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3)^2 \, \mathrm{d}\boldsymbol{\xi} + \dfrac{1}{2} \displaystyle\int_{\hat{\Gamma}_s} \hat{\rho}_0 g (\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3)^2 \, \mathrm{d}\sigma$
Barotropic case (Sec. 2.3)	$\dfrac{\partial^2 \psi}{\partial t^2} - \hat{c}_0^2 \Delta_\xi \psi + g \dfrac{\partial \psi}{\partial \xi^3} = 0,$ $\hat{\mathbf{U}}_1 = \nabla_\xi \psi$	$\nabla_\xi \psi \cdot \hat{\mathbf{n}}_b = \hat{u}_{b,1}, \quad \hat{\Gamma}_b$ $\dfrac{\partial^2 \psi}{\partial t^2} + g \dfrac{\partial \psi}{\partial \xi^3} = 0, \quad \hat{\Gamma}_s$	$\mathcal{E}_{bar} = \dfrac{1}{2} \displaystyle\int_{\hat{\Omega}} \hat{\rho}_0 \left(\dfrac{1}{\hat{c}_0^2} \left(\dfrac{\partial \psi}{\partial t} \right)^2 + \lvert \nabla \psi \rvert^2 \right) \mathrm{d}\boldsymbol{\xi}$ $+ \dfrac{1}{2} \displaystyle\int_{\hat{\Gamma}_s} \dfrac{\hat{\rho}_0}{g} \left(\dfrac{\partial \psi}{\partial t} \right)^2 \mathrm{d}\sigma$
Incompressible limit (Sec. 3.2)	$\hat{\rho}_0 \dfrac{\partial^2 \hat{\mathbf{U}}_{1,0}}{\partial t^2} - g \hat{\rho}_0' (\hat{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3) \, \mathbf{e}_3 + g \nabla_\xi \hat{\varphi}_0 = 0,$ $\nabla_\xi \cdot \hat{\mathbf{U}}_{1,0} = 0$	$\hat{\mathbf{U}}_{1,0} \cdot \hat{\mathbf{n}}_b = \hat{u}_{b,1}, \quad \hat{\Gamma}_b$ $\nabla_\xi \cdot \hat{\mathbf{U}}_{1,0} = 0, \quad \hat{\Gamma}_s$ $\hat{\varphi}_0 = g \hat{\rho}_0 \hat{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3, \quad \hat{\Gamma}_s$	$\mathcal{E}_{incomp} = \dfrac{1}{2} \displaystyle\int_{\hat{\Omega}} \left(\hat{\rho}_0 \left| \dfrac{\partial \hat{\mathbf{U}}_{1,0}}{\partial t} \right|^2 - \hat{\rho}_0' \lvert \hat{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3 \rvert^2 \right) \mathrm{d}\boldsymbol{\xi}$ $+ \dfrac{1}{2} \displaystyle\int_{\hat{\Gamma}_s} \hat{\rho}_0 \lvert \hat{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3 \rvert^2 \, \mathrm{d}\sigma$
Incompressible limit, homogeneous density (Sec. 3.2)	$\Delta_\xi \hat{\varphi}_0 = 0,$ $\dfrac{\partial^2 \hat{\mathbf{U}}_{1,0}}{\partial t^2} = - \dfrac{g}{\hat{\rho}_0} \nabla_\xi \hat{\varphi}_0$	$\nabla_\xi \hat{\varphi}_0 \cdot \hat{\mathbf{n}}_b = - \hat{\rho}_0 g \, \hat{u}_{b,1}, \quad \hat{\Gamma}_b$ $\dfrac{\partial^2 \hat{\varphi}_0}{\partial t^2} + g \dfrac{\partial \hat{\varphi}_0}{\partial \xi^3} = 0, \quad \hat{\Gamma}_s$	$\mathcal{E}_{Poisson} = \dfrac{1}{2} \left(\displaystyle\int_{\hat{\Omega}} \lvert \nabla_\xi \hat{\varphi} \rvert^2 \, \mathrm{d}\boldsymbol{\xi} + \displaystyle\int_{\hat{\Gamma}_s} \dfrac{1}{g} (\dfrac{\partial \hat{\varphi}_0}{\partial t})^2 \, \mathrm{d}\sigma \right)$
Acoustic limit (Sec. 3.3)	$\dfrac{\partial^2 \hat{\psi}_0}{\partial t^2} - \hat{\rho}_0 \hat{c}_0^2 \nabla_\xi \cdot \left(\hat{\rho}_0^{-1} \nabla_\xi \hat{\psi}_0 \right) = 0,$ $\hat{\mathbf{U}}_1 = \nabla_\xi \hat{\psi}_0$	$\nabla_\xi \hat{\psi}_0 \cdot \hat{\mathbf{n}}_b = \hat{u}_{b,1}, \quad \hat{\Gamma}_b$ $\hat{\psi}_0 = 0, \quad \hat{\Gamma}_s$	$\mathcal{E}_{acoustic} = \dfrac{1}{2} \displaystyle\int_{\hat{\Omega}} \dfrac{1}{\hat{\rho}_0} \left(\dfrac{1}{\hat{c}_0^2} \dfrac{\partial \hat{\psi}_0}{\partial t} + \lvert \nabla \hat{\psi}_0 \rvert^2 \right) \mathrm{d}\boldsymbol{\xi}$

Table 1: Summary of the equations, boundary conditions and energy for the general model and its simplifications.