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Local-in-time existence of strong solutions to an averaged thick sprays model

V. Fournet*, C. Buet† and B. Després‡

Abstract

We propose a new system for the modelling of thick sprays. We prove that this system verifies conservation properties together with a maximum principle regarding the volume fraction of the gas. Our main result is that the barotropic version of this system is locally in time well-posed in H^s.

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1 Introduction

Sprays are defined as disperse liquid (such as droplet) or solid (such as dust) phase evolving in an underlying gas. Such models are usually presented as a coupling between a kinetic equation (for example, the Vlasov equation or the Vlasov-Boltzmann equation) and an hyperbolic (or Navier-Stokes) system and were introduced in [Wil85, CP83]. A classification of different types of spray have been proposed in [O'R81]. See [Des10] for a recent review of the different way to model sprays.

One type of sprays are the so-called *thin sprays*, in which the total volume occupied by the particles is negligible compared to the volume occupied by the gas. In this case, the coupling is usually made through a drag force, sometimes called the *Brinkman force*. Such models have been heavily studied in the purely hyperbolic setting [BD06, Mat10] and in the Navier-Stokes setting, that is, with parabolic terms [Cho16, EHKM21, HKMM20].

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In this work, we are interested in so-called *thick sprays* models, in which the total volume occupied by the particles is no longer negligible compared to the volume of the gas. The coupling is made not only through a drag force, but also through the volume fraction of the gas α. On the mathematical side, less is known about thick sprays models compared with think sprays models. See nevertheless recent works [BDM03, BDD21].

Hereafter the gas is described with a system of compressible Euler equations, for which the variables are the density $\varrho := \varrho(t, \boldsymbol{x}) \geq 0$, the velocity $\boldsymbol{u} := \boldsymbol{u}(t, \boldsymbol{x}) \in \mathbf{R}^3$ and the internal energy $e := (t, \boldsymbol{x}) \geq 0$. The particles are described with a phase space density $f := f(t, \boldsymbol{x}, \boldsymbol{v}) \geq 0$ following a Vlasov equation. It is also possible to use a Vlasov-Boltzmann equation, but numerical results may suggest that adding a collision operator is not necessary for accurate simulation [BDGN12]. Various other effects such as coalescence and fragmentation can also be taken into account, we chose here to neglect all these effects. The force acting on the particles, that shall be denoted $m_\star \Gamma$, is usually decomposed in two parts: one related to drag or friction between the particles and the gas. The other one is related, since the total volume of the particles is not negligible, to the pressure of the gas:

$$m_\star \Gamma = -m_\star \boldsymbol{\nabla} p - D_\star (\boldsymbol{v} - \boldsymbol{u}).$$

Depending of the modelisation, the coefficient $D_\star$ can be treated as a constant or as a function of $\boldsymbol{v} - \boldsymbol{u}$. We shall limit ourselves to the constant case. The constant $m_\star$ can be interpreted as the mass of the particles and is linked to their radius r by the formula (we assume here that the particles have a density equal to 1)

$$m_\star = \frac{4}{3} \pi r^3.$$

The spray is assumed to be monodisperse, meaning that all particles have the same radius. With those hypotheses, a possible thick sprays model [BDM03] can be written

$$\begin{cases} \partial_t(\alpha\varrho) + \boldsymbol{\nabla} \cdot (\alpha \varrho \boldsymbol{u}) = 0 \\ \partial_t(\alpha\varrho \boldsymbol{u}) + \boldsymbol{\nabla} \cdot (\alpha \varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{\nabla} p = m_\star \boldsymbol{\nabla} p \displaystyle\int_{\mathbf{R}^3} f \, \mathrm{d}v + D_\star \int_{\mathbf{R}^3} (\boldsymbol{v} - \boldsymbol{u}) f \, \mathrm{d}v \\ \partial_t(\alpha\varrho e) + \boldsymbol{\nabla} \cdot (\alpha \varrho e \boldsymbol{u}) + p(\partial_t \alpha + \boldsymbol{\nabla} \cdot (\alpha \boldsymbol{u})) = D_\star \displaystyle\int_{\mathbf{R}^3} |\boldsymbol{v} - \boldsymbol{u}|^2 f \, \mathrm{d}v \\ \partial_t f + \boldsymbol{v} \cdot \boldsymbol{\nabla}_x f + \boldsymbol{\nabla}_v \cdot (\Gamma f) = 0 \\ \alpha = 1 - m_\star \displaystyle\int_{\mathbf{R}^3} f \, \mathrm{d}v \\ m_\star \Gamma = -m_\star \boldsymbol{\nabla} p - D_\star (\boldsymbol{v} - \boldsymbol{u}) \end{cases} \tag{1.1}$$

This system suffers from some mathematical issues. In [Cho16], it is proved that smooth solutions blow up in finite time, which is to be expected because of the Euler part. Moreover, the term $\boldsymbol{\nabla}_x p \cdot \boldsymbol{\nabla}_v f$ in the Vlasov equation makes impossible to use standard theory for weak solution of such system (in particular, the Diperna-Lions theory of renormalized solutions does not apply here). To the best of our knowledge, even before the blow up phenomena, local-in-time well-posedness result is still lacking. It is even suspected in [BD06] that the system (1.1) is ill-posed locally in time. All those reasons motivate us to modify the model, with regularisation-convolution of certain terms specific to thick sprays models. We propose the following thick sprays model with regularisation

$$\begin{cases} \partial_t(\alpha\varrho) + \boldsymbol{\nabla} \cdot (\alpha \varrho \boldsymbol{u}) = 0 \\ \partial_t(\alpha\varrho \boldsymbol{u}) + \boldsymbol{\nabla} \cdot (\alpha \varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{\nabla} p = m_\star \boldsymbol{\nabla} p \displaystyle\int_{\mathbf{R}^3} \langle f \rangle \, \mathrm{d}v + D_\star \int_{\mathbf{R}^3} (\boldsymbol{v} - \boldsymbol{u}) f \, \mathrm{d}v \\ \partial_t(\alpha\varrho e) + \boldsymbol{\nabla} \cdot (\alpha \varrho e \boldsymbol{u}) + p(\partial_t \alpha + \boldsymbol{\nabla} \cdot (\alpha \boldsymbol{u})) = D_\star \displaystyle\int_{\mathbf{R}^3} |\boldsymbol{v} - \boldsymbol{u}|^2 f \, \mathrm{d}v \\ \partial_t f + \boldsymbol{v} \cdot \boldsymbol{\nabla}_x f + \boldsymbol{\nabla}_v \cdot (\Gamma f) = 0 \\ \alpha = 1 - m_\star \displaystyle\int_{\mathbf{R}^3} \langle f \rangle \, \mathrm{d}v \\ m_\star \Gamma = -m_\star \langle \boldsymbol{\nabla} p \rangle - D_\star (\boldsymbol{v} - \boldsymbol{u}) \end{cases} \tag{1.2}$$

where the convolution operator $\langle\cdot\rangle$ is introduced in the second, fifth and sixth line of the initial model (1.1). A possible "physical" form of the convolution operator is justified in section 2.

This work is organized as follows. In section 2, we justify the convolution operator and explain the modifications of the usual thick sprays equations leading to (1.2). In section 3, we show that the system verifies conservation properties for the total mass, the total momentum, and the total energy of the system, and is equipped with an entropy balance law. We also show that under reasonable assumptions, the volume fraction α stays in $(0, 1]$ for all times, provided that the initial condition verifies this property. Finally, in section 4, we prove the main result of this work on the local-in-time well-posedness in H^s of the barotrope version of (1.2).

2 Justification of the convolution operator

The idea of a convolution operator is easy to conceive on the equation that defines to force $m_\star\Gamma$, ignoring the friction force. Let us consider a single spherical particle of radius $r > 0$ with center coordinate $\boldsymbol{x}_\star$. In a surrounding gas with a pressure field p, the gas acts on the particle with the force

$$F = -\int_{\mathbf{S}^2(\boldsymbol{x}_\star,r)} p\boldsymbol{n}\,\mathrm{d}x.$$

In a thought experiment, the pressure can be extended inside the particle. Assuming that the pressure is differentiable in the particle, one obtains using Stokes's theorem

$$F = -\int_{\mathbf{S}^3(\boldsymbol{x}_\star,r)} \boldsymbol{\nabla} p(\boldsymbol{x})\,\mathrm{d}x = -m_\star \int_{\mathbf{R}^3} w(\boldsymbol{x}-\boldsymbol{x}_\star)\boldsymbol{\nabla} p(\boldsymbol{x})\,\mathrm{d}x = m_\star(w \star \boldsymbol{\nabla} p)(\boldsymbol{x}_\star)$$

where the convolution kernel is

$$w(\boldsymbol{y}) = \frac{1}{m_\star}\mathbf{1}_{|\boldsymbol{y}|<r}(\boldsymbol{y}).$$

Using the notation $\langle\cdot\rangle = w \star \cdot$ for the convolution operator, the force is rewritten as

$$m_\star\Gamma = -m_\star\langle\boldsymbol{\nabla} p\rangle.$$

This formula is valid independently of the extension of p inside the particle, provided it is differentiable.

Lemma 1. *The kernel w verifies $w \in \mathrm{BV}(\mathbf{R}^3)$, $\|w\|_{L^1} = 1$ and $\mathrm{TV}(w) = \frac{3}{r}$.*

Proof. We recall the definition [EG15] of the total variation of a function

$$\mathrm{TV}(f) := \sup\left(\int_{\mathbf{R}^3} f\,\mathrm{div}\,\varphi\,\mathrm{d}x, \quad \varphi \in \mathscr{C}^1_c(\mathbf{R}^3;\mathbf{R}^n), \|\varphi\|_{L^\infty} \leq 1\right),$$

Since w is the indicator function of a sphere, then $w \in \mathrm{BV}(\mathbf{R}^3)$. The kernel w has unit L^1-norm because $m_\star = \frac{4}{3}\pi r^3$. Finally the total variation of an indicator function is the perimeter of its support, therefore $\mathrm{TV}(w) = \frac{4\pi r^2}{m_\star} = \frac{3}{r}$. □

In order to introduce this principle in the original system (1.1), one needs to reintroduce the friction in the force $m_\star\Gamma$ and to modify other equations in a way that preserve the global conservation properties. Considering additionally that the modifications should

be kept to the minimum, we are led to propose the following system

$$\begin{cases} \partial_t(\alpha\varrho) + \boldsymbol{\nabla}\cdot(\alpha\varrho\boldsymbol{u}) = 0 & (2.1)\\ \partial_t(\alpha\varrho\boldsymbol{u}) + \boldsymbol{\nabla}\cdot(\alpha\varrho\boldsymbol{u}\otimes\boldsymbol{u}) + \boldsymbol{\nabla}p = m_\star\boldsymbol{\nabla}p\displaystyle\int_{\mathbf{R}^3}\langle f\rangle\,\mathrm{d}v + D_\star\int_{\mathbf{R}^3}(\boldsymbol{v}-\boldsymbol{u})f\,\mathrm{d}v & (2.2)\\ \partial_t(\alpha\varrho e) + \boldsymbol{\nabla}\cdot(\alpha\varrho e\boldsymbol{u}) + p(\partial_t\alpha + \boldsymbol{\nabla}\cdot(\alpha\boldsymbol{u})) = D_\star\displaystyle\int_{\mathbf{R}^3}|\boldsymbol{v}-\boldsymbol{u}|^2 f\,\mathrm{d}v & (2.3)\\ \partial_t f + \boldsymbol{v}\cdot\boldsymbol{\nabla}_x f + \boldsymbol{\nabla}_v\cdot(\Gamma f) = 0 & (2.4)\\ \alpha = 1 - m_\star\displaystyle\int_{\mathbf{R}^3}\langle f\rangle\,\mathrm{d}v & (2.5)\\ m_\star\Gamma = -m_\star\langle\boldsymbol{\nabla}p\rangle - D_\star(\boldsymbol{v}-\boldsymbol{u}) & (2.6) \end{cases}$$

Let us comment in more details the equations (2.1)-(2.6). Equations (2.1), (2.3) and (2.4) are unchanged. Equation (2.2) is modified with the convolution kernel on the right-hand-side since we found that it is a way to recover the conservation of the total momentum, as justified in Proposition 2. Equation (2.5) is also modified by compatibility with the conservation of the total energy. In the last equation (2.6), the friction is reintroduced without a convolution because there is no conservation issue related to this term. Notice that when the size of the support of w goes to 0 (ignoring the fact that $m_\star$ is dependent of the size of the particle r), meaning that w converges to a Dirac mass, one recovers the system (1.1). To close the system, we assume that p follows a perfect gas law

$$p = (\gamma-1)\varrho e, \quad \gamma > 1. \tag{2.7}$$

and the energy law

$$e = C_v T, \tag{2.8}$$

with $T > 0$ the temperature and $C_v > 0$ is a constant.

3 Properties of the system

In this section, we present basic properties of the new model (1.2).

3.1 Conservation properties

To obtain the conservation properties, we will make use of the classical formula: let f, g two functions and w even, then

$$\int_{\mathbf{R}^d}(f\star w)g\,\mathrm{d}x = \int_{\mathbf{R}^d} f(g\star w)\,\mathrm{d}x \tag{3.1}$$

which is easily shown using a change of variable and the fact that w is even.

Proposition 2. *Formally, one has the following*

(i)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^3}\alpha\varrho\,\mathrm{d}x = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbf{R}^3\times\mathbf{R}^3} f\,\mathrm{d}x\mathrm{d}v = 0,$$

(ii)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbf{R}^3}\alpha\varrho\boldsymbol{u}\,\mathrm{d}x + \iint_{\mathbf{R}^3\times\mathbf{R}^3} f\boldsymbol{v}\,\mathrm{d}x\mathrm{d}v\right) = 0,$$

(iii)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbf{R}^3}\alpha\varrho|\boldsymbol{u}|^2\,\mathrm{d}x + \frac{1}{2}\iint_{\mathbf{R}^3\times\mathbf{R}^3}|\boldsymbol{v}|^2 f\,\mathrm{d}x\mathrm{d}v + \int_{\mathbf{R}^3}\alpha\varrho e\,\mathrm{d}x\right) = 0.$$

Proof. By formally we mean that all functions are smooth and integrable. The proof of (i) is obvious because the equations are already in divergence form.

To obtain (ii) one multiplies the Vlasov equation (2.4) by $\boldsymbol{v}$. Integration yields

$$m_\star \partial_t \int_{\mathbf{R}^3} f\boldsymbol{v}\,\mathrm{d}v + m_\star \boldsymbol{\nabla}_x \cdot \int_{\mathbf{R}^3} f\boldsymbol{v}\otimes\boldsymbol{v}\,\mathrm{d}v = -m_\star\langle\boldsymbol{\nabla}p\rangle\int_{\mathbf{R}^3} f\,\mathrm{d}v - D_\star\int_{\mathbf{R}^3}(\boldsymbol{v}-\boldsymbol{u})f\,\mathrm{d}v.$$

Summing with the fluid momentum equation (2.2), one gets

$$\begin{aligned}&\partial_t\left(\alpha\varrho\boldsymbol{u} + m_\star\int_{\mathbf{R}^3} f\boldsymbol{v}\,\mathrm{d}v\right) + \boldsymbol{\nabla}\cdot\left(\alpha\varrho\boldsymbol{u}\otimes\boldsymbol{u} + m_\star\int_{\mathbf{R}^3} f\boldsymbol{v}\otimes\boldsymbol{v}\,\mathrm{d}v\right) + \boldsymbol{\nabla}p\\ &\quad = m_\star\left(\boldsymbol{\nabla}p\int_{\mathbf{R}^3}\langle f\rangle\,\mathrm{d}v - \langle\boldsymbol{\nabla}p\rangle\int_{\mathbf{R}^3} f\,\mathrm{d}v\right).\end{aligned} \tag{3.2}$$

Using formula (3.1), one has $\int_{\mathbf{R}^3}\left(\boldsymbol{\nabla}p\int_{\mathbf{R}^3}\langle f\rangle\,\mathrm{d}v - \langle\boldsymbol{\nabla}p\rangle\int_{\mathbf{R}^3} f\,\mathrm{d}v\right)\mathrm{d}x = 0$, so integrating (3.2) in x yields (ii).

To obtain (iii) one multiplies the momentum equation (2.2) by $\boldsymbol{u}$. One obtains

$$\partial_t\left(\alpha\varrho\frac{|\boldsymbol{u}|^2}{2}\right) + \boldsymbol{\nabla}\cdot\left(\alpha\varrho\boldsymbol{u}\frac{|\boldsymbol{u}|^2}{2}\right) + \boldsymbol{u}\cdot\boldsymbol{\nabla}p = m_\star\boldsymbol{u}\cdot\boldsymbol{\nabla}p\cdot\int_{\mathbf{R}^3}\langle f\rangle\,\mathrm{d}v + D_\star\int_{\mathbf{R}^3}\boldsymbol{u}\cdot(\boldsymbol{v}-\boldsymbol{u})f\,\mathrm{d}v.$$

Multiplying the Vlasov equation (2.4) by $m_\star\frac{|\boldsymbol{v}|}{2}$ and integrating, one gets

$$\begin{aligned}&m_\star\partial_t\left(\int_{\mathbf{R}^3} f\frac{|\boldsymbol{v}|^2}{2}\,\mathrm{d}v\right) + m_\star\boldsymbol{\nabla}_x\cdot\left(\int_{\mathbf{R}^3} f\boldsymbol{v}\frac{|\boldsymbol{v}|^2}{2}\,\mathrm{d}v\right) =\\ &\quad -m_\star\langle\boldsymbol{\nabla}p\rangle\cdot\int_{\mathbf{R}^3} f\boldsymbol{v}\,\mathrm{d}v - D_\star\int_{\mathbf{R}^3}\boldsymbol{v}\cdot(\boldsymbol{v}-\boldsymbol{u})f\,\mathrm{d}v.\end{aligned}$$

Then, summation of these equations with the internal energy equation (2.3) yields

$$\begin{aligned}&\partial_t\left(\alpha\varrho\frac{|\boldsymbol{u}|^2}{2} + m_\star\int_{\mathbf{R}^3} f\frac{|\boldsymbol{v}|^2}{2}\,\mathrm{d}v + \alpha\varrho e\right) + \boldsymbol{\nabla}\cdot\left(\alpha\varrho\boldsymbol{u}\frac{|\boldsymbol{u}|^2}{2} + m_\star\int_{\mathbf{R}^3} f\boldsymbol{v}\frac{|\boldsymbol{v}|^2}{2}\,\mathrm{d}v + \alpha\varrho e\boldsymbol{u}\right)\\
&= -p\partial_t\alpha - p\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{u}) - \boldsymbol{u}\cdot\boldsymbol{\nabla}p - m_\star\boldsymbol{u}\cdot\boldsymbol{\nabla}p\int_{\mathbf{R}^3}\langle f\rangle\,\mathrm{d}v - m_\star\langle\boldsymbol{\nabla}p\rangle\cdot\int_{\mathbf{R}^3} f\boldsymbol{v}\,\mathrm{d}v\\
&\quad + D_\star\int_{\mathbf{R}^3}\boldsymbol{u}\cdot(\boldsymbol{v}-\boldsymbol{u})f\,\mathrm{d}v - D_\star\int_{\mathbf{R}^3}\boldsymbol{v}\cdot(\boldsymbol{v}-\boldsymbol{u})f\,\mathrm{d}v + D_\star\int_{\mathbf{R}^3}|\boldsymbol{v}-\boldsymbol{u}|^2 f\,\mathrm{d}v\\
&= -p\partial_t\alpha - p\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{u}) - \boldsymbol{u}\cdot\boldsymbol{\nabla}p - m_\star\boldsymbol{u}\cdot\boldsymbol{\nabla}p\int_{\mathbf{R}^3}\langle f\rangle\,\mathrm{d}v - m_\star\langle\boldsymbol{\nabla}p\rangle\cdot\int_{\mathbf{R}^3} f\boldsymbol{v}\,\mathrm{d}v\\
&= -p\partial_t\alpha - p\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{u}) - \boldsymbol{u}\cdot\boldsymbol{\nabla}p + (1-\alpha)\boldsymbol{u}\cdot\boldsymbol{\nabla}p - m_\star\langle\boldsymbol{\nabla}p\rangle\cdot\int_{\mathbf{R}^3} f\boldsymbol{v}\,\mathrm{d}v\\
&= -p\partial_t\alpha - p\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{u}) - \alpha\boldsymbol{u}\cdot\boldsymbol{\nabla}p - m_\star\langle\boldsymbol{\nabla}p\rangle\cdot\int_{\mathbf{R}^3} f\boldsymbol{v}\,\mathrm{d}v\\
&= -\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{u}p) - p\partial_t\alpha - m_\star\langle\boldsymbol{\nabla}p\rangle\cdot\int_{\mathbf{R}^3} f\boldsymbol{v}\,\mathrm{d}v\\
&= -\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{u}p) - m_\star p\boldsymbol{\nabla}\cdot\int_{\mathbf{R}^3}\langle f\rangle\boldsymbol{v}\,\mathrm{d}v - m_\star\langle\boldsymbol{\nabla}p\rangle\cdot\int_{\mathbf{R}^3} f\boldsymbol{v}\,\mathrm{d}v\\
&= -\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{u}p) - m_\star\boldsymbol{\nabla}p\cdot\int_{\mathbf{R}^3}\langle f\rangle\boldsymbol{v}\,\mathrm{d}v - m_\star p\boldsymbol{\nabla}\cdot\int_{\mathbf{R}^3}\langle f\rangle\boldsymbol{v}\,\mathrm{d}v\\
&\quad + m_\star\left(\boldsymbol{\nabla}p\cdot\int_{\mathbf{R}^3}\langle f\rangle\boldsymbol{v}\,\mathrm{d}v - \langle\boldsymbol{\nabla}p\rangle\cdot\int_{\mathbf{R}^3} f\boldsymbol{v}\,\mathrm{d}v\right)\\
&= -\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{u}p) - m_\star\boldsymbol{\nabla}\cdot\left(p\int_{\mathbf{R}^3}\langle f\rangle\boldsymbol{v}\,\mathrm{d}v\right) + m_\star\left(\boldsymbol{\nabla}p\int_{\mathbf{R}^3}\langle f\rangle\boldsymbol{v}\,\mathrm{d}v - \langle\boldsymbol{\nabla}p\rangle\int_{\mathbf{R}^3} f\boldsymbol{v}\,\mathrm{d}v\right).\end{aligned}$$

Using again formula (3.1), we have $\int_{\mathbf{R}^3}\left(\boldsymbol{\nabla}p\int_{\mathbf{R}^3}\langle f\rangle\boldsymbol{v}\,\mathrm{d}v - \langle\boldsymbol{\nabla}p\rangle\int_{\mathbf{R}^3} f\boldsymbol{v}\,\mathrm{d}v\right)\mathrm{d}x = 0$, so integrating in x yields (iii). □

3.2 Entropy property

Following the analysis made in [BDD21], we show that the system (2.1)-(2.6) closed by the equation of state (2.7) is equipped with an entropy inequality.

Proposition 3. *Formally, one has the entropy inequality*

$$\partial_t(\alpha\varrho S) + \boldsymbol{\nabla}\cdot(\alpha\varrho S\boldsymbol{u}) = \frac{D_\star}{T}\int_{\mathbf{R}^3}|\boldsymbol{v}-\boldsymbol{u}|^2 f\,\mathrm{d}v \geq 0. \tag{3.3}$$

Proof. The density equation (2.1) yields

$$\varrho(\partial_t\alpha + \boldsymbol{\nabla}\cdot(\alpha\boldsymbol{u})) + \alpha \mathrm{D}_t\varrho = 0 \iff \partial_t\alpha + \boldsymbol{\nabla}\cdot(\alpha\boldsymbol{u}) = \alpha\varrho\mathrm{D}_t\tau,$$

with $\tau = 1/\varrho > 0$ the specific volume, and $\mathrm{D}_t = \partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$. The internal energy equation (2.3) can be rewritten as

$$\alpha\varrho(\mathrm{D}_t e + p\mathrm{D}_t\tau) = D_\star\int_{\mathbf{R}^3}|\boldsymbol{v}-\boldsymbol{u}|^2 f\,\mathrm{d}v.$$

With the perfect gas pressure law (2.7) and the energy law (2.8), the entropy S is

$$S = C_v\ln(e\varrho^{1-\gamma}).$$

The second principle of thermodynamics writes

$$T\mathrm{d}S = \mathrm{d}e + p\mathrm{d}\tau,$$

so that

$$\alpha\varrho\mathrm{D}_t S = \frac{D_\star}{T}\int_{\mathbf{R}^3}|\boldsymbol{v}-\boldsymbol{u}|^2 f\,\mathrm{d}v,$$

which is rewritten as (3.3). □

3.3 Positivity of the volume fraction of the gas for smooth solutions

In this section, we study the positivity of the volume fraction α, which is that if $0 < \alpha \leq 1$ at time $t = 0$, then the inequality stays true for all times $t > 0$. Although is to clear that $\alpha \leq 1$ (thanks to the fact that $f \geq 0$ is propagated by the characteristic curves), it is less immediate to prove that $\alpha > 0$. In [BDD21], the authors proved that this property holds for the thick sprays equations (1.1) under reasonable assumptions. We prove here the same result for (1.2), under similar assumptions.

Proposition 4. *Assume that a solution of the system (1.2) is defined on the whole space* $\mathbf{R}^3$ *and is smooth on* $[0, T_{end}[$ *for some* $T_{end} > 0$. *We assume*

- *For all* $(\boldsymbol{x},\boldsymbol{v}) \in \mathbf{R}^3\times\mathbf{R}^3$, $\quad f(0,\boldsymbol{x},\boldsymbol{v}) > 0$.
- *One has* $0 < \varrho_- = \inf_{\boldsymbol{x}\in\mathbf{R}^3}\varrho(0,\boldsymbol{x}) \leq \varrho_+ = \sup_{\boldsymbol{x}\in\mathbf{R}^3}\varrho(0,\boldsymbol{x}) < +\infty$.
- *One has* $-\infty < S_- = \inf_{\boldsymbol{x}\in\mathbf{R}^3} S(0,\boldsymbol{x})$.
- *One has* $0 < \alpha_- = \inf_{\boldsymbol{x}\in\mathbf{R}^3}\alpha(0,\boldsymbol{x}) \leq 1$.

We assume the following regularity of the velocity variables

$$\boldsymbol{u} \in W^{1,\infty}([0,T_{\mathrm{end}}[\times\mathbf{R}^3), \quad \frac{\int_{\mathbf{R}^3} f\boldsymbol{v}\,\mathrm{d}v}{\int_{\mathbf{R}^3}\langle f\rangle\,\mathrm{d}v} \in L^\infty([0,T_{\mathrm{end}}[\times\mathbf{R}^3).$$

We finally assume that the pressure vanishes at infinity, that is for all $\varepsilon > 0$, *there exists* $A > 0$ *such that*

$$0 < p(t,\boldsymbol{x}) = (\gamma-1)\varrho(t,\boldsymbol{x})e(t,\boldsymbol{x}) < \varepsilon, \quad \text{for } 0 \leq t < T_{end} \text{ and } |\boldsymbol{x}| > A.$$

Then there exists a constant $C > 0$ depending on $T_{\rm end}$, ϱ_+, ϱ_-, α_-, S_-, A (corresponding to $\varepsilon = 1$), $\|\boldsymbol{u}\|_{W^{1,\infty}([0,T_{\rm end}[\times\mathbf{R}^3)}$ and $\left\|\frac{\int f\boldsymbol{v}\,\mathrm{d}v}{\int\langle f\rangle\,\mathrm{d}v}\right\|_{L^\infty([0,T_{\rm end}[\times\mathbf{R}^3)}$ so that the following estimate holds :

$$0 < C \leq \alpha(t,\boldsymbol{x}) \leq 1, \quad t \in [0, T_{end}[, \quad \boldsymbol{x} \in \mathbf{R}^3.$$

Proof. The momentum equation (2.2) can be rewritten as

$$\alpha\boldsymbol{\nabla}p = -\alpha\varrho \mathrm{D}_t\boldsymbol{u} + D_\star\frac{1-\alpha}{m_\star}\left(\frac{\int_{\mathbf{R}^3} f\boldsymbol{v}\,\mathrm{d}v}{\int_{\mathbf{R}^3}\langle f\rangle\,\mathrm{d}v} - \boldsymbol{u}\frac{\int_{\mathbf{R}^3} f\,\mathrm{d}v}{\int_{\mathbf{R}^3}\langle f\rangle\,\mathrm{d}v}\right)$$

with $\mathrm{D}_t = \partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$. Using the regularity $\boldsymbol{u} \in W^{1,\infty}$ and the assumptions above, we find

$$\|\alpha\boldsymbol{\nabla}p\|_{L^\infty} \leq C\left(1 + \|1-\alpha\|_{L^\infty}\right).$$

Following the fact that p follows a perfect gas law, we have the identity

$$\alpha\boldsymbol{\nabla}p = \left[(\gamma-1)^{1/\gamma}e^{S/C_v\gamma}\alpha\varrho\right]\frac{1}{p^{1/\gamma}}\boldsymbol{\nabla}p = \left[\frac{(\gamma-1)^{1/\gamma}e^{S/(C_v\gamma)}}{1-1/\gamma}\alpha\varrho\right]\boldsymbol{\nabla}(p^{1-1/\gamma}).$$

Now, using again $\boldsymbol{u} \in W^{1,\infty}$ and a classical treatment of the characteristic curves of the continuity equation, we obtain for some $C_-, C_+ > 0$,

$$C_- = \inf_{\boldsymbol{x}\in\mathbf{R}^3}(\alpha\varrho)(t,\boldsymbol{x}) \leq \sup_{\boldsymbol{x}\in\mathbf{R}^3}(\alpha\varrho)(t,\boldsymbol{x}) = C_+, \quad 0 \leq t < T_{\rm end}. \tag{3.4}$$

Therefore, because S is lower bounded.

$$\|\boldsymbol{\nabla}(p^{1-1/\gamma})\|_{L^\infty} \leq C\left(1 + \|1-\alpha\|_{L^\infty}\right).$$

We then use the fact that the pressure vanishes at infinity

$$\|p^{1-1/\gamma}\|_{L^\infty} \leq C\left(1 + \|1-\alpha\|_{L^\infty}\right).$$

We then obtain that the density ϱ is bounded, using the perfect gas law for the pressure and the boundedness of the entropy S,

$$\|\varrho^{\gamma-1}\|_{L^\infty} \leq C\left(1 + \|1-\alpha\|_{L^\infty}\right).$$

Finally, using (3.4)

$$\left\|\frac{1}{\alpha}\right\|_{L^\infty} \leq C\left(1 + \|1-\alpha\|_{L^\infty}^{1/(\gamma-1)}\right) \leq C.$$

□

4 Local in time well-posedness

In this section, we consider the barotropic version of system (1.2), meaning that we suppose that the pressure depends only on the density of the fluid ϱ, that is

$$p = \varrho^\gamma, \quad \gamma > 1,$$

so that the energy equation (2.3) is not needed. We also assume, without loss of generality, that $D_\star = 0$ and $m_\star = 1$. The treatment of the friction term presents no difficulty because it does not contain any derivatives, so that the treatment of this term is very similar to what is done in [BD06]. The barotropic system writes

$$\begin{cases} \partial_t(\alpha\varrho) + \boldsymbol{\nabla}\cdot(\alpha\varrho\boldsymbol{u}) = 0 \\ \partial_t(\alpha\varrho\boldsymbol{u}) + \boldsymbol{\nabla}\cdot(\alpha\varrho\boldsymbol{u}\otimes\boldsymbol{u}) + \boldsymbol{\nabla}p = \boldsymbol{\nabla}p\displaystyle\int_{\mathbf{R}^3}\langle f\rangle\,\mathrm{d}v \\ \partial_t f + \boldsymbol{v}\cdot\boldsymbol{\nabla}_x f - \langle\boldsymbol{\nabla}_x p\rangle\cdot\boldsymbol{\nabla}_v f = 0 \\ \alpha = 1 - \displaystyle\int_{\mathbf{R}^3}\langle f\rangle\,\mathrm{d}v \\ p = p(\varrho) = \varrho^\gamma. \end{cases} \tag{4.1}$$

The goal of this section is to prove the following theorem:

Theorem 5. *Let $\Omega =]0,+\infty[\times\mathbf{R}^3$, $s \in \mathbf{N}$ such that $s > 3/2+1$ and Ω_1, Ω_2 two open sets of Ω such that $\overline{\Omega}_1 \subset \Omega_2$ and Ω_1 and Ω_2 are relatively compact in Ω. Let $(\varrho_0, \varrho_0 \boldsymbol{u}_0) : \mathbf{R}^3 \to \Omega_1$ satisfying $\varrho_0 - 1 \in H^s(\mathbf{R}^3)$ and $\boldsymbol{u}_0 \in H^s(\mathbf{R}^3)$. Let $f_0 : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}_+$ be a function in $\mathscr{C}^1_c(\mathbf{R}^3 \times \mathbf{R}^3) \cap H^s(\mathbf{R}^3 \times \mathbf{R}^3)$ satisfying*

$$\|f_0\|_{L^\infty} < \frac{1}{2^4 \|w\|_{L^1} V_M(0)^3}.$$

Then, one can find $T \in]0,1[$ such that there exists a solution $(\varrho, \varrho\boldsymbol{u}, f)$ of the system (4.1) belonging to $\mathscr{C}^1([0,T]\times\mathbf{R}^3, \Omega_2) \times \mathscr{C}^1_c([0,T]\times\mathbf{R}^3\times\mathbf{R}^3, \mathbf{R}_+)$. Moreover this solution is unique.

Remark. This result could be extended to the whole system (1.2) with an energy equation. It could also be extended if p a sufficiently well-behaved function [BD06] not necessarily a power function of ϱ. It is also true if w is a non-negative convolution kernel in $\mathrm{BV}(\mathbf{R}^3)$.

The proof is based on the following idea. We want to combine classical theory of local-in-time solution for symmetrisable hyperbolic system of conservation laws (see [Maj84]) and the theory of characteristics for the control of H^s norms of f and its support (like for the Vlasov-Poisson system). This idea has already been used in [BD06, Mat10]. Finding appropriate symmetriser for the system (4.1) is not obvious. Our strategy is to expand the derivatives on the left-hand-side of the equations, then to treat the term containing derivatives of α as source terms. The left-hand-side then becomes the well-known Euler equations of gas dynamics for which a symmetriser is classical. To apply the proof [Maj84], the source term needs to be of degree 0, that is, no derivative. An issue is that the initial thick sprays equations (1.1) have derivatives in the source term. This is an asset of the convolution operator in systems (1.2) and (4.1), because it avoids the loss of regularity in the source term in the original thick sprays equations (1.1).

Proposition 6. *The system (4.1) is equivalent, for strong solution, to*

$$\begin{cases} \partial_t \mathbf{U} + \boldsymbol{\nabla} \cdot \mathbf{F}(\mathbf{U}) = b(\mathbf{U}, f) \\ \partial_t f + \boldsymbol{v} \cdot \boldsymbol{\nabla}_x f - \langle \boldsymbol{\nabla}_x p \rangle \cdot \boldsymbol{\nabla}_v f = 0 \end{cases} \tag{4.2}$$

with

$$\mathbf{U} = \begin{pmatrix} \varrho \\ \varrho \boldsymbol{u} \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} \varrho \boldsymbol{u} \\ \varrho \boldsymbol{u} \otimes \boldsymbol{u} + p\mathrm{Id} \end{pmatrix}$$

$$\text{and } b(\mathbf{U}, f) = \begin{pmatrix} \dfrac{\varrho \boldsymbol{u} \cdot \boldsymbol{\nabla}_x \int_{\mathbf{R}^3} \langle f \rangle \, \mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f \rangle \, \mathrm{d}v} - \dfrac{\varrho \boldsymbol{\nabla}_x \cdot \int_{\mathbf{R}^3} \langle f \rangle \boldsymbol{v} \, \mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f \rangle \, \mathrm{d}v} \\ \dfrac{\varrho \boldsymbol{u} \otimes \boldsymbol{u} \int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f \rangle \, \mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f \rangle \, \mathrm{d}v} - \dfrac{\varrho \boldsymbol{u} \boldsymbol{\nabla}_x \cdot \int_{\mathbf{R}^3} \langle f \rangle \boldsymbol{v} \, \mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f \rangle \, \mathrm{d}v} \end{pmatrix}.$$

Furthermore, the hyperbolic part of the system can be symmetrized as

$$S(\mathbf{U}) \partial_t \mathbf{U} + \sum_{i=1}^{3} (SA_i)(\mathbf{U}) \partial_{x_i} \mathbf{U} = S(\mathbf{U}) b(\mathbf{U}, f),$$

with

$$S(\mathbf{U}) = \begin{pmatrix} p'(\varrho) + |\boldsymbol{u}|^2 & -\boldsymbol{u}^T \\ -\boldsymbol{u} & \mathrm{Id} \end{pmatrix},$$

and $A_i = \partial_{\mathbf{U}} \mathbf{F}_i$ for $1 \le i \le 3$. Moreover, the symmetric positive definite matrix $S(\mathbf{U})$ is a smooth function of $\mathbf{U}$. For all relatively compact set Ω_1, there exists a constant $c > 0$ such that $\forall \mathbf{U} \in \Omega_1$

$$c\mathrm{Id} \le S(\mathbf{U}) \le c^{-1}\mathrm{Id}.$$

Finally the matrices $SA_i(\mathbf{U})$ are symmetric.

Proof. The derivation of (4.2) is a just matter of simple computation. For the symmetrisation of the system, we refer to [BD06]. □

We now introduce some notations. Let ϱ be a smooth enough function, then the Vlasov equation in (4.1) is a linear transport equation

$$\partial_t f + \boldsymbol{v} \cdot \boldsymbol{\nabla}_x f - \langle \boldsymbol{\nabla} \varrho^\gamma \rangle \cdot \boldsymbol{\nabla}_v f = 0.$$

It has a unique solution, computed by the method of characteristics

$$f(t, \boldsymbol{x}, \boldsymbol{v}) = f_0(\mathbf{X}(0; \boldsymbol{x}, \boldsymbol{v}, t), \mathbf{V}(0; \boldsymbol{x}, \boldsymbol{v}, t)),$$

where the characteristic curves are defined by

$$\begin{aligned} \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}(t; \boldsymbol{x}, \boldsymbol{v}, s) &= \mathbf{V}(t; \boldsymbol{x}, \boldsymbol{v}, s) \\ \mathbf{X}(s; \boldsymbol{x}, \boldsymbol{v}, s) &= \boldsymbol{x} \\ \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}(t; \boldsymbol{x}, \boldsymbol{v}, s) &= -\langle \boldsymbol{\nabla} \varrho^\gamma(\mathbf{X}(t; \boldsymbol{x}, \boldsymbol{v}, s), t) \rangle \\ \mathbf{V}(s; \boldsymbol{x}, \boldsymbol{v}, s) &= \boldsymbol{v}. \end{aligned}$$

If f_0 has a compact support, then $f(t, \cdot, \cdot)$ also has a compact support for all t. We denote

$$X_M(t) = \sup_{(\boldsymbol{x}, \boldsymbol{v}) \in \mathbf{R}^3 \times \mathbf{R}^3, f(t, \boldsymbol{x}, \boldsymbol{v}) > 0} |\boldsymbol{x}|,$$

and

$$V_M(t) = \sup_{(\boldsymbol{x}, \boldsymbol{v}) \in \mathbf{R}^3 \times \mathbf{R}^3, f(t, \boldsymbol{x}, \boldsymbol{v}) > 0} |\boldsymbol{v}|.$$

Then the following holds: $\operatorname{supp} f(t, \cdot, \cdot) \subset B(0, X_M(t)) \times B(0, V_M(t))$.

In all the sequel, we will use the following notations ($s \in \mathbf{N}$, $T > 0$, and α is a multi-index):

$$\begin{aligned} \|h\|_{H^s} &= \sum_{|\alpha| \leq s} \|\partial^\alpha h\|_{L^2} \\ \|h\|_{H^s, T} &= \sup_{t \in [0,T]} \|h\|_{H^s}(t) \\ \|h\|_{L^2, T} &= \sup_{t \in [0,T]} \|h\|_{L^2}(t). \end{aligned}$$

In particular the notation ∂^α will always denote a derivative in the x variable.

4.1 Iterative approximation scheme

We explain here the main steps of the proof of Theorem 5. We fix $s \in \mathbf{N}$ an integer such that $s > 3/2 + 1$. To construct a solution of the system (4.1), it is equivalent to construct a solution to the symmetrised quasi-linear system

$$\begin{cases} S(\mathbf{U}) \partial_t \mathbf{U} + \displaystyle\sum_{i=1}^{3} (SA_i)(\mathbf{U}) \partial_{x_i} \mathbf{U} & = S(\mathbf{U}) b(\mathbf{U}, f), \\ \partial_t f + \boldsymbol{v} \cdot \boldsymbol{\nabla}_x f - \langle \boldsymbol{\nabla}_x p \rangle \cdot \boldsymbol{\nabla}_v f & = 0, \end{cases} \tag{4.3}$$

The proof proceeds via a classical iteration scheme. We first work with smooth and compactly supported initial data

$$\varrho_0 - 1 \in \mathcal{D}(\mathbf{R}^3), \quad \boldsymbol{u}_0 \in \mathcal{D}(\mathbf{R}^3), \quad f_0 \in \mathcal{D}(\mathbf{R}^3 \times \mathbf{R}^3). \tag{4.4}$$

Later, we will use a mollification process to prove the case of all initial data in H^s. We ask that f_0 is small, more precisely, we assume

$$\|f_0\|_{L^\infty} < \frac{1}{2^4 \|w\|_{L^1} V_M(0)^3}. \tag{4.5}$$

We note $\Omega =]0,+\infty[\times\mathbf{R}^3$ and Ω_1 a relatively compact open set of Ω such that $\mathbf{U}_0 := \begin{pmatrix} \varrho_0 \\ \varrho_0 \boldsymbol{u}_0 \end{pmatrix} \in \Omega_1$.

We will construct the solution of (4.3) through the following iteration process: for $k=0$, one sets $\theta_0 = +\infty$, $(\mathbf{U}^0(t), f^0(t)) = (\mathbf{U}_0, f_0)$. Then, given the quantities θ_k, $\mathbf{U}^k$ and f^k, one defines $(\mathbf{U}^{k+1}, f^{k+1})$ as the solution of the linear system

$$S(\mathbf{U}^k)\partial_t \mathbf{U}^{k+1} + \sum_{i=1}^{3} (SA_i)(\mathbf{U}^k)\partial_{x_i}\mathbf{U}^{k+1} = S(\mathbf{U}^k) b(\mathbf{U}^k, f^k), \tag{4.6}$$

$$\mathbf{U}^{k+1}(\boldsymbol{x}, 0) = \mathbf{U}_0(\boldsymbol{x}), \tag{4.7}$$

$$\partial_t f^{k+1} + \boldsymbol{v}\cdot\boldsymbol{\nabla}_x f^{k+1} - \langle \boldsymbol{\nabla} p^k\rangle \cdot \boldsymbol{\nabla}_v f^{k+1} = 0, \tag{4.8}$$

$$f^{k+1}(\boldsymbol{x}, \boldsymbol{v}, 0) = f_0(\boldsymbol{x}, \boldsymbol{v}). \tag{4.9}$$

For now, it is not obvious that the sequence $(\mathbf{U}^k, f^k)_{k\in\mathbf{N}}$ is well-defined.

Let Ω_2 a relatively compact open subset of $\mathbf{R}^3$ such that $\overline{\Omega_1} \subset \Omega_2$. From the Sobolev embedding for $s > 3/2+1$, $H^s(\mathbf{R}^3) \hookrightarrow L^\infty(\mathbf{R}^3)$, there exists a constant $R > 0$ depending on Ω_1, Ω_2 and s, such that, if $\|\mathbf{U} - \mathbf{U}_0\|_{H^s} \leq R$, then $\mathbf{U} \in \Omega_2$. Then, one defines θ_{k+1} as the supremum of times $\theta < \theta_k$ such that $\|\mathbf{U}^{k+1} - \mathbf{U}_0\|_{H^s,\theta} \leq R$.

The system (4.6) is linear in $\mathbf{U}^{k+1}$, symmetric and has smooth coefficients on $[0,\theta_k[$, therefore it admits a smooth solution on $[0,\theta_k[$. For equation (4.8), since it is a linear transport equation with smooth coefficients on $[0,\theta_k[$, it admits a smooth solution that can be explicitly computed by the method of characteristics. Finally, $\theta_{k+1} > 0$ since $\mathbf{U}^{k+1}$ is smooth, and $\mathbf{U}_0 \in \Omega_1$. The sequence $(\mathbf{U}^k, f^k)_{k\in\mathbf{N}}$ is then well-defined.

We will restrict further the lifetime of the solution. One defines T_k as the supremum of times $T \in [0,\theta_k[$ such that

$$\|\mathbf{U}^k - \mathbf{U}_0\|_{H^s,T} \leq R \tag{4.10}$$

$$\forall t \in [0,T], \quad X_M^k(t) \leq 2X_M(0), \tag{4.11}$$

$$\forall t \in [0,T], \quad V_M^k(t) \leq 2V_M(0) \tag{4.12}$$

$$\left\| \int_{\mathbf{R}^3} \langle f^k\rangle \,\mathrm{d}v \right\|_{L^\infty,T} \leq 2^4 \|w\|_{L^1} V_M(0)^3 \|f_0\|_{L^\infty}. \tag{4.13}$$

and such that $T_{k+1} \leq T_k$. In particular,

$$\left\| \frac{1}{1 - \int_{\mathbf{R}^3}\langle f^k\rangle \,\mathrm{d}v} \right\|_{L^\infty} \leq \left\| \frac{1}{1 - 2^4\|w\|_{L^1} V_M(0)^3 f_0} \right\|_{L^\infty}. \tag{4.14}$$

and one consider the sequence $(\mathbf{U}^k, f^k)_{k\in\mathbf{N}}$ on the time interval $[0,T_k[$. We emphasize that the number 2 in the inequalities (4.11)-(4.13) is only of a cosmetic nature. One could replace the number 2 by any $a > 1$ and it would not change the proof.

The proof is then made of three parts:

- First, we prove that there exists a time $T_\star > 0$ such that $T_k \geq T_\star$ for all $k \in \mathbf{N}$. This is the subject of the Proposition 14, for which we will need Lemmas 11, 12 and 13. Note that without such result, the sequence of lifespan $(T_k)_{k\in\mathbf{N}}$ could converges to 0.
- Next, we prove in Proposition 15 that the sequence $(\mathbf{U}^k, f^k)_{k\in\mathbf{N}}$ converges in $L^\infty(0,T_{\star\star}; L^2(\mathbf{R}^3)) \times L^\infty(0,T_{\star\star}; L^2(\mathbf{R}^3\times\mathbf{R}^3))$, where $0 < T_{\star\star} \leq T_\star$.
- Finally, we conclude the proof of Theorem 5 and we prove that the limit $(\mathbf{U}, f)$ is a solution of (4.3) in $\mathscr{C}^1([0,T]\times\mathbf{R}^3, \Omega_2) \times \mathscr{C}^1_c([0,T]\times\mathbf{R}^3\times\mathbf{R}^3, \mathbf{R}_+)$ and the solution is unique. We also explain why the proof works in the case of all initial data in H^s.

We write some preliminary lemmas which we will use in this work.

Lemma 7. *Let $S, A_i \in \mathscr{C}^1([0,T[\times\mathbf{R}^3, \mathcal{M}_3(\mathbf{R}))$ be smooth matrices such that S and SA_i are symmetric, We suppose moreover that there exists $c>0$ such that $c\mathrm{Id} \leq S(t,\boldsymbol{x}) \leq c^{-1}\mathrm{Id}$. Then, every smooth and compactly supported vectors $\mathbf{W}$ and $\mathbf{F}$ satisfying the equation $S\partial_t\mathbf{W} + \sum_i SA_i\partial_{x_i}\mathbf{W} = \mathbf{F}$ with the initial data $\mathbf{W}(0,\boldsymbol{x}) = \mathbf{W}_0(\boldsymbol{x})$ verify the energy estimate*

$$\|\mathbf{W}\|_{L^2} \leq c^{-1}\left(\|\mathbf{W}_0\|_{L^2} + \frac{1}{2}\left\|\partial_t S + \sum_i \partial_{x_i}(SA_i)\right\|_{L^\infty,T}\int_0^t \|\mathbf{W}\|_{L^2}(\tau)\,\mathrm{d}\tau + \int_0^t \|\mathbf{F}\|_{L^2}(\tau)\,\mathrm{d}\tau\right). \tag{4.15}$$

Proof. By multiplying by $\mathbf{W}^T$ and integrating, one has

$$\frac{1}{2}\partial_t\int_{\mathbf{R}^3}\mathbf{W}^T S\mathbf{W}\,\mathrm{d}x = \frac{1}{2}\int_{\mathbf{R}^3}\mathbf{W}^T(\partial_t S + \sum_i \partial_{x_i}(SA_i))\mathbf{W}\,\mathrm{d}x + \int_{\mathbf{R}^3}\mathbf{W}^T\mathbf{F}\,\mathrm{d}x$$

then one integrates with respect to t and uses the estimate $W^T SW \geq cW^T W$. □

Lemma 8 (See [Maj84])**.** *Let $\boldsymbol{g} : \Omega \subset \mathbf{R}^3 \to \mathbf{R}^3$ a smooth vector-valued function, $\boldsymbol{u} : \mathbf{R}^3 \to \Omega_1 \subset \Omega$ with Ω_1 relatively compact in Ω. Assume that $\boldsymbol{u} \in H^s(\mathbf{R}^3)\cap L^\infty(\mathbf{R}^3)$. Then for $|\alpha| \leq s$, one has the inequality*

$$\|\partial^\alpha \boldsymbol{g}(\boldsymbol{u})\|_{L^2} \leq C(s)\sup_{\boldsymbol{v}\in\Omega_1}\left(\sup_{|\beta|\leq s-1}|\partial^\beta \boldsymbol{g}(\boldsymbol{v})|\right)\|\boldsymbol{u}\|_{L^\infty}^{s-1}\|\boldsymbol{u}\|_{H^s}.$$

Lemma 9 (See [Maj84])**.** *Let $f, g \in H^s(\mathbf{R}^3)\cap L^\infty(\mathbf{R}^3)$, one has the inequality*

$$\|\partial^\alpha(fg)\|_{L^2} \leq C(s)(\|f\|_{L^\infty}\|g\|_{H^s} + \|f\|_{H^s}\|g\|_{L^\infty}).$$

Lemma 10. *Let $f \in H^s(\mathbf{R}^3)$ and $g \in \mathrm{BV}(\mathbf{R}^3)$. Then*

$$\|\boldsymbol{\nabla}(f\star g)\|_{L^2} \leq \|f\|_{L^2}\mathrm{TV}(g).$$

In particular

$$\|\boldsymbol{\nabla}(f\star g)\|_{H^s} \leq \|f\|_{H^s}\mathrm{TV}(g).$$

Proof. The proof is done via an approximation argument. The function g belongs to $\mathrm{BV}(\mathbf{R}^3)$, therefore [EG15] (Theorem 5.3), there exists a sequence $(g_k)_{k\in\mathbf{N}} \in \mathrm{BV}(\mathbf{R}^3)\cap\mathscr{C}^\infty(\mathbf{R}^3)$ such that $g_k \xrightarrow{L^1} g$ and $\mathrm{TV}(g_k) \to \mathrm{TV}(g)$. In particular, $g_k \in W^{1,1}(\mathbf{R}^3)$ for all k, and $\mathrm{TV}(g_k) = \|\boldsymbol{\nabla}g_k\|_{L^1} \to \mathrm{TV}(g)$. Applying Young's convolution inequality to f and g_k, we obtain $\|\boldsymbol{\nabla}(f\star g_k)\|_{L^2} \leq \|f\|_{L^2}\|\boldsymbol{\nabla}g_k\|_{L^1}$, passing to the limit we obtain $\|\boldsymbol{\nabla}(f\star g)\|_{L^2} \leq \|f\|_{L^2}\mathrm{TV}(g)$. Replacing f by $\partial^\alpha f$ with $|\alpha| \leq s$ in the previous inequality, we obtain $\|\boldsymbol{\nabla}(f\star g)\|_{H^s} \leq \|f\|_{H^s}\mathrm{TV}(g)$.

□

4.2 A priori estimates

The goal is to prove that the sequence $(T_k)_{k\in\mathbf{N}}$ has a strictly positive lower bound $T_\star$. In all this section, one considers initial data that such (4.4)-(4.5) hold and define the sequences θ_k, $\mathbf{U}^k$, f^k by (4.6)-(4.9) and T_k by (4.10)-(4.12).

Lemma 11. *For all $k \geq 0$,*

$$\|\partial_t\mathbf{U}^{k+1}\|_{H^{s-1},T_{k+1}} \leq C(s, f_0, \mathbf{U}_0, \Omega_2, R, w)\mathrm{TV}(w). \tag{4.16}$$

Proof. Let $t \in [0, T_{k+1}[$, then

$$\|\partial_t \mathbf{U}^{k+1}\|_{H^{s-1}}(t) \leq \sum_i \|A_i(\mathbf{U}^k)\partial_{x_i}\mathbf{U}^{k+1}\|_{H^{s-1},T_{k+1}} + \|b(\mathbf{U}^k, f^k)\|_{H^{s-1},T_{k+1}}.$$

One has to control each term, let's start with b. One has,

$$\|b(\mathbf{U}^k, f^k)\|_{H^{s-1},T_{k+1}} = \sum_{|\alpha|\leq s-1} \left\| \begin{pmatrix} \partial^\alpha \left(\frac{\varrho^k \int_{\mathbf{R}^3} (\boldsymbol{u}^k - \boldsymbol{v}) \cdot \boldsymbol{\nabla}_x \langle f^k \rangle \, \mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \, \mathrm{d}v} \right) \\ \partial^\alpha \left(\frac{\varrho^k \boldsymbol{u}^k \int_{\mathbf{R}^3} (\boldsymbol{u}^k - \boldsymbol{v}) \cdot \boldsymbol{\nabla}_x \langle f^k \rangle \, \mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \, \mathrm{d}v} \right) \end{pmatrix} \right\|_{L^2,T_{k+1}}$$

Applying Lemma 9 for $|\alpha| \leq s-1$,

$$\begin{aligned} &\left\| \partial^\alpha \left(\frac{\varrho^k \boldsymbol{u}^k \cdot \int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \, \mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \, \mathrm{d}v} \right) \right\|_{L^2} \\ &\leq C(s) \left(\|\varrho^k \boldsymbol{u}^k\|_{H^{s-1}} \left\| \frac{\int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \, \mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \, \mathrm{d}v} \right\|_{L^\infty} + \|\varrho^k \boldsymbol{u}^k\|_{L^\infty} \left\| \frac{\int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \, \mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \, \mathrm{d}v} \right\|_{H^{s-1}} \right). \end{aligned} \tag{4.17}$$

Thanks to (4.10), one has

$$\|\varrho^k \boldsymbol{u}^k\|_{H^{s-1}} \leq \|\mathbf{U}^k\|_{H^{s-1}} \leq \|\mathbf{U}_0\|_{H^{s-1}} + \|\mathbf{U}^k - \mathbf{U}_0\|_{H^{s-1}} \leq C(\mathbf{U}_0, R).$$

Thanks to the Sobolev embedding $H^{s-1}(\mathbf{R}^3) \hookrightarrow L^\infty(\mathbf{R}^3)$, one has

$$\|\varrho^k \boldsymbol{u}^k\|_{L^\infty} \leq C(\Omega_2) \|\varrho^k \boldsymbol{u}^k\|_{H^{s-1}} \leq C(\Omega_2, \mathbf{U}_0, R).$$

Thanks to (4.10)-(4.13), one has

$$\begin{aligned} \left\| \int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \, \mathrm{d}v \right\|_{L^\infty} &\leq \mathrm{TV}(w) \left\| \int_{\mathbf{R}^3} f^k \, \mathrm{d}v \right\|_{L^\infty} \\ &\leq \mathrm{TV}(w) 2^3 (V_M^k)^3 \|f^k\|_{L^\infty} \\ &\leq \mathrm{TV}(w) 2^3 (V_M^k)^3 \|f_0\|_{L^\infty} \\ &\leq C(f_0) \mathrm{TV}(w). \end{aligned}$$

So that, using (4.5) and (4.14)

$$\left\| \frac{\int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \, \mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \, \mathrm{d}v} \right\|_{L^\infty} \leq C(f_0, w) \mathrm{TV}(w).$$

It remains to bound the last term in (4.17) $\left\| \frac{\int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \, \mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \, \mathrm{d}v} \right\|_{H^{s-1}}$. Again applying Lemma 9, one has for $|\alpha| \leq s-1$:

$$\begin{aligned} \left\| \partial^\alpha \left(\frac{\int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \, \mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \, \mathrm{d}v} \right) \right\|_{L^2} &\leq C(s) \left(\left\| \int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \, \mathrm{d}v \right\|_{H^{s-1}} \left\| \frac{1}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \, \mathrm{d}v} \right\|_{L^\infty} \right. \\ &\quad \left. + \left\| \int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \, \mathrm{d}v \right\|_{L^\infty} \left\| \frac{1}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \, \mathrm{d}v} \right\|_{H^{s-1}} \right) \\ &\leq C(s) \left(C(f_0, w) \mathrm{TV}(w) \right. \\ &\quad \left. + C(f_0) \mathrm{TV}(w) \left\| \frac{1}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \, \mathrm{d}v} \right\|_{H^{s-1}} \right) \end{aligned}$$

For the term $\left\|\frac{1}{1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v}\right\|_{H^{s-1}}$, one has

$$\begin{aligned}\left\|\frac{1}{1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v}\right\|_{H^{s-1}} &= \sum_{|\alpha|\leq s-1}\left\|\partial^\alpha\left(\frac{1}{1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v}\right)\right\|_{L^2}\\ &= \sum_{|\alpha|\leq s-1}\left\|\frac{P_\alpha}{(1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v)^{q(\alpha)}}\right\|_{L^2}\end{aligned}$$

where P_α is a polynomial in $\partial^\alpha(1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v)$, and $q(\alpha)$ is an integer than depend on α. One can bound the numerator P_α thanks to the inequality

$$\begin{aligned}\left\|\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v\right\|_{H^s} &\leq (V_M^k)^{3/2}\|\langle f^k\rangle\|_{H^s}\\ &\leq \mathrm{TV}(w)C(f_0)\end{aligned}$$

and the denominator thanks to (4.13). As a consequence, one obtains

$$\left\|\partial^\alpha\left(\frac{\varrho^k\boldsymbol{u}^k\cdot\int_{\mathbf{R}^3}\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v}{1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v}\right)\right\|_{L^2}\leq C(\mathbf{U}_0,f_0,\Omega_2,R,w)\mathrm{TV}(w)\tag{4.18}$$

Next, one has the term

$$\begin{aligned}&\left\|\partial^\alpha\left(\frac{\varrho^k\int_{\mathbf{R}^3}\boldsymbol{v}\cdot\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v}{1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v}\right)\right\|_{L^2}\\ &\leq C(s)\left(\|\varrho^k\|_{H^{s-1}}\left\|\frac{\int_{\mathbf{R}^3}\boldsymbol{v}\cdot\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v}{1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v}\right\|_{L^\infty}+\|\varrho^k\|_{L^\infty}\left\|\frac{\int_{\mathbf{R}^3}\boldsymbol{v}\cdot\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v}{1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v}\right\|_{H^{s-1}}\right)\end{aligned}$$

We have

$$\left\|\int_{\mathbf{R}^3}\boldsymbol{v}\cdot\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v\right\|_{L^\infty}\leq 2^3\|f^k\|_{L^\infty}(V_M^k)^4\mathrm{TV}(w)\leq C(f_0,w)\mathrm{TV}(w)$$

and, using Lemma 10

$$\left\|\int_{\mathbf{R}^3}\boldsymbol{v}\cdot\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v\right\|_{H^{s-1}}\leq C(V_M)\left\|\int_{\mathbf{R}^3}\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v\right\|_{H^{s-1}}\leq C(f_0)\mathrm{TV}(w)$$

So

$$\left\|\partial^\alpha\left(\frac{\varrho^k\int_{\mathbf{R}^3}\boldsymbol{v}\cdot\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v}{1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v}\right)\right\|_{L^2}\leq C(\mathbf{U}_0,f_0,\Omega_2,R,w)\mathrm{TV}(w).\tag{4.19}$$

It is clear thanks to the previous inequalities that we have the bound

$$\left\|\frac{\varrho^k\boldsymbol{u}^k\left(\int_{\mathbf{R}^3}\boldsymbol{v}\cdot\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v\right)}{1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v}\right\|_{H^{s-1}}\leq C(s,f_0,R,\mathbf{U}_0,\Omega_2,w)\mathrm{TV}(w).\tag{4.20}$$

For the term

$$\left\|\frac{\varrho^k\boldsymbol{u}^k\otimes\boldsymbol{u}^k\int_{\mathbf{R}^3}\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v}{1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v}\right\|_{H^{s-1}}=\left\|\frac{\varrho^k\boldsymbol{u}^k\left(\boldsymbol{u}^k\cdot\int_{\mathbf{R}^3}\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v\right)}{1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v}\right\|_{H^{s-1}}$$

we use the fact that Ω_2 is relatively compact and so ϱ^k is bounded from below by a constant that depends only on Ω_2. Writing

$$\begin{aligned}&\left\|\partial^\alpha\left(\frac{\varrho^k\boldsymbol{u}^k\left(\boldsymbol{u}^k\cdot\int_{\mathbf{R}^3}\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v\right)}{1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v}\right)\right\|_{L^2}\\ &\leq C(s)\left(\|\varrho^k\boldsymbol{u}^k\|_{H^{s-1}}\left\|\frac{\boldsymbol{u}^k\cdot\int_{\mathbf{R}^3}\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v}{1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v}\right\|_{L^\infty}+\|\varrho^k\boldsymbol{u}^k\|_{L^\infty}\left\|\frac{\boldsymbol{u}^k\cdot\int_{\mathbf{R}^3}\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v}{1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v}\right\|_{H^{s-1}}\right)\end{aligned}$$

and

$$\|\boldsymbol{u}^k\|_{L^\infty} = \left\| \frac{\varrho^k \boldsymbol{u}^k}{\varrho^k} \right\|_{L^\infty} \leq C(\Omega_2) \|\varrho^k \boldsymbol{u}^k\|_{L^\infty},$$

one obtains

$$\left\| \frac{\boldsymbol{u}^k \cdot \int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \, \mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \, \mathrm{d}v} \right\|_{L^\infty} \leq C(s, \mathbf{U}_0, f_0, R, \Omega_2, w) \mathrm{TV}(w).$$

Therefore

$$\begin{aligned} &\left\| \partial^\alpha \left(\frac{\boldsymbol{u}^k \cdot \int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \, \mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \, \mathrm{d}v} \right) \right\|_{L^2} \\ &\quad \leq C(s) \left(\|\boldsymbol{u}^k\|_{H^{s-1}} \left\| \frac{\int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \, \mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \, \mathrm{d}v} \right\|_{L^\infty} + \|\boldsymbol{u}^k\|_{L^\infty} \left\| \frac{\int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \, \mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \, \mathrm{d}v} \right\|_{H^{s-1}} \right). \end{aligned}$$

Writing

$$\|\partial^\alpha \boldsymbol{u}^k\|_{L^2} \leq C(s) \left(\|\varrho^k \boldsymbol{u}^k\|_{H^{s-1}} \left\| \frac{1}{\varrho^k} \right\|_{L^\infty} + \|\varrho^k \boldsymbol{u}^k\|_{L^\infty} \left\| \frac{1}{\varrho^k} \right\|_{H^{s-1}} \right),$$

one notices that

$$\begin{aligned} \left\| \partial^\alpha \left(\frac{1}{\varrho^k} \right) \right\|_{L^2} &= \left\| \frac{P_\alpha(\partial^\alpha \varrho^k)}{(\varrho^k)^{q(\alpha)}} \right\|_{L^2} \\ &\leq \frac{1}{\|(\varrho^k)^{q(\alpha)}\|_{L^\infty}} \|P_\alpha(\partial^\alpha \varrho^k)\|_{L^2} \\ &\leq C(\Omega_2, R, \mathbf{U}_0) \end{aligned}$$

where P_α is a polynomial in $\partial^\alpha(\varrho^k)$ and $q(\alpha)$ an integer depending on α. One gets

$$\left\| \frac{\varrho^k \boldsymbol{u}^k \left(\boldsymbol{u} \cdot \int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \, \mathrm{d}v \right)}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \, \mathrm{d}v} \right\|_{H^{s-1}} \leq C(s, f_0, R, \mathbf{U}_0, \Omega_2, w) \mathrm{TV}(w). \tag{4.21}$$

In the end, combining the inequalities (4.18)-(4.21)

$$\|b(\mathbf{U}^k, f^k)\|_{H^{s-1}, T_{k+1}} \leq C(s, \mathbf{U}_0, f_0, w, R, \Omega_2) \mathrm{TV}(w).$$

We turn to the term

$$\|A_i(\mathbf{U}^k) \partial_{x_i} \mathbf{U}^{k+1}\|_{H^{s-1}, T_{k+1}} = \sum_{|\alpha| \leq s-1} \|\partial^\alpha (A_i(\mathbf{U}^k) \partial_{x_i} \mathbf{U}^{k+1})\|_{L^2, T_{k+1}}.$$

Using Lemma 8 and the Sobolev embedding $H^s(\mathbf{R}^3) \hookrightarrow L^\infty(\mathbf{R}^3)$, one has for $|\alpha| \leq s-1$,

$$\begin{aligned} \|\partial^\alpha (A_i(\mathbf{U}^k) \partial_{x_i} \mathbf{U}^{k+1})\|_{L^2, T_{k+1}} \leq & \|\partial^\alpha ((A_i(\mathbf{U}^k) - A_i(\overline{\mathbf{U}_0})) \partial_{x_i} \mathbf{U}^{k+1})\|_{L^2, T_{k+1}} \\ & + \|A_i(\overline{\mathbf{U}_0}) \partial^\alpha (\partial_{x_i} \mathbf{U}^{k+1})\|_{L^2, T_{k+1}} \\ \leq & C(s) (\|A_i(\mathbf{U}^k) - A_i(\overline{\mathbf{U}_0})\|_{H^s, T_{k+1}} \|\partial_{x_i} \mathbf{U}^{k+1}\|_{L^\infty, T_{k+1}} \\ & + \|A_I(\overline{\mathbf{U}_0})\|_{L^\infty, T_{k+1}} \|\partial_{x_i} \mathbf{U}^{k+1}\|_{H^{s-1}, T_{k+1}}) \\ \leq & C(s, \Omega_2, R, \mathbf{U}_0). \end{aligned}$$

This completes the proof. □

Lemma 12. *For all $k \geq 0$ and $T \in [0, \inf(1, T_{k+1})[$*

$$\sup_{t \in [0,T]} \|\mathbf{U}^{k+1} - \mathbf{U}_0\|_{H^s}(t) \leq T C(s, R, \Omega_2, \mathbf{U}_0, f_0, w, \mathrm{TV}(w)) e^{C(s, R, f_0, \mathbf{U}_0, \Omega_2, w, \mathrm{TV}(w))T}. \tag{4.22}$$

Proof. The function $\mathbf{W}^{k+1} = \mathbf{U}^{k+1} - \mathbf{U}_0$ satisfies

$$S(\mathbf{U}^k)\partial_t \mathbf{W}^{k+1} + \sum_i (SA_i)(\mathbf{U}^k)\partial_{x_i}\mathbf{W}^{k+1} = S(\mathbf{U}^k)b(\mathbf{U}^k, f^k) + \mathbf{H}^k,$$
$$\mathbf{W}^{k+1}(x,0) = 0,$$

with $\mathbf{H}^k = -\sum_i (SA_i)(\mathbf{U}^k)\partial_{x_i}\mathbf{U}_0$. We look for a bound in H^s of $\mathbf{W}^{k+1}$. The function $\partial^\alpha \mathbf{W}^{k+1}$ satifies

$$S(\mathbf{U}^k)\partial_t\partial^\alpha\mathbf{W}^{k+1} + \sum_i (SA_i)(\mathbf{U}^k)\partial_{x_i}\partial^\alpha\mathbf{W}^{k+1} = S(\mathbf{U}^k)\partial^\alpha(S^{-1}(\mathbf{U}^k)\mathbf{H}^k + b(\mathbf{U}^k, f^k)) + \mathbf{F}_\alpha$$

with $\mathbf{F}_\alpha = S(\mathbf{U}^k)\sum_i \left(A_i(\mathbf{U}^k)\partial_{x_i}\partial^\alpha\mathbf{W}^{k+1} - \partial^\alpha(A_i(\mathbf{U}^k)\partial_{x_i}\mathbf{W}^{k+1})\right)$. Moreover, one has

$$\begin{aligned}\frac{1}{2}\partial_t\int_{\mathbf{R}^3}(\partial^\alpha\mathbf{W}^{k+1})^T S(\mathbf{U}^k)\partial^\alpha\mathbf{W}^{k+1}\,\mathrm{d}x = \frac{1}{2}\int_{\mathbf{R}^3}(\partial^\alpha\mathbf{W}^{k+1})^T\Bigg(&\partial_t S(\mathbf{U}^k)\\ &+\sum_i \partial_{x_i}(SA_i)(\mathbf{U}^k)\Bigg)\partial^\alpha\mathbf{W}^{k+1}\,\mathrm{d}x\\ +\int_{\mathbf{R}^3}(\partial^\alpha\mathbf{W}^{k+1})^T S(\mathbf{U}^k)\partial^\alpha(S^{-1}(\mathbf{U}^k)\mathbf{H}^k&\\ +b(\mathbf{U}^k, f^k))\,\mathrm{d}x + \int_{\mathbf{R}^3}(\partial^\alpha\mathbf{W}^{k+1})^T\mathbf{F}_\alpha\,\mathrm{d}x.&\end{aligned}$$

Up to time T_k, $\mathbf{U}^k$ takes its values in Ω_2 on which S and SA_i are smooth. One can bound the derivatives of S and SA_i at any order by a constant that depends on Ω_2. One also uses the Sobolev embedding $H^{s-1}(\mathbf{R}^3) \hookrightarrow L^\infty(\mathbf{R}^3)$ and Lemma 11 to obtain the estimates

$$\begin{aligned}\|\partial_t\mathbf{U}^k\|_{L^\infty,T_k} &\le C(s)\|\partial_t\mathbf{U}^k\|_{H^{s-1},T_k} \le C(s,\Omega_2,R,\mathbf{U}_0,f_0,w)\mathrm{TV}(w),\\ \|\partial_{x_i}\mathbf{U}^k\|_{L^\infty,T_k} &\le C(s)\|\partial_{x_i}\mathbf{U}^k\|_{H^{s-1},T_k}\\ &\le C(s)\left(\|\mathbf{U}^k - \mathbf{U}_0\|_{H^s,T_k} + \|\mathbf{U}_0 - \bar{\mathbf{U}}_0\|_{H^s,T_k}\right) \le C(s,R,\mathbf{U}_0).\end{aligned}$$

So that

$$\|\partial_t S(\mathbf{U}^k) + \sum_i \partial_{x_i}(SA_i)(\mathbf{U}^k)\|_{L^\infty,T_{k+1}} \le C(s,\Omega_2,R,\mathbf{U}_0,f_0,w,\mathrm{TV}(w)). \tag{4.23}$$

One now uses the following classical inequality: if $h \in H^s(\mathbf{R}^3)$, $\boldsymbol{\nabla} h \in L^\infty(\mathbf{R}^3)$, $g \in H^{s-1}(\mathbf{R}^3)\cap L^\infty(\mathbf{R}^3)$ and $|\alpha| \le s$, then [Maj84]

$$\|\partial^\alpha(hg) - h\partial^\alpha g\|_{L^2} \le C(s)\left(\|\boldsymbol{\nabla} h\|_{L^\infty}\|g\|_{H^{s-1}} + \|g\|_{L^\infty}\|h\|_{H^s}\right).$$

Then, for $|\alpha| \le s$

$$\begin{aligned}A_i(\mathbf{U}^k)\partial_{x_i}\partial^\alpha\mathbf{W}^{k+1} - \partial^\alpha(A_i(\mathbf{U}^k)\partial_{x_i}\mathbf{W}^{k+1}) = &(A_i(\mathbf{U}^k) - A_i(\bar{\mathbf{U}}_0))\partial^\alpha\partial_{x_i}\mathbf{W}^{k+1}\\ &-\partial^\alpha((A_i(\mathbf{U}^k) - A_i(\bar{\mathbf{U}}_0))\partial_{x_i}\mathbf{W}^{k+1}).\end{aligned}$$

According to the previous inequality, one has

$$\begin{aligned}\|\mathbf{F}_\alpha\|_{L^2,T_{k+1}} \le &\|S(\mathbf{U}^k)\|_{L^\infty,T_{k+1}} C(s)\sum_i (\|\partial^\alpha(A_i(\mathbf{U}^k) - A_i(\bar{\mathbf{U}}_0))\|_{L^\infty,T_{k+1}}\\ &\times\|\partial_{x_i}\mathbf{W}^{k+1}\|_{H^{s-1},T_{k+1}} + \|\partial_{x_i}\mathbf{W}^{k+1}\|_{L^\infty,T_{k+1}}\|A_i(\mathbf{U}^k) - A_i(\bar{\mathbf{U}}_0)\|_{H^s,T_{k+1}})\\ \le &C(s,\Omega_2,R,\mathbf{U}_0).\end{aligned}$$

One has using Lemma 9

$$\begin{aligned}\|S(\mathbf{U}^k)\partial^\alpha(S^{-1}(\mathbf{U}^k)\mathbf{H}^k\|_{L^2,T_{k+1}}\leq&\|S(\mathbf{U}^k)\|_{L^\infty,T_k}\sum_i\|\partial^\alpha(A_i(\mathbf{U}^k)\partial_{x_i}\mathbf{U}_0\|_{L^2,T_k}\\ \leq&C(s,\Omega_2)\sum_i\|A_i(\mathbf{U}^k)-A_i(\bar{\mathbf{U}}_0\|_{L^\infty,T_k}\|\partial_{x_i}\mathbf{U}_0\|_{H^s,T_k}\\ &+\sum_i\|\partial_{x_i}\mathbf{U}_0\|_{L^\infty,T_k}\|A_i(\mathbf{U}^k)-A_i(\bar{\mathbf{U}}_0)\|_{H^s,T_k}+C\|\partial_{x_i}\mathbf{U}_0\|_{H^s,T_k}\\ \leq&C(s,\Omega_2,R,\mathbf{U}_0).\end{aligned}$$

It yields

$$\|S(\mathbf{U}^k)\partial^\alpha(b(\mathbf{U}^k,f^k))\|_{L^2,T_{k+1}}\leq C(s,R,f_0,\Omega_2,w,\mathrm{TV}(w)). \tag{4.24}$$

One now uses the estimate (4.15). For $t\in[0,T_{k+1}]$, one obtains

$$\begin{aligned}\|\partial^\alpha\mathbf{W}^{k+1}\|_{L^2}(t)\leq&C(\Omega_2)[\|\partial^\alpha\mathbf{W}^{k+1}\|_{L^2}(0)+\frac{1}{2}\|\partial_t S+\sum_i\partial_{x_i}(SA_i)\|_{L^\infty,T_{k+1}}\int_0^t\|\partial^\alpha\mathbf{W}^{k+1}\|_{L^2}(\tau)\,\mathrm{d}\tau\\ &+\int_0^t\left(\|\mathbf{F}_\alpha\|_{L^2}+\|S(\mathbf{U}^k)\partial^\alpha(S^{-1}(\mathbf{U}^k)\mathbf{H}^k)\|_{L^2}+\|S(\mathbf{U}^k)\partial^\alpha(b(\mathbf{U}^k,f^k))\|_{L^2}\right)\,\mathrm{d}\tau.\end{aligned}$$

Then one gets

$$\begin{aligned}\|\partial^\alpha\ \mathbf{W}^{k+1}\|_{L^2}(t)\leq&[\|\partial^\alpha\mathbf{W}^{k+1}\|_{L^2}(0)+C(s,\Omega_2,R,\mathbf{U}_0,f_0)\int_0^t\|\partial^\alpha\mathbf{W}^{k+1}\|_{L^2}(\tau)\,\mathrm{d}\tau\\ &+\int_0^t(C(s,\Omega_2,R,\mathbf{U}_0)+C(s,\Omega_2,R,\mathbf{U}_0)+C(s,\mathbf{U}_0,f_0,w,\mathrm{TV}(w),R,\Omega_2))\,\mathrm{d}\tau].\end{aligned}$$

Summing for all $|\alpha|\leq s$ these estimates, one ends up with

$$\begin{aligned}\|\mathbf{W}^{k+1}\|_{H^s}(t)\leq&C(\Omega_2)\left(\|\mathbf{W}^{k+1}\|_{H^s}(0)+C(s,\Omega_2,R,\mathbf{U}_0,f_0)\int_0^t\|\mathbf{W}^{k+1}\ \|_{H^s}(\tau)\,\mathrm{d}\tau\right)\\ &+tC(s,R,\Omega_2,\mathbf{U}_0,f_0,w,\mathrm{TV}(w)).\end{aligned}$$

Thanks to Gronwall's lemma, one has for all $t\in[0,T]$ with $T\leq T_{k+1}$

$$\|\mathbf{W}^{k+1}\|_{H^s}(t)\leq C(\Omega_2)\left(\|\mathbf{W}^{k+1}\|_{H^s}(0)+TC(s,R,\Omega_2,\mathbf{U}_0,f_0,w,\mathrm{TV}(w))\right)e^{C(s,R,f_0,\mathbf{U}_0,\Omega_2,w,\mathrm{TV}(w))T}.$$

Choosing $T\leq 1$, and since $\mathbf{W}^{k+1}(x,0)=0$, one obtains the claim. □

Lemma 13. *The following inequalities hold for all $T\in[0,\inf(1,T_{k+1})[$*

$$\sup_{t\in[0,T]}V_M^{k+1}(t)\leq V_M(0)+C(\Omega_2,w,\gamma,\mathbf{U}_0)\mathrm{TV}(w)T, \tag{4.25}$$

$$\sup_{t\in[0,T]}X_M^{k+1}(t)\leq X_M(0)+C(f_0,\Omega_2,w,\gamma,\mathbf{U}_0)\mathrm{TV}(w)T, \tag{4.26}$$

$$\left\|\int_{\mathbf{R}^3}\langle f^{k+1}\rangle\,\mathrm{d}v\right\|_{L^\infty,T}\leq 2^3\|w\|_{L^1}\|f_0\|_{L^\infty}(V_M(0)+C(\Omega_2,w,\gamma,\mathbf{U}_0)T)^3. \tag{4.27}$$

Proof. Recall the characteristic curves of the Vlasov equation

$$\frac{\mathrm{d}\mathbf{X}^{k+1}}{\mathrm{d}t}(t;\boldsymbol{x},\boldsymbol{v},s)=\mathbf{V}^{k+1}(t;\boldsymbol{x},\boldsymbol{v},s), \tag{4.28}$$

$$\mathbf{X}^{k+1}(s;\boldsymbol{x},\boldsymbol{v},s)=\boldsymbol{x} \tag{4.29}$$

$$\frac{\mathrm{d}\mathbf{V}^{k+1}}{\mathrm{d}t}(t;\boldsymbol{x},\boldsymbol{v},s)=-\langle\boldsymbol{\nabla}p(\mathbf{X}^{k+1}(t;\boldsymbol{x},\boldsymbol{v},s),t)\rangle \tag{4.30}$$

$$\mathbf{V}^{k+1}(s;\boldsymbol{x},\boldsymbol{v},s)=\boldsymbol{v}. \tag{4.31}$$

One has, writting in an implicity way (4.28)-(4.30),

$$\mathbf{X}^{k+1}(t;\boldsymbol{x},\boldsymbol{v},s) = \boldsymbol{x} + \int_s^t \mathbf{V}^{k+1}(\tau;\boldsymbol{x},\boldsymbol{v},s)\,\mathrm{d}\tau, \tag{4.32}$$

$$\mathbf{V}^{k+1}(t;\boldsymbol{x},\boldsymbol{v},s) = \boldsymbol{v} - \int_s^t \langle \boldsymbol{\nabla} p^k(\mathbf{X}^{k+1}(\tau,\boldsymbol{x},\boldsymbol{v},s),\tau)\rangle\,\mathrm{d}\tau. \tag{4.33}$$

We recall the notations

$$X_M^{k+1}(t) = \sup_{(\boldsymbol{x},\boldsymbol{v})\in\mathbf{R}^3\times\mathbf{R}^3, f^{k+1}(t,\boldsymbol{x},\boldsymbol{v})>0} |\boldsymbol{x}|,$$

$$V_M^{k+1}(t) = \sup_{(\boldsymbol{x},\boldsymbol{v})\in\mathbf{R}^3\times\mathbf{R}^3, f^{k+1}(t,\boldsymbol{x},\boldsymbol{v})>0} |\boldsymbol{v}|.$$

Then

$$\mathbf{V}^{k+1}(0;\boldsymbol{x},\boldsymbol{v},t) = \boldsymbol{v} - \int_t^0 \langle \boldsymbol{\nabla} p^k(\mathbf{X}^{k+1}(\tau;\boldsymbol{x},\boldsymbol{v},t),\tau)\rangle\,\mathrm{d}\tau$$

and

$$|\boldsymbol{v}| \leq \mathbf{V}^{k+1}(0;\boldsymbol{x},\boldsymbol{v},t) + \int_0^t \|p^k\|_{L^\infty}\mathrm{TV}(w)\,\mathrm{d}\tau.$$

One obtains

$$V_M^{k+1}(t) \leq V_M(0) + \int_0^t \|p^k\|_{L^\infty}(\tau)\mathrm{TV}(w)\,\mathrm{d}\tau.$$

And it yields (4.25). One proceeds similarly for X_M^{k+1}. Using the formula (4.33), one has

$$\begin{aligned}\boldsymbol{x} &= \mathbf{X}^{k+1}(0;\boldsymbol{x},\boldsymbol{v},t) + \int_0^t \mathbf{V}^{k+1}(\tau;\boldsymbol{x},\boldsymbol{v},t)\,\mathrm{d}\tau \\ &= \mathbf{X}^{k+1}(0;\boldsymbol{x},\boldsymbol{v},t) + \boldsymbol{v}\int_0^t \mathrm{d}\tau - \int_0^t\int_t^\tau \langle \boldsymbol{\nabla} p^k(\mathbf{X}^{k+1}(\tilde{\tau};\boldsymbol{x},\boldsymbol{v},t),\tilde{\tau})\rangle\,\mathrm{d}\tilde{\tau}\,\mathrm{d}\tau,\end{aligned}$$

so

$$X_M^{k+1}(t) \leq X_M(0) + V_M(0)t + \int_0^t\int_\tau^t \|p^k\|_{L^\infty}(\tilde{\tau})\mathrm{TV}(w)\,\mathrm{d}\tilde{\tau}\,\mathrm{d}\tau.$$

And one obtains (4.26).

Moreover, one has

$$\begin{aligned}\left\|\int_{\mathbf{R}^3}\langle f^{k+1}\rangle\,\mathrm{d}v\right\|_{L^\infty,T} &\leq \|w\|_{L^1}\left\|\int_{\mathbf{R}^3} f^{k+1}\,\mathrm{d}v\right\|_{L^\infty,T} \\ &\leq 2^3\|w\|_{L^1}\|f^{k+1}\|_{L^\infty,T}(V_M^{k+1})^3 \\ &\leq 2^3\|w\|_{L^1}\|f_0\|_{L^\infty}(V_M(0) + C(\Omega_2,w,\gamma,\mathbf{U}_0)\mathrm{TV}(w)T)^3.\end{aligned}$$

□

Proposition 14. *There exists $T_\star \in]0,1]$ which depends only upon Ω_1, Ω_2, s, $\mathbf{U}_0$, w and f_0 such that, for all $k \in \mathbf{N}$, $T_k \geq T_\star$.*

Proof. From Lemma 12, one has

$$\sup_{t\in[0,T]} \|\mathbf{U}^{k+1} - \mathbf{U}_0\|_{H^s}(t) \leq TC(s,R,\Omega_2,\mathbf{U}_0,f_0,w,\mathrm{TV}(w))e^{C(s,R,f_0,\mathbf{U}_0,\Omega_2,w,\mathrm{TV}(w))T}.$$

The right-hand-side of this inequality defines a continuous function of T independent of k, with value 0 if $T = 0$. Therefore, for all k there exists $T_k > T_1 > 0$ such that

$$\sup_{t\in[0,T]} \|\mathbf{U}^{k+1} - \mathbf{U}_0\|_{H^s}(t) \leq T_1 C(s,R,\Omega_2,\mathbf{U}_0,f_0,w,\mathrm{TV}(w))e^{C(s,R,f_0,\mathbf{U}_0,\Omega_2,w,\mathrm{TV}(w))T_1} \leq R.$$

For the same reasons, using Lemma 13, for all k there exists T_2, T_3 and T_4 in $]0,T_k[$ such that

$$\sup_{t\in[0,T]} V_M^{k+1}(t) \le V_M(0) + C(\Omega_2, w, \gamma, \mathbf{U}_0)\mathrm{TV}(w)T_2 \le 2V_M(0),$$
$$\sup_{t\in[0,T]} X_M^{k+1}(t) \le X_M(0) + C(f_0, \Omega_2, w, \gamma, \mathbf{U}_0)\mathrm{TV}(w)T_3 \le 2X_M(0),$$
$$\left\| \int_{\mathbf{R}^3} \langle f^{k+1}\rangle \,\mathrm{d}v \right\|_{L^\infty,T} \le 2^3\|w\|_{L^1}\|f_0\|_{L^\infty}(V_M(0) + C(\Omega_2, w, \gamma, \mathbf{U}_0)T_4)^3 \le 2^4\|w\|_{L^1}\|f_0\|_{L^\infty}V_M(0)^3.$$

Let $T_\star = \min(T_1, T_2, T_3, T_4)$ then for all $k \in \mathbf{N}$, $T_k \ge T_\star > 0$ and the bounds (4.10)-(4.13) are verified. □

4.3 Convergence of the iterative process

Proposition 15. *We consider initial data such that (4.4)-(4.5) hold and the associated sequences $\mathbf{U}^k$, f^k defined by (4.6)-(4.9). Let $T_\star$ given by Proposition 14.*

Then one can find $T_{\star\star} \in]0,T_\star[$ such that, for $k \ge 2$,

$$\|\mathbf{U}^{k+1} - \mathbf{U}^k\|_{L^2,T_{\star\star}} \le \frac{1}{4}\|\mathbf{U}^k - \mathbf{U}^{k-1}\|_{L^2,T_{\star\star}} + \frac{1}{4}\|\mathbf{U}^{k-1} - \mathbf{U}^{k-2}\|_{L^2,T_{\star\star}}, \tag{4.34}$$
$$\|f^k - f^{k-1}\|_{L^2,T_{\star\star}} \le C(f_0, \Omega_2, \gamma)\mathrm{TV}(w)\|\mathbf{U}^{k-1} - \mathbf{U}^{k-2}\|_{L^2,T_{\star\star}}. \tag{4.35}$$

Proof. Let $k \ge 2$. The function $\mathbf{U}^{k+1} - \mathbf{U}^k$ is solution of the system

$$S(\mathbf{U}^k)\partial_t(\mathbf{U}^{k+1} - \mathbf{U}^k)$$
$$+\sum_{i=1}^{3}((SA_i(\mathbf{U}^{k-1}) - (SA_i)(\mathbf{U}^k)\partial_{x_i}(\mathbf{U}^{k+1} - \mathbf{U}^k) = b(\mathbf{U}^k, f^k) - b(\mathbf{U}^{k-1}, f^{k-1}) + \mathbf{F}_k,$$

with

$$\mathbf{F}_k = (S(\mathbf{U}^{k-1}) - S(\mathbf{U}^k))\partial_t\mathbf{U}^k + \sum_{i=1}^{3}((SA_i(\mathbf{U}^{k-1}) - (SA_i)(\mathbf{U}^k))\partial_{x_i}\mathbf{U}^k.$$

Thanks to Lemma 7, one can write

$$\begin{aligned}
&\|\mathbf{U}^{k+1} - \mathbf{U}^k\|_{L^2}(t)\\
\le& C(\frac{1}{2}\|\partial_t S(\mathbf{U}^k) + \sum_i \partial_{x_i}(SA_i)(\mathbf{U}^k)\|_{L^\infty,T_\star}\int_0^t \|\mathbf{U}^{k+1} - \mathbf{U}^k\|_{L^2}(\tau)\,\mathrm{d}\tau\\
&+ \int_0^t (\|\mathbf{F}_k\|_{L^2}(\tau) + \|b(\mathbf{U}^k, f^k) - b(\mathbf{U}^{k-1}, f^{k-1})\|_{L^2}(\tau))\,\mathrm{d}\tau).
\end{aligned}$$

Then, by Gronwall's lemma, inequalities (4.10)-(4.16) and the fact that S and SA_i are smooth on $\overline{\Omega_2}$ which is compact, one has

$$\begin{aligned}
\|\mathbf{U}^{k+1} - \mathbf{U}^k\|_{L^2}(t) \le& Ce^{\left(C(\Omega_2)\|\partial_t S(\mathbf{U}^k) + \sum_i \partial_{x_i}(SA_i)(\mathbf{U}^k)\|_{L^\infty,T_\star}\right)T_\star}\\
&\times \int_0^t (\|\mathbf{F}_k\|_{L^2}(\tau) + \|b(\mathbf{U}^k, f^k) - b(\mathbf{U}^{k-1}, f^{k-1})\|_{L^2}(\tau)\,\mathrm{d}\tau\\
\le& C(\Omega_2)e^{C(s,\Omega_2,R,\mathbf{U}_0,f_0)T_\star}T_\star\\
&\times (\|\mathbf{F}_k\|_{L^2,T_\star} + \|b(\mathbf{U}^k, f^k) - b(\mathbf{U}^{k-1}, f^{k-1})\|_{L^2,T_\star}).
\end{aligned}$$

To bound to right-hand-side of this inequality, one first notices that the first term verifies $\|\mathbf{F}_k\|_{L^2} \leq C(s, \Omega_2, R, \mathbf{U}_0, \mathrm{TV}(w))\|\mathbf{U}^k - \mathbf{U}^{k-1}\|_{L^2,T_\star}$. It remains to bound the term

$$\begin{aligned}
&\|b(\mathbf{U}^k, f^k) - b(\mathbf{U}^{k-1}, f^{k-1})\|_{L^2,T_\star} \\
&= \left\| \begin{pmatrix} \dfrac{\varrho^k \int_{\mathbf{R}^3} (\boldsymbol{u}^k - \boldsymbol{v}) \cdot \boldsymbol{\nabla}_x \langle f^k \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \,\mathrm{d}v} - \dfrac{\varrho^{k-1} \int_{\mathbf{R}^3} (\boldsymbol{u}^{k-1} - \boldsymbol{v}) \cdot \boldsymbol{\nabla}_x \langle f^{k-1} \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^{k-1} \rangle \,\mathrm{d}v} \\ \dfrac{\varrho^k \boldsymbol{u}^k \int_{\mathbf{R}^3} (\boldsymbol{u}^k - \boldsymbol{v}) \cdot \boldsymbol{\nabla}_x \langle f^k \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \,\mathrm{d}v} - \dfrac{\varrho^{k-1} \boldsymbol{u}^{k-1} \int_{\mathbf{R}^3} (\boldsymbol{u}^{k-1} - \boldsymbol{v}) \cdot \boldsymbol{\nabla}_x \langle f^{k-1} \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^{k-1} \rangle \,\mathrm{d}v} \end{pmatrix} \right\|_{L^2,T_\star}.
\end{aligned}$$

One has

$$\begin{aligned}
&\left\| \frac{\varrho^k \boldsymbol{u}^k \cdot \int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \,\mathrm{d}v} - \frac{\varrho^{k-1} \boldsymbol{u}^{k-1} \cdot \int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^{k-1} \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^{k-1} \rangle \,\mathrm{d}v} \right\|_{L^2,T_\star} \\
&\leq \left\| (\varrho^k \boldsymbol{u}^k - \varrho^{k-1} \boldsymbol{u}^{k-1}) \cdot \frac{\int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \,\mathrm{d}v} \right\|_{L^2,T_\star} \\
&+ \left\| \varrho^{k-1} \boldsymbol{u}^{k-1} \cdot \left(\frac{\int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \,\mathrm{d}v} - \frac{\int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^{k-1} \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^{k-1} \rangle \,\mathrm{d}v} \right) \right\|_{L^2,T_\star} \\
&\leq \|\varrho^k \boldsymbol{u}^k - \varrho^{k-1} \boldsymbol{u}^{k-1}\|_{L^2,T_\star} \left\| \frac{\int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \,\mathrm{d}v} \right\|_{L^\infty,T_\star} \\
&+ \|\varrho^{k-1} \boldsymbol{u}^{k-1}\|_{L^\infty} \left\| \frac{1}{1 - 2^4 \|w\|_{L^1} V_M(0)^3 f_0} \right\|_{L^\infty} \left\| \int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k - f^{k-1} \rangle \,\mathrm{d}v \right\|_{L^2,T_\star} \\
&\leq C(f_0) \mathrm{TV}(w) \left\| \frac{1}{1 - 2^4 \|w\|_{L^1} V_M(0)^3 f_0} \right\|_{L^\infty} \|\mathbf{U}^k - \mathbf{U}^{k-1}\|_{L^2,T_\star} \\
&+ C(f_0, R) \mathrm{TV}(w) \left\| \frac{1}{1 - 2^4 \|w\|_{L^1} V_M(0)^3 f_0} \right\|_{L^\infty} \|f^k - f^{k-1}\|_{L^2,T_\star} \\
&\leq C(f_0, w) \mathrm{TV}(w) \|\mathbf{U}^k - \mathbf{U}^{k-1}\|_{L^2,T_\star} \\
&+ C(f_0, R, w) \mathrm{TV}(w) \|f^k - f^{k-1}\|_{L^2,T_\star}.
\end{aligned}$$

Similarly, because of the inequality

$$\left\| \int_{\mathbf{R}^3} (f^k - f^{k-1}) \boldsymbol{v} \,\mathrm{d}v \right\|_{L^2,T_\star} \leq C(f_0) \|f^k - f^{k-1}\|_{L^2,T_\star},$$

one has

$$\begin{aligned}
&\left\| \frac{\varrho^k \int_{\mathbf{R}^3} \boldsymbol{v} \cdot \boldsymbol{\nabla}_x \langle f^k \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \,\mathrm{d}v} - \frac{\varrho^{k-1} \int_{\mathbf{R}^3} \boldsymbol{v} \cdot \boldsymbol{\nabla}_x \langle f^{k-1} \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^{k-1} \rangle \,\mathrm{d}v} \right\|_{L^2,T_\star} \\
&\leq C(f_0) \mathrm{TV}(w) \left\| \frac{1}{1 - 2^4 \|w\|_{L^1} V_M(0)^3 f_0} \right\|_{L^\infty} \|\mathbf{U}^k - \mathbf{U}^{k-1}\|_{L^2,T_\star} \\
&+ C(f_0, R) \mathrm{TV}(w) \left\| \frac{1}{1 - 2^4 \|w\|_{L^1} V_M(0)^3 f_0} \right\|_{L^\infty} \|f^k - f^{k-1}\|_{L^2,T_\star} \\
&\leq C(f_0, w) \mathrm{TV}(w) \|\mathbf{U}^k - \mathbf{U}^{k-1}\|_{L^2,T_\star} + C(f_0, R, w) \|f^k - f^{k-1}\|_{L^2,T_\star}.
\end{aligned}$$

In the same way

$$\begin{aligned}
&\left\| \frac{\varrho^k \boldsymbol{u}^k \int_{\mathbf{R}^3} \boldsymbol{v} \cdot \boldsymbol{\nabla}_x \langle f^k \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \,\mathrm{d}v} - \frac{\varrho^{k-1} \boldsymbol{u}^{k-1} \int_{\mathbf{R}^3} \boldsymbol{v} \cdot \boldsymbol{\nabla}_x \langle f^{k-1} \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^{k-1} \rangle \,\mathrm{d}v} \right\|_{L^2,T_\star} \\
&\leq C(f_0) \mathrm{TV}(w) \left\| \frac{1}{1 - 2^4 \|w\|_{L^1} V_M(0)^3 f_0} \right\|_{L^\infty} \|\mathbf{U}^k - \mathbf{U}^{k-1}\|_{L^2,T_\star} \\
&+ C(f_0, R) \mathrm{TV}(w) \left\| \frac{1}{1 - 2^4 \|w\|_{L^1} V_M(0)^3 f_0} \right\|_{L^\infty} \|f^k - f^{k-1}\|_{L^2,T_\star} \\
&\leq C(f_0, w) \mathrm{TV}(w) \|\mathbf{U}^k - \mathbf{U}^{k-1}\|_{L^2,T_\star} + C(f_0, R, w) \|f^k - f^{k-1}\|_{L^2,T_\star}.
\end{aligned}$$

For the final term, one notices that

$$\begin{aligned}
&\left\| \frac{\varrho^k \boldsymbol{u}^k \int_{\mathbf{R}^3} \boldsymbol{u}^k \cdot \boldsymbol{\nabla}_x \langle f^k \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \,\mathrm{d}v} - \frac{\varrho^{k-1} \boldsymbol{u}^{k-1} \int_{\mathbf{R}^3} \boldsymbol{u}^{k-1} \cdot \boldsymbol{\nabla}_x \langle f^{k-1} \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^{k-1} \rangle \,\mathrm{d}v} \right\|_{L^2,T_\star} \\
\leq &\|\varrho^k \boldsymbol{u}^k - \varrho^{k-1} \boldsymbol{u}^{k-1}\|_{L^2,T_\star} \left\| \frac{\int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \,\mathrm{d}v} \right\|_{L^\infty,T_\star} \|\boldsymbol{u}^k\|_{L^\infty,T_\star} \\
&+ \|\varrho^{k-1} \boldsymbol{u}^{k-1}\|_{L^\infty} \left\| \frac{1}{1 - 2^4 \|w\|_{L^1} V_M(0)^3 f_0} \right\|_{L^\infty} \\
&\times \max(\|\boldsymbol{u}^k\|_{L^\infty,T_\star}, \|\boldsymbol{u}^{k-1}\|_{L^\infty,T_\star}) \left\| \int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f^k - f^{k-1} \rangle \,\mathrm{d}v \right\|_{L^2,T_\star}.
\end{aligned}$$

Moreover, because that for all k, $\mathbf{U}^k \in \Omega_2$ which is relatively compact in $]0, +\infty[\times \mathbf{R}^3$, the quantity ϱ^k is bounded from below for all k by a constant that depends only on Ω_2. So that one has, for all $k \geq 2$

$$\|\boldsymbol{u}^k\|_{L^\infty} = \left\| \frac{\varrho^k \boldsymbol{u}^k}{\varrho^k} \right\|_{L^\infty} \leq C(\Omega_2) \|\varrho^k \boldsymbol{u}^k\|_{L^\infty}.$$

So

$$\begin{aligned}
&\left\| \frac{\varrho^k \boldsymbol{u}^k \int_{\mathbf{R}^3} \boldsymbol{u}^k \cdot \boldsymbol{\nabla}_x \langle f^k \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^k \rangle \,\mathrm{d}v} - \frac{\varrho^{k-1} \boldsymbol{u}^{k-1} \int_{\mathbf{R}^3} \boldsymbol{u}^{k-1} \cdot \boldsymbol{\nabla}_x \langle f^{k-1} \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f^{k-1} \rangle \,\mathrm{d}v} \right\|_{L^2,T_\star} \\
&\leq C(f_0, w, \Omega_2) \mathrm{TV}(w) \|\mathbf{U}^k - \mathbf{U}^{k-1}\|_{L^2,T_\star} + C(f_0, R, w, \Omega_2) \mathrm{TV}(w) \|f^k - f^{k-1}\|_{L^2,T_\star}.
\end{aligned}$$

In the end

$$\begin{aligned}
&\|b(\mathbf{U}^k, f^k) - b(\mathbf{U}^{k-1}, f^{k-1})\|_{L^2,T_\star} \\
&\leq C(f_0, w, \Omega_2) \mathrm{TV}(w) \|\mathbf{U}^k - \mathbf{U}^{k-1}\|_{L^2,T_\star} + C(f_0, R, w, \Omega_2) \mathrm{TV}(w) \|f^k - f^{k-1}\|_{L^2,T_\star}.
\end{aligned}$$

The function $f^k - f^{k-1}$ verifies

$$\partial_t (f^k - f^{k-1}) + \boldsymbol{v} \cdot \boldsymbol{\nabla}_x (f^k - f^{k-1}) - \langle \boldsymbol{\nabla} p^{k-1} \rangle \cdot \boldsymbol{\nabla}_v (f^k - f^{k-1}) = (\langle \boldsymbol{\nabla} p^{k-1} \rangle - \langle \boldsymbol{\nabla} p^{k-2} \rangle) \cdot \boldsymbol{\nabla}_v f^{k-1}.$$

Moreover $f^k(\boldsymbol{x}, \boldsymbol{v}, 0) - f^{k-1}(\boldsymbol{x}, \boldsymbol{v}, 0) = 0$, so

$$(f^k - f^{k-1})(\boldsymbol{x}, \boldsymbol{v}, t) = \int_0^t B(\mathbf{X}^{k-1}(\tau; \boldsymbol{x}, \boldsymbol{v}, t), \mathbf{V}^{k-1}(\tau; \boldsymbol{x}, \boldsymbol{v}, t), \tau) \,\mathrm{d}\tau$$

with

$$B = (\langle \boldsymbol{\nabla} p^{k-1} \rangle - \langle \boldsymbol{\nabla} p^{k-2} \rangle) \cdot \boldsymbol{\nabla}_v f^{k-1}.$$

One writes

$$\|f^k - f^{k-1}\|_{L^2}(t) \leq \int_0^t \|B\|_{L^2}(\tau) \,\mathrm{d}\tau,$$

then one has the following for B in the L^2-norm

$$\begin{aligned}\|B\|_{L^2} &\leq \|\boldsymbol{\nabla}_v f^{k-1}\|_{L^\infty}\|\langle\boldsymbol{\nabla} p^{k-1}\rangle - \langle\boldsymbol{\nabla} p^{k-2}\rangle\|_{L^2}\\ &\leq \|\boldsymbol{\nabla}_v f^{k-1}\|_{L^\infty}\mathrm{TV}(w)\|(\varrho^{k-1})^\gamma - (\varrho^{k-2})^\gamma\|_{L^2}.\end{aligned}$$

One needs to bound $\boldsymbol{\nabla}_v f^k$ in L^∞. The function $\boldsymbol{\nabla}_v f^k$ verifies

$$\partial_t(\boldsymbol{\nabla}_v f^k) + \boldsymbol{v}\cdot\boldsymbol{\nabla}_x\boldsymbol{\nabla}_v f^k - \langle\boldsymbol{\nabla}_x p^{k-1}\rangle\cdot\boldsymbol{\nabla}_v\boldsymbol{\nabla}_v f^k = -\boldsymbol{\nabla}_x f,$$

where $\cdot$ is understood as a matrix-vector product. Therefore, the characteristic method yields the formula

$$\begin{aligned}\boldsymbol{\nabla}_v f^k(\boldsymbol{x},\boldsymbol{v},t) &= \boldsymbol{\nabla}_v f_0(\mathbf{X}^{k-1}(0;\boldsymbol{x},\boldsymbol{v},t),\mathbf{V}^{k-1}(0;\boldsymbol{x},\boldsymbol{v},t))\\ &\quad - \int_0^t \boldsymbol{\nabla}_x f^k(\mathbf{X}^{k-1}(\tau;\boldsymbol{x},\boldsymbol{v},t),\mathbf{V}^{k-1}(\tau;\boldsymbol{x},\boldsymbol{v},t),\tau)\,\mathrm{d}\tau,\end{aligned}$$

which yields, for $t\in[0,T_\star[$

$$\|\boldsymbol{\nabla}_v f^k\|_{L^\infty}(t) \leq \|\boldsymbol{\nabla}_v f_0\|_{L^\infty} + \int_0^t \|\boldsymbol{\nabla}_x f^k\|_{L^\infty}(\tau)\,\mathrm{d}\tau.$$

Similarly, the function $\boldsymbol{\nabla}_x f^k$ verifies

$$\partial_t(\boldsymbol{\nabla}_x f^k) + \boldsymbol{v}\cdot\boldsymbol{\nabla}_x\boldsymbol{\nabla}_x f^k - \langle\boldsymbol{\nabla}_x p^{k-1}\rangle\cdot\boldsymbol{\nabla}_v\boldsymbol{\nabla}_x f^k = \langle\boldsymbol{\nabla}_x\boldsymbol{\nabla}_x p^{k-1}\rangle\cdot\boldsymbol{\nabla}_v f^k.$$

Again, applying the method of characteristic

$$\begin{aligned}&\boldsymbol{\nabla}_x f^k(\boldsymbol{x},\boldsymbol{v},t)\\ &= \boldsymbol{\nabla}_x f_0(\mathbf{X}^{k-1}(0;\boldsymbol{x},\boldsymbol{v},t),\mathbf{V}^{k-1}(0;\boldsymbol{x},\boldsymbol{v},t))\\ &\quad + \int_0^t \left\langle\boldsymbol{\nabla}_x\boldsymbol{\nabla}_x p^{k-1}(\mathbf{X}^{k-1}(\tau;\boldsymbol{x},\boldsymbol{v},t),\tau)\right\rangle\boldsymbol{\nabla}_v f^k(\mathbf{X}^{k-1}(\tau;\boldsymbol{x},\boldsymbol{v},t),\mathbf{V}^{k-1}(\tau;\boldsymbol{x},\boldsymbol{v},t),\tau)\,\mathrm{d}\tau\end{aligned}$$

and

$$\begin{aligned}\|\boldsymbol{\nabla}_x f^k\|_{L^\infty}(t) &\leq \|\boldsymbol{\nabla}_x f_0\|_{L^\infty} + \int_0^t \|\langle\boldsymbol{\nabla}_x\boldsymbol{\nabla}_x p^{k-1}\rangle\|_{L^\infty}(\tau)\|\boldsymbol{\nabla}_v f^k\|_{L^\infty}(\tau)\,\mathrm{d}\tau\\ &\leq \|\boldsymbol{\nabla}_x f_0\|_{L^\infty} + C(s,\gamma,R,\Omega_2,U_0\mathrm{TV}(w))\int_0^t \|\boldsymbol{\nabla}_v f^k\|_{L^\infty}(\tau)\,\mathrm{d}\tau.\end{aligned}$$

It is a consequence of Gronwall's lemma that for $t\in[0,T_\star[$

$$\|\boldsymbol{\nabla}_x f^k\|_{L^\infty}(t) + \|\boldsymbol{\nabla}_v f^k\|_{L^\infty}(t) \leq C(s,R,f_0,\Omega_2,\gamma,\mathrm{TV}(w)).$$

Then, applying the mean value theorem to $z\mapsto z^\gamma$ and using the fact that Ω_2 is relatively compact in Ω yields

$$\begin{aligned}\|B\|_{L^2}(t) &\leq \|\boldsymbol{\nabla}_v f^{k-1}\|_{L^\infty}(t)\mathrm{TV}(w)\sup_z\|\gamma z^{\gamma-1}\|_{L^\infty}\|\varrho^{k-1}-\varrho^{k-2}\|_{L^2}(t)\\ &\leq C(s,R,f_0,\Omega_2,\gamma,\mathrm{TV}(w))\mathrm{TV}(w)\|\mathbf{U}^{k-1}-\mathbf{U}^{k-2}\|_{L^2,T_\star}\end{aligned}$$

and

$$\|f^k - f^{k-1}\|_{L^2}(t) \leq tC(s,R,f_0,\Omega_2,\gamma,\mathrm{TV}(w))\mathrm{TV}(w)\|\mathbf{U}^{k-1}-\mathbf{U}^{k-2}\|_{L^2}(t).$$

Finally, using the inequality $T_\star\leq 1$,

$$\begin{aligned}&\|b(\mathbf{U}^k,f^k) - b(\mathbf{U}^{k-1},f^{k-1})\|_{L^2,T_\star}\\ &\leq C(f_0,w,\Omega_2)\mathrm{TV}(w)\|\mathbf{U}^k-\mathbf{U}^{k-1}\|_{L^2,T_\star} + C(s,R,f_0,\Omega_2,\gamma,\mathrm{TV}(w))\mathrm{TV}(w)\|\mathbf{U}^{k-1}-\mathbf{U}^{k-2}\|_{L^2,T_\star}.\end{aligned}$$

So that one obtains for all $t\in[0,T_\star[$

$$\|\mathbf{U}^{k-1}-\mathbf{U}^k\|_{L^2}(t) \leq C(R,f_0,\mathbf{U}_0,\Omega_2,w,\mathrm{TV}(w),\gamma)T_\star\left(\|\mathbf{U}^k-\mathbf{U}^{k-1}\|_{L^2,T_\star} + \|\mathbf{U}^{k-1}-\mathbf{U}^{k-2}\|_{L^2,T_\star}\right).$$

This inequality is also valid if one replaces $T_\star$ by $T_{\star\star}\in]0,T_\star[$. In particular, one can chose $T_{\star\star}$ such that

$$C(R,f_0,\mathbf{U}_0,\Omega_2,w,\mathrm{TV}(w),\gamma)T_{\star\star} < \frac{1}{4}.$$

This concludes the proof. □

4.4 Proof of Theorem 5

The proof of Theorem 5 will be divided into three parts. First we prove the existence in the case of smooth compactly supported initial data. Then we treat the uniqueness in this case. Finally we explain how to modify the proof to obtain the result for general initial data.

Proof of Theorem 5.
1) Existence for smooth compactly supported initial data. From Proposition 15, one has

$$\sum_{k\in\mathbf{N}} \|\mathbf{U}^{k+1}-\mathbf{U}^k\|_{L^2,T_\star\star} < +\infty.$$

Then the sequence $(\mathbf{U}^k)_k$ is a Cauchy sequence and converges in $L^\infty(0,T_\star\star;L^2(\mathbf{R}^3))$ to $\mathbf{U}$. Since $\mathbf{U}^k$ is smooth for all k, we have $\mathbf{U}\in\mathscr{C}(0,T_\star\star;L^2(\mathbf{R}^3))$. In the same way,

$$\sum_{k\in\mathbf{N}} \|f^{k+1}-f^k\|_{L^2,T_\star\star} < +\infty.$$

Then the sequence $(f^k)_k$ is a Cauchy sequence and converges in $L^\infty(0,T_\star\star;L^2(\mathbf{R}^3\times\mathbf{R}^3))$ to f. Since f^k is smooth for all k, we have $f\in\mathscr{C}(0,T_\star\star;L^2(\mathbf{R}^3\times\mathbf{R}^3))$.

Thanks to the inequality (4.10), one has $\|\mathbf{U}-\mathbf{U}_0\|_{H^s,T_\star\star}\le R$ and $\mathbf{U}\in L^\infty(0,T_\star\star;H^s(\mathbf{R}^3))$. In particular, thanks to the Sobolev embedding $H^s(\mathbf{R}^3)\hookrightarrow\mathscr{C}^1(\mathbf{R}^3)$ (remember that $s>3/2+1$), one has that $\mathbf{U}\in L^\infty(0,T_\star\star;\mathscr{C}^1(\mathbf{R}^3))$ takes its values in Ω_2. For the same reasons, $f\in L^\infty(0,T_\star\star;H^s(\mathbf{R}^3\times\mathbf{R}^3))$.

We emphasize however, that we do not yet have $\mathbf{U}^k\overset{\mathscr{C}(0,T_\star\star;H^s(\mathbf{R}^3))}{\longrightarrow}\mathbf{U}$. To get the convergence in $\mathscr{C}(0,T_\star\star;\mathscr{C}^1(\mathbf{R}^3))$, we use an interpolation argument. By Gagliardo-Nirenberg interpolation inequality, we have for all $s'\in]3/2+1,s[$:

$$\begin{aligned}
\|\mathbf{U}^{k+1}-\mathbf{U}^k\|_{H^{s'},T_\star\star} &\le C(s)\|\mathbf{U}^{k+1}-\mathbf{U}^k\|_{H^s,T_\star\star}^{s'/s}\|\mathbf{U}^{k+1}-\mathbf{U}^k\|_{L^2,T_\star\star}^{1-s'/s}\\
&\le C(s)\left(\|\mathbf{U}^{k+1}-\mathbf{U}_0\|_{H^s,T_\star\star}^{s'/s}+\|\mathbf{U}^{k}-\mathbf{U}_0\|_{H^s,T_\star\star}^{s'/s}\right)\|\mathbf{U}^{k+1}-\mathbf{U}^k\|_{L^2,T_\star\star}^{1-s'/s}\\
&\le C(s)2R^{s'/s}\|\mathbf{U}^{k+1}-\mathbf{U}^k\|_{L^2,T_\star\star}^{1-s'/s}.
\end{aligned}$$

So $(\mathbf{U}^k)_k$ converges to $\mathbf{U}$ in $\mathscr{C}(0,T_\star\star;H^{s'}(\mathbf{R}^3))$ for $s'\in]3/2+1,s[$. By the Sobolev embedding $H^{s'}(\mathbf{R}^3)\hookrightarrow\mathscr{C}^1(\mathbf{R}^3)$, we conclude that

$$\mathbf{U}^k\overset{\mathscr{C}(0,T_\star\star;\mathscr{C}^1(\mathbf{R}^3))}{\longrightarrow}\mathbf{U}.$$

Then $\left(\partial_{x_i}\mathbf{U}^k\right)_k$ converges to $\partial_{x_i}\mathbf{U}$ in $\mathscr{C}([0,T_\star\star]\times\mathbf{R}^3)$. The same is true for $\left(\sum_i A_i(\mathbf{U}^{k+1})\partial_{x_i}\mathbf{U}^k\right)_k$ because the A_i are smooth. We now turn to the convergence of

$$b(\mathbf{U}^k,f^k)=\begin{pmatrix}\dfrac{\varrho^k\int_{\mathbf{R}^3}(\boldsymbol{u}^k-\boldsymbol{v})\cdot\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v}{1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v}\\[2ex]\dfrac{\varrho^k\boldsymbol{u}^k\int_{\mathbf{R}^3}(\boldsymbol{u}^k-\boldsymbol{v})\cdot\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v}{1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v}\end{pmatrix}.$$

The support of the function f^k in v is uniformly contained in a compact set, so the sequences $\left(\int_{\mathbf{R}^3}f^k\,\mathrm{d}v\right)_k$ and $\left(\int_{\mathbf{R}^3}\boldsymbol{v}f^k\right)_k$ are bounded in $L^\infty(0,T_\star\star;H^s(\mathbf{R}^3))$, therefore they converge up to a subsequence in $L^\infty(0,T_\star\star;L^2(\mathbf{R}^3))$ towards $\int_{\mathbf{R}^3}f\,\mathrm{d}v$ and $\int_{\mathbf{R}^3}\boldsymbol{v}f\,\mathrm{d}v$. Then, using the inequality

$$\left|\boldsymbol{\nabla}_x\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v\right|\le \mathrm{TV}(w)\left\|\int_{\mathbf{R}^3}f^k\,\mathrm{d}v\right\|_{L^\infty},$$

we get that the sequences $\left(\int_{\mathbf{R}^3}\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v\right)_k$ and $\left(\int_{\mathbf{R}^3}\boldsymbol{v}\cdot\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v\right)_k$ converge in $L^\infty(0,T_\star\star;L^2(\mathbf{R}^3))$. Using again the Gagliardo-Nirenberg interpolation inequality, we obtain that $\left(\int_{\mathbf{R}^3}\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v\right)_k$ and $\left(\int_{\mathbf{R}^3}\boldsymbol{v}\cdot\boldsymbol{\nabla}_x\langle f^k\rangle\,\mathrm{d}v\right)_k$ converge in $L^\infty(0,T_\star\star;H^{s'}(\mathbf{R}^3))$ towards $\int_{\mathbf{R}^3}\boldsymbol{\nabla}_x\langle f\rangle\,\mathrm{d}v$ and

$\int_{\mathbf{R}^3} \boldsymbol{v}\cdot\boldsymbol{\nabla}_x\langle f\rangle\,\mathrm{d}v$ and therefore in $\mathscr{C}([0,T_{\star\star}]\times\mathbf{R}^3)$. Furthermore, since $\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v$ converges to $\int_{\mathbf{R}^3}\langle f\rangle\,\mathrm{d}v$ in $\mathscr{C}([0,T_{\star\star}]\times\mathbf{R}^3)$ (again using an interpolation argument) and using the inequality (4.13) we obtain that

$$\left\|\frac{1}{1-\int_{\mathbf{R}^3}\langle f\rangle\,\mathrm{d}v}\right\|_{L^\infty}<+\infty$$

and $\left(\frac{1}{1-\int_{\mathbf{R}^3}\langle f^k\rangle\,\mathrm{d}v}\right)_k$ converges to $\frac{1}{1-\int_{\mathbf{R}^3}\langle f\rangle\,\mathrm{d}v}$ in $\mathscr{C}([0,T_{\star\star}]\times\mathbf{R}^3)$. Finally $b(\mathbf{U}^k,f^k)$ converges towards $b(\mathbf{U},f)$ in $\mathscr{C}([0,T_{\star\star}]\times\mathbf{R}^3)$. We now prove that

$$\partial_t\mathbf{U}^k\overset{\mathscr{C}(0,T_{\star\star};\mathscr{C}^1(\mathbf{R}^3))}{\longrightarrow}\partial_t\mathbf{U}.$$

Passing to the limit in $\mathcal{D}'$ in the equation

$$\partial_t\mathbf{U}^k=-\sum_{i=1}^3 A_i(\mathbf{U}^k)\partial_{x_i}\mathbf{U}^{k+1}+b(\mathbf{U}^k,f^k)$$

we obtain that $\mathbf{U}$ solves

$$\partial_t\mathbf{U}=-\sum_{i=1}^3 A_i(\mathbf{U})\partial_{x_i}\mathbf{U}+b(\mathbf{U},f) \tag{4.36}$$

in $\mathcal{D}'$. Since the right-hand-side defines a function in $\mathscr{C}([0,T_{\star\star}]\times\mathbf{R}^3)$, we have $\partial_t\mathbf{U}\in\mathscr{C}([0,T_{\star\star}]\times\mathbf{R}^3)$ and $\mathbf{U}$ solves (4.36) in the classical sense.

We now turn to the Vlasov equation. We pass to the limit in $\mathcal{D}'$ in

$$\partial_t f^{k+1}+\boldsymbol{v}\cdot\boldsymbol{\nabla}_x f^{k+1}-\boldsymbol{\nabla}_x\langle(\varrho^k)^\gamma\rangle\cdot\boldsymbol{\nabla}_v f^{k+1}=0.$$

We already know that $(\varrho^k)_k$ converges to ϱ in $\mathscr{C}([0,T_{\star\star}]\times\mathbf{R}^3)$, so does $\left((\varrho^k)^\gamma\right)_k$ towards ϱ^γ, finally we have $\boldsymbol{\nabla}_x\langle(\varrho^k)^\gamma\rangle$ converges towards $\boldsymbol{\nabla}_x\langle\varrho^\gamma\rangle$ in $\mathscr{C}([0,T_{\star\star}]\times\mathbf{R}^3)$. So we can pass to the limit in the sense of distribution and f is a solution in $\mathcal{D}'$ of

$$\partial_t f+\boldsymbol{v}\cdot\boldsymbol{\nabla}_x f-\boldsymbol{\nabla}_x\langle\varrho^\gamma\rangle\cdot\boldsymbol{\nabla}_v f=0. \tag{4.37}$$

Now, using the fact that the characteristic curves of f are $\mathscr{C}^1$ because $\mathbf{U}$ is $\mathscr{C}^1$, we obtain that $f\in\mathscr{C}^1_c([0,T_{\star\star}[\times\mathbf{R}^3\times\mathbf{R}^3)$ and f is a solution of (4.37) in the classical sense. This concludes the proof in the case where the initials data are smooth with compact support.

2) Uniqueness. For the uniqueness, we consider two solutions $(\varrho^1,\varrho^1\boldsymbol{u}^1,f^1)$ and $(\varrho^2,\varrho^2\boldsymbol{u}^2,f^2)$ which are smooth in $[0,T]$. Then, using the same algebra as in Proposition 15, one has for some $T_{\star\star}\in]0,T[$,

$$\|\mathbf{U}^1-\mathbf{U}^2\|_{L^2,T_{\star\star}}\leq\frac{1}{4}\|\mathbf{U}^1-\mathbf{U}^2\|_{L^2,T_{\star\star}}+\frac{1}{4}\|\mathbf{U}^1-\mathbf{U}^2\|_{L^2,T_{\star\star}}$$

$$\|f^1-f^2\|_{L^2,T_{\star\star}}\leq C(\Omega_2,s,R,f_0)\|\mathbf{U}^1-\mathbf{U}^2\|_{L^2,T_{\star\star}}.$$

As a consequence, $\mathbf{U}^1=\mathbf{U}^2$ and $f^1=f^2$.

3) General initial data. We turn to the case where $\varrho_0-1\in H^s(\mathbf{R}^3)$, $\boldsymbol{u}\in H^s(\mathbf{R}^3)$ and $f_0\in\mathscr{C}^1_c(\mathbf{R}^3\times\mathbf{R}^3)\cap H^s(\mathbf{R}^3\times\mathbf{R}^3)$. We introduce an approximation of the identity φ_{ε_k} with $\varepsilon_k=2^{-k}\varepsilon_0$, with ε_0 small enough. Then we define $\mathbf{U}_0^k=\varphi_{\varepsilon_k}\star\mathbf{U}_0$ and $f_0^k=(\varphi_{\varepsilon_k}\otimes\varphi_{\varepsilon_k})\star f_0$. With such a choice, one has

$$\sum_{k\in\mathbf{N}}\|\mathbf{U}_0^{k+1}-\mathbf{U}_0^k\|_{L^2}<+\infty,\quad\sum_{k\in\mathbf{N}}\|f_0^{k+1}-f_0^k\|_{L^2}<+\infty.$$

The solution is then obtained by the same approximation scheme but we replace the initial data by $\mathbf{U}_0^{k+1}$ and f_0^{k+1}. The proof is then similar, expect for two small modifications. Firstly, one has to prove estimates on $\mathbf{U}^k-\mathbf{U}_0^0$ instead of $\mathbf{U}^k-\mathbf{U}_0$, this is where we need to chose ε_0 small enough. Secondly, the estimates (4.34)-(4.35) are replaced by

$$\|\mathbf{U}^{k+1}-\mathbf{U}^k\|_{L^2,T_{\star\star}}\leq\frac{1}{4}\|\mathbf{U}^k-\mathbf{U}^{k-1}\|_{L^2,T_{\star\star}}+\frac{1}{4}\|\mathbf{U}^{k-1}-\mathbf{U}^{k-2}\|_{L^2,T_{\star\star}}+C\|\mathbf{U}_0^{k+1}-\mathbf{U}_0^k\|_{L^2}$$

and

$$\|f^k-f^{k-1}\|_{L^2,T_{\star\star}}\leq C(f_0,\Omega_2,\mathrm{TV}(w),\gamma)\|\mathbf{U}^{k-1}-\mathbf{U}^{k-2}\|_{L^2,T_{\star\star}}+\|f_0^k-f_0^{k-1}\|_{L^2}.$$

□

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