Refined Rellich boundary inequalities for the derivatives of a harmonic function

Siddhant Agrawal, Thomas Alazard

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Abstract. The classical Rellich inequalities imply that the L^2-norms of the normal and tangential derivatives of a harmonic function are equivalent. In this note, we prove several refined inequalities, which make sense even if the domain is not Lipschitz. For two-dimensional domains, we obtain a sharp L^p-estimate for $1 < p \leq 2$ by using a Riemann mapping and interpolation argument.

1. Introduction

Let $d \geq 1$ and denote by $\mathbf{T}^d$ a d-dimensional torus. Given two real valued functions $h \in W^{1,\infty}(\mathbf{T}^d)$ and $\zeta \in H^{1/2}(\mathbf{T}^d)$, it is classical that there exists a unique variational solution ϕ to the following problem

$$\begin{cases} \Delta_{x,y}\phi = 0 \quad \text{in } \Omega = \{(x,y) \in \mathbf{T}^d \times \mathbf{R}\,;\, y < h(x)\}, \\ \phi(x,h(x)) = \zeta(x), \\ \lim\limits_{y \to -\infty} \sup\limits_{x \in \mathbf{T}^d} |\nabla_{x,y}\phi(x,y)| = 0. \end{cases} \tag{1.1}$$

We are interested by quantitative estimates for the trace of the normal derivative $\partial_N \phi$ on the boundary $\partial\Omega$, where the normal unit vector $N \in \mathbf{R}^{d+1}$ is defined by

$$N = \frac{1}{\sqrt{1+|\nabla h|^2}} \begin{pmatrix} -\nabla h \\ 1 \end{pmatrix}. \tag{1.2}$$

By construction, the variational solution is such that $\nabla_{x,y}\phi \in L^2(\Omega)$, so it is not obvious that one can consider the trace $\partial_N \phi|_{\partial\Omega}$. However, since $\Delta_{x,y}\phi = 0$, one can express the normal derivative in terms of the tangential derivatives and prove that $\sqrt{1+|\nabla h|^2}\partial_N\phi|_{\partial\Omega}$ is well-defined and belongs to $H^{-\frac{1}{2}}(\mathbf{T}^d)$.

In this paper, we are chiefly interested by another estimate, known as Rellich inequality, which plays a key role in the study of boundary value problems in Lipschitz domains. This inequality shows the equivalence between the L^2-norm of the tangential derivatives and the L^2-norm of the normal derivative (see [**7, 18, 8, 10, 4, 5, 15**]): there is constant $C > 0$, depending only on d and $\|\nabla h\|_{L^\infty}$ such that

$$\frac{1}{C}\int_{\partial\Omega} (\partial_N\phi)^2 \,\mathrm{d}\sigma \leq \int_{\mathbf{T}^d} |\nabla\zeta|^2 \,\mathrm{d}x \leq C\int_{\partial\Omega} (\partial_N\phi)^2 \,\mathrm{d}\sigma, \tag{1.3}$$

where $\mathrm{d}\sigma = \sqrt{1+|\nabla h|^2}\,\mathrm{d}x$ is the surface measure on $\partial\Omega$.

We are going to prove several estimates which clarifies the dependance on the domain. Hereafter, given a function $f = f(x, y)$ we use $f|_{y=h}$ as a short notation for the function $x \mapsto f(x, h(x))$.

THEOREM 1.1. *Let $d \geq 1$. For all $h \in C^1(\mathbf{T}^d)$ and for all $\zeta \in H^1(\mathbf{T}^d)$, the traces of the derivatives $(\nabla_{x,y}\phi)|_{y=h}$ are well-defined and belong to $L^2(\mathbf{T}^d)$. In addition, there holds*

$$\int_{\mathbf{T}^d} (\partial_N \phi)(x, h(x))^2 \, \mathrm{d}x \leq 40 \int_{\mathbf{T}^d} (1 + |\nabla h(x)|^2)^2 |\nabla \zeta(x)|^2 \, \mathrm{d}x, \tag{1.4}$$

and

$$\int_{\mathbf{T}^d} |(\nabla_{x,y}\phi)(x, h(x))|^2 \, \mathrm{d}x \leq 41 \int_{\mathbf{T}^d} (1 + |\nabla h(x)|^2)^2 |\nabla \zeta(x)|^2 \, \mathrm{d}x. \tag{1.5}$$

REMARK 1.2. (*i*) Compared to (1.3), the rather surprising feature of (1.4) and (1.5) is the fact that the right-hand sides can be estimated even if h is *not* a Lipschitz function. For example, we can write that

$$\int_{\mathbf{T}^d} (1 + |\nabla h|^2)^2 |\nabla \zeta|^2 \, \mathrm{d}x \leq 2 \, \|\nabla \zeta\|_{L^2}^2 + 2 \, \|\nabla \zeta\|_{L^\infty}^2 \, \|\nabla h\|_{L^4}^4 .$$

In the same vein, if $\zeta = h$, we obtain from (1.5) that

$$\|(\nabla_{x,y}\phi)|_{y=h}\|_{L^2} \leq 7\Big(\|\nabla h\|_{L^2} + \|\nabla h\|_{L^6}^3 \Big).$$

Notice that the case $\zeta = h$ is interesting for the Hele-Shaw equation (see [**6, 3, 14, 9**]).

(*ii*) One could extend the estimates (1.4) and (1.5) to the cases where h belongs to $W^{1,\infty}(\mathbf{T}^d)$ instead of $C^1(\mathbf{T}^d)$ by using the arguments in Nečas [**13**, Chapter 5], Brown [**4**] or McLean [**11**, Theorem 4.24].

Consider now the Dirichlet-to-Neumann operator $G(h)$ defined by

$$G(h)\zeta = \big(\partial_y \phi - \nabla h \cdot \nabla \phi\big) \, \big|_{y=h} = \sqrt{1 + |\nabla h|^2} \partial_N \phi \, \big|_{y=h}.$$

From the previous inequalities, we immediately obtain the following

COROLLARY 1.3. *Let $d \geq 1$. For all $h \in C^1(\mathbf{T}^d)$ and for all $\zeta \in H^1(\mathbf{T}^d)$, there holds $G(h)\zeta \in L^2(\mathbf{T}^d)$ together with the estimate*

$$\int_{\mathbf{T}^d} \frac{(G(h)\zeta)^2}{1 + |\nabla h|^2} \, \mathrm{d}x \leq 40 \int_{\mathbf{T}^d} (1 + |\nabla h|^2)^2 |\nabla \zeta|^2 \, \mathrm{d}x. \tag{1.6}$$

REMARK 1.4. In particular,

$$\int_{\mathbf{T}^d} (G(h)\zeta)^2 \, \mathrm{d}x \leq 40 \, (1 + \|\nabla h\|_{L^\infty}^2)^3 \int_{\mathbf{T}^d} |\nabla \zeta|^2 \, \mathrm{d}x. \tag{1.7}$$

As said above, compared to (1.7), the estimate (1.6) is quite surprising in that the right-hand side of the former might be finite even if ∇h is unbounded. In this case, we do not control the L^2-norm of $G(h)\zeta$ but only a weaker quantity.

In dimension $d = 1$, we can extend the above result in two directions. The first one is a stronger version of estimate (1.6) where the right-hand side does not involve h at all, while the second version generalizes to L^p estimates. If $d = 1$, we will denote simply by f_x the derivative $\partial_x f$.

THEOREM 1.5. *For all $h \in C^1(\mathbf{T})$ and for all $\zeta \in H^1(\mathbf{T})$ we have*

$$\int_{\mathbf{T}} \frac{(G(h)\zeta)^2}{1+h_x^2}\,\mathrm{d}x \leq 4\int_{\mathbf{T}} \zeta_x^2\,\mathrm{d}x, \tag{1.8}$$

and

$$\int_{\mathbf{T}} \frac{\zeta_x^2}{1+h_x^2}\,\mathrm{d}x \leq 4\int_{\mathbf{T}} (G(h)\zeta)^2\,\mathrm{d}x. \tag{1.9}$$

As a corollary, one can get a surprising geometric estimate.

COROLLARY 1.6. *Denote by κ the curvature of $\partial\Omega$ and by θ the angle the interface $\partial\Omega$ makes with the x-axis, defined by*

$$\kappa = \partial_x\left(\frac{h_x}{\sqrt{1+h_x^2}}\right) \quad , \quad \theta = \arctan(h_x).$$

Then, there holds

$$\|G(h)\kappa\|_{H^{-1}} \leq 2\,\|\theta_x\|_{L^2}\,.$$

PROOF. Notice that $\kappa = h_{xx}/(1+h_x^2)^{3/2}$. Since $G(h)$ is self-adjoint for the L^2-scalar product, for any function $\varphi \in H^1(\mathbf{T})$, we deduce from (1.8) that

$$\begin{aligned}\int_{\mathbf{T}} \varphi G(h)\kappa\,\mathrm{d}x = \int_{\mathbf{T}} \kappa G(h)\varphi\,\mathrm{d}x &\leq \left(\int_{\mathbf{T}} (1+h_x^2)\kappa^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathbf{T}} \frac{(G(h)\varphi)^2}{1+h_x^2}\,\mathrm{d}x\right)^{\frac{1}{2}}\\ &\leq 2\left(\int_{\mathbf{T}} \frac{h_{xx}^2}{(1+h_x^2)^2}\,\mathrm{d}x\right)^{\frac{1}{2}}\|\varphi_x\|_{L^2} = 2\left(\int_{\mathbf{T}} \theta_x^2\,\mathrm{d}x\right)^{\frac{1}{2}}\|\varphi_x\|_{L^2}\,,\end{aligned}$$

and the result follows. □

Our final result extends (1.8) and (1.9) to the L^p-setting. In dimension $d = 1$, the normal and tangential unit vectors are defined by

$$N = \frac{1}{\sqrt{1+h_x^2}}\begin{pmatrix}-h_x\\1\end{pmatrix} \quad , \quad T = \frac{1}{\sqrt{1+h_x^2}}\begin{pmatrix}1\\h_x\end{pmatrix}, \tag{1.10}$$

and the arc length measure on $\partial\Omega$ is $\mathrm{d}\sigma = \sqrt{1+h_x^2}\,\mathrm{d}x$.

THEOREM 1.7. *For all $1 < p \leq 2$, there exists a constant $C_p > 0$ such that, for all $h \in C^1(\mathbf{T})$ and for all $\zeta \in H^1(\mathbf{T})$, if ϕ is defined by (1.1), then the following two inequalities hold:*

$$\int_{\partial\Omega} \frac{|\partial_N\phi|^p}{(1+h_x^2)^{\frac{p-1}{2}}}\,\mathrm{d}\sigma \leq C_p\int_{\partial\Omega} |\partial_T\phi|^p(1+h_x^2)^{\frac{p-1}{2}}\,\mathrm{d}\sigma, \tag{1.11}$$

and

$$\int_{\partial\Omega} \frac{|\partial_T\phi|^p}{(1+h_x^2)^{\frac{p-1}{2}}}\,\mathrm{d}\sigma \leq C_p\int_{\partial\Omega} |\partial_N\phi|^p(1+h_x^2)^{\frac{p-1}{2}}\,\mathrm{d}\sigma. \tag{1.12}$$

REMARK 1.8. The estimates do not extend to $p = 1$, as can be seen by assuming that $h = 0$. Indeed, if $h = 0$ and $p = 1$, then

$$\int_{\partial\Omega} \frac{|\partial_N\phi|^p}{(1+h_x^2)^{\frac{p-1}{2}}}\,\mathrm{d}\sigma = \int_{\mathbf{T}} |\mathbb{H}\partial_x\zeta|\,\mathrm{d}x \quad , \quad \int_{\partial\Omega} |\partial_T\phi|^p(1+h_x^2)^{\frac{p-1}{2}}\,\mathrm{d}\sigma = \int_{\mathbf{T}} |\partial_x\zeta|\,\mathrm{d}x,$$

where $\mathbb{H}$ is the periodic Hilbert transform (see (3.4)) and hence we see that the estimates do not hold for $p = 1$, since $\mathbb{H}$ is not bounded on $L^1(\mathbf{T})$.

2. Refined Rellich estimates

In this section we prove Theorem 1.1 and Theorem 1.5.

2.1. Proof of Theorem 1.1

The proof is decomposed into four steps. We start by proving the quantitative estimates (1.4) and (1.5) under the additional assumption that the functions h and ζ are smooth, so that all calculations will be easily justified. Then, we will consider in the fourth step the general case by an approximation argument.

Step 1: Reduction to an estimate for $G(h)$. Assume that h and ζ belong to $C^\infty(\mathbf{T}^d)$. Then (1.1) is a classical elliptic boundary problem, which admits a unique solution $\phi \in C^\infty(\overline{\Omega})$ such that $\nabla_{x,y}\phi \in L^2(\Omega)$.

By definition of the Dirichlet-to-Neumann operator $G(h)$, there holds

$$G(h)\zeta = \big(\partial_y\phi - \nabla h\cdot\nabla\phi\big)\,\big|_{\,y=h} = \sqrt{1+|\nabla h|^2}\partial_N\phi\,\big|_{\,y=h}. \tag{2.1}$$

(Let us recall that ∇ denotes the gradient with respect to $x \in \mathbf{T}^d$.) We see that (1.4) is equivalent to

$$\int_{\mathbf{T}^d} \frac{(G(h)\zeta)^2}{1+|\nabla h|^2}\,\mathrm{d}x \le 40\int_{\mathbf{T}^d}(1+|\nabla h|^2)^2|\nabla\zeta|^2\,\mathrm{d}x. \tag{2.2}$$

Let us show that (1.5) also follows from (2.2). To do so, it is convenient to introduce the notations

$$\mathcal{V} = (\nabla\phi)|_{y=h}, \qquad \mathcal{B} = (\partial_y\phi)|_{y=h}.$$

Using (2.1), we have

$$G(h)\zeta = \mathcal{B} - \nabla h\cdot\mathcal{V}. \tag{2.3}$$

On the other hand, it follows from the chain rule that

$$\nabla\zeta = \nabla(\phi|_{y=h}) = \mathcal{V} + \mathcal{B}\nabla h.$$

By combining the previous identities, we see that $\mathcal{B}$ and $\mathcal{V}$ can be defined only in terms of h and ζ by means of the formulas

$$\mathcal{B} = \frac{G(h)\zeta + \nabla\zeta\cdot\nabla h}{1+|\nabla h|^2}, \qquad \mathcal{V} = \nabla\zeta - \mathcal{B}\nabla h. \tag{2.4}$$

It follows that

$$\begin{aligned} |(\nabla_{x,y}\phi)|_{y=h}|^2 &= ((\partial_y\phi)|_{y=h})^2 + \big|(\nabla\phi)|_{y=h}\big|^2 \\ &= \mathcal{B}^2 + |\mathcal{V}|^2 \\ &= \frac{(G(h)\zeta)^2}{1+|\nabla h|^2} + |\nabla\zeta|^2 - \frac{(\nabla h\cdot\nabla\zeta)^2}{1+|\nabla h|^2}. \end{aligned} \tag{2.5}$$

This shows that (1.5) will follow directly from (2.2).

Therefore, both estimates of the theorem will be proved if we show (2.2).

Step 2: An intermediate Rellich type estimate.

To prove (2.2), we begin by establishing a Rellich type estimate which allows to estimate the L^2-norm of $G(h)\zeta$ in terms of $\mathcal{V} = (\nabla\phi)_{|y=h}$.

Proposition 2.1. *There holds*

$$\int_{\mathbf{T}^d}(G(h)\zeta)^2\,\mathrm{d}x \le \int_{\mathbf{T}^d}(1+|\nabla h|^2)|\mathcal{V}|^2\,\mathrm{d}x. \tag{2.6}$$

PROOF. By squaring the identity (2.3) we get

$$(G(h)\zeta)^2 = \mathcal{B}^2 - 2\mathcal{B}\nabla h \cdot \mathcal{V} + (\nabla h \cdot \mathcal{V})^2.$$

Since $(\nabla h \cdot \mathcal{V})^2 \leq |\nabla h|^2 |\mathcal{V}|^2$, this implies

$$(G(h)\zeta)^2 \leq \mathcal{B}^2 - |\mathcal{V}|^2 - 2\mathcal{B}\nabla h \cdot \mathcal{V} + (1 + |\nabla h|^2)\mathcal{V}^2. \tag{2.7}$$

So,

$$\int_{\mathbf{T}^d} (G(h)\zeta)^2 \, \mathrm{d}x \leq \int_{\mathbf{T}^d} (1 + |\nabla h|^2) |\mathcal{V}|^2 \, \mathrm{d}x + R,$$

where

$$R = \int_{\mathbf{T}^d} \Big(\mathcal{B}^2 - |\mathcal{V}|^2 - 2\mathcal{B}\nabla h \cdot \mathcal{V} \Big) \, \mathrm{d}x. \tag{2.8}$$

We see that, to obtain (2.6), it is sufficient to prove that $R = 0$. It is interesting to observe that the latter result is a consequence of the classical Rellich identity. It can be proven by multiplying the equation $\Delta_{x,y}\phi = 0$ by $\partial_y \phi$ and then integrating by parts. We will give an alternative proof, following [**1**], which consists in observing that R is the flux associated to a vector field. Indeed,

$$R = \int_{\partial\Omega} X \cdot N \, \mathrm{d}\sigma$$

where $X \colon \Omega \to \mathbf{R}^{d+1}$ is given by

$$X = (2(\partial_y \phi)\nabla\phi; (\partial_y \phi)^2 - |\nabla\phi|^2).$$

Then the key observation is that this vector field satisfies $\mathrm{div}_{x,y} X = 0$ since

$$\partial_y \big((\partial_y \phi)^2 - |\nabla\phi|^2 \big) + 2 \, \mathrm{div} \, \big((\partial_y \phi)\nabla\phi \big) = 2(\partial_y \phi)\Delta_{x,y}\phi = 0,$$

as can be verified by an elementary computation. Now, we see that the cancellation $R = 0$ comes from the Stokes' theorem. To rigorously justify this point, we truncate Ω in order to work in a smooth bounded domain. Given a parameter $\beta > 0$, set

$$\Omega_\beta = \{(x, y) \in \mathbf{T}^d \times \mathbf{R} \, ; -\beta < y < h(x)\}.$$

An application of the divergence theorem in Ω_β gives that

$$0 = \iint_{\Omega_\beta} \mathrm{div}_{x,y} X \, \mathrm{d}y \, \mathrm{d}x = R + \int_{\{y=-\beta\}} X \cdot n \, \mathrm{d}\sigma.$$

Recall that the potential ϕ satisfies (1.1)

$$\lim_{y \to -\infty} \sup_{x \in \mathbf{T}^d} |\nabla_{x,y}\phi(x, y)| = 0.$$

Therefore, X converges to 0 uniformly when y goes to $-\infty$. So, by sending β to $+\infty$, we obtain the expected result $R = 0$ which completes the proof of the proposition. □

Step 3: Proof of (2.2).

Introduce the function $\varepsilon \colon \mathbf{T}^d \to [0, +\infty)$ defined by

$$\varepsilon(x) := \frac{1}{8(1 + |\nabla h(x)|^2)}.$$

Introduce also the functions

$$\lambda(x) = 1 + \varepsilon(x) \quad , \quad \Lambda(x) = 1 + \frac{1}{\varepsilon(x)}.$$

Directly from the identity (2.4) for $\mathcal{B}$ and the elementary inequality

$$|a+b|^2 \le \lambda(x)|a|^2 + \Lambda(x)|b|^2 \qquad (\text{for any } (a,b,x) \in \mathbf{R}^d \times \mathbf{R}^d \times \mathbf{T}^d),$$

we have the pointwise inequalities

$$\begin{aligned}
|\nabla\zeta - \mathcal{B}\nabla h|^2 &\le \Lambda\,|\nabla\zeta|^2 + \lambda\mathcal{B}^2\,|\nabla h|^2 \\
&\le \Lambda\,|\nabla\zeta|^2 + \lambda\frac{|\nabla h|^2}{(1+|\nabla h|^2)^2}(G(h)\zeta + \nabla\zeta\cdot\nabla h)^2 \\
&\le \Lambda\,|\nabla\zeta|^2 + \lambda^2\frac{|\nabla h|^2}{(1+|\nabla h|^2)^2}(G(h)\zeta)^2 + \lambda\Lambda\frac{|\nabla h|^4}{(1+|\nabla h|^2)^2}\,|\nabla\zeta|^2\,.
\end{aligned}$$

Hence, it follows from (2.6) that we have an estimate of the form:

$$\int_{\mathbf{T}^d} \gamma(G(h)\zeta)^2\,\mathrm{d}x \le \int_{\mathbf{T}^d} \delta\,|\nabla\zeta|^2\,\mathrm{d}x,$$

where

$$\gamma := 1 - \lambda^2\frac{|\nabla h|^2}{1+|\nabla h|^2}, \qquad \delta := (1+|\nabla h|^2)\left(\Lambda + \lambda\Lambda\frac{|\nabla h|^4}{(1+|\nabla h|^2)^2}\right).$$

Then, we notice that

$$\delta \le (1+|\nabla h|^2)(\Lambda+\lambda\Lambda) \le (1+|\nabla h|^2)\left(4+\frac{2}{\varepsilon}\right) \le 20(1+|\nabla h|^2)^2.$$

On the other hand, we have

$$\gamma = 1 - \lambda^2\frac{|\nabla h|^2}{1+|\nabla h|^2} = \frac{1-(2\varepsilon+\varepsilon^2)\,|\nabla h|^2}{1+|\nabla h|^2} \ge \frac{1}{2}\cdot\frac{1}{1+|\nabla h|^2},$$

where we used the pointwise inequality $(2\varepsilon+\varepsilon^2)\,|\nabla h|^2 \le 3\varepsilon|\nabla h|^2 \le 1/2$. It follows that

$$\frac{1}{2}\int_{\mathbf{T}^d}\frac{(G(h)\zeta)^2}{1+|\nabla h|^2}\,\mathrm{d}x \le \int_{\mathbf{T}^d} 20(1+|\nabla h|^2)^2|\nabla\zeta|^2\,\mathrm{d}x.$$

This implies the wanted result (2.2) and hence concludes the proof of the theorem.

Step 4: The general case. We now assume only that $h \in C^1(\mathbf{T}^d)$ and $\zeta \in H^1(\mathbf{T}^d)$.

Introduce two sequences of smooth functions $\{h_n\}_{n\in\mathbf{N}}$ and $\{\zeta_n\}_{n\in\mathbf{N}}$ such that $\|h_n - h\|_{W^{1,\infty}}$ and $\|\zeta_n - \zeta\|_{H^1}$ converge to 0 when n goes to $+\infty$. Then it follows from variational arguments (see [**2**, Section 3]) that $G(h_n)\zeta_n$ converges to $G(h)\zeta$ in $H^{-1/2}(\mathbf{T}^d)$.

On the other hand, it follows from (2.2) applied with (h,ζ) replaced by (h_n,ζ_n) that the sequence $\{G(h_n)\zeta_n\}_{n\in\mathbf{N}}$ is bounded in $L^2(\mathbf{T}^d)$, indeed

$$\int_{\mathbf{T}^d}(G(h_n)\zeta_n)^2\,\mathrm{d}x \le 40\,(1+\|\nabla h_n\|_{L^\infty}^2)^3\int_{\mathbf{T}^d}|\nabla\zeta_n|^2\,\mathrm{d}x.$$

It follows that there exists a subsequence $\{G(h_{n'})\zeta_{n'}\}$ converging weakly in $L^2(\mathbf{T}^d)$. Therefore, by uniqueness of the limit in the space of distributions, we see that $G(h)\zeta$ belongs to $L^2(\mathbf{T}^d)$. Given (2.5), this in turn implies that $(\partial_N\phi)|_{y=h}$ and $(\nabla_{x,y}\phi)|_{y=h}$ are well defined and belong to $L^2(\mathbf{T}^d)$.

It remains to prove the estimates. Notice that $(G(h_n)\zeta_n)/\sqrt{1+|\nabla h_n|^2}$ converges weakly in L^2 to $G(h)\zeta/\sqrt{1+|\nabla h|^2}$. Therefore, the L^2-norm of the latter is bounded by

$$\liminf \left\|(G(h_n)\zeta_n)/\sqrt{1+|\nabla h_n|^2}\right\|_{L^2}.$$

This establishes the estimate (2.2). Using again (2.5), this in turn implies the estimate (1.5) which completes the proof.

2.2. Proof of Theorem 1.5

We will do the computations for smooth h and ζ. We can then extend the estimates to $h \in C^1(\mathbf{T})$ and $\zeta \in H^1(\mathbf{T})$ by the same logic as in the proof of Theorem 1.1.

We know from the proof of Proposition 2.1 that the quantity R defined in (2.8) is zero, i.e.

$$\int_{\mathbf{T}} \left(\mathcal{B}^2 - \mathcal{V}^2 - 2h_x\mathcal{B}\mathcal{V}\right) \mathrm{d}x = 0.$$

Now as we are in one dimension, the equations (2.4) simplify

$$\begin{aligned}
\mathcal{B} &= \frac{h_x}{1+h_x^2}\zeta_x + \frac{1}{1+h_x^2}G(h)\zeta,\\
\mathcal{V} &= \frac{1}{1+h_x^2}\zeta_x - \frac{h_x}{1+h_x^2}G(h)\zeta.
\end{aligned}$$

Substituting it in the above formula and simplifying we get

$$\int_{\mathbf{T}} \left\{ -\frac{\zeta_x^2}{1+h_x^2} + \frac{(G(h)\zeta)^2}{1+h_x^2} + \frac{2h_x\zeta_x G(h)\zeta}{1+h_x^2} \right\} \mathrm{d}x = 0. \tag{2.9}$$

Now using Young's inequality $ab \le \frac{a^2}{2} + \frac{b^2}{2}$ gives

$$\begin{aligned}
\int_{\mathbf{T}} \frac{(G(h)\zeta)^2}{1+h_x^2}\,\mathrm{d}x &\le \int_{\mathbf{T}} \frac{\zeta_x^2}{1+h_x^2}\,\mathrm{d}x + \frac{1}{2}\int_{\mathbf{T}} \frac{(G(h)\zeta)^2}{1+h_x^2}\,\mathrm{d}x + \frac{1}{2}\int_{\mathbf{T}} \frac{4h_x^2\zeta_x^2}{1+h_x^2}\,\mathrm{d}x\\
&\le \frac{1}{2}\int_{\mathbf{T}} \frac{(G(h)\zeta)^2}{1+h_x^2}\,\mathrm{d}x + \int_{\mathbf{T}} \frac{(1+2h_x^2)\zeta_x^2}{1+h_x^2}\,\mathrm{d}x\\
&\le \frac{1}{2}\int_{\mathbf{T}} \frac{(G(h)\zeta)^2}{1+h_x^2}\,\mathrm{d}x + 2\int_{\mathbf{T}} |\zeta_x|^2\,\mathrm{d}x.
\end{aligned}$$

The estimate (1.8) now follows. The proof of (1.9) follows the same logic.

3. Riemann mapping and Rellich estimates

In this section, we prove Theorem 1.7.

We will do the computations for smooth h and ζ. We can then extend the estimates to $h \in C^1(\mathbf{T})$ and $\zeta \in H^1(\mathbf{T})$ by the same logic as in the proof of Theorem 1.1.

Note that the estimate (1.8), which reads

$$\int_{\mathbf{T}} \frac{(G(h)\zeta)^2}{1+h_x^2}\,\mathrm{d}x \le 4\int_{\mathbf{T}} \zeta_x^2\,\mathrm{d}x$$

can be rewritten as

$$\int_{\partial\Omega} \frac{(\partial_N \phi)^2}{(1+h_x^2)^{\frac{1}{2}}}\,\mathrm{d}\sigma \leq 4\int_{\partial\Omega} (\partial_T\phi)^2(1+h_x^2)^{\frac{1}{2}}\,\mathrm{d}\sigma, \tag{3.1}$$

which is the wanted estimate (1.11) for $p=2$. We will deduce that (1.11) holds for $1<p<2$ by an interpolation argument. To do so, we will exploit the existence of a Riemann mapping to reduce the problem to the study of harmonic functions in a half-space.

We first consider the 2π periodic version of Ω by considering the domain $\widetilde{\Omega} = \left\{(x,y)\in\mathbf{R}^2 \,\middle|\, \exists n\in\mathbb{Z} \text{ so that } (x-2n\pi,y)\in\Omega\right\}$. Let $P_- = \left\{(x,y)\in\mathbf{R}^2 \,\middle|\, y<0\right\}$ be the lower half plane and let $\Psi : P_- \to \widetilde{\Omega}$ be a Riemann mapping. As the boundary $\partial\widetilde{\Omega}$ is a Jordan curve, by Carathéodory's theorem the map Ψ extends continuously to a homeomorphism on the boundary. Let Z be the boundary value of Ψ and so $Z:\mathbf{R}\to\partial\widetilde{\Omega}$ is a homeomorphism. We will denote the coordinates on this $\mathbf{R}$ by α so we will use quantities like $Z(\alpha), \partial_\alpha$ etc.

Now as Ψ is a Riemann map from $P_- \to \widetilde{\Omega}$, we see that $z\mapsto \Psi(k(z-c))$ for $k>0$ and $c\in\mathbf{R}$ are all the Riemann maps from $P_-\to\widetilde{\Omega}$. Therefore we let $Z(0)=(0,h(0))$ and $Z(2\pi)=(2\pi,h(2\pi))=Z(0)+2\pi$. Now consider $\Psi_1 : P_-\to\widetilde{\Omega}$ given by $\Psi_1(z)=\Psi(z+2\pi)-2\pi$. Clearly Ψ_1 is a Riemann map with $\Psi_1(0)=\Psi(0)$ and so there exists $k>0$ so that $\Psi_1(z)=\Psi(z+2\pi)-2\pi=\Psi(kz)$. If $k\neq 1$, then we get a contradiction by plugging in $z=\frac{2\pi}{k-1}$ in this equation. Hence $\Psi_1=\Psi$ and therefore $\Psi(z+2\pi)=\Psi(z)+2\pi$.

As Ψ is a Riemann map, we see that $\Psi_z\neq 0$ in P_- and as P_- is simply connected, we see that $\log(\Psi_z)$ is well defined if we fix the value of $\log(\Psi_z(-i))$ (the choice one makes is immaterial). Now the smoothness of the domain $\widetilde{\Omega}$ implies that $\log(\Psi_z)$ extends continuous to $\overline{P}_-$ (see Theorem 3.5 in [**16**]. The proof given there is for the unit disc but the same proof also works for the half plane). In particular this means that there exists $c_1,c_2>0$ such that $c_1\leq|Z_\alpha(\alpha)|\leq c_2$ for all $\alpha\in\mathbf{R}$. Now we define $g:\mathbf{R}\to\mathbf{R}$ by

$$g = \mathrm{Im}(\log(Z_\alpha)). \tag{3.2}$$

Notice that g is 2π periodic.

As the slope of the interface is bounded, we can define $\theta(x)=\arctan(h_x(x))$, where θ is now the angle the interface makes with the x-axis. Hence we see that

$$e^{i\theta(\mathrm{Re}(Z(\alpha)))} = e^{ig(\alpha)}.$$

Therefore $1+h_x(\mathrm{Re}(Z(\alpha)))^2 = 1+\tan(g(\alpha))^2$. We also note that $\tan(g)$ is a bounded function.

Now let $\widetilde{\phi}$: $P_-\to\mathbf{R}$ be the pullback of ϕ, given by

$$\widetilde{\phi}(z)=\phi(\Psi(z)),$$

with its boundary value being $\widetilde{\zeta}$, i.e. $\widetilde{\phi}(\alpha)=\widetilde{\zeta}(\alpha)=\zeta(Z(\alpha))$. As Ψ is conformal, we see that $\widetilde{\phi}$ is also a harmonic function and on the boundary we have

$$(\partial_T\phi)(Z(\alpha)) = \frac{1}{|Z_\alpha|}(\partial_\alpha\widetilde{\phi})(\alpha) = \frac{1}{|Z_\alpha|}(\partial_\alpha\widetilde{\zeta})(\alpha).$$

If n is the unit outward normal of P_-, then we also see that

$$(\partial_N \phi)(Z(\alpha)) = \frac{1}{|Z_\alpha|}(\partial_n \widetilde{\phi})(\alpha) = \frac{1}{|Z_\alpha|}(|D|\widetilde{\zeta})(\alpha)$$

where $|D| = \sqrt{-\Delta}$. We can also see that the pullback of the measure $\mathrm{d}\sigma$ on $\partial\Omega$ is the measure $|Z_\alpha|\,\mathrm{d}\alpha$ on $\mathbf{T}$. Hence (3.1) is equivalent to

$$\int_{\mathbf{T}} \frac{\big||D|\widetilde{\zeta}\big|^2}{|Z_\alpha|(1+\tan^2(g))^{\frac{1}{2}}}\,\mathrm{d}\alpha \le 4\int_{\mathbf{T}} \frac{\big|\partial_\alpha \widetilde{\zeta}\big|^2(1+\tan^2(g))^{\frac{1}{2}}}{|Z_\alpha|}\,\mathrm{d}\alpha. \tag{3.3}$$

If $\mathcal{F}(f)$ is the Fourier transform of f, then the periodic Hilbert transform $\mathbb{H} : L^2(\mathbf{T}) \to L^2(\mathbf{T})$ is given by the relation

$$\mathcal{F}(\mathbb{H}f)(n) = -isgn(n)\mathcal{F}(f)(n) \qquad \text{for } n \in \mathbb{Z}, \tag{3.4}$$

where $sgn(n) = 1$ if $n > 0$, $sgn(n) = -1$ if $n < 0$ and $sgn(0) = 0$. Hence

$$||D|\widetilde{\zeta}| = |\mathbb{H}\partial_\alpha \widetilde{\zeta}|.$$

Therefore we see that (3.3) is equivalent to the statement that the map $\mathbb{H} : L^2(\mathbf{T}, v\,\mathrm{d}\alpha) \to L^2(\mathbf{T}, u\,\mathrm{d}\alpha)$ is bounded, where the weights u and v are defined by

$$u = \frac{(1+\tan^2(g))^{-\frac{1}{2}}}{|Z_\alpha|} \quad \text{and} \quad v = \frac{(1+\tan^2(g))^{\frac{1}{2}}}{|Z_\alpha|}.$$

Note that there exists constants $c_3, c_4 > 0$ such that $c_3 \le u, v \le c_4$ on all of $\mathbf{T}$ due to the properties of $\tan(g)$ and Z_α mentioned above.

Now we know that $\mathbb{H} : L^1(\mathbf{T}, \mathrm{d}\alpha) \to L^{1,\infty}(\mathbf{T}, \mathrm{d}\alpha)$ is bounded, where we recall that $f \in L^{1,\infty}$ if we have $\|f\|_{1,\infty} = \sup_{t>0} t|\{x \in \mathbf{T} \,|\, |f(x)| > t\}| < \infty$ (see Corollary 3.16 in [**12**]). Hence by real interpolation of operators with change of measures (namely, by using Theorem 2.9 from [**17**] with $T = \mathbb{H}$, $p_0 = q_0 = 1$, $p_1 = q_1 = 2$, $M = N = \mathbf{T}$, $d\mu_0 = d\nu_0 = d\alpha$, $d\mu_1 = v\,d\alpha$ and $d\nu_1 = u\,d\alpha$) we see that, for all $1 < p < 2$,

$$\mathbb{H} : L^p\big(\mathbf{T}, v^{p-1}\,\mathrm{d}\alpha\big) \to L^p\big(\mathbf{T}, u^{p-1}\,\mathrm{d}\alpha\big) \quad \text{ is bounded.} \tag{3.5}$$

Therefore for $1 < p < 2$, there exists a constant $C_p > 0$ such that

$$\int_{\mathbf{T}} \frac{\big||D|\widetilde{\zeta}\big|^p}{|Z_\alpha|^{p-1}(1+\tan^2(g))^{\frac{p-1}{2}}}\,\mathrm{d}\alpha \le C_p \int_{\mathbf{T}} \frac{\big|\partial_\alpha \widetilde{\zeta}\big|^p(1+\tan^2(g))^{\frac{p-1}{2}}}{|Z_\alpha|^{p-1}}\,\mathrm{d}\alpha,$$

which is equivalent to

$$\int_{\partial\Omega} \frac{|\partial_N \phi|^p}{(1+h_x^2)^{\frac{p-1}{2}}}\,\mathrm{d}\sigma \le C_p \int_{\partial\Omega} |\partial_T \phi|^p (1+h_x^2)^{\frac{p-1}{2}}\,\mathrm{d}\sigma,$$

proving the first statement. The other statement also follows directly as (3.5) applied on the function $|D|\widetilde{\zeta}$ instead gets us

$$\int_{\mathbf{T}} \frac{\big|\partial_\alpha \widetilde{\zeta}\big|^p}{|Z_\alpha|^{p-1}(1+\tan^2(g))^{\frac{p-1}{2}}}\,\mathrm{d}\alpha \le C_p \int_{\mathbf{T}} \frac{\big||D|\widetilde{\zeta}\big|^p(1+\tan^2(g))^{\frac{p-1}{2}}}{|Z_\alpha|^{p-1}}\,\mathrm{d}\alpha,$$

which is equivalent to

$$\int_{\partial\Omega} \frac{|\partial_T \phi|^p}{(1+h_x^2)^{\frac{p-1}{2}}}\,\mathrm{d}\sigma \le C_p \int_{\partial\Omega} |\partial_N \phi|^p (1+h_x^2)^{\frac{p-1}{2}}\,\mathrm{d}\sigma.$$

This completes the proof.

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References

1. Thomas Alazard, *Stabilization of the water-wave equations with surface tension*, Ann. PDE **3** (2017), no. 2, Paper No. 17, 41.
2. Thomas Alazard, Nicolas Burq, and Claude Zuily, *On the Cauchy problem for gravity water waves*, Invent. Math. **198** (2014), no. 1, 71–163.
3. Thomas Alazard, Nicolas Meunier, and Didier Smets, *Lyapunov functions, identities and the Cauchy problem for the Hele-Shaw equation*, Comm. Math. Phys. **377** (2020), no. 2, 1421–1459.
4. Russell Brown, *The mixed problem for Laplace's equation in a class of Lipschitz domains*, Comm. Partial Differential Equations **19** (1994), no. 7-8, 1217–1233.
5. Simon N. Chandler-Wilde, Ivan G. Graham, Stephen Langdon, and Euan A. Spence, *Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering*, Acta Numer. **21** (2012), 89–305.
6. Héctor A. Chang-Lara, Nestor Guillen, and Russell W. Schwab, *Some free boundary problems recast as nonlocal parabolic equations*, Nonlinear Anal. **189** (2019), 11538, 60.
7. Björn E. J. Dahlberg, *Estimates of harmonic measure*, Arch. Rational Mech. Anal. **65** (1977), no. 3, 275–288.
8. Björn E. J. Dahlberg and Carlos E. Kenig, *Hardy spaces and the Neumann problem in L^p for Laplace's equation in Lipschitz domains*, Ann. of Math. (2) **125** (1987), no. 3, 437–465.
9. Hongjie Dong, Francisco Gancedo, and Huy Q. Nguyen, *Global well-posedness for the one-phase muskat problem*, Preprint (2020), arXiv:2103.02656.
10. Wen Jie Gao, *Layer potentials and boundary value problems for elliptic systems in Lipschitz domains*, J. Funct. Anal. **95** (1991), no. 2, 377–399.
11. William McLean, *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, Cambridge, 2000.
12. Camil Muscalu and Wilhelm Schlag, *Classical and multilinear harmonic analysis. Vol. I*, Cambridge Studies in Advanced Mathematics, vol. 137, Cambridge University Press, Cambridge, 2013.
13. Jindřich Nečas, *Direct methods in the theory of elliptic equations*, Springer Monographs in Mathematics, Springer, Heidelberg, 2012, Translated from the 1967 French original by Gerard Tronel and Alois Kufner, Editorial coordination and preface by Šárka Nečasová and a contribution by Christian G. Simader.
14. Huy Q. Nguyen and Benoît Pausader, *A paradifferential approach for well-posedness of the Muskat problem*, Arch. Ration. Mech. Anal. **237** (2020), no. 1, 35–100.
15. Katharine A. Ott and Russell M. Brown, *The mixed problem for the Laplacian in Lipschitz domains*, Potential Anal. **38** (2013), no. 4, 1333–1364.
16. Ch. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 299, Springer-Verlag, Berlin, 1992.
17. Elias M. Stein and Guido Weiss, *Interpolation of operators with change of measures*, Trans. Amer. Math. Soc. **87** (1958), 159–172.
18. Gregory Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal. **59** (1984), no. 3, 572–611.

Siddhant Agrawal
Instituto de Ciencias Matemáticas (ICMAT),
C/ Nicolás Cabrera, 13-15 (Campus Cantoblanco)
28049 Madrid
Spain

Thomas Alazard
Université Paris-Saclay, ENS Paris-Saclay, CNRS,
Centre Borelli UMR9010, avenue des Sciences,
F-91190 Gif-sur-Yvette
France