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Functional convex order for the scaled McKean-Vlasov processes

Yating Liu *

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Abstract

We establish the functional convex order results for two scaled McKean-Vlasov processes $X = (X_t)_{t \in [0, T]}$ and $Y = (Y_t)_{t \in [0, T]}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ by

$$\begin{cases} dX_t = b(t, X_t, \mu_t)dt + \sigma(t, X_t, \mu_t)dB_t, & X_0 \in L^p(\mathbb{P}), \\ dY_t = b(t, Y_t, \nu_t)dt + \theta(t, Y_t, \nu_t)dB_t, & Y_0 \in L^p(\mathbb{P}), \end{cases}$$

where $p \geq 2$, for every $t \in [0, T]$, μ_t, ν_t denote the probability distribution of X_t, Y_t respectively and the drift coefficient $b(t, x, \mu)$ is affine in x (scaled). If we make the convexity and monotony assumption (only) on σ and if $\sigma \preceq \theta$ with respect to the partial matrix order, the convex order for the initial random variable $X_0 \preceq_{cv} Y_0$ can be propagated to the whole path of process X and Y . That is, if we consider a convex functional F defined on the path space with polynomial growth, we have $\mathbb{E}F(X) \leq \mathbb{E}F(Y)$; for a convex functional G defined on the product space involving the path space and its marginal distribution space, we have $\mathbb{E}G(X, (\mu_t)_{t \in [0, T]}) \leq \mathbb{E}G(Y, (\nu_t)_{t \in [0, T]})$ under appropriate conditions. The symmetric setting is also valid, that is, if $\theta \preceq \sigma$ and $Y_0 \leq X_0$ with respect to the convex order, then $\mathbb{E}F(Y) \leq \mathbb{E}F(X)$ and $\mathbb{E}G(Y, (\nu_t)_{t \in [0, T]}) \leq \mathbb{E}G(X, (\mu_t)_{t \in [0, T]})$. The proof is based on several forward and backward dynamic programming principles and the convergence of the Euler scheme of the McKean-Vlasov equation.

Keywords: Convergence rate of the Euler scheme, Diffusion process, Functional convex order, McKean-Vlasov equation

1 Introduction

Let $U, V : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ be two integrable random variables. We say that U is dominated by V for the convex order - denoted by $U \preceq_{cv} V$ - if for any convex function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$, such that $\mathbb{E}\varphi(U)$ and $\mathbb{E}\varphi(V)$ are well defined in $(-\infty, +\infty]$,

$$\mathbb{E}\varphi(U) \leq \mathbb{E}\varphi(V). \quad (1.1)$$

Note that if U is integrable, then $\mathbb{E}\varphi(U)$ is always well-defined in $(-\infty, +\infty]$ by considering $\varphi^\pm(U)$ with $\varphi^\pm(x) := \max(\pm\varphi(x), 0)$ since φ^- is upper bounded by an affine function. For $p \in [1, +\infty)$, let $\mathcal{P}_p(\mathbb{R}^d)$ denote the set of probability distributions on \mathbb{R}^d with p -th finite moment.

Hence, the above definition of the convex order has the obvious equivalent version for two probability distributions $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$: we say that the distribution μ is dominated by ν for the convex order - denoted by $\mu \preceq_{cv} \nu$ - if, for every convex function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$, $\int_{\mathbb{R}^d} \varphi(\xi)\mu(d\xi) \leq \int_{\mathbb{R}^d} \varphi(\xi)\nu(d\xi)$.

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Also note that, as U and V have a finite first moment, then

$$U \preceq_{cv} V \implies \mathbb{E} U = \mathbb{E} V \quad (1.2)$$

by simply considering the two linear functions $\varphi(x) = \pm x$. In fact, the connection between the distributions of U and V , say μ and ν , is much stronger than this necessary condition or the elementary domination inequality $\text{var}(U) \leq \text{var}(V)$ when $U, V \in L^2(\mathbb{P})$. Indeed, a special case of Kellerer's theorem ([Kel72, HR12]) shows that $\mu \preceq_{cv} \nu$ if and only if there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$ and a couple (\tilde{U}, \tilde{V}) such that $U \sim \mu$, $\tilde{V} \sim \nu$ and $\tilde{\mathbb{E}}(\tilde{V} | \tilde{U}) = \tilde{U}$. Similarly Strassen's theorem ([Str65]) establishes the equivalence with the existence of a martingale Markovian kernel ⁽¹⁾ $K(x, dy)$ such that $\nu(dy) = \int_{\mathbb{R}^d} K(x, dy)\mu(dx)$ and $\int_{\mathbb{R}^d} yK(x, dy) = x$ for every $x \in \mathbb{R}^d$.

The functional convex order for two Brownian martingale diffusion processes having a form $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$ has been studied in [Pag16], [ACJ19a] and [JP19] (among other references). Such functional convex order results have applications in quantitative finance to establish robust bounds for various option prices including those written on path-dependent payoffs. In this paper, we extend such functional convex order results to the McKean-Vlasov equation, which was originally introduced in [McK67] as a stochastic model naturally associated to a class of non-linear PDEs. Nowadays, it refers to the whole family of stochastic differential equations whose coefficients not only depend on the position of the process X_t at time t but also on its probability distribution $\mathbb{P}_{X_t} = \mathbb{P} \circ X_t^{-1}$. Thanks to this specific structure, the McKean-Vlasov equations have become widely used to model phenomena in Statistical Physics (see e.g. [MA01]), in mathematical biology (see e.g. [BFFT12] and [BFT15]), but also in social sciences and in quantitative finance often motivated by the development of the Mean-Field Games (see e.g. [CD13], [CD15], [CL18] and [CD18a]). Moreover, results in this paper can be used to establish the convex bounds and the convex partitions, which may be extended to applications within the framework of Mean-Field Games in a future work (see further Section 3). For example, the modeling of the gas storage (see e.g. [Gas19]), or the stochastic control of McKean-Vlasov type when the control appears in the volatility coefficient (see e.g. [BDL11], [Yon13]).

We consider now a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual condition and an (\mathcal{F}_t) -standard Brownian motion $(B_t)_{t \geq 0}$ defined on this space and valued in \mathbb{R}^q . Let $\mathbb{M}_{d \times q}(\mathbb{R})$ denote the set of matrices with d rows and q columns equipped with the operator norm $\|\cdot\|$ defined by $\|A\| := \sup_{|z|_q \leq 1} |Az|$, where $|\cdot|$ denotes the canonical Euclidean norm on \mathbb{R}^d generated by the canonical inner product $\langle \cdot, \cdot \rangle$. Let $X = (X_t)_{t \in [0, T]}$ and $Y = (Y_t)_{t \in [0, T]}$ be two d -dimensional McKean-Vlasov processes, respective solutions to

$$dX_t = b(t, X_t, \mu_t) dt + \sigma(t, X_t, \mu_t) dB_t, \quad X_0 \in L^p(\mathbb{P}), \quad (1.3)$$

$$dY_t = b(t, Y_t, \nu_t) dt + \theta(t, Y_t, \nu_t) dB_t, \quad Y_0 \in L^p(\mathbb{P}), \quad (1.4)$$

where $p \geq 2$, $b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d) \rightarrow \mathbb{R}^d$, $\sigma, \theta : [0, T] \times \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d) \rightarrow \mathbb{M}_{d \times q}$ and, for every $t \in [0, T]$, μ_t and ν_t respectively denote the probability distribution of X_t and Y_t .

In this paper, we will only consider the *scaled* McKean-Vlasov processes, which means the drift function $b : (t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d) \mapsto b(t, x, \mu) \in \mathbb{R}^d$ is affine in x (see further Assumption II-(1)).

We define a *partial order* between two matrices in $\mathbb{M}_{d \times q}$ as follows:

$$\forall A, B \in \mathbb{M}_{d \times q}, \quad A \preceq B \quad \text{if} \quad BB^\top - AA^\top \text{ is a positive semi-definite matrix,} \quad (1.5)$$

where A^\top stands for the transpose of the matrix A . Moreover, we introduce the L^p -Wasserstein distance \mathcal{W}_p on $\mathcal{P}_p(\mathbb{R}^d)$ defined for any $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$ by

$$\mathcal{W}_p(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} d(x, y)^p \pi(dx, dy) \right)^{\frac{1}{p}}$$

¹For every $x \in \mathbb{R}^d$, $K(x, dy)$ is a probability measure on $(\mathbb{R}^d, \text{Bor}(\mathbb{R}^d))$ and the function $x \mapsto K(x, A)$ is Borel for every fixed Borel set A of \mathbb{R}^d .

$$= \inf \left\{ \left[\mathbb{E} |X - Y|^p \right]^{\frac{1}{p}}, X, Y : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}^d, \text{Bor}(\mathbb{R}^d)) \text{ with } \mathbb{P}_X = \mu, \mathbb{P}_Y = \nu \right\}, \quad (1.6)$$

where in the first line of (1.6), $\Pi(\mu, \nu)$ denotes the set of all probability measures on $(\mathbb{R}^d \times \mathbb{R}^d, \text{Bor}(\mathbb{R}^d)^{\otimes 2})$ with marginals μ and ν .

Throughout this paper, we make the following two assumptions on the coefficients b , σ and the starting values X_0 and Y_0 . Both depend on an integrability exponent $p \in [2, +\infty)$.

Assumption I. Assume $\|X_0\|_p \vee \|Y_0\|_p < +\infty$. The functions b , σ and θ are ρ -Hölder continuous in t and Lipschitz continuous in x and in μ in the following sense: for every $s, t \in [0, T]$ with $s \leq t$, there exist a positive constant \tilde{L} such that

$$\begin{aligned} \forall x \in \mathbb{R}^d, \forall \mu \in \mathcal{P}_p(\mathbb{R}^d), \\ |b(t, x, \mu) - b(s, x, \mu)| \vee \|\sigma(t, x, \mu) - \sigma(s, x, \mu)\| \vee \|\theta(t, x, \mu) - \theta(s, x, \mu)\| \\ \leq \tilde{L}(1 + |x| + \mathcal{W}_p(\mu, \delta_0))(t - s)^\rho, \end{aligned} \quad (1.7)$$

where δ_0 denotes the Dirac mass at 0; for every $t \in [0, T]$, there exists $L > 0$ such that

$$\begin{aligned} \forall x, y \in \mathbb{R}^d, \forall \mu, \nu \in \mathcal{P}_p(\mathbb{R}^d), \\ |b(t, x, \mu) - b(t, y, \nu)| \vee \|\sigma(t, x, \mu) - \sigma(t, y, \nu)\| \vee \|\theta(t, x, \mu) - \theta(t, y, \nu)\| \\ \leq L(|x - y| + \mathcal{W}_p(\mu, \nu)). \end{aligned} \quad (1.8)$$

Assumption II. (1) The function b is affine in x and constant in μ w.r.t the convex order in the sense that for every $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$ with $\mu \preceq_{cv} \nu$, we have

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad b(t, x, \mu) = b(t, x, \nu). \quad (1.9)$$

(2) For every fixed $t \in \mathbb{R}_+$ and $\mu \in \mathcal{P}_p(\mathbb{R}^d)$, the function $x \mapsto \sigma(t, x, \mu)$ is convex in the sense that

$$\forall x, y \in \mathbb{R}^d, \forall \lambda \in [0, 1], \quad \sigma(t, \lambda x + (1 - \lambda)y, \mu) \preceq \lambda \sigma(t, x, \mu) + (1 - \lambda) \sigma(t, y, \mu). \quad (1.10)$$

(3) For every fixed $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the function $\mu \mapsto \sigma(t, x, \mu)$ is non-decreasing with respect to the convex order, that is,

$$\forall \mu, \nu \in \mathcal{P}_p(\mathbb{R}^d), \quad \mu \preceq_{cv} \nu \implies \sigma(t, x, \mu) \preceq \sigma(t, x, \nu). \quad (1.11)$$

(4) For every $(t, x, \mu) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d)$, we have

$$\sigma(t, x, \mu) \preceq \theta(t, x, \mu). \quad (1.12)$$

(5) $X_0 \preceq_{cv} Y_0$.

Remark 1.1. Note that if Assumption II is satisfied with some $p_0 \geq 1$ (especially when $p_0 = 1$) then it is satisfied for any $p \geq p_0$ by the restrictions of b and σ to $[0, T] \times \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d)$ since $\mathcal{P}_p(\mathbb{R}^d) \subset \mathcal{P}_{p_0}(\mathbb{R}^d)$. Idem for Assumption I, except of course for the integrability of X_0 and Y_0 , since $\mathcal{W}_{p_0} \leq \mathcal{W}_p$. More generally we will often use without specific mention that the restriction to $\mathcal{P}_p(\mathbb{R}^d)$ of a \mathcal{W}_1 -continuous (resp. Lipschitz) functional $\Phi : \mathcal{P}_1(\mathbb{R}^d) \rightarrow \mathbb{R}$ is \mathcal{W}_p -continuous (resp. Lipschitz).

Let E denote a separable Banach space equipped with the norm $|\cdot|_E$. A function $f : (E, |\cdot|_E) \rightarrow \mathbb{R}$ has an r -polynomial growth for some $r \geq 0$ if there exists a constant $C \in \mathbb{R}_+^*$ such that for every $x \in E$, $|f(x)| \leq C(1 + |x|_E^r)$. Moreover, let

$\mathcal{C}([0, T], \mathcal{P}_p(\mathbb{R}^d)) := \{(\mu_t)_{t \in [0, T]}\}$ such that the mapping $t \mapsto \mu_t$

is continuous from $[0, T]$ to $(\mathcal{P}_p(\mathbb{R}^d), \mathcal{W}_p)$ (1.13)

equipped with the distance

$$d_{\mathcal{C}}((\mu_t)_{t \in [0, T]}, (\nu_t)_{t \in [0, T]}) := \sup_{t \in [0, T]} \mathcal{W}_p(\mu_t, \nu_t) \quad (1.14)$$

be the space in which the marginal distribution of $X = (X_t)_{t \in [0, T]}$ and $Y = (Y_t)_{t \in [0, T]}$ have values. The continuity of $t \mapsto \mu_t = \mathbb{P}_{X_t}$ will be proved later in Lemma 5.2.

The main theorem of this paper is the following.

Theorem 1. *Let $p \in [2, +\infty)$. Assume I and II are in force. Let $X := (X_t)_{t \in [0, T]}$, $Y := (Y_t)_{t \in [0, T]}$ denote the solutions of the McKean-Vlasov equations (1.3) and (1.4) respectively. For every $t \in [0, T]$, let μ_t, ν_t denote the probability distributions of X_t and Y_t respectively. Then, we have*

(a) Functional convex order. *For any convex function $F : (\mathcal{C}([0, T], \mathbb{R}^d), \|\cdot\|_{\text{sup}}) \rightarrow \mathbb{R}$ with p -polynomial growth, one has*

$$\mathbb{E} F(X) \leq \mathbb{E} F(Y). \quad (1.15)$$

(b) Extended functional convex order. *For any function*

$$G : (\alpha, (\eta_t)_{t \in [0, T]}) \in \mathcal{C}([0, T], \mathbb{R}^d) \times \mathcal{C}([0, T], \mathcal{P}_1(\mathbb{R}^d)) \mapsto G(\alpha, (\eta_t)_{t \in [0, T]}) \in \mathbb{R}$$

satisfying the following conditions:

(i) G is convex in α ,

(ii) G has a p -polynomial growth in the sense that

$$\begin{aligned} & \exists C \in \mathbb{R}_+ \text{ such that } \forall (\alpha, (\eta_t)_{t \in [0, T]}) \in \mathcal{C}([0, T], \mathbb{R}^d) \times \mathcal{C}([0, T], \mathcal{P}_p(\mathbb{R}^d)), \\ & G(\alpha, (\eta_t)_{t \in [0, T]}) \leq C[1 + \|\alpha\|_{\text{sup}}^p + \sup_{t \in [0, T]} \mathcal{W}_p^p(\eta_t, \delta_0)], \end{aligned} \quad (1.16)$$

(iii) G is continuous in $(\eta_t)_{t \in [0, T]}$ with respect to the distance $d_{\mathcal{C}}$ defined in (1.14) and non-decreasing in $(\eta_t)_{t \in [0, T]}$ with respect to the convex order in the sense that

$$\begin{aligned} & \forall \alpha \in \mathcal{C}([0, T], \mathbb{R}^d), \forall (\eta_t)_{t \in [0, T]}, (\tilde{\eta}_t)_{t \in [0, T]} \in \mathcal{C}([0, T], \mathcal{P}_p(\mathbb{R}^d)) \text{ s.t. } \forall t \in [0, T], \eta_t \preceq_{cv} \tilde{\eta}_t, \\ & G(\alpha, (\eta_t)_{t \in [0, T]}) \leq G(\alpha, (\tilde{\eta}_t)_{t \in [0, T]}), \end{aligned}$$

one has

$$\mathbb{E} G(X, (\mu_t)_{t \in [0, T]}) \leq \mathbb{E} G(Y, (\nu_t)_{t \in [0, T]}). \quad (1.17)$$

The proof is postponed to Section 5 (and Section 4 for preliminary discrete time results). The symmetric case of Theorem 1 remains true, that is, if we replace Assumption II by Assumption II' where conditions (4) and (5) are replaced respectively by (4') and (5') as follows:

(4') For every $(t, x, \mu) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d)$, we have $\theta(t, x, \mu) \preceq \sigma(t, x, \mu)$.

(5') $Y_0 \preceq_{cv} X_0$,

then we have the following result, whose proof is very similar to that of Theorem 1.

Theorem 2 (Symmetric setting). *Let $p \in [2, +\infty)$. Under Assumption I and II', for every functions $F : \mathcal{C}([0, T], \mathbb{R}^d) \rightarrow \mathbb{R}$ and $G : \mathcal{C}([0, T], \mathbb{R}^d) \times \mathcal{C}([0, T], \mathcal{P}_p(\mathbb{R}^d)) \mapsto \mathbb{R}$ respectively satisfying the conditions in Theorem 1 - (a) and (b), then*

$$\mathbb{E} F(Y) \leq \mathbb{E} F(X) \quad \text{and} \quad \mathbb{E} G(Y, (\nu_t)_{t \in [0, T]}) \leq \mathbb{E} G(X, (\mu_t)_{t \in [0, T]}).$$

Theorem 1 directly implies the following results.

Corollary 1.1. *Let $X := (X_t)_{t \in [0, T]}$, $Y := (Y_t)_{t \in [0, T]}$ denote the solutions of the McKean-Vlasov equations (1.3) and (1.4) respectively. For every $t \in [0, T]$, let μ_t, ν_t denote the probability distributions of X_t and Y_t respectively. Under Assumption I and II, we have :*

(a) Marginal convex order. *For every $t \in [0, T]$, $\mu_t \preceq_{cv} \nu_t$.*

(b) Convexity with respect to the initial value. *Let $X^x = (X_t^x)_{t \in [0, T]}$ denote the McKean-Vlasov process defined by (1.3) starting with the initial value $X_0 = x$. Then for every functionals F and G respectively satisfying conditions from Theorem 1-(a) and (b), the functions*

$$x \mapsto \mathbb{E} F(X^x) \quad \text{and} \quad x \mapsto \mathbb{E} G(X^x, (\mu_t)_{t \in [0, T]}) \text{ are convex.}$$

The proof of Corollary 1.1 is postponed to Section 4. It also has an obvious version under Assumption II'.

In fact, as far as marginal convex order is concerned, it is also possible to dissociate convexity in x and monotonicity in μ that is replace Assumption II-(3) by the following assumption:

(3') *For every fixed $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the function $\theta(t, x, \cdot)$ is non-decreasing in μ with respect to the convex order in the sense that*

$$\forall \mu, \nu \in \mathcal{P}_p(\mathbb{R}^d), \mu \preceq_{cv} \nu, \quad \theta(t, x, \mu) \preceq \theta(t, x, \nu).$$

Then, we have the same result in Corollary 1.1-(a). This is the purpose of Proposition 4.3 in Section 4.

This paper is organized as follows. Section 2 contains comments on the Assumption I and II including necessary and sufficient conditions on the monotonicity with respect to the convex order in terms of the linear functional derivative. Next, in Section 3, we show two applications of Theorem 1 and 2 in the framework of the stochastic differential equation and the stochastic optimal control. The proof of the main theorem is constructed in Section 4-5. Our strategy of proof is to first establish the propagation of convex order for the marginal distribution of the Euler scheme of the McKean-Vlasov equation (see Section 4) and then rely on it to establish in a backward way the functional convex order for the whole trajectory (Section 5). To be more precise, in Section 4, we show the convex order result for $(\bar{X}_{t_m}^M)_{m=0, \dots, M}$ and $(\bar{Y}_{t_m}^M)_{m=0, \dots, M}$ defined by the Euler schemes (see further (2.1) and (2.2)). We first prove that the Euler scheme propagates the marginal convex order, namely, for every $m = 0, \dots, M$, $\bar{X}_{t_m}^M \preceq_{cv} \bar{Y}_{t_m}^M$. Then we prove the functional convex order

$$\mathbb{E} F(\bar{X}_{t_0}^M, \dots, \bar{X}_{t_M}^M) \leq \mathbb{E} F(\bar{Y}_{t_0}^M, \dots, \bar{Y}_{t_M}^M) \quad (1.18)$$

for any convex function $F : (\mathbb{R}^d)^{M+1} \rightarrow \mathbb{R}$ with p -polynomial growth, by using a backward dynamic programming principle. Next, in Section 5, we prove Theorem 1, the functional convex order result for the stochastic processes and their probability distributions based on (1.18) by applying the convergence of the Euler schemes of the McKean-Vlasov equation. At the end, in Appendix A, we propose a detailed proof of the convergence rate of the Euler scheme for the McKean-Vlasov equation in the general setting

$$dX_t = b(t, X_t, \mu_t)dt + \sigma(t, X_t, \mu_t)dB_t,$$

where b, σ are Lipschitz in (x, μ) and ρ -Hölder in t .

Generalization in dimension 1. In one dimension, it is possible to consider more general drift b (convex in x and non-decreasing in μ for convex ordering) if we restrict to monotone (non-decreasing) convex order. This idea originated from Hajek's theorem in ([Haj85]) established for Brownian diffusions by other methods. However our approach based on the Euler scheme cannot be adapted straightforwardly: a truncated version of the scheme is necessary to complete the proofs which adds significant some technicalities.

This extension for the McKean-Vlasov equations is developed in a devoted paper [LP21].

2 Comments on the assumptions

In this section, we give some comments on the assumptions made in this paper. In Section 2.1, we prove that Assumption I implies the convergence of the Euler scheme for the McKean-Vlasov equations and in Section 2.2, we give some necessary and sufficient conditions for Assumption II-(1) and (3).

2.1 Comments on Assumption I

Let

$$\mathcal{C}([0, T], \mathbb{R}^d) := \{f : [0, T] \rightarrow \mathbb{R}^d \text{ continuous function}\}$$

equipped with the uniform norm $\|f\|_{\text{sup}} := \sup_{t \in [0, T]} |f(t)|$. Assumption I guarantees the existence and strong uniqueness of the respective solutions of (1.3) and (1.4) in $L^p_{\mathcal{C}([0, T], \mathbb{R}^d)}(\Omega, \mathcal{F}, \mathbb{P})$ (see [Liu19, Section 5.1], [Lac18, Theorem 3.3]) and the convergence of the following Euler scheme. Let $M \in \mathbb{N}^*$ and let $h = \frac{T}{M}$. For $m = 0, \dots, M$, we define $t_m^M := h \cdot m = \frac{T}{M} \cdot m$. When there is no ambiguity, we write t_m instead of t_m^M . Let $Z_m := \frac{1}{\sqrt{h}}(B_{t_{m+1}} - B_{t_m})$, $m = 1, \dots, M$, be i.i.d random variables having probability distribution $\mathcal{N}(0, \mathbf{I}_q)$, independent of X_0 and Y_0 . The Euler schemes of equations (1.3) and (1.4) are defined by

$$\bar{X}_{t_{m+1}}^M = \bar{X}_{t_m}^M + h \cdot b(t_m, \bar{X}_{t_m}^M, \bar{\mu}_{t_m}^M) + \sqrt{h} \cdot \sigma(t_m, \bar{X}_{t_m}^M, \bar{\mu}_{t_m}^M) Z_{m+1}, \quad \bar{X}_0^M = X_0 \quad (2.1)$$

$$\bar{Y}_{t_{m+1}}^M = \bar{Y}_{t_m}^M + h \cdot b(t_m, \bar{Y}_{t_m}^M, \bar{\nu}_{t_m}^M) + \sqrt{h} \cdot \theta(t_m, \bar{Y}_{t_m}^M, \bar{\nu}_{t_m}^M) Z_{m+1}, \quad \bar{Y}_0^M = Y_0 \quad (2.2)$$

where for every $m = 0, \dots, M$, $\bar{\mu}_{t_m}^M$ and $\bar{\nu}_{t_m}^M$ respectively denote the probability distribution of $\bar{X}_{t_m}^M$ and $\bar{Y}_{t_m}^M$. Moreover, we classically define the *genuine* (or continuous time) Euler scheme $\bar{X} = (\bar{X}_t^M)_{t \in [0, T]}$, $\bar{Y} = (\bar{Y}_t^M)_{t \in [0, T]}$ as follows: for every $t \in [t_m, t_{m+1})$,

$$\bar{X}_t^M := \bar{X}_{t_m}^M + b(t_m, \bar{X}_{t_m}^M, \bar{\mu}_{t_m}^M)(t - t_m) + \sigma(t_m, \bar{X}_{t_m}^M, \bar{\mu}_{t_m}^M)(B_t - B_{t_m}), \quad (2.3)$$

$$\bar{Y}_t^M := \bar{Y}_{t_m}^M + b(t_m, \bar{Y}_{t_m}^M, \bar{\nu}_{t_m}^M)(t - t_m) + \theta(t_m, \bar{Y}_{t_m}^M, \bar{\nu}_{t_m}^M)(B_t - B_{t_m}). \quad (2.4)$$

When there is no ambiguity, we write \bar{X}_m and \bar{X}_t instead of $\bar{X}_{t_m}^M$ and \bar{X}_t^M to simplify the notation.

The value $p \in [2, +\infty)$ in Assumption I such that $\|X_0\|_p \vee \|Y_0\|_p < +\infty$ and in the Lipschitz condition (1.8) is crucial for the moment controls of the processes X , Y , $(\bar{X}_t)_{t \in [0, T]}$ and $(\bar{Y}_t)_{t \in [0, T]}$ and the L^p -strong convergence result for the continuous Euler scheme (2.3) and (2.4). For convenience, we state the following proposition only for X and $(\bar{X}_t)_{t \in [0, T]}$ but the results remain true for Y and $(\bar{Y}_t)_{t \in [0, T]}$. The proof of Proposition 2.1 is postponed to Appendix A.

Proposition 2.1. *Assume Assumption I is in force.*

(a) *There exists a constant C depending on $p, d, \sigma, \theta, T, L$ such that, for every $t \in [0, T]$ and for every $M \geq 1$,*

$$\left\| \sup_{u \in [0, t]} |X_u| \right\|_p \vee \left\| \sup_{u \in [0, t]} |\bar{X}_u^M| \right\|_p \leq C(1 + \|X_0\|_p). \quad (2.5)$$

Moreover, there exists a constant κ depending on $L, b, \sigma, \|X_0\|_p, p, d, T$ such that for any $s, t \in [0, T]$, $s \leq t$,

$$\forall M \geq 1, \quad \|\bar{X}_t^M - \bar{X}_s^M\|_p \vee \|X_t - X_s\|_p \leq \kappa \sqrt{t - s}.$$

(b) There exists a constant \tilde{C} depending on $p, d, T, L, \tilde{L}, \rho, \|X_0\|_p$ such that

$$\left\| \sup_{t \in [0, T]} |X_t - \bar{X}_t^M| \right\|_p \leq \tilde{C} h^{\frac{1}{2} \wedge \rho}.$$

2.2 Comments on Assumption II

Assumption II contains technical conditions. The drift b is assumed to be affine and Lipschitz continuous in x , i.e. b has the following form

$$b(t, x, \mu) = \alpha(t)x + \beta(t, \mu). \quad (2.6)$$

In fact, Jensen's inequality implies that for every $\mu \in \mathcal{P}_1(\mathbb{R}^d)$, $\delta_{\int \xi \mu(d\xi)} \preceq_{cv} \mu$ as for every convex function f , $f(\int \xi \mu(d\xi)) \leq \int f(\xi) \mu(d\xi)$. Hence the condition in (1.9) implies $b(t, x, \mu) = b(t, x, \delta_{\int \xi \mu(d\xi)})$ so that the drift (2.6) is equivalent to the following drift

$$\tilde{b}(t, x, \mu) = \alpha(t)x + \tilde{\beta}\left(t, \int_{\mathbb{R}^d} \xi \mu(d\xi)\right) \quad (2.7)$$

with $\tilde{\beta}(t, \int_{\mathbb{R}^d} \xi \mu(d\xi)) := \beta(t, \delta_{\int \xi \mu(d\xi)}) = \beta(t, \mu)$.

Now we give necessary and sufficient conditions and a criterion based on the linear functional derivative to establish monotonicity with respect to the convex order of a function $\Phi(\mu)$, as it appears in Assumption II-(3). We will consider the case of probability measures on $\mathcal{P}_2(\mathbb{R}^d)$ for simplicity but what follows can be straightforwardly adapted to $\mathcal{P}_p(\mathbb{R}^d)$ for a $p \in [1, +\infty)$. The proof of the following proposition is postponed to Appendix B.

Proposition 2.2. *Let $\Phi : (\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2) \rightarrow \mathbb{R}$ be a continuous function.*

(a) Φ is non-decreasing with respect to the convex order if and only if, for every $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, $\mu \preceq_{cv} \nu$,

$$\liminf_{\varepsilon \rightarrow 0^+} \frac{\Phi(\mu + \varepsilon(\nu - \mu)) - \Phi(\mu)}{\varepsilon} \geq 0.$$

(b) Characterization when Φ is smooth. *Assume $\Phi : (\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2) \rightarrow \mathbb{R}$ is linearly functionally differentiable with linear functional derivative $\frac{\delta \Phi}{\delta m}$ defined on $\mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$ in the sense of [CD18a, Definition 5.43] ⁽²⁾. Then, the following conditions are equivalent.*

(i) *The function Φ is non-decreasing w.r.t. the convex order on $\mathcal{P}_2(\mathbb{R}^d)$.*

(ii) *For every $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ with $\mu \preceq_{cv} \nu$, $\int_{\mathbb{R}^d} \frac{\delta \Phi}{\delta m}(\mu)(x) d(\nu - \mu)(x) \geq 0$.*

(c) *A convexity based criterion. In particular, if, for every $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $x \mapsto \frac{\delta \Phi}{\delta m}(\mu)(x)$ is convex, then Φ is non-decreasing for the convex ordering on $\mathcal{P}_2(\mathbb{R}^d)$.*

Remark 2.1. The converse of the above criterion i.e. (b) implies the convexity of $\frac{\delta \Phi}{\delta m}(\mu)(x)$ in x for every μ seems not clear although we have no obvious counterexample.

Example 2.1. (a) Elementary examples of such monotonic functions Ψ on $\mathcal{P}_2(\mathbb{R}^d)$ for convex ordering are functions of the form

$$\Psi(\mu) = \chi \left(\int_{\mathbb{R}^d} \psi(\xi) \mu(d\xi) \right)$$

²i.e. $\frac{\delta \Phi}{\delta m}(\mu)(x)$ is jointly continuous in (μ, x) and, for any \mathcal{W}_2 -bounded subset $\mathcal{K} \subset \mathcal{P}_2(\mathbb{R}^d)$, $x \mapsto \frac{\delta \Phi}{\delta m}(\mu)(x)$ has at most quadratic growth in x uniformly in $\mu \in \mathcal{K}$, and satisfies

$$\Phi(\mu') - \Phi(\mu) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta \Phi}{\delta m}(t\mu' + (1-t)\mu)(x) d(\mu' - \mu)(x) dt.$$

Note that such a quantity is defined up to a real constant (not depending upon x).

where $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a convex function with at most quadratic growth at infinity and $\chi : \mathbb{R} \rightarrow \mathbb{R}$ is nondecreasing. In one dimension, typical examples of such functions ψ are $\psi(\xi) = \mathbb{E} [(aZ + b\xi)^+]^2$, $Z^+ \in L^2$, $a \in \mathbb{R}_+$, $b \in \mathbb{R}$ or $\psi(x) = \mathbb{E} |aZ + b\xi|^\gamma$, $1 \leq \gamma \leq 2$ with $Z \in L^\gamma$, $a, b \in \mathbb{R}$. Any positive linear combination of such functions Ψ is of course still non-decreasing for the convex ordering. This leads to consider the more general family of functions

$$\Psi(\mu) = \int_E \chi \left(\int_{\mathbb{R}^d} \psi(\xi, u) \mu(d\xi) \right) \pi(du) \quad (2.8)$$

where π is a non-negative σ -finite measure on a measure space (E, \mathcal{E}) , $\psi(\cdot, u)$ is convex with quadratic growth $|\psi(x, u)| \leq \kappa(u)(1 + |x|^2)$, $u \in E$, such that $\chi : \mathbb{R} \rightarrow \mathbb{R}$ is non-decreasing and $\int_E \left| \chi \left((1 + \int |\xi|^2 \mu(d\xi)) \kappa(u) \right) \right| d\pi(u) < +\infty$.

Note that if χ in (2.8) is continuously differentiable, one has, under appropriate integrability conditions not detailed here,

$$\frac{\delta \Psi}{\delta m}(\mu)(x) = \int_E \chi' \left(\int_{\mathbb{R}^d} \psi(\xi, u) \mu(d\xi) \right) \psi(x, u) \pi(du)$$

which is clearly a convex function in x for every distribution μ since $\chi' \geq 0$.

(b) Let $W : \mathbb{R}^d \rightarrow \mathbb{R}$ be a convex function with at most quadratic growth at infinity. Then the function Φ defined on $\mathcal{P}_2(\mathbb{R}^d)$ by

$$\Phi(\mu) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(x - y) \mu(dx) \mu(dy)$$

is well defined and non decreasing for convex ordering. This can be easily checked directly since both $W(x - \cdot)$ and $W(\cdot - y)$ are convex functions. Nevertheless, one can also check that its linear functional derivative is given by

$$\frac{\delta \Phi}{\delta m}(\mu)(x) = \frac{1}{2} \int_{\mathbb{R}^d} (W(x - y) + W(y - x)) \mu(dy),$$

and is convex in x for every $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

3 Applications

This section contains two applications of the main results.

3.1 Application: convex partitioning and convex bounding

Theorem 1 and Theorem 2 show that we can *upper and lower bound*, with respect to the functional convex order, a scaled McKean-Vlasov process by two scaled McKean-Vlasov processes satisfying Assumption II-(1), (2), (3), or we can *separate*, with respect to the functional convex order, two scaled McKean-Vlasov processes by a scaled McKean-Vlasov process satisfying Assumption II-(1), (2), (3). That is, if we consider the following scaled McKean-Vlasov equations satisfying Assumption I

$$\begin{aligned} dX_t^{\sigma_1} &= (a(t)X_t^{\sigma_1} + \beta(t, \mathbb{E} X_t^{\sigma_1}))dt + \sigma_1(t, X_t^{\sigma_1}, \mu_t^{\sigma_1})dB_t, & X_0^{\sigma_1} &\in L^p(\mathbb{P}), \\ dY_t^{\theta_1} &= (a(t)Y_t^{\theta_1} + \beta(t, \mathbb{E} Y_t^{\theta_1}))dt + \theta_1(t, Y_t^{\theta_1}, \nu_t^{\theta_1})dB_t, & Y_0^{\theta_1} &\in L^p(\mathbb{P}), \\ dX_t^{\sigma_2} &= (a(t)X_t^{\sigma_2} + \beta(t, \mathbb{E} X_t^{\sigma_2}))dt + \sigma_2(t, X_t^{\sigma_2}, \mu_t^{\sigma_2})dB_t, & X_0^{\sigma_2} &\in L^p(\mathbb{P}), \\ dY_t^{\theta_2} &= (a(t)Y_t^{\theta_2} + \beta(t, \mathbb{E} Y_t^{\theta_2}))dt + \theta_2(t, Y_t^{\theta_2}, \nu_t^{\theta_2})dB_t, & Y_0^{\theta_2} &\in L^p(\mathbb{P}), \end{aligned}$$

and if σ_1 and σ_2 satisfy Assumption II-(2), (3), $X_0^{\sigma_1} \preceq_{cv} Y_0^{\theta_1} \preceq_{cv} X_0^{\sigma_2} \preceq_{cv} Y_0^{\theta_2}$ and

$$\sigma_1 \preceq \theta_1 \preceq \sigma_2 \preceq \theta_2, \quad (3.1)$$

then we have the following two types of inequalities:

– Convex bounding

$$\begin{cases} \mathbb{E} F(X^{\sigma_1}) \leq \mathbb{E} F(Y^{\theta_1}) \leq \mathbb{E} F(X^{\sigma_2}), \\ \mathbb{E} G(X^{\sigma_1}, (\mu_t^{\sigma_1})_{t \in [0, T]}) \leq \mathbb{E} G(Y^{\theta_1}, (\nu_t^{\theta_1})_{t \in [0, T]}) \leq \mathbb{E} G(X^{\sigma_2}, (\mu_t^{\sigma_2})_{t \in [0, T]}), \end{cases} \quad (3.2)$$

– Convex partitioning

$$\begin{cases} \mathbb{E} F(Y^{\theta_1}) \leq \mathbb{E} F(X^{\sigma_2}) \leq \mathbb{E} F(Y^{\theta_2}), \\ \mathbb{E} G(Y^{\theta_1}, (\nu_t^{\theta_1})_{t \in [0, T]}) \leq \mathbb{E} G(X^{\sigma_2}, (\mu_t^{\sigma_2})_{t \in [0, T]}) \leq \mathbb{E} G(Y^{\theta_2}, (\nu_t^{\theta_2})_{t \in [0, T]}) \end{cases} \quad (3.3)$$

for any two applications F and G satisfying conditions of Theorem 1.

Remark that Assumption II-(2) and (3) are only made on σ of the equation of X . Consequently, in this application, we can choose two “simple” functions σ_i , $i = 1, 2$ to construct the convex partitioning and convex bounding. For example, we can choose two convex functions σ_i , $i = 1, 2$ which do not depend on μ and satisfy (3.1). In this a case, the results in (3.2) and (3.3) make a comparison between a McKean-Vlasov equation and regular Brownian diffusions, the latter ones being much easier to simulate.

3.2 Application in stochastic control problem and mean field games

In this subsection, we give two examples to explain how to apply Theorem 1 and Theorem 2 in the framework of the stochastic control problem and mean field games.

► The construction of this first example for the stochastic control problem is based on the dynamic and cost function in [CDL13, Section 4] but the same idea can be applied to other similar examples. Consider the following two one-dimensional McKean-Vlasov dynamics:

$$\begin{aligned} dX_t^{x, \alpha} &= [a_t X_t^{x, \alpha} + \bar{a}_t \mathbb{E} X_t^{x, \alpha} + b_t \alpha_t + \beta_t] dt + \sigma dB_t, & X_0 &= x \in \mathbb{R} \\ dY_t^{x, \alpha} &= [a_t Y_t^{x, \alpha} + \bar{a}_t \mathbb{E} Y_t^{x, \alpha} + b_t \alpha_t + \beta_t] dt + \theta(t, Y_t^{x, \alpha}, \nu_t) dB_t, & Y_0 &= x \end{aligned}$$

equipped with the respective cost function \mathcal{J}^X and \mathcal{J}^Y defined by

$$\begin{aligned} \mathcal{J}^X(x, \alpha) &= \mathbb{E} \left[\int_0^T \left[\frac{1}{2} (m_t X_t^{x, \alpha} + \bar{m}_t \mathbb{E} X_t^{x, \alpha})^2 + \frac{1}{2} n_t \alpha_t^2 \right] dt + \frac{1}{2} (q X_T^{x, \alpha} + \bar{q} \mathbb{E} X_T^{x, \alpha})^2 \right] \\ \mathcal{J}^Y(x, \alpha) &= \mathbb{E} \left[\int_0^T \left[\frac{1}{2} (m_t Y_t^{x, \alpha} + \bar{m}_t \mathbb{E} Y_t^{x, \alpha})^2 + \frac{1}{2} n_t \alpha_t^2 \right] dt + \frac{1}{2} (q Y_T^{x, \alpha} + \bar{q} \mathbb{E} Y_T^{x, \alpha})^2 \right] \end{aligned}$$

with admissible controls $\alpha = (\alpha_t)_{t \in [0, T]}$ (see [CDL13, Section 2], where for every $t \in [0, T]$, ν_t is the probability distribution of $Y_t^{x, \alpha}$, $a_t, \bar{a}_t, b_t, \beta_t, m_t, \bar{m}_t, n_t$ are deterministic ρ -Hölder continuous functions of $t \in [0, T]$, q and \bar{q} are deterministic constants and the function $t \mapsto n_t$ is positive. Assume moreover that θ satisfies Assumption I and

$$\forall (t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d), \quad 0 \leq \theta(t, x, \mu) \leq \sigma.$$

We know from [CDL13, Proposition 4.1 and the remark that follows] that the optimal control minimizing $\mathcal{J}^X(x, \cdot)$ has a closed form given by

$$\alpha^{*, X}(t, X_t) = -\frac{b_t}{n_t} \eta_t X_t - \frac{b_t}{n_t} \chi_t \quad (3.4)$$

where $(\eta_t)_{0 \leq t \leq T}$ and $(\chi_t)_{0 \leq t \leq T}$ are determined by solving a Riccati equation depending only on $b_t, n_t, a_t, \bar{a}_t, m_t, \bar{m}_t, \beta_t, q$ and \bar{q} (see (57) still in [CDL13]). If the coefficient $\frac{b_t}{n_t} \eta_t$ and $\frac{b_t}{n_t} \chi_t$ in (3.4) are still ρ -Hölder continuous,

then Theorem 2 directly implies

$$\inf_{\alpha} \mathcal{J}^X(x, \alpha) = \mathcal{J}^X(x, (\alpha^{*,X}(t, X_t))_{t \in [0, T]}) \geq \mathcal{J}^Y(x, (\alpha^{*,X}(t, Y_t))_{t \in [0, T]}) \geq \inf_{\alpha} \mathcal{J}^Y(x, \alpha). \quad (3.5)$$

Consequently, if we define the value function by

$$v^X(x) := \inf_{\alpha} \mathcal{J}^X(x, \alpha) \quad \text{and} \quad v^Y(x) := \inf_{\alpha} \mathcal{J}^Y(x, \alpha),$$

then (3.4) directly implies $v^X(x) \geq v^Y(x)$. Moreover, it follows from Corollary 1.1 that the function $x \mapsto v^X(x)$ is convex.

► Similarly, in the framework of mean field games, a McKean-Vlasov equation appears when the number of agents tends to infinity. If we consider the following one-dimensional (limiting) dynamics

$$dX_t = \alpha_t dt + \sigma(\mu_t) dB_t, \quad X_0 \in L^p(\mathbb{P}), \quad p \geq 2$$

where $\sigma \geq 0$ is \mathcal{W}_p -Lipschitz continuous and $(\mu_t)_{t \in [0, T]}$ is a flow of square integrable probability measures on \mathbb{R} , associated with the following cost function

$$\mathcal{J}^X(\alpha) = \mathbb{E} \left[\frac{1}{2} |c_g X_T + g \left(\int x \mu_T(dx) \right)|^2 + \int_0^T \left[\frac{1}{2} |c_f X_t + f \left(\int x \mu_t(dx) \right)|^2 + \frac{1}{2} |\alpha_t|^2 \right] dt \right].$$

Theorems in this paper may provide some information on the value of the game. Assume that the Lasry-Lions monotonicity condition is satisfied, then the equilibrium of the mean field game is unique with a McKean-Vlasov dynamics (μ_t is the distribution of X_t) given by

$$dX_t = -[\eta_t X_t + h(t, \mathbb{E} X_t)] dt + \sigma(\mu_t) dB_t \quad (3.6)$$

by taking $\alpha_t = -[\eta_t X_t + h(t, \mathbb{E} X_t)]$ (see e.g. [CD18b, Section 3.5.2]). In (3.6), $(\eta_t)_{t \in [0, T]}$ is the solution of a Riccati equation depending only on c_f and c_g and h depends on f, g, c_f and c_g . If σ is non-decreasing w.r.t. the convex order like in the previous example, results in this paper allow us to compare the value of the cost function $\mathcal{J}(\alpha)$ of the system (3.6) with that of the following system $(Y_t)_{t \in [0, T]}$

$$dY_t = -[\eta_t Y_t + h(t, \mathbb{E} Y_t)] dt + \theta(\nu_t) dB_t, \quad Y_0 \in L^p(\mathbb{P}), \quad p \geq 2, \quad (3.7)$$

when

$$0 \leq \sigma \leq \theta \quad \text{and} \quad X_0 \preceq_{cv} Y_0.$$

In fact, if we denote by ν_t the probability distribution of Y_t for every $t \in [0, T]$, by applying Theorem 1 and (1.2), we know that $\mu_t \preceq \nu_t$ for every $t \in [0, T]$ so that $\mathbb{E} X_t = \mathbb{E} Y_t$. Thus $g(\int x \mu_T(dx)) = g(\int x \nu_T(dx))$ and $f(\int x \mu_t(dx)) = f(\int x \nu_t(dx))$, $t \in [0, T]$. Let $\alpha \mapsto \mathcal{J}^X(\alpha)$ and $\alpha \mapsto \mathcal{J}^Y(\alpha)$ denote the cost function for the system (3.6) and (3.7) respectively. Then the difference between $\mathcal{J}^X(\alpha)$ and $\mathcal{J}^Y(\alpha)$ only depends on terms containing $\mathbb{E}[X_t^2]$, $\mathbb{E}[Y_t^2]$, $t \in [0, T]$, that is, on the variances of X_t and Y_t since they have the same mean.

4 Convex order results for the Euler scheme

In this section, we will discuss the convex order results for the random variables $\bar{X}_{t_m}^M$ and $\bar{Y}_{t_m}^M$, $m = 0, \dots, M$ defined by the Euler scheme (2.1) and (2.2). In order to simplify the notations, we rewrite (2.1) and (2.2) by setting

$$\bar{X}_m := \bar{X}_{t_m}^M, \quad \bar{Y}_m := \bar{Y}_{t_m}^M, \quad \bar{\mu}_m := \bar{\mu}_{t_m}^M \quad \text{and} \quad \bar{\nu}_m := \bar{\nu}_{t_m}^M.$$

It reads

$$\bar{X}_{m+1} = b_m(\bar{X}_m, \bar{\mu}_m) + \sigma_m(\bar{X}_m, \bar{\mu}_m)Z_{m+1}, \quad \bar{X}_0 = X_0, \quad (4.1)$$

$$\bar{Y}_{m+1} = b_m(\bar{Y}_m, \bar{\nu}_m) + \theta_m(\bar{Y}_m, \bar{\nu}_m)Z_{m+1}, \quad \bar{Y}_0 = Y_0, \quad (4.2)$$

where for every $m = 0, \dots, M$,

$$b_m(x, \mu) := x + h \cdot b(t_m, x, \mu), \quad \sigma_m(x, \mu) := \sqrt{h} \cdot \sigma(t_m, x, \mu), \quad \theta_m(x, \mu) := \sqrt{h} \cdot \theta(t_m, x, \mu). \quad (4.3)$$

First note that it follows from Proposition 2.1(a) that

$$\forall m = 0, \dots, M, \quad \bar{\mu}_m, \bar{\nu}_m \in \mathcal{P}_p(\mathbb{R}^d)$$

(hence lie in $\mathcal{P}_1(\mathbb{R}^d)$). Then it follows from Assumption II that $X_0, Y_0, b_m, \sigma_m, \theta_m, m = 0, \dots, M$, satisfy its discrete time counterpart.

Assumption II_{disc}. (1) The function $b_m, m = 0, \dots, M$, are affine in x and

$$\forall \mu, \nu \in \mathcal{P}_p(\mathbb{R}^d), \quad \mu \preceq_{cv} \nu \quad b_m(x, \mu) = b_m(x, \nu). \quad (4.4)$$

(2) The functions $\sigma_m, m = 0, \dots, M$, are convex in x :

$$\forall x, y \in \mathbb{R}^d, \forall \lambda \in [0, 1], \quad \sigma_m(\lambda x + (1 - \lambda)y, \mu) \preceq \lambda \sigma_m(x, \mu) + (1 - \lambda)\sigma_m(y, \mu). \quad (4.5)$$

(3) The functions $\sigma_m, m = 0, \dots, M$, are non-decreasing in μ with respect to the convex order :

$$\forall \mu, \nu \in \mathcal{P}_p(\mathbb{R}^d), \quad \mu \preceq_{cv} \nu, \quad \sigma_m(x, \mu) \preceq \sigma_m(x, \nu). \quad (4.6)$$

(4) We have the following order between σ_m and $\theta_m, m = 0, \dots, M$:

$$\forall (x, \mu) \in \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d), \quad \sigma_m(x, \mu) \preceq \theta_m(x, \mu). \quad (4.7)$$

(5) $\bar{X}_0 \preceq_{cv} \bar{Y}_0$ ⁽³⁾.

At this stage let us mention that we will extensively use the following elementary characterization of convex ordering between two integrable \mathbb{R}^d -valued random variables or their distributions.

Lemma 4.1 (Lemma A.1 in [ACJ19b]). *Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$. We have $\mu \preceq_{cv} \nu$ if and only if for every convex function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ with (at most) linear growth in the sense that there exists a real constant $C > 0$ such that, for every $x \in \mathbb{R}^d$, $|\varphi(x)| \leq C(1 + |x|)$,*

$$\int_{\mathbb{R}^d} \varphi(x) \mu(dx) \leq \int_{\mathbb{R}^d} \varphi(x) \nu(dx).$$

This characterization allows us to restrict the proofs to convex functions with linear growth to establish the convex ordering.

The main result of this section is the following proposition.

Proposition 4.1. *Under Assumption I and II, for any convex function $F : (\mathbb{R}^d)^{M+1} \rightarrow \mathbb{R}$ with p -polynomial growth in the sense that*

$$\forall x = (x_0, \dots, x_M) \in (\mathbb{R}^d)^{M+1}, \exists C > 0, \text{ such that } |F(x)| \leq C \left(1 + \sup_{0 \leq i \leq M} |x_i|^p\right), \quad (4.8)$$

we have $\mathbb{E} F(\bar{X}_0, \dots, \bar{X}_M) \leq \mathbb{E} F(\bar{Y}_0, \dots, \bar{Y}_M)$.

³Since $\bar{X}_0 = X_0$ and $\bar{Y}_0 = Y_0$ (see the definition (2.1) and (2.2)).

Before proving Proposition 4.1, we first show in the next section that the Euler scheme defined in (4.1) and (4.2) propagates the marginal convex order step by step, i.e. $\bar{X}_m \preceq_{cv} \bar{Y}_m$, for any fixed $m \in \{0, \dots, M\}$ by a forward induction.

4.1 Marginal convex order for the Euler scheme

Let $\mathcal{C}_{cv}(\mathbb{R}^d, \mathbb{R}) := \{\varphi : \mathbb{R}^d \rightarrow \mathbb{R} \text{ convex function}\}$. For every $m = 0, \dots, M-1$, we define an operator $Q_{m+1} : \mathcal{C}_{cv}(\mathbb{R}^d, \mathbb{R}) \rightarrow \mathcal{C}(\mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \times \mathbb{M}_{d \times q}, \mathbb{R})$ associated with Z_{m+1} defined in (4.1) and (4.2) by

$$(x, \mu, u) \in \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \times \mathbb{M}_{d \times q} \longmapsto (Q_{m+1} \varphi)(x, \mu, u) := \mathbb{E} \left[\varphi(b_m(x, \mu) + u Z_{m+1}) \right]. \quad (4.9)$$

For every $m = 0, \dots, M$, let \mathcal{F}_m denote the σ -algebra generated by $(X_0, Y_0, Z_1, \dots, Z_m)$. The main result in this section is the following.

Proposition 4.2. *Let $(\bar{X}_m)_{m=0, \dots, M}$, $(\bar{Y}_m)_{m=0, \dots, M}$ be random variables defined by (4.1) and (4.2). Under Assumption II_{disc}, we have*

$$\bar{X}_m \preceq_{cv} \bar{Y}_m, \quad m = 0, \dots, M.$$

The proof of Proposition 4.2 relies on the following two lemmas whose proofs are postponed to Appendix C.

Lemma 4.2 (Revisited Jensen's Lemma). *Let $\varphi \in \mathcal{C}_{cv}(\mathbb{R}^d, \mathbb{R})$ with linear growth and let $\mu \in \mathcal{P}_1(\mathbb{R}^d)$. Then, for every $m = 1, \dots, M$,*

- (i) *the function $(x, u) \mapsto (Q_m \varphi)(x, \mu, u)$ is (finite and) convex.*
- (ii) *for any fixed $x \in \mathbb{R}^d$, the function $u \mapsto (Q_m \varphi)(x, \mu, u)$ attains its minimum at $\mathbf{0}_{d \times q}$, where $\mathbf{0}_{d \times q}$ is the zero-matrix of size $d \times q$,*
- (iii) *for any fixed $x \in \mathbb{R}^d$, the function $u \mapsto (Q_m \varphi)(x, \mu, u)$ is non-decreasing with respect to the partial order of $d \times q$ matrix (1.5).*

Lemma 4.3. *Let $\varphi \in \mathcal{C}_{cv}(\mathbb{R}^d, \mathbb{R})$ with linear growth. Then for a fixed $\mu \in \mathcal{P}_1(\mathbb{R}^d)$, the function $x \mapsto \mathbb{E} \left[\varphi(b_m(x, \mu) + \sigma_m(x, \mu) Z_{m+1}) \right]$ is convex with linear growth for every $m = 0, \dots, M-1$.*

Proof of Proposition 4.2. Assumption II_{disc} directly implies $\bar{X}_0 \preceq_{cv} \bar{Y}_0$. Assume $\bar{X}_m \preceq_{cv} \bar{Y}_m$ or, equivalently, $\bar{\mu}_m \preceq_{cv} \bar{\nu}_m$. Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a convex function with linear growth. Then,

$$\begin{aligned} \mathbb{E} [\varphi(\bar{X}_{m+1})] &= \mathbb{E} \left[\varphi(b_m(\bar{X}_m, \bar{\mu}_m) + \sigma_m(\bar{X}_m, \bar{\mu}_m) Z_{m+1}) \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\varphi(b_m(\bar{X}_m, \bar{\mu}_m) + \sigma_m(\bar{X}_m, \bar{\mu}_m) Z_{m+1}) \mid \mathcal{F}_m \right] \right] \\ &= \int_{\mathbb{R}^d} \bar{\mu}_m(dx) \mathbb{E} \left[\varphi(b_m(x, \bar{\mu}_m) + \sigma_m(x, \bar{\mu}_m) Z_{m+1}) \right] \\ &\quad \text{(the integrability is due to Proposition 2.1 and Lemma 4.3)} \\ &\leq \int_{\mathbb{R}^d} \bar{\mu}_m(dx) \mathbb{E} \left[\varphi(b_m(x, \bar{\nu}_m) + \sigma_m(x, \bar{\nu}_m) Z_{m+1}) \right] \\ &\quad \text{(by Assumption (4.4), (4.6) and Lemma 4.2, since } \bar{\mu}_m \preceq_{cv} \bar{\nu}_m \text{)} \\ &\leq \int_{\mathbb{R}^d} \bar{\nu}_m(dx) \mathbb{E} \left[\varphi(b_m(x, \bar{\nu}_m) + \sigma_m(x, \bar{\nu}_m) Z_{m+1}) \right] \\ &\quad \text{(by Lemma 4.3, since } \bar{\mu}_m \preceq_{cv} \bar{\nu}_m \text{)} \\ &\leq \int_{\mathbb{R}^d} \bar{\nu}_m(dx) \mathbb{E} \left[\varphi(b_m(x, \bar{\nu}_m) + \theta_m(x, \bar{\nu}_m) Z_{m+1}) \right] \end{aligned}$$

$$\begin{aligned} & \text{(by Assumption (4.7) and Lemma 4.2)} \\ & = \mathbb{E} [\varphi(\bar{Y}_{m+1})]. \end{aligned}$$

Thus $\bar{X}_{m+1} \preceq_{cv} \bar{Y}_{m+1}$ by applying Lemma 4.1. One concludes by a forward induction. \square

Proof of Corollary 1.1. (a) This is a straightforward consequence of Theorem 1(a) by setting, f being a convex function with linear growth and $t \in [0, T]$, $F(\alpha) = f(\alpha(t))$, $\alpha \in \mathcal{C}([0, T], \mathbb{R})$.

A direct proof is as follows: we only need to prove $\mu_T \preceq_{cv} \nu_T$ since for any $t \in [0, T]$ the proof for $\mu_t \preceq_{cv} \nu_t$ is the same by considering $(X_s)_{s \in [0, t]}$ and $(Y_s)_{s \in [0, t]}$. Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a convex function with linear growth. Proposition 4.2 implies that for every $h = \frac{T}{M} \geq 0$, $\bar{X}_M \preceq_{cv} \bar{Y}_M$. Thus for every $h > 0$, $\mathbb{E} \varphi(\bar{X}_M) \leq \mathbb{E} \varphi(\bar{Y}_M)$. It follows from Proposition 2.1 that $\mathbb{E} \varphi(\bar{X}_M) \rightarrow \mathbb{E} \varphi(X_T)$ and $\mathbb{E} \varphi(\bar{Y}_M) \rightarrow \mathbb{E} \varphi(Y_T)$ as $h \rightarrow 0$ since we assumed $p \geq 2$. Hence $\mathbb{E} \varphi(X_T) \leq \mathbb{E} \varphi(Y_T)$ by letting $h \rightarrow 0$, i.e. $X_T \preceq_{cv} Y_T$ by applying Lemma 4.1.

(b) For every $x, y \in \mathbb{R}^d$ and $\lambda \in [0, 1]$, we have $\delta_{\lambda x + (1-\lambda)y} \preceq_{cv} \lambda \delta_x + (1-\lambda)\delta_y$ and one concludes by applying Theorem 1 with $\theta = \sigma$, $X_0 \sim \delta_{\lambda x + (1-\lambda)y}$ and $Y_0 \sim \lambda \delta_x + (1-\lambda)\delta_y$. \square

The next proposition prove that we can dissociate the assumption on the convexity and monotonicity in Assumption II - (2), (3) to obtain the same marginal convex order as in Corollary 1.1 - (a). This seems to be specific to marginal convex ordering.

Proposition 4.3. *Let $\mu_t, \nu_t, t \in [0, T]$, respectively denote the marginal distributions of the solution processes X and Y in (1.3), (1.4). If we replace Assumption II-(3) by the following condition:*

(3') *For every fixed $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the function $\theta(t, x, \cdot)$ is non-decreasing in μ with respect to the convex order in the sense that*

$$\forall \mu, \nu \in \mathcal{P}_p(\mathbb{R}^d), \quad \mu \preceq_{cv} \nu \implies \theta(t, x, \mu) \preceq_{cv} \theta(t, x, \nu), \quad (4.10)$$

then for every $t \in [0, T]$, $\mu_t \preceq_{cv} \nu_t$.

Proof of Proposition 4.3. First, remark that the proofs of Lemma 4.2 and 4.3 do not depend on the condition (1.11) and (4.10) and the induction step of Proposition 4.2 remains true under the new assumption (3'). Assume that $\bar{X}_m \preceq_{cv} \bar{Y}_m$. If we consider a convex function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ with linear growth, then

$$\begin{aligned} \mathbb{E} [\varphi(\bar{X}_{m+1})] &= \mathbb{E} \left[\varphi(b_m(\bar{X}_m, \bar{\mu}_m) + \sigma_m(\bar{X}_m, \bar{\mu}_m)Z_{m+1}) \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\varphi(b_m(\bar{X}_m, \bar{\mu}_m) + \sigma_m(\bar{X}_m, \bar{\mu}_m)Z_{m+1}) \mid \mathcal{F}_m \right] \right] \\ &= \int_{\mathbb{R}^d} \bar{\mu}_m(dx) \mathbb{E} \left[\varphi(b_m(x, \bar{\mu}_m) + \sigma_m(x, \bar{\mu}_m)Z_{m+1}) \right] \\ &\leq \int_{\mathbb{R}^d} \bar{\nu}_m(dx) \mathbb{E} \left[\varphi(b_m(x, \bar{\mu}_m) + \sigma_m(x, \bar{\mu}_m)Z_{m+1}) \right] \\ &\quad \text{(by Lemma 4.3, as } \bar{\mu}_m \preceq_{cv} \bar{\nu}_m \text{)} \\ &\leq \int_{\mathbb{R}^d} \bar{\nu}_m(dx) \mathbb{E} \left[\varphi(b_m(x, \bar{\mu}_m) + \theta_m(x, \bar{\mu}_m)Z_{m+1}) \right] \\ &\quad \text{(by Assumption (4.7) and Lemma 4.2-(iii))} \\ &\leq \int_{\mathbb{R}^d} \bar{\nu}_m(dx) \mathbb{E} \left[\varphi(b_m(x, \bar{\nu}_m) + \theta_m(x, \bar{\nu}_m)Z_{m+1}) \right] \\ &\quad \text{(by Assumption (4.4), (4.10) and Lemma 4.2, since } \bar{\mu}_m \preceq_{cv} \bar{\nu}_m \text{)} \\ &= \mathbb{E} [\varphi(\bar{Y}_{m+1})]. \end{aligned}$$

Thus one has $\bar{X}_m \preceq_{cv} \bar{Y}_m$ for every $m = 0, \dots, M$ by a forward induction. Hence we can conclude by applying the convergence of the Euler scheme as in the proof of Corollary 1.1(a). \square

4.2 Global convex order for the Euler scheme

We will prove Proposition 4.1 in this section. For any $K \in \mathbb{N}^*$, we consider the norm on $(\mathbb{R}^d)^K$ defined by $\|x\| := \sup_{1 \leq i \leq K} |x_i|$ for every $x = (x_1, \dots, x_K) \in (\mathbb{R}^d)^K$, where $|\cdot|$ denotes the canonical Euclidean norm on \mathbb{R}^d . For any $m_1, m_2 \in \mathbb{N}^*$ with $m_1 \leq m_2$, we denote by $x_{m_1:m_2} := (x_{m_1}, x_{m_1+1}, \dots, x_{m_2}) \in (\mathbb{R}^d)^{m_2-m_1+1}$. Similarly, we denote by $\mu_{m_1:m_2} := (\mu_{m_1}, \dots, \mu_{m_2}) \in (\mathcal{P}_1(\mathbb{R}^d))^{m_2-m_1+1}$. We recursively define a sequence of functions ⁽⁴⁾

$$\Phi_m : (\mathbb{R}^d)^{m+1} \times (\mathcal{P}_1(\mathbb{R}^d))^{M-m+1} \rightarrow \mathbb{R}, \quad m = 0, \dots, M$$

in a backward way as follows:

► Set

$$\Phi_M(x_{0:M}; \mu_M) := F(x_0, \dots, x_M) \quad (4.11)$$

where $F : (\mathbb{R}^d)^{M+1} \rightarrow \mathbb{R}$ is a convex function with p -polynomial growth (4.8).

► For $m = 0, \dots, M-1$, set

$$\begin{aligned} \Phi_m(x_{0:m}; \mu_{m:M}) &:= (Q_{m+1} \Phi_{m+1}(x_{0:m}, \cdot; \mu_{m+1:M}))(x_m, \mu_m, \sigma_m(x_m, \mu_m)) \\ &= \mathbb{E} \left[\Phi_{m+1}(x_{0:m}, b_m(x_m, \mu_m) + \sigma_m(x_m, \mu_m) Z_{m+1}; \mu_{m+1:M}) \right]. \end{aligned} \quad (4.12)$$

The functions $\Phi_m, m = 0, \dots, M$, share the following properties.

Lemma 4.4. *For every $m = 0, \dots, M$,*

- (i) *for a fixed $\mu_{m:M} \in (\mathcal{P}_1(\mathbb{R}^d))^{M-m+1}$, the function $\Phi_m(\cdot; \mu_{m:M})$ is convex and has a p -polynomial growth in $x_{0:m}$ so that Φ_m is well-defined.*
- (ii) *for a fixed $x_{0:m} \in (\mathbb{R}^d)^{m+1}$, the function $\Phi_m(x_{0:m}; \cdot)$ is non-decreasing in $\mu_{m:M}$ with respect to the convex order in the sense that for any $\mu_{m:M}, \nu_{m:M} \in (\mathcal{P}_1(\mathbb{R}^d))^{M-m+1}$ such that $\mu_i \preceq_{cv} \nu_i, i = m, \dots, M$,*

$$\Phi_m(x_{0:m}; \mu_{m:M}) \leq \Phi_m(x_{0:m}; \nu_{m:M}). \quad (4.13)$$

Proof. (i) The function Φ_m is convex in $x_{0:M}$ owing to the hypotheses on F . Now assume that $x_{0:m+1} \mapsto \Phi_{m+1}(x_{0:m+1}; \mu_{m+1:M})$ is convex. For any $x_{0:m}, y_{0:m} \in (\mathbb{R}^d)^{m+1}$ and $\lambda \in [0, 1]$, it follows that

$$\begin{aligned} &\Phi_m(\lambda x_{0:m} + (1-\lambda)y_{0:m}; \mu_{m:M}) \\ &= \mathbb{E} \Phi_{m+1} \left(\lambda x_{0:m} + (1-\lambda)y_{0:m}, b_m(\lambda x_m + (1-\lambda)y_m, \mu_m) \right. \\ &\quad \left. + \sigma_m(\lambda x_m + (1-\lambda)y_m, \mu_m) Z_{m+1}; \mu_{m+1:M} \right) \\ &\leq \mathbb{E} \Phi_{m+1} \left(\lambda x_{0:m} + (1-\lambda)y_{0:m}, \lambda b_m(x_m, \mu_m) + (1-\lambda)b_m(y_m, \mu_m) \right. \\ &\quad \left. + [\lambda \sigma_m(x_m, \mu_m) + (1-\lambda)\sigma_m(y_m, \mu_m)] Z_{m+1}; \mu_{m+1:M} \right) \\ &\quad \text{(by Assumption (4.5) and Lemma 4.2 since } \Phi_{m+1}(x_{0:m}, \cdot; \mu_{m+1:M}) \text{ is convex)} \\ &\leq \lambda \mathbb{E} \left[\Phi_{m+1}(x_{0:m}, b_m(x_m, \mu_m) + \sigma_m(x_m, \mu_m) Z_{m+1}; \mu_{m+1:M}) \right] \end{aligned}$$

⁴We formally consider the case where Φ_M may depend on μ_M in view of the proof of Proposition 5.1 and item (b) of Theorem 1.

$$\begin{aligned}
& + (1 - \lambda) \mathbb{E} \left[\Phi_{m+1}(y_{0:m}, b_m(y_m, \mu_m) + \sigma_m(y_m, \mu_m) Z_{m+1}; \mu_{m+1:M}) \right] \\
& \quad (\text{since } x_{0:m+1} \mapsto \Phi_{m+1}(x_{0:m+1}; \mu_{m+1:M}) \text{ is convex}) \\
& = \lambda \Phi_m(x_{0:m}; \mu_{m:M}) + (1 - \lambda) \Phi_m(y_{0:m}; \mu_{m:M}).
\end{aligned}$$

Thus the function $\Phi_m(\cdot; \mu_{m:M})$ is convex and one concludes by a backward induction.

The function Φ_M has a p -polynomial growth by the assumption made on F . Now assume that Φ_{m+1} has a p -polynomial growth. As Assumption I implies that b_m and σ_m have linear growth (see further (A.1)), it is obvious that Φ_m has p -polynomial growth. Thus one concludes by a backward induction.

(ii) Firstly, it is obvious that for any $\mu_M, \nu_M \in \mathcal{P}_1(\mathbb{R}^d)$ such that $\mu_M \preceq_{cv} \nu_M$, we have

$$\Phi_M(x_{0:M}; \mu_M) = F(x_{0:M}) = \Phi_M(x_{0:M}; \nu_M).$$

Assume that $\Phi_{m+1}(x_{0:m+1}; \cdot)$ is non-decreasing with respect to the convex order of $\mu_{m+1:M}$. For any $\mu_{m:M}, \nu_{m:M} \in (\mathcal{P}_1(\mathbb{R}^d))^{M-m+1}$ such that $\mu_i \preceq_{cv} \nu_i, i = m, \dots, M$, we have

$$\begin{aligned}
\Phi_m(x_{0:m}; \mu_{m:M}) & = \mathbb{E} \left[\Phi_{m+1}(x_{0:m}, b_m(x_m, \mu_m) + \sigma_m(x_m, \mu_m) Z_{m+1}; \mu_{m+1:M}) \right] \\
& \leq \mathbb{E} \left[\Phi_{m+1}(x_{0:m}, b_m(x_m, \nu_m) + \sigma_m(x_m, \nu_m) Z_{m+1}; \mu_{m+1:M}) \right] \\
& \quad (\text{by Assumption (4.6), (4.4) and Lemma 4.2 since } \Phi_{m+1}(x_{0:m}, \cdot; \mu_{m+1:M}) \text{ is convex}) \\
& \leq \mathbb{E} \left[\Phi_{m+1}(x_{0:m}, b_m(x_m, \nu_m) + \sigma_m(x_m, \nu_m) Z_{m+1}; \nu_{m+1:M}) \right] \\
& \quad (\text{by the assumption on } \Phi_{m+1}) \\
& = \Phi_m(x_{0:m}; \nu_{m:M}).
\end{aligned}$$

Then one concludes by a backward induction. \square

As F has an p -polynomial growth, then the integrability of $F(\bar{X}_0, \dots, \bar{X}_M)$ and $F(\bar{Y}_0, \dots, \bar{Y}_M)$ is guaranteed by Proposition 2.1 since $X_0, Y_0 \in L^p(\mathbb{P})$. We define for every $m = 0, \dots, M$,

$$\mathcal{X}_m := \mathbb{E} [F(\bar{X}_0, \dots, \bar{X}_M) \mid \mathcal{F}_m].$$

Recall the notation $\bar{\mu}_m := \mathbb{P}_{\bar{X}_m} := \mathbb{P} \circ \bar{X}_m^{-1}, m = 0, \dots, M$.

Lemma 4.5. *For every $m = 0, \dots, M$, $\Phi_m(\bar{X}_{0:m}; \bar{\mu}_{m:M}) = \mathcal{X}_m$.*

Proof. It is obvious that $\Phi_M(\bar{X}_{0:M}; \bar{\mu}_M) = F(\bar{X}_0, \dots, \bar{X}_M) = \mathcal{X}_M$. Assume that $\Phi_{m+1}(\bar{X}_{0:m+1}; \bar{\mu}_{m+1:M}) = \mathcal{X}_{m+1}$. Then

$$\begin{aligned}
\mathcal{X}_m & = \mathbb{E} [\mathcal{X}_{m+1} \mid \mathcal{F}_m] = \mathbb{E} [\Phi_{m+1}(\bar{X}_{0:m+1}; \bar{\mu}_{m+1:M}) \mid \mathcal{F}_m] \\
& = \mathbb{E} [\Phi_{m+1}(\bar{X}_{0:m}, b_m(\bar{X}_m, \bar{\mu}_m) + \sigma_m(\bar{X}_m, \bar{\mu}_m) Z_{m+1}; \bar{\mu}_{m+1:M}) \mid \mathcal{F}_m] \\
& = (Q_{m+1} \Phi_{m+1}(\bar{X}_{0:m}, \cdot; \bar{\mu}_{m+1:M}))(\bar{X}_m, \bar{\mu}_m, \sigma_m(\bar{X}_m, \bar{\mu}_m)) = \Phi_m(\bar{X}_{0:m}; \bar{\mu}_{m:M}).
\end{aligned}$$

Then a backward induction completes the proof. \square

Similarly, we define $\Psi_m : (\mathbb{R}^d)^{m+1} \times (\mathcal{P}_1(\mathbb{R}^d))^{M-m+1} \rightarrow \mathbb{R}, m = 0, \dots, M$ by

$$\begin{aligned}
\Psi_M(x_{0:M}; \mu_M) & := F(x_{0:M}) \\
\Psi_m(x_{0:m}; \mu_{m:M}) & := (Q_{m+1} \Psi_{m+1}(x_{0:m}, \cdot; \mu_{m+1:M}))(x_m, \mu_m, \theta_m(x_m, \mu_m)) \\
& = \mathbb{E} \left[\Psi_{m+1}(x_{0:m}, b_m(x_m, \mu_m) + \theta_m(x_m, \mu_m) Z_{m+1}; \mu_{m+1:M}) \right]. \tag{4.14}
\end{aligned}$$

Recall the notation $\bar{\nu}_m := \mathbb{P}_{\bar{Y}_m}$. By using the same method of proof as for Lemma 4.5, we get

$$\Psi_m(\bar{Y}_{0:M}; \bar{\nu}_{m:M}) = \mathbb{E} [F(\bar{Y}_0, \dots, \bar{Y}_M) \mid \mathcal{F}_m].$$

Proof of Proposition 4.1. We first prove by a backward induction that for every $m = 0, \dots, M$, $\Phi_m \leq \Psi_m$.

It follows from the definition of Φ_M and Ψ_M that $\Phi_M = \Psi_M$. Assume now $\Phi_{m+1} \leq \Psi_{m+1}$. For any $x_{0:m} \in (\mathbb{R}^d)^{m+1}$ and $\mu_{m:M} \in (\mathcal{P}_1(\mathbb{R}^d))^{M-m+1}$, we have

$$\begin{aligned} & \Phi_m(x_{0:m}; \mu_{m:M}) \\ &= \mathbb{E} [\Phi_{m+1}(x_{0:m}, b_m(x_m, \mu_m) + \sigma_m(x_m, \mu_m)Z_{m+1}; \mu_{m+1:M})] \\ &\leq \mathbb{E} [\Phi_{m+1}(x_{0:m}, b_m(x_m, \mu_m) + \theta_m(x_m, \mu_m)Z_{m+1}; \mu_{m+1:M})] \\ &\quad (\text{by Assumption (4.7) and Lemma 4.2, since Lemma 4.4 shows that } \Phi_{m+1} \text{ is convex in } x_{0:m+1}) \\ &\leq \mathbb{E} [\Psi_{m+1}(x_{0:m}, b_m(x_m, \mu_m) + \theta_m(x_m, \mu_m)Z_{m+1}; \mu_{m+1:M})] = \Psi_m(x_{0:m}; \mu_{m:M}). \end{aligned}$$

Thus, the backward induction is completed and

$$\forall m = 0, \dots, M, \quad \Phi_m \leq \Psi_m. \quad (4.15)$$

Consequently,

$$\begin{aligned} \mathbb{E} [F(\bar{X}_0, \dots, \bar{X}_M)] &= \mathbb{E} \Phi_0(\bar{X}_0; \bar{\mu}_{0:M}) && (\text{by Lemma 4.5}) \\ &\leq \mathbb{E} \Phi_0(\bar{Y}_0; \bar{\mu}_{0:M}) && (\text{by Lemma 4.4-(i) since } \bar{X}_0 = X_0 \preceq_{cv} Y_0 = \bar{Y}_0) \\ &\leq \mathbb{E} \Phi_0(\bar{Y}_0; \bar{\nu}_{0:M}) && (\text{by Lemma 4.4-(ii) and Proposition 4.2}) \\ &\leq \mathbb{E} \Psi_0(\bar{Y}_0; \bar{\nu}_{0:M}) && (\text{by (4.15)}) \\ &= \mathbb{E} [F(\bar{Y}_0, \dots, \bar{Y}_M)]. \end{aligned} \quad \square$$

5 Functional convex order for the McKean-Vlasov process

This section is devoted to prove Theorem 1-(a). Recall that $t_m^M = m \cdot \frac{T}{M}$, $m = 0, \dots, M$. We define two interpolators as follows.

Definition 5.1. (i) For every integer $M \geq 1$, we define the piecewise affine interpolator $i_M : x_{0:M} \in (\mathbb{R}^d)^{M+1} \mapsto i_M(x_{0:M}) \in \mathcal{C}([0, T], \mathbb{R}^d)$ by

$$\forall m = 0, \dots, M-1, \forall t \in [t_m^M, t_{m+1}^M], \quad i_M(x_{0:M})(t) = \frac{M}{T} [(t_{m+1}^M - t)x_m + (t - t_m^M)x_{m+1}].$$

(ii) For every $M \geq 1$, we define the functional interpolator $I_M : \mathcal{C}([0, T], \mathbb{R}^d) \rightarrow \mathcal{C}([0, T], \mathbb{R}^d)$ by

$$\forall \alpha \in \mathcal{C}([0, T], \mathbb{R}^d), \quad I_M(\alpha) = i_M(\alpha(t_0^M), \dots, \alpha(t_M^M)).$$

It is obvious that

$$\forall x_{0:M} \in (\mathbb{R}^d)^{M+1}, \quad \|i_M(x_{0:M})\|_{\text{sup}} = \max_{0 \leq m \leq M} |x_m| \quad (5.1)$$

since the norm $|\cdot|$ is convex. Consequently,

$$\forall \alpha \in \mathcal{C}([0, T], \mathbb{R}^d), \quad \|I_M(\alpha)\|_{\text{sup}} \leq \|\alpha\|_{\text{sup}}. \quad (5.2)$$

Moreover, for any $\alpha \in \mathcal{C}([0, T], \mathbb{R}^d)$, we have

$$\|I_M(\alpha) - \alpha\|_{\text{sup}} \leq w(\alpha, \frac{T}{M}), \quad (5.3)$$

where w denotes the uniform continuity modulus of α . The proof of Theorem 1-(a) relies on the following lemma.

Lemma 5.1 (Lemma 2.2 in [Pag16]). *Let $X^M, M \geq 1$, be a sequence of continuous processes weakly converging towards X as $M \rightarrow +\infty$ for the $\|\cdot\|_{\text{sup}}$ -norm topology. Then, the sequence of interpolating processes $\tilde{X}^M = I_M(X^M), M \geq 1$ is weakly converging toward X for the $\|\cdot\|_{\text{sup}}$ -norm topology.*

Proof of Theorem 1-(a). Let $M \in \mathbb{N}^*$. Let $(\bar{X}_{t_m}^M)_{m=0,\dots,M}$ and $(\bar{Y}_{t_m}^M)_{m=0,\dots,M}$ denote the Euler scheme defined in (2.1) and (2.2). Let $\bar{X}^M := (\bar{X}_t^M)_{t \in [0,T]}$, $\bar{Y}^M := (\bar{Y}_t^M)_{t \in [0,T]}$ denote the continuous Euler scheme of $(X_t)_{t \in [0,T]}$, $(Y_t)_{t \in [0,T]}$ defined by (2.3) and (2.4). By Proposition 2.1, there exists a constant \tilde{C} such that

$$\begin{aligned} \left\| \sup_{t \in [0,T]} |\bar{X}_t^M| \right\|_p \vee \left\| \sup_{t \in [0,T]} |X_t| \right\|_p &\leq \tilde{C}(1 + \|X_0\|_p) < +\infty, \\ \left\| \sup_{t \in [0,T]} |\bar{Y}_t^M| \right\|_p \vee \left\| \sup_{t \in [0,T]} |Y_t| \right\|_p &\leq \tilde{C}(1 + \|Y_0\|_p) < +\infty \end{aligned} \quad (5.4)$$

as $1 \leq r \leq p$ and $X_0, Y_0 \in L^p(\mathbb{P})$. Hence, $F(X)$ and $F(Y)$ are in $L^1(\mathbb{P})$ since F has a r -polynomial growth.

We define a function $F_M : (\mathbb{R}^d)^{M+1} \rightarrow \mathbb{R}$ by

$$x_{0:M} \in (\mathbb{R}^d)^{M+1} \mapsto F_M(x_{0:M}) := F(i_M(x_{0:M})). \quad (5.5)$$

The function F_M is obviously convex since i_M is a linear application. Moreover, F_M has also an r -polynomial growth (on \mathbb{R}^{M+1}) by (5.1).

Furthermore, we have $I_M(\bar{X}^M) = i_M((\bar{X}_{t_0}^M, \dots, \bar{X}_{t_M}^M))$ by the definition of the continuous Euler scheme and the interpolators i_M and I_M , so that

$$F_M(\bar{X}_{t_0}^M, \dots, \bar{X}_{t_M}^M) = F(i_M((\bar{X}_{t_0}^M, \dots, \bar{X}_{t_M}^M))) = F(I_M(\bar{X}^M)).$$

It follows from Proposition 4.1 that

$$\begin{aligned} \mathbb{E} F(I_M(\bar{X}^M)) &= \mathbb{E} F(i_M(\bar{X}_0^M, \dots, \bar{X}_M^M)) = \mathbb{E} F_M(\bar{X}_0^M, \dots, \bar{X}_M^M) \\ &\leq \mathbb{E} F_M(\bar{Y}_0^M, \dots, \bar{Y}_M^M) = \mathbb{E} F(i_M(\bar{Y}_0^M, \dots, \bar{Y}_M^M)) = \mathbb{E} F(I_M(\bar{Y}^M)). \end{aligned} \quad (5.6)$$

The function F is $\|\cdot\|_{\text{sup}}$ -continuous since it is convex with $\|\cdot\|_{\text{sup}}$ -polynomial growth (see Lemma 2.1.1 in [Luc06]). Moreover the process \bar{X}^M weakly converges for the sup-norm topology to X as $M \rightarrow +\infty$ as a consequence of Proposition 2.1. Then $I_M(\bar{X}^M)$ weakly converges for the sup-norm topology to X owing to Lemma 5.1. This proves that $F(I_M(\bar{X}^M))$ weakly converges toward $F(X)$ and, similarly, that $F(I_M(\bar{Y}^M))$ weakly converges toward $F(Y)$. Moreover, as F has a p -polynomial growth, we have

$$F(I_M(\bar{X}^M)) \leq C(1 + \|I_M(\bar{X}^M)\|_{\text{sup}}^p) \leq C(1 + \|\bar{X}^M\|_{\text{sup}}^p)$$

where the last inequality follows from (5.2). It follows from Proposition 2.1 that

$$\mathbb{E} \|\bar{X}^M\|_{\text{sup}}^p \rightarrow \mathbb{E} \|X\|_{\text{sup}}^p \quad \text{as } M \rightarrow +\infty.$$

Then one derives that $\mathbb{E} F(I_M(\bar{X}^M)) \rightarrow \mathbb{E} F(X)$ as $M \rightarrow +\infty$. The same reasoning shows that $\mathbb{E} F(I_M(\bar{Y}^M)) \rightarrow \mathbb{E} F(Y)$. Finally, one derives by letting $M \rightarrow +\infty$ in inequality (5.6) that

$$\mathbb{E} F(X) \leq \mathbb{E} F(Y). \quad \square$$

Remark 5.1. The functional convex order result, in a general setting, can be used to establish a robust option price bound (see e.g. [ACJ19a]). However, in the McKean-Vlasov setting, the functional convex order result Theorem 1 is established by using the *theoretical* Euler scheme (2.1) and (2.2) which can not be directly simulated so that there are still some work to do to produce simulable approximations which are consistent for the convex order. One simulable approximation of the McKean-Vlasov equation is the particle method (see e.g. [BT97], [AKH02] and [Liu19, Section 7.1] among many other references), which, in the context of this paper, can be written as follows: for $n = 1, \dots, N$,

$$\begin{cases} \bar{X}_{t_{m+1}}^{n,N} = \bar{X}_{t_m}^{n,N} + h \cdot b(t_m, \bar{X}_{t_m}^{n,N}, \frac{1}{N} \sum_{n=1}^N \delta_{\bar{X}_{t_m}^{n,N}}) + \sqrt{h} \cdot \sigma(t_m, \bar{X}_{t_m}^{n,N}, \frac{1}{N} \sum_{n=1}^N \delta_{\bar{X}_{t_m}^{n,N}}) Z_{m+1}^n, \\ \bar{Y}_{t_{m+1}}^{n,N} = \bar{Y}_{t_m}^{n,N} + h \cdot b(t_m, \bar{Y}_{t_m}^{n,N}, \frac{1}{N} \sum_{n=1}^N \delta_{\bar{Y}_{t_m}^{n,N}}) + \sqrt{h} \cdot \theta(t_m, \bar{Y}_{t_m}^{n,N}, \frac{1}{N} \sum_{n=1}^N \delta_{\bar{Y}_{t_m}^{n,N}}) Z_{m+1}^n, \end{cases} \quad (5.7)$$

where $\bar{X}_0^{n,N}, 1 \leq n \leq N, \overset{i.i.d.}{\sim} X_0, \bar{Y}_0^{n,N}, 1 \leq n \leq N \overset{i.i.d.}{\sim} Y_0, t_m = t_m^M := m \cdot \frac{T}{M}, M \in \mathbb{N}^*$ and $Z_m^n, 0 \leq n \leq N, 0 \leq m \leq M, \overset{i.i.d.}{\sim} \mathcal{N}(0, \mathbf{I}_q)$.

Unfortunately, this scheme (5.7) based on particles does not propagate nor preserve the convex order as in Proposition 4.2 since we cannot obtain for a convex function φ that,

$$\frac{1}{N} \sum_{n=1}^N \varphi(X_{t_m}^{n,N}(\omega)) \leq \frac{1}{N} \sum_{n=1}^N \varphi(Y_{t_m}^{n,N}(\omega)), \quad a.s.$$

under the condition that $X_{t_m}^{n,N} \preceq_{cv} Y_{t_m}^{n,N}, n = 1, \dots, N$, even if the random variables $X_{t_m}^{n,N}, n = 1, \dots, N$ and $Y_{t_m}^{n,N}, n = 1, \dots, N$ were both i.i.d. (see again [ACJ19a]).

5.1 Extension of the functional convex order result

This section is devoted to the proof of Theorem 1-(b). We first discuss the marginal distribution space for the strong solutions $X = (X_t)_{t \in [0, T]}$ and $Y = (Y_t)_{t \in [0, T]}$ of equations (1.3) and (1.4). By Proposition 2.1, $X, Y \in L^p_{\mathcal{C}([0, T], \mathbb{R}^d)}(\Omega, \mathcal{F}, \mathbb{P})$ then their probability distributions μ, ν naturally lie in

$$\begin{aligned} & \mathcal{P}_p(\mathcal{C}([0, T], \mathbb{R}^d)) \\ & := \left\{ \mu \text{ probability distribution on } \mathcal{C}([0, T], \mathbb{R}^d) \text{ s.t. } \int_{\mathcal{C}([0, T], \mathbb{R}^d)} \|\alpha\|_{\text{sup}}^p \mu(d\alpha) < +\infty \right\}. \end{aligned}$$

We define an L^p -Wasserstein distance \mathbb{W}_p on $\mathcal{P}_p(\mathcal{C}([0, T], \mathbb{R}^d))$ by

$$\begin{aligned} & \forall \mu, \nu \in \mathcal{P}_p(\mathcal{C}([0, T], \mathbb{R}^d)), \\ & \mathbb{W}_p(\mu, \nu) := \left[\inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathcal{C}([0, T], \mathbb{R}^d) \times \mathcal{C}([0, T], \mathbb{R}^d)} \|x - y\|_{\text{sup}}^p \pi(dx, dy) \right]^{\frac{1}{p}}, \end{aligned} \quad (5.8)$$

where $\Pi(\mu, \nu)$ denotes the set of probability measures on $\mathcal{C}([0, T], \mathbb{R}^d) \times \mathcal{C}([0, T], \mathbb{R}^d)$ with respective marginals μ and ν . The space $\mathcal{P}_p(\mathcal{C}([0, T], \mathbb{R}^d))$ equipped with \mathbb{W}_p is complete and separable since $(\mathcal{C}([0, T], \mathbb{R}^d), \|\cdot\|_{\text{sup}})$ is a Polish space (see [Bo108]).

Now, we prove that for any stochastic process $X = (X_t)_{t \in [0, T]} \in L^p_{\mathcal{C}([0, T], \mathbb{R}^d)}(\Omega, \mathcal{F}, \mathbb{P})$, its marginal distribution $(\mu_t)_{t \in [0, T]}$ lies in $\mathcal{C}([0, T], \mathcal{P}_p(\mathbb{R}^d))$. For any $t \in [0, T]$, we define $\pi_t : \mathcal{C}([0, T], \mathbb{R}^d) \rightarrow \mathbb{R}^d$ by $\alpha \mapsto \pi_t(\alpha) = \alpha_t$ and we define

$$\iota : \mu \in \mathcal{P}_p(\mathcal{C}([0, T], \mathbb{R}^d)) \longmapsto \iota(\mu) := (\mu \circ \pi_t^{-1})_{t \in [0, T]} = (\mu_t)_{t \in [0, T]} \in \mathcal{C}([0, T], \mathcal{P}_p(\mathbb{R}^d)).$$

Lemma 5.2. *The application ι is well-defined.*

The proof of Lemma 5.2 is postponed in Appendix D. For the functional convex order result for the

Euler scheme, like in Section 4, we have the following proposition.

Proposition 5.1. *Let $\bar{X}_{0:M}, \bar{Y}_{0:M}, \bar{\mu}_{0:M}, \bar{\nu}_{0:M}$ be respectively random variables and probability distributions defined by (2.1) and (2.2). Under Assumption I and II, for any function*

$$\tilde{G} : (x_{0:M}, \eta_{0:M}) \in (\mathbb{R}^d)^{M+1} \times (\mathcal{P}_p(\mathbb{R}^d))^{M+1} \longmapsto \tilde{G}(x_{0:M}, \eta_{0:M}) \in \mathbb{R}$$

satisfying the following conditions (i), (ii) and (iii)

(i) \tilde{G} is convex in $x_{0:M}$,

(ii) \tilde{G} is non-decreasing in $\mu_{0:M}$ with respect to the convex order in the sense that

$$\forall x_{0:M} \in (\mathbb{R}^d)^{M+1} \text{ and } \forall \mu_{0:M}, \nu_{0:M} \in (\mathcal{P}_p(\mathbb{R}^d))^{M+1} \text{ s.t. } \mu_i \preceq_{cv} \nu_i, 0 \leq i \leq M, \\ \tilde{G}(x_{0:M}, \mu_{0:M}) \leq \tilde{G}(x_{0:M}, \nu_{0:M}),$$

(iii) \tilde{G} has a p -polynomial growth in the sense that

$$\exists C \in \mathbb{R}_+ \text{ s.t. } \forall (x_{0:M}, \mu_{0:M}) \in (\mathbb{R}^d)^{M+1} \times (\mathcal{P}_p(\mathbb{R}^d))^{M+1}, \\ \tilde{G}(x_{0:M}, \mu_{0:M}) \leq C \left[1 + \sup_{0 \leq m \leq M} |x_m|^p + \sup_{0 \leq m \leq M} \mathcal{W}_p^p(\mu_m, \delta_0) \right], \quad (5.9)$$

we have

$$\mathbb{E} \tilde{G}(\bar{X}_0, \dots, \bar{X}_M, \bar{\mu}_0, \dots, \bar{\mu}_M) \leq \mathbb{E} \tilde{G}(\bar{Y}_0, \dots, \bar{Y}_M, \bar{\nu}_0, \dots, \bar{\nu}_M). \quad (5.10)$$

The proof of Proposition 5.1 is quite similar to that of Proposition 4.1. We just need to replace the definition of Φ_m and Ψ_m in (4.11), (4.12) and (4.14) by the following $\Phi'_m, \Psi'_m : (\mathbb{R}^d)^{m+1} \times (\mathcal{P}_p(\mathbb{R}^d))^{m+1} \rightarrow \mathbb{R}$, $m = 0, \dots, M$, defined by

$$\forall (x_{0:m}, \mu_{0:M}) \in (\mathbb{R}^d)^{m+1} \times (\mathcal{P}_p(\mathbb{R}^d))^{M+1}, \\ \Phi'_M(x_{0:M}; \mu_{0:M}) = \tilde{G}(x_{0:M}, \mu_{0:M}), \\ \Phi'_m(x_{0:m}; \mu_{0:M}) = (Q_{m+1} \Phi'_{m+1}(x_{0:m}, \cdot; \mu_{0:M}))(x_m, \mu_m, \sigma_m(x_m, \mu_m)), \\ \Psi'_M(x_{0:M}; \mu_{0:M}) = \tilde{G}(x_{0:M}, \mu_{0:M}), \\ \Psi'_m(x_{0:m}; \mu_{0:M}) = (Q_{m+1} \Psi'_{m+1}(x_{0:m}, \cdot; \mu_{0:M}))(x_m, \mu_m, \theta_m(x_m, \mu_m)).$$

The key step to prove Theorem 1(b) starting from (5.10) is how to define the ‘‘interpolator’’ of the marginal distributions $(\bar{\mu}_t)_{t \in [0, T]}$ and $(\bar{\nu}_t)_{t \in [0, T]}$. Let $\lambda \in [0, 1]$. For any two random variables X_1, X_2 with respective probability distributions $\mu_1, \mu_2 \in \mathcal{P}_p(\mathbb{R}^d)$, we define the linear combination of μ_1, μ_2 , denoted by $\lambda\mu_1 + (1 - \lambda)\mu_2$, by

$$\forall A \in \mathcal{B}(\mathbb{R}^d), \quad (\lambda\mu_1 + (1 - \lambda)\mu_2)(A) := \lambda\mu_1(A) + (1 - \lambda)\mu_2(A). \quad (5.11)$$

In fact, $\lambda\mu_1 + (1 - \lambda)\mu_2$ is the distribution of the random variable

$$\mathbb{1}_{\{U \leq \lambda\}} X_1 + \mathbb{1}_{\{U > \lambda\}} X_2, \quad (5.12)$$

where U is a random variable with probability distribution $\mathcal{U}([0, 1])$, independent of (X_1, X_2) . Then it is obvious that $\lambda\mu_1 + (1 - \lambda)\mu_2 \in \mathcal{P}_p(\mathbb{R}^d)$. Moreover, with the help of the random variable (5.12), one proves that the application $\lambda \in [0, 1] \mapsto \lambda\mu_1 + (1 - \lambda)\mu_2 \in \mathcal{P}_p(\mathbb{R}^d)$ is continuous with respect to \mathcal{W}_p for a fixed $(\mu_1, \mu_2) \in (\mathcal{P}_p(\mathbb{R}^d))^2$.

From (5.11), we can extend the definition of the interpolator i_M (respectively I_M) on the probability

distribution space $(\mathcal{P}_p(\mathbb{R}^d))^{M+1}$ (resp. $\mathcal{C}([0, T], \mathcal{P}_p(\mathbb{R}^d))$) as follows

$$\begin{aligned} \forall m = 0, \dots, M-1, \forall t \in [t_m^M, t_{m+1}^M], \\ \forall \mu_{0:M} \in (\mathcal{P}_p(\mathbb{R}^d))^{M+1}, \quad i_M(\mu_{0:M})(t) = \frac{M}{T} [(t_{m+1}^M - t)\mu_m + (t - t_m^M)\mu_{m+1}], \\ \forall (\mu_t)_{t \in [0, T]} \in \mathcal{C}([0, T], \mathcal{P}_p(\mathbb{R}^d)), \quad I_M((\mu_t)_{t \in [0, T]}) = i_M(\mu_{t_0^M}, \dots, \mu_{t_M^M}). \end{aligned}$$

We consider now $\bar{X}^M = (\bar{X}_t^M)_{t \in [0, T]}$, $\bar{Y}^M = (\bar{Y}_t^M)_{t \in [0, T]}$ defined by (2.3) and (2.4) with respective probability distributions $\bar{\mu}^M, \bar{\nu}^M \in \mathcal{P}_p(\mathcal{C}([0, T], \mathbb{R}^d))$ (see (2.5)). Let $(\bar{\mu}_t^M)_{t \in [0, T]} = \iota(\bar{\mu}^M)$ and $(\bar{\nu}_t^M)_{t \in [0, T]} = \iota(\bar{\nu}^M)$. We define now for every $t \in [0, T]$, $\tilde{\mu}_t^M := I_M((\bar{\mu}_t^M)_{t \in [0, T]})_t$. By the same idea as (5.12), for every $t \in [t_m^M, t_{m+1}^M]$, $\tilde{\mu}_t^M$ is the probability distribution of the random variable

$$\tilde{X}_t^M := \mathbb{1}_{\left\{U_m \leq \frac{M(t_{m+1}^M - t)}{T}\right\}} \bar{X}_{t_m}^M + \mathbb{1}_{\left\{U_m > \frac{M(t_{m+1}^M - t)}{T}\right\}} \bar{X}_{t_{m+1}}^M,$$

where (U_0, \dots, U_M) is independent to the Brownian motion $(B_t)_{t \in [0, T]}$ in (1.3), (1.4) and then to (Z_0, \dots, Z_M) in (2.1), (2.2).

Now we prove that $(\tilde{\mu}_t^M)_{t \in [0, T]}$ converges to the weak solution $(\mu_t)_{t \in [0, T]}$ of (1.3) with respect to the distance $d_{\mathcal{C}}$ defined in (1.14). We know from Proposition 2.1-(c) that for any $p \geq 2$

$$\sup_{t \in [0, T]} \mathcal{W}_p(\mu_t, \tilde{\mu}_t^M) \rightarrow 0 \quad \text{as } M \rightarrow +\infty. \quad (5.13)$$

It follows that

$$\begin{aligned} \forall m \in \{0, \dots, M\}, \forall t \in [t_m^M, t_{m+1}^M], \\ \mathcal{W}_p^p(\bar{\mu}_t^M, \tilde{\mu}_t^M) \leq \mathbb{E} \left| \bar{X}_t^M - \tilde{X}_t^M \right|^p \\ = \mathbb{E} \left| \bar{X}_t^M - \mathbb{1}_{\left\{U_m \leq \frac{M(t_{m+1}^M - t)}{T}\right\}} \bar{X}_{t_m}^M - \mathbb{1}_{\left\{U_m > \frac{M(t_{m+1}^M - t)}{T}\right\}} \bar{X}_{t_{m+1}}^M \right|^p \\ \leq \mathbb{E} \left| \bar{X}_t^M - \bar{X}_{t_m}^M \right|^p + \mathbb{E} \left| \bar{X}_t^M - \bar{X}_{t_{m+1}}^M \right|^p. \end{aligned}$$

We derive from Proposition 2.1-(b) that

$$\forall s, t \in [t_m^M, t_{m+1}^M], s < t, \quad \mathbb{E} \left| \bar{X}_t^M - \bar{X}_s^M \right|^p \leq (\kappa \sqrt{t-s})^p \leq \kappa^p \left(\frac{T}{M}\right)^{\frac{p}{2}} \rightarrow 0, \quad \text{as } M \rightarrow +\infty.$$

Thus, we have $\sup_{t \in [0, T]} \mathcal{W}_p^p(\bar{\mu}_t^M, \tilde{\mu}_t^M) \rightarrow 0$ as $M \rightarrow +\infty$. Hence,

$$\begin{aligned} d_{\mathcal{C}}((\tilde{\mu}_t^M)_{t \in [0, T]}, (\mu_t)_{t \in [0, T]}) &= \sup_{t \in [0, T]} \mathcal{W}_p^p(\mu_t, \tilde{\mu}_t^M) \leq \sup_{t \in [0, T]} \mathcal{W}_p^p(\bar{\mu}_t^M, \mu_t) + \sup_{t \in [0, T]} \mathcal{W}_p^p(\bar{\mu}_t^M, \tilde{\mu}_t^M) \\ &\rightarrow 0 \quad \text{as } M \rightarrow +\infty. \end{aligned} \quad (5.14)$$

Proof of Theorem 1-(b). We define for every $(x_{0:M}, \eta_{0:M}) \in (\mathbb{R}^d)^{M+1} \times (\mathcal{P}_p(\mathbb{R}^d))^{M+1}$, $G_M(x_{0:M}, \eta_{0:M}) := G(i_M(x_{0:M}), i_M(\eta_{0:M}))$. For every fixed $M \in \mathbb{N}^*$, we have

$$\begin{aligned} \mathbb{E} G(I_M(\bar{X}^M), (\tilde{\mu}_t^M)_{t \in [0, T]}) &= \mathbb{E} G\left(I_M(\bar{X}^M), I_M((\bar{\mu}_t^M)_{t \in [0, T]})\right) \\ &= \mathbb{E} G(i_M(\bar{X}_{t_0}^M, \dots, \bar{X}_{t_M}^M), i_M(\bar{\mu}_{t_0}^M, \dots, \bar{\mu}_{t_M}^M)) = \mathbb{E} G_M(\bar{X}_{t_0}^M, \dots, \bar{X}_{t_M}^M, \bar{\mu}_0^M, \dots, \bar{\mu}_{t_M}^M) \\ &\leq \mathbb{E} G_M(\bar{Y}_{t_0}^M, \dots, \bar{Y}_{t_M}^M, \bar{\nu}_{t_0}^M, \dots, \bar{\nu}_{t_M}^M) \quad (\text{by Proposition 5.1}) \\ &= \mathbb{E} G(i_M(\bar{Y}_{t_0}^M, \dots, \bar{Y}_{t_M}^M), i_M(\bar{\nu}_{t_0}^M, \dots, \bar{\nu}_{t_M}^M)) = \mathbb{E} G(I_M(\bar{Y}^M), (\bar{\nu}_t^M)_{t \in [0, T]}). \end{aligned}$$

Using the continuity assumption on G (see Theorem 1-(b)-(iii)) and the convergence in (5.14) imply weak convergence of both sequences of random variables. Then using that G has at most p -polynomial growth in both space and measure arguments, one concludes like for claim (a) that $\mathbb{E} G(I_M(\bar{X}^M), (\tilde{\mu}_t^M)_{t \in [0, T]}) \rightarrow \mathbb{E} G(X, (\mu_t)_{t \in [0, T]})$ (idem for Y). Combining these two properties, we finally obtain (1.17) by letting $M \rightarrow +\infty$. \square

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Appendix A Convergence rate of the Euler scheme for the McKean-Vlasov equation

We prove Proposition 2.1 in this section. Under Assumption I, the functions b and σ have a linear growth in x and in μ in the sense that there exists a constant $C_{b,\sigma,L,T}$ depending on b, σ, L and T such that for any $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d)$,

$$|b(t, x, \mu)| \vee \|\sigma(t, x, \mu)\| \leq C_{b,\sigma,L,T}(1 + |x| + \mathcal{W}_p(\mu, \delta_0)), \quad (\text{A.1})$$

since for any $x \in \mathbb{R}^d$ and for any $\mu \in \mathcal{P}_p(\mathbb{R}^d)$, we have for every $t \in [0, T]$,

$$|b(t, x, \mu)| \leq |b(t, 0, \delta_0)| + L(|x| + \mathcal{W}_p(\mu, \delta_0)) \leq (|b(t, 0, \delta_0)| \vee L)(1 + |x| + \mathcal{W}_p(\mu, \delta_0))$$

and $\|\sigma(t, x, \mu)\| \leq (\|\sigma(t, 0, \delta_0)\| \vee L)(1 + |x| + \mathcal{W}_p(\mu, \delta_0))$ by applying (1.8) so that one can take *e.g.* $C_{b,\sigma,L,T} := \sup_{t \in [0, T]} |b(t, 0, \delta_0)| \vee \sup_{t \in [0, T]} \|\sigma(t, 0, \delta_0)\| \vee L$.

Moreover, the definition of continuous time Euler scheme (2.3) implies that $\bar{X} := (\bar{X}_t)_{t \in [0, T]}$ is a $\mathcal{C}([0, T], \mathbb{R}^d)$ -valued stochastic process. Let $\bar{\mu}$ denote the probability distribution of \bar{X} and for every $t \in [0, T]$, let $\bar{\mu}_t$ denote the marginal distribution of \bar{X}_t . Then $(\bar{X}_t)_{t \in [0, T]}$ is the solution of

$$\begin{cases} d\bar{X}_t = b(\underline{t}, \bar{X}_{\underline{t}}, \bar{\mu}_{\underline{t}})dt + \sigma(\underline{t}, \bar{X}_{\underline{t}}, \bar{\mu}_{\underline{t}})dB_t, \\ \bar{X}_0 = X_0, \end{cases} \quad (\text{A.2})$$

where for every $t \in [t_m, t_{m+1})$, $\underline{t} := t_m$.

Now we recall a variant of Gronwall's Lemma (see Lemma 7.3 in [Pag18] for a proof) and two important technical tools used throughout the proof: the generalized Minkowski Inequality and the Burkölder-Davis-Gundy Inequality. We refer to [Pag18, Section 7.8] and [RY99, Chapter IV - Section 4] for proofs (among many others).

Lemma A.1 (“À la Gronwall” Lemma). *Let $f : [0, T] \rightarrow \mathbb{R}_+$ be a Borel, locally bounded, non-negative and non-decreasing function and let $\psi : [0, T] \rightarrow \mathbb{R}_+$ be a non-negative non-decreasing function satisfying*

$$\forall t \in [0, T], f(t) \leq A \int_0^t f(s)ds + B \left(\int_0^t f^2(s)ds \right)^{\frac{1}{2}} + \psi(t),$$

where A, B are two positive real constants. Then, for any $t \in [0, T]$,

$$f(t) \leq 2e^{(2A+B^2)t}\psi(t).$$

Proposition A.1 (The Generalized Minkowski Inequality). *For any (bi-measurable) process $X = (X_t)_{t \geq 0}$, for every $p \in [1, \infty)$ and for every $T \in [0, +\infty]$,*

$$\left\| \int_0^T X_t dt \right\|_p \leq \int_0^T \|X_t\|_p dt. \quad (\text{A.3})$$

Theorem 3 (Burkölder-Davis-Gundy Inequality (continuous time)). *For every $p \in (0, +\infty)$, there exists two real constants $C_p^{BDG} > c_p^{BDG} > 0$ such that, for every continuous local martingale $(X_t)_{t \in [0, T]}$ null at 0 with sharp bracket process $(\langle X \rangle_t)_{t \in [0, T]}$,*

$$c_p^{BDG} \left\| \sqrt{\langle X \rangle_T} \right\|_p \leq \left\| \sup_{t \in [0, T]} |X_t| \right\|_p \leq C_p^{BDG} \left\| \sqrt{\langle X \rangle_T} \right\|_p.$$

In particular, if (B_t) is an (\mathcal{F}_t) -standard Brownian motion and $(H_t)_{t \geq 0}$ is an (\mathcal{F}_t) -progressively mea-

surable process having values in $\mathbb{M}_{d \times q}(\mathbb{R})$ such that $\int_0^T \|H_t\|^2 dt < +\infty$ \mathbb{P} -a.s., then the d -dimensional local martingale $\int_0^\cdot H_s dB_s$ satisfies

$$\left\| \sup_{t \in [0, T]} \left| \int_0^t H_s dB_s \right| \right\|_p \leq C_{d,p}^{BDG} \left\| \sqrt{\int_0^T \|H_t\|^2 dt} \right\|_p. \quad (\text{A.4})$$

where $C_{d,p}^{BDG}$ only depends on p, d .

Proof of Proposition 2.1-(a). (a) If X is the unique strong solution of (1.3), then its probability distribution μ is the unique weak solution. We define two new coefficient functions depending on $\nu(\mu) = (\mu_t)_{t \in [0, T]}$ by

$$\tilde{b}(t, x) := b(t, x, \mu_t) \quad \text{and} \quad \tilde{\sigma}(t, x) := \sigma(t, x, \mu_t).$$

Now we discuss the continuity in t of \tilde{b} and $\tilde{\sigma}$. In fact,

$$\begin{aligned} \left| \tilde{b}(t, x) - \tilde{b}(s, x) \right| &\leq |b(t, x, \mu_t) - b(s, x, \mu_s)| \\ &\leq |b(t, x, \mu_t) - b(s, x, \mu_t)| + |b(s, x, \mu_t) - b(s, x, \mu_s)| \\ &\leq |b(t, x, \mu_t) - b(s, x, \mu_t)| + \mathcal{W}_p(\mu_t, \mu_s), \end{aligned} \quad (\text{A.5})$$

and we have a similar inequality for $\tilde{\sigma}$. Moreover, we know from Assumption I that b and σ are continuous in t and from Lemma 5.2 that $\nu(\mu) = (\mu_t)_{t \in [0, T]} \in \mathcal{C}([0, T], \mathcal{P}_p(\mathbb{R}^d))$. Hence, \tilde{b} and $\tilde{\sigma}$ are continuous in t . Moreover, it is obvious that \tilde{b} and $\tilde{\sigma}$ are still Lipschitz continuous in x . Consequently, X is also the unique strong solution of the following stochastic differential equation

$$dX_t = \tilde{b}(t, X_t)dt + \tilde{\sigma}(t, X_t)dB_t, \quad X_0 \text{ same as in (1.3)}.$$

Hence, the inequality

$$\left\| \sup_{u \in [0, t]} |X_u| \right\|_p \leq C_{p,d,b,\sigma} e^{C_{p,d,b,\sigma} t} (1 + \|X_0\|_p)$$

can be obtained by the usual method for the regular stochastic differential equation for which we refer to [Pag18, Proposition 7.2 and (7.12)] among many other references.

Next, we prove the inequality for $\left\| \sup_{u \in [0, t]} |\bar{X}_u^M| \right\|_p$.

We go back to the discrete Euler scheme

$$\bar{X}_{t_{m+1}}^M = \bar{X}_{t_m}^M + h \cdot b(t_m, \bar{X}_{t_m}^M, \bar{\mu}_{t_m}^M) + \sqrt{h} \sigma(t_m, \bar{X}_{t_m}^M, \bar{\mu}_{t_m}^M) Z_{m+1}.$$

We write \bar{X}_{t_m} instead of $\bar{X}_{t_m}^M$ in the following. By Minkowski's inequality, we have

$$\|\bar{X}_{t_{m+1}}\|_p = \|\bar{X}_{t_m}\|_p + h \|b(t_m, \bar{X}_{t_m}, \bar{\mu}_{t_m})\|_p + \sqrt{h} \left\| \left\| \sigma(t_m, \bar{X}_{t_m}, \bar{\mu}_{t_m}) \right\| |Z_{m+1}| \right\|_p.$$

As Z_{m+1} is independent of the σ -algebra generated by $\bar{X}_{t_0}, \dots, \bar{X}_{t_m}$, one can apply the linear growth result in (A.1) and obtain

$$\|\bar{X}_{t_{m+1}}\|_p = \|\bar{X}_{t_m}\|_p + C_{b,\sigma,L,T} (h + c_p h^{1/2}) (1 + \|\bar{X}_{t_m}\|_p + \mathcal{W}_p(\delta_0, \bar{\mu}_{t_m})),$$

where $C_{b,\sigma,L,T}$ and c_p are two real constants. As $\mathcal{W}_p(\delta_0, \bar{\mu}_{t_m}) \leq \|\bar{X}_{t_m}\|_p$, there exists constants C_1 and C_2 such that

$$\|\bar{X}_{t_{m+1}}\|_p \leq C_1 \|\bar{X}_{t_m}\|_p + C_2,$$

which in turn implies by induction that $\max_{m=0,\dots,M} \|\bar{X}_{t_m}\|_p < +\infty$ since

$$\|\bar{X}_0\|_p = \|X_0\|_p < +\infty.$$

For every $t \in [t_m, t_{m+1})$, it follows from the definition (2.3) that

$$\|\bar{X}_t^M\|_p \leq \|\bar{X}_{t_m}\|_p + (t - t_m) \|b(t_m, \bar{X}_{t_m}, \bar{\mu}_{t_m})\|_p + \left\| \int_{t_m}^t \sigma(s, \bar{X}_s, \bar{\mu}_s) dB_s \right\|_p.$$

We write \bar{X}_t instead of \bar{X}_t^M in the following when there is no ambiguity.

As $B_t - B_{t_m}$ is independent to $\sigma(\mathcal{F}_s, s \leq t_m)$, it follows that

$$\begin{aligned} \|\bar{X}_t\|_p &\leq \|\bar{X}_{t_m}\|_p + C_{b,\sigma,L,T} (1 + \|\bar{X}_{t_m}\|_p + \mathcal{W}_p(\delta_0, \bar{\mu}_{t_m})) (h + c_{d,p}(t - t_m)^{1/2}) \\ &\leq C_1 \|\bar{X}_{t_m}\|_p + C_2, \end{aligned}$$

where C_1 and C_2 are two constants. Finally, for a fixed $M \geq 1$,

$$\sup_{t \in [0, T]} \|\bar{X}_t^M\|_p < +\infty. \tag{A.6}$$

Consequently,

$$\begin{aligned} &\left\| \sup_{u \in [0, t]} |\bar{X}_u^M| \right\|_p \\ &\leq \|X_0\|_p + \left\| \int_0^t |b(s, \bar{X}_s, \bar{\mu}_s)| ds \right\|_p + \left\| \sup_{u \in [0, t]} \left| \int_0^u \sigma(s, \bar{X}_s, \bar{\mu}_s) dB_s \right| \right\|_p \\ &\quad (\text{by Minkowski's Inequality}) \\ &\leq \|X_0\|_p + \int_0^t \|b(s, \bar{X}_s, \bar{\mu}_s)\|_p ds + C_{d,p}^{BDG} \left\| \sqrt{\int_0^t \|\sigma(s, \bar{X}_s, \bar{\mu}_s)\|^2 ds} \right\|_p \\ &\quad (\text{by Lemma A.1 and (A.4)}) \\ &\leq \|X_0\|_p + \int_0^t C_{b,\sigma,L,T} \|1 + |\bar{X}_s| + \mathcal{W}_p(\bar{\mu}_s, \delta_0)\|_s ds \\ &\quad + C_{d,p,L}^{BDG} \left\| \sqrt{\int_0^t |1 + |\bar{X}_s| + \mathcal{W}_p(\bar{\mu}_s, \delta_0)|^2 ds} \right\|_p \quad (\text{by (A.1)}) \\ &\leq \|X_0\|_p + \int_0^t C_{b,\sigma,L,T} (1 + 2 \|\bar{X}_s\|_p) ds \\ &\quad + C_{d,p,L}^{BDG} \left\| \sqrt{\int_0^t 4(1 + |\bar{X}_s|^2 + \mathcal{W}_p^2(\bar{\mu}_s, \delta_0)) ds} \right\|_p \\ &\leq \|X_0\|_p + \int_0^t C_{b,\sigma,L,T} (1 + 2 \|\bar{X}_s\|_p) ds \\ &\quad + C_{d,p,L}^{BDG} \left\| \sqrt{4[t + \int_0^t |\bar{X}_s|^2 ds + \int_0^t \mathcal{W}_p^2(\bar{\mu}_s, \delta_0) ds]} \right\|_p \\ &\leq \|X_0\|_p + \int_0^t C_{b,\sigma,L,T} (1 + 2 \|\bar{X}_s\|_p) ds \\ &\quad + C_{d,p,L}^{BDG'} \left\| \sqrt{t} + \sqrt{\int_0^t |\bar{X}_s|^2 ds} + \sqrt{\int_0^t \mathcal{W}_p^2(\bar{\mu}_s, \delta_0) ds} \right\|_p \end{aligned}$$

$$\begin{aligned}
&\leq \|X_0\|_p + \int_0^t C_{b,\sigma,L,T}(1 + 2\|\bar{X}_{\underline{s}}\|_p)ds \\
&\quad + C_{d,p,L}^{BDG'} \left[\sqrt{t} + \left\| \sqrt{\int_0^t |\bar{X}_{\underline{s}}|^2 ds} \right\|_p + \sqrt{\int_0^t \mathcal{W}_p^2(\bar{\mu}_{\underline{s}}, \delta_0) ds} \right] \\
&\leq \|X_0\|_p + \int_0^t C_{b,\sigma,L,T}(1 + 2\|\bar{X}_{\underline{s}}\|_p)ds \\
&\quad + C_{d,p,L}^{BDG'} \left[\sqrt{t} + \left\| \int_0^t |\bar{X}_{\underline{s}}|^2 ds \right\|_{\frac{p}{2}}^{\frac{1}{2}} + \left(\int_0^t \mathcal{W}_p^2(\bar{\mu}_{\underline{s}}, \delta_0) ds \right)^{\frac{1}{2}} \right] \\
&\leq \|X_0\|_p + \int_0^t C_{b,\sigma,L,T}(1 + 2\|\bar{X}_{\underline{s}}\|_p)ds \\
&\quad + C_{d,p,L}^{BDG'} \left[\sqrt{t} + \left[\int_0^t \left\| |\bar{X}_{\underline{s}}|^2 \right\|_{\frac{p}{2}} ds \right]^{\frac{1}{2}} + \left[\int_0^t \mathcal{W}_p^2(\bar{\mu}_{\underline{s}}, \delta_0) ds \right]^{\frac{1}{2}} \right] \\
&\quad \text{(by Lemma A.1 since } \frac{p}{2} \geq 1\text{)}.
\end{aligned}$$

It follows from $\left\| |\bar{X}_{\underline{s}}|^2 \right\|_{\frac{p}{2}} = [\mathbb{E} |\bar{X}_{\underline{s}}|^{2 \cdot \frac{p}{2}}]^{\frac{2}{p}} = \|\bar{X}_{\underline{s}}\|_p^2$ and

$$\left[\int_0^t \mathcal{W}_p^2(\bar{\mu}_{\underline{s}}, \delta_0) ds \right]^{\frac{1}{2}} \leq \left[\int_0^t \|\mathcal{W}_p(\bar{\mu}_{\underline{s}}, \delta_0)\|_p^2 ds \right]^{\frac{1}{2}} \leq \left[\int_0^t \|\bar{X}_{\underline{s}}\|_p^2 ds \right]^{\frac{1}{2}}$$

that

$$\begin{aligned}
\left\| \sup_{u \in [0,t]} |\bar{X}_u^M| \right\|_p &\leq \|X_0\|_p + \int_0^t C_{b,\sigma,L,T}(1 + 2\|\bar{X}_{\underline{s}}\|_p) ds \\
&\quad + C_{d,p,L}^{BDG'} \left(\sqrt{t} + \left[\int_0^t \|\bar{X}_{\underline{s}}\|_p^2 ds \right]^{\frac{1}{2}} \right). \tag{A.7}
\end{aligned}$$

Hence, (A.7) implies that, for every $M \geq 1$, one has $\left\| \sup_{u \in [0,T]} |\bar{X}_u^M| \right\|_p < +\infty$ by applying (A.6).

In order to establish the uniformity in M , we come back to (A.7). As $\|\bar{X}_{\underline{s}}\|_p \leq \left\| \sup_{u \in [0,s]} |\bar{X}_u| \right\|_p$, it follows that

$$\begin{aligned}
\left\| \sup_{u \in [0,t]} |\bar{X}_u^M| \right\|_p &\leq \|X_0\|_p + C_{b,\sigma,L,T}(t + C_{d,p,L}^{BDG'} \sqrt{t}) \\
&\quad + C_{b,\sigma,L,T} \left\{ \int_0^t \left\| \sup_{u \in [0,s]} |\bar{X}_u| \right\|_p ds + C_{d,p,L}^{BDG'} \left[\int_0^t \left\| \sup_{u \in [0,s]} |\bar{X}_u| \right\|_p^2 ds \right]^{\frac{1}{2}} \right\}.
\end{aligned}$$

Hence,

$$\left\| \sup_{u \in [0,t]} |\bar{X}_u^M| \right\|_p \leq 2e^{(2C_{b,\sigma,L,T} + C_{d,p,L}^{BDG'^2})t} (\|X_0\|_p + C_{b,\sigma,L,T}(t + C_{d,p,L}^{BDG'} \sqrt{t})),$$

by applying Lemma A.1. Thus one concludes the proof by taking

$$C_{p,d,b,\sigma} = (2C_{b,\sigma,L,T} + C_{d,p,L}^{BDG'^2}) \vee 2C_{b,\sigma,L,T}(T + C_{d,p,L}^{BDG'} \sqrt{T}) \vee 2.$$

Next, it follows from $|X_t - X_s| = \left| \int_s^t b(u, X_u, \mu_u) du + \int_s^t \sigma(u, X_u, \mu_u) dB_u \right|$ that,

$$\begin{aligned}
\|X_t - X_s\|_p &\leq \left\| \int_s^t b(u, X_u, \mu_u) du \right\|_p + \left\| \int_s^t \sigma(u, X_u, \mu_u) dB_u \right\|_p \\
&\leq \int_s^t \|b(u, X_u, \mu_u)\|_p du + C_{d,p}^{BDG} \left\| \int_s^t \|\sigma(u, X_u, \mu_u)\|^2 du \right\|_{\frac{p}{2}}^{\frac{1}{2}} \\
&\quad (\text{by Lemma A.1 and Lemma 3}) \\
&\leq \int_s^t C_{b,\sigma,L,T} \left[1 + \|X_u\|_p + \|\mathcal{W}_p(\mu_p, \delta_0)\|_p \right] du \\
&\quad + C_{d,p}^{BDG} \left\| \int_s^t C_{b,\sigma,L,T} \left[1 + \|X_u\|_p + \|\mathcal{W}_p(\mu_p, \delta_0)\|_p \right]^2 du \right\|_{\frac{p}{2}}^{\frac{1}{2}} \quad (\text{by (A.1)}) \\
&\leq \int_s^t C_{b,\sigma,L,T} \left[1 + 2\|X_u\|_p \right] du + 4C_{d,p}^{BDG} \cdot C_{b,\sigma,L,T} \left\| \int_s^t \left[1 + \|X_u\|_p^2 + \mathcal{W}_p^2(\mu_p, \delta_0) \right] du \right\|_{\frac{p}{2}}^{\frac{1}{2}} \\
&\leq \int_s^t C_{b,\sigma,L,T} \left[1 + 2\|X_u\|_p \right] du \\
&\quad + 4C_{d,p}^{BDG} \cdot C_{b,\sigma,L,T} \left[(t-s) + \left\| \int_s^t |X_u|^2 du \right\|_{\frac{p}{2}} + \left\| \int_s^t \mathcal{W}_p^2(\mu_u, \delta_0) du \right\|_{\frac{p}{2}} \right]^{\frac{1}{2}} \\
&\leq \int_s^t C_{b,\sigma,L,T} \left[1 + 2\|X_u\|_p \right] du \\
&\quad + 4C_{d,p}^{BDG} \cdot C_{b,\sigma,L,T} \left[\sqrt{t-s} + \left[\int_s^t \left\| |X_u|^2 \right\|_{\frac{p}{2}} du \right]^{\frac{1}{2}} + \left[\int_s^t \|\mathcal{W}_p^2(\mu_u, \delta_0)\|_{\frac{p}{2}} du \right]^{\frac{1}{2}} \right] \\
&\leq \int_s^t C_{b,\sigma,L,T} \left[1 + 2 \left\| \sup_{u \in [0,T]} |X_u| \right\|_p \right] du \\
&\quad + 4C_{d,p}^{BDG} \cdot C_{b,\sigma,L,T} \left\{ \sqrt{t-s} + \sqrt{\int_s^t \|X_u\|_p^2 du} + \sqrt{\int_s^t \|\mathcal{W}_p(\mu_u, \delta_0)\|_p^2 du} \right\} \\
&\leq C_{b,\sigma,L,T} \left[1 + 2 \left\| \sup_{u \in [0,T]} |X_u| \right\|_p \right] (t-s) \\
&\quad + 4C_{d,p}^{BDG} \cdot C_{b,\sigma,L,T} \left\{ \sqrt{t-s} + 2\sqrt{t-s} \left\| \sup_{u \in [0,T]} |X_u| \right\|_p \right\} \\
&\leq \left\{ C_{b,\sigma,L,T} \left[1 + 2 \left\| \sup_{u \in [0,T]} |X_u| \right\|_p \right] \sqrt{T} \right. \\
&\quad \left. + 4C_{d,p}^{BDG} \cdot C_{b,\sigma,L,T} \left[1 + 2 \left\| \sup_{u \in [0,T]} |X_u| \right\|_p \right] \right\} \sqrt{t-s}.
\end{aligned}$$

Owing to the result in (a), $\left\| \sup_{u \in [0,T]} |X_u| \right\|_p \leq C_{p,d,b,\sigma} e^{C_{p,d,b,\sigma} t} (1 + \|X_0\|_p)$, then one can conclude by setting

$$\begin{aligned}
\kappa = C_{L,b,\sigma,\|X_0\|,p,d,T} &:= C_{b,\sigma,L,T} \left[1 + 2C_{p,d,b,\sigma} e^{C_{p,d,b,\sigma} t} (1 + \|X_0\|_p) \right] \sqrt{T} \\
&\quad + 4C_{d,p}^{BDG} \cdot C_{b,\sigma,L,T} \left[1 + 2C_{p,d,b,\sigma} e^{C_{p,d,b,\sigma} t} (1 + \|X_0\|_p) \right]. \quad \square
\end{aligned}$$

Proof of Proposition 2.1-(b). We write \bar{X}_t and $\bar{\mu}_t$ instead of \bar{X}_t^M and $\bar{\mu}_t^M$ to simplify the notation in this

proof. For every $s \in [0, T]$, set

$$\varepsilon_s := X_s - \bar{X}_s = \int_0^s [b(u, X_u, \mu_u) - b(\underline{u}, \bar{X}_{\underline{u}}, \bar{\mu}_{\underline{u}})] du + \int_0^s [\sigma(u, X_u, \mu_u) - \sigma(\underline{u}, \bar{X}_{\underline{u}}, \bar{\mu}_{\underline{u}})] dB_u,$$

and let

$$f(t) := \left\| \sup_{s \in [0, t]} |\varepsilon_s| \right\|_p = \left\| \sup_{s \in [0, t]} |X_s - \bar{X}_s| \right\|_p.$$

It follows from Proposition 2.1-(a) that $\bar{X} = (\bar{X}_t)_{t \in [0, T]} \in L^p_{\mathcal{C}([0, T], \mathbb{R}^d)}(\Omega, \mathcal{F}, \mathbb{P})$. Consequently, $\bar{\mu} \in \mathcal{P}_p(\mathcal{C}([0, T], \mathbb{R}^d))$ and $\nu(\mu) = (\mu_t)_{t \in [0, T]} \in \mathcal{C}([0, T], \mathcal{P}_p(\mathbb{R}^d))$ by applying Lemma 5.2. Hence,

$$\begin{aligned} f(t) &= \left\| \sup_{s \in [0, t]} |X_s - \bar{X}_s| \right\|_p \\ &\leq \left\| \int_0^t |b(s, X_s, \mu_s) - b(\underline{s}, \bar{X}_{\underline{s}}, \bar{\mu}_{\underline{s}})| ds + \sup_{s \in [0, t]} \left| \int_0^s (\sigma(u, X_u, \mu_u) - \sigma(\underline{u}, \bar{X}_{\underline{u}}, \bar{\mu}_{\underline{u}})) dB_u \right| \right\|_p \\ &\leq \int_0^t \|b(s, X_s, \mu_s) - b(\underline{s}, \bar{X}_{\underline{s}}, \bar{\mu}_{\underline{s}})\|_p ds + C_{d,p}^{BDG} \left\| \sqrt{\int_0^t \|\sigma(s, X_s, \mu_s) - \sigma(\underline{s}, \bar{X}_{\underline{s}}, \bar{\mu}_{\underline{s}})\|^2 ds} \right\|_p \\ &= \int_0^t \|b(s, X_s, \mu_s) - b(\underline{s}, \bar{X}_{\underline{s}}, \bar{\mu}_{\underline{s}})\|_p ds + C_{d,p}^{BDG} \left\| \int_0^t \|\sigma(s, X_s, \mu_s) - \sigma(\underline{s}, \bar{X}_{\underline{s}}, \bar{\mu}_{\underline{s}})\|^2 ds \right\|_{\frac{p}{2}}^{\frac{1}{2}} \\ &\leq \int_0^t \|b(s, X_s, \mu_s) - b(\underline{s}, \bar{X}_{\underline{s}}, \bar{\mu}_{\underline{s}})\|_p ds + C_{d,p}^{BDG} \left[\int_0^t \|\sigma(s, X_s, \mu_s) - \sigma(\underline{s}, \bar{X}_{\underline{s}}, \bar{\mu}_{\underline{s}})\|^2 ds \right]_{\frac{p}{2}}^{\frac{1}{2}} \\ &= \int_0^t \|b(s, X_s, \mu_s) - b(\underline{s}, \bar{X}_{\underline{s}}, \bar{\mu}_{\underline{s}})\|_p ds + C_{d,p}^{BDG} \left[\int_0^t \|\sigma(s, X_s, \mu_s) - \sigma(\underline{s}, \bar{X}_{\underline{s}}, \bar{\mu}_{\underline{s}})\|_p^2 ds \right]_{\frac{p}{2}}^{\frac{1}{2}} \\ &\leq \int_0^t \|b(s, X_s, \mu_s) - b(\underline{s}, X_s, \mu_s)\|_p ds + \int_0^t \|b(\underline{s}, X_s, \mu_s) - b(\underline{s}, \bar{X}_{\underline{s}}, \bar{\mu}_{\underline{s}})\|_p ds \\ &\quad + C_{d,p}^{BDG} \left[\int_0^t \|\sigma(s, X_s, \mu_s) - \sigma(\underline{s}, X_s, \mu_s)\| + \|\sigma(\underline{s}, X_s, \mu_s) - \sigma(\underline{s}, \bar{X}_{\underline{s}}, \bar{\mu}_{\underline{s}})\|_p^2 ds \right]_{\frac{p}{2}}^{\frac{1}{2}}, \end{aligned} \tag{A.8}$$

where the last term of (A.8) can be upper-bounded by

$$\begin{aligned} &C_{d,p}^{BDG} \left[\int_0^t \|\sigma(s, X_s, \mu_s) - \sigma(\underline{s}, X_s, \mu_s)\| + \|\sigma(\underline{s}, X_s, \mu_s) - \sigma(\underline{s}, \bar{X}_{\underline{s}}, \bar{\mu}_{\underline{s}})\|_p^2 ds \right]_{\frac{p}{2}}^{\frac{1}{2}} \\ &\leq C_{d,p}^{BDG} \left[\int_0^t [\|\sigma(s, X_s, \mu_s) - \sigma(\underline{s}, X_s, \mu_s)\|_p + \|\sigma(\underline{s}, X_s, \mu_s) - \sigma(\underline{s}, \bar{X}_{\underline{s}}, \bar{\mu}_{\underline{s}})\|_p]^2 ds \right]_{\frac{p}{2}}^{\frac{1}{2}} \\ &\leq \sqrt{2} C_{d,p}^{BDG} \left[\int_0^t \|\sigma(s, X_s, \mu_s) - \sigma(\underline{s}, X_s, \mu_s)\|_p^2 ds \right]_{\frac{p}{2}}^{\frac{1}{2}} \\ &\quad + \sqrt{2} C_{d,p}^{BDG} \left[\int_0^t \|\sigma(\underline{s}, X_s, \mu_s) - \sigma(\underline{s}, \bar{X}_{\underline{s}}, \bar{\mu}_{\underline{s}})\|_p^2 ds \right]_{\frac{p}{2}}^{\frac{1}{2}}. \end{aligned} \tag{A.9}$$

It follows that

$$\begin{aligned} &\int_0^t \|b(s, X_s, \mu_s) - b(\underline{s}, X_s, \mu_s)\|_p ds + \sqrt{2} C_{d,p}^{BDG} \left[\int_0^t \|\sigma(s, X_s, \mu_s) - \sigma(\underline{s}, X_s, \mu_s)\|_p^2 ds \right]_{\frac{p}{2}}^{\frac{1}{2}} \\ &\leq \int_0^t \left\| (s - \underline{s})^\rho \tilde{L}(1 + |X_s| + \mathcal{W}_p(\mu_s, \delta_0)) \right\|_p ds \end{aligned}$$

$$\begin{aligned}
& + \sqrt{2}C_{d,p}^{BDG} \left[\int_0^t \|(s - \underline{s})^\rho \tilde{L}(1 + |X_s| + \mathcal{W}_p(\mu_s, \delta_0))\|_p^2 ds \right]^{\frac{1}{2}} \quad (\text{by Assumption I}) \\
& \leq h^\rho T \tilde{L}(1 + 2 \|\sup_{s \in [0, T]} |X_s|\|_p) + \sqrt{2}h^\rho \tilde{L}C_{d,p}^{BDG} \left[T(2 + 4 \|\sup_{s \in [0, T]} |X_s|\|_p^2) \right]^{\frac{1}{2}} \\
& \leq h^\rho T \tilde{L}(1 + 2 \|\sup_{s \in [0, T]} |X_s|\|_p) + \sqrt{2}h^\rho \tilde{L}C_{d,p}^{BDG} \left[\sqrt{2T} + 2\sqrt{T} \|\sup_{s \in [0, T]} |X_s|\|_p \right] \quad (\text{A.10})
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^t \|b(\underline{s}, X_s, \mu_s) - b(\underline{s}, \bar{X}_{\underline{s}}, \bar{\mu}_{\underline{s}})\|_p ds + \sqrt{2}C_{d,p}^{BDG} \left[\int_0^t \|\sigma(\underline{s}, X_s, \mu_s) - \sigma(\underline{s}, \bar{X}_{\underline{s}}, \bar{\mu}_{\underline{s}})\|_p^2 ds \right]^{\frac{1}{2}} \\
& \leq \int_0^t \|L(|X_s - \bar{X}_{\underline{s}}| + \mathcal{W}_p(\mu_s, \bar{\mu}_{\underline{s}}))\|_p ds \\
& \quad + \sqrt{2}C_{d,p}^{BDG} \left[\int_0^t \|L(|X_s - \bar{X}_{\underline{s}}| + \mathcal{W}_p(\mu_s, \bar{\mu}_{\underline{s}}))\|_p^2 ds \right]^{\frac{1}{2}} \\
& \leq \int_0^t 2L \|X_s - \bar{X}_{\underline{s}}\|_p ds + \sqrt{2}C_{d,p}^{BDG} \left[\int_0^t 4L^2 \|X_s - \bar{X}_{\underline{s}}\|_p^2 ds \right]^{\frac{1}{2}} \\
& \leq \int_0^t 2L [\|X_s - X_{\underline{s}}\|_p + \|X_{\underline{s}} - \bar{X}_{\underline{s}}\|_p] ds \\
& \quad + \sqrt{2}C_{d,p}^{BDG} \left[\int_0^t 4L^2 [\|X_s - X_{\underline{s}}\|_p + \|X_{\underline{s}} - \bar{X}_{\underline{s}}\|_p]^2 ds \right]^{\frac{1}{2}} \\
& \leq \int_0^t 2L [\kappa\sqrt{h} + \|X_{\underline{s}} - \bar{X}_{\underline{s}}\|_p] ds + \sqrt{2}C_{d,p}^{BDG} \left[\int_0^t 4L^2 [\kappa\sqrt{h} + \|X_{\underline{s}} - \bar{X}_{\underline{s}}\|_p]^2 ds \right]^{\frac{1}{2}} \\
& \quad (\text{by applying Proposition 2.1-(a)}) \\
& \leq 2Lt\kappa\sqrt{h} + 4C_{d,p}^{BDG}L\sqrt{t}\kappa\sqrt{h} + 2L \int_0^t f(s)ds + \sqrt{2}C_{d,p}^{BDG}4L \left[\int_0^t f(s)^2 ds \right]^{\frac{1}{2}}. \quad (\text{A.11})
\end{aligned}$$

Let $\tilde{\kappa}(T, \|X_0\|_p) = C_{p,d,b,\sigma} e^{C_{p,d,b,\sigma} T} (1 + \|X_0\|_p)$, which is the right hand side of results in Proposition 2.1-(a). A combination of (A.8), (A.9), (A.10) and (A.11) leads to

$$\begin{aligned}
f(t) & = \left\| \sup_{s \in [0, t]} |X_s - \bar{X}_s| \right\|_p \\
& \leq h^\rho T \tilde{L}(1 + 2 \|\sup_{s \in [0, T]} |X_s|\|_p) + \sqrt{2}h^\rho \tilde{L}C_{d,p}^{BDG} \left[\sqrt{2T} + 2\sqrt{T} \|\sup_{s \in [0, T]} |X_s|\|_p \right] \\
& \quad + 2Lt\kappa\sqrt{h} + \sqrt{2}C_{d,p}^{BDG} 2\sqrt{2}L\sqrt{t}\kappa\sqrt{h} + 2L \int_0^t f(s)ds + \sqrt{2}C_{d,p}^{BDG} 4L \left[\int_0^t f(s)^2 ds \right]^{\frac{1}{2}} \\
& \leq h^{\frac{1}{2} \wedge \rho} \psi(T) + 2L \int_0^t f(s)ds + \sqrt{2}C_{d,p}^{BDG} 4L \left[\int_0^t f(s)^2 ds \right]^{\frac{1}{2}},
\end{aligned}$$

where

$$\begin{aligned}
\psi(T) & = T^{\rho - \rho \wedge \frac{1}{2}} \left[T \tilde{L}(1 + 2\tilde{\kappa}(T, \|X_0\|_p)) + \sqrt{2} \tilde{L} C_{d,p}^{BDG} (\sqrt{2T} + 2\sqrt{T} \tilde{\kappa}(T, \|X_0\|_p)) \right] \\
& \quad + T^{\frac{1}{2} - \rho \wedge \frac{1}{2}} \left[2LT\kappa + 4C_{d,p}^{BDG} L\sqrt{T}\kappa \right].
\end{aligned}$$

Then it follows from lemma A.1 that $f(t) \leq 2e^{(4L+16C_{d,p}^{BDG}L^2)T} \cdot \psi(T)h^{\rho \wedge \frac{1}{2}}$. Then we can conclude the

proof by letting $\tilde{C} = 2e^{(4L+16C_{d,p}^{BDG^2}L^2)T} \cdot \psi(T)$. \square

The proof of Proposition 2.1-(b) directly derives the following result.

Corollary A.1. *Let $\bar{X} := (\bar{X}_t)_{t \in [0, T]}$ denote the process defined by the continuous time Euler scheme (2.3) with step $h = \frac{T}{M}$ and let $X := (X_t)_{t \in [0, T]}$ denote the unique solution of the McKean-Vlasov equation (1.3). Then under Assumption I, one has*

$$\mathbb{W}_p(\bar{X}, X) \leq \left\| \sup_{t \in [0, T]} |X_t - \bar{X}_t| \right\|_p \leq \tilde{C} h^{\frac{1}{2} \wedge \rho}, \quad (\text{A.12})$$

where \tilde{C} is the same as in Proposition 2.1-(a).

Appendix B Proof of Proposition 2.2

Lemma B.1. *Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$. We have $\mu \preceq_{cv} \nu$ if and only if the application $t \in [0, 1] \mapsto (1-t)\mu + t\nu$ is non-decreasing w.r.t. the convex order.*

Proof of Lemma B.1. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be convex function with linear growth and let $t \in [0, 1]$. Then if $\mu \preceq_{cv} \nu$,

$$\int_{\mathbb{R}^d} f(\xi) \mu(d\xi) \leq (1-t) \int_{\mathbb{R}^d} f(\xi) \mu(d\xi) + t \int_{\mathbb{R}^d} f(\xi) \nu(d\xi) \leq \int_{\mathbb{R}^d} f(\xi) \nu(d\xi).$$

Then one can conclude that $\mu \preceq_{cv} (1-t)\mu + t\nu \preceq_{cv} \nu$ for every $t \in [0, 1]$. Now let $0 \leq s < t$. It follows from what precedes that

$$(1-s)\mu + s\nu = (1-\frac{s}{t})\mu + \frac{s}{t}((1-t)\mu + t\nu) \preceq_{cv} (1-t)\mu + t\nu.$$

The proof for the converse direction is trivial. \square

Proof of Proposition 2.2. (a) The direct sense follows from Lemma B.1 since $\mu \preceq_{cv} (1-\varepsilon)\mu + \varepsilon\nu$ for $\varepsilon \in [0, 1]$.

For the converse, we proceed as follows. Let $\mu \preceq_{cv} \nu$, note that $\varepsilon \mapsto \Phi(\mu + \varepsilon(\nu - \mu))$ is continuous since $\varepsilon \mapsto \mu + \varepsilon(\nu - \mu)$ from $[0, 1]$ to $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)$ is continuous. Consequently $\tilde{\Phi}(\varepsilon) := \Phi(\mu + \varepsilon(\nu - \mu)) - \Phi(\mu) - \varepsilon(\Phi(\nu) - \Phi(\mu))$ is also continuous and satisfies $\tilde{\Phi}(0) = \tilde{\Phi}(1) = 0$. Hence, $\tilde{\Phi}$ attains its maximum at some $\varepsilon_0 \in [0, 1]$. This in turn implies that

$$\begin{aligned} 0 &\geq \liminf_{\varepsilon \rightarrow \varepsilon_0^+} \frac{\tilde{\Phi}(\varepsilon) - \tilde{\Phi}(\varepsilon_0)}{\varepsilon - \varepsilon_0} \\ &= \liminf_{\eta \rightarrow 0^+} \frac{\Phi(\mu + \varepsilon_0(\nu - \mu) + \eta(\nu - \mu)) - \Phi(\mu + \varepsilon_0(\nu - \mu))}{\eta} - (\Phi(\nu) - \Phi(\mu)). \end{aligned}$$

Then set $\tilde{\mu} = \mu + \varepsilon_0(\nu - \mu)$ and $\tilde{\nu} = \nu$. One has $\tilde{\mu} \preceq_{cv} \tilde{\nu}$ owing to Lemma B.1 and note that $\tilde{\nu} - \tilde{\mu} = (1-\varepsilon_0)(\nu - \mu)$. Consequently, $\mu + \varepsilon_0(\nu - \mu) + \eta(\nu - \mu) = \tilde{\mu} + \frac{\eta}{1-\varepsilon_0}(\tilde{\nu} - \tilde{\mu})$. Applying the assumption made on Φ to $\tilde{\mu}$ and $\tilde{\nu}$ implies

$$\liminf_{\eta \rightarrow 0^+} \frac{\Phi(\mu + \varepsilon_0(\nu - \mu) + \eta(\nu - \mu)) - \Phi(\mu + \varepsilon_0(\nu - \mu))}{\eta} \geq 0.$$

Finally, this yields $\Phi(\nu) \geq \Phi(\mu)$.

(b) Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\mu \preceq_{cv} \nu$. Then, for every $\varepsilon \in [0, 1]$, $\mu \preceq_{cv} (1 - \varepsilon)\mu + \varepsilon\nu$ so that, as Φ is linearly functionally differentiable,

$$\begin{aligned} \frac{\Phi((1 - \varepsilon)\mu + \varepsilon\nu) - \Phi(\mu)}{\varepsilon} &= \int_0^1 \int_{\mathbb{R}^d} \frac{\delta\Phi}{\delta m}((1 - t)\mu + t((1 - \varepsilon)\mu + \varepsilon\nu))(x) d[\nu - \mu](x) dt \\ &= \int_0^1 \int_{\mathbb{R}^d} \frac{\delta\Phi}{\delta m}((1 - \varepsilon t)\mu + \varepsilon t\nu)(x) d[\nu - \mu](x) dt. \end{aligned}$$

The function $u \mapsto (1 - u)\mu + u\nu$, $u \in [0, 1]$, being \mathcal{W}_2 -continuous, it follows from the definition of the linear functional derivative, that

$$\frac{\delta\Phi}{\delta m}((1 - u)\mu + u\nu)(x) \rightarrow \frac{\delta\Phi}{\delta m}(\mu)(x) \quad \text{as } u \rightarrow 0 \quad \text{for every } x \in \mathbb{R}^d.$$

Moreover, $\mathcal{K} = \{(1 - u)\mu + u\nu, u \in [0, 1]\}$ being clearly \mathcal{W}_2 -bounded, one has, still by this definition,

$$\forall x \in \mathbb{R}^d, \quad \sup_{u \in [0, 1]} \left| \frac{\delta\Phi}{\delta m}((1 - u)\mu + u\nu)(x) \right| \leq C_{\mathcal{K}}(1 + |x|^2)$$

for some real constant $C_{\mathcal{K}}$. Consequently, Lebesgue's dominated convergence applied with both $\mu \otimes dt$ and $\nu \otimes dt$ implies

$$\int_0^1 dt \int_{\mathbb{R}^d} \frac{\delta\Phi}{\delta m}((1 - t\varepsilon)\mu + t\varepsilon\nu)(x) d[\nu - \mu](x) \rightarrow \int_{\mathbb{R}^d} \frac{\delta\Phi}{\delta m}(\mu)(x) d[\nu - \mu](x) \quad \text{as } \varepsilon \rightarrow 0.$$

This shows that the function $\varepsilon \in [0, 1] \mapsto \Phi(\mu + \varepsilon(\nu - \mu))$ is right differentiable at $\varepsilon = 0$ with

$$\frac{d}{d\varepsilon} [\Phi(\mu + \varepsilon(\nu - \mu))]_{|\varepsilon=0} = \int_{\mathbb{R}^d} \frac{\delta\Phi}{\delta m}(\mu)(x) d[\nu - \mu](x).$$

Then the equivalence between (i) and (ii) is a direct consequence of the characterization (a).

(c) This claim is a straightforward consequence of (b). □

Appendix C Proof of Lemma 4.2 and Lemma 4.3

The proof of Lemma 4.2 relies on the following lemma.

Lemma C.1. (see [JP19, Lemma 3.2] and [Fad19]) Let $Z \sim \mathcal{N}(0, \mathbf{I}_q)$. If $u_1, u_2 \in \mathbb{M}_{d \times q}$ with $u_1 \preceq u_2$, then $u_1 Z \preceq_{cv} u_2 Z$.

Proof of Lemma C.1. ⁽⁵⁾ We define $M_1 := u_1 Z$ and $M_2 := M_1 + \sqrt{u_2 u_2^* - u_1 u_1^*} \tilde{Z}$, where \sqrt{A} denotes the square root of a positive semi-definite matrix A and $\tilde{Z} \sim \mathcal{N}(0, \mathbf{I}_d)$, \tilde{Z} is independent to Z . Hence the probability distribution of M_2 is $\mathcal{N}(0, u_2 u_2^*)$, which is the distribution of $u_2 Z$.

For any convex function φ , we have, owing to the conditional Jensen inequality,

$$\begin{aligned} \mathbb{E} [\varphi(M_2)] &= \mathbb{E} [\varphi(M_1 + \sqrt{u_2 u_2^* - u_1 u_1^*} \cdot \tilde{Z})] \\ &= \mathbb{E} \left[\mathbb{E} [\varphi(M_1 + \sqrt{u_2 u_2^* - u_1 u_1^*} \cdot \tilde{Z}) \mid Z] \right] \\ &\geq \mathbb{E} \left[\varphi(\mathbb{E} [M_1 + \sqrt{u_2 u_2^* - u_1 u_1^*} \cdot \tilde{Z} \mid Z]) \right] \\ &= \mathbb{E} \left[\varphi(M_1 + \mathbb{E} [\sqrt{u_2 u_2^* - u_1 u_1^*} \cdot \tilde{Z}]) \right] = \mathbb{E} \varphi(M_1). \end{aligned} \tag{C.1}$$

⁵This proof is reproduced from [JP19] for convenience.

Hence, $u_1 Z \preceq_{cv} u_2 Z$ owing to the equivalence of convex order of the random variable and its probability distribution. \square

Proof of Lemma 4.2. (i) Let $(Q_m^\mu \varphi)(\cdot, \cdot) := (Q_m \varphi)(\cdot, \mu, \cdot)$ to simplify the notation. For every $(x_1, u_1), (x_2, u_2) \in \mathbb{R}^d \times \mathbb{M}_{d \times q}$ and $\lambda \in [0, 1]$,

$$\begin{aligned}
& (Q_m^\mu \varphi)(\lambda(x_1, u_1) + (1 - \lambda)(x_2, u_2)) \\
&= \mathbb{E} \left[\varphi \left(b_{m-1}(\lambda x_1 + (1 - \lambda)x_2, \mu) + (\lambda u_1 + (1 - \lambda)u_2) Z_m \right) \right] \\
&= \mathbb{E} \left[\varphi \left(\lambda b_{m-1}(x_1, \mu) + (1 - \lambda)b_{m-1}(x_2, \mu) + \lambda u_1 Z_m + (1 - \lambda)u_2 Z_m \right) \right] \\
&\quad (\text{as } b_m \text{ is affine in } x, \text{ see Assumption II}_{disc}\text{-}(5)) \\
&\leq \lambda \mathbb{E} \left[\varphi(b_{m-1}(x_1, \mu) + u_1 Z_m) \right] + (1 - \lambda) \mathbb{E} \left[\varphi(b_{m-1}(x_2, \mu) + u_2 Z_m) \right] \\
&\quad (\text{by the convexity of } \varphi \text{ and the linearity of the expectation}) \\
&= \lambda (Q_m^\mu \varphi)(x_1, u_1) + (1 - \lambda) (Q_m^\mu \varphi)(x_2, u_2).
\end{aligned}$$

Hence, $(Q_m \varphi)(\cdot, \mu, \cdot)$ is convex.

(ii) If we fix $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d)$, then for any $u \in \mathbb{M}_{d \times q}$,

$$\begin{aligned}
(Q_m \varphi)(x, \mu, u) &= \mathbb{E} \left[\varphi(b_{m-1}(x, \mu) + u Z_m) \right] \\
&\geq \varphi \left(\mathbb{E}[b_{m-1}(x, \mu) + u Z_m] \right) \\
&= \varphi(b_{m-1}(x, \mu) + \mathbf{0}_{d \times 1}) = (Q_m \varphi)(x, \mu, \mathbf{0}_{d \times q}).
\end{aligned}$$

(iii) For every fixed $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d)$, it is obvious that $\varphi(b_{m-1}(x, \mu) + \cdot)$ is also a convex function. Thus, Lemma C.1 directly implies that if $u_1 \preceq u_2$, then

$$\mathbb{E} \varphi(b_{m-1}(x, \mu) + u_1 Z_m) \leq \mathbb{E} \varphi(b_{m-1}(x, \mu) + u_2 Z_m),$$

which is equivalent to $Q_m \varphi(x, \mu, u_1) \leq Q_m \varphi(x, \mu, u_2)$. \square

Proof of Lemma 4.3. Let $x, y \in \mathbb{R}^d$ and $\lambda \in [0, 1]$. For every $m = 0, \dots, M - 1$, we have

$$\begin{aligned}
& \mathbb{E} \left[\varphi \left(b_m(\lambda x + (1 - \lambda)y, \mu) + \sigma_m(\lambda x + (1 - \lambda)y, \mu) Z_{m+1} \right) \right] \\
&\leq \mathbb{E} \left[\varphi \left(\lambda b_m(x, \mu) + (1 - \lambda)b_m(y, \mu) + \lambda \sigma_m(x, \mu) Z_{m+1} + (1 - \lambda) \sigma_m(y, \mu) Z_{m+1} \right) \right] \\
&\quad (\text{by Assumption (4.5), (4.4) and Lemma 4.2}) \\
&\leq \lambda \mathbb{E} \left[\varphi(b_m(x, \mu) + \sigma_m(x, \mu) Z_{m+1}) \right] + (1 - \lambda) \mathbb{E} \left[\varphi(b_m(y, \mu) + \sigma_m(y, \mu) Z_{m+1}) \right] \\
&\quad (\text{by the convexity of } \varphi \text{ and the linearity of the expectation}).
\end{aligned}$$

The function $x \mapsto \mathbb{E} \left[\varphi(b_m(x, \mu) + \sigma_m(x, \mu) Z_{m+1}) \right]$ obviously has a linear growth since Assumption I implies that b_m and σ_m have a linear growth (see (A.1)). \square

Appendix D Proof of Lemma 5.2

Proof of Lemma 5.2. For any $\mu \in \mathcal{P}_p(\mathcal{C}([0, T], \mathbb{R}^d))$, there exists $X : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathcal{C}([0, T], \mathbb{R}^d)$ such that $\mathbb{P}_X = \mu$ and $\mathbb{E} \|X\|_{\text{sup}}^p < +\infty$ so that $\sup_{t \in [0, T]} \mathbb{E} |X_t|^p < +\infty$. Hence, for any $t \in [0, T]$, we have $\mu_t \in \mathcal{P}_p(\mathbb{R}^d)$.

For a fixed $t \in [0, T]$, choose $(t_n)_{n \in \mathbb{N}^*} \in [0, T]^{\mathbb{N}^*}$ such that $t_n \rightarrow t$. Then, for \mathbb{P} -almost any $\omega \in \Omega$, $X_{t_n}(\omega) \rightarrow X_t(\omega)$ since $X(\omega)$ has \mathbb{P} -a.s. continuous paths. Moreover,

$$\sup_n \|X_{t_n}\|_p \vee \|X_t\|_p \leq \left\| \sup_{0 \leq s \leq T} |X_s| \right\|_p < +\infty,$$

Hence, $\|X_{t_n} - X_t\|_p \rightarrow 0$ owing to the dominated convergence theorem, which implies that $\mathcal{W}_p(\mu_{t_n}, \mu_t) \rightarrow 0$ as $n \rightarrow +\infty$. Hence, $t \mapsto \mu_t$ is a continuous application i.e. $\iota(\mu) = (\mu_t)_{t \in [0, T]} \in \mathcal{C}([0, T], \mathcal{P}_p(\mathbb{R}^d))$. \square