THE YAMABE INVARIANT OF THE GRAVITATIONAL MONOPOLE

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Abstract. We show the Yamabe invariant is non-positive due to the presence of the Gravitational Monopole equations.

1. Introduction

In this short note we announce some results on the Yamabe invariant of a Gravitational monopole, the notion of the Gravitational monopole was first introduced in [cf.1]. We shall give details of the proofs in a different paper.

Let (M^n, g) a smooth, compact-oriented Riemannian n-manifold. As we know the Einstein-Hilbert action

$$g \mapsto \int s_g dv_g =: \mathcal{I}_g \tag{1.1}$$

with s_g the scalar curvature and dv_g is the volume form with respect to the metric g gives the Einstein equations at the critical points of the action $\mathcal{I}_g$ with a volume constraint. The *Yamabe problem* is to minimize $\mathcal{I}_g$ in a fixed conformal class of metric. Due to such a constraint, critical points of $\mathcal{I}_g$ have constant scalar curvature.

Let us define now the Yamabe invariant $Y(g)$,

$$Y(g) := \inf_{\widetilde{g}=e^{2w}g} \mathrm{vol}(\widetilde{g})^{-\left(\frac{n-2}{n}\right)} \int s_{\widetilde{g}} dv_{\widetilde{g}}. \tag{1.2}$$

Due to the extensive work of Yamabe, Aubin, Trudinger, and Schoen, we now know that every conformal class on a compact manifold admits a metric that achieves $Y(g)$ and therefore has constant scalar curvature.

Now in particular for $n = 4$, M^4 is an oriented four-dimensional manifold. The Hodge $*$-operator induces a splitting of the space of two-forms

$$\bigwedge^2 = \bigwedge^2_+ \bigoplus \bigwedge^2_- \tag{1.3}$$

into the subspace of self-dual 2-forms $\bigwedge^2_+$ and anti-self-dual 2-forms $\bigwedge^2_-$. This decomposition induces a splitting of the Weyl curvature (defined as a trace-free endomorphism of the $\bigwedge^2$) into its self-dual and anti-self-dual components $W^{\pm}$. We have the Hirzebruch signature formula

$$12\pi^2\sigma(M^4) = \int_{M^4} (|W^+|^2 - |W^-|^2) dv_g. \tag{1.4}$$

Let us define the *Weyl functionals* $\mathcal{W}[g]$, $\mathcal{W}^{\pm}[g]$ by

$$\mathcal{W}[g] := \int |W_g|^2 dv_g, \ \mathcal{W}_{\pm}[g] := \int |W_g^{\pm}|^2 dv_g. \tag{1.5}$$

Due to the Hirzebruch signature formula (1.4), the study of the Weyl functional $\mathcal{W}[g]$ is equivalent to the study of the *self-dual functional* $\mathcal{W}_+[g]$.

The signature formula implies the following fact,

Proposition 1.

$$\mathcal{W}_+[g] \geq \max\{12\pi^2\sigma(M^4), 0\}. \tag{1.6}$$

The condition

$$\mathcal{W}_+[g] = \max\{12\pi^2\sigma(M^4), 0\}. \tag{1.7}$$

is true if and only if $W^+ \equiv 0$ or $W^- \equiv 0$, that is g is half-conformally flat.

The following formula is useful

$$|W|^2 = |W^+|^2 + |W^-|^2, \tag{1.8}$$

also since W is conformally invariant, $W^{\pm}$ are conformally invariant pieces of W.

The following result is due to Taubes: for any (closed, compact, orientable) four-dimensional manifold N^4, the manifold $M^4 = N^4 \# k\mathbb{C}P^2$ admits anti-self-dual metrics (i.e. $W^- = 0$) for all k sufficiently large.

It is a classical result that using the conformal invariance of the Bach tensor, one concludes that any metric which is (locally) conformally Einstein is critical for $\mathcal{W}_+$.

2. The Gravitational Monopole Equations

In [1], the following equations are introduced (sometimes we omit the mapping c, and denote by " $\cdot$ " the Clifford multiplication (or composition) and the dimension 4 for the convenience of computations):

$$\begin{aligned} \nabla\!\!\!/\,\psi &= (d + d^*)\psi = 0, \\ c(W_g^+) &= \frac{1}{4}\langle e_i \cdot e_j \psi, \psi\rangle e^i \wedge e^j, \end{aligned} \tag{2.1}$$

One can rewrite (2.1) in the following form

$$\begin{aligned} \nabla\!\!\!/\,\psi &= 0, \\ c(W_g^+) &= \psi^* \otimes \psi - \frac{|\psi|^2}{2}\mathrm{Id}, \end{aligned} \tag{2.2}$$

therefore we have the following proposition

Proposition 2.

$$|W^+|^2 = \frac{|\psi|^4}{8}. \tag{2.3}$$

3. Constraint on the Yamambe invariant due to the Gravitational Monopole Equations.

We have the following propositions.

Proposition 3. *Let (M^4, g_0) be a compact, oriented, four-dimensional manifold with a Spin$^{\mathbb{C}}$-structure. Let E the trace-free part of the Ricci tensor. We also assume the validity of (2.2) on M^4. We further assume*

$$\int 3|\psi|^4 dv_{g_0} = 32\pi^2(2\chi(M^4) + 3\sigma(M^4)), \tag{3.1}$$

then there is a metric $g = e^{2w}g_0$ such that

$$6\Delta s_g + 3|\psi|^4 + 12|E|^2 = s_g^2. \tag{3.2}$$

It was also shown in [2] without the existence of the Gravitational monopole hypothesis that

Proposition 4. *Let (M^4, g_0) be a compact, oriented, four-dimensional manifold. Let E the trace-free part of the Ricci tensor. Assume*

$$\int |W^+|^2 dv_{g_0} = \frac{4}{3}\pi^2(2\chi(M^4) + 3\sigma(M^4)), \tag{3.3}$$

then there is a metric $g = e^{2w} g_0$ such that

$$\Delta s_g = -4|W^+|^2 - 2|E|^2 + \frac{1}{6} s_g^2. \tag{3.4}$$

Also,

(1) *if $Y(g_0) > 0$, then $s_g > 0$.*
(2) *if $Y(g_0) = 0$, then g is Ricci-flat anti-self-dual metric.*

Since the Gravitational monopole puts the restriction on the scalar curvature $s_{g_0} \leq 0$ of M^4 [cf. 1], and since Proposition (4) implies whenever $Y(g_0) > 0$, then $s_g > 0$, we therefore conclude, the following theorem:

Theorem 3.1. *Let (M^4, g_0) be a compact, oriented, four-dimensional manifold with a Spin$^{\mathbb{C}}$-structure. Let E the trace-free part of the Ricci tensor. We also assume the validity of (2.2) on M^4. We further assume*

$$\int 3|\psi|^4 dv_{g_0} = 32\pi^2(2\chi(M^4) + 3\sigma(M^4)), \tag{3.5}$$

then there is a metric $g = e^{2w} g_0$ such that

$$Y(g_0) \leq 0. \tag{3.6}$$

Since $Y(g_0) = 0$ implies Ricci-flatness and anti-self-duality, it requires a special attention.

Proposition 5. *Let (M^4, g_0) be a compact, oriented, four-dimensional manifold with a Spin$^{\mathbb{C}}$-structure. Let E the trace-free part of the Ricci tensor. We also assume the validity of (2.2) on M^4. We further assume*

$$0 < \int 3|\psi|^4 dv_{g_0} < 32\pi^2(2\chi(M^4) + 3\sigma(M^4)), \tag{3.7}$$

then there is a metric $g = e^{2w} g_0$ such that

(1)
$$Y(g_0) \leq 0.$$
(2) *There is no metric g_0 satisfying (3.7) with $Y(g_0) = 0$, therefore,*
$$Y(g_0) < 0.$$
(3)
$$24\Delta s_g + 3(4 + \epsilon_1)|\psi|^4 + 48|E|^2 + 4s_g^2 = 0,$$
for some $\epsilon_1 > 0$.

Proposition 6. *Let (M^4, g_0) be a compact, oriented, four-dimensional manifold with a Spin$^{\mathbb{C}}$-structure. Let E the trace-free part of the Ricci tensor. We also assume the validity of (2.2) on M^4. We further assume*

$$\int |\psi|^4 dv_{g_0} = 16\pi^2(2\chi(M^4) + 3\sigma(M^4)), \tag{3.8}$$

then

(1)
$$Y(g_0) \leq 0.$$
(2) *there is a metric $g = e^{2w} g_0$ such that [cf. 2]*
$$\Delta s_g = -\frac{3}{2}|E|^2 + \frac{1}{8} s_g^2.$$

Proposition 7. *Let (M^4, g_0) be a compact, oriented, four-dimensional manifold with a Spin$^{\mathbb{C}}$-structure. Let E the trace-free part of the Ricci tensor. We also assume the validity of (2.2) on M^4. We further assume*

$$0 < \int |\psi|^4 dv_{g_0} = 16\pi^2(2\chi(M^4) + 3\sigma(M^4)), \tag{3.9}$$

then

(1)
$$Y(g_0) \leq 0.$$
(2) *there is a metric $g = e^{2w} g_0$ such that [cf. 2]*
$$\Delta s_g + \epsilon_2 |\psi|^4 + 12|E|^2 = s_g^2,$$
for some $\epsilon_2 > 0$.

References

[1] Bhattacharya.,Gourab *Gravitational Monopoles*, https://hal.archives-ouvertes.fr/hal-03816743 .

[2] Gursky., J. Matthew; *The Weyl functional, de Rham cohomology, and Kähler-Einstein metrics*, Annals of Mathematics, 148 (1998), 315-337.

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