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THE YAMABE INVARIANT OF THE GRAVITATIONAL MONOPOLE.

GOURAB BHATTACHARYA

ABSTRACT. We show the Yamabe invariant is non-positive due to the presence of the Gravitational Monopole equations.

1. INTRODUCTION

In this short note we announce some results on the Yamabe invariant of a Gravitational monopole, the notion of the Gravitational monopole was first introduced in [cf.1]. We shall give details of the proofs in a different paper.

Let (M^n, g) a smooth, compact-oriented Riemannian n -manifold. As we know the Einstein-Hilbert action

$$(1.1) \quad g \mapsto \int s_g dv_g =: \mathcal{I}_g$$

with s_g the scalar curvature and dv_g is the volume form with respect to the metric g gives the Einstein equations at the critical points of the action \mathcal{I}_g with a volume constraint. The *Yamabe problem* is to minimize \mathcal{I}_g in a fixed conformal class of metric. Due to such a constraint, critical points of \mathcal{I}_g have constant scalar curvature.

Let us define now the Yamabe invariant $Y(g)$,

$$(1.2) \quad Y(g) := \inf_{\tilde{g}=e^{2w}g} \text{vol}(\tilde{g})^{-\left(\frac{n-2}{n}\right)} \int s_{\tilde{g}} dv_{\tilde{g}}.$$

Due to the extensive work of Yamabe, Aubin, Trudinger, and Schoen, we now know that every conformal class on a compact manifold admits a metric that achieves $Y(g)$ and therefore has constant scalar curvature.

Now in particular for $n = 4$, M^4 is an oriented four-dimensional manifold. The Hodge $*$ -operator induces a splitting of the space of two-forms

$$(1.3) \quad \bigwedge^2 = \bigwedge^2_+ \oplus \bigwedge^2_-$$

into the subspace of self-dual 2-forms \bigwedge^2_+ and anti-self-dual 2-forms \bigwedge^2_- . This decomposition induces a splitting of the Weyl curvature (defined as a trace-free endomorphism of the \bigwedge^2) into its self-dual and anti-self-dual components W^\pm . We have the Hirzebruch signature formula

$$(1.4) \quad 12\pi^2 \sigma(M^4) = \int_{M^4} (|W^+|^2 - |W^-|^2) dv_g.$$

Let us define the *Weyl functionals* $\mathcal{W}[g]$, $\mathcal{W}^\pm[g]$ by

$$(1.5) \quad \mathcal{W}[g] := \int |W_g|^2 dv_g, \quad \mathcal{W}_\pm[g] := \int |W_g^\pm|^2 dv_g.$$

Due to the Hirzebruch signature formula (1.4), the study of the Weyl functional $\mathcal{W}[g]$ is equivalent to the study of the *self-dual functional* $\mathcal{W}_+[g]$.

The signature formula implies the following fact,

Proposition 1.

$$(1.6) \quad \mathcal{W}_+[g] \geq \max\{12\pi^2\sigma(M^4), 0\}.$$

The condition

$$(1.7) \quad \mathcal{W}_+[g] = \max\{12\pi^2\sigma(M^4), 0\}.$$

is true if and only if $W^+ \equiv 0$ or $W^- \equiv 0$, that is g is half-conformally flat.

The following formula is useful

$$(1.8) \quad |W|^2 = |W^+|^2 + |W^-|^2,$$

also since W is conformally invariant, W^\pm are conformally invariant pieces of W .

The following result is due to Taubes: for any (closed, compact, orientable) four-dimensional manifold N^4 , the manifold $M^4 = N^4 \# k\mathbb{C}P^2$ admits anti-self-dual metrics (i.e. $W^- = 0$) for all k sufficiently large.

It is a classical result that using the conformal invariance of the Bach tensor, one concludes that any metric which is (locally) conformally Einstein is critical for \mathcal{W}_+ .

2. THE GRAVITATIONAL MONOPOLE EQUATIONS

In [1], the following equations are introduced (sometimes we omit the mapping c , and denote by " \cdot " the Clifford multiplication (or composition) and the dimension 4 for the convenience of computations):

$$(2.1) \quad \begin{aligned} \nabla \psi &= (d + d^*)\psi = 0, \\ c(W_g^+) &= \frac{1}{4} \langle e_i \cdot e_j \psi, \psi \rangle e^i \wedge e^j, \end{aligned}$$

One can rewrite (2.1) in the following form

$$(2.2) \quad \begin{aligned} \nabla \psi &= 0, \\ c(W_g^+) &= \psi^* \otimes \psi - \frac{|\psi|^2}{2} \text{Id}, \end{aligned}$$

therefore we have the following proposition

Proposition 2.

$$(2.3) \quad |W^+|^2 = \frac{|\psi|^4}{8}.$$

3. CONSTRAINT ON THE YAMAMBE INVARIANT DUE TO THE GRAVITATIONAL MONOPOLE EQUATIONS.

We have the following propositions.

Proposition 3. *Let (M^4, g_0) be a compact, oriented, four-dimensional manifold with a $\text{Spin}^{\mathbb{C}}$ -structure. Let E the trace-free part of the Ricci tensor. We also assume the validity of (2.2) on M^4 . We further assume*

$$(3.1) \quad \int 3|\psi|^4 dv_{g_0} = 32\pi^2(2\chi(M^4) + 3\sigma(M^4)),$$

then there is a metric $g = e^{2w}g_0$ such that

$$(3.2) \quad 6\Delta s_g + 3|\psi|^4 + 12|E|^2 = s_g^2.$$

It was also shown in [2] without the existence of the Gravitational monopole hypothesis that

Proposition 4. *Let (M^4, g_0) be a compact, oriented, four-dimensional manifold. Let E the trace-free part of the Ricci tensor. Assume*

$$(3.3) \quad \int |W^+|^2 dv_{g_0} = \frac{4}{3}\pi^2(2\chi(M^4) + 3\sigma(M^4)),$$

then there is a metric $g = e^{2w}g_0$ such that

$$(3.4) \quad \Delta s_g = -4|W^+|^2 - 2|E|^2 + \frac{1}{6}s_g^2.$$

Also,

- (1) if $Y(g_0) > 0$, then $s_g > 0$.
- (2) if $Y(g_0) = 0$, then g is Ricci-flat anti-self-dual metric.

Since the Gravitational monopole puts the restriction on the scalar curvature $s_{g_0} \leq 0$ of M^4 [cf. 1], and since Proposition (4) implies whenever $Y(g_0) > 0$, then $s_g > 0$, we therefore conclude, the following theorem:

Theorem 3.1. *Let (M^4, g_0) be a compact, oriented, four-dimensional manifold with a $Spin^C$ -structure. Let E the trace-free part of the Ricci tensor. We also assume the validity of (2.2) on M^4 . We further assume*

$$(3.5) \quad \int 3|\psi|^4 dv_{g_0} = 32\pi^2(2\chi(M^4) + 3\sigma(M^4)),$$

then there is a metric $g = e^{2w}g_0$ such that

$$(3.6) \quad Y(g_0) \leq 0.$$

Since $Y(g_0) = 0$ implies Ricci-flatness and anti-self-duality, it requires a special attention.

Proposition 5. *Let (M^4, g_0) be a compact, oriented, four-dimensional manifold with a $Spin^C$ -structure. Let E the trace-free part of the Ricci tensor. We also assume the validity of (2.2) on M^4 . We further assume*

$$(3.7) \quad 0 < \int 3|\psi|^4 dv_{g_0} < 32\pi^2(2\chi(M^4) + 3\sigma(M^4)),$$

then there is a metric $g = e^{2w}g_0$ such that

- (1)
$$Y(g_0) \leq 0.$$
- (2) *There is no metric g_0 satisfying (3.7) with $Y(g_0) = 0$, therefore,*

$$Y(g_0) < 0.$$
- (3)
$$24\Delta s_g + 3(4 + \epsilon_1)|\psi|^4 + 48|E|^2 + 4s_g^2 = 0,$$

for some $\epsilon_1 > 0$.

Proposition 6. *Let (M^4, g_0) be a compact, oriented, four-dimensional manifold with a $Spin^C$ -structure. Let E the trace-free part of the Ricci tensor. We also assume the validity of (2.2) on M^4 . We further assume*

$$(3.8) \quad \int |\psi|^4 dv_{g_0} = 16\pi^2(2\chi(M^4) + 3\sigma(M^4)),$$

then

- (1)
$$Y(g_0) \leq 0.$$
- (2) *there is a metric $g = e^{2w}g_0$ such that [cf. 2]*

$$\Delta s_g = -\frac{3}{2}|E|^2 + \frac{1}{8}s_g^2.$$

Proposition 7. *Let (M^4, g_0) be a compact, oriented, four-dimensional manifold with a $Spin^C$ -structure. Let E the trace-free part of the Ricci tensor. We also assume the validity of (2.2) on M^4 . We further assume*

$$(3.9) \quad 0 < \int |\psi|^4 dv_{g_0} = 16\pi^2(2\chi(M^4) + 3\sigma(M^4)),$$

then

(1)

$$Y(g_0) \leq 0.$$

(2) *there is a metric $g = e^{2w} g_0$ such that [cf. 2]*

$$\Delta s_g + \epsilon_2 |\psi|^4 + 12|E|^2 = s_g^2,$$

for some $\epsilon_2 > 0$.

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