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Gourab Bhattacharya. THE YAMABE INVARIANT OF THE GRAVITATIONAL MONOPOLE. 2022. hal-03894018

## HAL Id: hal-03894018 https://cnrs.hal.science/hal-03894018

Preprint submitted on 12 Dec 2022

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#### THE YAMABE INVARIANT OF THE GRAVITATIONAL MONOPOLE.

GOURAB BHATTACHARYA

ABSTRACT. We show the Yamabe invariant is non-positive due to the presence of the Gravitational Monopole equations.

#### 1. INTRODUCTION

In this short note we announce some results on the Yamabe invariant of a Gravitational monopole, the notion of the Gravitational monopole was first introduced in [cf.1]. We shall give details of the proofs in a different paper.

Let  $(M^n, g)$  a smooth, compact-oriented Riemannian n-manifold. As we know the Einstein-Hilbert action

(1.1) 
$$g \mapsto \int s_g dv_g =: \mathcal{I}_g$$

with  $s_g$  the scalar curvature and  $dv_g$  is the volume form with respect to the metric g gives the Einstein equations at the critical points of the action  $\mathcal{I}_g$  with a volume constraint. The Yamabe problem is to minimize  $\mathcal{I}_g$  in a fixed conformal class of metric. Due to such a constraint, critical points of  $\mathcal{I}_g$  have constant scalar curvature.

Let us define now the Yamabe invariant Y(g),

(1.2) 
$$Y(g) := \inf_{\widetilde{g} = e^{2w}g} \operatorname{vol}(\widetilde{g})^{-\left(\frac{n-2}{n}\right)} \int s_{\widetilde{g}} dv_{\widetilde{g}}.$$

Due to the extensive work of Yamabe, Aubin, Trudinger, and Schoen, we now know that every conformal class on a compact manifold admits a metric that achieves Y(g) and therefore has constant scalar curvature.

Now in particular for n = 4,  $M^4$  is an oriented four-dimensional manifold. The Hodge \*-operator induces a splitting of the space of two-forms

(1.3) 
$$\bigwedge^2 = \bigwedge^2_+ \bigoplus \bigwedge^2_-$$

into the subspace of self-dual 2-forms  $\bigwedge_{+}^{2}$  and anti-self-dual 2-forms  $\bigwedge_{-}^{2}$ . This decomposition induces a splitting of the Weyl curvature (defined as a trace-free endomorphism of the  $\bigwedge^{2}$ ) into its self-dual and anti-self-dual components  $W^{\pm}$ . We have the Hirzebruch signature formula

(1.4) 
$$12\pi^2 \sigma(M^4) = \int_{M^4} (|W^+|^2 - |W^-|^2) dv_g$$

Let us define the Weyl functionals  $\mathcal{W}[g], \mathcal{W}^{\pm}[g]$  by

(1.5) 
$$\mathcal{W}[g] := \int |W_g|^2 dv_g, \ \mathcal{W}_{\pm}[g] := \int |W_g^{\pm}|^2 dv_g.$$

Due to the Hirzebruch signature formula (1.4), the study of the Weyl functional  $\mathcal{W}[g]$  is equivalent to the study of the self-dual functional  $\mathcal{W}_+[g]$ .

The signature formula implies the following fact,

Proposition 1.

(1.6) 
$$\mathcal{W}_{+}[g] \ge \max\{12\pi^{2}\sigma(M^{4}), 0\}.$$

The condition

(1.7) 
$$\mathcal{W}_{+}[g] = \max\{12\pi^{2}\sigma(M^{4}), 0\}.$$

is true if and only if  $W^+ \equiv 0$  or  $W^- \equiv 0$ , that is g is half-conformally flat.

The following formula is useful

(1.8) 
$$|W|^2 = |W^+|^2 + |W^-|^2,$$

also since W is conformally invariant,  $W^{\pm}$  are conformally invariant pieces of W.

The following result is due to Taubes: for any (closed, compact, orientable) four-dimensional manifold  $N^4$ , the manifold  $M^4 = N^4 \# k \mathbb{C}P^2$  admits anti-self-dual metrics (i.e.  $W^- = 0$ ) for all k sufficiently large.

It is a classical result that using the conformal invariance of the Bach tensor, one concludes that any metric which is (locally) conformally Einstein is critical for  $W_+$ .

#### 2. The Gravitational Monopole Equations

In [1], the following equations are introduced (sometimes we omit the mapping c, and denote by " $\cdot$ " the Clifford multiplication (or composition) and the dimension 4 for the convenience of computations):

(2.1)  

$$\nabla \psi = (d + d^*)\psi = 0,$$

$$c(W_g^+) = \frac{1}{4} \langle e_i \cdot e_j \psi, \psi \rangle e^i \wedge e^j,$$

One can rewrite (2.1) in the following form

 $\nabla \psi = 0.$ 

(2.2)

$$c(W_g^+) = \psi^* \otimes \psi - \frac{|\psi|^2}{2} \mathrm{Id},$$

therefore we have the following proposition

#### Proposition 2.

(2.3) 
$$|W^+|^2 = \frac{|\psi|^4}{8}.$$

3. Constraint on the Yamambe invariant due to the Gravitational Monopole Equations.

We have the following propositions.

**Proposition 3.** Let  $(M^4, g_0)$  be a compact, oriented, four-dimensional manifold with a  $Spin^{\mathbb{C}}$ -structure. Let E the trace-free part of the Ricci tensor. We also assume the validity of (2.2) on  $M^4$ . We further assume

(3.1) 
$$\int 3|\psi|^4 dv_{g_0} = 32\pi^2 (2\chi(M^4) + 3\sigma(M^4)),$$

then there is a metric  $g = e^{2w}g_0$  such that

(3.2) 
$$6\Delta s_g + 3|\psi|^4 + 12|E|^2 = s_g^2$$

It was also shown in [2] without the existence of the Gravitational monopole hypothesis that

**Proposition 4.** Let  $(M^4, g_0)$  be a compact, oriented, four-dimensional manifold. Let E the trace-free part of the Ricci tensor. Assume

(3.3) 
$$\int |W^+|^2 dv_{g_0} = \frac{4}{3}\pi^2 (2\chi(M^4) + 3\sigma(M^4)),$$

 $\mathbf{2}$ 

then there is a metric  $g = e^{2w}g_0$  such that

(3.4) 
$$\Delta s_g = -4|W^+|^2 - 2|E|^2 + \frac{1}{6}s_g^2$$

Also,

- (1) if  $Y(g_0) > 0$ , then  $s_q > 0$ .
- (2) if  $Y(q_0) = 0$ , then g is Ricci-flat anti-self-dual metric.

Since the Gravitational monopole puts the restriction on the scalar curvature  $s_{g_0} \leq 0$  of  $M^4$  [cf. 1], and since Proposition (4) implies whenever  $Y(g_0) > 0$ , then  $s_q > 0$ , we therefore conclude, the following theorem:

**Theorem 3.1.** Let  $(M^4, g_0)$  be a compact, oriented, four-dimensional manifold with a  $Spin^{\mathbb{C}}$ -structure. Let E the trace-free part of the Ricci tensor. We also assume the validity of (2.2) on  $M^4$ . We further assume

(3.5) 
$$\int 3|\psi|^4 dv_{g_0} = 32\pi^2 (2\chi(M^4) + 3\sigma(M^4)),$$

then there is a metric  $g = e^{2w}g_0$  such that

Since  $Y(g_0) = 0$  implies Ricci-flatness and anti-self-duality, it requires a special attention.

**Proposition 5.** Let  $(M^4, g_0)$  be a compact, oriented, four-dimensional manifold with a  $Spin^{\mathbb{C}}$ -structure. Let E the trace-free part of the Ricci tensor. We also assume the validity of (2.2) on  $M^4$ . We further assume

(3.7) 
$$0 < \int 3|\psi|^4 dv_{g_0} < 32\pi^2 (2\chi(M^4) + 3\sigma(M^4)),$$

then there is a metric  $g = e^{2w}g_0$  such that

(1)

(2) There is no metric  $g_0$  satisfying (3.7) with  $Y(g_0) = 0$ , therefore,

 $Y(g_0) < 0.$ 

 $Y(q_0) < 0.$ 

(3)

$$24\Delta s_g + 3(4+\epsilon_1)|\psi|^4 + 48|E|^2 + 4s_g^2 = 0,$$

for some  $\epsilon_1 > 0$ .

**Proposition 6.** Let  $(M^4, g_0)$  be a compact, oriented, four-dimensional manifold with a  $Spin^{\mathbb{C}}$ -structure. Let E the trace-free part of the Ricci tensor. We also assume the validity of (2.2) on  $M^4$ . We further assume

(3.8) 
$$\int |\psi|^4 dv_{g_0} = 16\pi^2 (2\chi(M^4) + 3\sigma(M^4)),$$

then

(1)

(2) there is a metric  $g = e^{2w}g_0$  such that [cf. 2]

$$\Delta s_g = -\frac{3}{2}|E|^2 + \frac{1}{8}s_g^2.$$

 $Y(q_0) \le 0.$ 

**Proposition 7.** Let  $(M^4, g_0)$  be a compact, oriented, four-dimensional manifold with a  $Spin^{\mathbb{C}}$ -structure. Let E the trace-free part of the Ricci tensor. We also assume the validity of (2.2) on  $M^4$ . We further assume

(3.9) 
$$0 < \int |\psi|^4 dv_{g_0} = 16\pi^2 (2\chi(M^4) + 3\sigma(M^4)),$$

then

(1)

$$Y(g_0) \le 0.$$

(2) there is a metric  $g = e^{2w}g_0$  such that [cf. 2]

$$\Delta s_g + \epsilon_2 |\psi|^4 + 12|E|^2 = s_g^2,$$

for some  $\epsilon_2 > 0$ .

#### References

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- [2] Gursky., J. Matthew; The Weyl functional, de Rham cohomology, and Kähler-Einstein metrics, Annals of Mathematics, 148 (1998), 315-337.

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