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APPLICATION OF WAIST INEQUALITY TO ENTROPY AND MEAN DIMENSION

RUXI SHI AND MASAKI TSUKAMOTO

Abstract. Waist inequality is a fundamental inequality in geometry and topology. We apply it to the study of entropy and mean dimension of dynamical systems. We consider equivariant continuous maps $\pi : (X, T) \to (Y, S)$ between dynamical systems and assume that the mean dimension of the domain (X, T) is larger than the mean dimension of the target (Y, S). We exhibit several situations for which the maps π necessarily have positive conditional metric mean dimension. This study has interesting consequences to the theory of topological conditional entropy. In particular it sheds new light on a celebrated result of Lindenstrauss and Weiss about minimal dynamical systems non-embeddable in $[0, 1]^{\mathbb{Z}}$.

1. Introduction

1.1. **Topological conditional entropy.** Waist inequality is a deep geometric-topological inequality which incorporates the ideas of isoperimetric inequality with the Borsuk–Ulam theorem in a highly nontrivial way. It states that, for $n \geq m$, every continuous map $f : S^n \to \mathbb{R}^m$ has some fiber of "large volume" in an appropriate sense. We will review it in Section 2 below. Its origin goes back to the work of Almgren [Al65] on geometric measure theory. Gromov mentioned a version of the waist inequality in his famous "filling" paper [Gro83, Appendix I (F)]. Later he developed its full theory in [Gro03].

Variants of the waist inequality have appeared in several interesting subjects: convex geometry [Ver06, GMT05, Kla17], generalizations of Morse theory [Gro09], combinatorics [Gro10], knots [Par11, GG12] and complex analysis [Kla18]. The purpose of this paper is to further add a new item to this list; we explain applications of the waist inequality to *topological dynamics*.

Our starting point is the theory of topological conditional entropy. We first quickly prepare basic definitions. Details about topological conditional entropy can be found in the book of Downarowicz [Dow11, Section 6].

A pair (X, T) is called a **dynamical system** if X is a compact metrizable space and $T : X \to X$ is a homeomorphism. Let d be a metric on X. For a natural number N we

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define a new metric d_N on X by

$$d_N(x,y) = \max_{0\le n<N} d\left(T^n x, T^n y\right).$$

For a positive number ε and a subset $E \subset X$ we denote by $\#\left(E, d_N, \varepsilon\right)$ the minimum number of n such that there exists an open covering $\{U_1, \dots, U_n\}$ of E satisfying $\mathrm{Diam}(U_i, d_N) < \varepsilon$ for all $1 \le i \le n$. Here $\mathrm{Diam}(U_i, d_N)$ is the diameter of U_i with respect to the metric d_N. When $E = \emptyset$, we set $\#\left(E, d_N, \varepsilon\right) = 0$.

We define the **topological entropy** of (X,T) by

$$h_{\mathrm{top}}(X,T) = \lim_{\varepsilon\to 0}\left(\lim_{N\to\infty} \frac{\log \#\left(X, d_N, \varepsilon\right)}{N}\right).$$

This is a topological invariant of dynamical systems. Namely its value is independent of the choice of the metric d. For a closed subset $E \subset X$ we also define

$$h_{\mathrm{top}}(E,T) = \lim_{\varepsilon\to 0}\left(\limsup_{N\to\infty} \frac{\log \#\left(E, d_N, \varepsilon\right)}{N}\right).$$

Let (X,T) and (Y,S) be dynamical systems. Let $\pi : X \to Y$ be an equivariant continuous map. Here "equivariant" means $\pi \circ T = S \circ \pi$. We often denote such a map by $\pi : (X,T) \to (Y,S)$ for clarifying the maps T and S. Let d be a metric on X. We define the **topological conditional entropy** of π by

$$h_{\mathrm{top}}(\pi,T) = \lim_{\varepsilon\to 0}\left(\lim_{N\to\infty} \frac{\sup_{y\in Y} \log \#\left(\pi^{-1}(y), d_N, \varepsilon\right)}{N}\right).$$

This definition is due to [Dow11, Lemma 6.8.2]. It coincides with the supremum of *fiberwise entropy* [Dow11, Theorem 6.8.3]:

$$h_{\mathrm{top}}(\pi,T) = \sup_{y\in Y} h_{\mathrm{top}}\left(\pi^{-1}(y), T\right).$$

In the book of Downarowicz [Dow11] the topological conditional entropy $h_{\mathrm{top}}(\pi,T)$ is denoted by $\mathbf{h}(T|S)$.

Bowen [Bow71, Theorem 17] proved that for any equivariant continuous map $\pi : (X,T) \to (Y,S)$ we have

$$h_{\mathrm{top}}(X,T) \le h_{\mathrm{top}}(Y,S) + h_{\mathrm{top}}(\pi,T). \tag{1·1}$$

This inequality has a clear intuitive meaning; if (X,T) is "larger than" (Y,S) then some fiber of π must be "large".

We are interested in the case that both $h_{\mathrm{top}}(X,T)$ and $h_{\mathrm{top}}(Y,S)$ are infinite. In this case Bowen's inequality (1·1) provides no information about the fibers of π. Typical examples of such infinite entropy systems are given by *shifts on infinite dimensional cubes*. Let a be a natural number, and let $[0,1]^a$ be the a-dimensional cube. We consider the bi-infinite product of $[0,1]^a$:

$$\left([0,1]^a\right)^{\mathbb{Z}} = \cdots \times [0,1]^a \times [0,1]^a \times [0,1]^a \times \cdots.$$

We define the shift $\sigma : ([0,1]^a)^{\mathbb{Z}} \to ([0,1]^a)^{\mathbb{Z}}$ by

$$\sigma\left((x_n)_{n\in\mathbb{Z}}\right) = (x_{n+1})_{n\in\mathbb{Z}}, \quad (\text{each } x_n \in [0,1]^a).$$

The pair $\left(([0,1]^a)^{\mathbb{Z}}, \sigma\right)$ is a dynamical system which has infinite topological entropy. This system is called the **full-shift on the alphabet** $[0,1]^a$. (Later we will often consider two full-shifts on different alphabets simultaneously. But we use the same letter σ for shift maps of both systems. Hopefully this will not cause any confusion.)

The following theorem is a starting point of our investigation.

Theorem 1.1. *Let $a > b$ be two natural numbers. Let $\left(([0,1]^a)^{\mathbb{Z}}, \sigma\right)$ and $\left(([0,1]^b)^{\mathbb{Z}}, \sigma\right)$ be the full-shifts on the alphabets $[0,1]^a$ and $[0,1]^b$ respectively. Then for any equivariant continuous map $\pi : ([0,1]^a)^{\mathbb{Z}} \to \left([0,1]^b\right)^{\mathbb{Z}}$ we have*

$$h_{\mathrm{top}}(\pi, \sigma) = +\infty.$$

This theorem has the same spirit as that of Bowen's inequality (1·1): Since we assumed $a > b$, the shift $([0,1]^a)^{\mathbb{Z}}$ is "larger than" the shift $\left([0,1]^b\right)^{\mathbb{Z}}$, and hence some fiber of $\pi : ([0,1]^a)^{\mathbb{Z}} \to \left([0,1]^b\right)^{\mathbb{Z}}$ must be "large". However this result does not follow from Bowen's inequality (1·1) because both the shifts $([0,1]^a)^{\mathbb{Z}}$ and $\left([0,1]^b\right)^{\mathbb{Z}}$ have infinite topological entropy.

Theorem 1.1 might look an innocent statement. However we can add it a small twist so that it becomes more interesting:

Theorem 1.2. *Let $a > b$ be two natural numbers. There exists a minimal closed subset X of $\left(([0,1]^a)^{\mathbb{Z}}, \sigma\right)$ such that for any equivariant continuous map $\pi : (X, \sigma) \to \left(([0,1]^b)^{\mathbb{Z}}, \sigma\right)$ we have*

$$h_{\mathrm{top}}(\pi, \sigma) = +\infty.$$

Here "minimal" means that $\sigma(X) = X$ and that every orbit of (X, σ) is dense in X.

Letting $a = 2$ and $b = 1$ in the theorem, we get a minimal closed subset X of the shift $([0,1]^2)^{\mathbb{Z}}$ such that for any equivariant continuous map $\pi : X \to [0,1]^{\mathbb{Z}}$ we have $h_{\mathrm{top}}(\pi, \sigma) = +\infty$. In particular, (X, σ) *does not embed in the shift* $[0,1]^{\mathbb{Z}}$. This provides an answer to Auslander's question in [Aus88, p. 193] from a new angle; Auslander asked whether there exists a minimal dynamical system which does not embed in the shift $[0,1]^{\mathbb{Z}}$. This problem was first solved in the celebrated paper of Lindenstrauss–Weiss [LW00] by using mean dimension theory. We briefly review their work below.

Mean dimension is a topological invariant of dynamical systems introduced by Gromov [Gro99]. Let (X, T) be a dynamical system with a metric d on X. For $\varepsilon > 0$ and a natural number N we define $\mathrm{Widim}_\varepsilon(X, d_N)$ as the minimum number of n such that there exist an n-dimensional finite simplicial complex P and an ε-embedding $f : (X, d_N) \to P$. Here

f is called an ε-**embedding** if we have $\mathrm{Diam}(f^{-1}(p), d_N) < \varepsilon$ for all $p \in P$. We define the **mean dimension** of (X, T) by

$$\mathrm{mdim}(X, T) := \lim_{\varepsilon \to 0} \left(\lim_{N \to \infty} \frac{\mathrm{Widim}_\varepsilon(X, d_N)}{N} \right).$$

This is a topological invariant, i.e. independent of the choice of d. For a closed subset $E \subset X$ we also define

$$\mathrm{mdim}(E, T) := \lim_{\varepsilon \to 0} \left(\limsup_{N \to \infty} \frac{\mathrm{Widim}_\varepsilon(E, d_N)}{N} \right).$$

If a dynamical system (X, T) embeds in another (Y, S) then we have $\mathrm{mdim}(X, T) \leq \mathrm{mdim}(Y, S)$. The full-shift $\left(([0,1]^a)^{\mathbb{Z}}, \sigma \right)$ has mean dimension a. Lindenstrauss-Weiss [LW00, Proposition 3.5] constructed a minimal closed subset $X \subset ([0,1]^2)^{\mathbb{Z}}$ such that (X, σ) has mean dimension strictly larger than one. Then such X does not embed in $[0,1]^{\mathbb{Z}}$ because $[0,1]^{\mathbb{Z}}$ has mean dimension one. So this provides a negative answer to Auslander's question.

Notice that the statement "X *does not embed in* $[0,1]^{\mathbb{Z}}$" means that every equivariant continuous map $\pi : X \to [0,1]^{\mathbb{Z}}$ has some fiber of cardinality strictly larger than one. Theorem 1.2 above provides a substantially stronger conclusion that some fiber of π must have infinite entropy. Indeed the construction of the minimal closed subset X in Theorem 1.2 is the same as that of Lindenstrauss-Weiss [LW00, Proposition 3.5]. But we can deduce a stronger conclusion with the aid of the waist inequality.

1.2. **Conditional metric mean dimension.** As we already saw at the end of the last subsection, our study has a connection to mean dimension theory. It turns out that the most fruitful framework is provided by a conditional version of *metric mean dimension*. Metric mean dimension is a notion introduced by Lindenstrauss–Weiss [LW00, §4] for better understanding the relation between mean dimension and entropy. Its conditional version was introduced by Liang [Lia21, Definition 4.1].

Let (X, T) be a dynamical system with a metric d on X. We define the **upper and lower metric mean dimensions** of (X, T, d) by

$$\overline{\mathrm{mdim}}_{\mathrm{M}}(X, T, d) = \limsup_{\varepsilon \to 0} \left(\lim_{N \to \infty} \frac{\log \# (X, d_N, \varepsilon)}{N \log(1/\varepsilon)} \right),$$

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(X, T, d) = \liminf_{\varepsilon \to 0} \left(\lim_{N \to \infty} \frac{\log \# (X, d_N, \varepsilon)}{N \log(1/\varepsilon)} \right).$$

These are metric-dependent quantities. When the upper and lower limits coincide, we denote the common value by $\mathrm{mdim}_{\mathrm{M}}(X, T, d)$. If $\underline{\mathrm{mdim}}_{\mathrm{M}}(X, T, d) > 0$ then $h_{\mathrm{top}}(X, T) = +\infty$. So, in a sense, metric mean dimension evaluates *how infinite topological entropy is*.

Lindenstrauss–Weiss [LW00, Theorem 4.2] proved

$$\mathrm{mdim}(X, T) \leq \underline{\mathrm{mdim}}_{\mathrm{M}}(X, T, d).$$

See also the paper of Lindenstrauss [Lin99] for a deeper study of metric mean dimension.

Let (X,T) and (Y,S) be dynamical systems, and let $\pi : X \to Y$ be an equivariant continuous map . Let d be a metric on X. We define the **upper and lower conditional metric mean dimensions** of π by

$$\overline{\mathrm{mdim}}_{\mathrm{M}}(\pi,T,d) = \limsup_{\varepsilon\to 0}\left(\lim_{N\to\infty}\frac{\sup_{y\in Y}\log \#\left(\pi^{-1}(y),d_N,\varepsilon\right)}{N\log(1/\varepsilon)}\right),$$
$$\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,T,d) = \liminf_{\varepsilon\to 0}\left(\lim_{N\to\infty}\frac{\sup_{y\in Y}\log \#\left(\pi^{-1}(y),d_N,\varepsilon\right)}{N\log(1/\varepsilon)}\right).$$

These are also metric-dependent quantities. When the upper and lower limits coincide, we denote the common value by $\mathrm{mdim}_{\mathrm{M}}(\pi,T,d)$. If $\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,T,d) > 0$ then $h_{\mathrm{top}}(\pi,T) = +\infty$. So conditional metric mean dimension evaluates *how infinite topological conditional entropy is*.

Let a be a natural number. For $x = (x_1,\dots,x_a) \in \mathbb{R}^a$ we set

$$\|x\|_\infty := \max_{1\le i\le a}|x_i|.$$

Let $([0,1]^a)^{\mathbb{Z}}$ be the full-shift on the alphabet $[0,1]^a$. We define a metric D on it by

$$D\left((x_n)_{n\in\mathbb{Z}},(y_n)_{n\in\mathbb{Z}}\right) = \sum_{n\in\mathbb{Z}} 2^{-|n|}\|x_n - y_n\|_\infty, \quad (x_n,y_n\in[0,1]^a). \tag{1·2}$$

Theorem 1.3. *Let $a > b$ be two natural numbers. Let $\left(([0,1]^a)^{\mathbb{Z}},\sigma\right)$ and $\left(\left([0,1]^b\right)^{\mathbb{Z}},\sigma\right)$ be the full-shifts on the alphabets $[0,1]^a$ and $[0,1]^b$ respectively. Then for any equivariant continuous map $\pi : ([0,1]^a)^{\mathbb{Z}} \to \left([0,1]^b\right)^{\mathbb{Z}}$ we have*

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,\sigma,D) \ge a - b.$$

Here D is the metric defined by (1·2).

Theorem 1.1 in §1.1 immediately follows from this theorem because $\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,\sigma,D) > 0$ implies $h_{\mathrm{top}}(\pi,\sigma) = +\infty$.

Example 1.4. The estimate in Theorem 1.3 is sharp: For $a > b$, let $f : [0,1]^a \to [0,1]^b$ be the natural projection to the first b coordinates. We define $\pi : ([0,1]^a)^{\mathbb{Z}} \to \left([0,1]^b\right)^{\mathbb{Z}}$ by

$$\pi\left((x_n)_{n\in\mathbb{Z}}\right) = \left(f(x_n)\right)_{n\in\mathbb{Z}}.$$

Then

$$\mathrm{mdim}_{\mathrm{M}}(\pi,\sigma,D) = a - b.$$

We can also prove a statement similar to Theorem 1.3 for maps from $([0,1]^a)^{\mathbb{Z}}$ to *arbitrary* dynamical systems:

Theorem 1.5. *Let a be a natural number and let (Y,S) be a dynamical system. For any equivariant continuous map $\pi:\left(([0,1]^a)^{\mathbb{Z}},\sigma\right)\to(Y,S)$ we have*

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,\sigma,D)\geq a-2\mathrm{mdim}(Y,S). \tag{1·3}$$

Here $\mathrm{mdim}(Y,S)$ is the mean dimension of (Y,S).

The factor 2 in the right-hand side of (1·3) looks a bit unsatisfactory. So we propose:

Problem 1.6. In the setting of Theorem 1.5, can one conclude the following stronger inequality?

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,\sigma,D)\geq a-\mathrm{mdim}(Y,S).$$

See also Remark 4.2 (2).

Here the target system (Y,S) is arbitrary, but the domain is only $([0,1]^a)^{\mathbb{Z}}$. It seems difficult for our current technology to try a similar question for maps $\pi:(X,T)\to(Y,S)$ with arbitrary dynamical systems (X,T). Nevertheless we would like to propose the following problem because the statement looks beautiful (if it is true).

Problem 1.7. Let (X,T) and (Y,S) be dynamical systems with

$$\mathrm{mdim}(X,T)>\mathrm{mdim}(Y,S).$$

Let $\pi:(X,T)\to(Y,S)$ be an equivariant continuous map. Can one always conclude that the topological conditional entropy of π is infinite? When $\mathrm{mdim}(Y,S)=0$, this can be proved by using [Tsu, Theorem 1.5]. But when $\mathrm{mdim}(Y,S)>0$, the problem is widely open.

Although it is currently difficult to study maps from general dynamical systems, our method works well for some subsystems of $([0,1]^a)^{\mathbb{Z}}$. We can prove the following theorem.

Theorem 1.8. *Let a be a natural number. Let s be a real number with $0\leq s<a$. There exists a minimal closed subset X of $\left(([0,1]^a)^{\mathbb{Z}},\sigma\right)$ satisfying the following two conditions.*

1. $\mathrm{mdim}(X,\sigma)=\mathrm{mdim}_{\mathrm{M}}(X,\sigma,D)=s$.
2. *For any natural number b and any equivariant continuous map $\pi:(X,\sigma)\to\left(([0,1]^b)^{\mathbb{Z}},\sigma\right)$ we have*

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,\sigma,D)\geq s-b.$$

Here D is a metric defined in (1·2).

When $s>b$, the inequality $\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,\sigma,D)\geq s-b>0$ implies $h_{\mathrm{top}}(\pi,\sigma)=+\infty$. Therefore Theorem 1.2 in §1.1 is an immediate corollary of Theorem 1.8.

1.3. **On an analogue of the Hurewicz theorem.** The Hurewicz theorem [HW41, p.91, Theorem VI 7] in classical dimension theory asserts that for any continuous map $f : X \to Y$ between compact metrizable spaces we have

$$\dim X \leq \dim Y + \sup_{y \in Y} \dim f^{-1}(y). \tag{1·4}$$

Here $\dim(\cdot)$ denotes topological dimension[1].

Let d be a metric on X. We define the **upper and lower Minkowski dimensions** of (X, d) by

$$\overline{\dim}_{\mathrm{M}}(X, d) = \limsup_{\varepsilon \to 0} \frac{\log \# (X, d, \varepsilon)}{\log(1/\varepsilon)}, \quad \underline{\dim}_{\mathrm{M}}(X, d) = \liminf_{\varepsilon \to 0} \frac{\log \# (X, d, \varepsilon)}{\log(1/\varepsilon)}.$$

When the upper and lower limits coincide, we denote the common value by $\dim_{\mathrm{M}}(X, d)$. The topological dimension is always bounded by the lower Minkowski dimension.

$$\dim X \leq \underline{\dim}_{\mathrm{M}}(X, d).$$

So $\dim f^{-1}(y) \leq \underline{\dim}_{\mathrm{M}} \left(f^{-1}(y), d\right)$, and hence we also have

$$\dim X \leq \dim Y + \sup_{y \in Y} \underline{\dim}_{\mathrm{M}} \left(f^{-1}(y), d\right) \tag{1·5}$$

for every continuous map $f : X \to Y$.

Theorems 1.3 and 1.5 in §1.2 are dynamical analogues of the inequality (1·5). Conditional metric mean dimension is an analogue of the term $\sup_{y \in Y} \underline{\dim}_{\mathrm{M}} \left(f^{-1}(y), d\right)$. Probably it is also natural to ask whether or not an analogue of the original Hurewicz theorem (1·4) holds for mean dimension. Indeed this question was already proposed by the second-named author [Tsu08] more than ten years ago and studied in detail by a recent paper [Tsu]. In this subsection we explain the relation between [Tsu] and the current paper.

The paper [Tsu, Theorem 3.9, Remark 3.10] proved the next theorem.

Theorem 1.9. *Let a be a natural number. There exists a zero-dimensional*[2] *dynamical system (Z, R) for which the following statement holds true: For any $\delta > 0$ there exist a dynamical system (Y, S) and an equivariant continuous map*

$$\pi : \left(([0,1]^a)^{\mathbb{Z}} \times Z, \sigma \times R \right) \to (Y, S)$$

satisfying

$$\mathrm{mdim}(Y, S) < \delta, \quad \sup_{y \in Y} \mathrm{mdim} \left(\pi^{-1}(y), \sigma \times R\right) = 0.$$

Here $\mathrm{mdim} \left(\pi^{-1}(y), \sigma \times R\right)$ is the mean dimension of the fiber $\pi^{-1}(y) \subset ([0,1]^a)^{\mathbb{Z}} \times Z$.

[1] Let d be a metric on X. We define the **topological dimension** by $\dim X = \lim_{\varepsilon \to 0} \mathrm{Widim}_{\varepsilon}(X, d)$. The value of $\dim X$ is independent of the choice of d

[2] "Zero-dimensional" means that the topological dimension of Z is zero.

In this statement, the system $\left(([0,1]^a)^{\mathbb{Z}} \times Z, \sigma \times R\right)$ has mean dimension a while $\mathrm{mdim}(Y,S) < \delta$ may be arbitrarily small. Nonetheless the map π has no mean dimension in the fiber direction. This implies that a direct analogue of the Hurewicz theorem (1·4) does not hold for mean dimension. In other words, mean dimension of the fibers $\mathrm{mdim}\,(\pi^{-1}(y), \sigma \times R)$ does not properly evaluate the complexity of the map π.

On the other hand, we can prove the following statement for conditional metric mean dimension:

Proposition 1.10. *Let a be a natural number. Let (Z,R) be a dynamical system with a metric ρ on Z. We define a metric d on the product $([0,1]^a)^{\mathbb{Z}} \times Z$ by*

$$d\left((x,z),(x',z')\right) = D(x,x') + \rho(z,z'),$$

where D is the metric on $([0,1]^a)^{\mathbb{Z}}$ defined by (1·2). Then for any dynamical system (Y,S) and any equivariant continuous map $\pi : \left(([0,1]^a)^{\mathbb{Z}} \times Z, \sigma \times R\right) \to (Y,S)$ we have

$$\underline{\mathrm{mdim}}_{\mathrm{M}}\,(\pi, \sigma \times R, d) \geq a - 2\mathrm{mdim}(Y,S).$$

Namely, we can detect a reasonable amount of complexity of the map π by using conditional metric mean dimension. This gives the authors more confidence that the conditional metric mean dimension is a useful tool for studying maps between dynamical systems.

Remark 1.11. In the setting of Proposition 1.10, we can also prove the following statement: Define an equivariant continuous map $\Pi : \left(([0,1]^a)^{\mathbb{Z}} \times Z, \sigma \times R\right) \to (Y \times Z, S \times R)$ by

$$\Pi(x,z) = (\pi(x,z), z)\,.$$

Then we have

$$\underline{\mathrm{mdim}}_{\mathrm{M}}\,(\Pi, \sigma \times R, d) \geq a - 2\mathrm{mdim}(Y,S).$$

This statement is a bit stronger (and arguably more natural) than Proposition 1.10. But the proof is more or less the same. So we omit the details of the proof.

1.4. **Organization of the paper.** We review the waist inequality in the next section. We prepare a simple fact on conditional metric mean dimension in §3. We prove the main technical theorem in §4. As its corollaries, we prove Theorems 1.3, 1.5 and Proposition 1.10 also in §4. We prove Theorem 1.8 in §5.

2. Waist inequality

In this section we review the waist inequality. Readers can find much more information in an excellent survey of Guth [Gut14]. Let $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ be the n-dimensional sphere. Here $|x|$ is the Euclidean norm. We naturally consider $S^0 \subset S^1 \subset$

$S^2 \subset \cdots \subset S^n$. For $A \subset S^n$ and $r > 0$ we denote by $A + r$ the r-neighborhood of A in the spherical metric.

The following inequality is the original waist inequality proved by Gromov [Gro03].

Theorem 2.1 (Gromov, 2003)**.** *Let $n \geq m$ be two natural numbers. For any continuous map $f : S^n \to \mathbb{R}^m$ there exists $t \in \mathbb{R}^m$ such that for every $r > 0$ we have*

$$\operatorname{vol}\left(f^{-1}(t) + r\right) \geq \operatorname{vol}\left(S^{n-m} + r\right). \tag{2·1}$$

Here $\operatorname{vol}(\cdot)$ *denotes the n-dimensional volume on the sphere S^n.*

For maps $f : S^n \to \mathbb{R}$, this statement follows from the isoperimetric inequality on the sphere. (See [AG15, Appendix] for the details.) For equidimensional maps $f : S^n \to \mathbb{R}^n$, if we additionally assume that every fiber of f has cardinality at most two, then the above inequality (2·1) for $r = \frac{\pi}{2}$ implies that $f^{-1}(t)$ consists of a pair of antipodal points. This is the statement of the Borsuk–Ulam theorem for such maps f. Therefore the waist inequality has close connections to both the isoperimetric inequality and the Borsuk–Ulam theorem. Its proof is a wonderful mixture of geometry (measure theory and convex geometry) and algebraic topology. Gromov's original paper [Gro03] is probably not very easily readable. Memarian's paper [Mem11] is helpful for understanding the details of the proof of the waist inequality.

As we already mentioned in the introduction, the ideas of the waist inequality have connections to several subjects. Here we review a connection to convex geometry developed by Klartag [Kla17]. This will be crucial for us.

For subsets A and B of $\mathbb{R}^n$ we define $A + B = \{x + y \mid x \in A, y \in B\}$. For $r > 0$ we set $rA = \{rx \mid x \in A\}$. A **convex body** of $\mathbb{R}^n$ is a compact convex subset of $\mathbb{R}^n$ with a non-empty interior. A convex body K is said to be **centrally-symmetric** if $-K = K$, where $-K := \{-x \mid x \in K\}$. A function $\varphi : \mathbb{R}^n \to [0, +\infty)$ is said to be **log-concave** if for any $x, y \in \mathbb{R}^n$ and $0 < \lambda < 1$

$$\varphi\left(\lambda x + (1-\lambda)y\right) \geq \varphi(x)^{\lambda} \varphi(y)^{1-\lambda}.$$

Klartag [Kla17] proved several interesting theorems connecting the waist inequality to convex geometry. In particular he proved, among many other things, the following version of the waist inequality [Kla17, Theorem 5.7].

Theorem 2.2 (Klartag, 2017)**.** *Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body. Let $1 \leq m \leq n$. Let μ be a probability measure supported in K which has a log-concave density function with respect to the Lebesgue measure. Then for any continuous map $f : K \to \mathbb{R}^m$ there exists $t \in \mathbb{R}^m$ satisfying*

$$\mu\left(f^{-1}(t) + rK\right) \geq \left(\frac{r}{2+r}\right)^m \quad \text{for all } 0 < r < 1.$$

For $A \subset \mathbb{R}^n$ and $r > 0$ we define $A +_\infty r$ as the r-neighborhood of A with respect to the ℓ^∞-norm:

$$A +_\infty r = \{x \in \mathbb{R}^n \mid \exists a \in A : \|x - a\|_\infty \leq r\}.$$

Corollary 2.3. *Let $1 \leq m \leq n$. Let μ be the Lebesgue measure restricted to the unit cube $[0,1]^n$. Then for any continuous map $f : [0,1]^n \to \mathbb{R}^m$ there exists $t \in \mathbb{R}^m$ satisfying*

$$\mu\left(f^{-1}(t) +_\infty r\right) \geq \left(\frac{r}{2}\right)^m \quad \text{for all } 0 < r < \frac{1}{2}.$$

Proof. Let $K = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$. This is a centrally-symmetric convex body. Let ν be the Lebesgue measure restricted to K. The density of ν is the characteristic function of K, which is obviously log-concave. We define $g : K \to \mathbb{R}^m$ by

$$g(x) = f\left(x - \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)\right).$$

By Klartag's waist inequality, there exists $t \in \mathbb{R}^n$ satisfying

$$\nu\left(g^{-1}(t) + rK\right) \geq \left(\frac{r}{2+r}\right)^m \quad \text{for all } 0 < r < 1.$$

We have

$$g^{-1}(t) +_\infty r = g^{-1}(t) + 2rK.$$

Hence, for $0 < r < \frac{1}{2}$

$$\nu\left(g^{-1}(t) +_\infty r\right) = \nu\left(g^{-1}(t) + 2rK\right) \geq \left(\frac{2r}{2+2r}\right)^m \geq \left(\frac{r}{2}\right)^m.$$

We have

$$f^{-1}(t) +_\infty r = \left(g^{-1}(t) +_\infty r\right) + \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right).$$

Thus, for $0 < r < \frac{1}{2}$

$$\mu\left(f^{-1}(t) +_\infty r\right) = \nu\left(g^{-1}(t) +_\infty r\right) \geq \left(\frac{r}{2}\right)^m.$$

□

For $r > 0$ and a compact subset $A \subset \mathbb{R}^n$ we define $\#\left(A, \|\cdot\|_\infty, r\right)$ as the minimum number of N such that there exists an open cover $\{U_1, U_2, \ldots, U_N\}$ of A satisfying $\mathrm{Diam}\left(U_i, \|\cdot\|_\infty\right) < r$ for all $1 \leq i \leq r$. Here $\mathrm{Diam}\left(U_i, \|\cdot\|_\infty\right) = \sup_{x,y \in U_i} \|x - y\|_\infty$. When A is empty, we define $\#\left(A, \|\cdot\|_\infty, r\right) = 0$.

Corollary 2.4. *Let $1 \leq m \leq n$. For any continuous map $f : [0,1]^n \to \mathbb{R}^m$ there exists $t \in \mathbb{R}^m$ satisfying*

$$\#\left(f^{-1}(t), \|\cdot\|_\infty, r\right) \geq \frac{1}{8^n}\left(\frac{1}{r}\right)^{n-m} \quad \text{for all } 0 < r < \frac{1}{2}.$$

Proof. Let μ be the Lebesgue measure restricted to $[0,1]^n$. By Corollary 2.3 there exists $t \in \mathbb{R}^m$ satisfying

$$\mu\left(f^{-1}(t) +_\infty r\right) \geq \left(\frac{r}{2}\right)^m \quad \text{for all } 0 < r < \frac{1}{2}.$$

Let $0 < r < \frac{1}{2}$. Set $N = \#\left(f^{-1}(t), \|\cdot\|_\infty, r\right)$. There exists an open cover $f^{-1}(t) \subset U_1 \cup U_2 \cup \cdots \cup U_N$ satisfying $\mathrm{Diam}\left(U_i, \|\cdot\|_\infty\right) < r$ for all $1 \leq i \leq N$. Pick a point $x_i \in U_i$ for each U_i. We have

$$U_i \subset x_i + [-r,r]^n := \left\{y \in \mathbb{R}^n \mid \|y - x_i\|_\infty \leq r\right\}.$$

Hence

$$f^{-1}(t) \subset \bigcup_{i=1}^N \left(x_i + [-r,r]^n\right).$$

We have

$$f^{-1}(t) +_\infty r = \bigcup_{i=1}^N \left(x_i + [-2r,2r]^n\right).$$

Then

$$\left(\frac{r}{2}\right)^m \leq \mu\left(f^{-1}(t) +_\infty r\right) \leq \sum_{i=1}^N \mu\left(x_i + [-2r,2r]^n\right) \leq N(4r)^n.$$

Therefore

$$N \geq (4r)^{-n} \cdot \left(\frac{r}{2}\right)^m \geq \frac{1}{8^n} \cdot \left(\frac{1}{r}\right)^{n-m}.$$

□

For a convenience of readers, here we recall basic notions used in the definition of mean dimension. Let ε be a positive number, and let (X,d) be a compact metric space. A continuous map $f : X \to Y$ from X to a topological space Y is called an ε-embedding if we have $\mathrm{Diam} f^{-1}(y) < \varepsilon$ for all $y \in Y$. We define $\mathrm{Widim}_\varepsilon(X,d)$ as the minimum number n such that there exist an n-dimensional finite simplicial complex P and an ε-embedding $f : X \to P$.

We will need the following lemma later. This was proved in [LW00, Lemma 3.2].

Lemma 2.5. *For $0 < \varepsilon < 1$ we have*

$$\mathrm{Widim}_\varepsilon\left([0,1]^n, \|\cdot\|_\infty\right) = n.$$

Here $\|\cdot\|_\infty$ is the metric defined by the ℓ^∞-norm.

This lemma is a basis of the fact that the full-shift $\left([0,1]^a\right)^{\mathbb{Z}}$ has mean dimension a.

3. A preliminary on conditional metric mean dimension

In this section we prepare a simple fact about conditional metric mean dimension.

Let (X,T) be a dynamical system with a metric d. Recall that, for each $N \geq 1$, we defined a new metric d_N on X by $d_N(x_1,x_2) = \max_{0\leq n<N} d\left(T^n x_1, T^n x_2\right)$. Let (Y,S) be another dynamical system with a metric d'. Let N be a natural number and let r be a positive real number. For $y \in Y$ we set

$$B_r\left(y, d'_N\right) = \left\{x \in Y \mid d'_N(x,y) \leq r\right\}.$$

Let $\pi : (X,T) \to (Y,S)$ be an equivariant continuous map. Recall that we defined the upper and lower conditional metric mean dimensions of π by

$$\overline{\mathrm{mdim}}_{\mathrm{M}}(\pi,T,d) = \limsup_{\varepsilon\to 0}\left(\lim_{N\to\infty}\frac{\sup_{y\in Y}\log \#\left(\pi^{-1}(y), d_N, \varepsilon\right)}{N\log(1/\varepsilon)}\right),$$
$$\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,T,d) = \liminf_{\varepsilon\to 0}\left(\lim_{N\to\infty}\frac{\sup_{y\in Y}\log \#\left(\pi^{-1}(y), d_N, \varepsilon\right)}{N\log(1/\varepsilon)}\right).$$

In this definition, the limit with respect to N exists because the quantity

$$\sup_{y\in Y}\log \#\left(\pi^{-1}(y), d_N, \varepsilon\right)$$

is subadditive in N, as we will see below:

Lemma 3.1. *In the above setting, we have the following two statements.*

1. *The quantity* $\sup_{y\in Y}\log \#\left(\pi^{-1}(y), d_N, \varepsilon\right)$ *is subadditive in* N.
2. *Let* ε *and* δ *be positive numbers. Then the quantity*

$$a_N := \sup_{y\in Y}\log \#\left(\pi^{-1}\left(B_\delta(y, d'_N)\right), d_N, \varepsilon\right)$$

is subadditive in N.

Proof. We only prove (2). The proof of (1) is similar. Let N_1 and N_2 be natural numbers. We have

$$B_\delta\left(y, d'_{N_1+N_2}\right) = B_\delta\left(y, d'_{N_1}\right) \cap S^{-N_1} B_\delta\left(S^{N_1} y, d'_{N_2}\right).$$

Then

$$\begin{aligned}\pi^{-1}\left(B_\delta\left(y, d'_{N_1+N_2}\right)\right) &= \pi^{-1}\left(B_\delta\left(y, d'_{N_1}\right)\right) \cap \pi^{-1}\left(S^{-N_1} B_\delta\left(S^{N_1} y, d'_{N_2}\right)\right)\\ &= \pi^{-1}\left(B_\delta\left(y, d'_{N_1}\right)\right) \cap T^{-N_1}\pi^{-1}\left(B_\delta\left(S^{N_1} y, d'_{N_2}\right)\right).\end{aligned}$$

Hence

$$\begin{aligned}&\#\left(\pi^{-1}\left(B_\delta\left(y, d'_{N_1+N_2}\right)\right), d_{N_1+N_2}, \varepsilon\right)\\ &\leq \#\left(\pi^{-1}\left(B_\delta\left(y, d'_{N_1}\right)\right), d_{N_1}, \varepsilon\right) \cdot \#\left(\pi^{-1}\left(B_\delta\left(S^{N_1} y, d'_{N_2}\right)\right), d_{N_2}, \varepsilon\right)\\ &\leq e^{a_{N_1}} \cdot e^{a_{N_2}}.\end{aligned}$$

Therefore we have $a_{N_1+N_2} \leq a_{N_1} + a_{N_2}$. □

Lemma 3.2. *The upper and lower conditional metric mean dimensions of $\pi : (X,T) \to (Y,S)$ are given by*

$$\overline{\mathrm{mdim}}_{\mathrm{M}}(\pi,T,d) = \limsup_{\varepsilon\to 0}\left\{\lim_{\delta\to 0}\left(\lim_{N\to\infty}\frac{\sup_{y\in Y}\log \#\left(\pi^{-1}\left(B_\delta\left(y,d'_N\right)\right),d_N,\varepsilon\right)}{N\log(1/\varepsilon)}\right)\right\},$$

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,T,d) = \liminf_{\varepsilon\to 0}\left\{\lim_{\delta\to 0}\left(\lim_{N\to\infty}\frac{\sup_{y\in Y}\log \#\left(\pi^{-1}\left(B_\delta\left(y,d'_N\right)\right),d_N,\varepsilon\right)}{N\log(1/\varepsilon)}\right)\right\}.$$

Proof. We only prove the formula for the lower conditional metric mean dimension. The upper case is similar. We set

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,T,d)' = \liminf_{\varepsilon\to 0}\left\{\lim_{\delta\to 0}\left(\lim_{N\to\infty}\frac{\sup_{y\in Y}\log \#\left(\pi^{-1}\left(B_\delta\left(y,d'_N\right)\right),d_N,\varepsilon\right)}{N\log(1/\varepsilon)}\right)\right\}.$$

It is obvious that $\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,T,d) \leq \underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,T,d)'$. We will prove the reverse inequality.

Suppose $\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,T,d) < a$ for some positive number a. We show $\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,T,d)' \leq a$. There exists a decreasing sequence $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots \to 0$ such that

$$\lim_{N\to\infty}\frac{\sup_{y\in Y}\log \#\left(\pi^{-1}(y),d_N,\varepsilon_k\right)}{N\log(1/\varepsilon_k)} < a.$$

There exists N_k for each k satisfying

$$\sup_{y\in Y}\#\left(\pi^{-1}(y),d_{N_k},\varepsilon_k\right) < \left(\frac{1}{\varepsilon_k}\right)^{aN_k}.$$

From the compactness of Y, we can find $\delta_k > 0$ such that

$$\sup_{y\in Y}\#\left(\pi^{-1}\left(B_{\delta_k}(y,d'_{N_k})\right),d_{N_k},\varepsilon_k\right) < \left(\frac{1}{\varepsilon_k}\right)^{aN_k}.$$

From the subadditivity (2) in Lemma 3.1

$$\lim_{N\to\infty}\frac{\sup_{y\in Y}\log \#\left(\pi^{-1}\left(B_{\delta_k}(y,d'_N)\right),d_N,\varepsilon_k\right)}{N} < a\log(1/\varepsilon_k).$$

Hence

$$\lim_{\delta\to 0}\lim_{N\to\infty}\frac{\sup_{y\in Y}\log \#\left(\pi^{-1}\left(B_\delta(y,d'_N)\right),d_N,\varepsilon_k\right)}{N} < a\log(1/\varepsilon_k).$$

Thus we have $\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,T,d)' \leq a$. This has shown $\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,T,d)' \leq \underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,T,d)$. □

4. Main technical theorem

Here we formulate the main technical result of this paper. All the theorems in §1 are more or less its corollaries. We prove Theorems 1.3, 1.5 and Proposition 1.10 in this section. Theorem 1.8 will be proved in §5.

Theorem 4.1 (Main Technical theorem)**.** *Let s be a nonnegative real number. Let (X,T) be a dynamical system with a metric d on X. Suppose that there exist sequences of natural numbers $\{N_n\}_{n=1}^{\infty}$, $\{M_n\}_{n=1}^{\infty}$ and continuous maps $\psi_n : [0,1]^{M_n} \to X$ $(n \geq 1)$ satisfying the following three conditions.*

- *$N_n \to +\infty$ as $n \to \infty$.*
- *The ratio M_n/N_n converges to s as $n \to \infty$.*
- *For any $x, y \in [0,1]^{M_n}$ we have*

$$\|x-y\|_{\infty} \leq d_{N_n}\left(\psi_n(x), \psi_n(y)\right).$$

Then we have the following three conclusions.

(1) $\mathrm{mdim}(X,T) \geq s$.

(2) *For any natural number b and any equivariant continuous map $\pi : (X,T) \to \left(\left([0,1]^b\right)^{\mathbb{Z}}, \sigma\right)$ (where σ is the shift map) we have*

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi, T, d) \geq s - b.$$

(3) *For any dynamical system (Y,S) and any equivariant continuous map $\pi : (X,T) \to (Y,S)$ we have*

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi, T, d) \geq s - 2\mathrm{mdim}(Y,S).$$

Proof. (1) For $0 < \varepsilon < 1$ we have

$$\mathrm{Widim}_{\varepsilon}(X, d_{N_n}) \geq \mathrm{Widim}_{\varepsilon}\left([0,1]^{M_n}, \|\cdot\|_{\infty}\right) = M_n \quad \text{by Lemma 2.5.}$$

Then

$$\mathrm{mdim}(X,T) = \lim_{\varepsilon \to 0}\left(\lim_{n\to\infty} \frac{\mathrm{Widim}_{\varepsilon}(X, d_{N_n})}{N_n}\right) \geq \lim_{n\to\infty} \frac{M_n}{N_n} = s.$$

(2) For two integers $\ell < m$ and a point $y = (y_n)_{n\in\mathbb{Z}}$ in $\left([0,1]^b\right)^{\mathbb{Z}}$ we denote

$$y|_{\ell}^{m} = (y_{\ell}, y_{\ell+1}, \ldots, y_{m-1}, y_m) \in \left([0,1]^b\right)^{m-\ell+1}.$$

We take a metric d' on $\left([0,1]^b\right)^{\mathbb{Z}}$. Let $0 < \varepsilon < 1/2$ and $\delta > 0$ be arbitrary. We take a natural number m such that for $y, z \in \left([0,1]^b\right)^{\mathbb{Z}}$

$$y|_{-m}^{m} = z|_{-m}^{m} \Longrightarrow d'(y,z) < \delta.$$

This implies that for any $N > 0$ and $y, z \in \left([0,1]^b\right)^{\mathbb{Z}}$

$$y|_{-m}^{N+m-1} = z|_{-m}^{N+m-1} \Longrightarrow d'_N(y,z) < \delta.$$

We consider a map

$$\pi_n : [0,1]^{M_n} \to \left([0,1]^b\right)^{N_n+2m}, \quad x \mapsto \pi\left(\psi_n(x)\right)|_{-m}^{N_n+m-1}.$$

We apply the waist inequality to this map. By Corollary 2.4 we can find a point $t \in \left([0,1]^b\right)^{N_n+2m}$ satisfying

$$\#\left(\pi_n^{-1}(t), \|\cdot\|_\infty, \varepsilon\right) \geq \frac{1}{8^{M_n}} \cdot \left(\frac{1}{\varepsilon}\right)^{M_n - b(N_n+2m)}. \tag{4·1}$$

(When $M_n \geq b(N_n+2m)$, this is a direct consequence of Corollary 2.4. When $M_n < b(N_n+2m)$, the right-hand side is less than one. So it is obviously true.) Here, of course, the choice of t depends on n. But we suppress its dependence on n in our notation for simplicity.

Take $t' \in \left([0,1]^b\right)^{\mathbb{Z}}$ with $t'|_{-m}^{N_n+m-1} = t$. We claim that

$$\psi_n\left(\pi_n^{-1}(t)\right) \subset \pi^{-1}\left(B_\delta\left(t', d'_{N_n}\right)\right). \tag{4·2}$$

Indeed, let $x \in \pi_n^{-1}(t)$. Then

$$\pi\left(\psi_n(x)\right)|_{-m}^{N_n+m-1} = \pi_n(x) = t = t'|_{-m}^{N_n+m-1}.$$

By the choice of m we have $d'_{N_n}\left(\pi(\psi_n(x)), t'\right) < \delta$. Hence $\pi(\psi_n(x)) \in B_\delta\left(t', d'_{N_n}\right)$ and $\psi_n(x) \in \pi^{-1}\left(B_\delta\left(t', d'_{N_n}\right)\right)$. This has proved (4·2).

By $\|x-y\|_\infty \leq d_{N_n}\left(\psi_n(x), \psi_n(y)\right)$ $(x, y \in [0,1]^{M_n})$

$$\begin{aligned}\#\left(\pi_n^{-1}(t), \|\cdot\|_\infty, \varepsilon\right) &\leq \#\left(\psi_n\left(\pi_n^{-1}(t)\right), d_{N_n}, \varepsilon\right) \\ &\leq \#\left(\pi^{-1}\left(B_\delta\left(t', d'_{N_n}\right)\right), d_{N_n}, \varepsilon\right) \quad \text{by (4·2).}\end{aligned}$$

By (4·1)

$$\sup_{y \in \left([0,1]^b\right)^{\mathbb{Z}}} \#\left(\pi^{-1}\left(B_\delta\left(y, d'_{N_n}\right)\right), d_{N_n}, \varepsilon\right) \geq \frac{1}{8^{M_n}} \cdot \left(\frac{1}{\varepsilon}\right)^{M_n - b(N_n+2m)}.$$

Taking the logarithm, we have

$$\sup_{y \in \left([0,1]^b\right)^{\mathbb{Z}}} \log \#\left(\pi^{-1}\left(B_\delta\left(y, d'_{N_n}\right)\right), d_{N_n}, \varepsilon\right) \geq -M_n \log 8 + (M_n - bN_n - 2bm) \log\left(\frac{1}{\varepsilon}\right).$$

We divide this by $N_n \log(1/\varepsilon)$ and let $n \to \infty$. By $\lim_{n\to\infty} M_n/N_n = s$

$$\lim_{N\to\infty} \frac{\sup_{y \in \left([0,1]^b\right)^{\mathbb{Z}}} \log \#\left(\pi^{-1}\left(B_\delta\left(y, d'_N\right)\right), d_N, \varepsilon\right)}{N \log(1/\varepsilon)} \geq s - b - \frac{s \log 8}{\log(1/\varepsilon)}.$$

This holds for arbitrary $0 < \varepsilon < 1/2$ and $\delta > 0$. So we can let $\delta \to 0$ and then $\varepsilon \to 0$. By Lemma 3.2

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi, T, d) \geq s - b.$$

(3) Let d' be a metric on Y. Let $0 < \varepsilon < 1/2$ and $\delta > 0$ be arbitrary. We can find $N(\delta) > 0$ such that for any natural number $N \geq N(\delta)$ there exist a finite simplicial complex K_N and a δ-embedding $\varphi_N : (Y, d'_N) \to K_N$ satisfying

$$\frac{\dim K_N}{N} < \mathrm{mdim}(Y, S) + \delta. \tag{4·3}$$

For each natural number n with $N_n \geq N(\delta)$ we consider a map

$$\pi_n := \varphi_{N_n} \circ \pi \circ \psi_n : [0,1]^{M_n} \to K_{N_n}.$$

We use a famous result of topological dimension theory [HW41, Theorem V.2]: The space K_N topologically embeds in the Euclidean space $\mathbb{R}^{2\dim K_N+1}$. (Since K_N is a finite simplicial complex, this is a rather elementary fact. See e.g. [GQT19, Lemma 2.6].) Therefore we can consider that

$$K_{N_n} \subset \mathbb{R}^{2\dim K_{N_n}+1}.$$

So π_n is a continuous map from $[0,1]^{M_n}$ to $\mathbb{R}^{2\dim K_{N_n}+1}$. We apply the waist inequality to this map. The rest of the argument is quite similar to the case (2).

By Corollary 2.4 we can find $t \in K_{N_n}$ (depending on n) such that

$$\#\left(\pi_n^{-1}(t), \|\cdot\|_\infty, \varepsilon\right) \geq \frac{1}{8^{M_n}} \cdot \left(\frac{1}{\varepsilon}\right)^{M_n - 2\dim K_{N_n} - 1}. \tag{4·4}$$

Pick $t' \in Y$ (depending on n) with $\varphi_{N_n}(t') = t$. We claim that

$$\psi_n\left(\pi_n^{-1}(t)\right) \subset \pi^{-1}\left(B_\delta\left(t', d'_{N_n}\right)\right). \tag{4·5}$$

Indeed, take $x \in \pi_n^{-1}(t)$. Then $\varphi_{N_n}\left(\pi\left(\psi_n(x)\right)\right) = \pi_n(x) = t = \varphi_{N_n}(t')$. Since φ_{N_n} is a δ-embedding with respect to d'_{N_n}, we have

$$d'_{N_n}\left(\pi\left(\psi_n(x)\right), t'\right) < \delta.$$

So $\pi\left(\psi_n(x)\right) \in B_\delta\left(t', d'_{N_n}\right)$ and hence $\psi_n(x) \in \pi^{-1}\left(B_\delta\left(t', d'_{N_n}\right)\right)$. This has shown (4·5).

Since $\|x-y\|_\infty \leq d_{N_n}\left(\psi_n(x), \psi_n(y)\right)$ $(x, y \in [0,1]^{M_n})$,

$$\begin{aligned}\#\left(\pi_n^{-1}(t), \|\cdot\|_\infty, \varepsilon\right) &\leq \#\left(\psi_n\left(\pi_n^{-1}(t)\right), d_{N_n}, \varepsilon\right) \\ &\leq \#\left(\pi^{-1}\left(B_\delta\left(t', d'_{N_n}\right)\right), d_{N_n}, \varepsilon\right) \quad \text{by (4·5)}.\end{aligned}$$

By (4·4) we have

$$\sup_{y \in Y} \#\left(\pi^{-1}\left(B_\delta(y, d'_{N_n})\right), d_{N_n}, \varepsilon\right) \geq \frac{1}{8^{M_n}} \cdot \left(\frac{1}{\varepsilon}\right)^{M_n - 2\dim K_{N_n} - 1}.$$

Taking the logarithm and dividing it by $N_n \log(1/\varepsilon)$, we have

$$\begin{aligned}\frac{\sup_{y\in Y} \log \#\left(\pi^{-1}\left(B_\delta(y, d'_{N_n})\right), d_{N_n}, \varepsilon\right)}{N_n \log(1/\varepsilon)} &\geq \frac{M_n}{N_n} - \frac{2\dim K_n}{N_n} - \frac{1}{N_n} - \frac{M_n \log 8}{N_n \log(1/\varepsilon)} \\ &\geq \frac{M_n}{N_n} - 2\mathrm{mdim}(Y, S) - 2\delta - \frac{1}{N_n} - \frac{M_n \log 8}{N_n \log(1/\varepsilon)}.\end{aligned}$$

Here we have used (4·3) in the second inequality.

We let $n \to \infty$. Noting $M_n/N_n \to s$, we get

$$\lim_{N\to\infty} \frac{\sup_{y\in Y} \log \#\left(\pi^{-1}\left(B_\delta(y, d'_N)\right), d_N, \varepsilon\right)}{N \log(1/\varepsilon)} \geq s - 2\mathrm{mdim}(Y, S) - 2\delta - \frac{s \log 8}{\log(1/\varepsilon)}.$$

We let $\delta \to 0$ and then $\varepsilon \to 0$. We conclude $\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi, T, d) \geq s - 2\mathrm{mdim}(Y, S)$ by Lemma 3.2. □

Remark 4.2. There are two points in the above proof which might be able to be improved in a future:

(1) In the above proof, we used the waist inequality in (4·1) and (4·4). We notice that they are slightly weaker than the original waist inequalities. In both the inequalities (4·1) and (4·4) the parameter ε is fixed. On the other hand, it can vary in the original waist inequalities. It is an interesting problem to discover how to apply the full-power of the waist inequality to the study of mean dimension. A related question is the following: For an equivariant continuous map $\pi : (X, T) \to (Y, S)$ between dynamical systems, we define the **fiber-wise lower conditional metric mean dimension** by

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi, T, d)_{\mathrm{fiber}} = \sup_{y \in Y} \left\{ \liminf_{\varepsilon \to 0} \left(\limsup_{N \to \infty} \frac{\log \# \left(\pi^{-1}(y), d_N, \varepsilon\right)}{N \log(1/\varepsilon)} \right) \right\}.$$

Can one replace $\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi, T, d)$ with $\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi, T, d)_{\mathrm{fiber}}$ in the statements (2) and (3) in Theorem 4.1?

(2) The multiplicative factor 2 of $2\mathrm{mdim}(Y, S)$ in the statement of Theorem 4.1 (3) comes form the embedding $K_{N_n} \subset \mathbb{R}^{2 \dim K_{N_n} + 1}$ used in the above proof. (See also Problem 1.6 in §1.2.) If we would like to reduce the factor 2 to a smaller value (hopefully, one), then we need to develop a method to bypass this embedding. It seems possible to find such a method if the local topology of the simplicial complex K_{N_n} is not very complicated. For example, if K_{N_n} is a smooth manifold then there exists a smooth map $f : K_{N_n} \to \mathbb{R}^{\dim K_{N_n}}$ for which every fiber has cardinality at most $4 \dim K_{N_n}$ [Gro10, p. 447]. Then we apply the waist inequality to $f \circ \pi_n$ and get the desired result. Another idea is to develop a waist inequality directly applicable to the map $\pi_n : [0,1]^{M_n} \to K_{N_n}$. But currently we do not know how to do it.

Theorems 1.3, 1.5 and Proposition 1.10 in §1 are immediate corollaries of the above main technical theorem. They are all included in the following statement.

Corollary 4.3 (= Theorems 1.3, 1.5 and Proposition 1.10)**.** *Let a be a natural number, and let $\left(([0,1]^a)^{\mathbb{Z}}, \sigma\right)$ be the full-shift on the alphabet $[0,1]^a$. We define a metric D on it by*

$$D\left((x_n)_{n \in \mathbb{Z}}, (y_n)_{n \in \mathbb{Z}}\right) = \sum_{n \in \mathbb{Z}} 2^{-|n|} \left\| x_n - y_n \right\|_{\infty}.$$

(1) *Let b be a natural number. For any equivariant continuous map $\pi : \left(([0,1]^a)^{\mathbb{Z}}, \sigma\right) \to \left(([0,1]^b)^{\mathbb{Z}}, \sigma\right)$ we have $\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi, \sigma, D) \geq a - b$.*

(2) *Let (Y,S) be a dynamical system. For any equivariant continuous map $\pi : \left(([0,1]^a)^{\mathbb{Z}}, \sigma\right) \to (Y,S)$ we have $\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,\sigma,D) \geq a - 2\mathrm{mdim}(Y,S)$.*

Notice that this statement is logically contained in the next claim (3) (letting Z to be one-point there). But we separately state it for making the exposition easier to understand.

(3) *Let (Z,R) be a dynamical system with a metric ρ. We define a metric d on the product $([0,1]^a)^{\mathbb{Z}} \times Z$ by $d((x,z),(x',z')) = D(x,x') + \rho(z,z')$. Then for any dynamical system (Y,S) and any equivariant continuous map $\pi : \left(([0,1]^a)^{\mathbb{Z}} \times Z, \sigma \times R\right) \to (Y,S)$ we have $\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi,\sigma\times R,d) \geq a - 2\mathrm{mdim}(Y,S)$.*

Proof. For each natural number n we define a map $\psi_n : ([0,1]^a)^n \to ([0,1]^a)^{\mathbb{Z}}$ by

$$\psi_n(x_0,x_1,\dots,x_{n-1})_k = \begin{cases} x_k & (0 \leq k \leq n-1) \\ 0 & (\text{otherwise}) \end{cases}.$$

This satisfies that for any $x,y \in ([0,1]^a)^n$

$$\|x-y\|_\infty \leq D_n(\psi_n(x),\psi_n(y)).$$

Then the statements (1) and (2) immediately follow from Theorem 4.1 with the parameters $N_n := n$ and $M_n := an$.

For (3), we fix a point $p \in Z$ and consider a map

$$\psi'_n : ([0,1]^a)^n \to ([0,1]^a)^{\mathbb{Z}} \times Z, \quad x \mapsto (\psi_n(x),p).$$

This satisfies that for any $x,y \in ([0,1]^a)^n$

$$\|x-y\|_\infty \leq d_n(\psi'_n(x),\psi'_n(y)).$$

We use Theorem 4.1 with these maps ψ'_n and get the statement (3). □

5. Construction of a minimal closed subset

We prove Theorem 1.8 in this section. The construction of a minimal closed subset itself is the same as that of [LW00, Proposition 3.5]. It uses an idea of *block-system* reviewed below.

5.1. **Review of block-system.** Let a be a natural number. As before, we define a metric D on the full-shift $([0,1]^a)^{\mathbb{Z}}$ by

$$D((x_n)_{n\in\mathbb{Z}},(y_n)_{n\in\mathbb{Z}}) = \sum_{n\in\mathbb{Z}} 2^{-|n|}\|x_n-y_n\|_\infty.$$

Let x be a point in $([0,1]^a)^{\mathbb{Z}}$. For integers $\ell < m$ we set

$$x|_\ell^m = (x_\ell, x_{\ell+1},\dots,x_{m-1},x_m) \in ([0,1]^a)^{m-\ell+1}.$$

Let N be a natural number, and let K be a closed subset of $([0,1]^a)^N$. We define a **block-system** $X(K)$ in $([0,1]^a)^{\mathbb{Z}}$ by

$$X(K) = \left\{ x \in ([0,1]^a)^{\mathbb{Z}} \middle| \, \exists \ell \in \mathbb{Z} : \forall n \in \mathbb{Z} : x|_{\ell+nN}^{\ell+(n+1)N-1} \in K \right\}.$$

This is a shift-invariant closed subset of $([0,1]^a)^{\mathbb{Z}}$.

Lemma 5.1.

$$\overline{\mathrm{mdim}}_{\mathrm{M}}\left(X(K), \sigma, D\right) \leq \frac{\overline{\dim}_{\mathrm{M}}(K, \|\cdot\|_\infty)}{N}.$$

Here σ is the shift-map, and $\overline{\dim}_{\mathrm{M}}(K, \|\cdot\|_\infty)$ is the upper Minkowski dimension of K with respect to the ℓ^∞-distance.

Proof. For $0 \leq \ell \leq N-1$ we set

$$X_\ell = \left\{ x \in ([0,1]^a)^{\mathbb{Z}} \middle| \, \forall n \in \mathbb{Z} : x|_{\ell+nN}^{\ell+(n+1)N-1} \in K \right\}.$$

We have

$$X(K) = X_0 \cup X_1 \cup X_2 \cup \cdots \cup X_{N-1}.$$

Let ε be a positive number. Take $m = m(\varepsilon) > 0$ satisfying $\sum_{|n|>m} 2^{-|n|} < \varepsilon/2$.

Let L be an arbitrary natural number. We set

$$s = \left\lfloor \frac{-\ell - m}{N} \right\rfloor, \quad t = \left\lceil \frac{L - \ell + m}{N} \right\rceil - 1.$$

We define $f_\ell : X_\ell \to K^{t-s+1}$ by

$$f_\ell(x) = \left(x|_{\ell+nN}^{\ell+(n+1)N-1} \right)_{s \leq n \leq t}.$$

Notice that

$$\bigcup_{n=s}^{t} [\ell + nN, \ell + (n+1)N) = [\ell + sN, \ell + (t+1)N) \supset [-m, L+m).$$

Hence, for any $x, y \in X_\ell$

$$D_L(x,y) < 3 \left\| f_\ell(x) - f_\ell(y) \right\|_\infty + \frac{\varepsilon}{2}.$$

Then

$$\#\left(X_\ell, D_L, \varepsilon\right) \leq \#\left(K^{t-s+1}, \|\cdot\|_\infty, \frac{\varepsilon}{9}\right) \leq \left(\#\left(K, \|\cdot\|_\infty, \frac{\varepsilon}{9}\right)\right)^{t-s+1}.$$

We have

$$t - s + 1 \leq \frac{L - \ell + m}{N} - \frac{-\ell - m}{N} + 1 + 1 \leq \frac{L + 2m}{N} + 2.$$

Hence

$$\#\left(X_\ell, D_L, \varepsilon\right) \leq \left(\#\left(K, \|\cdot\|_\infty, \frac{\varepsilon}{9}\right)\right)^{\frac{L+2m}{N}+2}.$$

Therefore

$$\#\left(X(K), D_L, \varepsilon\right) \leq N \cdot \left(\#\left(K, \|\cdot\|_\infty, \frac{\varepsilon}{9}\right)\right)^{\frac{L+2m}{N}+2}.$$

Taking the logarithm and dividing it by $L\log(1/\varepsilon)$, we get

$$\frac{\log \#\left(X(K), D_L, \varepsilon\right)}{L\log(1/\varepsilon)} \leq \frac{\log N}{L\log(1/\varepsilon)} + \left(\frac{1}{N} + \frac{2m}{NL} + \frac{2}{L}\right) \frac{\log \#\left(K, \|\cdot\|_\infty, \frac{\varepsilon}{9}\right)}{\log(1/\varepsilon)}.$$

We let $L \to +\infty$ and then let $\varepsilon \to 0$. We conclude that

$$\overline{\mathrm{mdim}}_{\mathrm{M}}\left(X(K), \sigma, D\right) \leq \frac{\overline{\dim}_{\mathrm{M}}(K, \|\cdot\|_\infty)}{N}.$$

□

5.2. **Proof of Theorem 1.8.** Now we prove Theorem 1.8. We write the statement again.

Theorem 5.2 (= Theorem 1.8)**.** *Let a be a natural number and s a real number with $0 \leq s < a$. There exists a minimal closed subset X of $\left(([0,1]^a)^{\mathbb{Z}}, \sigma\right)$ satisfying the following two conditions.*

(1) $\mathrm{mdim}\left(X, \sigma\right) = \mathrm{mdim}_{\mathrm{M}}\left(X, \sigma, D\right) = s$.
(2) *For any natural number b and any equivariant continuous map $\pi : (X, \sigma) \to \left(([0,1]^b)^{\mathbb{Z}}, \sigma\right)$ we have $\underline{\mathrm{mdim}}_{\mathrm{M}}\left(\pi, \sigma, D\right) \geq s - b$.*

Proof. As we already noted, the construction of X is the same as that of [LW00, Proposition 3.5]. The difference is that we analyze the resulting minimal closed subset by using the waist inequality.

We take and fix a sequence of rational numbers $0 < r_n < 1$ $(n \geq 1)$ satisfying

$$a\prod_{n=1}^{\infty}(1 - r_n) = s.$$

Set $N_1 = 1$ and $K_1 = [0,1]^a$. Starting from these, we inductively construct an increasing sequence of natural numbers N_n and closed subsets $K_n \subset ([0,1]^a)^{N_n}$ $(n \geq 1)$ satisfying the following conditions.

(i) There are natural numbers $p_n > q_n$ $(n \geq 1)$ larger than one satisfying $N_{n+1} = p_n N_n$ and $r_n = q_n/p_n$.
(ii) $K_{n+1} \subset (K_n)^{p_n}$. Moreover, there is a point $(w_1^{(n)}, w_2^{(n)}, \ldots, w_{q_n}^{(n)})$ in $(K_n)^{q_n}$ such that
  - $K_{n+1} = (K_n)^{p_n - q_n} \times \{(w_1^{(n)}, w_2^{(n)}, \ldots, w_{q_n}^{(n)})\}$.
  - For any $x \in K_n \times K_n$ there exists $1 \leq k < q_n$ satisfying

$$\left\| x - (w_k^{(n)}, w_{k+1}^{(n)}) \right\|_\infty < \frac{1}{n}.$$

A moment thought shows that such an inductive construction works well.

We consider the block-system $X_n = X(K_n)$ associated with K_n. We set

$$X = \bigcap_{n=1}^{\infty} X_n \subset ([0,1]^a)^{\mathbb{Z}}.$$

By the above condition (ii) the set X is minimal. We will show that it satisfies the required conditions (1) and (2).

We define a sequence of natural numbers M_n $(n \geq 1)$ by

$$M_1 = 1, \quad M_{n+1} = (p_n - q_n)M_n \quad (n \geq 1).$$

We have

$$M_{n+1} = p_n \left(1 - \frac{q_n}{p_n}\right) M_n = p_n(1 - r_n)M_n = p_n N_n (1 - r_n)\frac{M_n}{N_n} = N_{n+1}(1 - r_n)\frac{M_n}{N_n}.$$

So $\frac{M_{n+1}}{N_{n+1}} = (1 - r_n)\frac{M_n}{N_n}$ and hence

$$\frac{M_{n+1}}{N_{n+1}} = \prod_{k=1}^{n}(1 - r_k) \to \frac{s}{a} \quad (n \to \infty).$$

From the construction K_n is naturally homeomorphic to the cube $([0,1]^a)^{M_n}$. More precisely, $(K_n, \|\cdot\|_\infty)$ is isometric to $\left(([0,1]^a)^{M_n}, \|\cdot\|_\infty\right)$. In particular,

$$\dim_{\mathrm{M}}(K_n, \|\cdot\|_\infty) = aM_n.$$

By Lemma 5.1

$$\overline{\mathrm{mdim}}_{\mathrm{M}}(X_n, \sigma, D) \leq \frac{\dim_{\mathrm{M}}(K_n, \|\cdot\|_\infty)}{N_n} = \frac{aM_n}{N_n}.$$

Thus

$$\overline{\mathrm{mdim}}_{\mathrm{M}}(X, \sigma, D) \leq \lim_{n\to\infty} \frac{aM_n}{N_n} = s.$$

So we have $\overline{\mathrm{mdim}}_{\mathrm{M}}(X, \sigma, D) \leq s$.

It also follows from the construction that there exists a continuous map $\psi_n : ([0,1]^a)^{M_n} \to X$ satisfying

$$\left\|\psi_n(x)|_0^{N_n-1} - \psi_n(y)|_0^{N_n-1}\right\|_\infty = \|x - y\|_\infty \quad (x, y \in ([0,1]^a)^{M_n}).$$

In particular

$$\|x - y\|_\infty \leq D_{N_n}(\psi_n(x), \psi_n(y)).$$

Then we can apply to the maps ψ_n Theorem 4.1 with the parameters aM_n and N_n. Noting $\lim_{n\to\infty} \frac{aM_n}{N_n} = s$, we get the following conclusions:

- $\mathrm{mdim}(X, \sigma) \geq s$.
- For any natural number b and any equivariant continuous map $\pi : (X, \sigma) \to \left(([0,1]^b)^{\mathbb{Z}}, \sigma\right)$ we have $\underline{\mathrm{mdim}}_{\mathrm{M}}(\pi, \sigma, D) \geq s - b$.

Since mean dimension is bounded from above by metric mean dimension [LW00, Theorem 4.2], we have

$$s \leq \mathrm{mdim}(X, \sigma) \leq \underline{\mathrm{mdim}}_{\mathrm{M}}(X, \sigma, D) \leq \overline{\mathrm{mdim}}_{\mathrm{M}}(X, \sigma, D) \leq s.$$

Thus $\mathrm{mdim}(X, \sigma) = \mathrm{mdim}_{\mathrm{M}}(X, \sigma, D) = s$. So (X, σ, D) satisfies all the required conditions. □

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