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► **To cite this version:**

| Philippe Gille. Loop group schemes and Abhyankar's lemma. 2023. hal-03933634v1

HAL Id: hal-03933634

<https://cnrs.hal.science/hal-03933634v1>

Preprint submitted on 10 Jan 2023 (v1), last revised 30 Jul 2023 (v4)

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LOOP GROUP SCHEMES AND ABHYANKAR'S LEMMA

P. GILLE

ABSTRACT. We define the notion of reductive group schemes defined over the localization of a regular henselian ring A at a strict normal crossing divisor D . We provide a criterion for the existence for parabolic subgroups of a given type.

Keywords: Reductive group schemes, normal crossing divisor, parabolic subgroups.

MSC 2000: 14L15, 20G15, 20G35.

1. INTRODUCTION

In the reference [7], we investigated a theory of loop reductive group schemes over the ring of Laurent polynomials $k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Using Bruhat-Tits' theory, this permitted to relate the study of those group schemes to that of reductive algebraic groups over the field of iterated Laurent series $k((t_1)) \dots ((t_n))$. The main issue of this note is to start a similar approach for reductive group schemes defined over the localization A_D of a regular henselian ring A at a strict normal crossing divisor D and to relate with algebraic groups defined over a natural field associated to A and D , namely the completion K_v of the fraction field K with respect to the valuation arising from the blow-up of $\text{Spec}(A)$ at its maximal ideal. The example which connects the two viewpoints is $k[[t_1, \dots, t_n]][\frac{1}{t_1}, \dots, \frac{1}{t_n}]$ where $K_v \cong k(\frac{t_1}{t_n}, \dots, \frac{t_1}{t_{n-1}})((t_n))$

After defining the notion of loop reductive group schemes in this setting, we show that for this class of group schemes, the existence of parabolic subgroups over the localization A_D is controlled by the parabolic subgroups over K_v (Theorem 4.1).

Acknowledgements. We thank R. Parimala for her insight about the presented results.

2. TAME FUNDAMENTAL GROUP

2.1. Abhyankar's lemma. Let $X = \text{Spec}(A)$ be a regular local scheme. Let k be the residue field of A and $p \geq 0$ be its characteristic. We put $\widehat{\mathbb{Z}}' = \prod_{l \neq p} \mathbb{Z}_l$. Let K be the fraction field of A , and let K_s be a separable closure of K . It determines a base point $\xi : \text{Spec}(K) \rightarrow X$ so that we can deal with the Grothendieck fundamental group $\Pi_1(X, \xi)$ [10].

Date: January 10, 2023.

Let (f_1, \dots, f_r) be a regular sequence of A and consider the divisor $D = \sum D_i = \sum \text{div}(f_i)$, it has strict normal crossing. We put $U = X \setminus D = \text{Spec}(A_D)$.

We recall that a finite étale cover $V \rightarrow U$ is *tamely ramified* with respect to D if the associated étale K -algebra $L = L_1 \times \dots \times L_a$ is tamely ramified at the D_i 's, that is, for each i , there exists j_i such that for the Galois closure \tilde{L}_{j_i}/K of L_{j_i}/K , the inertia group associated to v_{D_i} has order prime to p [10, XIII.2.0].

Grothendieck and Murre defined the tame (*modéré* in French) fundamental group $\Pi_1^D(U, \xi)$ with respect to $U \subset X$ as defined in [10, XIII.2.1.3] and [8, §2]. This is a profinite quotient of $\Pi_1(U, \xi)$ whose quotients by open subgroups provides finite Galois tame cover of U .

We are given a finite étale tame cover $V \rightarrow U$. In this case Abhyankar's lemma states that there exists a flat Kummer cover $X' = \text{Spec}(A') \rightarrow X$ where

$$A' = A[T_1, \dots, T_r]/(T_1^{n_1} - f_1, \dots, T_r^{n_r} - f_r)$$

and the n_i 's are coprime to p such that $V' = V \times_X X' \rightarrow X'$ extends uniquely to a finite étale cover $Y' \rightarrow X'$ [10, XIII.5.2].

Lemma 2.1. *Let $V \rightarrow U$ be a finite étale cover which is tame. Then $\text{Pic}(V) = 0$.*

Proof. We use the some notation as above. We know that X' is regular [10, XIII.5.1] so a fortiori locally factorial. It follows that the restriction maps $\text{Pic}(X') \rightarrow \text{Pic}(V') \rightarrow \text{Pic}(V)$ are surjective [5, 21.6.11]. Since A' is finite over the local ring A , it is semilocal so that $\text{Pic}(A') = \text{Pic}(X') = 0$. Thus $\text{Pic}(V) = 0$ as desired. \square

From now on we assume that A is henselian. According to [5, 18.5.10], the finite A -ring A' is a finite product of henselian local rings. We observe that $A' \otimes_A k = k[T_1, \dots, T_r]/(T_1^{n_1}, \dots, T_r^{n_r})$ is a local Artinian algebra so that A' is connected. It follows that A' is a henselian local ring. Its maximal ideal is $\mathfrak{m}' = \mathfrak{m} \otimes_A A' + \langle T_1, \dots, T_r \rangle$ so that $A'/\mathfrak{m}' = k$. Since there is an equivalence of categories between finite étale covers of A (resp. A') and étale k -algebras [5, 18.5.15], the base change from A to A' provides an equivalence of categories between the category of finite étale covers of A and that of A' .

It follows that $Y' \rightarrow X'$ descends uniquely to a finite étale cover $\tilde{f} : \tilde{Y} \rightarrow X$. From now on, we assume that V is furthermore connected, it implies that

$$H^0(V, \mathcal{O}_V) = B[T_1, \dots, T_r]/(T_1^n - f_1, \dots, T_r^n - f_r)$$

where B is finite connected étale cover of A . It follows that $V \rightarrow U$ is a quotient of a Galois cover of the shape

$$B_n = B[T_1^{\pm 1}, \dots, T_r^{\pm 1}]/(T_1^n - f_1, \dots, T_r^n - f_r)$$

where B is Galois cover of A containing a primitive n -root of unity. We record that B_n is the localization at $T_1 \dots T_r$ of $B'_n = B[T_1, \dots, T_r]/(T_1^n - f_1, \dots, T_r^n - f_r)$. We

have

$$\mathrm{Gal}(B_n/A_D) = \left(\prod_{i=1}^r \mu_n(B) \right) \rtimes \mathrm{Gal}(B/A).$$

Passing to the limit we obtain an isomorphism

$$\pi_1^t(U, \xi) \cong \left(\prod_{i=1}^r \widehat{\mathbb{Z}}'(1) \right) \rtimes \pi_1(X, \xi).$$

We denote by $f : U^{sc,t} \rightarrow U$ the profinite étale cover associated to the quotient $\pi_1^t(U, \xi)$ of $\pi_1(U, \xi)$. According to [8, thm. 2.4.2], it is the universal tamely ramified cover of U . It is a localization of the inductive limit \widetilde{B}' of the B'_n . On the other hand we consider the inductive limit \widetilde{B} of the B 's and see that \widetilde{B}' is a \widetilde{B} -ring.

2.2. Blow-up. We follow a blowing-up construction arising from [5, lemma 15.1.1.6]. We denote by X the blow-up of $\mathrm{Spec}(A)$ at his closed point, this is a regular scheme [9, §8.1, th. 1.19] and the exceptional divisor $E \subset X$ is a Cartier divisor isomomorphic to \mathbb{P}_k^{r-1} . We denote by $R = \mathcal{O}_{X,\eta}$ the local ring at the generic point η of E . The ring R is a DVR of fraction field K and of residue field $F = k(E) = k(t_1, \dots, t_{r-1})$ where t_i is the image of $\frac{f_i}{f_r} \in R$ by the specialization map. We denote by $v : K^\times \rightarrow \mathbb{Z}$ the discrete valuation associated to R .

We deal now with a Galois extension B_n of A_D as above. Since B is a connected finite étale cover of A , B is regular and local; it is furthermore henselian [5, 18.5.10]. We denote by L the field of fraction of B and by L_n that of B_n . We have $[L_n : L] = n^r$. We want to extend the valuation v to L and to L_n .

We denote by $l = B/\mathfrak{m}_B$ the residue field of B , this is a finite Galois field extension of k . Also (t_1, \dots, t_r) is a system of parameters for B . We denote by $w : L^\times \rightarrow \mathbb{Z}$ the discrete valuation associated to the exceptional divisor of the blow-up of $\mathrm{Spec}(B)$ at its closed point. Then w extends v and L_w/K_v is an unramified extension of degree $[L : K]$ and of residual extension $F_l = l(t_1, \dots, t_{r-1})/k(t_1, \dots, t_{r-1})$.

On the other hand we denote by $w_n : L_n^\times \rightarrow \mathbb{Z}$ the discrete valuation associated to the exceptional divisor of the blow-up of $\mathrm{Spec}(B_n)$ at its closed point. We put $l_n = B_n/\mathfrak{m}_{B_n}$, we have $l = l_n$. The valuation $\frac{w_n}{n}$ on L_n extends w and its residual extension is $F_{l,n} = l\left(t_1^{1/n}, \dots, t_{r-1}^{1/n}\right)/k(t_1, \dots, t_{r-1})$ so that $[F_{l,n} : F_l] = n^{r-1}$. Furthermore the ramification index e_n of L_n/L is $\geq n$. Since $n^r \leq e_n [F_{l,n} : F_l] \leq [L_n : K] = n^r$ (where the last inequality is [2, §VI.3, prop. 2]) it follows that $e_n = n$. The same statement shows that the map $L_w \otimes_L L_n \rightarrow L_{w_n}$ is an isomorphism. To summarize L_{w_n}/L_w is tamely ramified of ramification index n and of degree n^r . All together we have $L_{w_n} = L_w \otimes_K L_n$ so that L_{w_n} is Galois over K_v of group $\prod_i \mu_n(B) \rtimes \mathrm{Gal}(B/A) = \prod_i \mu_n(l) \rtimes \mathrm{Gal}(l/k)$.

We denote by $\Delta : \mu_n(l) \subset \prod_i \mu_n(l)$ the diagonal subgroup. We put $L_{w_n}^\Delta = L_n^{\Delta(\mu_n(B))}$. Since t_r is a uniformizing parameter of K_v and since $\Delta(\zeta) \cdot t_r = \zeta \cdot t_r$ for each $\zeta \in \mu_n(B)$, it follows that $(L_{w_n})^\Delta$ is the maximal unramified extension of L_{w_n}/K_v .

2.3. Loop cocycles and loop torsors. Let G be an affine X -group scheme locally of finite presentation. A loop cocycle is an element of $Z^1(\pi_1^t(U), G(\tilde{B}))$ and it defines a Galois cocycle in $Z^1(\pi_1^t(U), G(U^{sc,t}))$. We denote by $Z_{loop}^1(\pi_1^t(U), G(U^{sc,t}))$ the image of the map $Z^1(\pi_1^t(U), G(\tilde{B})) \rightarrow Z^1(\pi_1^t(U), G(U^{sc,t}))$ and by $H_{loop}^1(U, G)$ the image of the map

$$Z^1(\pi_1^t(U), G(\tilde{B})) \rightarrow H^1(\pi_1^t(U), G(U^{sc,t})) \rightarrow H^1(U, G).$$

We say that a G -torsor E over U (resp. a fppf sheaf G -torsor) is a loop torsor if its class belongs to $H_{loop}^1(U, G) \subset H^1(U, G)$.

A given class $\gamma \in H_{loop}^1(U, G)$ is represented by a 1-cocycle $\phi : \text{Gal}(B_n/A_D) \rightarrow G(B)$ for some cover B_n/A as above. Its restriction $\phi^{ar} : \text{Gal}(B/A) \rightarrow G(B)$ to the subgroup $\text{Gal}(B/A)$ of $\text{Gal}(B_n/A)$ is called the ‘‘arithmetic part’’ and the other restriction $\phi^{geo} : \prod_i \mu_n(B) \rightarrow \mathfrak{G}(B)$ is called the geometric part. We observe that ϕ^{geo} is a B -group homomorphism.

Furthermore for $\sigma \in \text{Gal}(B/A)$ and $\tau \in \prod_i \mu_n(B)$ the computation of [7, page 16] shows that $\phi^{geo}(\sigma\tau\sigma^{-1}) = \phi^{ar}(\sigma)\sigma\phi(\tau)\phi^{ar}(\sigma)^{-1}$ so that ϕ^{geo} descends to a homomorphism of A -group schemes $\phi^{geo} : \mu_n^r \rightarrow \phi^{ar}G$. This provides a parameterization of loop cocycles.

Lemma 2.2. (1) For B_n/A as above, the map $\phi \mapsto (\phi^{ar}, \phi^{geo})$ provides a bijection between $Z_{loop}^1(\text{Gal}(B_n/A_D), G(B))$ and the couples (z, η) where $z \in Z^1(\text{Gal}(B/A), G(B))$ and $\eta : \prod_i \mu_n \rightarrow {}_zG$ is an A -group homomorphism.

(2) The map $\phi \mapsto (\phi^{ar}, \phi^{geo})$ provides a bijection between $Z_{loop}^1(\pi^1(U, \xi)^t, G(\tilde{B}))$ and the couples (z, η) where $z \in Z^1(\pi^1(X, \xi), G(\tilde{B}))$ and $\eta : \prod_{i=1}^r \tilde{\mathbb{Z}}^i \rightarrow {}_zG$ is an A -group homomorphism.

Proof. This is similar with [7, lemma 3.7]. □

We examine more closely the case of a finite étale X -group scheme \mathfrak{F} of constant degree d .

Lemma 2.3. (1) $\mathfrak{F}(\tilde{B}) = \mathfrak{F}(X^{sc}) = \mathfrak{F}(U^{sc,t})$.

(2) We assume that d is prime to p . We have $H_{loop}^1(U, \mathfrak{F}) = H^1(U, \mathfrak{F})$.

(3) We assume that d is prime to p . Let $f : \mathfrak{F} \rightarrow \mathfrak{H}$ be a homomorphism of A -group schemes (locally of finite type). Then $f_*\left(H^1(U, \mathfrak{F})\right) \subset H_{loop}^1(U, \mathfrak{H})$.

Proof. (1) We are given a cover B_n/A as above such that $\mathfrak{F}_{B_n} \cong \Gamma_{B_n}$ is finite constant. as above. Since B and B_n are connected, the map $\mathfrak{F}(B) \rightarrow \mathfrak{F}(B_n)$ reads as the

identity $\Gamma \cong \mathfrak{F}(B) \rightarrow \mathfrak{F}(B_n) \cong \Gamma$ so is bijective. By passing to the limit we get $\mathfrak{F}(\tilde{B}) = \mathfrak{F}(U^{sc,t})$.

(2) Let \mathfrak{E} be a \mathfrak{F} -torsor over U . This is a finite étale U -scheme. Since U is noetherian and connected, we have a decomposition $\mathfrak{E} = V_1 \times_U \cdots \times_U V_l$ where each V_i is a connected finite étale U -scheme of constant degree d_i . We have $d_1 + \cdots + d_l = d$ so that we can assume that d_1 is prime to p . We have then $\mathfrak{E}(V_1) \neq \emptyset$.

It follows that $f_1 : V_1 \rightarrow U$ is a finite étale cover so that there exists a factorization $U^{sc,t} \rightarrow V_1 \xrightarrow{h} U$ of f so that $\mathfrak{E}(U^{sc,t}) \neq \emptyset$. Therefore $[\mathfrak{E}]$ arises from $H^1(\pi_1^t(U, \xi), \mathfrak{F}(U^{sc,t})) \subset H^1(U, \mathfrak{F})$. It follows that $H^1(\pi_1^t(U, \xi), \mathfrak{F}(U^{sc,t})) \xrightarrow{\sim} H^1(U, \mathfrak{F})$. We use now (1) and obtain the desired bijection $H^1(\pi_1^t(U, \xi), \mathfrak{F}(B)) \xrightarrow{\sim} H^1(U, \mathfrak{F})$.

(3) This follows readily from (2). \square

2.4. Twisting by loop torsors. We assume that the A -group scheme G acts on an A -scheme Z . Let $\phi : (\prod_i^r \mu_n)(B) \rtimes \text{Gal}(B/A) \rightarrow G(B)$ be a loop cocycle. It gives rise to an A -action of μ_n^r on ${}_{\phi^{ar}}Z$. We denote by $({}_{\phi^{ar}}Z)^{\phi^{geo}}$ the fixed point locus for this action, it is representable by a closed A -subscheme of ${}_{\phi^{ar}}Z$ [4, A.8.10.(1)]. We have a closed embedding $({}_{\phi^{ar}}Z)^{\phi^{geo}} \times_X U \subset {}_{\phi}Z$ of U -schemes.

3. FIXED POINTS METHOD

Theorem 3.1. *Let $X = \text{Spec}(A)$ be a henselian regular local scheme and $U = X \setminus D$ as above. We denote by $v : K^\times \rightarrow \mathbb{Z}$ the discrete valuation associated to the exceptional divisor E of the blow-up of X at its closed point.*

Let G be an affine A -group scheme of finite presentation acting on a proper smooth A -scheme Z . Let ϕ be a loop cocycle for G . Then $Y = ({}_{\phi^{ar}}Z)^{\phi^{geo}}$ is a smooth proper A -scheme and the following are equivalent:

- (i) $({}_{\phi}Z)(K_v) \neq \emptyset$;
- (ii) $Y(k) \neq \emptyset$;
- (iii) $Y(U) \neq \emptyset$;
- (iv) $({}_{\phi}Z)(U) \neq \emptyset$.

This is quite similar with the fixed point theorem [7, §, thm. 7.1]. The following example makes the connection.

Example 3.2. We assume that $A = k[[t_1, \dots, t_r]]$ for a field k and $k[U] = k[[t_1, \dots, t_n]][\frac{1}{t_1}, \dots, \frac{1}{t_r}]$. We are given an affine algebraic k -group G acting on a smooth proper k -scheme Z . In this case $K = k((t_1, \dots, t_r))$ and A embeds in $k(\frac{t_1}{t_r}, \dots, \frac{t_{r-1}}{t_r})[[t_r]]$ so that K embeds in $k(\frac{t_1}{t_n}, \dots, \frac{t_{r-1}}{t_r})(t_r)$ which is nothing but the complete field K_v . If Q is a loop G -torsor over U , the statement is then that ${}^QZ(U) \neq \emptyset$ if and only if ${}^QZ(K_v) \neq \emptyset$. Taking a cocycle $\phi \in Z^1(\pi_1(U)^t, G(k_s))$ for E , this rephrases by the equivalence between $({}_{\phi}Z)(U) \neq \emptyset$ and $({}_{\phi}Z)(K_v) \neq \emptyset$.

What we have from [7, thm. 7.1] (in characteristic zero but this extends to this tame setting) is the equivalence between $(\phi Z)(k[t_1^{\pm 1}, \dots, t_r^{\pm 1}]) \neq \emptyset$ and $(\phi Z)(k((t_1)) \dots ((t_r))) \neq \emptyset$. Since $(\phi Z)(k[t_1^{\pm 1}, \dots, t_r^{\pm 1}]) \subset (\phi Z)(U)$ and $(\phi Z)(K_v) \subset (\phi Z)(k((t_1)) \dots ((t_r)))$, it follows that this special case of Theorem 3.1 is a consequence of the fixed point result of [7].

We proceed to the proof of Theorem 3.1.

Proof. According to [4, A.8.10.(1)], $Y = (\phi^{ar}(Z^{\phi^{geo}}))$ is a closed A -scheme of $\phi^{ar}Z$ so is proper. It is smooth over X according to point (2) of the same reference.

Let $\phi : \text{Gal}(B_n/A_D) \rightarrow G(B)$ be the loop 1-cocycle for some Galois cover B_n/A_D as above for some n prime to p . Up to replace G by $\phi^{ar}G$ and G by $\phi^{ar}Z$, we can assume that $\phi^{ar} = 1$ without lost of generality.

(ii) \implies (iii). Since Y_k is the special fiber of the smooth X -scheme Y , Hensel's lemma shows that $Y(A) \rightarrow Y(k)$ is onto. Since $Y(k)$ is not empty, it follows that $Y(A)$ is not empty and so is $Y(U)$.

(iii) \implies (iv). Since $Y(U) \subset \phi Z(U)$, $Y(U) \neq \emptyset$ implies that $\phi Z(U) \neq \emptyset$.

(iv) \implies (i). This is obvious.

(i) \implies (ii). We assume that $(\phi Z)(K_v) \neq \emptyset$. By definition we have

$$(\phi Z)(K_v) = \{z \in Z(L_{w_n}) \mid \phi(\sigma).\sigma(z) = z \ \forall \sigma \in \text{Gal}(L_n/K)\}$$

and our assumption is that this set is non-empty. Let O_{w_n} be the valuation ring of $Z(L_{w_n})$. Since Z is proper over X , we have a specialization map $Z(L_{w_n}) = Z(\mathcal{O}_{w_n}) \rightarrow Z_k(F_{l,n})$. We get that the set

$$\{z \in Z_k(F_{l,n}), \mid \phi(\sigma).\sigma(z) = z \ \forall \sigma \in \text{Gal}(L_{w_n}/K_v)\}$$

is not empty. Since we have an embedding

$$F_{l,n} = l(t_1^{1/n}, \dots, t_{r-1}) \hookrightarrow l((t_1^{1/n})) \dots ((t_{r-1}^{1/n}))$$

in a higher field of Laurent series successive specializations, along the coordinates $t_1^{1/n}, \dots, t_{r-1}^{1/n}$ show similarly that the set

$$(3.1) \quad \left\{ z \in (Z_k)(l) \mid \phi(\sigma).\sigma(z) = z \ \forall \sigma \in \text{Gal}(L_{w_n}/K_v) \right\}$$

is not empty. Since $\eta^{ar} = 1$, this set is $(Z_k)^{\eta^{geo}}(k)$. Thus $Y(k) = (Z_k)^{\eta^{geo}}(k)$ is non empty. \square

4. PARABOLIC SUBGROUPS OF LOOP REDUCTIVE GROUP SCHEMES

4.1. Chevalley groups. Let G_0 be Chevalley group defined over \mathbb{Z} . Let T_0 be a maximal split \mathbb{Z} -subtorus of G_0 together with a Borel subgroup B_0 containing it. We denote by Δ_0 the Dynkin diagram of (G_0, B_0, T_0) . We denote by $G_{0,ad}$ the adjoint

quotient of G_0 and by G_0^{sc} the simply connected covering of DG_0 . We have a map $\text{Aut}(G_0) \rightarrow \text{Aut}(G_0^{sc}) \xrightarrow{\sim} \text{Aut}(G_{0,ad})$ and a fundamental exact sequence

$$1 \rightarrow G_{0,ad} \rightarrow \text{Aut}(G_{0,ad}) \rightarrow \text{Out}(G_{0,ad}) \rightarrow 1$$

where $\text{Out}(G_{0,ad}) \xrightarrow{\sim} \text{Aut}(\Delta_0)$. We recall that there is a bijection $I \rightarrow P_{0,I}$ between the finite subsets of Δ_0 and the parabolic subgroups of G_0 containing B_0 [11, XXVI.3.8]; it is increasing for the inclusion order, in particular $B_0 = P_{0,\emptyset}$ and $G_0 = P_{0,\Delta_0}$. We consider the total scheme Par_{G_0} of parabolic subgroups of G_0 , it is a projective smooth \mathbb{Z} -scheme equipped with a type map $\mathbf{t} : \text{Par}_{G_0} \rightarrow \text{Of}(\Delta_0)$ where $\text{Of}(\Delta_0)$ stands for the finite constant scheme attached to the set of subsets of Δ_0 [11, XXVI.3]. The fiber at I is denoted by $\text{Par}_{G_0,I}$, it has connected fibers and is the scheme of parabolic subgroups of G_0 of type I . We have a natural action of $\text{Aut}(G_0)$ on Par_{G_0} . As in [6, §5.1], we denote by $\text{Aut}_I(G_0)$ the stabilizer of I for this action. By construction $\text{Aut}_I(G_0)$ acts on $\text{Par}_{G_0,I}$.

4.2. Definition. Let G be a reductive U -group scheme in the sense of Demazure-Grothendieck [11, XIX]. Since U is connected and G is locally splittable [11, XXII.2.2] for the étale topology, G is an étale form of a Chevalley group G_0 as above defined over \mathbb{Z} .

We say that G is a *loop group scheme* if the $\text{Aut}(G_0)$ -torsor $Q = \text{Isom}(G_0, G)$ (defined in [11, XXIV.1.9]) is a loop $\text{Aut}(G_0)$ -torsor. We denote by $G_{0,ad}$ the adjoint quotient of G_0 and by G_0^{sc} the simply connected covering of DG_0 . We have a map $\text{Aut}(G_0) \rightarrow \text{Aut}(G_0^{sc}) \xrightarrow{\sim} \text{Aut}(G_{0,ad})$ which permits to see G_{ad} (resp. G^{sc}) as twisted forms of $G_{0,ad}$ (resp. G_0^{sc}) so that G_{ad} and G^{sc} are also loop reductive group schemes. We consider the map $\text{Aut}(G_0) \rightarrow \text{Aut}(G_{0,ad}) \rightarrow \text{Out}(G_{0,ad}) \xrightarrow{\sim} \text{Aut}(\Delta_0)$.

If $\phi : \text{Gal}(B_n/A) \rightarrow \text{Aut}(G_0)(B)$ is a loop cocycle, we get an action of $\text{Gal}(B_n/A)$ on Δ_0 called the star action. If I is stable under the star action, we can twist $\text{Par}_{G_0,I}$ by ϕ and deal with the scheme ${}_{\phi}\text{Par}_{G_0,I}$ which is the scheme of parabolic subgroup schemes of G of type I .

4.3. Parabolics.

Theorem 4.1. *Assume that G is a loop U -group scheme and let $\phi : \text{Gal}(B_n/A_D) \rightarrow \text{Aut}(G_0)(B)$ be a loop cocycle such that $G \cong {}_{\phi}G_0$. Let $I \subset \Delta_0$ be a subset stable under the star action defined by ϕ . Then the following are equivalent:*

- (i) G admits a U -parabolic subgroup of type I ;
- (ii) the k -morphism $\eta_k^{geo} : \mu_n^r \rightarrow \text{Aut}({}_{\eta^{ar}}G_0)_k = ({}_{\eta^{ar}}\text{Aut}(G_0))_k$ normalizes a parabolic k -subgroup of ${}_{\eta^{ar}}G_{0,k}$ of type I ;
- (iii) G_{K_v} admits a parabolic subgroup of type I .

Proof. Without loss of generality we can assume that G is adjoint. Our assumption on the star action rephrases by saying that ϕ takes values in $\text{Aut}_I(G_0)$.

We apply Theorem 3.1 to the action of $\text{Aut}_I(G_0)$ on the proper A -scheme $\text{Par}_{G_0, I}$. We consider the A -scheme $Y = (\phi^{ar}\text{Par}_{G_0, I})^{\phi^{geo}}$. Theorem 3.1 shows that the following statements are equivalent.

- (i') $(\phi\text{Par}_{G_0, I})(U) \neq \emptyset$;
- (ii') $Y(k) \neq \emptyset$.
- (iii') $(\phi\text{Par}_{G_0, I})(K_v) \neq \emptyset$.

Clearly (i') is equivalent to condition (i) of the Theorem and similarly we have (iii') \iff (iii). It remains to establish the equivalence between (ii) and (ii').

Assume that $(\phi^{ar}\text{Par}_{G_0, I})^{\phi^{geo}}(k)$ is not empty and pick a k -point z . Then the stabilizer $(\phi^{ar}G_0)_z$ is a k -parabolic subgroup of $\phi^{ar}G_0$ of type I which is stabilized by the action ϕ_k^{geo} . In other words, ϕ_k^{geo} normalizes $(\phi^{ar}G)_z$. Conversely we assume that $\phi^{ar}G$ admits a k -parabolic subgroup of type I normalized by ϕ^{geo} .

Conversely assume that $\phi^{ar}G$ admits a k -parabolic subgroup of type I normalized by ϕ^{geo} . It defines then a point $z \in (\phi^{ar}\text{Par}_{G_0, I})(k)$ which is fixed by ϕ^{geo} . \square

4.4. An example. Assume that the residue field k is not of characteristic two and consider the diagonal quadratic form of dimension 2^r

$$q = \sum_{I \subset \{1, \dots, r\}} u_I t^I(x_I)^2$$

where $t_I = \prod_{i \in I} t_i$ and $u_I \in A^\times$. Then $\text{SO}(q)$ is a loop reductive group scheme over U . Since the projective quadric $\{q = 0\}$ is a scheme of parabolic subgroups of $\text{SO}(q)$, Theorem 4.1 shows that q is isotropic over A_D if and only if q is isotropic over K_v . The two dimensional case is related with [3, proof of Theorem 3.1].

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