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# LOOP GROUP SCHEMES AND ABHYANKAR'S LEMMA

SCHÉMAS EN GROUPES DE LACETS ET LEMME D'ABHYANKAR

PHILIPPE GILLE

**ABSTRACT.** We define the notion of loop reductive group schemes defined over the localization of a regular henselian ring  $A$  at a strict normal crossing divisor  $D$ . We provide a criterion for the existence of parabolic subgroups of a given type.

*Résumé.* On définit la notion de schémas en groupes réductifs de lacets au-dessus du localisé d'un anneau hensélien  $A$  en un diviseur à croisements normaux stricts  $D$ . On établit un critère pour qu'un tel schéma en groupes admette un sous-schéma en groupes paraboliques d'un type donné.

**Keywords:** Reductive group schemes, normal crossing divisor, parabolic subgroups.

**MSC 2000:** 14L15, 20G15, 20G35.

## Version française abrégée

Soit  $A$  un anneau local hensélien régulier muni d'un système de paramètres  $f_1, \dots, f_r$ . On note  $k$  le corps résiduel de  $A$  et  $D$  le diviseur  $D = \text{div}(f_1) + \dots + \text{div}(f_r)$ , il est à croisements normaux stricts. On pose  $X = \text{Spec}(A)$  et  $U = X \setminus D = \text{Spec}(A_D)$ . La théorie d'Abhyankar décrit les revêtements finis étales connexes de  $U = X \setminus D = \text{Spec}(A_D)$  qui sont modérément ramifiés le long de  $D$  [10, XIII.2]. Un tel objet est dominé par un revêtement galoisien de la forme

$$B_n = B[T_1^{\pm 1}, \dots, T_r^{\pm 1}] / (T_1^n - f_1, \dots, T_r^n - f_r)$$

où  $n$  désigne un entier  $\geq 1$  premier à la caractéristique de  $k$  et  $B$  est une  $A$ -algèbre galoisienne contenant une racine primitive  $n$ -ième de l'unité. Le groupe de Galois  $\text{Gal}(B_n/A_D)$  est le produit semi-direct  $\mu_n(B)^r \rtimes \text{Gal}(B/A)$  où  $\mu_n(B)^r$  agit par multiplication sur les  $T_1, \dots, T_r$ . Si  $G$  est un  $A$ -schéma en groupes localement de présentation finie, un 1-cocycle  $z : \text{Gal}(B_n/A_D) \rightarrow G(B_n)$  est dit de *lacets* (loop en anglais) s'il est à valeurs dans  $G(B) \subset G(B_n)$ . Cette terminologie est inspirée par l'analogie avec le cas des polynômes de Laurent [7, ch. 3].

On note  $\widehat{X}$  l'éclaté de  $X = \text{Spec}(A)$  en son point fermé, c'est un schéma régulier [9, §8.1, th. 1.19] et le diviseur exceptionnel  $E \subset \widehat{X}$  est un diviseur de Cartier isomorphe à  $\mathbb{P}_k^{r-1}$ . On note alors  $R = \mathcal{O}_{\widehat{X}, \eta}$  l'anneau local au point générique  $\eta$  de  $E$ . Cet anneau  $R$  est de valuation discrète, de corps des fractions  $K$ , et de corps résiduel

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$F = k(E) = k(t_1, \dots, t_{r-1})$  où  $t_i$  désigne l'image de  $\frac{f_i}{f_r} \in R$  par l'application de spécialisation  $R \rightarrow F$ . On note alors  $v : K^\times \rightarrow \mathbb{Z}$  la valuation discrète associée à  $R$  et  $K_v$  le complété de  $K$ . Le résultat principal de cette note est le suivant.

**Théorème** (extrait du th. 3.1). *On suppose que  $G$  agit sur un  $A$ -schéma propre et lisse  $Z$ . Soit  $\phi$  un 1-cocycle de lacets pour  $G$ . On note  ${}_\phi Z/U$  le tordu par  $\phi$  de  $Z \times_X U$ . Alors les assertions suivantes sont équivalentes:*

- (a)  $({}_\phi Z)(U) \neq \emptyset$ ;
- (b)  $({}_\phi Z)(K_v) \neq \emptyset$ .

C'est assez proche d'un résultat sur les polynômes de Laurent [7, §, thm. 7.1]. L'application principale concerne le cas d'un schéma en groupes réductifs de lacets. Par définition, un  $U$ -schéma en groupes réductifs  $G$  est *de lacets* si il est isomorphe à un tordu de sa forme déployée  $G_0$  par un 1-cocycle de lacets à valeurs dans le schéma en groupes des automorphismes  $\text{Aut}(G_0)$ . On applique alors le résultat ci-dessus à des  $A$ -schémas de sous-groupes paraboliques de  $G_0$  d'un type donné (th. 4.1). On en déduit par exemple que si  $G$  est un  $U$ -schéma en groupes réductifs *de lacets*, alors  $G$  admet un  $U$ -schéma en groupes de Borel si et seulement si le  $K_v$ -schéma en groupes  $G_{K_v}$  est quasi-déployé. Plus généralement l'isotropie de  $G$  est contrôlée par l'indice de Tits de  $G_{K_v}$ .

## 1. INTRODUCTION

In the reference [7], we investigated a theory of loop reductive group schemes over the ring of Laurent polynomials  $k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ . Using Bruhat-Tits' theory, this permitted to relate what the study of those group schemes to that of reductive algebraic groups over the field of iterated Laurent series  $k((t_1)) \dots ((t_n))$ . The main issue of this note is to start a similar approach for reductive group schemes defined over the localization  $A_D$  of a regular henselian ring  $A$  at a strict normal crossing divisor  $D$  and to relate with algebraic groups defined over a natural field associated to  $A$  and  $D$ , namely the completion  $K_v$  of the fraction field  $K$  with respect to the valuation arising from the blow-up of  $\text{Spec}(A)$  at its maximal ideal. The example which connects the two viewpoints is  $k[[t_1, \dots, t_n]][\frac{1}{t_1}, \dots, \frac{1}{t_n}]$  where  $K_v \cong k(\frac{t_1}{t_n}, \dots, \frac{t_1}{t_{n-1}})((t_n))$ .

After defining the notion of loop reductive group schemes in this setting, we show that for this class of group schemes, the existence of parabolic subgroups over the localization  $A_D$  is controlled by the parabolic subgroups over  $K_v$  (Theorem 4.1).

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## 2. TAME FUNDAMENTAL GROUP

**2.1. Abhyankar's lemma.** Let  $X = \text{Spec}(A)$  be a regular local scheme (not assumed henselian at this stage). Let  $k$  be the residue field of  $A$  and  $p \geq 0$  be its

characteristic. We put  $\widehat{\mathbb{Z}}' = \prod_{l \neq p} \mathbb{Z}_l$ . Let  $K$  be the fraction field of  $A$ , and let  $K_s$  be a separable closure of  $K$ . It determines a base point  $\xi : \text{Spec}(K) \rightarrow X$  so that we can deal with the Grothendieck fundamental group  $\Pi_1(X, \xi)$  [10].

Let  $(f_1, \dots, f_r)$  be a regular sequence of  $A$  and consider the divisor  $D = \sum D_i = \sum \text{div}(f_i)$ , it has strict normal crossing. We put  $U = X \setminus D = \text{Spec}(A_D)$ .

We recall that a finite étale cover  $V \rightarrow U$  is *tamely ramified* with respect to  $D$  if the associated étale  $K$ -algebra  $L = L_1 \times \dots \times L_a$  is tamely ramified at the  $D_i$ 's, that is, for each  $i$ , there exists  $j_i$  such that for the Galois closure  $\widetilde{L}_{j_i}/K$  of  $L_{j_i}/K$ , the inertia group associated to  $v_{D_i}$  has order prime to  $p$  [10, XIII.2.0].

Grothendieck and Murre defined the tame (*modéré* in French) fundamental group  $\Pi_1^D(U, \xi)$  with respect to  $U \subset X$  as defined in [10, XIII.2.1.3] and [8, §2]. This is a profinite quotient of  $\Pi_1(U, \xi)$  whose quotients by open subgroups provides finite Galois tame cover of  $U$ .

Let  $V \rightarrow U$  be a finite étale tame cover. In this case Abhyankar's lemma states that there exists a flat Kummer cover  $X' = \text{Spec}(A') \rightarrow X$  where

$$A' = A[T_1, \dots, T_r]/(T_1^{n_1} - f_1, \dots, T_r^{n_r} - f_r)$$

and the  $n_i$ 's are coprime to  $p$  such that  $V' = V \times_X X' \rightarrow X'$  extends uniquely to a finite étale cover  $Y' \rightarrow X'$  [10, XIII.5.2].

**Lemma 2.1.** *Let  $V \rightarrow U$  be a finite étale cover which is tame. Then  $\text{Pic}(V) = 0$ .*

*Proof.* We use the same notation as above. We know that  $X'$  is regular [10, XIII.5.1] so a fortiori locally factorial. It follows that the restriction maps  $\text{Pic}(X') \rightarrow \text{Pic}(V') \rightarrow \text{Pic}(V)$  are surjective [5, 21.6.11]. Since  $A'$  is finite over the local ring  $A$ , it is semilocal so that  $\text{Pic}(A') = \text{Pic}(X') = 0$ . Thus  $\text{Pic}(V) = 0$  as desired.  $\square$

From now on we assume that  $A$  is henselian. According to [5, 18.5.10], the finite  $A$ -ring  $A'$  is a finite product of henselian local rings. We observe that  $A' \otimes_A k = k[T_1, \dots, T_r]/(T_1^{n_1}, \dots, T_r^{n_r})$  is a local Artinian algebra so that  $A'$  is connected. It follows that  $A'$  is a henselian local ring. Its maximal ideal is  $\mathfrak{m}' = \mathfrak{m} \otimes_A A' + \langle T_1, \dots, T_r \rangle$  so that  $A'/\mathfrak{m}' = k$ . Since there is an equivalence of categories between finite étale covers of  $A$  (resp.  $A'$ ) and étale  $k$ -algebras [5, 18.5.15], the base change from  $A$  to  $A'$  provides an equivalence of categories between the category of finite étale covers of  $A$  and that of  $A'$ .

It follows that  $Y' \rightarrow X'$  descends uniquely to a finite étale cover  $\widetilde{f} : \widetilde{Y} \rightarrow X$ . From now on, we assume that  $V$  is furthermore connected, it implies that

$$H^0(V, \mathcal{O}_V) = B[T_1, \dots, T_r]/(T_1^n - f_1, \dots, T_r^n - f_r)$$

where  $B$  is a finite connected étale cover of  $A$ . It follows that  $V \rightarrow U$  is a quotient of a Galois cover of the shape

$$B_n = B[T_1^{\pm 1}, \dots, T_r^{\pm 1}]/(T_1^n - f_1, \dots, T_r^n - f_r)$$

where  $B$  is Galois cover of  $A$  containing a primitive  $n$ -th root of unity. We notice that  $B_n$  is the localization at  $T_1 \dots T_r$  of  $B'_n = B[T_1, \dots, T_r]/(T_1^n - f_1, \dots, T_r^n - f_r)$ . We have

$$\mathrm{Gal}(B_n/A_D) = \left( \prod_{i=1}^r \mu_n(B) \right) \rtimes \mathrm{Gal}(B/A).$$

Passing to the limit we obtain an isomorphism

$$\pi_1^t(U, \xi) \cong \left( \prod_{i=1}^r \widehat{\mathbb{Z}}'(1) \right) \rtimes \pi_1(X, \xi).$$

We denote by  $f : U^{sc,t} \rightarrow U$  the profinite étale cover associated to the quotient  $\pi_1^t(U, \xi)$  of  $\pi_1(U, \xi)$ . According to [8, thm. 2.4.2], it is the universal tamely ramified cover of  $U$ . It is a localization of the inductive limit  $\widetilde{B}'$  of the  $B'_n$ . On the other hand we consider the inductive limit  $\widetilde{B}$  of the  $B$ 's and observe that  $\widetilde{B}'$  is a  $\widetilde{B}$ -ring.

**2.2. Blow-up.** We follow a blowing-up construction arising from [5, lemma 15.1.1.6]. We denote by  $\widehat{X}$  the blow-up of  $X = \mathrm{Spec}(A)$  at his closed point, this is a regular scheme [9, §8.1, th. 1.19] and the exceptional divisor  $E \subset \widehat{X}$  is a Cartier divisor isomorphic to  $\mathbb{P}_k^{r-1}$ . We denote by  $R = \mathcal{O}_{\widehat{X}, \eta}$  the local ring at the generic point  $\eta$  of  $E$ . The ring  $R$  is a DVR of fraction field  $K$  and of residue field  $F = k(E) = k(t_1, \dots, t_{r-1})$  where  $t_i$  is the image of  $\frac{f_i}{f_r} \in R$  by the specialization map. We denote by  $v : K^\times \rightarrow \mathbb{Z}$  the discrete valuation associated to  $R$ .

We deal now with a Galois extension  $B_n$  of  $A_D$  as above. Since  $B$  is a connected finite étale cover of  $A$ ,  $B$  is regular and local; it is furthermore henselian [5, 18.5.10]. We denote by  $L$  the field of fraction of  $B$  and by  $L_n$  that of  $B_n$ . We have  $[L_n : L] = n^r$ . We want to extend the valuation  $v$  to  $L$  and to  $L_n$ .

We denote by  $l = B/\mathfrak{m}_B$  the residue field of  $B$ , this is a finite Galois field extension of  $k$ . Also  $(t_1, \dots, t_r)$  is a system of parameters for  $B$ . We denote by  $w : L^\times \rightarrow \mathbb{Z}$  the discrete valuation associated to the exceptional divisor of the blow-up of  $\mathrm{Spec}(B)$  at its closed point. Then  $w$  extends  $v$  and  $L_w/K_v$  is an unramified extension of degree  $[L : K]$  and of residual extension  $F_l = l(t_1, \dots, t_{r-1})/k(t_1, \dots, t_{r-1})$ .

On the other hand we denote by  $w_n : L_n^\times \rightarrow \mathbb{Z}$  the discrete valuation associated to the exceptional divisor of the blow-up of  $\mathrm{Spec}(B_n)$  at its closed point. We put  $l_n = B_n/\mathfrak{m}_{B_n}$ , we have  $l = l_n$ . The valuation  $\frac{w_n}{n}$  on  $L_n$  extends  $w$  and its residual extension is  $F_{l,n} = l\left(t_1^{1/n}, \dots, t_{r-1}^{1/n}\right)/k(t_1, \dots, t_{r-1})$  so that  $[F_{l,n} : F_l] = n^{r-1}$ . Furthermore the ramification index  $e_n$  of  $L_n/L$  is  $\geq n$ . Since  $n^r \leq e_n [F_{l,n} : F_l] \leq [L_n : K] = n^r$  (where the last inequality is [2, §VI.3, prop. 2]) it follows that  $e_n = n$ . The same statement shows that the map  $L_w \otimes_L L_n \rightarrow L_{w_n}$  is an isomorphism. To summarize  $L_{w_n}/L_w$  is tamely ramified of ramification index  $n$  and of degree  $n^r$ . Altogether we have  $L_{w_n} = L_w \otimes_K L_n$  so that  $L_{w_n}$  is Galois over  $K_v$  of group  $\prod_i \mu_n(B) \rtimes \mathrm{Gal}(B/A) = \prod_i \mu_n(l) \rtimes \mathrm{Gal}(l/k)$ .

We denote by  $\Delta$  the diagonal embedding  $\mu_n(l) \subset \prod_i \mu_n(l)$ . We put  $L_{w_n}^\Delta = L_n^{\Delta(\mu_n(B))}$ . Since  $t_r$  is a uniformizing parameter of  $K_v$  and since  $\Delta(\zeta) \cdot t_r = \zeta \cdot t_r$  for each  $\zeta \in \mu_n(B)$ , it follows that  $(L_{w_n})^\Delta$  is the maximal unramified extension of  $L_{w_n}/K_v$ .

**2.3. Loop cocycles and loop torsors.** Let  $G$  be an  $X$ -group scheme locally of finite presentation. A loop cocycle is an element of  $Z^1(\pi_1^t(U), G(\tilde{B}))$  and it defines a Galois cocycle in  $Z^1(\pi_1^t(U), G(U^{sc,t}))$ . We denote by  $Z_{loop}^1(\pi_1^t(U), G(U^{sc,t}))$  the image of the map  $Z^1(\pi_1^t(U), G(\tilde{B})) \rightarrow Z^1(\pi_1^t(U), G(U^{sc,t}))$  and by  $H_{loop}^1(U, G)$  the image of the map

$$Z^1(\pi_1^t(U), G(\tilde{B})) \rightarrow H^1(\pi_1^t(U), G(U^{sc,t})) \rightarrow H^1(U, G).$$

We say that a  $G$ -torsor  $E$  over  $U$  (resp. an fppf sheaf  $G$ -torsor) is a loop torsor if its class belongs to  $H_{loop}^1(U, G) \subset H^1(U, G)$ .

A given class  $\gamma \in H_{loop}^1(U, G)$  is represented by a 1-cocycle  $\phi : \text{Gal}(B_n/A_D) \rightarrow G(B)$  for some cover  $B_n/A$  as above. Its restriction  $\phi^{ar} : \text{Gal}(B/A) \rightarrow G(B)$  to the subgroup  $\text{Gal}(B/A)$  of  $\text{Gal}(B_n/A_D)$  is called the ‘‘arithmetic part’’ and the other restriction  $\phi^{geo} : \prod_i \mu_n(B) \rightarrow \mathfrak{G}(B)$  is called the geometric part. We observe that  $\phi^{geo}$  is a  $B$ -group homomorphism.

Furthermore for  $\sigma \in \text{Gal}(B/A)$  and  $\tau \in \prod_i \mu_n(B)$  the computation of [7, page 16] shows that  $\phi^{geo}(\sigma\tau\sigma^{-1}) = \phi^{ar}(\sigma) \sigma\phi(\tau) \phi^{ar}(\sigma)^{-1}$  so that  $\phi^{geo}$  descends to a homomorphism of  $A$ -group schemes  $\phi^{geo} : \mu_n^r \rightarrow \phi^{ar}G$ . This provides a parameterization of loop cocycles.

**Lemma 2.2.** (1) For  $B_n/A_D$  as above, the map  $\phi \mapsto (\phi^{ar}, \phi^{geo})$  provides a bijection between  $Z_{loop}^1(\text{Gal}(B_n/A_D), G(B))$  and the couples  $(z, \eta)$  where  $z \in Z^1(\text{Gal}(B/A), G(B))$  and  $\eta : \prod_i \mu_n \rightarrow {}_zG$  is an  $A$ -group homomorphism.

(2) The map  $\phi \mapsto (\phi^{ar}, \phi^{geo})$  provides a bijection between  $Z_{loop}^1(\pi^1(U, \xi)^t, G(\tilde{B}))$  and the couples  $(z, \eta)$  where  $z \in Z^1(\pi^1(X, \xi), G(\tilde{B}))$  and  $\eta : \prod_{i=1}^r \tilde{\mathbb{Z}}^1 \rightarrow {}_zG$  is an  $A$ -group homomorphism.

*Proof.* This is similar with [7, lemma 3.7].  $\square$

We examine more closely the case of a finite étale  $X$ -group scheme  $\mathfrak{F}$  of constant degree  $d$ .

**Lemma 2.3.** (1)  $\mathfrak{F}(\tilde{B}) = \mathfrak{F}(X^{sc}) = \mathfrak{F}(U^{sc,t})$ .

(2) We assume that  $d$  is prime to  $p$ . We have  $H_{loop}^1(U, \mathfrak{F}) = H^1(U, \mathfrak{F})$ .

(3) We assume that  $d$  is prime to  $p$ . Let  $f : \mathfrak{F} \rightarrow \mathfrak{H}$  be a homomorphism of  $A$ -group schemes (locally of finite type). Then  $f_*\left(H^1(U, \mathfrak{F})\right) \subset H_{loop}^1(U, \mathfrak{H})$ .

*Proof.* (1) We are given a cover  $B_n/A_D$  as above such that  $\mathfrak{F}_{B_n} \cong \Gamma_{B_n}$  is finite constant. as above. Since  $B$  and  $B_n$  are connected, the map  $\mathfrak{F}(B) \rightarrow \mathfrak{F}(B_n)$  reads as

the identity  $\Gamma \cong \mathfrak{F}(B) \rightarrow \mathfrak{F}(B_n) \cong \Gamma$  so is bijective. By passing to the limit we get  $\mathfrak{F}(\tilde{B}) = \mathfrak{F}(U^{sc,t})$ .

(2) Let  $\mathfrak{E}$  be a  $\mathfrak{F}$ -torsor over  $U$ . This is a finite étale  $U$ -scheme. Since  $U$  is noetherian and connected, we have a decomposition  $\mathfrak{E} = V_1 \times_U \cdots \times_U V_l$  where each  $V_i$  is a connected finite étale  $U$ -scheme of constant degree  $d_i$ . We have  $d_1 + \cdots + d_l = d$  so that we can assume that  $d_1$  is prime to  $p$ . We have then  $\mathfrak{E}(V_1) \neq \emptyset$ .

It follows that  $f_1 : V_1 \rightarrow U$  is a finite étale cover so that there exists a factorization  $U^{sc,t} \rightarrow V_1 \xrightarrow{h} U$  of  $f$  so that  $\mathfrak{E}(U^{sc,t}) \neq \emptyset$ . Therefore  $[\mathfrak{E}]$  arises from  $H^1(\pi_1^t(U, \xi), \mathfrak{F}(U^{sc,t})) \subset H^1(U, \mathfrak{F})$ . It follows that  $H^1(\pi_1^t(U, \xi), \mathfrak{F}(U^{sc,t})) \xrightarrow{\sim} H^1(U, \mathfrak{F})$ . We use now (1) and obtain the desired bijection  $H^1(\pi_1^t(U, \xi), \mathfrak{F}(B)) \xrightarrow{\sim} H^1(U, \mathfrak{F})$ .

(3) This follows readily from (2).  $\square$

**2.4. Twisting by loop torsors.** We assume that the  $A$ -group scheme  $G$  acts on an  $A$ -scheme  $Z$ . Let  $\phi : (\prod_i^r \mu_n)(B) \rtimes \text{Gal}(B/A) \rightarrow G(B)$  be a loop cocycle. It gives rise to an  $A$ -action of  $\mu_n^r$  on  ${}_{\phi ar}Z$ . We denote by  $({}_{\phi ar}Z)^{\phi^{geo}}$  the fixed point locus for this action, it is representable by a closed  $A$ -subscheme of  ${}_{\phi ar}Z$  [4, A.8.10.(1)]. We have a closed embedding  $({}_{\phi ar}Z)^{\phi^{geo}} \times_X U \subset {}_{\phi}Z$  of  $U$ -schemes.

### 3. FIXED POINTS METHOD

**Theorem 3.1.** *Let  $X = \text{Spec}(A)$  be a henselian regular local scheme and  $U = X \setminus D$  as above. We denote by  $v : K^\times \rightarrow \mathbb{Z}$  the discrete valuation associated to the exceptional divisor  $E$  of the blow-up of  $X$  at its closed point.*

*Let  $G$  be an affine  $A$ -group scheme of finite presentation acting on a proper smooth  $A$ -scheme  $Z$ . Let  $\phi$  be a loop cocycle for  $G$ . Then  $Y = ({}_{\phi ar}Z)^{\phi^{geo}}$  is a smooth proper  $A$ -scheme and the following are equivalent:*

- (i)  $({}_{\phi}Z)(K_v) \neq \emptyset$ ;
- (ii)  $Y(k) \neq \emptyset$ ;
- (iii)  $Y(U) \neq \emptyset$ ;
- (iv)  $({}_{\phi}Z)(U) \neq \emptyset$ .

This is quite similar with the fixed point theorem [7, §, thm. 7.1]. The following example makes the connection.

**Example 3.2.** We assume that  $A = k[[t_1, \dots, t_r]]$  for a field  $k$  and  $k[U] = k[[t_1, \dots, t_n]] \left[ \frac{1}{t_1}, \dots, \frac{1}{t_r} \right]$ . We consider an affine algebraic  $k$ -group  $G$  acting on a smooth proper  $k$ -scheme  $Z$ . In this case  $K = k((t_1, \dots, t_r))$  and  $A$  embeds in  $k\left(\frac{t_1}{t_r}, \dots, \frac{t_{r-1}}{t_r}\right)[[t_r]]$  so that  $K$  embeds in  $k\left(\frac{t_1}{t_n}, \dots, \frac{t_{r-1}}{t_r}\right)((t_r))$  which is nothing but the complete field  $K_v$ . If  $Q$  is a loop  $G$ -torsor over  $U$ , the statement is then that  ${}^QZ(U) \neq \emptyset$  if and only if  ${}^QZ(K_v) \neq \emptyset$ . Taking a cocycle  $\phi \in Z^1(\pi_1(U)^t, G(k_s))$  for  $E$ , this rephrases by the equivalence between  $({}_{\phi}Z)(U) \neq \emptyset$  and  $({}_{\phi}Z)(K_v) \neq \emptyset$ .

What we have from [7, thm. 7.1] (in characteristic zero but this extends to this tame setting) is the equivalence between  $(\phi Z)(k[t_1^{\pm 1}, \dots, t_r^{\pm 1}]) \neq \emptyset$  and  $(\phi Z)(k((t_1)) \dots ((t_r))) \neq \emptyset$ . Since  $(\phi Z)(k[t_1^{\pm 1}, \dots, t_r^{\pm 1}]) \subset (\phi Z)(U)$  and  $(\phi Z)(K_v) \subset (\phi Z)(k((t_1)) \dots ((t_r)))$ , it follows that this special case of Theorem 3.1 is a consequence of the fixed point result of [7].

We proceed to the proof of Theorem 3.1.

*Proof.* According to [4, A.8.10.(1)],  $Y = (\phi^{ar} Z^{\phi^{geo}})$  is a closed  $A$ -scheme of  $\phi^{ar} Z$  so it is proper. It is smooth over  $X$  according to point (2) of the same reference. Let  $\phi : \text{Gal}(B_n/A_D) \rightarrow G(B)$  be the loop 1-cocycle for some Galois cover  $B_n/A_D$  as above for some  $n$  prime to  $p$ . Up to replacing  $G$  by  $\phi^{ar} G$  and  $Z$  by  $\phi^{ar} Z$ , we can assume that  $\phi^{ar} = 1$  without loss of generality.

(ii)  $\implies$  (iii). Since  $Y_k$  is the special fiber of the smooth  $X$ -scheme  $Y$ , Hensel's lemma shows that  $Y(A) \rightarrow Y(k)$  is onto. Since  $Y(k)$  is not empty, it follows that  $Y(A)$  is not empty and so is  $Y(U)$ .

(iii)  $\implies$  (iv). Since  $Y(U) \subset \phi Z(U)$ ,  $Y(U) \neq \emptyset$  implies that  $\phi Z(U) \neq \emptyset$ .

(iv)  $\implies$  (i). This is obvious.

(i)  $\implies$  (ii). We assume that  $(\phi Z)(K_v) \neq \emptyset$ . By definition we have

$$(\phi Z)(K_v) = \{z \in Z(L_{w_n}) \mid \phi(\sigma).\sigma(z) = z \ \forall \sigma \in \text{Gal}(L_n/K)\}$$

and our assumption is that this set is non-empty. Let  $O_{w_n}$  be the valuation ring of  $Z(L_{w_n})$ . Since  $Z$  is proper over  $X$ , we have a specialization map  $Z(L_{w_n}) = Z(\mathcal{O}_{w_n}) \rightarrow Z_k(F_{l,n})$ . We get that the set

$$\{z \in Z_k(F_{l,n}) \mid \phi(\sigma).\sigma(z) = z \ \forall \sigma \in \text{Gal}(L_{w_n}/K_v)\}$$

is not empty. Since we have an embedding

$$F_{l,n} = l(t_1^{1/n}, \dots, t_{r-1}) \hookrightarrow l((t_1^{1/n})) \dots ((t_{r-1}^{1/n}))$$

in a higher field of Laurent series successive specializations, along the coordinates  $t_1^{1/n}, \dots, t_{r-1}^{1/n}$  show similarly that the set

$$(3.1) \quad \left\{ z \in (Z_k)(l) \mid \phi(\sigma).\sigma(z) = z \ \forall \sigma \in \text{Gal}(L_{w_n}/K_v) \right\}$$

is not empty. Since  $\eta^{ar} = 1$ , this set is  $(Z_k)^{\eta^{geo}}(k)$ . Thus  $Y(k) = (Z_k)^{\eta^{geo}}(k)$  is non empty.  $\square$

#### 4. PARABOLIC SUBGROUPS OF LOOP REDUCTIVE GROUP SCHEMES

**4.1. Chevalley groups.** Let  $G_0$  be Chevalley group defined over  $\mathbb{Z}$ . Let  $T_0$  be a maximal split  $\mathbb{Z}$ -subtorus of  $G_0$  together with a Borel subgroup  $B_0$  containing it. We denote by  $\Delta_0$  the Dynkin diagram of  $(G_0, B_0, T_0)$ . We denote by  $G_{0,ad}$  the adjoint



quotient of  $G_0$  and by  $G_0^{sc}$  the simply connected covering of  $DG_0$ . We have a map  $\text{Aut}(G_0) \rightarrow \text{Aut}(G_0^{sc}) \xrightarrow{\sim} \text{Aut}(G_{0,ad})$  and a fundamental exact sequence

$$1 \rightarrow G_{0,ad} \rightarrow \text{Aut}(G_{0,ad}) \rightarrow \text{Out}(G_{0,ad}) \rightarrow 1$$

where  $\text{Out}(G_{0,ad}) \xrightarrow{\sim} \text{Aut}(\Delta_0)$ . We recall that there is a bijection  $I \rightarrow P_{0,I}$  between the finite subsets of  $\Delta_0$  and the parabolic subgroups of  $G_0$  containing  $B_0$  [11, XXVI.3.8]; it is increasing for the inclusion order, in particular  $B_0 = P_{0,\emptyset}$  and  $G_0 = P_{0,\Delta_0}$ . We consider the total scheme  $\text{Par}_{G_0}$  of parabolic subgroups of  $G_0$ , it is a projective smooth  $\mathbb{Z}$ -scheme equipped with a type map  $\mathbf{t} : \text{Par}_{G_0} \rightarrow \text{Of}(\Delta_0)$  where  $\text{Of}(\Delta_0)$  stands for the finite constant scheme attached to the set of subsets of  $\Delta_0$  [11, XXVI.3]. The fiber at  $I$  is denoted by  $\text{Par}_{G_0,I}$ , it has connected fibers and is the scheme of parabolic subgroups of  $G_0$  of type  $I$ . We have a natural action of  $\text{Aut}(G_0)$  on  $\text{Par}_{G_0}$ . As in [6, §5.1], we denote by  $\text{Aut}_I(G_0)$  the stabilizer of  $I$  for this action. By construction  $\text{Aut}_I(G_0)$  acts on  $\text{Par}_{G_0,I}$ .

**4.2. Definition.** Let  $G$  be a reductive  $U$ -group scheme in the sense of Demazure-Grothendieck [11, XIX]. Since  $U$  is connected and  $G$  is locally splittable [11, XXII.2.2] for the étale topology,  $G$  is an étale form of a Chevalley group  $G_0$  as above defined over  $\mathbb{Z}$ .

We say that  $G$  is a *loop group scheme* if the  $\text{Aut}(G_0)$ -torsor  $Q = \text{Isom}(G_0, G)$  (defined in [11, XXIV.1.9]) is a loop  $\text{Aut}(G_0)$ -torsor. We denote by  $G_{0,ad}$  the adjoint quotient of  $G_0$  and by  $G_0^{sc}$  the simply connected covering of  $DG_0$ . We have a map  $\text{Aut}(G_0) \rightarrow \text{Aut}(G_0^{sc}) \xrightarrow{\sim} \text{Aut}(G_{0,ad})$  which permits to see  $G_{ad}$  (resp.  $G^{sc}$ ) as twisted forms of  $G_{0,ad}$  (resp.  $G_0^{sc}$ ) so that  $G_{ad}$  and  $G^{sc}$  are also loop reductive group schemes. We consider the map  $\text{Aut}(G_0) \rightarrow \text{Aut}(G_{0,ad}) \rightarrow \text{Out}(G_{0,ad}) \xrightarrow{\sim} \text{Aut}(\Delta_0)$ .

If  $\phi : \text{Gal}(B_n/A_D) \rightarrow \text{Aut}(G_0)(B)$  is a loop cocycle, we get an action of  $\text{Gal}(B_n/A_D)$  on  $\Delta_0$  called the star action. If  $I$  is stable under the star action, we can twist  $\text{Par}_{G_0,I}$  by  $\phi$  and deal with the scheme  ${}_\phi\text{Par}_{G_0,I}$  which is the scheme of parabolic subgroup schemes of  $G$  of type  $I$ .

### 4.3. Parabolics.

**Theorem 4.1.** *Assume that  $G$  is a loop reductive  $U$ -group scheme and let  $\phi : \text{Gal}(B_n/A_D) \rightarrow \text{Aut}(G_0)(B)$  be a loop cocycle such that  $G \cong {}_\phi G_0$ . Let  $I \subset \Delta_0$  be a subset stable under the star action defined by  $\phi$ . Then the following are equivalent:*

- (i)  $G$  admits a  $U$ -parabolic subgroup of type  $I$ ;
- (ii) the  $k$ -morphism  $\eta_k^{geo} : \mu_n^r \rightarrow \text{Aut}({}_{\eta^{ar}}G_0)_k = ({}_{\eta^{ar}}\text{Aut}(G_0))_k$  normalizes a parabolic  $k$ -subgroup of  ${}_{\eta^{ar}}G_{0,k}$  of type  $I$ ;
- (iii)  $G_{K_v}$  admits a parabolic subgroup of type  $I$ .

*Proof.* Without loss of generality we can assume that  $G$  is adjoint. Our assumption on the star action is rephrased by saying that  $\phi$  takes values in  $\text{Aut}_I(G_0)$ . We apply Theorem 3.1 to the action of  $\text{Aut}_I(G_0)$  on the proper  $A$ -scheme  $\text{Par}_{G_0,I}$ . We consider

the  $A$ -scheme  $Y = (\phi^{ar}\text{Par}_{G_0, I})^{\phi^{geo}}$ . Theorem 3.1 shows that the following statements are equivalent.

- (i')  $(\phi\text{Par}_{G_0, I})(U) \neq \emptyset$ ;
- (ii')  $Y(k) \neq \emptyset$ .
- (iii')  $(\phi\text{Par}_{G_0, I})(K_v) \neq \emptyset$ .

Clearly (i') is equivalent to condition (i) of the Theorem and similarly we have  $(iii') \iff (iii)$ . It remains to establish the equivalence between (ii) and (ii').

Assume that  $(\phi^{ar}\text{Par}_{G_0, I})^{\phi^{geo}}(k)$  is not empty and pick a  $k$ -point  $z$ . Then the stabilizer  $(\phi^{ar}G_0)_z$  is a  $k$ -parabolic subgroup of  $\phi^{ar}G_0$  of type  $I$  which is stabilized by the action  $\phi_k^{geo}$ . In other words,  $\phi_k^{geo}$  normalizes  $(\phi^{ar}G)_z$ . Conversely we assume that  $\phi^{ar}G$  admits a  $k$ -parabolic subgroup of type  $I$  normalized by  $\phi^{geo}$ . It defines then a point  $z \in (\phi^{ar}\text{Par}_{G_0, I})(k)$  which is fixed by  $\phi^{geo}$ . □

**4.4. An example.** Assume that the residue field  $k$  is not of characteristic 2 and consider the diagonal quadratic form of dimension  $2^r$

$$q = \sum_{I \subset \{1, \dots, r\}} u_I t^I (x_I)^2$$

where  $t_I = \prod_{i \in I} t_i$  and  $u_I \in A^\times$ . Then  $\text{SO}(q)$  is a loop reductive group scheme over  $U$ . Since the projective quadric  $\{q = 0\}$  is a scheme of parabolic subgroups of  $\text{SO}(q)$ , Theorem 4.1 shows that  $q$  is isotropic over  $A_D$  if and only if  $q$  is isotropic over  $K_v$ . The two dimensional case is related with [3, proof of Theorem 3.1].

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