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Anyons in a tight wave-guide and the Tonks-Girardeau gas

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We consider a many-body system of 2D anyons, free quantum particles with general statistics parameter $\alpha \in]0, 2[$. In the magnetic gauge picture they are described as bosons attached to Aharonov-Bohm fluxes of intensity $2\pi\alpha$, generating long-range magnetic forces. A dimensional reduction to 1D is obtained by imposing a strongly anisotropic trapping potential. This freezes the motion in the direction of strong trapping, leading to 1D physics along the weak direction. The latter is governed to leading order by the Tonks-Girardeau model of impenetrable bosons, independently of α .

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Quasi-particle excitations of many-body systems confined to reduced dimensionalities are not in principle constrained by the symmetry dichotomy which sorts all fundamental particles into bosons and fermions [21, 32, 43, 54]. In 2D, many-body quantum wave-functions may be classified by the phase picked upon exchanging two particles. It is of the form $e^{i\pi\alpha}$ for $\alpha \in [0, 2[$, the standard cases of bosons and fermions being recovered for $\alpha = 0, 1$ respectively. Equivalently one may think of these so-called anyons in terms of standard bosons (or fermions), coupled to infinitely thin magnetic flux tubes of strength $2\pi\alpha$ (or $2\pi(\alpha - 1)$). This point of view is referred to as the magnetic gauge picture [15–17, 25, 27, 34, 45, 55]. In 1D there does not seem to be a unique agreed-upon model for anyonic exchange statistics. Depending on how one proceeds to quantization, they have mainly been described [32, 33, 46, 47] as ordinary particles with either contact interactions (Lieb-Liniger model) or inverse-square interactions (Calogero-Sutherland model) but other formalisms exist [2, 3, 6, 7, 19, 28, 51]. It makes sense to ask which one, if any, of the different theoretically possible descriptions of 1D anyons, is singled out as the dimensional reduction of the 2D theory.

This may have practical consequences as well. The main candidates for real-world implementation of anyon statistics indeed remain the charge carriers of fractional quantum Hall systems [4, 22, 38] or their counterparts in cold atom emulations [10, 11, 14, 52, 56, 57]. See e.g. [20, 26, 31, 48] for reviews and [5, 44] for experimental evidence. Said charge carriers are 2D objects described in the bulk via the usual, aforementioned, anyon model [29, 38]. Much of fractional quantum Hall physics is however probed via the transport of charge carriers along 1D edge channels.

In this note we report on our investigation of the 1D limit of the 2D anyon model. Full mathematical details will be provided elsewhere. We find that the magnetic-gauge picture Hamiltonian for N 2D anyons of statistical parameter $\alpha \in]0, 2[$ converges, in the limit of a tight confinement along one spatial dimension, to the impenetrable boson model of the Tonks-Girardeau gas, soluble by Bose-Fermi mapping [18, 42]. Thus, at leading order, the physics does not depend on α and is given

by an extreme case of the Lieb-Liniger [36, 37] model. The behavior is always essentially fermionic, which might be interpreted in light of the enhanced effect of interactions in reduced dimensionalities. However, it is remarkable that the long-range magnetic interactions of the original model result in a limiting purely local theory. This finding seems consistent with [23] although our approach differs and seems more systematic. In particular it clarifies the vanishing of the long-range magnetic interaction. This comes about via a particular phase acquired by the 2D wave-functions, that almost gauges the interaction away when particles are aligned.

Models and main result

Consider a wave function $\Psi : (\mathbb{R}^2)^N \rightarrow \mathbb{C}$ with anyonic exchange behavior, i.e.

$$\Psi(\mathbf{x}_1, \dots, \mathbf{x}_j, \dots, \mathbf{x}_k, \dots, \mathbf{x}_N) = e^{i\alpha\pi} \Psi(\mathbf{x}_1, \dots, \mathbf{x}_k, \dots, \mathbf{x}_j, \dots, \mathbf{x}_N) \quad (1)$$

with $\alpha \in]0, 1[$ (by periodicity and complex conjugation, considering this range is sufficient). It is convenient to perform a singular gauge transformation

$$\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N) = \prod_{j < k} e^{i\alpha\phi_{jk}} \Phi(\mathbf{x}_1, \dots, \mathbf{x}_N) \quad \text{with } \phi_{jk} = \arg \frac{\mathbf{x}_j - \mathbf{x}_k}{|\mathbf{x}_j - \mathbf{x}_k|}$$

with Φ a bosonic wave function, symmetric under particle exchange. We have denoted $\arg(\cdot)$ the angle of a planar vector with the horizontal axis. Applying this transformation, one finds

$$\left\langle \Psi \left| \left(-i\nabla_{\mathbf{x}_j} \right)^2 \right| \Psi \right\rangle = \left\langle \Phi \left| D_{\mathbf{x}_j}^2 \right| \Phi \right\rangle$$

where the momentum operator for particle j has changed as

$$-i\nabla_{\mathbf{x}_j} \rightsquigarrow D_{\mathbf{x}_j} := -i\nabla_{\mathbf{x}_j} + \alpha \mathbf{A}(\mathbf{x}_j) \quad (2)$$

with, denoting $(x, y)^\perp = (-y, x) \in \mathbb{R}^2$,

$$\mathbf{A}(\mathbf{x}_j) := \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2}. \quad (3)$$

We adopt this point of view throughout the note, using (the Friedrichs extension [1, 12, 13, 39–41] of)

$$H_\varepsilon^{2D} := \sum_{j=1}^N \left(D_{x_j}^2 + V_\varepsilon(\mathbf{x}_j) \right) \quad (4)$$

acting on bosonic wave-functions as our starting point, where

$$V_\varepsilon(\mathbf{x}) = V_\varepsilon(x, y) = x^2 + \frac{y^2}{\varepsilon^2} \quad (5)$$

is a convenient way of enforcing 1D behavior along the horizontal axis in the limit $\varepsilon \rightarrow 0$. This is arguably a crude description (but, perhaps, also an instructive toy model) if one has a fractional quantum Hall edge in mind. There is however in principle no difficulty in imposing such a potential on emerging anyons in cold atoms systems. The choice of a harmonic trapping is mostly out of convenience. Our results remain true with different choices, but the harmonic trapping has the virtue of leading to exactly soluble limit models.

We denote

$$\lambda_k^{2D} = \min_{\substack{V_k \subset L^2(\mathbb{R}^{2N}) \\ \dim V_k = k}} \max_{\Psi \in V_k, \|\Psi\|_{L^2} = 1} \langle \Psi | H_\varepsilon^{2D} | \Psi \rangle \quad (6)$$

the eigenvalues of (4), defined by standard Courant-Fisher min-max formulae. Let Ψ_k be associated eigenfunctions, i.e.

$$H_\varepsilon^{2D} \Psi_k = \lambda_k^{2D} \Psi_k.$$

There can be degeneracies, in which case we count eigenvalues with their multiplicities.

For small $\varepsilon > 0$ one expects the motion in the two spatial directions to decouple (which is true only to some extent in this particular case, see below). The motion in the y direction will be frozen in the ground state of the harmonic oscillator

$$H_\varepsilon^{\text{HO}} := -\partial_y^2 + \frac{y^2}{\varepsilon^2}. \quad (7)$$

It turns out that the motion in the x direction reduces to the free Hamiltonian

$$H^{1D} := \sum_{j=1}^N -\partial_{x_j}^2 + x_j^2 \quad (8)$$

but acting on the domain

$$\mathcal{D}^{1D} := \{ \psi \in H^2(\mathbb{R}^N), \psi(x_j = x_k) \equiv 0 \text{ for all } j \neq k \}. \quad (9)$$

This restriction is equivalent to the addition of a delta pair-potential of infinite strength to (8). It is well-known that this impenetrable boson model can be mapped to a free fermionic one [18, 42]. In turn, this leads to an exact solution in the particular case above. However, our approach does not rely on this exact solution, and we could in fact have added extra

interactions to our model. For simplicity, we do not consider this explicitly.

Let e_ε and u_ε be respectively the lowest eigenvalue and eigenfunction of (7). Let $(\lambda_k^{1D})_{k \in \mathbb{N}}$ be the eigenvalues of (8), with associated eigenfunctions ψ_k , $k = 1, 2, \dots$. In the model above, λ_k^{1D} is a sum of eigenvalues of the harmonic oscillator $-\partial_x + x^2$ and

$$\psi_k = c_k \prod_{i < j} e^{\text{isgn}(x_i - x_j)} \det(v_i(x_j))$$

where v_i are the associated one-particle eigenfunctions and c_k is a normalization constant.

We state our main finding as a theorem.

Theorem 1 (Dimensional reduction for anyons).

We have, in the limit $\varepsilon \rightarrow 0$, for all $k \in \mathbb{N}$

$$\lambda_k^{2D} = N e_\varepsilon + \lambda_k^{1D} + o(1). \quad (10)$$

Moreover, one can choose the 2D and 1D eigenbases $(\Psi_k)_k$ and ψ_k in such a way that

$$\int_{\mathbb{R}^{2N}} \left| \Psi_k - \psi_k(x_1, \dots, x_N) \prod_{j=1}^N u_\varepsilon(y_j) \right|^2 \xrightarrow{\varepsilon \rightarrow 0} 0 \quad (11)$$

Although it seems from (11) that a standard decoupling between the two space directions takes place, the actual ansätze for the eigenfunctions Ψ_k leading to the correct energy are more subtle. Essentially they are of the form

$$\psi_k(x_1, \dots, x_N) \prod_{j=1}^N u_\varepsilon(y_j) \prod_{j < k} e^{-\alpha i S(x_j - x_k)} \quad (12)$$

where we denote

$$S(\mathbf{x}) = S(x, y) = \frac{y}{x}. \quad (13)$$

The above trial states have the correct bosonic symmetry because $S(\mathbf{x}) = S(-\mathbf{x})$, but they are not of the form “function of x times function of y ” that is more common in dimensional reductions. The phase factors $e^{-i\alpha S(x_j - x_k)}$ modify the energy dramatically, almost gauging away the original magnetic interaction (see below). For this effect it is crucial to take advantage of the finite, albeit small, extension of our wave-guide, as shown by the discussion below.

A case for the Calogero-Sutherland model

Before we sketch the proof of the above, it is instructive to examine an argument that would rather point in the direction of the Calogero model with inverse square interactions (which is also a proposed model for 1D anyons) as effective description. This will emphasize two things:

• That quantization and dimensional reduction do not commute in this particular case. Classical particles with the above magnetic interactions would experience Calogero like interactions if confined on a line.

• The role of the phase factors $e^{-i\alpha S(\mathbf{x}_j - \mathbf{x}_k)}$ in the main result. Indeed, if one chooses a simpler ansatz of the form

$$\psi_k = \phi(x_1, \dots, x_N) \prod_{j=1}^N u_\epsilon(y_j),$$

the 1D function ϕ indeed experiences a Calogero-type Hamiltonian.

The possible connection between 2D anyons and Calogero-like models have already been point out in a similar context [35, 53]. It also arises for lowest Landau level anyons [8, 23, 45] via a very different mechanism.

Consider N classical particles with magnetic interactions akin to those of (4). We constrain them to move on the line $y = 0$, like pearls on a necklace. Expanding the square in (4), the Hamilton function for this system is

$$\begin{aligned} H(\mathbf{x}_1, \dots, \mathbf{x}_N; \mathbf{p}_1, \dots, \mathbf{p}_N) &= \sum_{j=1}^N \left(|\mathbf{p}_j|^2 + x_j^2 \right) \\ &+ 2\alpha \sum_{j \neq k} \mathbf{p}_j \cdot \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2} \\ &+ \alpha^2 \sum_{j \neq k \neq \ell} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2} \cdot \frac{(\mathbf{x}_j - \mathbf{x}_\ell)^\perp}{|\mathbf{x}_j - \mathbf{x}_\ell|^2} \\ &+ \alpha^2 \sum_{j \neq k} \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|^2} \end{aligned}$$

where $\mathbf{p}_j = (p_j, 0)$ and $\mathbf{x}_j = (x_j, 0)$ are momenta and positions, respectively. The cross-term on the second line is clearly null. The term on the third line is null as well, as follows from grouping terms as in [24, Lemma 3.2]

$$\sum_{\text{cyclic in } 1,2,3} \frac{(\mathbf{x}_1 - \mathbf{x}_2)^\perp}{|\mathbf{x}_1 - \mathbf{x}_2|^2} \cdot \frac{(\mathbf{x}_1 - \mathbf{x}_3)^\perp}{|\mathbf{x}_1 - \mathbf{x}_3|^2} = \frac{1}{2\mathcal{R}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)^2}$$

with $\mathcal{R}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ the circumradius of the triangle with summits $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$. This is the radius of the circle on which the three points lie, which is infinite for aligned points. Hence the Hamilton function boils down to

$$\sum_{j=1}^N \left(p_j^2 + x_j^2 \right) + \alpha^2 \sum_{j \neq k} \frac{1}{(x_j - x_k)^2}$$

which, once quantized, gives a Calogero Hamiltonian, albeit not with the expected $\alpha(\alpha - 1)$ coefficient [8, 33, 46] in front of the two-body term for particles of statistics parameter α . Note that this reduction could in any case not be correct for all α because the 2D anyon energy is periodic in α , but the Calogero energy is not [8, 9, 49, 50].

Outline of the proof

We turn to sketching the correct argument, i.e. give the main insights of the proof of Theorem 1. Turning them into a rigorous mathematical proof is somewhat lengthy, and will be done elsewhere.

The crucial observation is that for particles close to the line $y = 0$, the vector potential of the Aharonov-Bohm fluxes in (4) can almost be gauged away. The vector potential

$$\mathbf{A}_0(\mathbf{x}) = \frac{\mathbf{x}^\perp}{|\mathbf{x}|^2} = \begin{pmatrix} -y \\ x \end{pmatrix} \frac{1}{x^2 + y^2}$$

for a unit Aharonov-Bohm flux at the origin has a non-zero curl and thus cannot be written as a gradient globally. But, with S defined as in (13),

$$\nabla S(\mathbf{x}) = \begin{pmatrix} -y/x^2 \\ 1/x \end{pmatrix} \Big|_{|y| \ll |x|} \simeq \mathbf{A}_0(\mathbf{x})$$

and more precisely

$$|\nabla S - \mathbf{A}_0| \leq C \frac{|y|}{x^2}. \quad (14)$$

The singularity around $x = 0$ of the right-hand side has to be tamed, as we explain below. We ignore this for the moment.

Consider a trial state Ψ for (4) and write it as

$$\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N) = U_\epsilon \prod_{j < k} e^{-aiS(\mathbf{x}_j - \mathbf{x}_k)} \Phi \quad (15)$$

$$U_\epsilon(\mathbf{x}_1, \dots, \mathbf{x}_N) = \prod_{j=1}^N u_\epsilon(y_j)$$

with a new unknown function Φ . A direct calculation yields

$$\begin{aligned} \langle \Psi | H_\epsilon^{2D} | \Psi \rangle &= N e_\epsilon + \sum_j \int_{\mathbb{R}^{2N}} U_\epsilon^2 x_j^2 |\Phi|^2 \\ &+ \sum_j \int_{\mathbb{R}^{2N}} U_\epsilon^2 \left| \left(-i\nabla_{\mathbf{x}_j} + \alpha \sum_{k \neq j} (\mathbf{A}_0 - \nabla S)(\mathbf{x}_j - \mathbf{x}_k) \right) \Phi \right|^2 \end{aligned} \quad (16)$$

and we now seek critical points of this functional of Φ . Expanding a square and using a Cauchy-Schwartz inequality we obtain

$$\int_{\mathbb{R}^{2N}} U_\epsilon^2 \left| \left(-i\nabla_{\mathbf{x}_j} + \alpha \sum_{k \neq j} (\mathbf{A}_0 - \nabla S)(\mathbf{x}_j - \mathbf{x}_k) \right) \Phi \right|^2 = \int_{\mathbb{R}^{2N}} U_\epsilon^2 |\nabla_{\mathbf{x}_j} \Phi|^2 + \text{Error}$$

with

$$\begin{aligned} |\text{Error}| &\lesssim \alpha^2 \int_{\mathbb{R}^{2N}} U_\epsilon^2 \left| \sum_{k \neq j} (\mathbf{A}_0 - \nabla S)(\mathbf{x}_j - \mathbf{x}_k) \right|^2 |\Phi|^2. \end{aligned} \quad (17)$$

In view of (14), we will conclude that

$$\begin{aligned} \langle \Psi | H_\varepsilon^{2D} | \Psi \rangle &\simeq N e_\varepsilon + \sum_j^N \int_{\mathbb{R}^{2N}} U_\varepsilon^2 x_j^2 |\Phi|^2 \\ &+ \sum_j^N \int_{\mathbb{R}^{2N}} U_\varepsilon^2 |\nabla_{\mathbf{x}_j} \Phi|^2 \quad (18) \end{aligned}$$

if Φ is sufficiently concentrated around $y_1 = \dots = y_N = 0$ and vanishes sufficiently fast when $x_j \sim x_k$ for $j \neq k$. From (18), the conclusion of the theorem follows by a standard dimensional reduction. It will clearly be favorable for Φ not to depend on y_1, \dots, y_N and we then recognize the energy functional corresponding to (8).

The heart of the proof is then variational. To construct trial states giving the correct energies we would like to insert $\Phi \equiv \psi_k$ in (15). The concentration around $y = 0$ is obvious. However this choice does not lead to a sufficiently fast decay around $x_j \sim x_k$ for ψ_k vanishes only proportionally to $|x_j - x_k|$ (fermionic behavior). Combined with (14) and inserting (17) leads to non-integrable singularities $\propto |x_j - x_k|^{-2}$. Fortunately, ψ_k still vanishes fast enough for a slight regularization to be sufficient. We can modify it around particle encounters so that the singularities become integrable without modifying too much the energy.

We then need to prove a lower bound of the correct form for the energy of a true eigenstate of the 2D model. The strong singularity of (14) at particle encounters is not so much an issue here: we can exclude a thin neighborhood of the problematic set before extracting the phase factors as in (16), and pass to the limit where the removed neighborhood shrinks in measure after the limit $\varepsilon \rightarrow 0$. We still need to ensure some form of vanishing around particle encounters to ensure that our limit model is indeed set on (9). This comes about via an adapted use of the following Hardy-like inequality: For any Ψ , with the modified momentum as in (2)

$$\begin{aligned} \sum_{j=1}^N \left\langle \Psi | \left(D_{\mathbf{x}_j} \right)^2 | \Psi \right\rangle &\geq \\ c_{\alpha,N} \int_{\mathbb{R}^{2N}} \sum_{j < k} \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|^2} |\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N)|^2 &\quad (19) \end{aligned}$$

where the best possible constant $c_{\alpha,N}$ depends only on α and N . Such an inequality originates from [30, 39, 40] where it is proved in particular that

- $c_{\alpha,N} \geq cN^{-1}$ with a universal $c > 0$ if α is an odd-numerator fraction.
- $c_{\alpha,2} > 0$ for any $\alpha \neq 0$.

We improve these results by proving that there exists a universal constant $c' > 0$ such that

$$c_{\alpha,N} \geq c' N^{-2} \text{ for any } \alpha \neq 0,$$

which leads to our main result by providing the desired vanishing in the whole parameter range. One can understand heuristically why by considering the contribution of the set where $|x_j - x_k| \lesssim \sqrt{\varepsilon}$ and $|y_\ell| \lesssim \sqrt{\varepsilon}$ for $\ell = 1 \dots N$ to the right-hand side of (19). One has $|\mathbf{x}_j - \mathbf{x}_k|^2 \lesssim \varepsilon$ on this set. If the limiting 1D function does not vanish for $x_j \sim x_k$, then $|\Psi|^2 \propto U_\varepsilon^2 \propto \varepsilon^{-N/2}$. The total contribution (volume times typical value of the integrand) would thus be of order $\varepsilon^{-1/2}$, much larger than the expected energies, of order unity after removal of the contribution of $N e_\varepsilon$ as in (10). Hence the energy bounds force the limiting 1D function to vanish upon particle encounters

Conclusions

We have studied 2D anyons of statistics parameter α in the magnetic gauge picture, i.e. seeing them as bosons with magnetic interactions. We imposed a dimensional reduction by ways of a strongly anisotropic trap.

In the 1D limit we found that a suitable choice of gauge removes long-range magnetic interactions. Their only remnant is a hard-core condition upon particle encounters, leading to the Girardeau-Tonks model of 1D bosons for any $\alpha \neq 0$. Non-trivial dependence on α might survive at sub-leading order, in which case it could be determined by perturbation theory around Girardeau's solution of the impenetrable 1D Bose gas.

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- [1] R. ADAMI AND A. TETA, *On the Aharonov-Bohm effect*, Lett. Math.Phys, 43 (1998), pp. 43–53.
- [2] U. AGLIETTI, L. GRIGUOLO, R. JACKIW, S.-Y. PI, AND D. SEMINARA, *Anyons and chiral solitons on a line*, Phys. Rev. Lett., 77 (1996), pp. 4406–4409.
- [3] I. ANDRIĆ, V. BARDEK, AND L. JONKE, *Quantum fluctuations of the chern-simons theory and dynamical dimensional reduction*, Phys. Rev. D, 59 (1999), p. 107702.
- [4] S. AROVAS, J. SCHRIEFFER, AND F. WILCZEK, *Fractional statistics and the quantum Hall effect*, Phys. Rev. Lett., 53 (1984), pp. 722–723.
- [5] H. BARTOLOMEL, M. KUMAR, R. BISOGNIN, A. MARGUERITE, J.-M. BERROIR, E. BOCQUILLON, B. PLAÇAIS, A. CAVANNA, Q. DONG, U. GENNSER, Y. JIN, AND G. FÈVE, *Fractional statistics in anyon collisions*, Science, 368 (2020), pp. 173–177.
- [6] M. T. BATCHELOR, X.-W. GUAN, AND N. OELKERS, *One-dimensional interacting anyon gas: Low-energy properties and haldane exclusion statistics*, Phys. Rev. Lett., 96 (2006), p. 210402.

- [7] S. J. BENETTON RABELLO, *1d generalized statistics gas: A gauge theory approach*, Phys. Rev. Lett., 76 (1996), pp. 4007–4009.
- [8] L. BRINK, T. HANSSON, AND M. VASILIEV, *Explicit solution to the n -body calogero problem*, Physics Letters B, 286 (1992), pp. 109–111.
- [9] F. CALOGERO, *Solution of the one-dimensional N -body problems with quadratic and/or inversely quadratic pair potentials*, J. Mathematical Phys., 12 (1971), pp. 419–436.
- [10] L. W. CLARK, B. M. ANDERSON, L. FENG, A. GAI, K. LEVIN, AND C. CHIN, *Observation of density-dependent gauge fields in a bose-einstein condensate based on micromotion control in a shaken two-dimensional lattice*, Phys. Rev. Lett., 121 (2018), p. 030402.
- [11] M. CORREGGI, R. DUBOSCQ, N. ROUGERIE, AND D. LUNDHOLM, *Vortex patterns in the almost-bosonic anyon gas*, Europhysics Letters, 126 (2019), p. 20005.
- [12] M. CORREGGI AND D. FERMI, *Magnetic perturbations of anyonic and Aharonov-Bohm Schrödinger operators*, J. Math. Phys., 62 (2021), p. 032101.
- [13] M. CORREGGI AND L. ODDIS, *Hamiltonians for two-anyon systems*, Rend. Mat. Appl., 39 (2018), pp. 277–292.
- [14] M. J. EDMONDS, M. VALIENTE, G. JUZELIŪNAS, L. SANTOS, AND P. ÖHBERG, *Simulating an interacting gauge theory with ultracold bose gases*, Phys. Rev. Lett., 110 (2013), p. 085301.
- [15] S. FORTE, *Quantum mechanics and field theory with fractional spin and statistics*, Rev. Mod. Phys., 64 (1992), pp. 193–236.
- [16] J. FRÖHLICH, *Statistics of fields, the Yang-Baxter equation, and the theory of knots and links*, in Nonperturbative Quantum Field Theory, G. 't Hooft et al., ed., Les Houches - Ecole d'Été de Physique Théorique, 1988, pp. 71–100.
- [17] ———, *Quantum statistics and locality*, in Proceedings of the Gibbs Symposium, Les Houches - Ecole d'Été de Physique Théorique, 1990, pp. 89–142.
- [18] M. GIRARDEAU, *Relationship between systems of impenetrable bosons and fermions in one dimension*, J. Mathematical Phys., 1 (1960), pp. 516–523.
- [19] M. D. GIRARDEAU, *Anyon-fermion mapping and applications to ultracold gases in tight waveguides*, Phys. Rev. Lett., 97 (2006), p. 100402.
- [20] M. O. GOERBIG, *Quantum Hall effects*. arXiv:0909.1998, 2009.
- [21] G. GOLDIN, R. MENIKOFF, AND D. SHARP, *Representations of a local current algebra in nonsimply connected space and the Aharonov-Bohm effect*, J. Math. Phys., 22 (1981), p. 1664.
- [22] B. I. HALPERIN, *Statistics of quasiparticles and the hierarchy of fractional quantized Hall states*, Phys. Rev. Lett., 52 (1984), pp. 1583–1586.
- [23] T. HANSSON, J. LEINAAS, AND J. MYRHEIM, *Dimensional reduction in anyon systems*, Nuclear Physics B, 384 (1992), pp. 559–580.
- [24] M. HOFFMANN-OSTENHOF, T. HOFFMANN-OSTENHOF, A. LAPTEV, AND J. TIDBLUM, *Many-particle Hardy Inequalities*, J. London Math. Soc., 77 (2008), pp. 99–114.
- [25] R. IENGO AND K. LECHNER, *Anyon quantum mechanics and Chern-Simons theory*, Phys. Rep., 213 (1992), pp. 179–269.
- [26] J. K. JAIN, *Composite fermions*, Cambridge University Press, 2007.
- [27] A. KHARE, *Fractional Statistics and Quantum Theory*, World Scientific, Singapore, 2nd ed., 2005.
- [28] A. KUNDU, *Exact solution of double δ function Bose gas through an interacting anyon gas*, Phys. Rev. Lett., 83 (1999), pp. 1275–1278.
- [29] G. LAMBERT, D. LUNDHOLM, AND N. ROUGERIE, *On quantum statistics transmutation via magnetic flux attachment*. arXiv:2201.03518, 2020.
- [30] S. LARSON AND D. LUNDHOLM, *Exclusion bounds for extended anyons*, Archive for Rational Mechanics and Analysis, 227 (2018), pp. 309–365.
- [31] R. B. LAUGHLIN, *Nobel lecture: Fractional quantization*, Rev. Mod. Phys., 71 (1999), pp. 863–874.
- [32] J. M. LEINAAS AND J. MYRHEIM, *On the theory of identical particles*, Nuovo Cimento B Serie, 37 (1977), pp. 1–23.
- [33] J. M. LEINAAS AND J. MYRHEIM, *Intermediate statistics for vortices in superfluid films*, Phys. Rev. B, 37 (1988), pp. 9286–9291.
- [34] A. LERDA, *Anyons*, Springer-Verlag, Berlin–Heidelberg, 1992.
- [35] S. LI AND R. K. BHADURI, *Mapping of the Sutherland hamiltonian to anyons on a ring*. arXiv:cond-mat/9404068, 1994.
- [36] E. H. LIEB, *Exact analysis of an interacting Bose gas. II. The excitation spectrum*, Phys. Rev. (2), 130 (1963), pp. 1616–1624.
- [37] E. H. LIEB AND W. LINIGER, *Exact analysis of an interacting Bose gas. I. The general solution and the ground state*, Phys. Rev. (2), 130 (1963), pp. 1605–1616.
- [38] D. LUNDHOLM AND N. ROUGERIE, *Emergence of fractional statistics for tracer particles in a Laughlin liquid*, Phys. Rev. Lett., 116 (2016), p. 170401.
- [39] D. LUNDHOLM AND J. P. SOLOVEJ, *Hardy and Lieb-Thirring inequalities for anyons*, Comm. Math. Phys., 322 (2013), pp. 883–908.
- [40] ———, *Local exclusion principle for identical particles obeying intermediate and fractional statistics*, Phys. Rev. A, 88 (2013), p. 062106.
- [41] ———, *Local exclusion and Lieb-Thirring inequalities for intermediate and fractional statistics*, Ann. Henri Poincaré, 15 (2014), pp. 1061–1107.
- [42] A. MINGUZZI AND P. VIGNOLO, *Strongly interacting trapped one-dimensional quantum gases: Exact solution*, AVS Quantum Sci., 4 (2022), p. 027102.
- [43] J. MYRHEIM, *Anyons*, in Topological aspects of low dimensional systems, A. Comtet, T. Jolicœur, S. Ouvry, and F. David, eds., vol. 69 of Les Houches - Ecole d'Été de Physique Théorique, 1999, pp. 265–413.
- [44] J. NAKAMURA, S. LIANG, G. C. GARDNER, AND M. J. MANFRA, *Direct observation of anyonic braiding statistics*, Nature Physics, 16 (2020), pp. 931–936.
- [45] S. OUVRY, *Anyons and lowest Landau level anyons*, Séminaire Poincaré, 11 (2007), pp. 77–107.
- [46] A. POLYCHRONAKOS, *Non-relativistic bosonization and fractional statistics*, Nuclear Physics B, 324 (1989), pp. 597–622.
- [47] T. POSSKE, B. TRAUZETTEL, AND M. THORWART, *Second quantization of leinaas-myrheim anyons in one dimension and their relation to the lieb-liniger model*, Phys. Rev. B, 96 (2017), p. 195422.
- [48] N. ROUGERIE, *Some contributions to many-body quantum mathematics*. arXiv:1607.03833, 2016. habilitation thesis.
- [49] B. SUTHERLAND, *Quantum Many-Body Problem in One Dimension: Ground State*, J. Mathematical Phys., 12 (1971), pp. 246–250.
- [50] B. SUTHERLAND, *Quantum many-body problem in one dimension: Thermodynamics*, Journal of Mathematical Physics, 12 (1971), pp. 251–256.
- [51] G. TANG, S. EGGERT, AND A. PELSTER, *Ground-state properties of anyons in a one-dimensional lattice*, New Journal of Physics, 17 (2015), p. 123016.
- [52] G. VALENTÍ-ROJAS, N. WESTERBERG, AND P. ÖHBERG, *Synthetic flux attachment*, Phys. Rev. Research, 2 (2020), p. 033453.
- [53] R. VATHSAN, *Reduction of anyons to one dimension and Calogero-Sutherland-type models*, International Journal of

- Modern Physics A, 13 (1998), pp. 4123–4132.
- [54] F. WILCZEK, *Magnetic flux, angular momentum, and statistics*, Phys. Rev. Lett., 48 (1982), p. 1144.
- [55] ———, *Fractional Statistics and Anyon Superconductivity*, World Scientific, Singapore, 1990.
- [56] E. YAKABOYLU, A. GHAZARYAN, D. LUNDHOLM, N. ROUGERIE, M. LEMESHKO, AND R. SEIRINGER, *Quantum impurity model for anyons*, Phys. Rev. B, 102 (2020), p. 144109.
- [57] E. YAKABOYLU AND M. LEMESHKO, *Anyonic statistics of quantum impurities in two dimensions*, Phys. Rev. B, 98 (2018), p. 045402.