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DIOPHANTINE SETS AND DIRICHLET IMPROVABILITY

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Abstract. This note pushes further the discussion about relations between Dirichlet improvable, badly approximable and singular points held in [1] by considering Diophantine sets extending the notion of badly approximability.

1. Introduction

The aim of this paper is to extend slightly the main result of [1], questioning relations between Dirichlet improvability, badly approximability and singularity. We first provide a short description of this setting and definitions.

For \( x \in \mathbb{R} \), we denote \( \langle x \rangle := \min \{ |x - z| \mid z \in \mathbb{Z} \} \) the distance from \( x \) to a nearest integer.

For \( x, y \in \mathbb{R}^n \), we denote \( x \cdot y := x_1y_1 + \cdots + x_ny_n \) the usual scalar product.

The celebrated Dirichlet’s Theorem, root of Diophantine approximation, reads as follows.

**Theorem 1.1** (Dirichlet, 1842). Let \( x \in \mathbb{R}^n \). For every \( Q > 1 \) there exists \( q \in \mathbb{Z}^n \) such that

\[
0 < |q| \leq Q \quad \text{and} \quad \langle x \cdot q \rangle \leq Q^{-n}.
\]

A main interest in Diophantine approximation is to question when and how Dirichlet’s theorem can be improved. This lead Davenport and Schmidt [6] to call a given \( x \in \mathbb{R}^n \) Dirichlet improvable if there exists \( \varepsilon \in (0, 1) \) such that for all sufficiently large \( Q \), there exists \( q \) with

\[
0 < |q| \leq Q \quad \text{and} \quad \langle x \cdot q \rangle \leq \varepsilon Q^{-n}.
\]

We denote by \( \text{DI}_n(\varepsilon) \) the set of \( x \) satisfying (DI), so that the set of Dirichlet improvable numbers is

\[
\text{DI}_n = \bigcup_{\varepsilon \in (0, 1)} \text{DI}_n(\varepsilon).
\]

Furthermore, \( x \in \mathbb{R}^n \) is called singular if it is in \( \text{DI}_n(\varepsilon) \) for \( \varepsilon \) arbitrarily small. Denote

\[
\text{Sing}_n := \bigcap_{\varepsilon \in (0, 1)} \text{DI}_n(\varepsilon).
\]

On the opposite, we say that \( x \in \mathbb{R}^n \) is badly approximable if there exists \( \varepsilon = \varepsilon(x) \in (0, 1) \) such that

\[
\langle q \cdot x \rangle \geq \varepsilon |q|^{-n}
\]

for all \( q \in \mathbb{R}^n \). We denote the set of badly approximable numbers \( \text{Bad}_n \).

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A very natural question is the size and relation between these sets. Regarding sizes, it is well known that \( \text{DI}_n \), \( \text{Bad}_n \) and \( \text{Sing}_n \) have 0 Lebesgue measure. The set \( \text{Bad}_n \) has full Hausdorff dimension, hence \( \text{DI}_n \) as well:
\[
\dim_H(\text{Bad}_n) = \dim_H(\text{DI}_n) = n.
\]
For \( \text{Sing}_n \), the study of the Hausdorff dimension is much more involved. The result requires the powerful variational principle in parametric geometry of numbers by Das, Fishman, Simmons and Urbański [7], extending to a wider setting (including ours: approximation to a linear form) a brilliant result by Cheung and Chevallier [4] for simultaneous approximation.
\[
\dim_H(\text{Sing}_n) = \frac{n^2}{n + 1}, \quad \text{for } n \geq 2.
\]
Regarding relations, we have the inclusion of the disjoint union \( \text{Bad}_n \sqcup \text{Sing}_n \subset \text{DI}_n \).

When \( n = 1 \), singular reals are the rationals and the last inclusion is an equality. In higher dimension, one can ask about the set
\[
\mathcal{FS}_n := \text{DI}_n \setminus (\text{Bad}_n \sqcup \text{Sing}_n).
\]
This set is studied in [1], where it is proved to be uncountable. However, it is probably far from the truth and one conjectures

**Conjecture 1.2.** \( \dim_H(\mathcal{FS}_n) = n \).

Actually, in [1] the result is finer than just uncountability. This requires the definition of exponents of Diophantine approximation. Namely, for \( x \in \mathbb{R}^m \), we define \( \omega(x) \) as the supremum of positive reals \( w \) such that
\[
0 < |q| \leq Q, \quad \langle q \cdot x \rangle \leq Q^{-w}
\]
has integer solution \( q \) for arbitrarily large \( Q \).

This exponent is usually referred to as exponent of Diophantine approximation to a linear form. It can take any value in the interval \([n, \infty] \). The main result in [1] asserts that there are uncountably many \( x \in \mathbb{R}^n \) in
\[
(\text{DI}_n(\varepsilon) \setminus \text{DI}_n(c_n\varepsilon)) \setminus (\text{Bad}_n \sqcup \text{Sing}_n)
\]
with prescribed exponent \( \omega(x) \in [n, \infty] \). Here \( c_n \) is an explicit constant. The exclusion of \( \text{Sing}_n \) is already included in the exclusion of \( \text{DI}_n(c_n\varepsilon) \), but we express this set to enlighten its link to \( \mathcal{FS}_n \).

The motivation of this paper is the obvious observation that \( x \notin \text{Bad}_n \) if \( \omega(x) > n \). This leads to consider the following notion: fix dimension \( n \) and let \( \varepsilon \in (0, 1) \) and finite \( w \geq n \). We define the \((\varepsilon, w)\)-Diophantine set
\[
\mathcal{D}_w(\varepsilon) := \{x \in \mathbb{R}^n \mid \langle q \cdot x \rangle \geq \varepsilon |q|^{-w} > 0, \quad \text{for all } q \in \mathbb{N}^n\}
\]
and denote \( \mathcal{D}_w := \bigcup_{\varepsilon \in (0, 1)} \mathcal{D}_w(\varepsilon) \). One can see that \( \mathcal{D}_n = \text{Bad}_n \).

These Diophantine sets play a role in dynamical systems. For example they describe the Diophantine condition in small divisors problems with applications to KAM theory, Aubry-Mather theory, conjugation of circle diffeomorphisms, and so on (see [2, 5, 10, 13, 20]).

Our main theorem reads as follows.

**Theorem 1.3.** Fix the dimension \( n \geq 2 \), a finite exponent \( w > n \), parameters \( \varepsilon, \nu \in (0, 1) \) and the constants \( c_n = e^{-20(n+1)^5(n+10)} \) and \( c_n' = e^{-20(n+1)^4(w+1)(n+10)} \). Then
- there exists uncountably many
  \[
  x \in (\text{DI}_n(\varepsilon) \setminus \text{DI}_n(c_n\varepsilon)) \setminus \mathcal{D}_w
  \]
  with \( \omega(x) = w \),
there exists uncountably many
\[ x \in (\text{DI}_n(\varepsilon) \setminus \text{DI}_n(c_n \varepsilon)) \cap (\text{D}_w(\nu) \setminus \text{D}_w(c'_n \nu)). \]

As discussed in [1], the constants \( c_n \) and \( c'_n \) are not optimized.

So far, we considered only approximation to a linear form. An analogous argumentation applies for simultaneous approximation, where we replace \( \text{DI} \) by
\[ 0 < q < Q \text{ and } \|qx\| \leq \varepsilon Q^{-1/m} \]
and \( \text{D}_w(\varepsilon) \) by
\[ \text{D}_\lambda(\varepsilon) := \{ x \in \mathbb{R}^m \mid \|qx\| \geq \varepsilon |q|^{-\lambda} > 0 \} \]
where \( \| \cdot \| \) is the distance to a nearest integer point. We get uncountably many points \( x \in \mathbb{R}^m \) in
\[ x \in (\text{DI}_n(\varepsilon) \setminus \text{DI}_n(c_n \varepsilon)) \setminus \text{D}_\lambda \]
with exponent of simultaneous approximation \( \lambda \) and uncountably many \( x \in \mathbb{R}^m \) in the intersection
\[ x \in (\text{DI}_n(\varepsilon) \setminus \text{DI}_n(c_n \varepsilon)) \cap (\text{D}_\lambda(\nu) \setminus \text{D}_\lambda(c'_n \nu)). \]

One could also consider approximation by rational subspaces of dimension exactly \( d \), for \( 0 \leq d < n \), as introduced by Laurent [9] following Schmidt [14]. However, their usual study via compounds convex bodies involves constants depending on \( x \) (See [12, Proposition 3.1] or [3, §4]) that seem to break our proof. See [1] for more discussion about the sets \( \text{DI}_n, \text{Bad}_n \) and \( \text{Sing}_n \) related to these approximation settings.

The proof of Theorem 1.3 relies on the parametric geometry of numbers and Roy’s fundamental theorem [11]. We provide a short introduction to parametric geometry of numbers in Section 2, and prove our main theorem in Section 3.

### 2. Parametric geometry of numbers

Parametric geometry of numbers was developed by Schmidt and Summerer [18, 19], answering a question of Schmidt [15]. It was pushed by a fundamental theorem of Roy [11], that was quantified and extended to a matrix setting by Das, Fishman, Simmons and Urbański [7].

Fix dimension \( n > 1 \) and \( x \in \mathbb{R}^n \). Consider the convex body with parameter \( q \geq 0 \)
\[ C_x(q) := \{ y \in \mathbb{R}^{n+1} \mid |y_1| \leq 1, |y \cdot (1, x)| \leq e^{-q} \} \]
and its \( d \)th successive minima
\[ \lambda_{x,d}(q) := \lambda_d(\mathbb{Z}^{n+1}, C_x(q)) \]
for any \( d \) between 1 and \( n+1 \). Following Schmidt and Summerer, we consider the successive minima map consisting of their logs
\[ L_x : [0, \infty) \to \mathbb{R}^{n+1} : q \mapsto L_x(q) := (L_{x,1}(q), \ldots, L_{x,n+1}(q)), \]
where \( L_{x,d}(q) := \log \lambda_{x,d}(q) \). It appears that the map \( L_x \) encodes Diophantine properties of \( x \).

**Proposition 2.1.** Fix dimension \( n \) and \( x \in \mathbb{R}^n \).
\[ x \in \text{DI}_n(\varepsilon) \text{ if and only if for all sufficiently large } q \]
\[ \frac{q}{n+1} - L_{x,1}(q) \geq \frac{\log(\varepsilon)}{n+1} \tag{2.1} \]
Continuous linear map \( q \) particular, for each 

\[
\liminf_{q \to \infty} \frac{L_{x,1}(q)}{q} = \frac{1}{1+\omega(x)}
\]


The following notion of system was introduced by Roy [12] Definition 4.5. The latter approach exactly the family of successive minima maps.

**Definition 2.2.** Be \( I \) a subinterval of \([0, \infty)\] with nonempty interior. A system on \( I \) is a continuous linear map \( P = (P_1, \ldots, P_{n+1}): I \to \mathbb{R}^{n+1} \) with the following properties.

(i) For each \( q \in I \), \( 0 \leq P_1(q) \leq P_2(q) \leq \cdots \leq P_{n+1}(q) \) and \( P_1(q) + P_2(q) + \cdots + P_{n+1}(q) = q \).

(ii) If \( I' \subset I \) is a nonempty open subinterval on which \( P \) is differentiable, then there exists integers \( r_1, r_2 \) with \( 1 \leq r_1 \leq r_2 \leq n+1 \) such that \( P_{r_1}, \ldots, P_{r_2} \) coincide on \( I' \) and have slope \( \frac{1}{r_2-r_1+1} \) while all other component \( P_i \) are constant on \( I' \).

(iii) If \( q \) is an interior point of \( I \) at which \( P \) is not differentiable, and if \( r_1, r_2, s_1, s_2 \) are integers such that 

\[
P_i'(q^-) = \frac{1}{r_2 - r_1 + 1} \quad (r_1 \leq i \leq r_2)
\]

and if \( r_1 \leq s_2 \) then \( P_{r_1}(q) = \cdots = P_{s_2}(q) \).


**Theorem 2.3** (Roy, 2015). Fix dimension \( n \geq 1 \) and \( q_0 > 0 \). For each \( x \in \mathbb{R}^n \), there exists a system \( P: [q_0, \infty) \to \mathbb{R}^{n+1} \) such that \( L_x - P \) is bounded on \([q_0, \infty)\). Conversely, for each system \( P: [q_0, \infty) \to \mathbb{R}^{n+1} \), there exists a \( x \in \mathbb{R}^n \) such that \( L_x - P \) is bounded. In particular, for each \( q \geq q_0 \) and a constant \( R_n \)

\[
|L_x - P| \leq R_n.
\]

The constant \( R_n \) induces the constants \( c_n = e^{4(n+1)R_n} \) and \( c'_n = e^{4(w+1)R_n} \) in Theorem 1.3.

We call two systems \( P_1 \) and \( P_2 \) non-equivalent if there exists \( q \) such that \( |P_1(q) - P_2(q)| > 2C_n \). No point in \( \mathbb{R}^n \) has successive minima map close to two non-equivalent systems in the sense of Theorem 2.3.

3. Proof of Theorem 1.3

Fix dimension \( n > 1 \), exponent \( w > n \) and \( \varepsilon, \nu \in (0, 1) \). In view of Theorem 2.3 and Proposition 2.1 the proof of Theorem 1.3 reduces to the construction of uncountably many non-equivalent systems satisfying

\[
\liminf_{q \to \infty} \frac{P_1(q)}{q} = \frac{1}{1+w},
\]

\[
\liminf_{q \to \infty} \left(\frac{q}{n+1} - P_1(q)\right) = -\frac{\log(\varepsilon)}{n+1} + 2R_n,
\]

Either \( \limsup_{q \to \infty} \left(\frac{q}{w+1} - P_1(q)\right) = -\frac{\log(\nu)}{w+1} + 2R_n \)

or \( \limsup_{q \to \infty} \left(\frac{q}{w+1} - P_1(q)\right) = \infty \).

Note that the second and third inequalities provide the definition of the constants \( c_n := e^{4(n+1)R_n} \) and \( c'_n = e^{4(w+1)R_n} \).
Choose a parameter $\delta \in (0, 1)$ that will provide uncountability. We first construct elementary systems $P_k^\delta$ on intervals $[q_k, q_{k+1}]$. Denote $\alpha := -\frac{\log(w)}{w+1} + 2R_n$ and $\beta_k := (-\frac{\log(w)}{w+1} + 2R_n)$ or $\log q_k$. Figure 1 illustrates the construction.

At $q_k$, we fix $P_1(q_k) = \cdots = P_n(q_k) = \frac{q_k}{n+1} - \alpha$ and $P_{n+1}(q_k) = \frac{q_k}{n+1} + n\alpha$.

Define

\[ p_k = q_k \frac{w+1}{n+1} - (w+1)(\alpha - \beta_k), \quad q_{k+1} = \frac{w}{n} q_k + \frac{(w-1)(n+1)}{n}(\alpha - \beta_k), \]

\[ r_k = q_k + (n^2-1)\alpha, \quad u_k = p_k - (n+1)\alpha, \]

\[ s_k^M = r_k + n \log q_k, \quad s_k^m = r_k + \log q_k, \]

\[ s_k = \delta s_k^m + (1-\delta)s_k^M \in [s_k^m, s_k^M], \quad t_k = s_k + (n-1)(s_k - r_k), \]

and note that $q_{k+1}$ does not depend on $\delta$.

On the interval $[q_k, r_k]$, the $n-1$ components $P_2 = \cdots = P_n$ coincide and have slope $\frac{1}{n-1}$ while $P_1$ and $P_{n+1}$ are constant. By definition of $r_k$, $P_2(r_k) = P_{n+1}(r_k)$.

On the interval $[r_k, s_k]$, $P_{n+1}$ has slope 1 while all other component $P_i$ are constant and on $[s_k, t_k]$ the $n-1$ components $P_2 = \cdots = P_n$ coincide and have slope $\frac{1}{n-1}$ while $P_1$ and $P_{n+1}$ are constant. By definition of $t_k$, $P_{n+1}(t_k) = P_2(t_k)$.

On the interval $[t_k, u_k]$, the $n$ components $P_2 = \cdots = P_{n+1}$ coincide and have slope $1/n$, while $P_1$ is constant.

On the interval $[u_k, p_k]$, $P_{n+1}$ has slope 1 while all other component $P_i$ are constant. By definition of $u_k$ and $p_k$, at $p_k$ we have $P_{n+1}(p_k) - P_n(p_k) = (n+1)\alpha$ and $P_1(p_k) = \frac{p_k}{w+1} - \beta_k$.

On the interval $[p_k, q_{k+1}]$, $P_1$ has slope 1 while all other components are constant. By definition of $q_{k+1}$ we have $P_1(q_{k+1}) = \cdots = P_n(q_{k+1}) = \frac{q_{k+1}}{n+1} - \alpha$ and $P_{n+1}(q_{k+1}) = \frac{q_{k+1}}{n+1} + n\alpha$.

\[ \text{Figure 1. Generic system } P_k^\delta, \text{ dotted are the extremal cases } \delta = 0 \text{ and } 1. \]

Choose $q_1$ large enough so that $t_1 < u_1$. Since $w > n$, the inductive sequence $(q_k)_{k \geq 1}$ tends to infinity and we define $P^\delta = \bigcup_{k \geq 1} P_k^\delta$ the concatenation of the systems $P_k^\delta$. It is a system, as properties (i) - (iii) holds, in particular at $q_k$. 
By construction,

$$\liminf_{q \to \infty} \frac{P_1^\delta(q)}{q} = \liminf_{k \to \infty} \frac{P_1^\delta(p_k)}{p_k} = \frac{1}{1 + w}$$

$$\min_{[q_k, q_{k+1}]} \frac{q}{n+1} - P_1^\delta(q) = \alpha = \frac{1}{n+1} \log(\varepsilon) + 2R_n$$

$$\limsup_{q \to \infty} \left( \frac{q}{w+1} - P_1^\delta(q) \right) = \limsup_{k \to \infty} \left( \frac{p_k}{w+1} - P_1^\delta(p_k) \right) = \limsup_{k \to \infty} \beta_k$$

and the requested properties hold. Furthermore, $$P_1^\delta(s_k^M) = P_{n+1}(q_k) + \delta \log q_k$$, so that

$$\|P^\delta - P^{\delta'}\| \geq |P_1^\delta(s_k^M) - P_1^{\delta'}(s_k^M)| = |\delta - \delta'| \log q_k.$$ 

Hence, for $$\delta \neq \delta'$$ the systems $$P^\delta$$ and $$P^{\delta'}$$ are non-equivalent. Uncountability follows. $\Box$

For simultaneous approximation, we rather use the setting and notation of Schmidt and Summerer for parametric geometry of numbers, studying the convex body

$$C^\prime_x(q) := \left\{ y \in \mathbb{R}^{m+1} \mid |y_1| \leq e^{mq}, \max_{1 \leq i \leq n} |y_ix_i - y_{i+1}| \leq e^{-q} \right\}$$

See for example [18, 19, 16, 17]. The analogous proof relies on the construction of dual systems depicted by Figure 2.
References


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